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Know dive

Logic

extensional representation

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Assertional languages - limitations

Observation 8.1 (Assertional languages, limitations). The description of domains and models using *only* assertions is very limited. One would like to have more flexible ways to describe them.

What assertional languages do not represent (definitions)

The existence of representation languages is motivated by the fact there is information that cannot be easily and intuitively in a world model.

Definitions:

- *Definitions characterizing our knowledge of the world* (cats are animals, bachelor is un unmarried man, car is a synonym of automobile, the brother of my father is my uncle, cars are vehicles with four wheels carrying people)
- *Definitions characterizing the world* (Bruno is my uncle)
- *Descriptions of general properties of the worlds* (all cars are produced by a car maker, quite different from maker(car1, BMW), maker(car2, VW))

Atomic formulas – modeling definitions

Definition 8.3 (Atomic assertions, complex assertions, atomic formulas) Given a language L = La ∪ Lc, La is defined as

$$\mathrm{L}a = \mathrm{L}A \cup \mathrm{L}AC$$

where: LA is an assertional language, that we also call a **language of atomic assertions**, and LAC is a **language of complex assertions**. La is a language of **atomic formulas.**

Observation (Atomic formulas) Assertions are atomic formulas, but some atomic formulas are not assertions. The key property of atomic formulas and assertions is that they are interpreted by an interpretation function.

In other words the meaning of atomic formulas, like that of assertions, can be computed directly from the domain.

Atomic formulas - example

If C1 and C2 are concepts, then (following the notation of LOD), C1 ⊓ C2 (interpreted as set intersection), is also a concept description, where Ci can be an atomic assertion as well as a complex assertion.

Examples of assertions in this language are (";" is used to separate different formulas):

”being a person” ; ”having blond hair” ; ”having a dog” ;

”having blond hair ⊓ ”being a person” ;

To be used in definitions

”being a blond person” = ”having blond hair” ⊓ ”being a person”

“being a car” = ”being a vehicle” ⊓ ”having foru wheels”

Interpretation Function

Definition (Interpretation function) Given a language L = La ∪ Lc, La with La = LA ∪ LAC. Let D be a domain. Then an **Interpretation Function** I for La is defined as

$$\mathrm{I} : \mathrm{L}a \rightarrow \mathrm{D} \ (\mathrm{I} \subseteq \mathrm{L}a \times \mathrm{D})$$

with

$$\mathrm{I} = \mathrm{I}A \circ \mathrm{I}AC$$

where

$$\mathrm{I}AC : \mathrm{L}a \rightarrow \mathrm{L}A \,, \qquad \mathrm{I}A : \mathrm{L}A \rightarrow \mathrm{D}$$

where: IA is an **interpretation function for atomic assertions** and IAC is an **interpretation function for complex assertions**. IA is as defined above.

Furthermore, we say that a fact f ∈ M is the **interpretation** of w ∈ La, and write

$$\mathrm{f} = \mathrm{I}(a) = a^{\mathrm{I}}$$

to mean that w is a linguistic description of f.We say that f is the **interpretation of** w, or, equivalently, that w **denotes** f.

What assertional languages do not represent (judgements)

The existence of representation languages is motivated by the fact there is information that cannot be easily and intuitively in a world model. E

Judgements:

- *Negative information* (what I do not perceive)
- *Partial information* (what I perceive partially)
- *Consequential information* (e.g., cause effect)
- *Equivalent information* (e.g., bidirectional cause effect)
- *Universal / existential statements* (e.g., all swans are wight)
- *… and much more!*

Reasoning problems (with respect a world model)

Observation (Models in world models are incomplete) Models say what is the case. But they say nothing about "the rest" (what they do not mention.

Consider the model described by the assertion "my T-shirt is green".

What about the assertion "my paints are grey?"

What about the assertion "my t-shirt is are grey?"

There is a fundamental distinction between partial knowledge and negative knowledge, to be captured

Representation languages – modeling judgements (and definitions)

Definition 8.1 (Representation language, atomic formulas, complex formulas, representation interpretation function)

Let W =< L$\mathcal{A}$, D, I$\mathcal{A}$ > be a world model with I$\mathcal{A}$: L$\mathcal{A}$ → D. Let La be such that L$\mathcal{A}$ ⊆ La and such that there is a **representation interpretation function** I : La → D, with I$\mathcal{A}$ ⊆ I.

