



UNIVERSITY  
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Dipartimento di Ingegneria e Scienza dell'Informazione



# LOD - The logic of Descriptions

Sept 13,2023

# LOD - The logic of Descriptions

- Introduction
- Domain
- Language
- Interpretation function
- Entailment
- Reasoning problems
- Entailment properties

# LOD - The logic of Descriptions

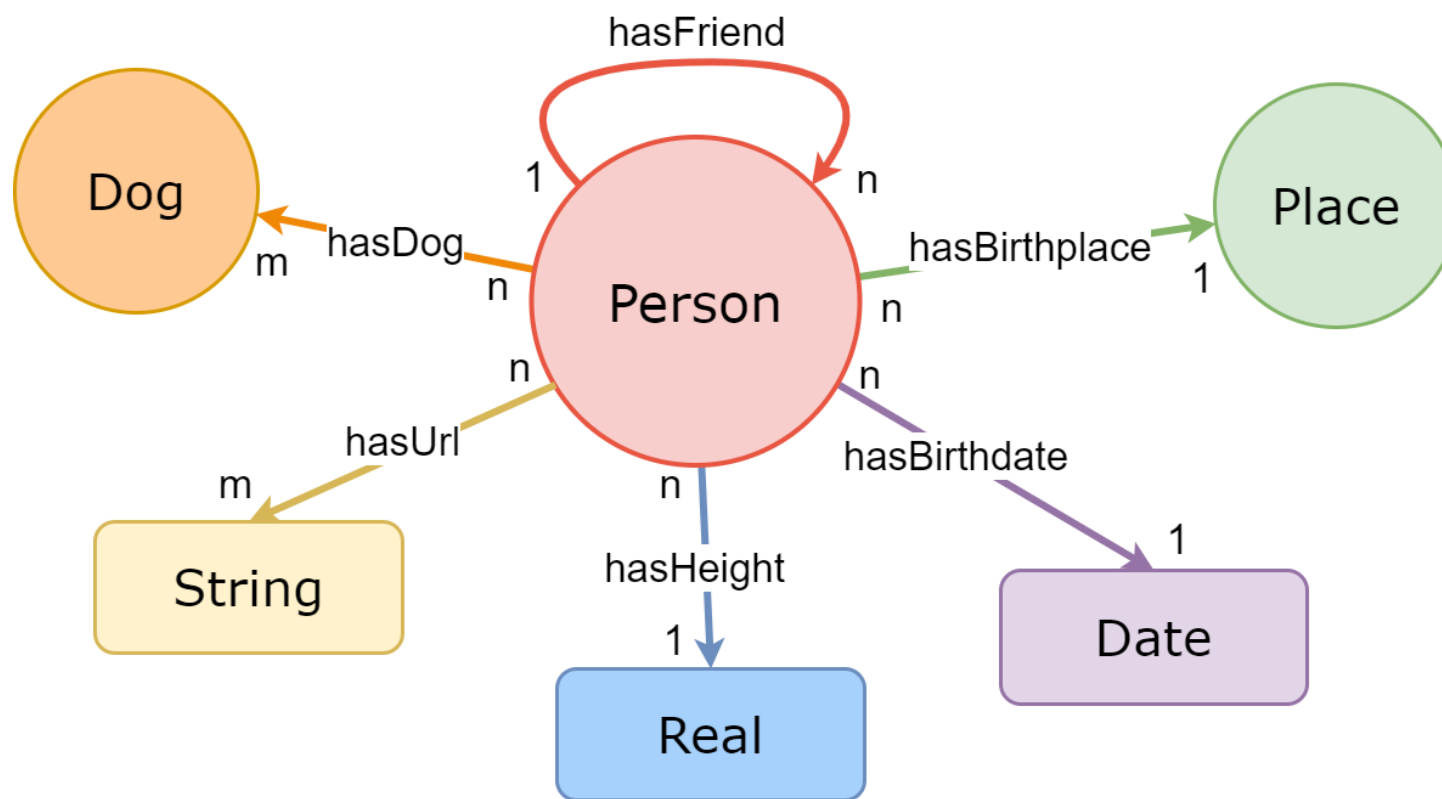
- The Logic of Descriptions (LOD) allows us to reason about the concepts and roles that describe entities in the world.
- Thus, we do not represent and reason about specific entities, but, in a more abstract way, about the classes associated to their properties.
- LOD allows to reason about ETG's.
- Any LOE EG is built with reference to a LOD ETG.
- LOD is conceptually similar to the Logics of Description (DL)

# LOD - The logic of Descriptions

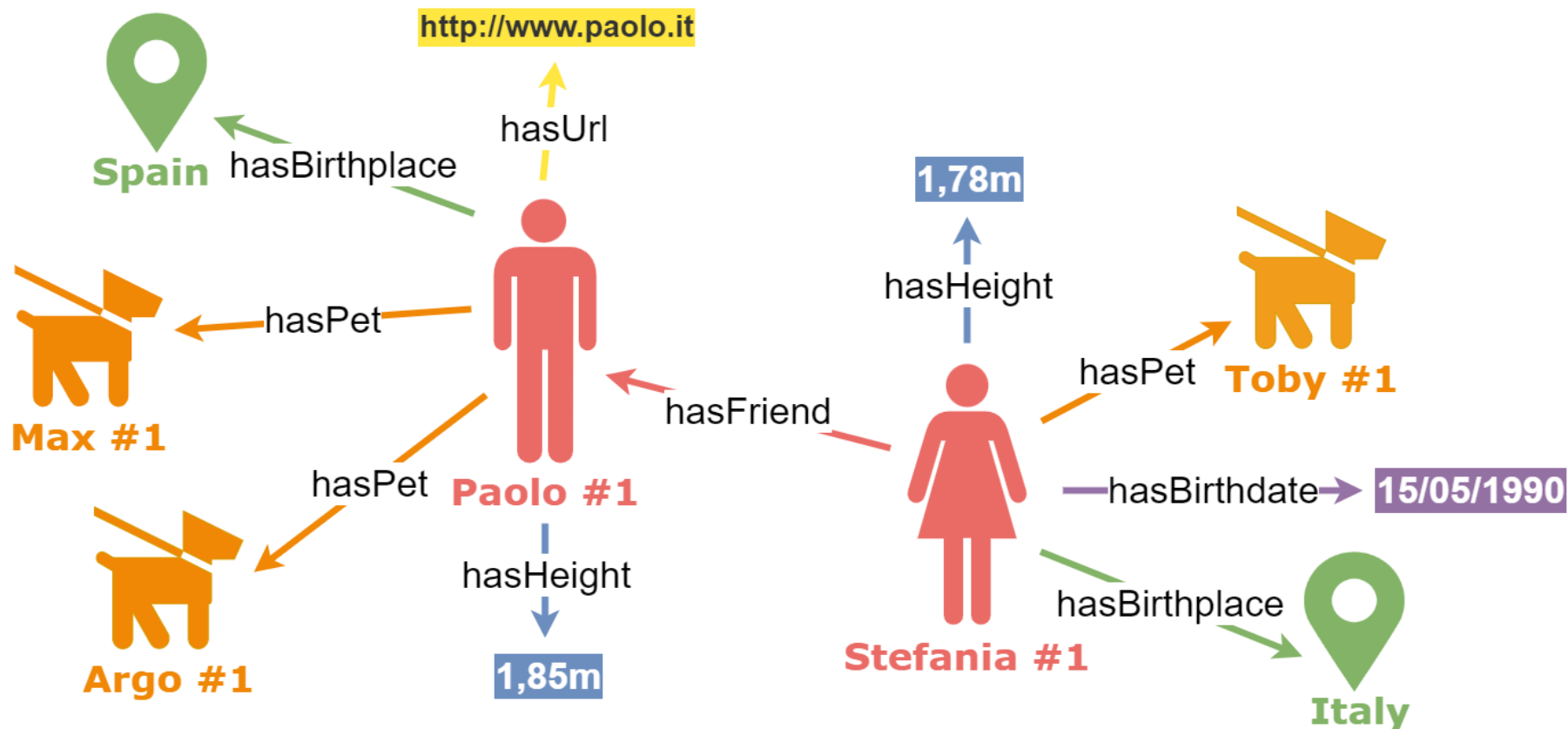
In LOD we have the following ETG fact elements:

