DOI 10.1007/s10598-017-9371-4

Computational Mathematics and Modeling, Vol. 28, No. 3, July, 2017

MATHEMATICAL MODELS OF INVESTMENT CYCLES

E. S. Kurkina

We propose and investigate three mathematical models describing production cycles. They incorporate various mechanisms of endogenous fluctuations in economic systems. The models are based on ODE systems. The first model is a Keynesian IS-LM model of business cycles. The interest rate is determined by the money market and influences the relationship between savings and investments, allowing funds to flow from one to the other and vice versa. In the second case, the fluctuation mechanism is associated with time lags between investment growth, capital growth, and rate of return on capital. As a result, the economy periodically "overheats", as rapid growth of capital suppresses the return rates, production becomes unprofitable, and investments sharply decline. Two models realizing this mechanism are proposed. One is a minimalist model based on a system of three ODEs. The other is an augmented model that sufficiently fully describes modern economic systems of developed countries and consists of nine ODEs and nine algebraic equations. It encompasses all the principal markets: labor market, capital market, financial market, and commodity market. Bifurcation analysis of the three models is carried out, oscillation regions are determined, and oscillation mechanisms are examined in detail. The model parameters are chosen so that the cycle periods are 12–17 years long.

Keywords: mathematical modeling, investment cycles, ordinary differential equations, bifurcation analysis, self-oscillations.

Introduction

Equilibrium is a prerequisite for stability and sustainable economic development in economic systems. However, in a market economy, despite attempts to reach equilibrium, the economic system undergoes cyclic oscillations about the potential GDP, when growth periods alternate with periods of decline and stagnation. McConnell and Brue identify the following phases of the business cycle: peak, recession, trough, and recovery.

It is commonly believed that the main reason for business cycles is mismatch between aggregate demand and aggregate supply, i.e., between aggregate expenditures and aggregate production, mismatch driven by the inertia of economic systems.

There are many different economic mechanisms that generate cyclic processes. Economic cycles may essentially differ in length, intensity, and scope. Some cycles concern a single enterprise or firm, other cycles cover a whole region, while yet others affect the entire world — the whole economic system. The best-known cycle types are listed in Table 1 [1].

The cycle generation mechanism is usually driven by time lags in the propagation of information relevant for decision making by economic agents and also in the implementation of these decisions, which may aim at contraction or conversely expansion of production.

Economic cycles superimpose on one another and interact with one another creating complex oscillatory processes in the economy. Different mechanisms producing different types of cyclic processes are often difficult to separate from the complex aperiodic oscillations of the system about the main economic development trend.

Faculty of Computational Mathematics and Cybernetics, Moscow State University, Moscow, Russia; e-mail: e.kurkina@rambler.ru.

Translated from Prikladnaya Matematika i Informatika, No. 53, 2016, pp. 67–92.

Table 1

Types of Economic Cycles

Type	Cycle length	Main features
Kitchin	2–4 years	Fluctuations of inventories, GNP, inflation, employment, demand
Juglar	7–12 years	Investment cycle → fluctuations of GNP, inflation, employment
Kuznets	16–25 years	Income → immigration→ residential construction, aggregate demand, income
Kondratieff	40–60 years	Global structural changes, technological progress

In this article we investigate industrial or business cycles associated with fluctuations of capital investment volumes and as a consequence with fluctuations in GNP, production, employment, commodity inventories, and so on. Such cycles with lengths of 7–12 years were originally discovered by the French economist C. Juglar in 1862. The time lag driving the fluctuations in these cycles is between the timing of investment decisions and the construction of corresponding production capacities, and also between construction and actual commissioning of production capacities. A further lag is observed between decline of demand and contraction of production volumes.

Kondratieff cycles are also classified as investment economic cycles, but unlike Juglar cycles they encompass all of global production, when aging technologies are periodically replaced with new, innovative technologies that determine the course of economic development for decades. Kondratieff cycles stretch for 45–60 years and are also referred to as long waves.

In this article, we propose and investigate three mathematical models of production cycles. They incorporate different mechanisms of endogenous fluctuations of the economic system. The models are based on ODE systems. The first model is the Keynesian IS-LM model of the business cycle. In this model, interest rates are formed in the money market and influence the relationship between savings and investments, with funds flowing from one to the other and vice versa.

In the second case, the fluctuation mechanism is associated with time lags between investment growth, capital growth, and rate of return on capital. As a result, the economy periodically "overheats", as rapid growth of capital suppresses the return rates, production becomes unprofitable, and investments sharply decline. Two models realizing this mechanism are proposed. One is a minimalist model based on a system of three ODEs. The other is an augmented model that sufficiently fully describes modern economic systems of developed countries and consists of nine ODEs and nine algebraic equations. It encompasses all the principal markets: labor market, capital market, financial market, commodity market.

Bifurcation analysis of the three models is carried out, oscillation regions are determined, and oscillation mechanisms are examined in detail. The model parameters are chosen so that the cycle periods are 12–17 years long.

1. IS-LM Business-Cycle Models

1. General Form of the Models. One of the most popular models describing the fluctuation of business investment cycles and demand for money is the IS-LM business-cycle model [1]. The theoretical foundation of this model was originally described by Keynes in his monumental *General Theory of Employment, Interest and Money* (1936), where Keynes uncovered the main functioning mechanisms of the economic macrosystem. In Chapter 22, "Notes on the Trade Cycle" [3], Keynes views "the trade cycle … as being occasioned by a cyclical change in the marginal efficiency of capital, though complicated and often aggravated by associated changes in the other significant short-period variables of the economic system." He also writes about the role of interest in the system's recovery from a crisis.

The IS-LM model remains relevant after all these years, and it has been further elaborated in the framework of the Keynesian theory and in a certain stage of the evolution of the monetarism. The recent crises have spurred a renewed interest in the traditional theory of the economic cycle driven by changes in macroeconomic demand. At the same time, serious doubts have been cast on the ability of neoclassical growth models to explain aggregate fluctuations of the economy. This has led to the neo-Keynesian paradigm as an alternative foundation for the understanding of the economic cycle. The main distinction between the classical and the neo-Keynesian paradigm is the presence of various nominal and real imbalances in the economic system in the neo-Keynesian paradigm.

These developments have spurred renewed interest in the Keynesian theory of the economic cycle, necessitating analysis of the dynamics of the IS-LM model with allowance for the concepts of economic synergetics, which can be applied to describe the emergence of new phenomena in complex nonlinear systems, such as endogenous self-oscillations. Numerous mathematical models have appeared, based on the Keynesian mechanism of the economic cycle. Many of these models are described by discrete finite-difference equations [5, 6, 9], others use delayed differential equations [7], and still others are based on ODE systems [2, 3].

Let us consider in more detail some issues associated with the functioning of the model described by an ODE system. In the initial mathematical formalization of the IL-LM model, the business cycle was described by a system of two ordinary differential equations for the variation of national income $Y(t)$ and interest $R(t)$ [2]:

$$\begin{cases} \dfrac{dY}{dt} = \alpha(I(Y,R) - S(Y,R)) = \alpha F(Y,R), \\[2ex] \dfrac{dR}{dt} = \beta(L(Y,R) - M), \end{cases} \tag{1}$$

where

$I(Y,R)$ is the investment demand function, which increases with the increase of national income $(dI/dY > 0)$ and decreases with the increase of interest $(dI/dR < 0)$;

$S(Y,R)$ is the savings function, which increases with the increase of both variables $(dS/dY > 0$, $dS/dR > 0)$;

$L(Y,R)$ is the aggregate money demand, which increases with the increase of national income $(dL/dY = L_Y > 0)$ and decreases with the increase of interest $(dL/dR = -L_R < 0)$;

M is the constant money supply;

α, $\beta > 0$ are parameters.

The model incorporates a mechanism that induces fluctuations. If the interest increases, the investment demand decreases and the savings volume increases. With the decrease in investments the growth of national income is arrested and switches to decline. The decrease of national income reduces the demand for money and the interest decreases. This in turn reduces the savings and increases investment demand. Investment growth raises the national income and the aggregate money demand. Money shortages begin to be felt, interest starts increasing, and its increase spurs savings — thus closing the cycle.

