UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 31 May 2001 9.00 to 12.00

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I and at most* ***five*** *questions from Section II. In Section II no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in two bundles, marked ***C*** *and* ***D*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet *must* ***bear your examination number and desk number.***

SECTION I

1C Algebra and Geometry

Show, using the summation convention or otherwise, that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$, for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

The function $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\Pi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n})$ where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. Show that Π is linear and find the elements of a matrix P such that $\Pi(\mathbf{x}) = P\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Find all solutions to the equation $\Pi(\mathbf{x}) = \mathbf{x}$. Evaluate $\Pi(\mathbf{n})$. Describe the function Π geometrically. Justify your answer.

2C Algebra and Geometry

Define what is meant by the statement that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ are linearly independent. Determine whether the following vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ are linearly independent and justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

For the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ taken from a real vector space V consider the statements

A) $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent,
B) $\exists\, \alpha, \beta, \gamma \in \mathbb{R} : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
C) $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$, not all $= 0 : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
D) $\exists\, \alpha, \beta \in \mathbb{R}$, not both $= 0 : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
E) $\exists\, \alpha, \beta \in \mathbb{R} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
F) $\nexists$ basis of V that contains all 3 vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

State if the following implications are true or false (no justification is required):

i)	A $\Rightarrow$ B,	vi)	B $\Rightarrow$ A,
ii)	A $\Rightarrow$ C,	vii)	C $\Rightarrow$ A,
iii)	A $\Rightarrow$ D,	viii)	D $\Rightarrow$ A,
iv)	A $\Rightarrow$ E,	ix)	E $\Rightarrow$ A,
v)	A $\Rightarrow$ F,	x)	F $\Rightarrow$ A.

3D Analysis I

What does it mean to say that $u_n \to l$ as $n \to \infty$?

Show that, if $u_n \to l$ and $v_n \to k$, then $u_n v_n \to lk$ as $n \to \infty$.

If further $u_n \neq 0$ for all n and $l \neq 0$, show that $1/u_n \to 1/l$ as $n \to \infty$.

Give an example to show that the non-vanishing of u_n for all n need not imply the non-vanishing of l.

4D Analysis I

Starting from the theorem that any continuous function on a closed and bounded interval attains a maximum value, prove Rolle's Theorem. Deduce the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(t) > 0$ for all t show that f is a strictly increasing function.

Conversely, if f is strictly increasing, is $f'(t) > 0$ for all t?

SECTION II

5C Algebra and Geometry

The matrix

$$A_\alpha = \begin{pmatrix} 1 & -1 & 2\alpha+1 \\ 1 & \alpha-1 & 1 \\ 1+\alpha & -1 & \alpha^2+4\alpha+1 \end{pmatrix}$$

defines a linear map $\Phi_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_\alpha(\mathbf{x}) = A_\alpha \mathbf{x}$. Find a basis for the kernel of Φ_α for all values of $\alpha \in \mathbb{R}$.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be bases of $\mathbb{R}^3$. Show that there exists a matrix S, to be determined in terms of $\mathcal{B}$ and $\mathcal{C}$, such that, for every linear mapping Φ, if Φ has matrix A with respect to $\mathcal{B}$ and matrix A' with respect to $\mathcal{C}$, then $A' = S^{-1}AS$.

For the bases

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad , \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\} \quad ,$$

find the basis transformation matrix S and calculate $S^{-1}A_0S$.

6C Algebra and Geometry

Assume that $\mathbf{x}_p$ is a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x}, \mathbf{b} \in \mathbb{R}^3$ and a real 3×3 matrix A. Explain why the general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = \mathbf{x}_p + \mathbf{h}$ where $\mathbf{h}$ is any vector such that $A\mathbf{h} = \mathbf{0}$.

Now assume that A is a real symmetric 3×3 matrix with three different eigenvalues λ_1, λ_2 and λ_3. Show that eigenvectors of A with respect to different eigenvalues are orthogonal. Let $\mathbf{x}_k$ be a normalised eigenvector of A with respect to the eigenvalue λ_k, $k = 1, 2, 3$. Show that the linear system

$$(A - \lambda_k I)\mathbf{x} \;=\; \mathbf{b} \;\; ,$$

where I denotes the 3×3 unit matrix, is solvable if and only if $\mathbf{x}_k.\mathbf{b} = 0$. Show that the general solution is given by

$$\mathbf{x} \;=\; \sum_{i \neq k} \tfrac{\mathbf{b}.\mathbf{x}_i}{\lambda_i - \lambda_k} \mathbf{x}_i + \beta \mathbf{x}_k, \quad \beta \in \mathbb{R}.$$

[*Hint: consider the components of* $\mathbf{x}$ *and* $\mathbf{b}$ *with respect to a basis of eigenvectors of* A.]

Consider the matrix A and the vector $\mathbf{b}$

$$A = \begin{pmatrix} -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \sqrt{2} + \sqrt{3} \\ -\sqrt{2} + \sqrt{3} \\ -2\sqrt{3} \end{pmatrix}.$$

Verify that $\frac{1}{\sqrt{3}}(1, 1, 1)^T$ and $\frac{1}{\sqrt{2}}(1, -1, 0)^T$ are eigenvectors of A. Show that $A\mathbf{x} = \mathbf{b}$ is solvable and find its general solution.

7C Algebra and Geometry

For $\alpha, \gamma \in \mathbb{R}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$ and $\beta\overline{\beta} \geqslant \alpha\gamma$ the equation $\alpha z\overline{z} - \beta\overline{z} - \overline{\beta}z + \gamma = 0$ describes a circle $C_{\alpha\beta\gamma}$ in the complex plane. Find its centre and radius. What does the equation describe if $\beta\overline{\beta} < \alpha\gamma$? Sketch the circles $C_{\alpha\beta\gamma}$ for $\beta = \gamma = 1$ and $\alpha = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1$.

Show that the complex function $f(z) = \beta\overline{z}/\overline{\beta}$ for $\beta \neq 0$ satisfies $f(C_{\alpha\beta\gamma}) = C_{\alpha\beta\gamma}$.

[*Hint:* $f(C) = C$ *means that* $f(z) \in C \ \forall z \in C$ *and* $\forall w \in C \ \exists z \in C$ *such that* $f(z) = w$.]

For two circles C_1 and C_2 a function $m(C_1, C_2)$ is defined by

$$m(C_1, C_2) \;=\; \max_{z \in C_1, w \in C_2} \left| z - w \right| \;\; .$$

Prove that $m(C_1, C_2) \leqslant m(C_1, C_3) + m(C_2, C_3)$. Show that

$$m(C_{\alpha_1\beta_1\gamma_1}, C_{\alpha_2\beta_2\gamma_2}) \;=\; \tfrac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{|\alpha_1\alpha_2|} + \tfrac{\sqrt{\beta_1\overline{\beta_1} - \alpha_1\gamma_1}}{|\alpha_1|} + \tfrac{\sqrt{\beta_2\overline{\beta_2} - \alpha_2\gamma_2}}{|\alpha_2|} \;\; .$$

8C Algebra and Geometry

Let $l_{\mathbf{x}}$ denote the straight line through $\mathbf{x}$ with directional vector $\mathbf{u} \neq \mathbf{0}$

$$l_{\mathbf{x}} = \left\{ \mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = \mathbf{x} + \lambda \mathbf{u}, \lambda \in \mathbb{R} \right\} .$$

Show that $l_{\mathbf{0}}$ is a subspace of $\mathbb{R}^3$ and show that $l_{\mathbf{x_1}} = l_{\mathbf{x_2}} \Leftrightarrow \mathbf{x_1} = \mathbf{x}_2 + \lambda \mathbf{u}$ for some $\lambda \in \mathbb{R}$.

