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Smooth unbiased multivariate probability simulators for maximum likelihood estimation of limited dependent variable models*

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We apply a new simulation method that solves the multidimensional probability integrals that arise in maximum likelihood estimation of a broad class of limited dependent variable models. The simulation method has four key features: the simulated choice probabilities are unbiased; they are a continuous and differentiable function of the parameters of the model; they are bounded between 0 and 1; and their computation takes an effort that is nearly linear in the dimension of the probability integral, independent of the magnitudes of the true probabilities. We also show that the new simulation method produces probability estimates with substantially smaller variance than those generated by acceptance–rejection methods or by Stern's (1992) method. The simulated probabilities can therefore be used to revive the Lerman and Manski (1981) procedure of approximating the likelihood function using simulated choice probabilities by overcoming its computational disadvantages.

1. Introduction

It has long been known that classical estimation of limited dependent variable (LDV) models is computationally intractable if one does not impose restrictive correlation structures on the unobservables, because of a need to evaluate high-dimensional integrals in likelihood and conditional moment conditions.

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Recently, McFadden (1989) and Pakes and Pollard (1989) have shown how the multivariate integration problems in a class of LDV models, namely discrete choice models, can be avoided through the method of simulated moments (MSM), which substitutes unbiased simulators for the choice probabilities in the conditional moment conditions. In a related line of work, Lerman and Manski (1981) proposed the method of simulated maximum likelihood (SML), which used (unbiased) frequency simulators of choice probabilities directly in likelihood function evaluation.

In this paper we apply a new simulation method for choice probabilities that builds on recent progress in Monte Carlo integration techniques by Geweke (1989), Hajivassiliou (1990), and Hajivassiliou and McFadden (1990) and can be employed in MSM and SML estimators. This method has four key features: the simulated choice probabilities are unbiased; they are a continuous and differentiable function of the parameters of the model; they are bounded away from 0 and 1; and their computation takes an effort that is nearly linear in the dimension of the probability integral. Most notably, and in contrast to other simulation methods proposed in the literature, the effort is independent of the true probabilities.

In section 2, we present the classical LDV model and explain the multivariate integration problems arising in maximum likelihood estimation. In section 3, we describe the new unbiased choice probability simulator, which we term the 'smooth recursive conditioning' simulator (SRC), and contrast its properties to the ones of the classic frequency simulator used in Lerman and Manski (1981) and to those of the Stern (1992) simulator. The properties of the SRC simulator are derived analytically.

In section 4, we present Monte Carlo results on the distribution of the three probability simulators. We find that the SRC method completely dominates the other two methods. We show that the SRC method produces probability estimates with substantially smaller variance than those generated by acceptance–rejection or by Stern's (1992) method. This is important because, as we show in section 5, it allows us to modify the Lerman and Manski (1981) simulation estimation method through the use of the SRC simulator to define a smooth simulated maximum likelihood (SSML) estimator. The small variance of the SRC simulator permits the insertion of the simulated probabilities in a likelihood function without the tough trade-off between a large bias and a heavy computational burden that has handicapped the Lerman and Manski (1981) procedure of approximating the likelihood function by frequency simulated choice probabilities. The SSML estimator has the additional advantage, likely to be important for applied researchers, that its implementation requires only a standard likelihood optimization package, modified suitably to use the SRC method in calculating the contributions to the likelihood function. Section 6 follows with some Monte Carlo results on the distribution of parameters estimated by SSML. These results show that due to the small variance of

the simulated probabilities, the bias intrinsic to the SSML procedure becomes negligible even for rather small numbers of replications, in sharp contrast to the Lerman and Manski (1981) procedure. Section 7 concludes with a brief summary.

2. Multidimensional integration problems in LDV models

A broad class of limited dependent variable models can be characterized by a combination of a linear latent variable model and a nonlinear relation between the latent variable and an observable variable that can be expressed as a set of inequalities on linear combinations of the latent variable. To be specific, the linear multivariate latent variable model,

$$u_i = X_i\beta + \varepsilon_i, \qquad i = 1, \ldots, I, \tag{1}$$

relates the latent $u \in \mathbb{R}^I$ to an $I \cdot K$ matrix of observed explanatory variables X, a K-dimensional parameter vector β, and a disturbance $\varepsilon \in \mathbb{R}^I$ with a covariance matrix Ω. The nonlinear function τ,

$$y_i = \tau(u_i), \qquad i = 1, \ldots, I, \tag{2}$$

relates the latent variable $u \in \mathbb{R}^I$ to an observed variable $y \in \mathbb{R}^I$ and can be described by inequalities between linear combinations of u_i:

$$\begin{aligned} T(y) &= \{u \in \mathbb{R}^I \mid y_i = \tau(u_i), i = 1, \ldots, I\} \\ &= \{u \in \mathbb{R}^I \mid a \leq Au \leq b\}. \end{aligned} \tag{3}$$

The limiting vectors a and $b \in \mathbb{R}^I_\infty$ and the $I \cdot I$ matrix A may depend on y and $X\beta$.[1]

The most prominent examples in this class are cross-sectional discrete choice models. In this case I denotes the number of alternatives, τ is defined by

$$\begin{aligned} \tau(u_i) = 1 \quad &\Leftrightarrow \quad i = \operatorname*{argmax}_j (u_j, j = 1, \ldots, I), \\ = 0 \quad &\text{otherwise}, \end{aligned} \tag{4}$$

and the inequalities in (3) are defined by

[1] The inequality in (3) is strict if a or b is infinite.

$$A = \begin{bmatrix} -1 & & & 1 & \\ & -1 & & 1 & \\ & & \cdots & & \\ & & -1 & 1 & \\ & & & 1 & \\ & & \cdots & & \\ & & & 1 & -1 \end{bmatrix},$$

$$a = \begin{bmatrix} X_1\beta - X_i\beta \\ X_2\beta - X_i\beta \\ \cdots \\ X_{i-1}\beta - X_i\beta \\ -\infty \\ \cdots \\ X_I\beta - X_i\beta \end{bmatrix}, \qquad b = \begin{bmatrix} \infty \\ \\ \vdots \\ \\ \\ \infty \end{bmatrix}. \tag{5}$$

The column of ones in the matrix A is in column $i = \text{argmax}_j (u_j, i = 1, \ldots, I)$ and the element $-\infty$ in the limiting vector a in the corresponding row.[2]

Hajivassiliou and McFadden (1990) elaborate on other examples in this class, such as multiperiod probit and tobit models and multimarket disequilibrium models. They develop the method of simulated scores (MSS) and report on an application to debt crises in less developed countries. Börsch-Supan et al. (1992) investigate a multiperiod-multinomial probit model in which I represents the number of periods times the number of alternatives, and they apply the model to living arrangement choices of the elderly.