The existence of cycles in this model was first conjectured by Torre in 1977 [8]. He also discovered the conditions for the existence of oscillations.

System (1) has at least one positive stationary state (Y^*, R^*) describing a static equilibrium in the IS-LM model. The stability and the type of the stationary state (Y^*, R^*) determine the trace Sp_A and the determinant Δ_A of the Jacoby matrix A on the stationary solution. The equilibrium is stable if

$$Sp_A = a_{11} + a_{22} = \alpha F'_Y + \beta L'_R < 0, \tag{2}$$

$$\Delta_A = a_{11}a_{22} - a_{12}a_{21} = \alpha\beta(F'_Y L'_R - F'_R L'_Y) > 0. \tag{3}$$

Let α be the active parameter. We investigate the conditions for the appearance of an Andronov–Hopf bifurcation in which a limit cycle of born from a complex focus. The boundary of the stability region of the stationary state and the self-oscillation region is determined from the relationships

$$\begin{cases} Sp_A = 0, \\ \Delta_A > 0. \end{cases} \tag{4}$$

The bifurcation value of the parameter is obtained from (2), (3), and (4):

$$\alpha_0 = -\beta L'_R / F', \qquad F'_Y L'_R - F'_R L'_Y > 0, \qquad F'_Y > 0. \tag{5}$$

Conditions (5) are necessary, but not sufficient. For the appearance of self-oscillations when

$$\alpha > \alpha_0 \tag{6}$$

it is necessary that either the first Lyapunov exponent is negative $l_1 < 0$, or the equilibrium is unique.

System (1) has been investigated in detail by Keynes's followers in the neighborhood of the equilibrium for the case when the right-hand sides contain quadratic nonlinearities [2, 8]. The amplitude and the period of the oscillations have been determined and formulas have been derived for the stability of the limit cycle. Subsequently it has been proved that, with a quadratic nonlinearity, a maximum of three limit cycles may be born from a singular point of complex focus type.

2. The Base Model. In the economic literature, general systems of the type of the model (1) are typically considered in the neighborhood of an equilibrium. A bifurcation analysis of the system is carried out and conclusions are drawn concerning stability or instability of the stationary state, the conditions for the appearance of an Andronov–Hopf bifurcation and birth of stable oscillations, etc. The conditions for the existence of a specific regime are imposed on particular functions entering the model or their derivatives and are stated in a fairly general form, as in (5) [2]. Following these conditions, we will try to construct a model that describes periodic oscillations of interest and national income.

The investment function is the main driver of the model dynamics. At small values of the national income y the investment function is assumed to grow exponentially. As y increases, the growth of $I(y)$ slows down and for large y it reaches a stationary path. For the function $I(y)$ we take [2]

$$I(y) = \frac{\exp(y)}{1+\exp(y)}, \qquad I(0) = 0.5, \quad I(y) \xrightarrow[y\to\infty]{} 1. \tag{7}$$

The other functions in model (1) are linear. We thus have

$$I(y,r) = \frac{1}{1+\exp(-y)} - \beta_1 r, \tag{8}$$

$$S(y,r) = l_1 y + \beta_2 r, \tag{9}$$

$$L(y,r) = l_2 y - \beta_3 r, \tag{10}$$

where $\beta_i > 0$, $l_i > 0$, $i = \overline{1,3}$. The Keynes model now takes the following form:

$$\begin{cases} \dfrac{dy}{\alpha \cdot dt} = \dfrac{1}{1+\exp(-y)} - l_1 y - r(\beta_1+\beta_2), \\ \dfrac{dr}{\beta \cdot dt} = l_2 y - r\beta_3 - l_s, \\ y(t_0) = y^0, \quad r(t_0) = r^0; \quad y, r > 0. \end{cases} \tag{11}$$

Let $\beta_{12} = \beta_1 + \beta_2$. Without loss of generality, we may take $\beta_3 = 1$, $\beta = 1$.
The stationary states y^*, r^* are obtained from the relationships

$$r^* = l_2 y^* - l_s > 0, \qquad \frac{1}{1+\exp(-y^*)} = l_1 y^* + r^* \beta_{12} > 0. \tag{12}$$

To determine stability, we write out the trace and the determinant:

$$Sp_A = a_{11} + a_{22} = \alpha \frac{\exp(-y)}{(1+\exp(-y))^2} - \alpha l_1 - 1;$$

$$\Delta_A = a_{11}a_{22} - a_{12}a_{21} = \alpha\left(\frac{-\exp(-y)}{(1+\exp(-y))^2} + l_1 + \beta_{12} l_2 \right). \tag{13}$$

Self-oscillations may arise under the following conditions:

$$Sp_A > 0 \Rightarrow \alpha \frac{\exp(-y)}{(1+\exp(-y))^2} > \alpha l_1 + 1, \tag{14}$$

$$\Delta_A > 0 \Rightarrow \alpha \frac{\exp(-y)}{(1+\exp(-y))^2} < \alpha l_1 + \alpha l_2 \beta_{12}. \tag{15}$$

Comparing expressions (14) and (15), we obtain the necessary condition for the appearance of self-oscillations:

$$\alpha l_2 \beta_{12} > 1. \tag{16}$$

Model (11) is nonlinear and dissipative, and a limit cycle may exist in this model. Carrying out parametric analysis of the model and choosing the parameters by conditions (12), (14), and (15), we easily find the self-oscillations in this model, which occur around the unique unstable positive stationary state, for instance, for $l_2 = 2$, $l_s = 0.1$, $l_1 = 0.1$, $\beta_{12} = 1$, $\alpha = 10$. The self-oscillations arise as a result of the Andronov–Hopf bifurcation.

However, these oscillations do not describe a physical process, as they extend into the negative region. Attempts to find parameters for the model (11) for which the interest and the national income oscillate in the positive region have been unsuccessful.

3. Modified Keynes Model. Let us formalize the function $I(y)$. In model (11), $I(y)$ (8) is monotone increasing from 0.5 to 1 with the increase of y. Retaining the properties of the function $I(y)$, we introduce parameters:

$$I(y) = \frac{l_0 K}{l_0 + (K - l_0)\exp(-a_1 y)}. \tag{17}$$

Here K is the capital. The function $I(y)$ may vary from l_0 to K. The other functions in the model have the previous form, but in the new model they affect the relative changes in national income and interest $\dot{y}/y$, $\dot{r}/r$, which is quite natural. As a result, the second equation becomes nonlinear and the modified model takes the form

$$\begin{cases} \dfrac{dy}{\alpha\, dt} = \dfrac{l_0 K}{(l_0 + (K - l_0)\exp(-a_1 y))} - l_1 y - r y(\beta_1 + \beta_2) = f_1(y, r), \\[2ex] \dfrac{dr}{\beta\, dt} = r(l_2 y - r\beta_3 - l_s) = f_2(y, r), \quad y > 0, \quad r > 0. \end{cases} \tag{18}$$

Although model (18) is qualitative, it contains a sufficient number of parameters that can be tuned to fit a specific economic system. Let us carry out a parametric analysis of the model.

Stationary States. We will determine the conditions when system (18) has positive equilibria. The second equation in the system has two stationary solutions: $r = 0$ and $r = l_2 y - l_s$. The zero solution is irrelevant for our purposes. Substituting the second solution in the first equation, we obtain a transcendental equation for y^*:

$$\frac{l_0 K}{(l_0 + (K - l_0)\exp(-a_1 y))} = y(l_2 y - l_s)\beta_{12} . \tag{19}$$

Depending on the value of the parameters, Eq. (19) may have one, two, or three solutions, all of them positive. The parameters α and β do not affect the number of stationary points and their location on the phase plane. However, they do affect the stability of the stationary state. The stability of the stationary state is determined by the eigenvalues of the Jacoby matrix, which are obtained from the characteristic equation

$$\lambda^2 - Sp_A \lambda + \Delta_A = 0 .$$

A stationary state is stable if the real parts of the eigenvalues are negative:

$$\lambda_i < 0 , \quad \text{or} \quad \mathrm{Re}(\lambda_i) < 0 , \quad i = 1,2 .$$

Here also $Sp_A < 0$, $\Delta > 0$.