For fixed $\mathbf{u} \neq \mathbf{0}$ let $\mathcal{L}$ be the set of all the parallel straight lines $l_{\mathbf{x}}$ ($\mathbf{x} \in \mathbb{R}^3$) with directional vector $\mathbf{u}$. On $\mathcal{L}$ an addition and a scalar multiplication are defined by

$$l_{\mathbf{x}} + l_{\mathbf{y}} = l_{\mathbf{x}+\mathbf{y}}, \ \alpha l_{\mathbf{x}} = l_{\alpha \mathbf{x}}, \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \ \alpha \in \mathbb{R} .$$

Explain why these operations are well-defined. Show that the addition is associative and that there exists a zero vector which should be identified.

You may now assume that $\mathcal{L}$ is a vector space. If $\{\mathbf{u}, \mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^3$ show that $\{l_{\mathbf{b}_1}, l_{\mathbf{b}_2}\}$ is a basis for $\mathcal{L}$.

For $\mathbf{u} = (1, 3, -1)^T$ a linear map $\Phi : \mathcal{L} \to \mathcal{L}$ is defined by

$$\Phi\left(l_{(1,-1,0)^T}\right) = l_{(2,4,-1)^T} \ , \ \Phi\left(l_{(1,1,0)^T}\right) = l_{(-4,-2,1)^T} \ .$$

Find the matrix A of Φ with respect to the basis $\{l_{(1,0,0)^T}, l_{(0,1,0)^T}\}$.

9D Analysis I

(i) If $a_0, a_1, \ldots$ are complex numbers show that if, for some $w \in \mathbb{C}, w \neq 0$, the set $\{|a_n w^n| : n \geq 0\}$ is bounded and $|z| < |w|$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Use this result to define the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$.

(ii) If $|a_n|^{1/n} \to R$ as $n \to \infty$ $(0 < R < \infty)$ show that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence equal to $1/R$.

(iii) Give examples of power series with radii of convergence 1 such that (a) the series converges at all points of the circle of convergence, (b) diverges at all points of the circle of convergence, and (c) neither of these occurs.

10D Analysis I

Suppose that f is a continuous real-valued function on $[a, b]$ with $f(a) < f(b)$. If $f(a) < v < f(b)$ show that there exists c with $a < c < b$ and $f(c) = v$.

Deduce that if f is a continuous function from the closed bounded interval $[a, b]$ to itself, there exists at least one fixed point, i.e., a number d belonging to $[a, b]$ with $f(d) = d$. Does this fixed point property remain true if f is a continuous function defined (i) on the open interval (a, b) and (ii) on $\mathbb{R}$? Justify your answers.

11D Analysis I

(i) Show that if $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable then, given $\epsilon > 0$, we can find some constant L and $\delta(\epsilon) > 0$ such that

$$|g(t) - g(\alpha) - g'(\alpha)\,(t - \alpha)| \;\leq\; L|t - \alpha|^2$$

for all $|t - \alpha| < \delta(\epsilon)$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable on $[a, b]$ (with one-sided derivatives at the end points), let f' and f'' be strictly positive functions and let $f(a) < 0 < f(b)$.

If $F(t) = t - (f(t)/f'(t))$ and a sequence $\{x_n\}$ is defined by $b = x_0, x_n = F(x_{n-1})$ $(n > 0)$, show that $x_0, x_1, x_2, \ldots$ is a decreasing sequence of points in $[a, b]$ and hence has limit α. What is $f(\alpha)$? Using part (i) or otherwise estimate the rate of convergence of x_n to α, i.e., the behaviour of the absolute value of $(x_n - \alpha)$ for large values of n.

12D Analysis I

Explain what it means for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$, and give an example of a bounded function that is not Riemann integrable.

Show each of the following statements is true for continuous functions f, but false for general Riemann integrable functions f.

(i) If $f : [a, b] \to \mathbb{R}$ is such that $f(t) \geq 0$ for all t in $[a, b]$ and $\int_a^b f(t)\,dt = 0$, then $f(t) = 0$ for all t in $[a, b]$.

(ii) $\int_a^t f(x)\,dx$ is differentiable and $\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$.

END OF PAPER

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Friday 1 June 2001 1.30 to 4.30

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SECTION I

1B Differential Equations

Find the solution to

$$\frac{dy(x)}{dx} + \tanh(x)\, y(x) \;=\; H(x)\ ,$$

in the range $-\infty < x < \infty$ subject to $y(0) = 1$, where $H(x)$ is the Heavyside function defined by

$$H(x) \;=\; \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}\ .$$

Sketch the solution.

2B Differential Equations

The function $y(x)$ satisfies the inhomogeneous second-order linear differential equation

$$y'' - y' - 2y \;=\; 18xe^{-x}\ .$$

Find the solution that satisfies the conditions that $y(0) = 1$ and $y(x)$ is bounded as $x \to \infty$.

3F Probability

The following problem is known as Bertrand's paradox. A chord has been chosen at random in a circle of radius r. Find the probability that it is longer than the side of the equilateral triangle inscribed in the circle. Consider three different cases:

a) the middle point of the chord is distributed uniformly inside the circle,

b) the two endpoints of the chord are independent and uniformly distributed over the circumference,

c) the distance between the middle point of the chord and the centre of the circle is uniformly distributed over the interval $[0, r]$.

[*Hint: drawing diagrams may help considerably.*]

4F Probability

The Ruritanian authorities decided to pardon and release one out of three remaining inmates, A, B and C, kept in strict isolation in the notorious Alkazaf prison. The inmates know this, but can't guess who among them is the lucky one; the waiting is agonising. A sympathetic, but corrupted, prison guard approaches A and offers to name, in exchange for a fee, another inmate (not A) who is doomed to stay. He says: "This reduces your chances to remain here from 2/3 to 1/2: will it make you feel better?" A hesitates but then accepts the offer; the guard names B.

Assume that indeed B will not be released. Determine the conditional probability

$$P\big(A \text{ remains } \bigm| B \text{ named}\big) \;=\; \tfrac{P(A\&B \text{ remain})}{P(B \text{ named})}$$

and thus check the guard's claim, in three cases:

a) when the guard is completely unbiased (i.e., names any of B and C with probability $1/2$ if the pair B, C is to remain jailed),

b) if he hates B and would certainly name him if B is to remain jailed,

c) if he hates C and would certainly name him if C is to remain jailed.

SECTION II

5B Differential Equations

The real sequence $y_k,\ k=1,2,\ldots$ satisfies the difference equation

$$y_{k+2} - y_{k+1} + y_k \ = \ 0 \ .$$

Show that the general solution can be written

$$y_k \ = \ a\cos\frac{\pi k}{3} \ + \ b\sin\frac{\pi k}{3} \ ,$$

where a and b are arbitrary real constants.