- An entity type (etype) is a class of entities (corresponding to the concept to which an entity belongs in a LOE EG);
- A datatype (dtype) is a class of (data) values (corresponding to the dtype to which a value belongs in a LOE EG);
- An Object Property describes a relation between two etypes (not between two entities, as in LOE)
- A Data Property, also called Attribute, describes a characteristic of an etype (not of an entity as in LOE);

# An example of ETG



# An example of EG for the previous ETG



# LoD – The Logic of Descriptions - definition

We formally define LOD as follows

$$\mathbf{LOD} = \langle \mathbf{ETG}, \models_{\mathbf{LOD}} \rangle$$

with

$$\mathbf{ETG} = \langle \mathbf{L}_{\mathbf{LOD}}, \mathbf{D}_{\mathbf{LOD}}, \mathbf{I}_{\mathbf{LOD}} \rangle$$

Below, any time no confusion arises, we drop the subscripts.

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# LoD – Domain/facts

## Definition (Domain, intensional definition)

$$Di = \langle E, \{C\}, \{R\} \rangle$$

where:

$$E = \{e\} \cup \{v\}$$

$$\{C\} = ET \cup DT$$

$$\{R\} = \{OR\} \cup \{DR\}$$

where  $E$  is a set of **entities** and **values**,  $ET = \{E_T\}$ ,  $E_T = \{e\}$  and  $DT = \{D_T\}$ ,  $D_T = \{v\}$  are **sets of entity types (etypes)** and **data types (dtypes)**, respectively, and  $OR$ ,  $DR$  are **(binary) object and data relations**.

**Observation.** LOD allows for the following facts:

- Every etype  $ET$  or dtype  $DT$  is a fact, that is  $ET \subseteq E$ ,  $DT \subseteq E$ .
- Every relation  $R$  populated by its two arguments is a fact, that is,  $OR \subseteq ET_1 \times ET_2$ ,  $DR \subseteq ET \times DT$ .

Facts only have one of the four possible forms above

# An example of domain of ETG (continued)

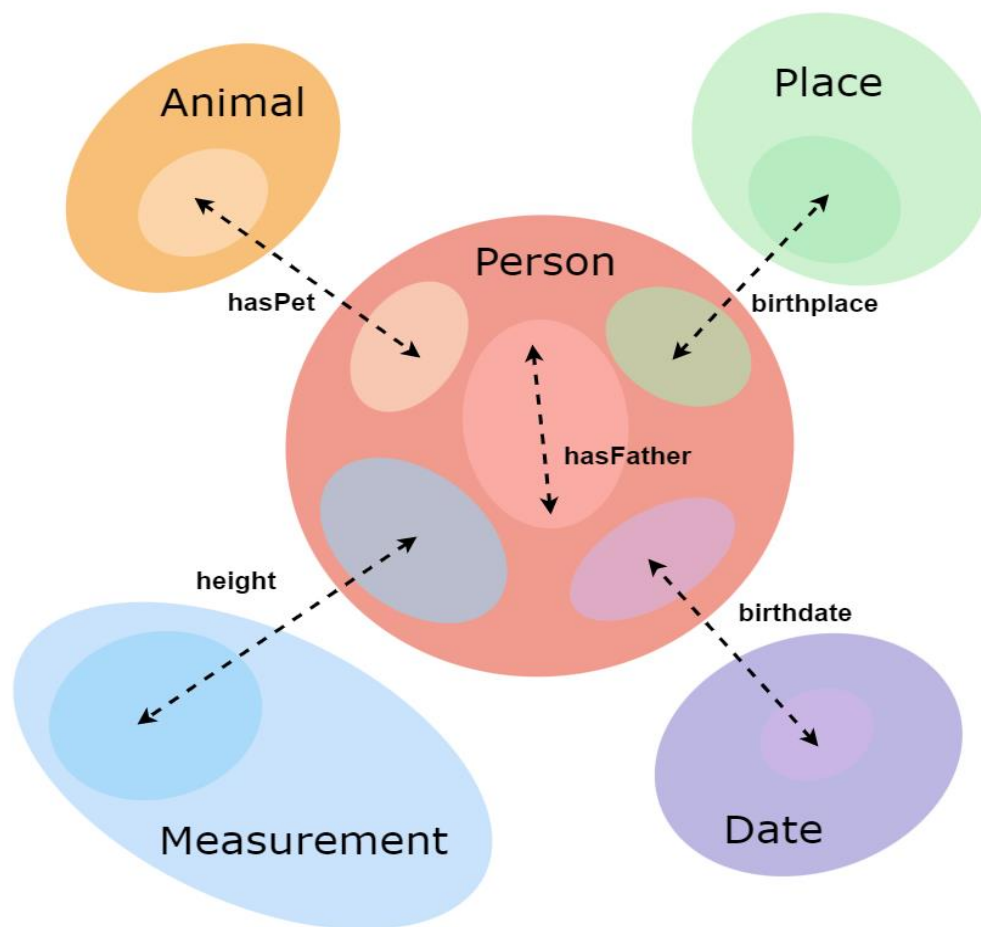
$$ET = \{P, D, L, \text{entity}, \dots\}$$
$$DT = \{\text{Real}, \text{String}, \text{dtype}, \dots\}$$
$$\{R\} = \{hF, hD, hH, hB, hL, hU, \dots\}$$

from which we construct the following facts in the domain:

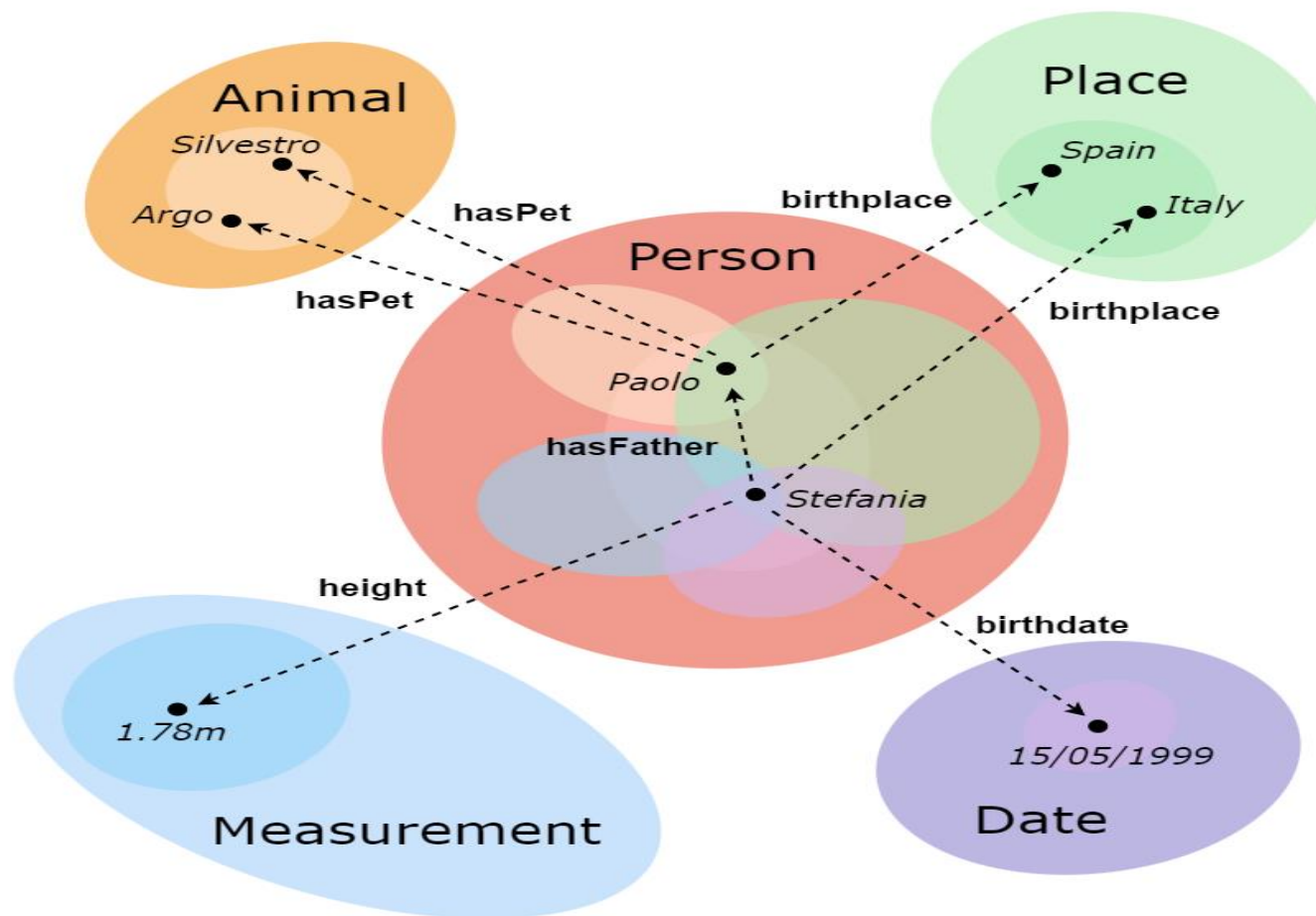
$$D = \{P \subseteq \text{entity}, \text{Real} \subseteq \text{dtype}, hF(P, P), D \subseteq \text{entity}, hD(P, D), hH(P, \text{Real}), \dots\}$$

with, e.g.,  $hF(P, P)$  standing for  $hF \subseteq P \times P$

# An example of ETG – Venn diagram (continued)



# An EG for the example ETG– Venn diagram



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# LoD – Language/wffs

## Definition 11.4 (The language L)

$$L = L_a \cup L_c$$

with

$$L_a = L_A \cup L_{Ac}$$

**Observation 11.2 (LOE *versus* LOD)** LOE allows for atomic assertions. LOD allows for (different) **atomic assertions** ( $L_A$ ), for **complex assertions** ( $L_{Ac}$ ) and also **complex formulas** ( $L_c$ ).

# LoD – Assertions

## Definition (The language of atomic assertions LA)

$$LA = \langle Aa, WA \rangle$$

where  $Aa$  is the alphabet and  $WA$  is the set of formation rules for generating complex assertions.