We take the following base set of parameters:

$$\begin{gathered} \beta_3 = 1 , \quad \beta = 7 , \quad l_1 = 0 , \quad l_2 = 0.35 , \quad l_s = 0.2 , \quad b_{12} = 2 , \\ l_0 = 0.005 , \quad K = 1 , \quad \alpha = 2 , \quad a_1 = 4 . \end{gathered} \tag{20}$$

Consider the dependence of the stationary solution of system (18) on one of the available parameters, keeping all other parameters fixed (we denote this stationary solution by p_1).

The boundary of the stability region of the stationary solution is either

$$Sp_A(p_1) = 0 , \quad \Delta(p_1) > 0 , \quad \text{in which case} \quad \mathrm{Re}\,\lambda_{1,2} = 0 ; \tag{21}$$

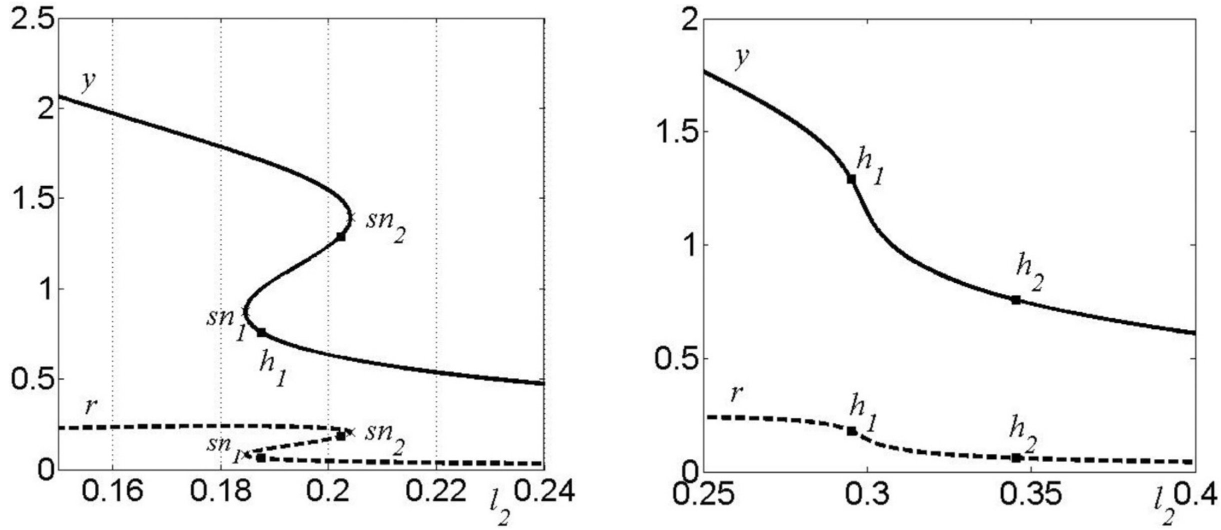
or

$$Sp_A(p_1) < 0 , \quad \Delta(p_1) = 0 , \quad \text{in which case} \quad \lambda_1 = 0 . \tag{22}$$

The first case is a necessary condition for the Andronov–Hopf bifurcation, the second case is a saddle-node bifurcation. In the first case, self-oscillations may appear when the parameter crosses the bifurcation value, while in the second case two stationary solutions — a saddle and a node — merge at the bifurcation point and subsequently disappear.

Figures 1a and 1b plot the parametric dependences of the stationary states on l_2 for two sets of parameters. The parameter values correspond to (20), except that $l_s = 0.08$ in both cases and in the second case also $\alpha = 3$.

In the first case (Fig. 1a), the system has a region with a multiplicity of stationary states. It is located between the saddle-node bifurcation points sn_1 and sn_2. Condition (21) holds at the bifurcation point h_1, but



(a) **(b)**

Fig. 1. The dependence of the stationary solutions on the parameter l_2 for two sets of parameter values.

no oscillations appear in this case. Investigations have shown that oscillations appear in the second case (Fig. 1b) when the parameter crosses the boundary (21) and the system has a unique unstable stationary state. The self-oscillation region lies between the Andronov–Hopf bifurcation points h_1 and h_2.

Let us now carry out a two-parametric analysis to construct the boundaries (21) and (22). Condition (21) in the two-parameter plane describes a neutrality region, which may act as an Andronov–Hopf bifurcation line on a part of the curve or on the entire curve. Condition (22) defines the boundary of the region of multiplicity of stationary solutions.

The relative position of the multiplicity and neutrality curves of the stationary states defines the parametric portrait of system (18). These curves partition the parameter space into regions that differ by the number of the stationary solutions and their stability type. Self-oscillations exist in the uniqueness and instability region of the stationary state.

Let us construct the boundaries (21) and (22), e.g., on the plane l_s and b_{12}. To this end, we apply the following algorithm of continuation in a parameter. The function $y(l_s, b_{12})$ on the stationary solution (21) (or (22)) is treated as an independent variable. From the equation $f_2(y, r, l_s, b_{12}) = 0$ we express r in terms of the other variables and substitute it in the equation $f_1(y, r, l_s, b_{12}) = 0$ and also in the expressions for the trace $Sp_A(y, r, l_s, b_{12}) = 0$ and the determinant $\Delta_A(y, r, l_s, b_{12}) = 0$. From the resulting equalities we express the parameter b_{12} in terms of the other variables to obtain

$$0 < y(l_s, b_{12}), \qquad r(y) = l_2 y - l_s; \tag{23}$$

$$f_1 = 0: \; b_{12} = \frac{I(y)}{y(l_2 y - l_s)} \tag{24}$$

On the neutrality line:

$$Sp_A = 0: \; b_{12} = \frac{\alpha I'_y(y)}{(l_2 y - l_s)} - \frac{\beta}{\alpha}. \tag{25}$$

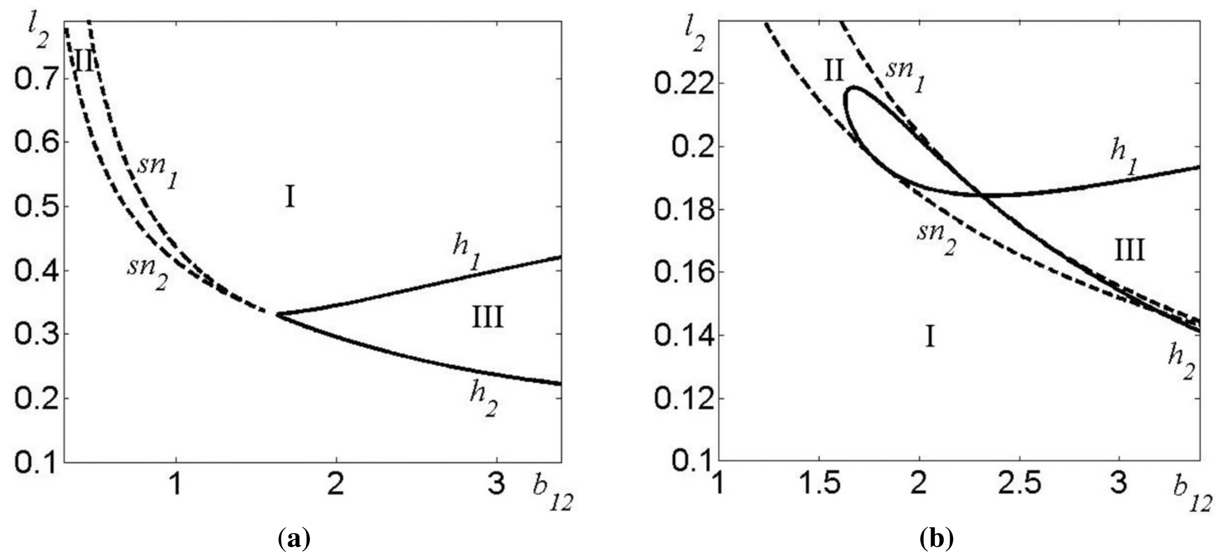


Fig. 2. Parametric portraits of system (18), (20): (a) in the plane (b_{12}, l_2), $l_s = 0.08$; (b) in the plane (b_{12}, l_s); III — self-oscillation region.