Now let y_k satisfy

$$y_{k+2} - y_{k+1} + y_k \ = \ \frac{1}{k+2} \ . \qquad (*)$$

Show that a particular solution of $(*)$ can be written in the form

$$y_k \ = \ \sum_{n=1}^{k} \frac{a_n}{k-n+1} \ ,$$

where

$$a_{n+2} - a_{n+1} + a_n = 0 \ , \qquad n \geq 1 \ ,$$

and $a_1 = 1,\ a_2 = 1$.

Hence, find the general solution to $(*)$.

6B Differential Equations

The function $y(x)$ satisfies the linear equation

$$y''(x) + p(x)y'(x) + q(x)y(x) \;=\; 0\;.$$

The Wronskian, $W(x)$, of two independent solutions denoted $y_1(x)$ and $y_2(x)$ is defined to be

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix}\;.$$

Let $y_1(x)$ be given. In this case, show that the expression for $W(x)$ can be interpreted as a first-order inhomogeneous differential equation for $y_2(x)$. Hence, by explicit derivation, show that $y_2(x)$ may be expressed as

$$y_2(x) \;=\; y_1(x)\int_{x_0}^{x} \frac{W(t)}{y_1(t)^2}\,dt\;, \qquad (*)$$

where the rôle of x_0 should be briefly elucidated.

Show that $W(x)$ satisfies

$$\frac{dW(x)}{dx} + p(x)W(x) \;=\; 0\;.$$

Verify that $y_1(x) = 1 - x$ is a solution of

$$xy''(x) - (1-x^2)y'(x) - (1+x)y(x) \;=\; 0\;. \qquad (\dagger)$$

Hence, using $(*)$ with $x_0 = 0$ and expanding the integrand in powers of t to order t^3, find the first three non-zero terms in the power series expansion for a solution, $y_2(x)$, of $(\dagger)$ that is independent of $y_1(x)$ and satisfies $y_2(0) = 0,\; {y_2}''(0) = 1$.

7B Differential Equations

Consider the linear system

$$\dot{\mathbf{z}} + A\mathbf{z} \;=\; \mathbf{h}\,, \qquad (*)$$

where

$$\mathbf{z}(t) \;=\; \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\,, \quad A \;=\; \begin{pmatrix} 1+a & -2 \\ 1 & -1+a \end{pmatrix}\,, \quad \mathbf{h}(t) \;=\; \begin{pmatrix} 2\cos t \\ \cos t - \sin t \end{pmatrix}\,,$$

where $\mathbf{z}(t)$ is real and a is a real constant, $a \geq 0$.

Find a (complex) eigenvector, $\mathbf{e}$, of A and its corresponding (complex) eigenvalue, l. Show that the second eigenvector and corresponding eigenvalue are respectively $\bar{\mathbf{e}}$ and $\bar{l}$, where the bar over the symbols signifies complex conjugation. Hence explain how the general solution to $(*)$ can be written as

$$\mathbf{z}(t) \;=\; \alpha(t)\,\mathbf{e} \;+\; \bar{\alpha}(t)\,\bar{\mathbf{e}}\,,$$

where $\alpha(t)$ is complex.

Write down a differential equation for $\alpha(t)$ and hence, for $a > 0$, deduce the solution to $(*)$ which satisfies the initial condition $\mathbf{z}(0) = \underset{\sim}{0}$.

Is the linear system resonant?

By taking the limit $a \to 0$ of the solution already found deduce the solution satisfying $\mathbf{z}(0) = \underset{\sim}{0}$ when $a = 0$.

8B Differential Equations

Carnivorous hunters of population h prey on vegetarians of population p. In the absence of hunters the prey will increase in number until their population is limited by the availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$\begin{aligned} \dot{p} \;&=\; p\left(1 - \frac{p}{a}\right) - \frac{ph}{a}\,, \\ \dot{h} \;&=\; \frac{h}{8}\left(\frac{p}{b} - 1\right)\,, \end{aligned} \qquad (*)$$

where a and b are positive constants, and $h(t)$ and $p(t)$ are non-negative functions of time, t. By giving an interpretation of each term explain briefly how these equations model the system described.

Consider these equations for $a = 1$. In the two cases $0 < b < 1/2$ and $b > 1$ determine the location and the stability properties of the critical points of $(*)$. In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

9F Probability

I play tennis with my parents; the chances for me to win a game against Mum (M) are p and against Dad (D) q, where $0 < q < p < 1$. We agreed to have three games, and their order can be DMD (where I play against Dad, then Mum then again Dad) or MDM. The results of games are independent.

Calculate under each of the two orders the probabilities of the following events:

a) that I win at least one game,

b) that I win at least two games,

c) that I win at least two games in succession (i.e., games 1 and 2 or 2 and 3, or 1, 2 and 3),

d) that I win exactly two games in succession (i.e., games 1 and 2 or 2 and 3, but not 1, 2 and 3),

e) that I win exactly two games (i.e., 1 and 2 or 2 and 3 or 1 and 3, but not 1, 2 and 3).

In each case a)– e) determine which order of games maximizes the probability of the event. In case e) assume in addition that $p + q > 3pq$.

10F Probability

A random point is distributed uniformly in a unit circle $\mathcal{D}$ so that the probability that it falls within a subset $\mathcal{A} \subseteq \mathcal{D}$ is proportional to the area of $\mathcal{A}$. Let R denote the distance between the point and the centre of the circle. Find the distribution function $F_R(x) = P(R < x)$, the expected value ER and the variance $\mathrm{Var}\ R = ER^2 - (ER)^2$.

Let Θ be the angle formed by the radius through the random point and the horizontal line. Prove that R and Θ are independent random variables.

Consider a coordinate system where the origin is placed at the centre of $\mathcal{D}$. Let X and Y denote the horizontal and vertical coordinates of the random point. Find the covariance $\mathrm{Cov}(X, Y) = E(XY) - EXEY$ and determine whether X and Y are independent.

Calculate the sum of expected values $E\frac{X}{R} + iE\frac{Y}{R}$. Show that it can be written as the expected value $Ee^{i\xi}$ and determine the random variable ξ.

11F Probability

Dipkomsky, a desperado in the wild West, is surrounded by an enemy gang and fighting tooth and nail for his survival. He has m guns, $m > 1$, pointing in different directions and tries to use them in succession to give an impression that there are several defenders. When he turns to a subsequent gun and discovers that the gun is loaded he fires it with probability $1/2$ and moves to the next one. Otherwise, i.e. when the gun is unloaded, he loads it with probability $3/4$ or simply moves to the next gun with complementary probability $1/4$. If he decides to load the gun he then fires it or not with probability $1/2$ and after that moves to the next gun anyway.

Initially, each gun had been loaded independently with probability p. Show that if after each move this distribution is preserved, then $p = 3/7$. Calculate the expected value EN and variance Var N of the number N of loaded guns under this distribution.

[*Hint: it may be helpful to represent* N *as a sum* $\sum_{1 \le j \le m} X_j$ *of random variables taking values* 0 *and* 1.]

