Computational Logic Exercises

Module VI – Logic of Propositions

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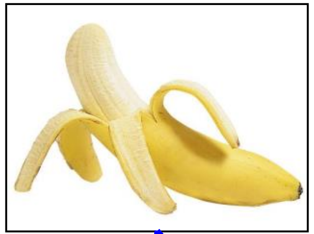
Modelling example

What is a banana?

OBSERVATION #1



OBSERVATION #2

THE WORLD

PROPOSITION = BANANA

MENTAL MODEL

I mean banana as something yellow and curve, no matter if partially opened

T F

FORMAL MODEL M1

T F

FORMAL MODEL M2

MENTAL MODEL

I mean banana as something yellow or red, which can be one or many together, no matter how big they are, and whether they are opened or not

Symbols used in LOP

Which of the following symbols are used in LOP?

$\sqcap$ $\neg$ $\top$ $\vee$ $\equiv$ $\sqcup$ $\sqsubseteq$ $\supset$$\bot$ $\wedge$ $\models$

ANSWER: $\neg$ $\vee$ $\wedge$$\supset$ $\equiv$ $\top$ $\bot$ $\models$

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Well formed formulas

Which of the following formulas are NOT syntactically correct in LOP?

a) **$\neg$ MonkeyLow $\vee$ BananaHigh**

b) **$\neg\neg$ MonkeyLow $\wedge$ BananaHigh**

c) **MonkeyLow $\neg \wedge$ BananaHigh**

d) **MonkeyLow $\supset \neg$ GetBanana**

e) **MonkeyLow $\equiv$ BananaHigh**

f) **$\neg$ (MonkeyLow $\vee$ BananaHigh) $\supset$ BananaHigh**

ANSWER: c

Translate natural language sentences into LOP

David or Bruno come to the party

$David \vee Bruno$

No matter what, Bruno will come to the party

$Bruno$

Bruno will come to the party, unless David is there

$\neg David \supset Bruno$ or equivalently $David \vee \neg Bruno$

Carlo comes to the party, therefore David comes too

$Carlo \supset David$

Either David or Bruno come to the party

$(David \wedge \neg Bruno) \vee (Bruno \wedge \neg David)$

Neither Carlo nor David will come to the party

$\neg Carlo \wedge \neg David$

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Translate natural language sentences into LOP

If David comes to the party then Bruno and Carlo come too

$David \supset (Bruno \wedge Carlo)$

Angelo will come to the party, provided that Bruno comes and Carlo does not

$(Bruno \wedge \neg Carlo) \supset Angelo$

Carlo comes to the party given that David doesn't come, but, if David comes, then Bruno doesn't come

$(Carlo \supset \neg David) \wedge (David \supset \neg Bruno)$

Carlo comes to the party only in case Angelo and Bruno do not come

$(\neg Angelo \wedge \neg Bruno) \equiv Carlo$

A necessary condition for Angelo coming to the party, is that Bruno and Carlo aren't coming

$(\neg Bruno \wedge \neg Carlo) \supset Angelo$

A necessary and sufficient condition for Angelo coming to the party, is that Bruno and Carlo aren't coming

$(\neg Bruno \wedge \neg Carlo) \equiv Angelo$

Modelling: Define the theory for a certain problem

Passing the exam. If you play and you study you'll pass the exams, while if you play and don't study you won't pass. Thus, if you play, then you study and you'll pass the exams, or you don't study and you won't pass it.

Define the set of Propositions {P, S, E} such that:

P = “you play”, S = “you study”; E = “you pass the exam”

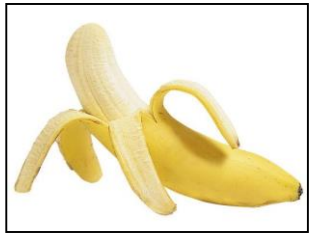
The Theory T can be formalized as follows:

$(P \wedge S) \supset E$

$(P \wedge \neg S) \supset \neg E$

$P \supset ((S \wedge E) \vee (\neg S \wedge \neg E))$

Modelling: Define the theory for a certain problem



The effect of bananas. Bananas may differ in many ways. However, there are red and yellow bananas. I like bananas, but I eat only yellow bananas. If I do not eat at least a banana I get crazy.

Define the set of Propositions {RedBanana, YellowBanana, EatBanana, Crazy, Banana}.