On the multiplicity line:

$$\Delta_A = \left(I'_y(y) - \beta_{12} r\right)\left(l_2 y - 2r - l_s\right) + l_2 \beta_{12} r y = 0,$$
$$b_{12} = \frac{I'_y(y)}{(2 l_2 y - l_s)}. \tag{26}$$

where

$$I'_y(y) = \frac{l_0 K (K - l_0) a_1 \exp(-a_1 y)}{(l_0 + (K - l_0)\exp(-a_1 y))^2}. \tag{27}$$

Eliminate the variable b_{12} from expressions (24), (25) and from expressions (24), (26) and use the resulting equalities to find the dependences $l_s(y)$ on the neutrality and multiplicity lines:

on the neutrality line:

$$l_s = l_2 y + \frac{\alpha}{\beta}\left(\frac{I(y)}{y} - \alpha I'_y(y)\right). \tag{28}$$

on the multiplicity line:

$$l_s = l_2 y \cdot \frac{y I'(y) - 2I(y)}{y I'(y) - I(y)}. \tag{29}$$

Let the variable y go with a certain increment over the values on some interval $y \in (0, y_{\max})$. From (28) and (29) we evaluate $l_s(y)$. For each value $l_s(y)$ we calculate from (25) and (26) the values of $b_{12}(l_s)$ and thus

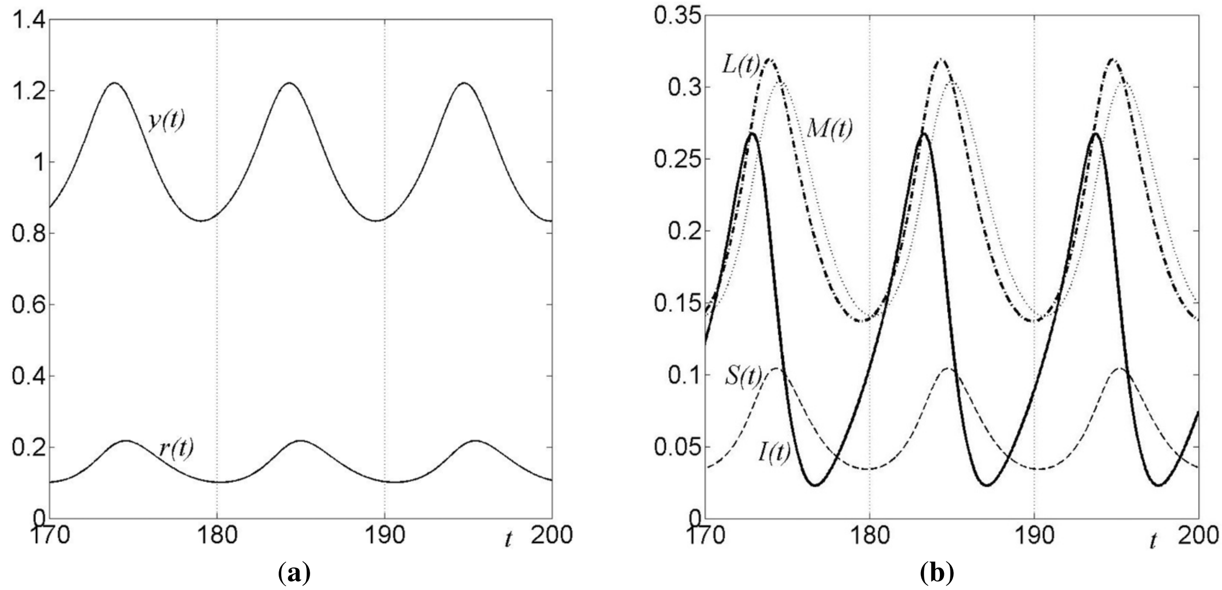


Fig. 3. Self-oscillations in system (18) with the parameters (20), $b_1 = 1.6$, $b_2 = 0.4$: (a) The functions $y(t)$ (national income) and $r(t)$ (interest); (b) Oscillations of investments $I(t)$, savings $S(t)$, money demand $L(t)$, and money supply $M(t)$.

construct the multiplicity and neutrality lines on the plane (b_{12}, l_s). (From (28), (29) we can easily express the variable $l_2(y)$ and similarly construct the parametric portrait of the system on the plane (b_{12}, l_2).) Figures 2a, 2b show examples of parametric portraits of system (18) in two planes. The multiplicity line is drawn by the broken curve and the neutrality line by the solid curve. In region I, we have a unique stable stationary state; in region II, we have three stationary states; in region III, a unique unstable stationary state encircled by a stable limit cycle.

Figure 3a shows the oscillations of the two phase variables of the model — national income and interest; Figure 3b shows the oscillations of investments $I(t)$, savings $S(t)$, demand for money $L(t)$, and money supply $M(t)$. The parameters α and β are chosen so that the oscillation period corresponds to the length of Juglar cycles (~ 10 years). We see that the saving function oscillates almost in time with interest oscillations, while the money demand moves synchronously with national income. Investment growth starts when the interest is low and it continues as long as interest is not too high. Money demand and supply alternately overtake one another: peak demand for money is followed with a small delay by a peak of money supply. The demand for money starts growing when investments increase markedly.

4. Economic Interpretation. The first model equation describes the condition of equilibrium in the commodity market; the second equation describes the equilibrium condition in the money market. The equilibrium condition in the commodity market specifies equality of investments and savings; the corresponding condition in the money market is equality of money demand and money supply (the money mass). Changes in the commodity market produce certain shifts in the money market and vice versa. According to Hicks, the equilibrium in both markets is determined simultaneously by both the interest and the income. When surplus money supply is observed in the money market, the agents start buying bonds, the demand for bonds increases, the bond price rises, and this lowers the interest. Conversely, if money demand increases in line with the increase in national income, agents start selling bonds in order to build up money reserves when cash is scarce, this leads to an increase in the supply of bonds, their price decreases, and the interest correspondingly increases. Changes in interest act to balance money demand and money supply.

Table 2

List of Principal Model Variables

Symbol (units)	Description	Symbol (units)	Description
F (money)	Liquid assets of banks and firms	**w** (money)	Mean wage
H (physical)	Commodity inventories	$\mathbf{G_{inv}}$ (dimensionless)	Investment level
K (physical)	Capital		
L (million workers)	Number of employed (labor)	**p** (money)	Price level
M (money)	Money holdings of consumers	**r** (dimensionless)	Interest

Investments are inversely related to interest. They increase when interest is low, national income and savings accordingly grow, and interest starts decreasing to stimulate the transition from S to I. Higher interest puts a brake on investment activity, which in turn leads to reduction of national income. If the interest is set too high, money holders will purchase securities. This leads to an increase in the demand for bonds and their prices rise. When money supply exceeds the demand for money, the interest decreases.

2. Mathematical Model of Production Cycle

Although system (18) adequately describes the mechanism of the observed oscillations in interest, national income, and investments, it constitutes a highly simplified model of the production cycle. On the ascending wave of the cycle, when massive investments flow into the economy, production growth is accompanied by growth of employment, consumption, commodity output, etc. To develop a more complete and better model, we need to introduce many more variables and to allow for balance relationships in all principal markets.

As the basis we take the macroeconomic dynamic model that describes the long-term behavior of the economy [10]. It has been slightly modified by adding an equation that describes the variation of interest; we have also corrected an error in the equation for wage level changes. In [10] wages start increasing if the number of employed exceeds full employment level; otherwise wages decrease, an obvious error.

Our production cycle model encompasses all the principal markets: labor, capital, financial, commodity market. It is based on a nonlinear ODE system for the variation of nine principal macroeconomic variables (see Table 2). Inflation is not included, to avoid trend movement. The model is augmented with algebraic equations obtained from balance-type relationships, which include eleven additional important variables (see Table 3). All variables are expressed either in physical units (number of items or pieces) or in money (monetary) units.