12F Probability

A taxi travels between four villages, W, X, Y, Z, situated at the corners of a rectangle. The four roads connecting the villages follow the sides of the rectangle; the distance from W to X and Y to Z is 5 miles and from W to Z and Y to X 10 miles. After delivering a customer the taxi waits until the next call then goes to pick up the new customer and takes him to his destination. The calls may come from any of the villages with probability $1/4$ and each customer goes to any other village with probability $1/3$. Naturally, when travelling between a pair of adjacent corners of the rectangle, the taxi takes the straight route, otherwise (when it travels from W to Y or X to Z or vice versa) it does not matter. Distances within a given village are negligible. Let D be the distance travelled to pick up and deliver a single customer. Find the probabilitites that D takes each of its possible values. Find the expected value ED and the variance Var D.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 5 June 2001 1.30 to 4.30

PAPER 3

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SECTION I

1F Algebra and Geometry

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 = 0$ if and only if $a = -d$ and $bc = -a^2$. Prove that $A^3 = 0$ if and only if $A^2 = 0$.

[*Hint: it is easy to check that* $A^2 - (a+d)A + (ad - bc)I = 0$.]

2D Algebra and Geometry

Show that the set of Möbius transformations of the extended complex plane $\mathbb{C} \cup \{\infty\}$ form a group. Show further that an arbitrary Möbius transformation can be expressed as the composition of maps of the form

$$f(z) = z + a, \quad g(z) = kz \text{ and } h(z) = 1/z \; .$$

3C Vector Calculus

For a real function $f(x,y)$ with $x = x(t)$ and $y = y(t)$ state the chain rule for the derivative $\frac{d}{dt} f(x(t), y(t))$.

By changing variables to u and v, where $u = \alpha(x)y$ and $v = y/x$ with a suitable function $\alpha(x)$ to be determined, find the general solution of the equation

$$x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y} = 6f \; .$$

4A Vector Calculus

Suppose that

$$u = y^2 \sin(xz) + xy^2 z\cos(xz), \quad v = 2xy\sin(xz), \quad w = x^2y^2\cos(xz).$$

Show that $u\,dx + v\,dy + w\,dz$ is an exact differential.

Show that

$$\int_{(0,0,0)}^{(\pi/2,1,1)} u\,dx + v\,dy + w\,dz = \frac{\pi}{2}.$$

SECTION II

5F Algebra and Geometry

Let A, B, C be 2×2 matrices, real or complex. Define the trace $\operatorname{tr} C$ to be the sum of diagonal entries $C_{11} + C_{22}$. Define the commutator $[A, B]$ to be the difference $AB - BA$. Give the definition of the eigenvalues of a 2×2 matrix and prove that it can have at most two distinct eigenvalues. Prove that

a) $\operatorname{tr}[A, B] = 0$,

b) $\operatorname{tr} C$ equals the sum of the eigenvalues of C,

c) if all eigenvalues of C are equal to 0 then $C^2 = 0$,

d) either $[A, B]$ is a diagonalisable matrix or the square $[A, B]^2 = 0$,

e) $[A, B]^2 = \alpha I$ where $\alpha \in \mathbb{C}$ and I is the unit matrix.

6E Algebra and Geometry

Define the notion of an *action* of a group G on a set X. Define *orbit* and *stabilizer*, and then, assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Show that the group of rotations of a cube has order 24.

7E Algebra and Geometry

State Lagrange's theorem. Use it to describe all groups of order p, where p is a fixed prime number.

Find all the subgroups of a fixed cyclic group $\langle x \rangle$ of order n.

8D Algebra and Geometry

(i) Let A_4 denote the alternating group of even permutations of four symbols. Let X be the 3-cycle (123) and P, Q be the pairs of transpositions (12)(34) and (13)(24). Find $X^3, P^2, Q^2, X^{-1}PX, X^{-1}QX$, and show that A_4 is generated by X, P and Q.

(ii) Let G and H be groups and let

$$G \times H = \{(g, h) : g \in G, h \in H\}.$$

Show how to make $G \times H$ into a group in such a way that $G \times H$ contains subgroups isomorphic to G and H.

If D_n is the dihedral group of order n and C_2 is the cyclic group of order 2, show that D_{12} is isomorphic to $D_6 \times C_2$. Is the group D_{12} isomorphic to A_4?

9C Vector Calculus

Explain, with justification, how the nature of a critical (stationary) point of a function $f(\mathbf{x})$ can be determined by consideration of the eigenvalues of the Hessian matrix H of $f(\mathbf{x})$ if H is non-singular. What happens if H is singular?

Let $f(x,y) = (y - x^2)(y - 2x^2) + \alpha x^2$. Find the critical points of f and determine their nature in the different cases that arise according to the values of the parameter $\alpha \in \mathbb{R}$.

10A Vector Calculus

State the rule for changing variables in a double integral.

Let D be the region defined by

$$\begin{cases} 1/x \le y \le 4x & \text{when } \frac{1}{2} \le x \le 1, \\ x \le y \le 4/x & \text{when } 1 \le x \le 2. \end{cases}$$

Using the transformation $u = y/x$ and $v = xy$, show that

$$\int_D \frac{4xy^3}{x^2 + y^2}\, dxdy \;=\; \frac{15}{2} \ln \frac{17}{2}.$$

11B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{r})$ in a closed region V bounded by a smooth surface S.

Let $\Omega(\mathbf{r})$ be a scalar field. By choosing $\mathbf{u} = \mathbf{c}\,\Omega$ for arbitrary constant vector $\mathbf{c}$, show that

$$\int_V \nabla\Omega \, dv \;=\; \int_S \Omega \, d\mathbf{S} \, . \qquad (*)$$

Let V be the bounded region enclosed by the surface S which consists of the cone $(x,y,z) = (r\cos\theta, r\sin\theta, r/\sqrt{3})$ with $0 \le r \le \sqrt{3}$ and the plane $z = 1$, where r, θ, z are cylindrical polar coordinates. Verify that $(*)$ holds for the scalar field $\Omega = (a - z)$ where a is a constant.

12B Vector Calculus

In $\mathbb{R}^3$ show that, within a closed surface S, there is at most one solution of Poisson's equation, $\nabla^2\phi = \rho$, satisfying the boundary condition on S

$$\alpha \frac{\partial \phi}{\partial n} + \phi = \gamma \ ,$$

where α and γ are functions of position on S, and α is everywhere non-negative.

Show that

$$\phi(x,y) = e^{\pm lx} \sin ly$$

are solutions of Laplace's equation $\nabla^2\phi = 0$ on $\mathbb{R}^2$.

Find a solution $\phi(x,y)$ of Laplace's equation in the region $0 < x < \pi$, $0 < y < \pi$ that satisfies the boundary conditions

$$\begin{array}{cccc} \phi = 0 & \text{on} & 0 < x < \pi & y = 0 \\ \phi = 0 & \text{on} & 0 < x < \pi & y = \pi \\ \phi + \partial\phi/\partial n = 0 & \text{on} & x = 0 & 0 < y < \pi \\ \phi = \sin(ky) & \text{on} & x = \pi & 0 < y < \pi \end{array}$$

where k is a positive integer. Is your solution the only possible solution?

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 4 June 2001 9.00 to 12.00

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I and at most* ***five*** *questions from Section II. In Section II no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in two bundles, marked ***A*** *and* ***E*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet must bear your examination number and desk number.

SECTION I

1E Numbers and Sets

(a) Show that, given a set X, there is no bijection between X and its power set.

(b) Does there exist a set whose members are precisely those sets that are not members of themselves? Justify your answer.

2E Numbers and Sets

Prove, by induction or otherwise, that

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+m}{m} = \binom{n+m+1}{m}.$$

Find the number of sequences consisting of zeroes and ones that contain exactly n zeroes and at most m ones.

