

## ANALYSIS I EXAMPLES 2

G.P. Paternain Lent 2012

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at [g.p.paternain@dpmms.cam.ac.uk](mailto:g.p.paternain@dpmms.cam.ac.uk).

1. Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = x$  if  $x \in \mathbb{Q}$  and  $f(x) = 1-x$  otherwise. Find  $\{a : f \text{ is continuous at } a\}$ .
  2. Write down the definition of “ $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ”. Prove that  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$  if, and only if,  $f(x_n) \rightarrow \infty$  for every sequence such that  $x_n \rightarrow \infty$ .
  3. Suppose that  $f(x) \rightarrow \ell$  as  $x \rightarrow a$  and  $g(y) \rightarrow k$  as  $y \rightarrow \ell$ . Must it be true that  $g(f(x)) \rightarrow k$  as  $x \rightarrow a$ ?
  4. Let  $f_n : [0, 1] \rightarrow [0, 1]$  be continuous,  $n \in \mathbb{N}$ . Let  $h_n(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$ . Show that  $h_n$  is continuous on  $[0, 1]$  for each  $n \in \mathbb{N}$ . Must  $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$  be continuous?
  5. The unit circle in  $\mathbb{C}$  is mapped to  $\mathbb{R}$  by a map  $e^{i\theta} \mapsto f(\theta)$ , where  $f : [0, 2\pi] \rightarrow \mathbb{R}$  is continuous and  $f(0) = f(2\pi)$ . Show that there exist two diametrically opposite points that have the same image.
  6. Let  $f(x) = \sin^2 x + \sin^2(x + \cos^7 x)$ . Assuming the familiar features of  $\sin$  without justification, prove that there exists  $k > 0$  such that  $f(x) \geq k$  for all  $x \in \mathbb{R}$ .
  7. Suppose that  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous, that  $f(0) = f(1) = 0$ , and that for every  $x \in (0, 1)$  there exists  $0 < \delta < \min\{x, 1-x\}$  with  $f(x) = (f(x-\delta) + f(x+\delta))/2$ . Show that  $f(x) = 0$  for all  $x$ .
  8. Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. Suppose that  $f((x+y)/2) \leq (f(x) + f(y))/2$  for all  $x, y \in [a, b]$ . Prove that  $f$  is continuous on  $(a, b)$ . Must it be continuous at  $a$  and  $b$  too?
  9. Prove that  $2x^5 + 3x^4 + 2x + 16 = 0$  has no real solutions outside  $[-2, -1]$  and exactly one inside.
  10. Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Which of (1)–(4) must be true?
    - (1) If  $f$  is increasing then  $f'(x) \geq 0$  for all  $x \in (a, b)$ .
    - (2) If  $f'(x) \geq 0$  for all  $x \in (a, b)$  then  $f$  is increasing.
    - (3) If  $f$  is strictly increasing then  $f'(x) > 0$  for all  $x \in (a, b)$ .
    - (4) If  $f'(x) > 0$  for all  $x \in (a, b)$  then  $f$  is strictly increasing.
- [Increasing means  $f(x) \leq f(y)$  if  $x < y$ , and *strictly increasing* means  $f(x) < f(y)$  if  $x < y$ .]
11. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable for all  $x$ . Prove that if  $f'(x) \rightarrow \ell$  as  $x \rightarrow \infty$  then  $f(x)/x \rightarrow \ell$ . If  $f(x)/x \rightarrow \ell$  as  $x \rightarrow \infty$ , must  $f'(x)$  tend to a limit?
  12. Let  $f(x) = x + 2x^2 \sin(1/x)$  for  $x \neq 0$  and  $f(0) = 0$ . Show that  $f$  is differentiable everywhere and that  $f'(0) = 1$ , but that there is no interval around 0 on which  $f$  is increasing.
  13. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function which has the intermediate value property: If  $f(a) < c < f(b)$ , then  $f(x) = c$  for some  $x$  between  $a$  and  $b$ . Suppose also that for every rational  $r$ , the set  $S_r$  of all  $x$  with  $f(x) = r$  is closed, that is, if  $x_n$  is any sequence in  $S_r$  with  $x_n \rightarrow a$ , then  $a \in S_r$ . Prove that  $f$  is continuous.