**Definition 11.6 (Alphabet  $Aa$ )** The alphabet of the atomic formula language contains the etype and dtype names and the names of the object and data properties:

$$Aa = \langle \emptyset, ET \cup DT, \{OP\} \cup \{DP\} \rangle$$

# Assertions – BNF production rules

$$\begin{aligned} \langle \text{assertion} \rangle &::= \langle \text{etype} \rangle \quad | \quad \langle \text{dtype} \rangle \\ &\quad \top \quad | \quad \perp \quad | \\ &\quad \exists \langle \text{objProp} \rangle (\langle \text{etype} \rangle) \quad | \\ &\quad \exists \langle \text{dataProp} \rangle (\langle \text{dtype} \rangle) \quad | \\ &\quad \forall \langle \text{objProp} \rangle (\langle \text{etype} \rangle) \quad | \\ &\quad \forall \langle \text{dataProp} \rangle (\langle \text{dtype} \rangle) \\ \langle \text{etype} \rangle &::= \text{ET1} \quad | \quad \dots \quad | \quad \text{ET}_n \\ \langle \text{dtype} \rangle &::= \text{DT1} \quad | \quad \dots \quad | \quad \text{DT}_n \\ \langle \text{objProp} \rangle &::= \text{OP1} \quad | \quad \dots \quad | \quad \text{OP}_n \\ \langle \text{dataProp} \rangle &::= \text{DP1} \quad | \quad \dots \quad | \quad \text{DP}_n \end{aligned}$$

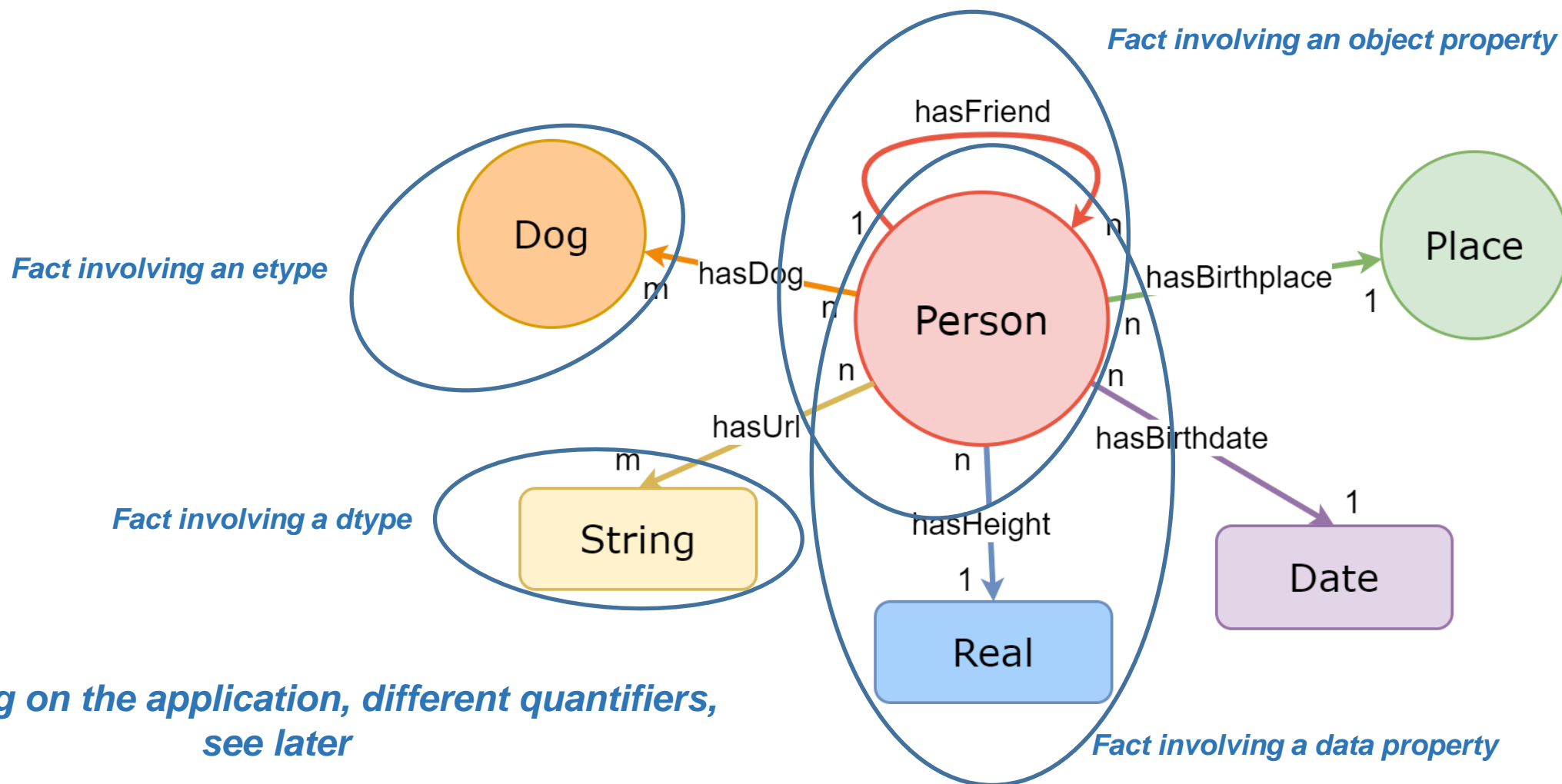
***Compare  
with  
LOE***



# Assertions – Example

- Person (***Intuition:** the set of entities – in the domain of interpretation – which are called persons*)
- $\exists \text{hasFriend.Person}$  (***Intuition:** the set of entities which have – at least – one friend who is a person*)
- Real (***Intuition:** the set of reals*)
- $\exists \text{hasHeight.Real}$  (***Intuition:** the set of entities which have their height - at least one – which measured as a real number*)
- $\forall \text{hasFriend.Person}$  (***Intuition:** the set of entities whose friends are only persons*)

## Example – how assertions represent ETG facts



# LoD – Atomic wffs

**Definition (The language of atomic formulas  $La$ )**

$$La = \langle LA, Wa \rangle$$

where  $LA$  is the language of (atomic) assertions and  $Wa$  is a set of formation rules for generating complex assertions.

**Definition 11.6 (Alphabet)** The alphabet consists of all the formulas in  $LA$

# Complex assertions – BNF production rules

$\langle \text{awff} \rangle ::= \langle \text{assertion} \rangle$

$\langle \text{awff} \rangle ::= \langle \text{awff} \rangle \sqcap \langle \text{awff} \rangle \mid$

$\langle \text{awff} \rangle \sqcup \langle \text{awff} \rangle \mid$

$\neg \langle \text{awff} \rangle$

# Complex assertions – Example

- $\text{Person} \sqcap \exists \text{hasFriend}.\text{Person}$  (**Intuition:** the set of entities which are persons and have a friend which is a person)
- $\text{Person} \sqcup \text{Dog}$  (**Intuition:** the set of entities which are a person or a dog)
- $\text{Person} \sqcap \neg(\exists \text{hasFriend}.\text{Person})$  (**Intuition:** the set of entities which are persons and which do not have a friend which is a person)

# Complex assertions – Example concept names

*Consider the following concept names:*

Vehicle, Boat, Bicycle, Car, Device, Wheel, Engine, Axle, Rotation,  
Water, Human, Driver, Adult, Child

