Heat Equation – Back to Basics

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The heat equation, with power generation is the following:

$$\frac{\partial u(x_i,t)}{\partial t} = \alpha \nabla^2 u(x_i,t) + \frac{\dot{q}(x_i,t)}{c_p \rho}$$

In this equation we have two functions which depend on space (x_i) and time (t):

- $u(x_i,t)$ Temperature with units K
- $\dot{q}(x_i,t)$ Power generation density with units $\mathrm{W/m^3}$

And the following physical parameters which depend on the material:

- α Thermal diffusivity with units $\mathrm{m^2/s}$
- c_p Specific heat capacity with units $\frac{\mathrm{J}}{\mathrm{kg\,K}}$
- ρ mass density with units $\mathrm{kg/m^3}$

1 Boundary conditions

Additionally, we consider what happens at a boundary. The heat flux (energy per unit area per unit time) is:

$$\Phi = \kappa \frac{\partial u}{\partial n}\bigg|_{\mathrm{bdry}}$$

where κ is another material parameter, the *thermal conductivity* with units $\frac{\mathrm{W}}{\mathrm{m\,K}}$. This is used in boundaries between materials as well as edge boundaries.

1.1 Material boundaries

Sometimes, there will be two materials that are next to each other. At the boundary surface, we will find that the temperature is continuous at all points (X,Y,Z) on the boundary.

$$u_1(X,Y,Z,t) = u_2(X,Y,Z,t)$$

And the heat flux is continuous on the boundary:

$$\kappa_1 \frac{\partial u_1}{\partial n}\bigg|_{(X,Y,Z)} = \kappa_2 \frac{\partial u_2}{\partial n}\bigg|_{(X,Y,Z)}$$

where u_1 and k_1 describe the temperature and thermal conductivity of material 1, and u_2 and k_2 describe the corresponding quantities of the other material.

1.2 Edge boundaries

In general, there are 4 types of boundary conditions that are usually considered for an ***outer*** boundary of an object:

1. Dirichelet
2. Neumann
3. Robin
4. Radiation

This last type is used infrequently for applications on earth, so I will not go into more detail on that one.

1.2.1 Dirichelet

For points (x_b, y_b, z_b) on the outer boundary of an object, the temperature is determined for us.

$$u(x_b, y_b, z_b, t) = f(x_b, y_b, z_b, t)$$

where f is a known function of space and/or time.

1.2.2 Neumann

This is for a perfectly insulated object. That is, for points (x_b, y_b, z_b) on the outer boundary of an object, the heat flux must be zero:

$$\left. \frac{\partial u(x, y, z, t)}{\partial n} \right|_{(x_b, y_b, z_b)} = 0$$

1.2.3 Robin

This describes convection. That is the heat flux depends on the difference between the surrounding temperature T and the temperature on the boundary.

$$\kappa \left. \frac{\partial u(x, y, z, t)}{\partial n} \right|_{(x_b, y_b, z_b)} = h(T - u(x_b, y_b, z_b, t))$$

where h is the *heat transfer coefficient* for a material with units W/m^2/K.

In Summary

The heat equation is:

$$\frac{\partial u(x_i,t)}{\partial t} = \alpha \nabla^2 u(x_i,t) + \frac{\dot{q}(x_i,t)}{c_p \rho}$$

where

- $u(x_i,t)$ describes the temperature with units of K
- $\dot{q}(x_i,t)$ describes the "internal" power generation density with units of $\mathrm{W/m^3}$

The heat flux through a surface is:

$$\kappa \frac{\partial u}{\partial n}$$

where n is the spatial coordinate in the direction of the surface normal. And the heat flux from a surface to the environment is described by:

$$h \Delta T$$

where ΔT is the difference in temperature between the surface and the environment. Finally, materials are described using the following properties:

Name	Symbol	Unit
Thermal diffusivity	α	$\mathrm{m^2/s}$
Specific heat capacity	c_p	$\frac{\mathrm{J}}{\mathrm{kg\,K}}$
Mass density	ρ	$\mathrm{kg/m^3}$
Thermal conductivity	κ	$\frac{\mathrm{W}}{\mathrm{m\,K}}$
Heat Transfer coefficient	h	$\mathrm{W/(m^2K)}$

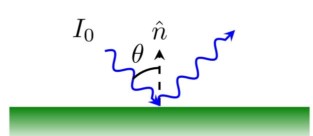
2 Power Generation

Often, this can take several forms. Of interest to me are the following:

- Power generation due to solar intensity
- Power generation due to a heat source/sink at a specific position.

2.1 Solar Intensity

For light with intensity I_0 incident on a surface as depicted below:



The power deposited in the surface is:

$$\dot{q} = \frac{I_0 \cos^2 \theta}{\delta}$$

where δ is the "penetration depth" of the light with units m.