The likelihood of an observation (y, X) associated with model (1)–(3) is

$$\ell(y, X; g, \beta) = \int_{u \in T(y)} g(u - X\beta)\,\mathrm{d}u, \tag{6}$$

where $g: \mathbb{R}^I \to \mathbb{R}$ denotes the joint multivariate density function of the disturbances ε_i, $i = 1, \ldots, I$, in the latent variable model (1), and $T(y)$ is the set of latent variables u obeying the nonlinear relation $\tau(\cdot)$ for a given observed dependent variable y, given by (3). Unless the joint density function g and the area of

[2] Hence, the dimension I can be reduced by one.

integration $T(y)$ are particularly benign, the integral in (6) will not have a closed form, necessitating approximation solutions.

The following section builds on recent progress in Monte Carlo integration techniques by Geweke (1989), Hajivassiliou (1990), and Hajivassiliou and McFadden (1990) and discusses the new simulation method to compute the multivariate integral $\ell(y, X; g, \beta)$ for models in the class of limited dependent variable models defined by (1)–(3).

3. Computing unbiased choice probabilities

Estimating the parameters in (6) is a formidable task because it requires, in the most general case, an evaluation of an I-dimensional integral for each observation and each iteration in the maximization process. The dimension I can become very large, particularly so in the case of multiperiod-multinomial discrete choice models.

Closed-form solutions for (6) only exist in some instances, whenever the specification of the joint density function g and the area of integration $T(y)$ are particularly benign, thereby contributing to the popularity of such specifications. The most prominent examples are those in which g is derived from the family of generalized extreme-value (GEV) distributions, e.g., the cross-sectional multinomial logit (MNL) or nested multinomial logit (NMNL) models. Closed-form solutions also exist if multiperiod variants of these models are combined with a one-factor random effect structure in which the random effects are also extreme-value distributed [e.g., McFadden (1984)]. But these GEV-type models have the disadvantage of requiring relatively rigid correlation structures, which must be assumed a priori. This is restrictive because the covariance matrix Ω of the ε in (1) can have up to $(I + 1) \times I/2 - 1$ deep parameters (the upper triangle of Ω including the diagonal, minus one element to scale the parameter vector β). These are many more parameters than GEV-type models can carry. Moreover, the specification of Ω is not constrained by hierarchical structures as it is in NMNL models.[3]

Numerical integration of (6) is computationally infeasible since the number of operations increases exponentially with the dimension I. Approximation methods for the multivariate normal distribution, such as the Clark approximation [Clark (1961), Daganzo (1981)] or its variant proposed by Langdon (1984), are tractable – their number of operations increases quadratically with I. They remain unsatisfactory, however, since their relatively large bias depends on the (unknown) covariance structure of the latent variable and cannot be controlled by increasing the number of observations.[4]

[3] Börsch-Supan (1989) investigates the estimates and error structure implied by NMNL models if the true model is multinomial probit.

[4] Horowitz, Sparmonn, and Daganzo (1984) report on the accuracy of the Clark approximation.

Simulation methods have been introduced to discrete choice models by Lerman and Manski (1981). The probability (6) is simulated by drawing pseudo-random realizations from the underlying error process (1). The main appeal of simulation methods is that, for any given number of draws, the number of operations increases essentially only linearly with the dimension I, thereby permitting the specification of quite general error structures. We proceed to discuss alternative simulators for the choice probabilities.

3.1. *The frequency simulator*

The most straightforward simulation method is to simulate the probability (6) by an observed frequency:

$$\frac{1}{R}\sum_{r=1}^{R} \mathbf{1}(a \leq A \cdot u_r \leq b), \quad \text{with } u_r \text{ drawn } \mathrm{N}(X\beta, \Omega), \tag{7}$$

for R draws or replications. The indicator function $\mathbf{1}(A)$ takes the value 1 when event A is true and 0 otherwise.

Lerman and Manski used this simulator in their simulated maximum likelihood approach. It has two drawbacks. If the true probability is smaller than 1, then for any given finite number of draws, R, the probability of a zero frequency count in (7) is strictly positive. Hence, the frequency simulator yields consistent likelihood estimates only if both sample size *and* number of draws per observation goes to infinity. This is reflected in the computational effort: a very large number of draws is required to obtain reasonably accurate estimates of small probabilities, which results in unacceptably long computer runs [Lerman and Manski (1981)]. The second drawback is the discontinuity of the frequency simulator. A small change in the parameter values will change the integer-valued frequency count only in discrete steps. This discontinuity impedes numerical optimization and requires a considerably more complex asymptotic theory to prove consistency and asymptotic normality of simulation estimators.

McFadden (1989) and Pakes and Pollard (1989) simultaneously developed the method of simulated moments (MSM) that circumvents the consistency problem. Rather than simulating the likelihood (6) directly and controlling the simulation error in each observation, their main idea is to simulate generalized moments and to rely on the law of large numbers to control the simulation error across a large number of observations. They also provide the appropriate asymptotic theory for simulation estimators. McFadden (1989) also suggests several alternative simulators with better numerical properties than the discontinuous frequency simulator.

3.2. The Stern simulator

One of the more promising multivariate probability simulators has been proposed by Stern (1992) for the multinomial probit model. This simulator is unbiased and smooth in the model parameters β and Ω.

Smoothness is a key feature of simulators because it keeps the number of random draws small. The intuition is as follows. With only one draw, a discontinuous probability simulator can map the parameters in the model (1)–(3) to only two outcomes, 0 or 1, while only a large number of draws addresses all values in the [0, 1] interval. Loosely speaking, a large number of draws serves to smooth the discontinuity of each single draw. In contrast, each draw of a continuous probability simulator can map the parameters of the model to any value in the interior of the [0, 1] interval, and no additional smoothing is necessary. Smoothness also permits the application of standard optimization routines and standard asymptotic theory. Since the smooth probability estimates are bounded away from 0 and 1, no numerical problems emerge when the probabilities are inserted in expressions such as $\log(p)$, $\log(1-p)$, $1/p$, or $1/(1-p)$. This is in marked contrast to the numerical problems afflicting frequency simulators like (7), since frequency simulators take the values 0 and 1 with positive probability.