*Formalize the following natural language statements:*

- Nothing (empty set):  $\perp$
- Everything (All the interpretation domain):  $\top$
- Humans and vehicles:  $\text{Human} \sqcap \text{Vehicle}$
- Vehicles and not boats:  $\text{Vehicle} \sqcap \neg \text{Boat}$
- Wheels or engines and humans:  $(\text{Wheel} \sqcup \text{Engine}) \sqcap \text{Human}$
- Adults or children:  $\text{Adult} \sqcup \text{Child}$

# Complex assertions – Example roles

*Consider the previous concept names plus the following role names:*

hasPart, poweredBy, capableOf, travelsOn, controls

*Formalize in DL the following natural language statements:*

1. Those vehicles that have wheels and are powered by an engine
2. Those vehicles that have wheels and are powered by a human
3. Those vehicles that travel on water
4. Those objects which have no wheels
5. Those objects which do not travel on water
6. Those devices that have an axle and are capable of rotation
7. Those humans who control a vehicle
8. The drivers of cars

# Complex assertions – Example roles

1.  $\text{Vehicle} \sqcap \exists \text{hasPart.Wheel} \sqcap \exists \text{poweredBy.Engine}$
2.  $\text{Vehicle} \sqcap \exists \text{hasPart.Wheel} \sqcap \exists \text{poweredBy.Human}$
3.  $\text{Vehicle} \sqcap \exists \text{travelsOn.Water}$
4.  $\forall \text{hasPart.} \neg \text{Wheel}$
5.  $\forall \text{travelsOn.} \neg \text{Water}$
6.  $\text{Device} \sqcap \exists \text{hasPart.Axle} \sqcap \exists \text{capableOf.Rotation}$
7.  $\text{Human} \sqcap \exists \text{controls.Vehicle}$
8.  $\text{Driver} \sqcap \exists \text{controls.Car}$



## LoD – complex wffs (the full language)

**Definition** (The language of complex formulas  $L_c$ )

$$L_c = \langle L_a, W_c \rangle$$

where  $L_a$  is the language of complex assertions from above and  $W_c$  is a set of formula constructors

**Definition (Alphabet)** The alphabet is all the atomic formulas (atomic and complex assertions) in  $L_a$

# Complex formulas – BNF production rules

$$\begin{aligned} \langle \text{cwff} \rangle &::= \langle \text{concept} \rangle \sqsubseteq \langle \text{awff} \rangle \mid \\ &\quad \langle \text{concept} \rangle \equiv \langle \text{awff} \rangle \end{aligned}$$

*where:*

- $\langle \text{concept} \rangle$  : we restrict  $\langle \text{concept} \rangle$  to be an etype
- $\sqsubseteq$  : subsumption relation
- $\equiv$  : equivalence relation

**NOTE:** In most common logics in the literature we have  $\langle \text{awff} \rangle$  instead of  $\langle \text{concept} \rangle$ .

# Complex formulas

- $\langle \text{concept} \rangle \sqsubseteq \langle \text{awff} \rangle$ 
  - A **concept inclusion** (formula)
  - *To be read  $\langle \text{concept} \rangle$  is subsumed by  $\langle \text{awff} \rangle$*
- $\langle \text{concept} \rangle \equiv \langle \text{awff} \rangle$ 
  - A **concept definition** (formula)
  - *To be read  $\langle \text{concept} \rangle$  is equivalent to  $\langle \text{awff} \rangle$*

# Complex formulas – concept inclusion examples

1. Boats have no wheels
2. Cars and bicycles do not travel on water
3. Drivers of cars are adults
4. Humans are not vehicles
5. Wheels or engines are not humans
6. Humans are either adults or children
7. Adults are not children

# Complex formulas – concept inclusion examples

1.  $\text{Boat} \sqsubseteq \forall \text{hasPart.} \neg \text{Wheel}$
2.  $\text{Car} \sqcup \text{Bicycle} \sqsubseteq \forall \text{travelsOn.} \neg \text{Water}$
3.  $\text{Driver} \sqcap \exists \text{controls.Car} \sqsubseteq \text{Adult}$
4.  $\text{Human} \sqsubseteq \neg \text{Vehicle}$
5.  $\text{Wheel} \sqcup \text{Engine} \sqsubseteq \neg \text{Human}$
6.  $\text{Human} \sqsubseteq \text{Adult} \sqcup \text{Child}$
7.  $\text{Adult} \sqsubseteq \neg \text{Child}$

# Complex formulas – definition examples

1. Cars are exactly those vehicles that have wheels and are powered by an engine
2. Bicycles are exactly those vehicles that have wheels and are powered by a human
3. Boats are exactly those vehicles that travel on water
4. Wheels are exactly those devices that have an axle and are capable of rotation
5. Drivers are exactly those humans who control a vehicle

# Complex formulas – definition examples

1.  $\text{Car} \equiv \text{Vehicle} \sqcap \exists \text{hasPart.Wheel} \sqcap \exists \text{poweredBy.Engine}$
2.  $\text{Bicycle} \equiv \text{Vehicle} \sqcap \exists \text{hasPart.Wheel} \sqcap \exists \text{poweredBy.Human}$
3.  $\text{Boat} \equiv \text{Vehicle} \sqcap \exists \text{travelsOn.Water}$
4.  $\text{Wheel} \equiv \text{Device} \sqcap \exists \text{hasPart.Axle} \sqcap \exists \text{capableOf.Rotation}$
5.  $\text{Driver} \equiv \text{Human} \sqcap \exists \text{controls.Vehicle}$