3A Dynamics

Derive the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{f(u)}{mh^2u^2}$$

for the motion of a particle of mass m under an attractive central force f, where $u = 1/r$ and r is the distance of the particle from the centre of force, and where mh is the angular momentum of the particle about the centre of force.

[*Hint: you may assume the expressions for the radial and transverse accelerations in the form* $\ddot{r} - r\dot{\theta}^2, 2\dot{r}\dot{\theta} + r\ddot{\theta}$.]

4A Dynamics

Two particles of masses m_1 and m_2 at positions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are subject to forces $\mathbf{F}_1 = -\mathbf{F}_2 = \mathbf{f}(\mathbf{x}_1 - \mathbf{x}_2)$. Show that the centre of mass moves at a constant velocity. Obtain the equation of motion for the relative position of the particles. How does the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

of the system enter?

SECTION II

5E Numbers and Sets

(a) Prove Wilson's theorem, that $(p-1)! \equiv -1 \pmod{p}$, where p is prime.

(b) Suppose that p is an odd prime. Express $1^2.3^2.5^2.\ldots.(p-2)^2 \pmod{p}$ as a power of -1.

[*Hint:* $k \equiv -(p-k) \pmod{p}$.]

6E Numbers and Sets

State and prove the principle of inclusion-exclusion. Use it to calculate $\phi(4199)$, where ϕ is Euler's ϕ-function.

In a certain large college, a survey revealed that 90% of the fellows detest at least one of the pop stars Hairy, Dirty and Screamer. 45% detest Hairy, 28% detest Dirty and 46% detest Screamer. If 27% detest only Screamer and 6% detest all three, what proportion detest Hairy and Dirty but not Screamer?

7E Numbers and Sets

(a) Prove that, if p is prime and a is not a multiple of p, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) The *order* of $a \pmod{p}$ is the least positive integer d such that $a^d \equiv 1 \pmod{p}$. Suppose now that $a^x \equiv 1 \pmod{p}$; what can you say about x in terms of d? Show that $p \equiv 1 \pmod{d}$.

(c) Suppose that p is an odd prime. What is the order of $x \pmod{p}$ if $x^2 \equiv -1 \pmod{p}$? Find a condition on $p \pmod 4$ that is equivalent to the existence of an integer x with $x^2 \equiv -1 \pmod{p}$.

8E Numbers and Sets

What is the Principle of Mathematical Induction? Derive it from the statement that every non-empty set of positive integers has a least element.

Prove, by induction on n, that $9^n \equiv 2^n \pmod 7$ for all $n \geq 1$.

What is wrong with the following argument?

"Theorem: $\sum_{i=1}^{n} i = n(n+1)/2 + 126$.

Proof: Assume that $m \geq 1$ and $\sum_{i=1}^{m} i = m(m+1)/2 + 126$. Add $m+1$ to both sides to get

$$\sum_{i=1}^{m+1} i \;=\; m(m+1)/2 + m + 1 + 126 \;=\; (m+1)(m+2)/2 + 126.$$

So, by induction, the theorem is proved."

9A Dynamics

The position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. Write down the equation of motion of the particle under a force $\mathbf{F} = -4m\omega^2\mathbf{x}$.

Find the motion of the particle in (x, y, z) coordinates with initial condition

$$\mathbf{x} = (1, 0, 0) \quad \text{and} \quad \dot{\mathbf{x}} = (0, 0, 0) \qquad \text{at } t = 0,$$

where $\boldsymbol{\omega} = (0, 0, \omega)$. Show that the particle has a maximum speed at $t = (2n+1)\pi/4\omega$, and find this speed.

[*Hint: you may find it useful to consider the combination* $\zeta = x + iy$.]

10A Dynamics

A spherical raindrop of radius $a(t) > 0$ and density ρ falls down at a velocity $v(t) > 0$ through a fine stationary mist. As the raindrop falls its volume grows at the rate $c\pi a^2 v$ with constant c. The raindrop is subject to the gravitational force and a resistive force $-k\rho\pi a^2 v^2$ with k a positive constant. Show a and v satisfy

$$\begin{aligned} \dot{a} &= \tfrac{1}{4}cv, \\ \dot{v} &= g - \tfrac{3}{4}(c+k)\frac{v^2}{a}. \end{aligned}$$

Find an expression for $\frac{d}{dt}(v^2/a)$, and deduce that as time increases v^2/a tends to the constant value $g/(\frac{7}{8}c + \frac{3}{4}k)$, and thence the raindrop tends to a constant acceleration which is less than $\frac{1}{7}g$.

11A Dynamics

A spacecraft of mass m moves under the gravitational influence of the Sun of mass M and with universal gravitation constant G. After a disastrous manoeuvre, the unfortunate spacecraft finds itself exactly in a parabolic orbit about the Sun: the orbit with zero total energy. Using the conservation of energy and angular momentum, or otherwise, show that in the subsequent motion the distance of the spacecraft from the Sun $r(t)$ satisfies

$$(r - r_0)(r + 2r_0)^2 = \tfrac{9}{2}GM(t - t_0)^2,$$

with constants r_0 and t_0.

12A Dynamics

Find the moment of inertia of a uniform solid cylinder of radius a, length l and total mass M about its axis.

The cylinder is released from rest at the top of an inclined plane of length L and inclination θ to the horizontal. The first time the plane is perfectly smooth and the cylinder slips down the plane without rotating. The experiment is then repeated after the plane has been roughened, so that the cylinder now rolls without slipping at the point of contact. Show that the time taken to roll down the roughened plane is $\sqrt{\frac{3}{2}}$ times the time taken to slip down the smooth plane.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

List of Courses

1/I/1C **Algebra and Geometry**

Show, using the summation convention or otherwise, that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$, for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

The function $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\Pi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n})$ where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$. Show that Π is linear and find the elements of a matrix P such that $\Pi(\mathbf{x}) = P\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

Find all solutions to the equation $\Pi(\mathbf{x}) = \mathbf{x}$. Evaluate $\Pi(\mathbf{n})$. Describe the function Π geometrically. Justify your answer.

1/I/2C **Algebra and Geometry**

Define what is meant by the statement that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ are linearly independent. Determine whether the following vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ are linearly independent and justify your answer.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

For the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ taken from a real vector space V consider the statements

A) $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly dependent,
B) $\exists\, \alpha, \beta, \gamma \in \mathbb{R} : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
C) $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$, not all $= 0 : \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = \mathbf{0}$,
D) $\exists\, \alpha, \beta \in \mathbb{R}$, not both $= 0 : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
E) $\exists\, \alpha, \beta \in \mathbb{R} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$,
F) $\nexists$ basis of V that contains all 3 vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

State if the following implications are true or false (no justification is required):

i) A $\Rightarrow$ B,	vi) B $\Rightarrow$ A,
ii) A $\Rightarrow$ C,	vii) C $\Rightarrow$ A,
iii) A $\Rightarrow$ D,	viii) D $\Rightarrow$ A,
iv) A $\Rightarrow$ E,	ix) E $\Rightarrow$ A,
v) A $\Rightarrow$ F,	x) F $\Rightarrow$ A.