As long as the underlying error distribution is multivariate normal, Stern's simulator can be generalized to the entire class of limited dependent variable models defined by (1)–(3) as follows. Let the latent variable u in (1) be normally distributed $\mathrm{N}(X\beta, \Omega)$. Define $M = A\Omega A'$ where A is defined by the inequality (3). Stern's basic idea is to divide Au into two components, one of them composed of independent random variables and the other with a covariance matrix that is as small as possible (in the positive definite sense):

$$Au = w_1 + w_2, \qquad \begin{matrix} w_1 \sim \mathrm{N}(0, D), \\ w_2 \sim \mathrm{N}(X\beta, M - D), \end{matrix} \qquad D = \mathrm{diag}(d_1, \ldots, d_I), \tag{8}$$

Stern proposes an eigenvalue-based algorithm to compute an optimal choice of D.

With this decomposition, the likelihood contribution (6) can be written as

$$\begin{aligned} \ell(y, X; \beta, \Omega) &= \mathrm{Prob}(a \le A \cdot u \le b) \qquad (9) \\ &= \mathrm{Prob}(a - w_1 \le w_2 \le b - w_1) \\ &= \int \mathrm{Prob}(a - w_2 \le w_1 \le b - w_2 \,|\, w_2) \cdot f(w_2)\, \mathrm{d}w_2 \\ &= \int H(w_2) \cdot f(w_2)\, \mathrm{d}w_2, \end{aligned}$$

where $f(w_2)$ is the multivariate density of w_2,

$$H(w_2) = \prod_{i=1}^{I} \{\Phi((b_i - X_i\beta - w_2)/d_i) - \Phi((a_i - X_i\beta - w_2)/d_i)\}, \quad (10)$$

and $\Phi(\cdot)$ denotes the standard Gaussian c.d.f.[5] Hence, $\ell(y, X; \beta, \Omega)$ can be simulated by

$$\frac{1}{R}\sum_{r=1}^{R} H(w_{2r}), \quad \text{with } w_{2r} \text{ drawn from } \mathrm{N}(X\beta, M - D), \quad (11)$$

for R replications.

Stern's method employs the additivity property of the normal distribution. Its accuracy, as measured by the variance of the simulated probabilities, depends crucially on the relative magnitudes of w_1 and w_2 in (8). If M is diagonal, $w_2 = 0$ and the Stern method is exact. When the correlation among the u_i's becomes stronger, however, the elements of D become smaller and less can be pulled into the independent part w_1. Indeed, the variance of the simulated ℓ is quite large if Ω has large off-diagonal elements.[6]

3.3. *The smooth recursive conditioning (SRC) simulator*

We now define a multivariate probability simulator that shares the unbiasedness and the smoothness of the Stern simulator, but that produces a substantially smaller variance of the simulated probabilities particularly for the interesting case of highly interdependent u_i. In contrast to the Stern method, it can also be generalized to some nonnormal distributions.

The method builds on recent work by Geweke, Hajivassiliou, and McFadden. Geweke (1989) has developed an algorithm to compute random variates from a multivariate, truncated normal distribution by reducing the multidimensional frequency simulator to a recursive sequence of univariate acceptance–rejection draws.[7] His recursive conditioning method has been modified by Hajivassiliou (1990) and by Hajivassiliou and McFadden (1990) to construct a smooth simulator of the score of the likelihood contribution (6), which yields a method of moments estimator in which the optimal instrument is X, the matrix of explanatory variables.[8] Unfortunately, Geweke's recursive conditioning method

[5] Due to normality, the density function g is completely characterized by the covariance matrix Ω.

[6] See section 4.

[7] Keane (1990) independently proposed a variant of this method for the special case of a multiperiod (panel-data) binary probit model.

[8] This was first pointed out by Paul Ruud.

does not produce *draws* of the correct multivariate truncated normal distribution, thereby hampering a direct simulation of the scores.

We will show, however, that the *probabilities* generated by the Geweke algorithm are unbiased. In the sequel, we will examine the smooth version of the Geweke algorithm, dubbed the smooth recursive conditioning (SRC) algorithm, when employed in the simulated likelihood approach of Lerman and Manski.

Geweke's recursive conditioning method works as follows for normally distributed u. Let L be the lower triangular Choleski factor of $M = A\Omega A'$,

$$LL' = A\Omega A'. \tag{12}$$

Instead of drawing from the original distribution of the latent variable, subject to the restrictions implied by the observed choice,

$$u \sim \mathrm{N}(X\beta, \Omega) \quad \text{s.t.} \quad a \leq A \cdot u \leq b, \tag{13}$$

we draw a random vector

$$e \sim \mathrm{N}(0, I) \qquad \text{s.t.} \quad a^* \equiv a - AX\beta \leq Le \leq b^* \equiv b - AX\beta. \tag{14}$$

The restrictions are recursive due to the triangular structure of L:

$$e_1 \sim \mathrm{N}(0, 1) \qquad \text{s.t.} \quad a_1^* \leq l_{11} \cdot e_1 \leq b_1^* \tag{15a}$$

$$\Leftrightarrow a_1^*/l_{11} \leq e_1 \leq b_1^*/l_{11},$$

$$e_2 \sim \mathrm{N}(0, 1) \qquad \text{s.t.} \quad a_2^* \leq l_{21} \cdot e_1 + l_{22} \cdot e_2 \leq b_2^* \tag{15b}$$

$$\Leftrightarrow (a_2^* - l_{21} \cdot e_1)/l_{22} \leq e_2$$

$$\leq (b_2^* - l_{21} \cdot e_1)/l_{22},$$

etc.

The e_i's can therefore be drawn sequentially by a univariate truncated simulator. Finally, the desired truncated random vector u^* is defined by

$$u^* = X\beta + A^{-1}Le. \tag{16}$$

The random vector u^* has covariance $A^{-1}LL'A^{-1\prime} = A^{-1}A\Omega A'A^{-1\prime} = \Omega$ and is subject to $A^{-1}a \leq A^{-1}Le \leq A^{-1}b \Leftrightarrow a \leq Au^* \leq b$.

