1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ when $x \in \mathbb{Q}$, and $f(x) = -x^2$ when $x \notin \mathbb{Q}$. At which points is f (a) continuous (b) differentiable?

2. Carefully define what it means that $f(x) \to \ell$ as $x \to \infty$. Prove that this happens if and only if $f(x_n) \to \ell$ for every sequence such that $x_n \to \infty$.

3. Let $f_n\colon [0,1] \to [0,1]$ be a continuous function for each $n \in \mathbb{N}$. Let $h_n(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$. Show that h_n is continuous on $[0,1]$ for each n. Must the function h defined by $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous on $[0,1]$?

4. Let $g\colon [0,1] \to [0,1]$ be a continuous function. Prove that there exists some $c \in [0,1]$ such that $g(c) = c$. Such a c is called a *fixed point* of g. Give an example of a bijection $h\colon [0,1] \to [0,1]$ with no fixed point. Give an example of a continuous bijection $k\colon (0,1) \to (0,1)$ with no fixed point.

5. A function f defined on a set A is *locally bounded* if every point in A has a neighbourhood on which f is bounded: for all $a \in A$ there exist $\delta > 0$ and $C \in \mathbb{R}$ such that if $x \in A$ and $|x - a| < \delta$ then $|f(x)| \leqslant C$. Show that every continuous function is locally bounded. Show that a locally bounded function on a closed bounded interval is bounded.

6. (i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable everywhere. For which x is f' continuous at x?
(ii) Give an example of a function $g\colon \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere such that g' is not bounded on the interval $(-\delta, \delta)$ for any $\delta > 0$.

7. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the inequality $|f(x) - f(y)| \leqslant |x - y|^2$ for every $x, y \in \mathbb{R}$. Show that f is constant.

8. Prove that the real polynomial $p(x) = 2x^5 + 3x^4 + 2x + 16$ takes the value 0 exactly once, and that the number where it takes that value is somewhere in the interval $[-2, -1]$.

9. Let $D \subset \mathbb{C}$ be a disc and $f\colon D \to \mathbb{R}$ be a continuous function. Show that the image $f(D)$ of f is an interval.

10. Let $f\colon [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1) = 0$. Suppose that for every $x \in (0,1)$ there exists $\delta > 0$ such that both $x - \delta$ and $x + \delta$ belong to $(0,1)$ and $f(x) = \frac{1}{2}\big(f(x-\delta) + f(x+\delta)\big)$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

11. Prove Cauchy's mean value theorem: let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions which are differentiable on the open interval (a,b); show that for some $c \in (a,b)$ the vectors $\big(f(b) - f(a), g(b) - g(a)\big)$ and $\big(f'(c), g'(c)\big)$ in $\mathbb{R}^2$ are parallel. Does this generalize to three or more functions?

12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Prove that if $f'(x) \to \ell$ as $x \to \infty$ then $f(x)/x \to \ell$ as $x \to \infty$. If $f(x)/x \to \ell$ as $x \to \infty$, does it follow that $f'(x) \to \ell$?

13. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by setting $f(x) = 0$ if x is irrational, and $f(x) = 1/q$ when $x = p/q$ for coprime integers p and q with $q > 0$. Prove that f is continuous at every irrational and discontinuous at every rational. $^{+}$ Does there exist a function $g\colon \mathbb{R} \to \mathbb{R}$ which is continuous at every rational and discontinuous at every irrational?

14. A function $f\colon I \to \mathbb{R}$ on an interval I is *convex* if

$$f\big((1-t)x + ty\big) \leqslant (1-t)f(x) + tf(y) \qquad \forall\, x, y \in I\ \forall\, t \in [0,1]\ .$$

Assume now that I is an open interval. Show the following.

(i) If f is convex then it is continuous.

(ii) Assume f is locally bounded and satisfies $f\big(\frac{1}{2}x + \frac{1}{2}y\big) \leqslant \frac{1}{2}f(x) + \frac{1}{2}f(y)$ for all $x, y \in I$. Show that f is continuous, and deduce that f is convex.

(iii) If f is convex then for each $c \in I$ there exists $m \in \mathbb{R}$ such that

$$m(x-c) + f(c) \leqslant f(x) \qquad \text{for all } x \in I\ ,$$

and if in addition f is differentiable at c then $f'(c)$ is the unique m that works. In general, must m be unique?

(iv) If f is differentiable on I, then f is convex if and only if f' is increasing.