1/II/5C **Algebra and Geometry**

The matrix

$$A_\alpha = \begin{pmatrix} 1 & -1 & 2\alpha+1 \\ 1 & \alpha-1 & 1 \\ 1+\alpha & -1 & \alpha^2+4\alpha+1 \end{pmatrix}$$

defines a linear map $\Phi_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_\alpha(\mathbf{x}) = A_\alpha \mathbf{x}$. Find a basis for the kernel of Φ_α for all values of $\alpha \in \mathbb{R}$.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be bases of $\mathbb{R}^3$. Show that there exists a matrix S, to be determined in terms of $\mathcal{B}$ and $\mathcal{C}$, such that, for every linear mapping Φ, if Φ has matrix A with respect to $\mathcal{B}$ and matrix A' with respect to $\mathcal{C}$, then $A' = S^{-1}AS$.

For the bases

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad , \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\} \quad ,$$

find the basis transformation matrix S and calculate $S^{-1}A_0S$.

1/II/6C **Algebra and Geometry**

Assume that $\mathbf{x}_p$ is a particular solution to the equation $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x}, \mathbf{b} \in \mathbb{R}^3$ and a real 3×3 matrix A. Explain why the general solution to $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x} = \mathbf{x}_p + \mathbf{h}$ where $\mathbf{h}$ is any vector such that $A\mathbf{h} = \mathbf{0}$.

Now assume that A is a real symmetric 3×3 matrix with three different eigenvalues λ_1, λ_2 and λ_3. Show that eigenvectors of A with respect to different eigenvalues are orthogonal. Let $\mathbf{x}_k$ be a normalised eigenvector of A with respect to the eigenvalue λ_k, $k = 1, 2, 3$. Show that the linear system

$$(A - \lambda_k I)\mathbf{x} \;=\; \mathbf{b} \;\; ,$$

where I denotes the 3×3 unit matrix, is solvable if and only if $\mathbf{x}_k.\mathbf{b} = 0$. Show that the general solution is given by

$$\mathbf{x} \;=\; \sum_{i \neq k} \tfrac{\mathbf{b}.\mathbf{x}_i}{\lambda_i - \lambda_k} \mathbf{x}_i + \beta \mathbf{x}_k, \quad \beta \in \mathbb{R}.$$

[*Hint: consider the components of* $\mathbf{x}$ *and* $\mathbf{b}$ *with respect to a basis of eigenvectors of* A.]

Consider the matrix A and the vector $\mathbf{b}$

$$A = \begin{pmatrix} -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{2}\sqrt{2} + \frac{1}{6}\sqrt{3} & -\frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{2}{3}\sqrt{3} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \sqrt{2} + \sqrt{3} \\ -\sqrt{2} + \sqrt{3} \\ -2\sqrt{3} \end{pmatrix}.$$

Verify that $\frac{1}{\sqrt{3}}(1, 1, 1)^T$ and $\frac{1}{\sqrt{2}}(1, -1, 0)^T$ are eigenvectors of A. Show that $A\mathbf{x} = \mathbf{b}$ is solvable and find its general solution.

1/II/7C Algebra and Geometry

For $\alpha, \gamma \in \mathbb{R}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$ and $\beta\overline{\beta} \geqslant \alpha\gamma$ the equation $\alpha z\overline{z} - \beta\overline{z} - \overline{\beta}z + \gamma = 0$ describes a circle $C_{\alpha\beta\gamma}$ in the complex plane. Find its centre and radius. What does the equation describe if $\beta\overline{\beta} < \alpha\gamma$? Sketch the circles $C_{\alpha\beta\gamma}$ for $\beta = \gamma = 1$ and $\alpha = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1$.

Show that the complex function $f(z) = \beta\overline{z}/\overline{\beta}$ for $\beta \neq 0$ satisfies $f(C_{\alpha\beta\gamma}) = C_{\alpha\beta\gamma}$.

[*Hint: $f(C) = C$ means that $f(z) \in C\ \forall z \in C$ and $\forall w \in C\ \ \exists z \in C$ such that $f(z) = w$.*]

For two circles C_1 and C_2 a function $m(C_1, C_2)$ is defined by

$$m(C_1, C_2) \;=\; \max_{z \in C_1, w \in C_2} \left| z - w \right| \;.$$

Prove that $m(C_1, C_2) \leqslant m(C_1, C_3) + m(C_2, C_3)$. Show that

$$m(C_{\alpha_1\beta_1\gamma_1}, C_{\alpha_2\beta_2\gamma_2}) \;=\; \tfrac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{|\alpha_1\alpha_2|} + \tfrac{\sqrt{\beta_1\overline{\beta_1} - \alpha_1\gamma_1}}{|\alpha_1|} + \tfrac{\sqrt{\beta_2\overline{\beta_2} - \alpha_2\gamma_2}}{|\alpha_2|} \;.$$

1/II/8C Algebra and Geometry

Let $l_{\mathbf{x}}$ denote the straight line through $\mathbf{x}$ with directional vector $\mathbf{u} \neq \mathbf{0}$

$$l_{\mathbf{x}} \;=\; \left\{ \mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = \mathbf{x} + \lambda\mathbf{u}, \lambda \in \mathbb{R} \right\} \;.$$

Show that $l_{\mathbf{0}}$ is a subspace of $\mathbb{R}^3$ and show that $l_{\mathbf{x}_1} = l_{\mathbf{x}_2} \Leftrightarrow \mathbf{x_1} = \mathbf{x}_2 + \lambda\mathbf{u}$ for some $\lambda \in \mathbb{R}$.

For fixed $\mathbf{u} \neq \mathbf{0}$ let $\mathcal{L}$ be the set of all the parallel straight lines $l_{\mathbf{x}}$ ($\mathbf{x} \in \mathbb{R}^3$) with directional vector $\mathbf{u}$. On $\mathcal{L}$ an addition and a scalar multiplication are defined by

$$l_{\mathbf{x}} + l_{\mathbf{y}} = l_{\mathbf{x}+\mathbf{y}},\ \alpha l_{\mathbf{x}} = l_{\alpha\mathbf{x}},\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^3,\ \alpha \in \mathbb{R} \;.$$

Explain why these operations are well-defined. Show that the addition is associative and that there exists a zero vector which should be identified.

You may now assume that $\mathcal{L}$ is a vector space. If $\{\mathbf{u}, \mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^3$ show that $\{l_{\mathbf{b}_1}, l_{\mathbf{b}_2}\}$ is a basis for $\mathcal{L}$.

For $\mathbf{u} = (1, 3, -1)^T$ a linear map $\Phi : \mathcal{L} \to \mathcal{L}$ is defined by

$$\Phi\left(l_{(1,-1,0)^T}\right) = l_{(2,4,-1)^T} \;,\; \Phi\left(l_{(1,1,0)^T}\right) = l_{(-4,-2,1)^T} \;.$$

Find the matrix A of Φ with respect to the basis $\{l_{(1,0,0)^T}, l_{(0,1,0)^T}\}$.

3/I/1F Algebra and Geometry

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 = 0$ if and only if $a = -d$ and $bc = -a^2$. Prove that $A^3 = 0$ if and only if $A^2 = 0$.

[*Hint: it is easy to check that* $A^2 - (a+d)A + (ad - bc)I = 0$.]