The Theory T can be formalized as follows:

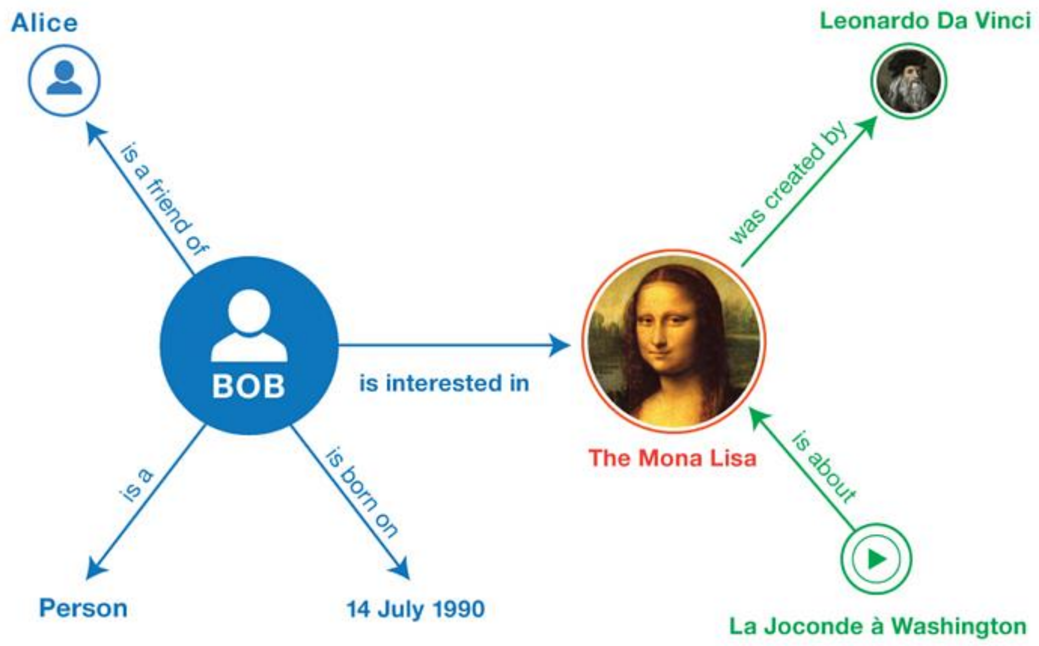
RedBanana ⊃ Banana

YellowBanana ⊃ Banana

EatBanana ⊃ YellowBanana

¬ EatBanana ⊃ Crazy

From a knowledge graph to a LOP theory



LOP THEORY: The LOP theory will include a proposition for each fact in the KG, and may contain some negations of propositions that are not expressed as facts in the KG. For instance:

Theory#1:

[wasCreatedBy(TheMonaLisa, LeonardoDaVinci)]

[friendOf(Bob, Alice)]

¬ [friendOf(Bob, Carol)]

…

Theory#2:

TheMonaLisa_wasCreatedBy_LeonardoDaVinci

Bob_friendOf_Alice

¬ Bob_friendOf_Carol

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Truth tables

Given the propositions A and B, calculate the Truth Table of the following formulas:

1. $A \wedge B$
2. $A \vee B$
3. $A \equiv B$

VARIABLES (1) (2) (3)

POSSIBLE ASSIGNEMENTS

A	B	$A \wedge B$	$A \vee B$	$A \equiv B$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	F	F	T

Finding models of formulas using Truth Tables

RECALL: an interpretation function I is a model for a formula P if and only if I(P) = true

List the models for the following formulas, given the propositions A, B, C:

1. $A \wedge \neg B$
2. $(A \wedge B) \vee (B \wedge C)$
3. $(A \vee B) \supset C$
4. $\neg A \equiv B \equiv C$

See here for (1).

Check yourself for (2), (3), (4).

	A	B	C	A ∧ ¬ B
	T	T	T	F
MODEL	T	F	T	T
	F	T	T	F
	F	F	T	F
	T	T	F	F
MODEL	T	F	F	T
	F	T	F	F
	F	F	F	F

Finding models of formulas using Truth Tables

List the models for the formula P: (A∨B) ⊃ ¬C

You can rewrite P as ¬ (A ∨ B) ∨ ¬ C
that is equivalent to (¬ A ∧ ¬ B) ∨ ¬ C

MODELS

A	B	C	P
T	T	T	F
T	F	T	F
F	T	T	F
F	F	T	T
T	T	F	T
T	F	F	T
F	T	F	T
F	F	F	T

Reasoning with Truth Tables

Prove that the formula P is **unsatistiable**, where P: $(A \land B) \land \neg B$