The draws of u^* are generally biased. This is evident from a simple two-dimensional example. Suppose $b_1 = b_2 = \infty$ as is the case in the probit model, and $l_{21} > 0$, corresponding to a positive correlation between u_1 and u_2. Draws

of e_1 according to the inequality in (15a) will ignore the constraint in (15b), hence will be too small on average. Given an e_1 too small, e_2, obeying the second constraint (15b), will be too large on average.

In contrast to this, the likelihood contribution (6), i.e., the probability of $a \leq A \cdot u \leq b$, is correctly simulated by the probability of $a^* \leq L \cdot e \leq b^*$, which in turn is the product of the probabilities that each e_i falls in the respective intervals given by (15):

$$\begin{aligned}
&\ell(y, X; \beta, \Omega) \qquad (17)\\
&= \text{Prob}(a_1^*/l_{11} \leq e_1 \leq b_1^*/l_{11})\\
&\quad \cdot \text{Prob}((a_2^* - l_{21} \cdot e_1)/l_{22} \leq e_2 \leq (b_2^* - l_{21} \cdot e_1)/l_{22} \,|\, e_1) \cdots\\
&\quad \cdot \text{Prob}((a_I^* - l_{I1} \cdot e_1 - \cdots - l_{Ii-1} \cdot e_{i-1})/l_{II} \leq e_I\\
&\quad \leq (b_I^* - l_{I1} \cdot e_1 - \cdots - l_{Ii-1} \cdot e_{i-1})/l_{II} \,|\, e_1, \ldots, e_{i-1})\\
&= Q_1 \cdot Q_2(e_1) \cdot Q_3(e_1, e_2) \cdots Q_I(e_1, \ldots, e_{I-1}).
\end{aligned}$$

This can be approximated by the simulator $\tilde{\ell}(y, X; \beta, \Omega; R)$

$$\frac{1}{R} \sum_{r=1}^{R} \prod_{i=1}^{I} Q_i(e_{1r}, \cdots, e_{i-1,r}) \quad \text{with } e_{ir} \text{ drawn from truncated N(0, 1)}, \tag{18}$$

where R denotes the number of replications.

Lemma. *The simulator $\tilde{\ell}(y, X; \beta, \Omega; R)$ defined by (18) is an unbiased estimator of $\ell(y, X; g, \beta)$.*

Proof. It is sufficient to show the Lemma for $R = 1$. The expected value of $\tilde{\ell}$ is

$$\mathrm{E}\tilde{\ell} = \int \tilde{\ell}(e) f(e)\, \mathrm{d}e,$$

where $f(e)$ denotes the density that generates the (biased) sequential truncated draws in (15):

$$\begin{aligned}
f(e) &= \prod_{i=1}^{I} \phi(e_i)/Q_i(e_1, \ldots, e_{i-1}) \quad \text{on the set} \quad T(y) = \{a^* \leq L \cdot e \leq b^*\},\\
&= 0 \qquad\qquad\qquad\qquad \text{elsewhere}.
\end{aligned}$$

By the definition of $\tilde{\ell}$, (18),

$$\mathrm{E}\tilde{\ell} = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{I} Q_i \right) \cdot \left(\prod_{i=1}^{I} \phi(e_i)/Q_i \right) \mathrm{d}e_1 \cdots \mathrm{d}e_I$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{I} \phi(e_i)\,\mathrm{d}e = \int_{T(y)} \prod_{i=1}^{I} \phi(e_i)\,\mathrm{d}e$$

$$= \mathrm{Prob}(a^* \leq Le \leq b^*) = \ell(y, X; \beta, \Omega). \quad \blacksquare$$

The univariate truncated normal variates e_i in (15) can be drawn smoothly according to a straightforward application of the integral transform theorem as proposed by Hajivassiliou (1990) and by Hajivassiliou and McFadden (1990). Let X be distributed according to the univariate uniform distribution on [0, 1]. Then

$$Z \equiv G^{-1}(X) = \Phi^{-1}[(\Phi(b) - \Phi(a)) \cdot X + \Phi(a)] \tag{19}$$

is distributed N(0, 1) s.t. $a \leq Z < b$, since the corresponding c.d.f. is

$$G(z) = [\Phi(z) - \Phi(a)]/[\Phi(b) - \Phi(a)], \tag{20}$$

where Φ denotes the univariate normal cumulative distribution function. Note that Z is a continuous and differentiable function of the parameters a and b.

The combination of the Geweke recursive conditioning method, the above Lemma, and the smooth univariate truncated variable generation algorithms produces an unbiased multivariate probability simulator for (6) that is smooth, i.e., a continuous and differentiable function of the model parameters β and Ω. This smoothness is essential for computational efficiency as was pointed out at the beginning of the previous subsection.

Apart from an initial Choleski decomposition and several matrix multiplications, most computational effort is in drawing the univariate truncated normal variates according to the steps in (15).[9] This effort is linear in I, the dimension of the probability integral (6). Moreover, if $M = A\Omega A'$ is a diagonal matrix, L is also diagonal. In this case, $Q_i(e_1, \ldots, e_{i-1})$ is independent of $e_1, \ldots, e_{i-1}$, and the simulator (18) is exact. This is also true for the Stern simulator (11). The

[9] For reliable results, it is crucial to compute the cumulative normal distribution function and its inverse with high accuracy.

simulator (18), however, avoids the main disadvantage of (11), namely its poor performance for highly correlated u_i.

We finally note, that the simulator (18) is applicable also for some nonnormal distributions, as long as the univariate draws from the conditional one-dimensional distributions correspond to the multivariate distribution of ε in (1). This useful property is not shared by the Stern simulator which relies on the very special additivity property of the normal distribution.