The model contains a large number of economic parameters. Many of the parameters have natural economic meaning and may be found in the literature. This sets our model apart from the phenomenological model (18), where all the parameters have to be fitted. The parameter values are chosen based on those in the European Union study [10] and adjusted so that the oscillation period corresponds to Juglar production cycles. The base set of parameters in Table 4 is suitable for developed countries with a stable economy.

Table 3
List of Supplementary Model Variables

Symbol (units)	Description	Symbol (units)	Description
C (physical)	Consumption	**R** (money)	Gross profit
D (physical)	Aggregate demand (sales)	**Y** (physical)	Output produced
Div (money)	Dividends	$\mathbf{R_n}$ (money)	Net profit
I (physical)	Investments	**S** (money)	Savings
$\mathbf{L_{max}}$ (million workers)	Working-age population	$\mathbf{L_d}$ (million workers)	Optimal demand for labor

Table 4
Base Parameter Values

Symbol	Description	Value
A	Aggregate productivity	$9.3 * 0.03$
e_{full}	Equilibrium employment	0.9
y_{save}	Consumer savings	0.3
V	Profit rate	0.03 year^{-1}
τ_{dep}	Depreciation life	5 years
α_{inv}	Investment coefficient	0.9–2
α_p	Price coefficient	0.036
α_F	Proportion of liquid assets in use	0.8 year^{-1}
α_M	Proportion of money in use	0.8 year^{-1}
τ_{empl}	Time characteristic of employment	0.1 years
τ_{wage}	Time characteristic of wages	0.1 years
g_{max}	Maximum investment level	0.8
α_r	Interest coefficient	0.1 year^{-1}
L	Number of employed	~ 160 millions
Q_1	Liquid money reserve held by firms and consumers ($M + F = \text{const} = Q_1$)	95 relative money units
Q_2	$M - \dfrac{ln(r)}{\alpha_r} = const = Q_2$	95 relative money units
Q_3	$L - \dfrac{\tau_{wage}}{\tau_{empl}} \ln(w) = \dfrac{Q_3}{\tau_{empl}}$	82.635 years

3. Model Description

1. Commodity Market. In accordance with the classical theory, the dynamics of commodity markets is determined by the difference between production and demand. It may be either positive or negative. The price increases or decreases depending on the situation in the commodity market, working to restore the system to the equilibrium state with $Y = D$ and $H = 0$. This implies the existence of a "conservation law" in the model: any commodity produced is ultimately sold, and any commodity bought is ultimately produced.

The demand D is equal to the sum of consumption C and investments I:

$$D = C + I. \tag{30}$$

Commodity inventories vary depending on the difference between production and demand:

$$dH/dt = Y - D. \tag{31}$$

The price dynamics is specified by the equation

$$dp/dt = -p\alpha_p H/D. \tag{32}$$

Price level variation is determined by the state of commodity inventories over the time required for price adjustment. If the inventories are positive $(H > 0)$, i.e., unsold goods are held in stock, the price will decline. If $H < 0$, i.e., demand exceeds supply, the price will rise.

2. Production Function. The output Y is produced according to the Cobb–Douglas production function

$$Y = f(L, K) = AL^{\lambda}K^{1-\lambda}, \tag{33}$$

where $\lambda = 2/3$, K is the capital, L is the number of employed.

3. Labor Market. In the considered model manufacturers are guided by the setting of the optimal labor demand L_{d} $(L_e = L_{\text{d}})$, if it does not exceed the value of $L_{\max}$ corresponding to full employment, otherwise $(L_e = L_{\max})$.

The variation of employment is determined by the difference between the effective demand for labor and the current employment level:

$$dL/dt = \left(-\frac{1}{\tau_{\text{empl}}}\right)(L - L_e). \tag{34}$$

Changes in labor-to-capital ratio require the producers to adapt to the new conditions by changing the organization of production, finding new workers or conversely partially reducing the labor force and other costs. This is allowed for by the parameter τ_{empl}, which characterizes the time of convergence of L to L_e.

The optimal employment is obtained from the relationship

$$w/p = dY(L_d, K)/dL. \tag{35}$$

From (35) and (33) we obtain

$$L_d = 8p^3KA^3/(27w^3) . \tag{36}$$

The demand for effective employment (labor) is obtained from the condition

$$L_e = \mathrm{Min}(L_\mathrm{d}, L_\mathrm{max}) . \tag{37}$$

Attracting migrant workers from other countries enables firms to solve labor shortages when $L_\mathrm{d} > L_\mathrm{max}$. In this case, the effective demand for labor is always equal to optimal demand.

We assume that the mean wage is determined entirely by employment. If the number of workers exceeds the optimal value L_d, the wages decrease. If, conversely, the employment is below optimal demand, wages tend to increase. In view of these observations, we obtain the following equation for the variation of wages:

$$\frac{dw}{dt} = \frac{w}{\tau_\mathrm{wage}}(L_\mathrm{d} - L) . \tag{38}$$

The parameter τ_wage allows for the elasticity of the labor market.

4. Consumer Behavior. The consumer income in the model is the sum of wages and dividends $(wL + \mathrm{Div})$. A large part of income is spent by consumers on consumption C.

The variation of consumer money balances is described by the equation

$$\frac{dM}{dt} = (wL + \mathrm{Div}) - (pC + S) . \tag{39}$$

Increasing income gradually leads to increasing consumption and savings. The allocation of the available income between consumption and savings depends on the parameter y_save and on the interest. It is defined by the relationships

$$\begin{aligned} C &= \left(1 - y_\mathrm{save}(1+r)\right)\left(\frac{1}{p}\right)\alpha_M M , \\ S &= \left(1+r\right) y_\mathrm{save}\alpha_M M . \end{aligned} \tag{40}$$

Savings increase as the interest rises.

5. Productive Capital. The growth of capital is described by the classical Solow equation

$$dK/dt = \left(-\frac{1}{\tau_\mathrm{dep}}\right)K + I . \tag{41}$$

The variation of liquid asset balances of producers is determined by the difference between profit earned plus available savings and dividends paid, on the one hand, and investment outflows, on the other:

$$\frac{dF}{dt} = \Pi + S - \mathrm{Div} - pI \, . \tag{42}$$

The gross profit Π and the net profit Π_n increase with the increase of sales and decrease as labor costs grow:

$$\Pi = pD - wL \, , \tag{43}$$

$$\Pi_n = \Pi - \left(\frac{1}{\tau_{\mathrm{dep}}}\right) pK = pD - wL - \left(\frac{1}{\tau_{\mathrm{dep}}}\right) pK \, . \tag{44}$$

The allocation of funds to dividends and investments is determined by the following relationships:

$$pI + \mathrm{Div} = \alpha_F F \, . \tag{45}$$

The allocation to investments and dividends depends on the investment proportion G_{inv}:

$$I = G_{\mathrm{inv}}\left(\frac{1}{p}\right)\alpha_F F \, . \tag{46}$$

The productive investment coefficient G_{inv} depends on net return to capital Π_n in comparison with the interest r. If the expected net return to capital $\Pi_n/(pK)$ is higher than r, investments grow; if, conversely, the expected profit is lower than r, investments shrink, while dividends relatively increase, contributing to growth of savings and consumption.

Noting that the coefficient G_{inv} varies from 0 to $g_{\max}$, we write the following equations for the variation of G_{inv}:

$$\frac{dG_{\mathrm{inv}}}{dt} = \alpha_{\mathrm{inv}}\left(g_{\max} - G_{\mathrm{inv}}\right)\left(\left(\frac{\Pi_n}{pK}\right) - r\right), \quad \left(\frac{\Pi_n}{pK}\right) \geq r \, , \tag{47}$$

$$\frac{dG_{\mathrm{inv}}}{dt} = \alpha_{\mathrm{inv}} G_{\mathrm{inv}}\left(\left(\frac{\Pi_n}{pK}\right) - r\right), \quad \left(\frac{\Pi_n}{pK}\right) < r \, . \tag{48}$$

6. Interest. Noting that the interest depends on money balances and increases with the increase in the demand for money, we obtain the following equation for the variation of the interest:

$$dr/dt = \alpha_r r\left((wL + \mathrm{Div}) - (pC + S)\right). \tag{49}$$

Here the demand for money is equal to the sum of money required to pay wages and dividends, and money supply is measured by the outflows to consumption and savings. The coefficient α_r reflects the influence of the money mass on the variation of the interest. If outflows exceed revenues, the interest decreases; conversely, if the demand for money is greater than money supply, the interest increases.