Then, a **representation language** L is defined as

$$L = \{w\} = La \cup Lc, \text{ with } La \subset Lc.$$

where:

- I$\mathcal{A}$ is as before,
- w ∈ L is a **(well-formed) formula**,
- w ∈ La is an **atomic (well-formed) formula** and
- w ∈ Lc is a **complex (well-formed) formula**.

La and Lc are the **language of atomic formulas** and **of complex formulas**, respectively

Complex formulas - example

We can build complex atomic formulas as follows. If $A1$ and $A2$ are formulas, then (following the notation of LOP), $A1$ xor $A2$, is also a formula where Ai can be an atomic as well as a complex formula. Examples:

"Sofia is a person" ;

"Sofia is a person" xor "Sofia is a person";

"Sofia is a person" xor "Paolo is a man" ;

("Sofia is a person" xor "Paolo is a man") xor "Paolo is a dog" ;

. . . and so on, with indefinitely long complex formulas.

The intuition is that $A1$ xor $A2$ contains one and only one fact between the facts denoted by $A1$ and $A2$.

Entailment

Definition (Entailment relation) Let M ⊆ D and w ∈ L be a formula. Then |=, to be read "**entails**", is an **entailment relation** defined as

$$|= \subseteq D \times L$$

We also write

$$M \models w \qquad\qquad (M \models T)$$

where M |= T stands for M |= w for all w ∈ T . We say that M **entails** w, or also that M **entails** T .

Entailment of an atomic formula

Definition (Entailment of an atomic formula) If w is an atomic formula then we have

$$M \models w \text{ if and only if } I(w) \in M$$

Observation (Entailment of atomic formulas) Entailment of atomic formulas reduces to their interpretation.

Observation (Entailment of complex formulas) Entailment of complex formulas operates in two steps, similarly to how interpretation functions operate on complex atomic formulas. In the first step, it reduces the entailment of a complex formula to that of its component atomic formulas. In the second step it applies the interpretation function o atomic formulas.

Entailment and interpretation (observations)

Observation (Entailment relation) Interpretation is a function. Entailment is a relation. It is a many-to-many relation. There may be multiple theories that denote a model and, symmetrically, for the same theory there may be multiple models entailed by it (the latter property being the one which makes entailment a relation).

Observation (Entailment relation and interpretation function). Interpretation functions operate on formulas and facts. The intuition is that entailment relations operate on theories (set of formulas) and models. In certain logics, in particular those used to formalize decision making allow to model *partial knowledge*, that is, the fact that a person does not have complete knowledge about the world, a situation which is intrinsic to human knowledge.

Entailment - example

Consider complex formulas of the form $A1$ xor $A2$, where Ai is any formula. Let us assume that $A1$ and $A2$ are atomic formulas. Then $A1$ xor $A2$ will be denoted by a model M containing the denotation of $A1$ or by one containing the denotation of $A2$. In formulas:

I($A1$) |= $A1$

I($A1$) |= $A1$ xor $A2$

I($A1$) not|= $A1$ xor $A2$

{I($A1$), I($A2$)} not|= $A1$ xor $A2$

Logical entailment (= reasoning)

Definition (Logical entailment) Let M ⊆ D be a model and T1, T2 ⊆ L be two theories and w ∈ L a formula. Then we write

T1 |={M} T2 (T1 |={M} w)

and say that T1 **(logically) entails** T2 (w) with respect to the **set of models** {M} if

for all M ∈ {M}, if M |= T1 then M |= T2 (M |= w)

Logical entailment - properties

Intuition (Reflexivity)

$$w \models w$$

Observation (Reflexivity) Every fact entails itself. Knowledge asserts itself as being knowledge. This is the essence of what knowledge is about.

Logical entailment – properties (cont.2)

Intuition (Cut)

$$\text{If } \Gamma \mid= w1 \text{ and } \Sigma \cup \{w1\} \mid= w2 \text{ then } \Gamma \cup \Sigma \mid= w2$$

Observation (Cut) There are two ways to interpret cut.

The first and most common is that reasoning can be made efficient by dropping intermediate irrelevant results.

The second is transitivity, namely the fact that reasoning con be composed by chaining independent reasoning sessions, something that people do all the time during their everyday life.