4. Monte Carlo results: Distribution of probabilities

To evaluate the relative accuracy of the three simulators, we take four examples from Stern (1992). They exemplify choice probabilities generated from a multinomial probit model with five choices. Apart from the obvious reason of comparability of our results to those of Stern, we chose to analyze the four examples of the multinomial probit model because this model exhibits in an extreme form the main problem of discrete probabilities in all canonical LDV models, given that *every* element of the latent variable vector is limited. It should be obvious, for example, that a multivariate or multiperiod Tobit model exhibits less 'discreteness', in that for some dimensions or periods the full, uncensored values of the latent variables are observed. Hence, the likelihood contributions in every canonical LDV model can be written as the product of a term for the fully observed latent variables, conditionally on the limited (discrete) variables, times a multinomial discrete choice term for the probability of the limited dependent variables. Since the fully observed latent variables have a continuous (conditional) distribution, the integration problem arises solely from the discrete part. A multinomial choice problem has no continuous components, and hence exhibits the greatest difficulties, for a given dimension of the latent vector, in evaluating the likelihood function. Moreover, the four examples of the five-dimensional multinomial probit model we study here correspond closely to the econometric models analyzed in the empirical studies of Börsch-Supan et al. (1992), Hausman and Wise (1978), and Keane (1990).

In the case of the multinomial choice model of the selected examples, the latent variable u_i in (1) represents the utility of choice i. Because only relative utilities matter in a choice among discrete alternatives, the actual dimension of the problem is $I = 4$, and instead of specifying $X\beta$ and Ω we assume different values for $\Delta X\beta$ and $\Delta\Omega$, defined as first differences with respect to, say, the first alternative. With this normalization, the probability of choosing the first alternative equals the probability that all Δu_i, $i = 1, \ldots, 4$, are nonpositive.

Table 1 presents the distribution of simulated choice probabilities for the three simulators defined by (7), the frequency simulator; (11), the simulator proposed by Stern (1992) with his optimal choice of D; and (18), the SRC simulator proposed in subsection 3.3. Each distribution is based on 10,000

draws: 100 realizations of each probability estimate, where in turn each estimate is based on $R = 100$, the number of replications. The reported statistics (means, standard deviations, and percentiles) refer to the empirical distributions of the 10,000 Monte Carlo probability realizations from each method. Finally, the 'true' probability values were calculated using Legendre quadrature with 40 points in the polynomial approximations [see Abramowitz and Stegun (1964)].

Example 1 features mild correlation and a very small choice probability. The frequency simulator has a very large standard deviation, about 75 percent of the mean. The Stern simulator (11) performs better with a standard deviation about 2.2 times smaller than the frequency simulator (7). However, the SRC simulator (18) is substantially more accurate. Its standard deviation is more than 11 times smaller than in the Stern case, and 25 times smaller than the frequency simulator.

Table 1

Distribution of probability simulators (alternative specifications of $\Delta X\beta$ and $\Delta\Omega$).[a]

	Percentile	Simulator		
		FREQ	*STERN*	*SRC*
Example 1	1%	0.00	0.00894	0.02290
$\Delta X\beta = [-1.0, -0.75, -0.5, -0.2]$	5%	0.00	0.01226	0.02297
	10%	0.00	0.01417	0.02327
$\Delta\Omega = \begin{bmatrix} 1.0 & & & \\ 0.2 & 1.0 & & \\ 0.3 & 0.4 & 1.0 & \\ 0.1 & 0.3 & 0.5 & 1.0 \end{bmatrix}$	25%	0.01	0.01837	0.02364
	33%	0.01	0.01957	0.02377
	Median	0.02	0.02351	0.02398
	66%	0.03	0.02744	0.02432
	75%	0.03	0.02882	0.02453
	90%	0.05	0.03268	0.02499
	95%	0.06	0.03598	0.02540
	99%	0.09	0.04923	0.02599
'True' probability = 0.2403	Mean	0.0228	0.02378	0.02409
	Std. dev.	0.0171	0.00773	0.00068
Example 2	1%	0.05	0.10447	0.14243
$\Delta X\beta = [0.0, 0.0, 0.0, 0.0]$	5%	0.10	0.12312	0.14331
	10%	0.12	0.12625	0.14555
$\Delta\Omega = \begin{bmatrix} 1.0 & & & \\ 0.2 & 1.0 & & \\ 0.2 & 0.4 & 1.0 & \\ 0.2 & 0.4 & 0.6 & 1.0 \end{bmatrix}$	25%	0.13	0.13959	0.14722
	33%	0.14	0.14488	0.14827
	Median	0.14	0.15286	0.14956
	66%	0.17	0.15804	0.15187
	75%	0.18	0.16433	0.15347
	90%	0.20	0.17661	0.15646
	95%	0.21	0.18357	0.15877
	99%	0.24	0.21171	0.16221
'True' probability = 0.1497	Mean	0.1506	0.15515	0.15037
	Std. dev.	0.0355	0.01978	0.00444

Table 1 (continued)

	Percentile	Simulator		
		FREQ	*STERN*	*SRC*
Example 3	1%	0.52	0.54609	0.62888
$\Delta X\beta = [1.0, 1.0, 1.0, 1.0]$	5%	0.55	0.56634	0.63375
	10%	0.57	0.57878	0.63640
$\Delta\Omega = \begin{bmatrix} 1.0 & & & \\ 0.9 & 1.0 & & \\ 0.0 & 0.0 & 1.0 & \\ 0.0 & 0.0 & 0.95 & 1.0 \end{bmatrix}$	25%	0.62	0.61931	0.64394
	33%	0.63	0.63253	0.64513
	Median	0.65	0.64899	0.64747
	66%	0.67	0.66387	0.65216
	75%	0.68	0.68304	0.65275
	90%	0.72	0.70928	0.65725
	95%	0.73	0.72285	0.65974
	99%	0.75	0.73334	0.66536
'True' probability = 0.6473	Mean	0.6463	0.64706	0.64773
	Std. dev.	0.0536	0.04573	0.00773
Example 4	1%	0.41	0.41884	0.47243
$\Delta X\beta = [1.5, 0.75, 0.5, 0.75]$	5%	0.42	0.44022	0.47612
	10%	0.43	0.44936	0.48010
$\Delta\Omega = \begin{bmatrix} 1.0 & & & \\ 0.5 & 1.0 & & \\ 0.2 & 0.5 & 1.0 & \\ 0.1 & 0.2 & 0.5 & 1.0 \end{bmatrix}$	25%	0.47	0.47882	0.48758
	33%	0.49	0.48395	0.49021
	Median	0.50	0.49919	0.49611
	66%	0.52	0.51789	0.50192
	75%	0.53	0.52536	0.50694
	90%	0.56	0.55565	0.51593
	95%	0.59	0.56592	0.52341
	99%	0.63	0.59952	0.53192
'True' probability = 0.4873	Mean	0.5023	0.50147	0.49716
	Std. dev.	0.0481	0.03940	0.01394

[a] Reported statistics calculated over 10,000 Monte Carlo realizations. *FREQ* is the frequency simulator defined by eq. (7); *STERN* is the simulator (11) proposed by Stern (1992); and *SRC* is the smooth simulator defined by eqs. (17)–(19).