RECALL: a formula is unsatistiable if it is false for all assignments (there are no models)

A	B	P
T	T	F
T	F	F
F	T	F
F	F	F

Reasoning with Truth Tables

Prove that the formula P is **valid**, where P: $\neg(A \supset B) \supset (A \wedge \neg B)$

RECALL: a formula is valid if it is true for all assignments (all assignments are models for it)

Notice that you can rewrite it as $\neg\neg(A \supset B) \vee (A \wedge \neg B)$

that is equivalent to$(\neg A \vee B) \vee (A \wedge \neg B)$

A	B	P
T	T	T
T	F	T
F	T	T
F	F	T

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Modeling and solving a problem with LOP

Knights and Knaves. A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet two inhabitants: Zoey and Mel.
(1) Zoey tells you that Mel is a knave.
(2) Mel says 'Neither Zoey nor I are knaves'.
Can you determine what are they? (who is a knight and who is a knave?)

PROOF:
Let use define two propositions Z = "Zoey is a knight" and M = "Mel is a knight".
The two sentences above can be translated in LOP as follows:
(1) $Z \supset \neg M$
(2) $M \supset Z \wedge M$

We can use truth tables to prove that there are only two possible answers:
- Both lie, i.e. they are both knaves, i.e. I(Z) = F and I(M) = F
- Zoey tells the true (is a Knight) and Mel lies (is a knave), i.e. I(Z) = T and I(M) = F

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Prove equivalence

Let A, B, C, D, E be propositions. Prove by using basic facts that:

$((A \vee \neg E) \wedge (\neg B \vee \neg D)) \supset C \equiv (\neg A \wedge E) \vee ((B \wedge D) \vee C)$

PROOF:

$((A \vee \neg E) \wedge (\neg B \vee \neg D)) \supset C$	$\equiv$	$(A \vee \neg E) \supset ((\neg B \vee \neg D) \supset C)$	*by Implication and conjunction*
	$\equiv$	$(A \vee \neg E) \supset (\neg(\neg B \vee \neg D) \vee C)$	*by Implication and disjunction*
	$\equiv$	$(A \vee \neg E) \supset ((B \wedge D) \vee C)$	*by De Morgan laws*
	$\equiv$	$\neg (A \vee \neg E) \vee ((B \wedge D) \vee C)$	*by Implication and disjunction*
	$\equiv$	$(\neg A \wedge E) \vee ((B \wedge D) \vee C)$	*by De Morgan laws*

Try yourself for:

1. $\neg(A \vee (A \supset B)) \equiv \neg A \wedge \neg B$
2. $\neg(A \supset B) \wedge (\neg B \wedge \neg C) \equiv A \wedge \neg(B \vee C)$
3. $\neg(A \supset \neg B) \wedge (\neg B \wedge \neg C) \equiv \perp$
4. $(C \supset (A \supset \neg B)) \supset (C \supset \neg A)$

Prove entailment

RECALL: $\Gamma \vDash \psi$ iff all models satisfying the formulas in Γ also satisfy ψ

Let A, B, C be propositions. If $A \vDash B \land C$, then $A \vDash B$ or $A \vDash C$ or both?

PROOF:
If $A \vDash B \land C$, for all I such that I(A) = True it should be I(B ∧ C) = True.
However, by definition this means that I(B) = True and I(C) = True.
Therefore both $A \vDash B$ and $A \vDash C$.

Prove entailment

RECALL: If for all and only the atomic propositions P occurring in a formula A we have $I(P)$ = $I'(P)$,
Then $I \vDash A$ iff $I' \vDash A$. That is:

- The truth value of atomic propositions which occur in A fully determines the truth value of A
- The truth value of the atomic propositions which do not occur in A play no role in the computation of the truth value of A;

EXERCISE: Let X, Y, Z be atomic propositions. Let us take A = (X ∧ ¬Z) and interpretations I = {X, Y} and I' = {X}
Then $I \vDash A$ iff $I' \vDash A$.

PROOF:
RECALL: I = {X, Y} means that I(X) = T, I(Y) = T, I(Z) = F, while I' = {X} means that I(X) = T, I(Y) = F, I(Z) = F

I(A) = I(X ∧ ¬Z) = I(X) ∧ I(¬Z) = T
I'(A) = I'(X ∧ ¬Z) = I'(X) ∧ I'(¬Z) = T

Reasoning by applying entailment properties

Let A and B be propositions.
Prove that the formula P is **valid**, where P: $(A \supset B) \supset (\neg A \lor B)$

PROOF:
A way to prove **validity** is to show that **P** $\models$ T.
This can be done by applying well known tautologies (e.g. De Morgan).