2.2 Heat source/heat sink

Oftentimes, what happens in reality is that a blower with extract (or deposit) a volume of air from (or to) a room, for example through a return or a vent and model this (for point sources) as follows:

$$\dot{q} = \pm\beta\rho c_p u(x, y, z, t)\delta(x - x_s)\delta(y - y_s)\delta(z - z_s)$$

where the $\delta(x - x_s)$ are the Dirac delta functions and (x_s, y_s, z_s) are coordinates of the point source/sink and β is the blower strength (volume/time or m^3/s) of the air handler.

2.3 No Power Generation

Note, that with no power generation, the "Temperature Units" don't matter. It's only when we are working with power sources/sinks that we need to use SI units.

Let

$$u(x, y, z, t) = af(x, y, z, t) + b$$

for some $a, b \in \mathbb{R}$. Furthermore let $u(x, y, z, t)$ solve the heat equation with $\dot{q}(x, y, z, t) = 0$. Therefore:

$$\frac{\partial u}{\partial t} = \alpha\nabla^2 u \to \frac{\partial}{\partial t}(af(x, y, z, t) + b) = \alpha\nabla^2 \left(af(x, y, z, t) + b\right)$$

Because of the properties of the derivative, we find

$$\frac{\partial f}{\partial t} = \alpha\nabla^2 f$$

So absolute temperature is not required.

3 Example solutions

For all of these examples, I'm going to consider a 3D, rectangular prism that goes from $(0, 0, 0)$ to (L, W, H).

In general, I'm going to stay in Cartesian coordinates, as such, I'll assume a separable solution:

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

We can substitute this into the heat equation to find:

$$\frac{\frac{dT}{dt}}{\alpha T} = \frac{\frac{d^2X}{dx}}{X} + \frac{\frac{d^2Y}{dy}}{Y} + \frac{\frac{d^2Z}{dz}}{Z}$$

Since we have `(function of time) = (function of space)`, both have to be equal to a constant. By convention, we choose that constant to be $-\lambda$. Let's start by looking at the right hand side of that equation:

$$\frac{\frac{d^2X}{dx}}{X} + \frac{\frac{d^2Y}{dy}}{Y} + \frac{\frac{d^2Z}{dz}}{Z} = -\lambda$$

We can re-arrange this as follows:

$$\frac{\frac{d^2X}{dx}}{X} = -\frac{\frac{d^2Y}{dy}}{Y} - \frac{\frac{d^2Z}{dz}}{Z} - \lambda$$

Here we have `(function of x) = (function of y) + (function of z) + constant` and by the same logic as above, we get that they have to be equal to a constant, let's call this k_x^2.

Following that pattern we can rewrite the heat equation as the following four ODEs:

$$\begin{aligned} T'(t) + \alpha\lambda T(t) &= 0 \\ X''(x) + k_x^2 X(x) &= 0 \\ Y''(y) + k_y^2 Y(y) &= 0 \\ Z''(z) + k_z^2 Z(z) &= 0 \end{aligned}$$

where $\lambda = k_x^2 + k_y^2 + k_z^2$. And the general solutions can be:

If all k_{xyz} are real and all $k_{xyx} \neq 0$:

$$\begin{aligned} T(t) &= \exp(-\alpha\lambda t) \\ X(x) &= A_x \sin(k_x x) + B_x \cos(k_x x) \\ Y(y) &= A_y \sin(k_y y) + B_y \cos(k_y y) \\ Z(z) &= A_z \sin(k_z z) + B_z \cos(k_z z) \\ \lambda &= k_x^2 + k_y^2 + k_z^2 \end{aligned}$$

If all $k_{xyx} = 0$ (and $\lambda = 0$):

$$\begin{aligned} T(t) &= 1 \\ X(x) &= A_x x + B_x \\ Y(y) &= A_y y + B_y \\ Z(z) &= A_z z + B_z \end{aligned}$$

If all k_{xyz} are imaginary and all $k_{xyx} \neq 0$, and $\lambda < 0$:

$$\begin{aligned} T(t) &= \exp(-\alpha\lambda t) \\ X(x) &= A_x \sinh(k_x x) + B_x \cosh(k_x x) \\ Y(y) &= A_y \sinh(k_y y) + B_y \cosh(k_y y) \\ Z(z) &= A_z \sinh(k_z z) + B_z \cosh(k_z z) \\ \lambda &= k_x^2 + k_y^2 + k_z^2 \end{aligned}$$

3.1 Example 1: Dirichelet BC's

Assume the temperature is 0 everywhere on the boundary

$$0 = u(0, y, z, t) = u(l, y, z, t) = u(x, 0, z, t) = u(x, w, z, t) = u(x, y, 0, t) = u(x, y, h, t)$$