Example 2 has a slightly higher correlation and is less diagonally dominant than example 1. The standard deviation of the Stern simulator is now only 1.8 times that of the frequency simulator, while the SRC simulator has an 8 times higher accuracy.

Example 3 features some very large correlation coefficients. In this case, the relative accuracy of the Stern method is only slightly better than the frequency simulator: its standard deviation is only 1.2 times smaller. However, the SRC simulator outperforms both the frequency and Stern's simulator with a 6 times smaller standard deviation than the Stern simulator.

Example 4 features a probability close to 0.5 and only mild correlation. In this benign case, the performance of the three simulators is very similar. Still, the

SRC simulator has a standard deviation that is 2.8 times smaller than the Stern method and 3.4 times smaller than the frequency simulator.

Table 2 investigates the performance of the simulators (11) and (18) with different numbers of replications. We use example 4, the most unfavorable for the SRC simulator relative to the Stern simulator. The distributions are based on 100 realizations of the simulators, each realization based on $R = 1, 3, 10$ and 100 replications.

Table 2
Distribution of probability simulators (example 4, alternative numbers of replications).[a,b]

	$R = 1$		$R = 3$	
Percentile	*STERN*	*SRC*	*STERN*	*SRC*
1%	0.00005	0.25053	0.06723	0.33782
5%	0.00268	0.30877	0.16197	0.37485
10%	0.01432	0.33497	0.23206	0.40125
25%	0.13740	0.39466	0.35429	0.44558
33%	0.25300	0.42191	0.39944	0.46212
Median	0.48756	0.48385	0.50775	0.49755
66%	0.71905	0.54611	0.60033	0.52762
75%	0.83464	0.59162	0.63660	0.54490
90%	0.97089	0.68864	0.77384	0.59608
95%	0.98954	0.73999	0.83626	0.62534
99%	0.99935	0.82959	0.93800	0.67204
Mean	0.49074	0.49833	0.50051	0.49733
Std. dev.	0.35042	0.13387	0.20397	0.07477
	$R = 10$		$R = 100$	
Percentile	*STERN*	*SRC*	*STERN*	*SRC*
1%	0.24452	0.40795	0.41710	0.46144
5%	0.32358	0.43241	0.43796	0.47241
10%	0.35651	0.44397	0.45230	0.47816
25%	0.42664	0.46831	0.47190	0.48690
33%	0.44939	0.47662	0.48068	0.48993
Median	0.49934	0.49706	0.49631	0.49530
66%	0.54535	0.51461	0.50949	0.50132
75%	0.57498	0.52474	0.51849	0.50530
90%	0.63815	0.54879	0.54024	0.51299
95%	0.68004	0.56590	0.55198	0.51712
99%	0.75349	0.61200	0.58402	0.52711
Mean	0.49901	0.49716	0.49571	0.49560
Std. dev.	0.10945	0.04152	0.03513	0.01362

[a] Reported statistics calculated over 10,000 Monte Carlo realizations. *STERN* is the simulator (11) proposed by Stern (1992) and *SRC* is the smooth simulator defined by eqs. (17)–(19).
[b] 'True' probability = 0.4873.

Table 3
Eight-dimensional case, AR1 plus RE.[a]

$$\varepsilon_{it} = \eta_i + \zeta_{it}, \quad \zeta_{it} = \rho_{i,t-1} + \nu_{it}, \quad \rho = 0.9, \quad \sigma_\eta = 2.0, \quad \sigma_\zeta = 1.0, \quad t = 1, \ldots, 8$$

$$\Delta X = \begin{bmatrix} 4.0 \\ 4.2 \\ 4.4 \\ 4.6 \\ 4.8 \\ 5.0 \\ 5.2 \\ 5.4 \end{bmatrix}, \quad \Delta\Omega = \begin{bmatrix} 5.000 & & & & & & & \\ 4.900 & 5.000 & & & & & & \\ 4.810 & 4.900 & 5.000 & & & & & \\ 4.729 & 4.810 & 4.900 & 5.000 & & & & \\ 4.656 & 4.729 & 4.810 & 4.900 & 5.000 & & & \\ 4.590 & 4.656 & 4.729 & 4.810 & 4.900 & 5.000 & & \\ 4.531 & 4.590 & 4.656 & 4.729 & 4.810 & 4.900 & 5.000 & \\ 4.478 & 4.531 & 4.590 & 4.656 & 4.729 & 4.810 & 4.900 & 5.000 \end{bmatrix}$$

'True' probability = 0.005512

Percentile	Simulator		
	FREQ	*STERN*	*SRC*
Min	0.000000	0.000000	0.003177
1%	0.000000	0.000000	0.003864
5%	0.000000	0.000000	0.004387
10%	0.000000	0.000000	0.004669
25%	0.000000	0.000000	0.005104
Median	0.000000	0.000000	0.005523
75%	0.000000	0.000000	0.005944
90%	0.000000	0.000000	0.006352
95%	0.000000	0.002870	0.006580
99%	0.100000	0.100000	0.007004
Max	0.200000	0.153600	0.007338
Mean	0.004605	0.004515	0.005519
Std. dev.	0.021440	0.020680	0.000656

[a] Reported statistics calculated over 10,000 Monte Carlo realizations. *FREQ* is the frequency simulator defined by eq. (7); *STERN* is the simulator (11) proposed by Stern (1987); and *SRC* is the smooth simulator defined by eqs. (17)–(19).

The standard deviation of the simulators decreases sharply with the number of replications. But even with one single draw, the SRC simulator has a coefficient of variation of less than 0.27, while that of the Stern simulator is more than 0.71. The relative accuracy stays approximately constant: across all replication numbers, the standard deviation of the SRC simulator is at least 2.6 times smaller than that of the Stern simulator.

