UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS   **Part IB & II (General)**

Tuesday, 30 May, 2017   9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

*You may submit answers to no more than **six** questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

*Write on **one** side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

*Each question has a number and a letter (for example, **3B**).*

*Answers must be tied up in **separate** bundles, marked **A, B or C** according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

*A **separate** green master cover sheet listing all the questions attempted **must** also be completed.*

Every cover sheet must bear your examination number and desk number.

> **You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

1B

(a) State the divergence theorem for a vector field $\boldsymbol{G}$. [2]

(b) Let A denote the *open* surface

$$x^2 + y^2 = 2z^2, \qquad 0 \leqslant z < h.$$

Sketch the surface A. [3]

(c) By applying the divergence theorem to a suitable *closed* surface, or otherwise, calculate

$$\int_A \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{A},$$

where $\mathrm{d}\boldsymbol{A}$ is the unit area element pointing out of A, and

$$\boldsymbol{G} = \begin{pmatrix} x^3 + 2xy \\ y^3 + \sin x \\ z \end{pmatrix}.$$

[15]

2B

Consider the equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{2v}{t+1},$$

where $v(x,t)$ is defined on $0 \leqslant x \leqslant \pi$ and is subject to the initial and boundary conditions

$$v(0,t) = 0, \ \ v(\pi,t) = f(t), \ \ v(x,0) = h(x),$$

for some functions $f(t)$ and $h(x)$.

(a) Using the substitution $v = (t+1)^2\, u$, show that u satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

and state the boundary and initial conditions satisfied by u. [5]

(b) Now consider the specific case when the functions f and h are given by

$$f(t) = 3(t+1)^2\,, \qquad h(x) = \frac{\sin(2x) + 3x}{\pi}\,.$$

Using the method of separation of variables, construct the solution $v(x,t)$.

[13]

[*Hint: You may find it helpful to use the substitution $u(x,t) = w(x,t) + \gamma\, x$, for a suitably chosen constant γ.*]

(c) For $t \gg 1$, show that

$$v \sim \frac{3\,x\,t^2}{\pi}\,.$$

[2]

3B

An amplifier outputs a signal $x(t)$ given by the initial-value problem

$$\frac{d^2x}{dt^2} + 2q\frac{dx}{dt} + (q^2 + 4)\,x = f(t)\,, \qquad x(0) = \frac{dx}{dt}(0) = 0\,, \qquad (\star)$$

for some constant $q > 0$ and input function $f(t)$.

(a) Show that the Green's function $G(t, \tau)$ for this problem is

$$G(t, \tau) = \begin{cases} 0 & 0 \leqslant t < \tau, \\ \frac{1}{2}e^{-q(t-\tau)}\sin\left[2(t-\tau)\right] & \tau \leqslant t\,. \end{cases}$$

[7]

Write down the general solution $x(t)$ of equation $(\star)$ in terms of an integral. [2]

(b) Now consider the specific case $q = 0$ and

$$f(t) = \begin{cases} t_0 & 0 \leqslant t < t_0, \\ 0 & t_0 \leqslant t, \end{cases}$$

where $t_0 > 0$ is a constant. Calculate the solution of equation $(\star)$ in this case. [8]

Find all values of t_0 for which $x(t) = 0$ for all $t \geqslant t_0$. [3]

4A

(a) The Fourier transform of a function $f(t)$ is given by

$$\widetilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, \mathrm{d}t.$$

Write down the corresponding expression for the inverse Fourier transform. [2]

(b) Consider the convolution of the functions f and g

$$h(z) = \int_{-\infty}^{\infty} f(t)g(z-t)\, \mathrm{d}t.$$

Prove that the Fourier transform of h is given by the product of the Fourier transforms of f and g. [5]

(c) Find the Fourier transform of

$$f(\gamma, p, t) = \begin{cases} e^{-\gamma t}\sin pt & t > 0\,, \\ 0 & t \leqslant 0\,, \end{cases}$$

where $\gamma > 0$ and p are fixed parameters.

[*Hint: Write $\sin pt$ in terms of exponential functions.*] [6]

(d) The current $I(t)$ flowing through a system is related to the applied voltage $V(t)$ by the equation

$$I(t) = \int_{-\infty}^{\infty} K(t-u)V(u)\, \mathrm{d}u,$$

where

$$K(\tau) = a_1 f(\gamma_1, p_1, \tau) + a_2 f(\gamma_2, p_2, \tau).$$

Here the function $f(\gamma, p, t)$ is as given in part (c), and all the $a_i, \gamma_i > 0$ and p_i are fixed parameters. By considering the Fourier transform of $I(t)$, find the relationship that must hold between a_1 and a_2 if the net charge Q, defined by

$$Q = \int_{-\infty}^{\infty} I(t')\, \mathrm{d}t'\,,$$

is to be zero for an arbitrary applied voltage.