3/I/2D Algebra and Geometry

Show that the set of Möbius transformations of the extended complex plane $\mathbb{C} \cup \{\infty\}$ form a group. Show further that an arbitrary Möbius transformation can be expressed as the composition of maps of the form

$$f(z) = z + a, \quad g(z) = kz \text{ and } h(z) = 1/z \ .$$

3/II/5F Algebra and Geometry

Let A, B, C be 2×2 matrices, real or complex. Define the trace $\operatorname{tr} C$ to be the sum of diagonal entries $C_{11} + C_{22}$. Define the commutator $[A, B]$ to be the difference $AB - BA$. Give the definition of the eigenvalues of a 2×2 matrix and prove that it can have at most two distinct eigenvalues. Prove that

a) $\operatorname{tr}[A, B] = 0$,

b) $\operatorname{tr} C$ equals the sum of the eigenvalues of C,

c) if all eigenvalues of C are equal to 0 then $C^2 = 0$,

d) either $[A, B]$ is a diagonalisable matrix or the square $[A, B]^2 = 0$,

e) $[A, B]^2 = \alpha I$ where $\alpha \in \mathbb{C}$ and I is the unit matrix.

3/II/6E Algebra and Geometry

Define the notion of an *action* of a group G on a set X. Define *orbit* and *stabilizer*, and then, assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Show that the group of rotations of a cube has order 24.

3/II/7E Algebra and Geometry

State Lagrange's theorem. Use it to describe all groups of order p, where p is a fixed prime number.

Find all the subgroups of a fixed cyclic group $\langle x \rangle$ of order n.

3/II/8D **Algebra and Geometry**

(i) Let A_4 denote the alternating group of even permutations of four symbols. Let X be the 3-cycle (123) and P, Q be the pairs of transpositions $(12)(34)$ and $(13)(24)$. Find $X^3, P^2, Q^2, X^{-1}PX, X^{-1}QX$, and show that A_4 is generated by X, P and Q.

(ii) Let G and H be groups and let

$$G \times H = \{(g, h) : g \in G, h \in H\}.$$

Show how to make $G \times H$ into a group in such a way that $G \times H$ contains subgroups isomorphic to G and H.

If D_n is the dihedral group of order n and C_2 is the cyclic group of order 2, show that D_{12} is isomorphic to $D_6 \times C_2$. Is the group D_{12} isomorphic to A_4?

1/I/3D **Analysis I**

What does it mean to say that $u_n \to l$ as $n \to \infty$?

Show that, if $u_n \to l$ and $v_n \to k$, then $u_n v_n \to lk$ as $n \to \infty$.

If further $u_n \neq 0$ for all n and $l \neq 0$, show that $1/u_n \to 1/l$ as $n \to \infty$.

Give an example to show that the non-vanishing of u_n for all n need not imply the non-vanishing of l.

1/I/4D **Analysis I**

Starting from the theorem that any continuous function on a closed and bounded interval attains a maximum value, prove Rolle's Theorem. Deduce the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(t) > 0$ for all t show that f is a strictly increasing function.

Conversely, if f is strictly increasing, is $f'(t) > 0$ for all t?

1/II/9D **Analysis I**

(i) If $a_0, a_1, \ldots$ are complex numbers show that if, for some $w \in \mathbb{C}, w \neq 0$, the set $\{|a_n w^n| : n \geq 0\}$ is bounded and $|z| < |w|$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Use this result to define the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$.

(ii) If $|a_n|^{1/n} \to R$ as $n \to \infty$ $(0 < R < \infty)$ show that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence equal to $1/R$.

(iii) Give examples of power series with radii of convergence 1 such that (a) the series converges at all points of the circle of convergence, (b) diverges at all points of the circle of convergence, and (c) neither of these occurs.

1/II/10D **Analysis I**

Suppose that f is a continuous real-valued function on $[a,b]$ with $f(a) < f(b)$. If $f(a) < v < f(b)$ show that there exists c with $a < c < b$ and $f(c) = v$.

Deduce that if f is a continuous function from the closed bounded interval $[a,b]$ to itself, there exists at least one fixed point, i.e., a number d belonging to $[a,b]$ with $f(d) = d$. Does this fixed point property remain true if f is a continuous function defined (i) on the open interval (a,b) and (ii) on $\mathbb{R}$? Justify your answers.

1/II/11D Analysis I

(i) Show that if $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable then, given $\epsilon > 0$, we can find some constant L and $\delta(\epsilon) > 0$ such that

$$|g(t) - g(\alpha) - g'(\alpha)\,(t - \alpha)| \;\leq\; L|t - \alpha|^2$$

for all $|t - \alpha| < \delta(\epsilon)$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable on $[a, b]$ (with one-sided derivatives at the end points), let f' and f'' be strictly positive functions and let $f(a) < 0 < f(b)$.

If $F(t) = t - (f(t)/f'(t))$ and a sequence $\{x_n\}$ is defined by $b = x_0, x_n = F(x_{n-1})$ $(n > 0)$, show that $x_0, x_1, x_2, \ldots$ is a decreasing sequence of points in $[a, b]$ and hence has limit α. What is $f(\alpha)$? Using part (i) or otherwise estimate the rate of convergence of x_n to α, i.e., the behaviour of the absolute value of $(x_n - \alpha)$ for large values of n.

1/II/12D Analysis I

Explain what it means for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$, and give an example of a bounded function that is not Riemann integrable.

Show each of the following statements is true for continuous functions f, but false for general Riemann integrable functions f.

(i) If $f : [a, b] \to \mathbb{R}$ is such that $f(t) \geq 0$ for all t in $[a, b]$ and $\int_a^b f(t)\,dt = 0$, then $f(t) = 0$ for all t in $[a, b]$.

(ii) $\int_a^t f(x)\,dx$ is differentiable and $\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$.

2/I/1B **Differential Equations**

Find the solution to

$$\frac{dy(x)}{dx} + \tanh(x)\, y(x) \;=\; H(x)\ ,$$

in the range $-\infty < x < \infty$ subject to $y(0) = 1$, where $H(x)$ is the Heavyside function defined by

$$H(x) \;=\; \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \ .$$

Sketch the solution.

2/I/2B **Differential Equations**

The function $y(x)$ satisfies the inhomogeneous second-order linear differential equation

$$y'' - y' - 2y \;=\; 18xe^{-x}\ .$$

Find the solution that satisfies the conditions that $y(0) = 1$ and $y(x)$ is bounded as $x \to \infty$.

2/II/5B **Differential Equations**

The real sequence y_k, $k = 1, 2, \ldots$ satisfies the difference equation

$$y_{k+2} - y_{k+1} + y_k = 0 .$$

Show that the general solution can be written

$$y_k = a \cos \frac{\pi k}{3} + b \sin \frac{\pi k}{3} ,$$

where a and b are arbitrary real constants.

Now let y_k satisfy

$$y_{k+2} - y_{k+1} + y_k = \frac{1}{k+2} . \qquad (*)$$

Show that a particular solution of $(*)$ can be written in the form

$$y_k = \sum_{n=1}^{k} \frac{a_n}{k-n+1} ,$$

where

$$a_{n+2} - a_{n+1} + a_n = 0 , \qquad n \geq 1 ,$$

and $a_1 = 1,\ a_2 = 1$.

Hence, find the general solution to $(*)$.