$(A \supset B) \supset (\neg A \lor B)$	$\equiv$	$\neg(A \supset B) \lor (\neg A \lor B)$	*by Implication and disjunction*
	$\equiv$	$\neg(\neg A \lor B) \lor (\neg A \lor B)$	*by Implication and disjunction*
	$\equiv$	T	*by The law of the excluded middle*

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Prove entailment

Let A, B, C, D, E be propositions, and given that:

$$P = (A \lor B) \land (\neg C \lor \neg D \lor E)$$
$$Q1 = A \lor B$$
$$Q2 = (A \lor B \lor C) \land ((B \land C \land D) \supset E)$$
$$Q3 = (A \lor B) \land (\neg D \lor E)$$

Does $P \models Q_i$?

PROOF:

Let $X = A \lor B$, $Y = \neg D \lor E$, then we can rewrite:

$P = X \land (\neg C \lor Y)$; $Q1 = X$; $Q2 = (X \lor C) \land (\neg B \lor \neg C \lor Y)$; $Q3 = X \land Y$

$P \models Q1$ is obvious.

Since $X \models X \lor C$ and $(\neg C \lor Y) \models (\neg B \lor \neg C \lor Y)$, then $P \models Q2$.

Since $Y \models (\neg C \lor Y)$, then $Q3 \models P$ (and not the vice versa).

Prove entailment using Truth Tables

Let A, B, C, D, E be propositions, and given that:

$$
\begin{aligned}
P &= (A \lor B) \land (\neg C \lor \neg D \lor E) \\
Q1 &= A \lor B \\
Q2 &= (A \lor B \lor C) \land ((B \land C \land D) \supset E) \\
Q3 &= (A \lor B) \land (\neg D \lor E)
\end{aligned}
$$

(1) List all truth assignments such that $P \models Qi$

(2) Is there any assignment such that $P \models Qi$ for all i?

Solution to (1): First compute the truth tables for all the propositions above. Then, list all rows for which both P and Qi are true.

Solution to (2): Check whether there is any assignment for which all the sentences above are true.

Models and theories

Let A and B be propositions. Let T1= {(A ⊃ B)} and T2= {¬A ∧ B}.
Say which of the following sentences is true.

a) T1 has 2 models
b) T2 has 1 model
c) T1 and T2 have 2 models in common
d) Does T1 ⊨ T2?
e) Does T2 ⊨ T1?
f) T1 and T2 are maximal theories for M = {B}

	A	B	A ⊃ B	¬A ∧ B
M1	T	T	T	F
M2	T	F	F	F
M3	F	T	T	T
M4	F	F	T	F

ANSWER

b, e, f

Models and theories

Let A and B be propositions. Let T1= {(A ≡ B)} and T2= {¬A, B}.
Say which of the following sentences is true.

a) T1 has 2 models
b) T2 has 1 model
c) T1 and T2 have 1 model in common
d) Does T1 ⊨ T2?
e) Does T2 ⊨ T1?
f) T2 is a maximal theory for M = {B}
g) T1 is a maximal theory for M = {A, B}
h) The minimal model of T2 is M = {B}
i) The minimal model of T1 is M = {A, B}

	A	B	A ≡ B	¬A
M1	T	T	T	F
M2	T	F	F	F
M3	F	T	F	T
M4	F	F	T	T

ANSWER

The models of T1 are {M1, M4}. The models of T2 are {M3}. Note that M1 = {A, B}, M4 = {} and M3 = {B}.
Therefore the true sentences are: a, b, f, h

Models and theories

RECALL: A minimal Model M of T is the intersection of all models of T.

Let A, B, C be propositions. Given the theory T = {¬A ∨ ¬B, C ∧ ¬B}, say which of the following sentences is true.

a) T has 2 models
b) T has 1 model
c) There is no minimal model of T
d) T satisfies M = {A}
e) The minimal model is M = {C}
f) The minimal model is M = {A, C}

ANSWER

a, e

A	B	C	¬A ∨ ¬ B	C ∧ ¬ B
T	T	T	F	F
T	F	T	T	T
F	T	T	T	F
F	F	T	T	T
T	T	F	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

Models and theories

RECALL: A minimal Model M of T is the intersection of all models of T.