[*Hint:* $\int_{-\infty}^{\infty} \exp[i\omega t']\, \mathrm{d}t' = 2\pi\delta(\omega).$] [7]

5C

(a) When is an $n \times n$ matrix A diagonalisable? Give an example of a non-diagonalizable $n \times n$ matrix (for some n). What is a Hermitian matrix? Show that the eigenvalues of a Hermitian matrix are real, and that the corresponding eigenvectors are orthogonal. [5]

(b) Diagonalise the matrix
$$A = \begin{pmatrix} 2 & -a & 0 \\ -a & 2 & 0 \\ 0 & 0 & c \end{pmatrix},$$
where $a > 0$ and $c > 0$ are real numbers and finds its eigenvectors. [6]

Sketch the surface
$$\mathbf{x}^T A \mathbf{x} = 1,$$

where $\mathbf{x} = (x, y, z)$, specifying the principal axes and, where appropriate, the semi-axis lengths. Note that different values of a may correspond to different surfaces. [9]

6C

(a) Let A and B be $n \times n$ Hermitian matrices, each with n distinct eigenvalues. Show that:

  (i) the matrix $H = i(AB - BA)$ is Hermitian; [4]

  (ii) the eigenvectors of A and B are identical if and only if $AB = BA$; [6]

  (iii) the matrix $N = A + iB$ is diagonalisable. [5]

(b) Suppose C is a unitary matrix, A is a Hermitian matrix, and p is a positive integer. Show that $(C^{-1}AC)^p$ has real eigenvalues. [5]

7A

(a) Use the Cauchy-Riemann relations to show that, for any analytic function $f(x, y) = u(x, y) + i\, v(x, y)$, the relation $|\nabla u| = |\nabla v|$ must hold. [2]

(b) Find the most general analytic function $f(z)$ of the variable $z = x + i\, y$ whose imaginary part is

$$(y \cos y + x \sin y) \exp x.$$

(*Your final expression for $f(z)$ should be in terms of z, not x and y.*) [10]

(c) Find the radii of convergence of the following Taylor series:

(i)
$$\sum_{n=2}^{\infty} \frac{z^n}{\ln n}\,;$$

[3]

(ii)
$$\sum_{n=1}^{\infty} \left(\frac{n+p}{n}\right)^{n^2} z^n\,, \quad \text{with } p \text{ real.}$$

[5]

[*Hint: You may want to use the following result:*

$$a^n = e^{n \ln a}\,,$$

for some real a.]

8C

(a) Find the power series solution of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\,y = 0\,, \tag{$\star$}$$

where λ is a real parameter, about the point $x = 0$, and find suitable recurrence relations for the coefficients. For what values of λ does ($\star$) have a polynomial solution? Find the solutions corresponding to two eigenvalues λ of your choice. [10]

(b) Consider the hypergeometric equation

$$x(1-x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + [\gamma - (1+\alpha+\beta)x]\frac{\mathrm{d}y}{\mathrm{d}x} - \alpha\,\beta\,y = 0\,,$$

where α, β and γ are real constants. Assuming a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n\,x^{n+\sigma} \quad \text{with} \quad a_0 \neq 0\,,$$

show that

$$\sigma = 0 \quad \text{or} \quad \sigma = 1-\gamma\,,$$

and that

$$a_n = \frac{(n+\sigma+\alpha-1)(n+\sigma+\beta-1)}{(n+\sigma)(n+\sigma+\gamma-1)}a_{n-1}$$

for all $n \geqslant 1$. [10]

9B

(a) The Euler–Lagrange equation for extrema of the functional

$$D[y] = \int_a^b f(x, y, y') \, \mathrm{d}x \,,$$

where $y' = \mathrm{d}y/\mathrm{d}x$, is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{$\star$}$$

If $f = f(y, y')$ does not depend explicitly on x show that ($\star$) can be written as

$$\frac{\mathrm{d}h}{\mathrm{d}x} = 0 \,.$$

for some h, which you should determine. [3]

(b) A forest lies in the (x, y) plane. A new path through the forest is proposed, starting at $(x, y) = (-1, 1)$ and ending at $(x, y) = (1, 1)$. The density of undergrowth in the forest is given by $g(y)$, such that the total undergrowth D to be destroyed by the new path is

$$D = \int_{\mathcal{P}} g(y) \, \mathrm{d}s \,,$$

where $\mathrm{d}s$ is the arc-length element along the path $\mathcal{P}$.

  (i) Given that the path always travels in the positive x direction, show that the path $y(x)$ that minimises the destruction of undergrowth satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g}{\sqrt{1 + y'^2}}\right) = 0 \,.$$

[4]

  (ii) In the specific case when $g = y^{-1}$, calculate the path $y(x)$. [9]

  (iii) Sketch the path, and determine its length. [4]

10B

Consider the problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \epsilon \left[x \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} - u \right] = -\lambda u, \quad 0 \leqslant x \leqslant \pi, \quad u'(0) = u(\pi) = 0, \qquad (\star)$$

where $\epsilon \geqslant 0$ is a parameter, λ is a real constant, and $u' = \mathrm{d}u/\mathrm{d}x$. Express $(\star)$ in the form

$$\mathcal{L}u = \lambda u, \qquad\qquad (\star\star)$$

where $\mathcal{L}$ is an operator in Sturm–Liouville form. [2]

Now consider the functional

$$I[v] = \int_0^\pi \left(p\, v'^2 + q\, v^2 \right)\, \mathrm{d}x,$$

where $v(x)$ satisfies $v'(0) = v(\pi) = 0$, and is subject to the constraint

$$\int_0^\pi w\, v^2\, \mathrm{d}x = 1,$$

for smooth functions $p(x) > 0$, $q(x) \geqslant 0$ and $w(x) > 0$. Show that, for a particular choice of the functions p, q and w, which should be specified, finding extrema of I is equivalent to finding solutions of $(\star)$. Explain why the stationary values of I are the eigenvalues λ of equation $(\star\star)$. You may use the Euler–Lagrange equation without proof. [8]

When $\epsilon = 0$, show that the smallest eigenvalue of $(\star\star)$ is $\lambda_0 = 1/4$, and the associated normalised eigenfunction is

$$U_0(x) = \sqrt{\frac{2}{\pi}} \cos\left(\frac{x}{2}\right).$$

Using $U_0(x)$ as a trial function, find an upper bound for the lowest eigenvalue λ of equation $(\star\star)$ when $\epsilon > 0$. [10]

END OF PAPER