Logical entailment – properties (cont.3)

Intuition (Compactness)

If $\Gamma \models w$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models w$

Observation (Compactness) Consider infinity as the possibility of describing another fact in the process of reasoning. Thus, for instance, natural numbers are infinite and, no matter how many numbers have already been used so far, it is always possible to provide a new one. Compactness says that logical consequence must be computed using a finite set of assumptions. Logical consequence for anhypothetically infinite set of formulas is not a behaviour that is considered of interest.

Logical entailment – properties (cont.4)

Intuition (Monotonicity)

$$\text{If } \Gamma \models w \text{ then } \Gamma \cup \Sigma \models w$$

Observation (Monotonicity) Monotonicity implements a fundamental and intuitive property of knowledge, for instance of scientific knowledge. If knowledge increases then what can be derived from it via reasoning can only increase. At most it can stay the same if the new piece of knowledge was implied by what is already known.

Logical entailment – properties (cont.5)

Intuition (NonMonotonicity)

$$\Gamma \models w \text{ and } \Gamma \cup \Sigma \text{ not} \models w$$

Observation (NonMonotonicity) Monotonicity is a property which most often does not hold. This is extensively the case with commonsense reasoning, a topic extensively studied in AI.

How many times getting to know something new has forced us to change our mind? Historical AI example:the belief that all birds fly can be defeated by the fact that penguins are birds and they do not fly.

Historical scientific knowledge example: the discovery that it is the earth rotating around the sun, and not vice versa.

Pratical point of view: the logics used in mathematical reasoning and in formal methods, as applied to, e.g., programming languages, are monotonic, while most logics defined in AI are nonmonotonic. Negation by failure, as implemented in relational DBs is nonmonotonic.

Logics

Definition (Logic). *L*, defined as

$$\mathcal{L} =< L, D, I, \models >,$$

is a **logic**.

World models and Logics- The roles of D, L, I$\mathcal{A}$, M, TA, |=

World models

$$\begin{tikzcd}
a & T_A \arrow[l, "\in"'] \arrow[r, "\subseteq"] \arrow[d, dashed, "I_A"] & L_A \arrow[d, "I_A"] \\
\mathfrak{f} & M \arrow[l, "\in"'] \arrow[r, "\subseteq"] & D
\arrow[from=1-1, to=2-1, "I_A"]
\end{tikzcd}$$

Logics

$$\begin{tikzcd}
a & T \arrow[l, "\in"'] \arrow[r, "\subseteq"] \arrow[d, dashed, "\models"] & L_a \arrow[d, "I"] \\
\mathfrak{f} & M \arrow[l, "\in"'] \arrow[r, "\subseteq"] & D
\arrow[from=1-1, to=2-1, "I"]
\end{tikzcd}$$

Idea: extend world models by logical reasoning

Representation languages - example

The following are examples of representation languages:

- All the natural languages, as used by people in their everyday life;
- The language of arithmetics which describes how to perform plus and minus operations on natural numbers. The language of arithmetic is a simplified natural language which allows to mention, among others, numbers, variables, plus, minus, times, and also to compose phrases in more complex phrases;
- Relational database (DB) languages do not extend to representation languages;
- Entity-relationship (ER) languages do not extend to representation languages
- KGs do not extend to representation languages.

Logics, models and theories

Logics provide the general framework within which logical theories, asserted in representation languages, and models can be defined and compared. Given a logic

$$\mathcal{L} =< L, D, I, \models >,$$

we have

$$M = \{f\} \subseteq D$$

$$T = \{w\} \subseteq L$$

Note how the notions of model and domain are the same as with world models

Representation languages - observations

Given a representation language L, a **theory** T is (still) defined as

$$T = \{w\} \subseteq L$$

But

the interpretation function applies ONLY to atomic formulas

Logics, models and theories – The practice

1. Select a Logic (crucial representation choice)

$$\mathcal{L} =< L, D, I, \models >,$$

2. Agree on L, I (… and therefore D)

3. Agree on $\models$ (… and therefore reasoning principles)

4. Construct TA = $\{a\} \subseteq LA$

5. The model M = {f} ⊆ D is automatically defined

NOTE: Agreement is on linguistic representation, based on a shared understanding of what language means, and on reasoning mechanism (shared understanding?)

NOTE 2: agreement must be formalized

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