*7. **The Mathematical Model.*** Summing all the relationships above, we obtain a balanced model that describes the long-term dynamics of the economic system. The model consists of nine ordinary differential equations

$$\frac{dX}{dt} = N(X), \tag{50}$$

or

$$\frac{dF}{dt} = \Pi + S - \mathrm{Div} - pI, \tag{50.1}$$

$$\frac{dH}{dt} = Y - D, \tag{50.2}$$

$$\frac{dK}{dt} = \left(-\frac{1}{\tau_{\mathrm{dep}}}\right)K + I, \tag{50.3}$$

$$\frac{dL}{dt} = \left(-\frac{1}{\tau_{\mathrm{empl}}}\right) * (L - L_e), \tag{50.4}$$

$$\frac{dM}{dt} = (wL + \mathrm{Div}) - (pC + S), \tag{50.5}$$

$$\frac{dp}{dt} = -\frac{p\alpha_p H}{D}, \tag{50.6}$$

$$\frac{dw}{dt} = \frac{w}{\tau_{\mathrm{wage}}}(L_{\mathrm{d}} - L), \tag{50.7}$$

$$\frac{dG_{\mathrm{inv}}}{dt} = \alpha_{\mathrm{inv}}(g_{\max} - G_{\mathrm{inv}})\left(\left(\frac{\Pi_n}{pK}\right) - r\right), \quad \left(\frac{\Pi_n}{pK}\right) \geq r, \tag{50.8a}$$

or

$$\frac{dG_{\mathrm{inv}}}{dt} = \alpha_{\mathrm{inv}}G_{\mathrm{inv}}\left(\left(\frac{\Pi_n}{pK}\right) - r\right), \quad \left(\frac{\Pi_n}{pK}\right) < r, \tag{50.8b}$$

$$dr/dt \;=\; \alpha_r r\big((wL+\mathrm{Div})-(pC+S)\big) \tag{50.9}$$

and is augmented with algebraic equality constraints:

$$D \;=\; C+I\,, \qquad Y \;=\; f(L,K) \;=\; AL^{\lambda}K^{1-\lambda}\,,$$

$$L_{\mathrm{d}} \;=\; 8p^3KA^3/(27w^3)\,, \qquad Le \;=\; \mathrm{Min}(L_{\max}, L_{\mathrm{d}})\,,$$

$$C \;=\; \big(1-y_{\mathrm{save}}(1+r)\big)\left(\frac{1}{p}\right)\alpha_M M\,, \qquad S \;=\; y_{\mathrm{save}}\,(1+r)\,\alpha_M M\,, \tag{50.10}$$

$$\Pi \;=\; pD-wL\,, \qquad pI+\mathrm{Div} \;=\; \alpha_F F\,, \qquad I \;=\; G_{\mathrm{inv}}\left(\frac{1}{p}\right)\alpha_F F\,,$$

$$\Pi_n \;=\; \Pi-\left(\frac{1}{\tau_{\mathrm{dep}}}\right)pK \;=\; pD-wL-\left(\frac{1}{\tau_{\mathrm{dep}}}\right)pK\,.$$

Note that three of the nine equations in system (50) can be eliminated by expressing the corresponding variables M, r, and L in terms of other variables. Indeed, in the phase space on the planes (F,M), (M,r), and (L,w), the system is conservative, because

$$dF/dt+dM/dt \;=\; 0\,, \qquad dM/dt-d\ln(r)/(\alpha_r dt) \;=\; 0\,,$$

$$\tau_{\mathrm{empl}} * \frac{dL}{dt}-\tau_{\mathrm{wage}} * d\ln(w)/dt \;=\; 0\,.$$

Hence we obtain three conservation laws:

$$F+M \;=\; \mathrm{const} \;=\; Q_1 \;>\; 0\,, \qquad M-\frac{\ln(r)}{\alpha_r} \;=\; \mathrm{const} \;=\; Q_2>0\,, \tag{51.1}$$

$$\tau_{\mathrm{empl}}L-\tau_{\mathrm{wage}}\ln(w) \;=\; \mathrm{const} \;=\; Q_3 \;>\; 0\,. \tag{51.2}$$

The first equality in (51.1) describes the conservation of money circulation. The liquid assets of banks and firms may be converted into consumer money through payment of dividends, and may be recovered by the firms through investments, but the total sum remains unchanged. The second equality in (51.1) describes the effect of the interest on the quantity of money in circulation. When the interest drops, the quantity of money held by consumers decreases; in accordance with the first conservation law, consumers transfer their balances to liquid assets of enterprises through investments.

The conservation law (51.2) closely couples the level of wages with the number of employed: the more employed, the lower the wage, and conversely, the smaller the number of employed, the higher the wage. The wages start decreasing when the number of employed exceeds the optimal demand for labor.

8. Stationary State. We start the investigation of the model with the study of the stationary states as a function of the parameters. The equilibrium is described by a system of nonlinear algebraic equations:

$$N(X) = 0 .$$

From Eq. (50.6) we obtain $H = 0$.

From the conservation laws (51.1) we obtain an expression for the variables M and r in terms of the variable F:

$$M = Q_1 - F , \qquad r = \exp\left(\alpha_r (M - Q_2)\right). \tag{52}$$

From Eqs. (50.5), (50.7) and equality (38), we obtain the expression for L:

$$L = L_e = L_\mathrm{d} = 8p^3 K A^3/(27w^3). \tag{53}$$

From conservation law (51.2) we express L in terms of w:

$$L = (\tau_\mathrm{wage} \ln(w) + Q_3)/\tau_\mathrm{empl} . \tag{54}$$

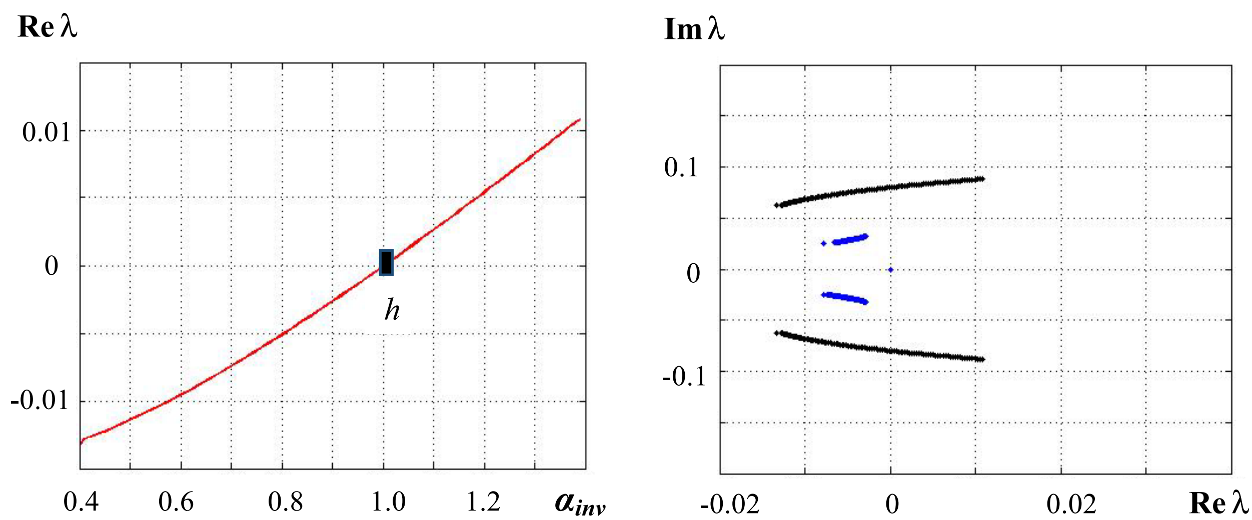
Expressions (53) and (54) are used to eliminate the variable L .