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# Interpretation of atomic formulas

$$I(\top) = \mathcal{D}$$

$$I(\perp) = \emptyset$$

$$I(A \sqcap B) = I(A) \cap I(B)$$

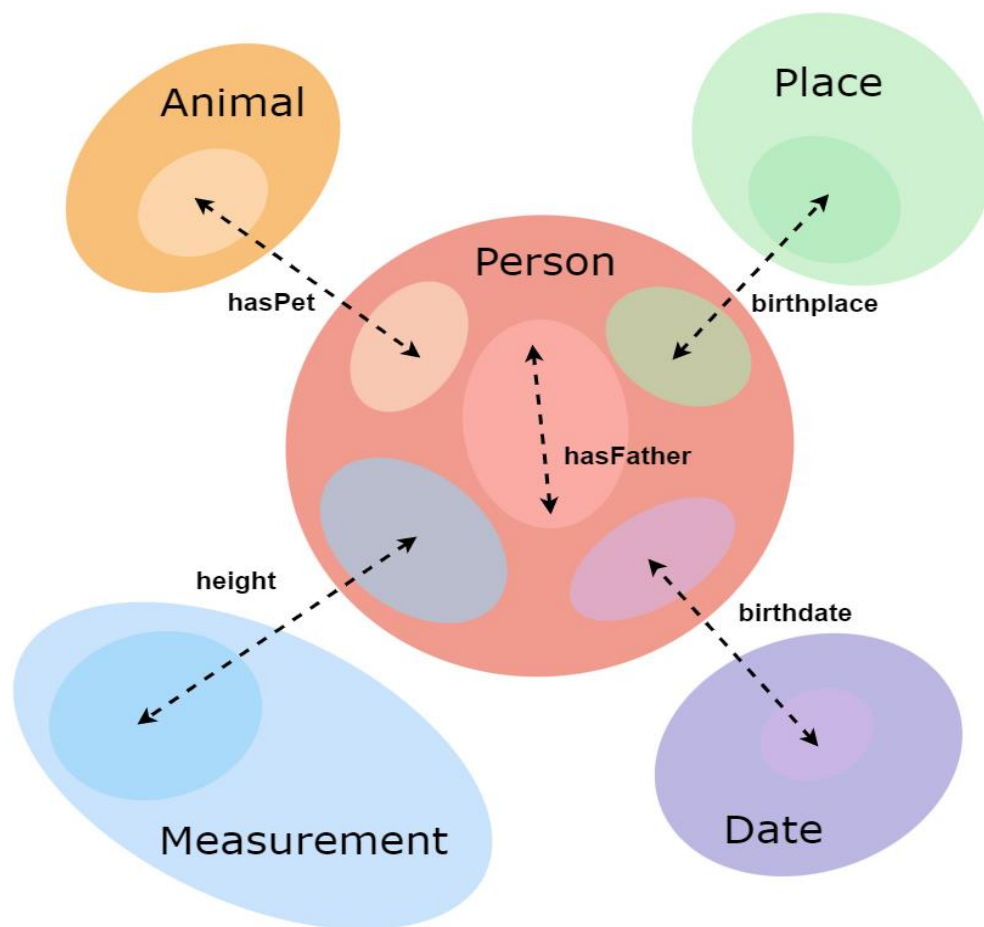
$$I(A \sqcup B) = I(A) \cup I(B)$$

$$I(\neg A) = \mathcal{D} \setminus I(A)$$

$$I(\exists R.A) = \{d \in \mathcal{D} \mid \text{there is an } e \in \mathcal{D} \text{ with } (d, e) \in I(R) \text{ and } e \in I(A) \}$$

$$I(\forall R.A) = \{d \in \mathcal{D} \mid \text{for all } e \in \mathcal{D} \text{ if } (d, e) \in I(R) \text{ then } e \in I(A) \}$$

# Interpretation function (Venn diagram) – example above



**Most often we assume  
both  
universal and existential  
quantifier**

**The first  
does not imply the  
second  
(when premise of the  
first  
is never satisfied)**

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# Entailment relation

## Definition (Entailment $\models$ )

- $M \models w1 \sqsubseteq w2$  iff  $I(w1) \subseteq I(w2)$
- $M \models w1 \equiv w2$  iff  $I(w1) = I(w2)$   
iff  $I(w1) \subseteq I(w2)$  and  $I(w2) \subseteq I(w1)$

With  $w1, w2 \in L$ ;

NOTE:  $w1$  is not necessarily an assertion

# Entailment relation (extended)

## Definition (Entailment $\models$ )

1.  $M \models w1 \sqsubseteq w2$  iff  $I(w1) \subseteq I(w2)$
2.  $M \models w1 \equiv w2$  iff  $I(w1) = I(w2)$   
iff  $I(w1) \subseteq I(w2)$  and  $I(w2) \subseteq I(w1)$
3.  $M \models w1 \supseteq w2$  iff  $I(w2) \subseteq I(w1)$
4.  $M \models w1 \perp w2$  iff  $I(w1) \cap I(w2) \subseteq \emptyset$

## with

- $w1, w2 \in L$ ;
- $w1 \supseteq w2$  a notational variant of  $w2 \sqsubseteq w1$ ;
- $w1 \perp w2$  a notation for  $w1 \sqcap w2 \sqsubseteq \perp$  (a special case of  $w1 \sqsubseteq w2$ )

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# Reasoning problems (definition)

- Model checking
- Satisfiability with respect to  $T$
- Subsumption with respect to  $T$
- Equivalence with respect to  $T$
- Disjointness with respect to  $T$

# Model checking

**Model checking.** Given  $I$ ,  $M$  is a model of a theory  $T$ , where  $T$  is a set of complex formulas, if the following two conditions hold.

- If  $C \sqsubseteq D \in T$ , then  $I(C) \subseteq I(D)$
- If  $C \equiv D \in T$ , then  $I(C) = I(D)$

**NOTE:** A model checking problem



# Satisfiability

**Satisfiability with respect to T.** A complex assertion  $C$  is satisfiable with respect to  $T$  if there exists an interpretation function  $I$  of  $T$  such that  $I(C)$  is nonempty (i.e.,  $I(C)$  is a model).

In this case we say also that  $I$  is a model of  $C$ , with respect to  $T$ .

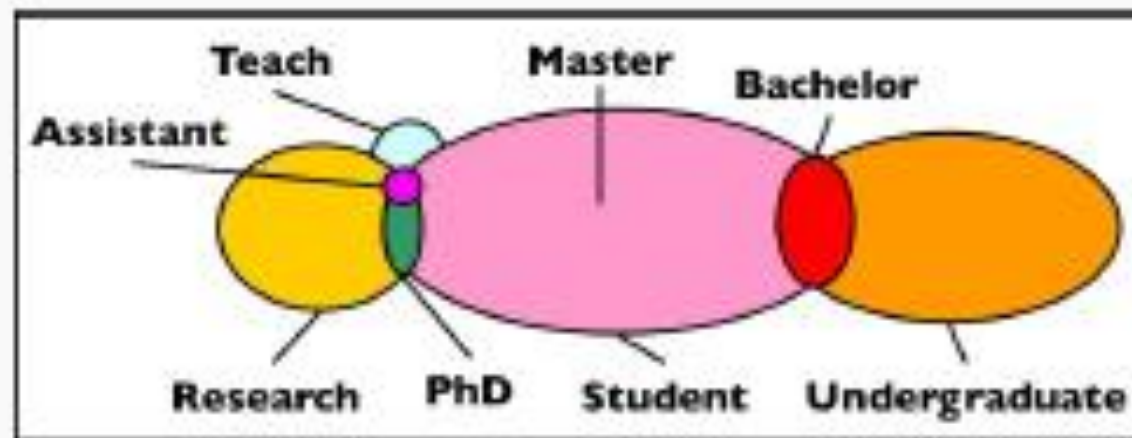
**NOTE 1:**  $T$  can also be empty (as with all the next reasoning problems)