Table 3 presents results on the Monte Carlo performance of the three simulators for a case with nine discrete alternatives in the choice set and the more complicated error correlation structure studied in Hajivassiliou (1990), in

Table 4

Accuracy of SRC simulator at small probabilities.[a]

$$\Delta X\beta = [-x, -x, 0], \qquad \Delta\Omega = \begin{bmatrix} 3.0 & & \\ 0.7, & 2.0 & \\ 0.5 & 0.3 & 1.0 \end{bmatrix}$$

		Continuous recursive probability simulator					
x	Numerical integration	$R = 2$	$R = 5$	$R = 10$	$R = 20$	$R = 50$	$R = 100$
0.0	0.18838e + 00	0.1893e + 00 (0.0329e + 00)	0.1898e + 00 (0.0215e + 00)	0.1902e + 00 (0.0145e + 00)	0.1901e + 00 (0.0103e + 00)	0.1891e + 00 (0.0057e + 00)	0.1888e + 00 (0.0038e + 00)
1.0	0.69869e − 01	0.7042e − 01 (0.1340e − 01)	0.7056e − 01 (0.0864e − 01)	0.7062e − 01 (0.0572e − 01)	0.7055e − 01 (0.0402e − 01)	0.7018e − 01 (0.0226e − 01)	0.7003e − 01 (0.0148e − 01)
2.0	0.15701e − 01	0.1587e − 01 (0.0333e − 01)	0.1589e − 01 (0.0212e − 01)	0.1588e − 01 (0.0138e − 01)	0.1586e − 01 (0.0096e − 01)	0.1578e − 01 (0.0054e − 01)	0.1574e − 01 (0.0036e − 01)
3.0	0.20252e − 02	0.2052e − 02 (0.0469e − 02)	0.2053e − 02 (0.0296e − 02)	0.2051e − 02 (0.0190e − 02)	0.2046e − 02 (0.0131e − 02)	0.2035e − 02 (0.0075e − 02)	0.2030e − 02 (0.0050e − 02)
4.0	0.14505e − 03	0.1473e − 03 (0.0362e − 03)	0.1473e − 03 (0.0227e − 03)	0.1470e − 03 (0.0144e − 03)	0.1465e − 03 (0.0099e − 03)	0.1458e − 03 (0.0057e − 03)	0.1454e − 03 (0.0038e − 03)
5.0	0.56534e − 05	0.5749e − 05 (0.1503e − 05)	0.5748e − 05 (0.0936e − 05)	0.5730e − 05 (0.0593e − 05)	0.5711e − 05 (0.0402e − 05)	0.5682e − 05 (0.0236e − 05)	0.5664e − 05 (0.0157e − 05)
6.0	0.11847e − 06	0.1206e − 06 (0.0331e − 06)	0.1206e − 06 (0.0205e − 06)	0.1201e − 06 (0.0129e − 06)	0.1197e − 06 (0.0087e − 06)	0.1191e − 06 (0.0051e − 06)	0.1187e − 06 (0.0034e − 06)
7.0	0.13251e − 08	0.1351e − 08 (0.0387e − 08)	0.1349e − 08 (0.0239e − 08)	0.1344e − 08 (0.0150e − 08)	0.1338e − 08 (0.0100e − 08)	0.1332e − 08 (0.0059e − 08)	0.1327e − 08 (0.0040e − 08)

[a] Reported values for the SRC method are means over 10,000 Monte Carlo realizations, with standard deviations appearing in parentheses.

Hajivassiliou and McFadden (1990), and in Börsch-Supan et al. (1992). Specifically, we model the error vector as possessing a random factor η, plus an autoregressive component of order 1, i.e., $\varepsilon_{it} = \eta_i + \zeta_{it}$, where $E\eta_i \equiv \mu$, and $\zeta_{it} = \rho \cdot \zeta_{i,t-1} + \nu_{it}$, with $\rho = 0.9$, $\sigma_\eta = 2$, and $\sigma_\zeta = 1$. The resulting variance–covariance matrix for the (differenced) error term and the chosen values for the vector μ are given in table 3. In this case, the 'true' choice probability is 0.005512. The findings once again confirm the superior performance of the SRC simulator in terms of average bias, standard deviation, and mean-squared-error. In particular, not only does the SRC simulator exhibit a substantially smaller bias compared to *FREQ* and *STERN*, but it is always positive and its standard deviation is about 30 times smaller. In contrast, *FREQ* and *STERN* take the value 0 very frequently: for these two simulators, all percentiles up to 95% are 0. This feature of *FREQ* and *STERN* implies that they cannot be used successfully in a simulated maximum likelihood procedure.

Finally, table 4 investigates the performance of the SRC simulator at very small probabilities. Even for probabilities very close to 0, few replications are sufficient to generate accurate probability estimates.

5. Estimation procedure: Smooth simulated maximum likelihood estimation (SSML)

Of central interest is the estimation of the parameters β and Ω from a sample of observations (y_n, X_n), $n = 1, \ldots, N$. The loglikelihood function corresponding to the multivariate LDV model (1)–(3), with likelihood probabilities given by (6), is

$$\mathscr{L}(\beta, \Omega) = \sum_{n=1}^{N} \log \ell(y, X; \beta, \Omega). \tag{21}$$

The simulated maximum likelihood (SML) approach replaces $\ell(y, X; \beta, \Omega)$ by a probability simulator $\tilde{\ell}$, e.g., defined by (7), (11), or (18). However, because of the nonlinearity of the logarithm,

$$E\tilde{\ell} = \ell \Rightarrow E(\log\tilde{\ell}) - \log(E\ell) \approx -\operatorname{var}(\tilde{\ell})/2\ell^2 < 0. \tag{22}$$

Hence, the SML approach is biased as long as $\tilde{\ell}$ has a positive variance, i.e., for any finite number of replications in case of the three simulators defined above. This is – in a less extreme form – the consistency problem that was mentioned in subsection 3.1 when the frequency simulator was discussed. The bias goes to infinity for very small probabilities ℓ. In the discrete choice context, the likelihood contribution ℓ is the predicted choice probability of an

observation. In a well-specified model, this probability should be large on average, and the bias therefore relatively small.

Because the bias depends crucially on the variance of the probability simulator, it is important to keep this variance small. As the results of the preceding section have shown, the SRC simulator (18) is for this reason a much more suitable simulator than the Stern simulator (11) or the frequency simulator (7).