With this in mind, the boundary conditions become:

$$0 = X(0) = X(L) = Y(0) = Y(W) = Z(0) = Z(H)$$

If we apply these conditions, we find:

- $\lambda = 0$ is not a meaningful solution because $A_x = A_y = A_z = B_x = B_y = B_z = 0$
- $\lambda < 0$ is not a meaningful solution because $A_x = A_y = A_z = B_x = B_y = B_z = 0$
- $\lambda > 0$ is the only possible solution with

$$B_x = B_y = B_z = 0 \quad k_x = \frac{i\pi}{L} \quad k_y = \frac{j\pi}{W} \quad k_z = \frac{k\pi}{H}$$

where i, j, k are non-zero, positive integers. As a result, we get:

$$u(x, y, z, t) = \sum_{klm} a_{klm} \exp\left(-\alpha\lambda_{klm} t\right) \cos\left(\frac{k\pi x}{L}\right) \cos\left(\frac{l\pi y}{W}\right) \cos\left(\frac{m\pi z}{H}\right)$$

$$\lambda_{klm} = \frac{k^2\pi^2}{L^2} + \frac{l^2\pi^2}{W^2} + \frac{m^2\pi^2}{H^2}$$

where a_{klm} are arbitrary constants that depend on the initial condition. Given some initial condition:

$$u(x, y, z, 0) = f(x, y, z)$$

We find:

$$a_{klm} = \frac{8}{LWH} \int_0^L dx \int_0^W dy \int_0^H dz f(x, y, z) \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi y}{W}\right) \sin\left(\frac{m\pi z}{H}\right)$$

Note: If $f(x, y, z) = u_0$, we can find the a_{klm} values as follows:

$$a_{klm} = \begin{cases} 0 & \text{if any } klm \text{ even} \\ u_0 \frac{4}{k\pi}\frac{4}{l\pi}\frac{4}{m\pi} & \text{if all } klm \text{ odd} \end{cases}$$

So:

$$u(x, y, z, t) = \sum_{klm} \frac{64u_0}{\pi^3(2k+1)(2l+1)(2m+1)} \exp\left(-\alpha\lambda_{klm} t\right) \cos\left(\frac{k\pi x}{L}\right) \cos\left(\frac{l\pi y}{W}\right) \cos\left(\frac{m\pi z}{H}\right)$$

$$\lambda_{klm} = \frac{(2k+1)^2\pi^2}{L^2} + \frac{(2l+1)^2\pi^2}{W^2} + \frac{(2m+1)^2\pi^2}{H^2}$$

3.2 Example 2: Neumann BC's

Assume Neumann BC's

$$0 = u_x(0, y, z, t) = u_x(L, y, z, t) = u_y(x, 0, z, t) = u_y(x, W, z, t) = u_z(x, y, 0, t) = u_z(x, y, H, t)$$

Assume a separable solution:

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

With this in mind, the boundary conditions become:

$$X'(0) = 0X'(L) = 0Y'(0) = 0Y'(W) = 0Z'(0) = 0Z'(H) = 0$$

If we apply these conditions, we find:

- $\lambda = 0$ is a possible solution with $A_x = A_y = A_z = 0$.
- $\lambda < 0$ is not a meaningful solution because $A_x = A_y = A_z = B_x = B_y = B_z = 0$
- $\lambda > 0$ is also a possible solution.

$$B_x = B_y = B_z = 0 \, k_x = \frac{k\pi}{L} \, k_y = \frac{l\pi}{W} \, k_z = \frac{m\pi}{H}$$

where k, l, m are non-zero, positive integers. As a result, we get:

$$u(x, y, z, t) = \frac{a_0}{8} + \sum_{klm} a_{klm} \exp\left(-\alpha\lambda_{klm} t\right) \cos\left(\frac{k\pi x}{L}\right) \cos\left(\frac{l\pi y}{W}\right) \cos\left(\frac{m\pi z}{H}\right)$$

$$\lambda_{klm} = \frac{k^2\pi^2}{L^2} + \frac{l^2\pi^2}{W^2} + \frac{m^2\pi^2}{H^2}$$

Initial condition:

$$u(x,y,z,0) = f(x,y,z)$$

This implies

$$a_0 = \frac{8}{LWH} \int_0^L dx \int_0^W dy \int_0^H dz f(x,y,z)$$

and

$$a_{klm} = \frac{8}{LWH} \int_0^L dx \int_0^W dy \int_0^H dz f(x,y,z) \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi y}{W}\right) \sin\left(\frac{m\pi z}{H}\right)$$

If $f(x,y,z) = u_0$, then $a_0 = u_0$ and $a_{klm} = 0$ for all k, l, m, so this is a static system.