Thus, for the stationary solutions, we have a system of nonlinear algebraic equations in the five unknowns $(F, K, p, w, G_\mathrm{inv})$, which is solved by Newton's method:

$$\begin{aligned}
&\text{(i)} \quad R + S - \mathrm{Div} - p * I = 0 ; \\
&\text{(ii)} \quad Y - D = 0 ; \\
&\text{(iii)} \quad I - \frac{K}{t_\mathrm{dep}} = 0 ; \\
&\text{(iv)} \quad L_d - L = 0 ; \\
&\text{(v)} \quad \left(\frac{\Pi_n}{pK}\right) - r = 0 .
\end{aligned} \tag{55}$$

Investigations show that the model has a unique stationary state for all admissible parameter values. The stationary state may be stable or unstable. The main parameters influencing the stability of the stationary solution are the coefficients α_inv and γ_save , which determine the investments and the saving. The stationary value is independent of α_inv , but it depends on γ_save . For the base set of parameters from Table 3, the variables have the following values on the stationary solution:

$$\begin{gathered}
F \cong 37 , \quad M \cong 58 , \quad H = 0 , \quad K \cong 43.5 , \quad L \cong 163.6 , \\
p \cong 1.56 , \quad w \cong 0.19 , \quad G_\mathrm{inv} \cong 0.46 , \quad r \cong 0.025 .
\end{gathered} \tag{56}$$



(a) **(b)**

Fig. 4. Bifurcation diagrams: (a) Eigenvalues of the Jacoby matrix versus the parameter α_{inv}; (b) Crossing of the imaginary axis by the pair of complex eigenvalues as α_{inv} varies.

Investigation of the eigenvalues of the (9×9) Jacoby matrix on the stationary solution of system (50) has shown that as the parameter α_{inv} (or γ_{save}) increases, the real part of the pair of complex eigenvalues goes through zero (Fig. 4) for $\alpha_{cr} \cong 1.02$. The stationary solution becomes unstable due to a supercritical Hopf bifurcation and stable self-oscillations develop in the system.

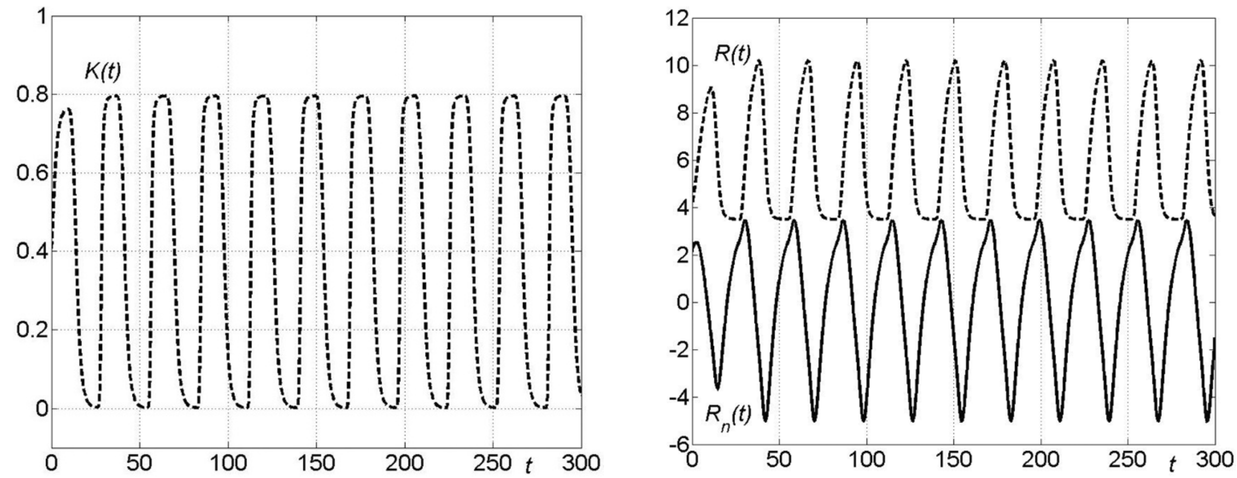
9. The Oscillation Mechanism. Thus, for $\alpha_{\text{inv}} > \alpha_{cr}$ the system displays oscillations. To elucidate the oscillation mechanism, we proceed as follows. Choose some α_{inv} from the oscillation region. Set each of the model variables constant one after another and assign to them some mean values, while excluding the corresponding differential equations from the model (50). The variables and equations remaining in system (50) are those responsible for self-oscillations.

Our investigations have shown that only three nonlinear differential equations (50.1), (50.3), and (50.8) are responsible for self-oscillations. They describe time dependence and coupling of three principal variables (F, K, G_{inv}). The number of employed in production L, the wage w, the interest r, the consumer money balances M, stored inventories H, and the price p are assumed constant. We set

$$H = 0, \quad L = L_{\text{d}} = 162, \quad p = 1, \quad r = 0.03, \quad M = 55, \quad w = 0.038.$$

The expressions for I, S, D, Div, etc., entering the equations are described by the same algebraic equations (50.10). Figure 5 shows the oscillations in capital and in gross and net profit in the minimalist model with three ODEs.

Analyzing the oscillations, we conclude that the oscillation mechanism is embedded in the dependence of the investment volume on production efficiency. Investments increase if the return on capital exceeds the profitability rate (or the interest) in accordance with Eq. (50.8a). Investment growth leads to increase of capital (Eq. (50.3)), decrease of dividend payout, and increase of aggregate demand. As a result, firm assets F



(a) **(b)**

Fig. 5. The form of self-oscillations in the minimalist model: (a) capital $K(t)$, (b) gross and net profit $R(t)$ and $R_n(t)$.

employed in production start increasing (Eq. (50.1)). Capital growth *leads* generation of net profit: there is a time lag between the capital peak and the profit peak, so that at some times the economy is "overheated" and production is unprofitable (the rate of return on capital decreases). As a result, investments start decreasing in accordance with Eq. (50.8b), capital growth is arrested, enterprises shut down or freeze, production decline is observed. The net profit becomes negative. Firm assets shrink. Production is reduced until it becomes profitable again. Then investment growth is resumed, followed by recovery of production (the capital K). The cycle is closed.

Thus, the oscillation mechanism in this model is different from the mechanism previously considered for the IS-LM model, where changes in interest are regulated by the ratio of investment volumes and savings balances, allowing one to flow into another and vice versa. In model (50), oscillations exist when the interest rate is constant and the savings balances are unchanged. Here oscillations arise because investment growth produces faster capital growth than profit growth, and production efficiency begins to decline.

10. Production Cycles in the Full Model. Let us now analyze the oscillations in the full model (50). Allowing for the variation of all other variables leads to a better description of production cycles than in the minimalist model. Changes in investments lead to growth or shrinking of the capital, which in turn affects employment, wages, commodity inventories, and other variables, inducing oscillations. Time variation of other variables excluded from the minimalist model also affects the principal variables in model (50), altering the shape of the cycle, smoothing or exacerbating the decline of production as a function of parameter values. The eight graphs in Figs. 6a–6h illustrate the oscillations in all the principal variables from Tables 2–3 for the base set of parameters (Table 3) and $\alpha_{\text{inv}} = 2$. Let us analyze the simulation results.

1. Economic Revival. Economic revival starts with the revival of the business activity recovery phase. In reality, this is manifested in signing of new business contracts, gradual increase of the demand for labor reducing unemployment, growth of consumer demand, and as a consequence decrease of commodity inventories. Investments start to grow, leading to growth of capital; liquid assets of enterprises increase, triggering increase in dividends. Savings and consumption are still at a low level and continue to contract. Prices remain low and continue to drop for a while.

