1. Let $f\colon [a,b] \to \mathbb{R}$ be an integrable function with $f(x) \geqslant 0$ for every x. Assume further that f is continuous. Show that $\int_a^b f(t)\,\mathrm{d}t = 0$ iff $f(x) = 0$ for every x. Does this hold without the assumption of continuity?

2. Give an example of a function $f\colon [0,1] \to \mathbb{R}$ such that $|f|$ is integrable but f is not. Give an example of a sequence $f_n\colon [0,1] \to [0,1]$ of integrable functions such that the function $f(x) = \sup_n f_n(x)$ is not integrable.

3. Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = 0$ when x is irrational, and $f(x) = 1/q$ when $x = p/q$ is a rational written in its lowest terms. Prove that f is integrable on $[0,1]$. What is $\int_0^1 f(x)\,\mathrm{d}x$?

4. Let $a < b$ and f be an integrable function on $[a,b]$. Show that for every closed subinterval $I \subset [a,b]$ of positive length and every $\varepsilon > 0$ there exists a closed subinterval $J \subset I$ of positive length such that $\sup_J f - \inf_J f < \varepsilon$. Use this to show that if $f(x) > 0$ for every x then $\int_a^b f(x)\,\mathrm{d}x > 0$.

5. Let f be a continuous function on $[a,b]$ and let $a < c < d < b$. Prove that

$$\lim_{h\to 0} \int_c^d \frac{f(x+h) - f(x)}{h}\,\mathrm{d}x = f(d) - f(c)\ .$$

6. Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Let $G(x,t) = t(x-1)$ when $t \leqslant x$ and $x(t-1)$ when $t \geqslant x$. Let $g(x) = \int_0^1 G(x,t)f(t)\,\mathrm{d}t$. Show that $g''(x)$ exists for $x \in (0,1)$ and equals $f(x)$.

7. Which of the following improper integrals converge?

(i) $\displaystyle\int_1^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x\ .$

(ii) $\displaystyle\int_0^\infty x^p \exp(-x^q)\,\mathrm{d}x$ (where $p, q > 0$).

(iii) $\displaystyle\int_0^\infty \sin(x^2)\,\mathrm{d}x\ .$

8. Give an example of a continuous function $f\colon [0,\infty) \to [0,\infty)$ such that $\int_0^\infty f(x)\,\mathrm{d}x$ exists but f is unbounded.

9. Let $f\colon [1,\infty) \to \mathbb{R}$ be a decreasing, non-negative function. Show that the series $\sum\limits_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int\limits_1^\infty f$ converges. Compute the limits

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{n+k}\ .$$

10. For each non-negative integer n let $I_n(\theta) = \int_{-1}^{1}(1-x^2)^n \cos(\theta x)\,\mathrm{d}x$. Prove that $\theta^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2}$ for all $n \geqslant 2$, and hence that $\theta^{2n+1} I_n(\theta) = n!(P_n(\theta)\sin\theta + Q_n(\theta)\cos\theta)$ for some pair P_n, Q_n of polynomials of degree at most $2n$ with integer coefficients. Deduce that π is irrational.

11. For each $n \in \mathbb{N}$ let $u_n = \int_0^{\pi/2} \sin 2nx \,\cot x\,\mathrm{d}x$ and $v_n = \int_0^{\pi/2} \frac{\sin 2nx}{x}\,\mathrm{d}x$. Prove that $u_n = \pi/2$. By considering the limit of v_n and of $u_n - v_n$ as $n \to \infty$, show that $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$.

12. A function $f\colon [a,b] \to \mathbb{R}$ is said to have *bounded variation* if there exists $K \geqslant 0$ such that for every dissection $\mathcal{D}\colon a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$, we have $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leqslant K$. Prove that a function of bounded variation is integrable. Is the converse true?

13. For a dissection $\mathcal{D}\colon a = x_0 < x_1 < \cdots < x_n = b$ of a closed bounded interval $[a,b]$ define $|\mathcal{D}| = \max_k(x_k - x_{k-1})$. Assume that $f\colon [a,b] \to \mathbb{R}$ is integrable and $(\mathcal{D}_n)$ is a sequence of dissections of $[a,b]$ with $|\mathcal{D}_n| \to 0$ as $n \to \infty$. Prove that $U_{\mathcal{D}_n}(f) - L_{\mathcal{D}_n}(f) \to 0$ as $n \to \infty$. Deduce that if $\mathcal{D}_n$ is the sequence $a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{m_n}^{(n)} = b$ and $\xi_k^{(n)} \in [x_{k-1}^{(n)}, x_k^{(n)}]$ for each $1 \leqslant k \leqslant m_n$, then

$$\sum_{k=1}^{m_n} f(\xi_k^{(n)})(x_k^{(n)} - x_{k-1}^{(n)}) \to \int_a^b f(t)\,\mathrm{d}t \qquad \text{as } n \to \infty\ .$$

14$^+$. Let $f\colon [0,1] \to \mathbb{R}$ be a function that is differentiable everywhere (with one-sided derivatives at the endpoints) with a derivative f' that is bounded. Must f' be integrable?