**NOTE 2:** A satisfiability problem ( $I$  builds the model)

# Satisfiability

Consider the Tbox

$$T = \begin{cases} \text{Undergraduate} \sqsubseteq \neg \text{Teach} \\ \text{Bachelor} \equiv \text{Student} \sqcap \text{Undergraduate} \\ \text{Master} \equiv \text{Student} \sqcap \neg \text{Undergraduate} \\ \text{PhD} \equiv \text{Master} \sqcap \text{Research} \\ \text{Assistant} \equiv \text{PhD} \sqcap \text{Teach} \end{cases}$$



# Satisfiability

## Example 1

Is  $\text{Bachelor} \sqcap \text{PhD}$  satisfied by  $\mathcal{T}$ ? (No)

The problem can be formalized as:

$$\mathcal{T} \models \text{Bachelor} \sqcap \text{PhD}$$

**Proof:**

$\text{Bachelor} \sqcap \text{PhD}$

$\equiv (\text{Student} \sqcap \text{Undergraduate}) \sqcap (\text{Master} \sqcap \text{Research})$

$\equiv (\text{Student} \sqcap \text{Undergraduate}) \sqcap ((\text{Student} \sqcap \neg \text{Undergraduate}) \sqcap \text{Research})$

$\equiv \text{Student} \sqcap \text{Undergraduate} \sqcap \neg \text{Undergraduate} \sqcap \text{Research}$

$\equiv \text{Student} \sqcap \perp \sqcap \text{Research}$

# Subsumption

**Subsumption with respect to T** A complex assertion  $C$  is subsumed by a complex assertion  $D$  with respect to  $T$  if

$$I(C) \subseteq I(D)$$

for every  $I$  (used to build models for  $T$ , with  $T$  possibly empty) .

In this case we write

$$C \sqsubseteq_T D$$

or

$$T \models C \sqsubseteq D$$

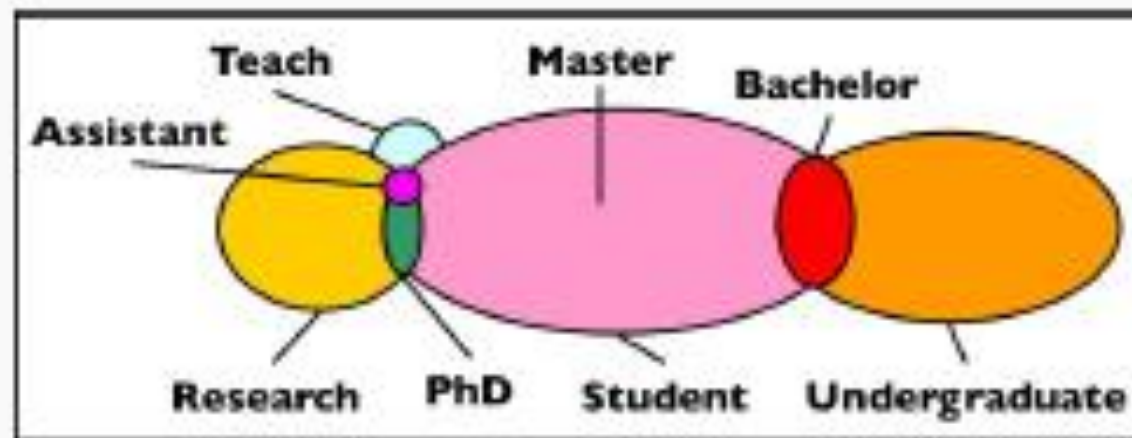
**NOTE:** A validity problem

# Subsumption

Consider the Tbox

$$T = \begin{cases} \text{Undergraduate} \sqsubseteq \neg \text{Teach} \\ \text{Bachelor} \equiv \text{Student} \sqcap \text{Undergraduate} \\ \text{Master} \equiv \text{Student} \sqcap \neg \text{Undergraduate} \\ \text{PhD} \equiv \text{Master} \sqcap \text{Research} \\ \text{Assistant} \equiv \text{PhD} \sqcap \text{Teach} \end{cases}$$

It should be  
checked  
for all models of T



# Subsumption

## Example 2

Is  $\text{PhD} \sqsubseteq \text{Student}$  satisfiable? (Yes)

The problem can be formalized as:

$$\mathcal{T} \models \text{PhD} \sqsubseteq \text{Student}$$

**Proof:**

$\text{PhD}$

$\equiv \text{Master} \sqcap \text{Research}$

$\equiv (\text{Student} \sqcap \neg \text{Undergraduate}) \sqcap \text{Research}$

$\sqsubseteq \text{Student}$

# Equivalence

**Equivalence with respect to T.** Two complex assertions C and D are equivalent with respect to T if

$$I(C) = I(D)$$

for every model I (used to build models for T, with T possibly empty) .