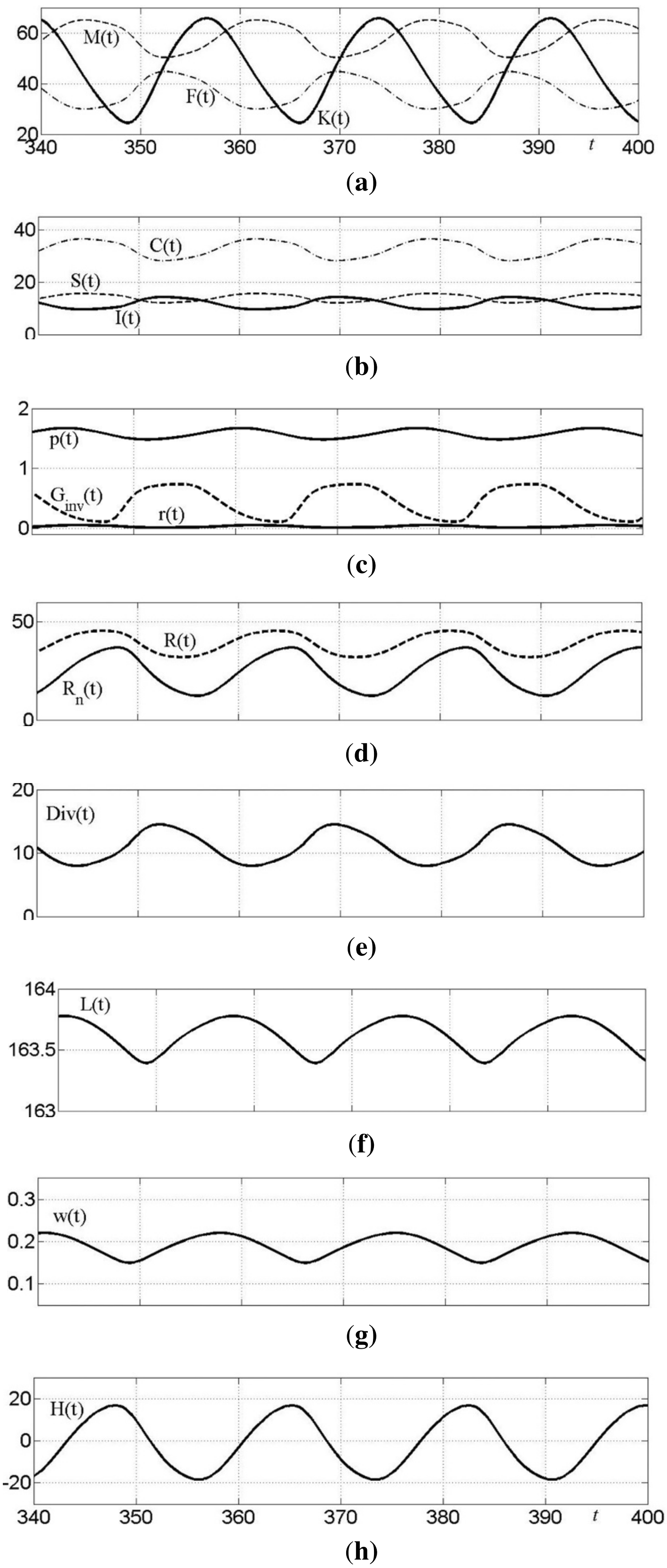


Fig. 6. Self-oscillations in the full model (50): eight principal economic variables, base set of parameters, $\alpha_{inv} = 2$.

2. Economic Growth. Then economic growth starts. It is characterized by rapid growth of capital, increase of the production of goods and services. Expansion of production requires more workers and employment correspondingly increases. Wages also rise until they reach the optimal level. Consumption and savings start increasing, consumer money balances, which continued to shrink in the initial growth phase, start increasing in the second phase of growth, while firm liquid assets shrink. The interest starts increasing. Growth of demand is observed, commodity shortages develop, prices begin to rise.

Fast growth of capital produces "overheating" of the economy, returns to capital decrease, and growth of capital slows down.

3. The Peak. Then the economy reaches the peak of its growth, when production is at full capacity and production volume is at its maximum. The number of employed and wages also approach their maximum levels. Savings and consumption increase, prices are still increasing, commodity shortages reach their maximum (H reaches a negative minimum). Investments continue to decline, dividends shrink. Net profit hits its minimum.

4. Recession. Lack of profits, unprofitable production, and shrinking investments lead to the closing of enterprises, decrease of production and employment, decline in wages and consumption. Economic recession sets in in the market economy. Inventories of commodities, for which shortages were previously experienced, gradually begin to build up in stores. Prices also start dropping. The lowest point of recession is economic crisis.

5. Economic Crisis. The crisis is characterized by sharp contraction of production, closing of many enterprises, over-production, and surplus inventories. It is accompanied by price decreases and growth of unemployment. At the same time, the economic crisis creates a new impulse for economic growth. It creates incentives for cost savings, increase of profits, renewal of capital using new technologies. Production becomes profitable again, investments resume their growth.

The crisis is the end of the preceding period of economic growth and the start of the next period. This is the most important component of the self-regulation mechanism in market economies. Recession is followed by a period of depression (stagnation), which is characterized by stagnation of the market economy, weak demand for consumer goods and services, high loads on enterprises, mass unemployment, drop in consumer standard of living, among other factors due to decline in wages in the face of continuing price increases. In this period, households adapt to the new conditions and requirements in the economy.

11. Cycle Length. The original model on which system (50) is based has been proposed in [10]. In this study, the cycle is about 50–60 years long, and should the system be jolted out of the cycle orbit by some random factors, the convergence back to the cycle should take millennia. Although the cycle period matches Kondratieff's long waves, they cannot be described by this model. Kondratieff waves emerged under capitalism at the very end of the 18th century (in 1780s–1790s). A total of five such waves have been observed. The sixth Kondratieff wave is expected to start very soon, in 2018–2020. The model parameters fitted for a current European economic system have changed dramatically during these 230 cycles, and in all probability will continue changing.

The model [10] is more appropriated for describing Juglar's shorter investment cycles, and we have accordingly modified it as described above. We have corrected the equation for wages, added an equation for interest, and altered the parameters so that the cycle period is now 15–17 years. In the improved model, the cycle reaches the orbit in a realistic time scale comparable with the cycle period.

Supported by the Russian Foundation for Basic Research grant 15–01–07944.

REFERENCES

1. T. Yu. Matveeva, *An Introduction to Macroeconomics: A Textbook* [in Russian], 6th ed., Vysshaya Shkola Ekonomiki, Moscow (2008).
2. W.-B. Zhang, *Synergetic Economics: Time and Change in Nonlinear Economics* [Russian translation], Mir, Moscow (1999).
3. J. M. Keynes, *General Theory of Employment, Interest, and Money* [Russian translation], Gelios ARV, Moscow (2002).
4. R. Franke and T. Asada, "A Keynes–Goodwin model of the business cycle," *J. Econom. Behav. Organ.*, **24**, 273–295 (1994).
5. H. F. Naish, "Keynesian real business cycles framework," *J. Econom. Behav. Organ.*, **27**, 183–211 (1995).
6. L. Karnizova, "The spirit of capitalism and expectation-driven business cycles," *J. Monetary Econ.,* **57(6)**, 739–752 (2010).
7. A. Abta, A. Kaddar, and H. Talibi Alaoui, "Stability of limit cycle in a delayed IS-LM business cycle model," *Appl. Math. Sci.*, **2(50)**, 2459 – 2471 (2008).
8. V. Torre, "Existence of limit cycles and control in complete Keynesian systems by theory of bifurcations," *Econometrica*, **45**, 457–1466 (1977).
9. T. Puu and I. Sushko, "A business cycle model with cubic nonlinearity," *Chaos Solitons Fractals*, **19**, 597–612 (2004).
10. S. Hallegattea, M. Ghilc, P. Dumasa, and J.-Ch. Hourcadea, "Business cycles, bifurcations and chaos in a neo-classical model with investment dynamics," *J. Econom. Behav. Organ.*, 57–77 (2008).