Let A, B, C be propositions. Given the theory T = {A ∧ ¬B, A ∧ C, C ∧ ¬B}, say which of the following sentences is true.

a) T has 2 models
b) T has 1 model
c) There is no minimal model of T
d) T satisfies M = {A}
e) The minimal model is M = {A}
f) The minimal model is M = {A, C}

ANSWER

b, d, f

A	B	C	A ∧ ¬ B	A ∧ C	C ∧ ¬ B
T	T	T	F	T	F
T	F	T	T	T	T
F	T	T	F	F	F
F	F	T	F	F	T
T	T	F	F	F	F
T	F	F	T	F	F
F	T	F	F	F	F
F	F	F	F	F	F

DPLL

Input: A set of clauses ϕ.

Output: A truth value indicating whether ϕ is satisfiable.

function DPLL(ϕ)
    while there is a unit clause $\{l\}$ in ϕ do
        $\phi \leftarrow$ unit-propagate(l, ϕ);
    while there is a literal l that occurs pure in ϕ do
        $\phi \leftarrow$ pure-literal-assign(l, ϕ);
    if ϕ is empty then return true;
    if ϕ contains an empty clause then return false;
    $l \leftarrow$ select-literal(ϕ);
    DPLL($\phi \wedge \{l\}$) or **DPLL**($\phi \wedge \{\neg(l)\}$);

Unit Propagation

Pure Literal Elimination

Consistency Check

Empty Close Test

Splitting rule

Convert a formula in CNF

$(A \vee B) \supset \neg A$

$CNF(\neg (A \vee B)) \otimes CNF(\neg A)$

$(CNF(\neg A) \wedge CNF(\neg B)) \otimes CNF(\neg A)$

$(\neg A \wedge \neg B) \otimes \neg A$

$(\neg A \vee \neg A) \wedge (\neg B \vee \neg A)$

$\neg A \wedge (\neg B \vee \neg A)$

In terms of clauses becomes $\{\{\neg A\}, \{\neg B, \neg A\}\}$

$(C \supset \neg A) \wedge \neg(B \wedge \neg A)$

$CNF(C \supset \neg A) \wedge CNF(\neg(B \wedge \neg A))$

$(CNF(\neg C) \otimes CNF(\neg A)) \wedge (CNF(\neg B) \otimes CNF(\neg\neg A))$

$(\neg C \vee \neg A) \wedge (\neg B \vee A)$

In terms of clauses becomes $\{\{\neg C, \neg A\}, \{\neg B, A\}\}$

Use DPLL to prove satisfiability

$B \wedge \neg A \wedge (\neg C \vee A) \wedge (B \vee C)$

In terms of clauses is: $\{\{B\}, \{\neg A\}, \{\neg C, A\}, \{B, C\}\}$

$\{\{B\}, \{\neg A\}, \{\neg C, A\}, \{B, C\}\}$
$\{\{\top\}, \{\neg A\}, \{\neg C, A\}, \{\top, C\}\}$
$\{\{\neg A\}, \{\neg C, A\}\}$
$\{\{\top\}, \{\neg C, \bot\}\}$
$\{\{\neg C\}\}$
$\{\{\top\}\}$
$\{\}$

Therefore the formula is satisfiable. A possible model $M = \{\neg A, B, \neg C\}$

Use DPLL to prove satisfiability

$(C \supset A) \land (C \supset B) \land \neg(A \land B)$

First convert it in CNF: $(\neg C \lor A) \land (\neg C \lor B) \land (\neg A \lor \neg B)$
In terms of clauses is: $\{\{\neg C, A\}, \{\neg C, B\}, \{\neg A, \neg B\}\}$

We can apply the pure literal:
$\{\{\neg C, A\}, \{\neg C, B\}, \{\neg A, \neg B\}\}$
$\{\{T, A\}, \{T, B\}, \{\neg A, \neg B\}\}$
$\{\{\neg A, \neg B\}\}$

We then select a literal for the splitting rule:
$\{\{\neg A\}, \{\neg A, \neg B\}\}$
$\{\{T\}, \{T, \neg B\}\}$
$\{\}$

Therefore the formula is satisfiable.

Use DPLL to prove satisfiability

{{¬A, C, D}, {¬B, F, D}, {¬B, ¬F, ¬C}, {¬D, ¬B}, {B, ¬C, ¬A}, {B, F, C}, {B, ¬F, ¬D}, {A, E}, {A, F}, {¬F, C, ¬E}, {A, ¬C, ¬E}}

There are no unit clauses, nor pure literals. Let us then select the literal A and apply the splitting rule.