In this case we write

$$C \equiv_T D$$

or

$$T \models C \equiv D$$

**NOTE:** A validity problem

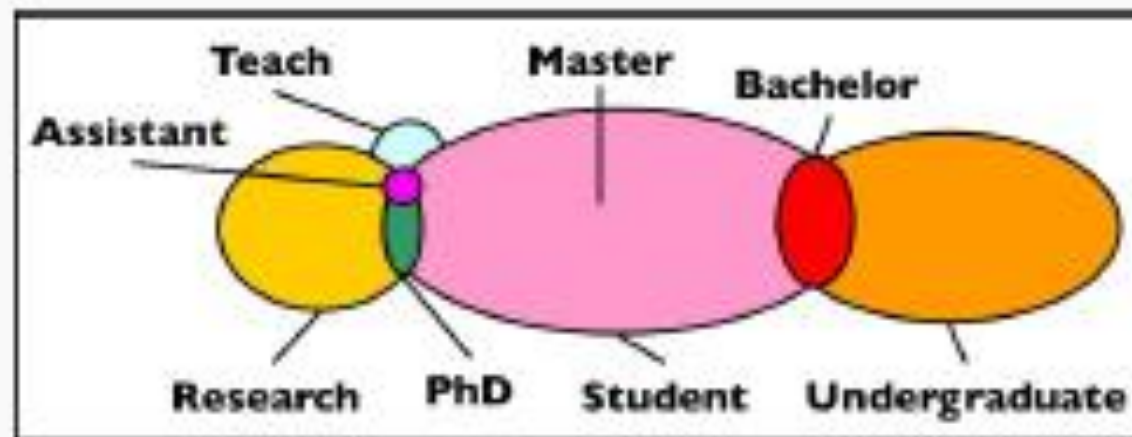


# Equivalence

Consider the Tbox

$$T = \begin{cases} \text{Undergraduate} \sqsubseteq \neg \text{Teach} \\ \text{Bachelor} \equiv \text{Student} \sqcap \text{Undergraduate} \\ \text{Master} \equiv \text{Student} \sqcap \neg \text{Undergraduate} \\ \text{PhD} \equiv \text{Master} \sqcap \text{Research} \\ \text{Assistant} \equiv \text{PhD} \sqcap \text{Teach} \end{cases}$$

It should be  
checked  
for all models of T





# Equivalence

## Example 3

Is  $\text{Student} \equiv \text{Bachelor} \sqcup \text{Master}$  consistent with  $\mathcal{T}$ ? (Yes)

The problem can be formalized as:

$$\mathcal{T} \models \text{Student} \equiv \text{Bachelor} \sqcup \text{Master}$$

**Proof:**

$\text{Bachelor} \sqcup \text{Master}$

$\equiv (\text{Student} \sqcap \text{Undergraduate}) \sqcup (\text{Student} \sqcap \neg \text{Undergraduate})$

$\equiv \text{Student} \sqcup (\text{Undergraduate} \sqcap \neg \text{Undergraduate})$

$\equiv \text{Student} \sqcup \top$

$\equiv \text{Student}$

# Disjointness

**Disjointness with respect to T.** Two complex assertions C and D are disjoint with respect to T if

$$I(C) \cap I(D) = \emptyset$$

for every I (used to build models for T, with T possibly empty) .

In this case we write

$$C \perp_T D$$

or

$$T \models C \perp D$$

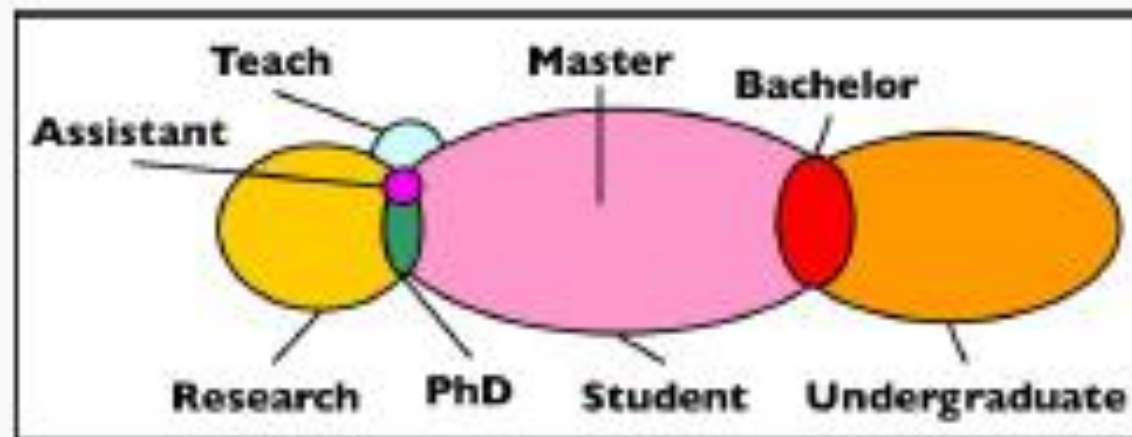
**NOTE:** A validity problem

# Disjointness

Consider the Tbox

$$T = \begin{cases} \text{Undergraduate} \sqsubseteq \neg \text{Teach} \\ \text{Bachelor} \equiv \text{Student} \sqcap \text{Undergraduate} \\ \text{Master} \equiv \text{Student} \sqcap \neg \text{Undergraduate} \\ \text{PhD} \equiv \text{Master} \sqcap \text{Research} \\ \text{Assistant} \equiv \text{PhD} \sqcap \text{Teach} \end{cases}$$

It should be  
checked  
for all models of T



# Disjointness

## Example 4

Is  $\text{Undergraduate} \sqcap \text{Assistant} \sqsubseteq \perp$  consistent with  $\mathcal{T}$ ? (Yes)

The problem can be formalized as:

$$\mathcal{T} \models \text{Undergraduate} \sqcap \text{Assistant} \sqsubseteq \perp$$

**Proof:**

$$\begin{aligned} & \text{Undergraduate} \sqcap \text{Assistant} \\ & \sqsubseteq \neg \text{Teach} \sqcap \text{Assistant} \\ & \equiv \neg \text{Teach} \sqcap (\text{PhD} \sqcap \text{Teach}) \\ & \equiv \perp \sqcap \text{PhD} \\ & \equiv \perp \end{aligned}$$

# Reasoning problems (Reduction)

- **Model checking.** Core Q/A functionality (see LOE)
- **Equivalence.**  $C \equiv_T D$  iff  $C \sqsubseteq_T D$  and  $D \sqsubseteq_T C$
- **Subsumption.**  $C \sqsubseteq_T D$  iff  $C \sqcap \neg D$  is *unsatisfiable* with respect to  $T$
- **Disjointness.**  $C \perp_T D$  iff  $C \sqcap D$  is *unsatisfiable*

## Observation

- LOD reasoning can be implemented as *LOD satisfiability* (see above)
- LOD satisfiability can be implemented as Truth Table satisfiability (see later)

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- **Entailment properties**

# Observations (Logical entailment – properties)

**Intuition (Reflexivity):**  $w \models w$  **YES!**

**Intuition (Cut):** If  $\Gamma \models w_1$  and  $\Sigma \cup \{w_1\} \models w_2$  then  $\Gamma \cup \Sigma \models w_2$  **YES!**

**Intuition (Compactness)**

If  $\Gamma \models w$  then there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \models w$  . **YES!**

**Intuition (Monotonicity):** If  $\Gamma \models w$  then  $\Gamma \cup \Sigma \models w$  **YES!**

**Intuition (NonMonotonicity)**  $\Gamma \models w$  and  $\Gamma \cup \Sigma \not\models w$  **NO!**



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