The smoothness of the SRC simulator is also a key feature for a convenient computation of the optimal parameter values of model (1)–(3). Because $\tilde{\ell}(y, X; \beta, \Omega; R)$ is a continuous and differentiable function of these parameters, the first-order conditions of maximizing (21) with respect to β and Ω are well-defined, and conventional numerical methods with a high speed of convergence such as one of the conjugate gradient methods or quadratic hillclimbing can be employed. This is in sharp contrast to the frequency simulator that generates a discontinuous objective function with the associated numerical problems. It is important to note that the SSML estimator has the additional advantage that its implementation requires only a standard likelihood optimization package, modified suitably to use the SRC method in the calculation of the likelihood contributions. This is in contrast to the more specialized software that is required for computing MSM and MSS estimators.

Finally, we stress that the computational effort of the SSML approach increases almost linearly with the dimension of the latent variable vector in (1). This near-linearity permits applications to discrete choice problems with large choice sets, to panel data models with a large number of panel waves, or a combination of the two.

6. Monte Carlo results: Distribution of SSML estimates

In order to assess the bias that is generated by taking the logarithm in (21), we investigate the distribution of the parameter estimates in a simple three-alternative probit model. We created a 500-observation data set (y, X) specified by model (1) and (4) with a dependent choice variable y and a single explanatory variable X. The explanatory variable was realized as independently and normally distributed draws with mean [1, 0, 0] and standard deviation [2, 2, 2]. This large variation in the explanatory variable was chosen to generate a wide range of predicted choice probabilities.

Given the explanatory variables X, the dependent choice variable y was defined by

$$y = \underset{j}{\operatorname{argmax}} (u_j | j = 1, \ldots, 3), \tag{23}$$

Table 5
Distribution of SSML estimators (alternative numbers of replications).

Monte Carlo data set (500 observations): $\beta = 0.925$, $\Omega = \begin{bmatrix} 2.0 & 0.7 & 0.0 \\ 0.7 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$

	Distribution of $\hat{\beta}_{SSML}$ (based on 100 Monte Carlo realizations)				
Percentiles	$R = 1$	$R = 3$	$R = 6$	$R = 10$	$R = 20$
1%	0.66167	0.76337	0.83462	0.86093	0.89159
5%	0.68356	0.78429	0.85367	0.86630	0.89466
10%	0.69626	0.80365	0.85692	0.87005	0.89841
25%	0.72614	0.81759	0.86567	0.88251	0.91280
33%	0.73202	0.82828	0.86841	0.89277	0.91642
Median	0.74736	0.83822	0.87698	0.90196	0.92434
66%	0.77041	0.84946	0.89082	0.91172	0.93363
75%	0.77597	0.86050	0.90186	0.91707	0.93605
90%	0.79580	0.87786	0.91930	0.93883	0.94622
95%	0.80411	0.90133	0.92769	0.94137	0.96520
99%	0.83851	0.92168	0.95149	0.96841	0.97846
Mean	0.74835	0.83921	0.88299	0.90235	0.92497
Std. dev.	0.03752	0.03210	0.02494	0.02414	0.01914

where the latent variable vector u was drawn from the multivariate normal distribution $N(X\beta, \Omega)$ with true parameters

$$\beta = 1.0, \qquad \Omega = \begin{bmatrix} 2.0 & 0.7 & 0.0 \\ 0.7 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}. \tag{24}$$

To ensure the positive definiteness of the estimated $\hat{\Omega}$ matrix, we write $\Omega = \Gamma\Gamma'$, where Γ is a lower triangular Choleski factor of Ω, and optimize with respect to the (unrestricted) elements of Γ.

Parameter estimation of the particular 500-observation sample using the Hausman–Wise (1978) method with high accuracy numerical integration yields

$$\hat{\beta} = 0.925, \qquad \hat{\Omega} = \begin{bmatrix} 2.062 & 0.695 & 0.0 \\ 0.695 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}. \tag{25}$$

At these parameters, the predicted choice probabilities range from less than 0.01 to above 0.999.

Table 5 presents the distribution of $\hat{\beta}_{\text{SSML}}$, estimated by inserting the SRC simulator into the loglikelihood function (22), for alternative numbers of replications.[10] The bias in the estimates of parameter β is substantial for very small numbers of replications. However, the bias is below its standard deviation for $R = 10$ and becomes negligible for $R = 20$. This is in sharp contrast to the Lerman and Manski method. They report a minimum number of draws $R_0 = 1000$, and they used much larger numbers for the accurate simulation of probabilities close to 0 and 1.[11]

7. Conclusions

We revived the Lerman and Manski (1981) procedure of approximating the likelihood function by simulated choice probabilities by employing a new probability simulator that is bounded away from 0 and 1, is smooth in the parameters to be estimated, and has a small variance relative to the frequency simulator employed by Lerman and Manski and the simulator by Stern (1992).

Due to the small variance of the SRC simulator, the bias intrinsic to the simulated maximum likelihood approach stays small. In our Monte Carlo experiment, 20 replications were sufficient to produce a negligible bias. The experiment was designed to represent the case of a large variation in the predicted choice probabilities, particularly small predicted choice probabilities. Because the bias is largest for small predicted choice probabilities, a well-specified model of real data should feature even less bias than our Monte Carlo example.

Due to the smoothness of the simulator, conventional and fast optimizers can be employed for the computation of maximum likelihood estimates. The method is feasible for discrete choice models with large choice sets and/or panel data applications with a large number of panel waves. Börsch-Supan et al. (1992) report on the application of the SML approach on a multinomial–multiperiod probit model with unobserved heterogeneity and autocorrelated errors, explaining living arrangement choices of elderly Americans. Börsch-Supan (1991) presents a similar analysis using German data. Börsch-Supan and Pfeiffer (1991) report on an application to labor supply. These applications show not only the feasibility of the proposed algorithm even for large panels but also the importance of a rich specification of the intertemporal error structure. The class of models (1)–(3) includes many other limited dependent variable models such as tobit and switching regression models with their multiperiod generalizations.

[10] The results for the SSML joint estimates of the Ω parameters lead to similar conclusions, and therefore we do not report them separately.

[11] Lerman and Manski (1981, fn. 10).

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