{{A}, {¬A, C, D}, {¬B, F, D}, {¬B, ¬F, ¬C}, {¬D, ¬B}, {B, ¬C, ¬A}, {B, F, C}, {B, ¬F, ¬D}, {A, E}, {A, F}, {¬F, C, ¬E}, {A, C, ¬E}}
{{⊤}, {⊥, C, D}, {¬B, F, D}, {¬B, ¬F, ¬C}, {¬D, ¬B}, {B, ¬C, ⊥}, {B, F, C}, {B, ¬F, ¬D}, {⊤, E}, {⊤, F}, {¬F, C, ¬E}, {⊤, C, ¬E}}
{{C, D}, {¬B, F, D}, {¬B, ¬F, ¬C}, {¬D, ¬B}, {B, ¬C}, {B, F, C}, {B, ¬F, ¬D}, {E}, {F}, {¬F, C, ¬E}, {C, ¬E}}
{{C, D}, {¬B, F, D}, {¬B, ¬F, ¬C}, {¬D, ¬B}, {B, ¬C}, {B, F, C}, {B, ¬F, ¬D}, {⊤}, {F}, {¬F, C, ⊥}, {C, ⊥}}
{{C, D}, {¬B, F, D}, {¬B, ¬F, ¬C}, {¬D, ¬B}, {B, ¬C}, {B, F, C}, {B, ¬F, ¬D}, {F}, {¬F, C}, {C}}
{{C, D}, {¬B, ⊤, D}, {¬B, ⊥, ¬C}, {¬D, ¬B}, {B, ¬C}, {B, ⊤, C}, {B, ⊥, ¬D}, {⊤}, {⊥, C}, {C}}
{{C, D}, {¬B, ¬C}, {¬D, ¬B}, {B, ¬C}, {B, ¬D}, {C}, {C}}
{{⊤, D}, {¬B, ⊥}, {¬D, ¬B}, {B, ⊥}, {B, ¬D}, {⊤}, {⊤}}
{{¬B}, {¬D, ¬B}, {B}, {B, ¬D}}
{{⊤}, {¬D, ⊤}, {⊥}, {⊥, ¬D}}
{{¬D}, {}, {¬D}}

There is an empty clause. Therefore it returns false. We need to check with the literal ¬A:

{{¬A}, {¬A, C, D}, {¬B, F, D}, {¬B, ¬F, ¬C}, {¬D, ¬B}, {B, ¬C, ¬A}, {B, F, C}, {B, ¬F, ¬D}, {A, E}, {A, F}, {¬F, C, ¬E}, {A, C, ¬E}}

…

Use DPLL to prove unsatisfiability

By using DPLL, prove the unsatisfiability of $\mathbf{(B \supset A) \land (\neg A \land B)}$

First convert it in CNF: $(\neg B \lor A) \land \neg A \land B$
In terms of clauses is: $\{\{\neg B, A\}, \{\neg A\}, \{B\}\}$

$\{\{\neg B, A\}, \{\neg A\}, \{B\}\}$
$\{\{\neg B, \bot\}, \{\top\}, \{B\}\}$
$\{\{\neg B\}, \{B\}\}$
$\{\{\top\}, \{\bot\}\}$
$\{\{\}\}$

There is an empty clause, therefore it returns false.
Therefore the formaula is unstatistiable.

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Use DPLL to prove validity

By using DPLL, prove the validity of **(A ⊃ B) ⊃ (¬B ⊃ ¬A)**

First negate the formula: ¬((A ⊃ B) ⊃ (¬B ⊃ ¬A))
Then convert it in CNF: (¬A ∨ B) ∧ ¬B ∧ A
In terms of clauses is: {{¬A, B}, {¬B}, {A}}

{{¬A, B}, {¬B}, {A}}
{{¬A, ⊥}, {⊤}, {A}}
{{¬A}, {A}}
{{}}

There is an empty clause, therefore it returns false.
Therefore the formaula is valid.

Use DPLL

- By using DPLL, prove the following formula is satisfiable but not valid: $\neg(A \vee B) \supset C$

  1. Prove satisfiability by computing CNF of the formula and by checking that the DPLL returns true.
  2. Prove not validity by checking that the negation of the formula is satisfiable, i.e. by computing CNF of the negation of the formula and by checking that the DPLL returns true.

Try it yourself.

- By using DPLL, prove the validity of **$(A \wedge B) \vee C \equiv (A \supset \neg B) \supset C$**

Try it yourself.