2/II/6B **Differential Equations**

The function $y(x)$ satisfies the linear equation

$$y''(x) + p(x)y'(x) + q(x)y(x) \;=\; 0\ .$$

The Wronskian, $W(x)$, of two independent solutions denoted $y_1(x)$ and $y_2(x)$ is defined to be

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}\ .$$

Let $y_1(x)$ be given. In this case, show that the expression for $W(x)$ can be interpreted as a first-order inhomogeneous differential equation for $y_2(x)$. Hence, by explicit derivation, show that $y_2(x)$ may be expressed as

$$y_2(x) \;=\; y_1(x) \int_{x_0}^{x} \frac{W(t)}{y_1(t)^2}\, dt\ , \qquad (*)$$

where the rôle of x_0 should be briefly elucidated.

Show that $W(x)$ satisfies

$$\frac{dW(x)}{dx} + p(x)W(x) \;=\; 0\ .$$

Verify that $y_1(x) = 1 - x$ is a solution of

$$xy''(x) - (1 - x^2)y'(x) - (1 + x)y(x) \;=\; 0\ . \qquad (\dagger)$$

Hence, using $(*)$ with $x_0 = 0$ and expanding the integrand in powers of t to order t^3, find the first three non-zero terms in the power series expansion for a solution, $y_2(x)$, of $(\dagger)$ that is independent of $y_1(x)$ and satisfies $y_2(0) = 0,\ y_2{}''(0) = 1$.

2/II/7B **Differential Equations**

Consider the linear system

$$\dot{\mathbf{z}} + A\mathbf{z} \;=\; \mathbf{h}\,, \qquad (*)$$

where

$$\mathbf{z}(t) \;=\; \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\,, \quad A \;=\; \begin{pmatrix} 1+a & -2 \\ 1 & -1+a \end{pmatrix}\,, \quad \mathbf{h}(t) \;=\; \begin{pmatrix} 2\cos t \\ \cos t - \sin t \end{pmatrix}\,,$$

where $\mathbf{z}(t)$ is real and a is a real constant, $a \geq 0$.

Find a (complex) eigenvector, $\mathbf{e}$, of A and its corresponding (complex) eigenvalue, l. Show that the second eigenvector and corresponding eigenvalue are respectively $\bar{\mathbf{e}}$ and $\bar{l}$, where the bar over the symbols signifies complex conjugation. Hence explain how the general solution to $(*)$ can be written as

$$\mathbf{z}(t) \;=\; \alpha(t)\,\mathbf{e} \;+\; \bar{\alpha}(t)\,\bar{\mathbf{e}}\,,$$

where $\alpha(t)$ is complex.

Write down a differential equation for $\alpha(t)$ and hence, for $a > 0$, deduce the solution to $(*)$ which satisfies the initial condition $\mathbf{z}(0) = \underset{\sim}{0}$.

Is the linear system resonant?

By taking the limit $a \to 0$ of the solution already found deduce the solution satisfying $\mathbf{z}(0) = \underset{\sim}{0}$ when $a = 0$.

2/II/8B **Differential Equations**

Carnivorous hunters of population h prey on vegetarians of population p. In the absence of hunters the prey will increase in number until their population is limited by the availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$\begin{aligned} \dot{p} \;&=\; p\left(1 - \frac{p}{a}\right) - \frac{ph}{a}\,, \\ \dot{h} \;&=\; \frac{h}{8}\left(\frac{p}{b} - 1\right)\,, \end{aligned} \qquad (*)$$

where a and b are positive constants, and $h(t)$ and $p(t)$ are non-negative functions of time, t. By giving an interpretation of each term explain briefly how these equations model the system described.

Consider these equations for $a = 1$. In the two cases $0 < b < 1/2$ and $b > 1$ determine the location and the stability properties of the critical points of $(*)$. In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

4/I/3A **Dynamics**

Derive the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{f(u)}{mh^2u^2}$$

for the motion of a particle of mass m under an attractive central force f, where $u = 1/r$ and r is the distance of the particle from the centre of force, and where mh is the angular momentum of the particle about the centre of force.

[*Hint: you may assume the expressions for the radial and transverse accelerations in the form* $\ddot{r} - r\dot{\theta}^2, 2\dot{r}\dot{\theta} + r\ddot{\theta}$.]

4/I/4A **Dynamics**

Two particles of masses m_1 and m_2 at positions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are subject to forces $\mathbf{F}_1 = -\mathbf{F}_2 = \mathbf{f}(\mathbf{x}_1 - \mathbf{x}_2)$. Show that the centre of mass moves at a constant velocity. Obtain the equation of motion for the relative position of the particles. How does the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

of the system enter?

4/II/9A **Dynamics**

The position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. Write down the equation of motion of the particle under a force $\mathbf{F} = -4m\omega^2\mathbf{x}$.

Find the motion of the particle in (x, y, z) coordinates with initial condition

$$\mathbf{x} = (1, 0, 0) \quad \text{and} \quad \dot{\mathbf{x}} = (0, 0, 0) \qquad \text{at } t = 0,$$

where $\boldsymbol{\omega} = (0, 0, \omega)$. Show that the particle has a maximum speed at $t = (2n + 1)\pi/4\omega$, and find this speed.

[*Hint: you may find it useful to consider the combination* $\zeta = x + iy$.]

4/II/10A Dynamics

A spherical raindrop of radius $a(t) > 0$ and density ρ falls down at a velocity $v(t) > 0$ through a fine stationary mist. As the raindrop falls its volume grows at the rate $c\pi a^2 v$ with constant c. The raindrop is subject to the gravitational force and a resistive force $-k\rho\pi a^2 v^2$ with k a positive constant. Show a and v satisfy

$$\begin{aligned} \dot{a} &= \tfrac{1}{4}cv, \\ \dot{v} &= g - \tfrac{3}{4}(c+k)\frac{v^2}{a}. \end{aligned}$$

Find an expression for $\frac{d}{dt}(v^2/a)$, and deduce that as time increases v^2/a tends to the constant value $g/(\frac{7}{8}c + \frac{3}{4}k)$, and thence the raindrop tends to a constant acceleration which is less than $\frac{1}{7}g$.

4/II/11A Dynamics

A spacecraft of mass m moves under the gravitational influence of the Sun of mass M and with universal gravitation constant G. After a disastrous manoeuvre, the unfortunate spacecraft finds itself exactly in a parabolic orbit about the Sun: the orbit with zero total energy. Using the conservation of energy and angular momentum, or otherwise, show that in the subsequent motion the distance of the spacecraft from the Sun $r(t)$ satisfies

$$(r - r_0)(r + 2r_0)^2 = \tfrac{9}{2}GM(t - t_0)^2,$$

with constants r_0 and t_0.

4/II/12A Dynamics

Find the moment of inertia of a uniform solid cylinder of radius a, length l and total mass M about its axis.

The cylinder is released from rest at the top of an inclined plane of length L and inclination θ to the horizontal. The first time the plane is perfectly smooth and the cylinder slips down the plane without rotating. The experiment is then repeated after the plane has been roughened, so that the cylinder now rolls without slipping at the point of contact. Show that the time taken to roll down the roughened plane is $\sqrt{\frac{3}{2}}$ times the time taken to slip down the smooth plane.

4/I/1E Numbers and Sets

(a) Show that, given a set X, there is no bijection between X and its power set.

(b) Does there exist a set whose members are precisely those sets that are not members of themselves? Justify your answer.

4/I/2E Numbers and Sets

Prove, by induction or otherwise, that

$$\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+m}{m} = \binom{n+m+1}{m}.$$

Find the number of sequences consisting of zeroes and ones that contain exactly n zeroes and at most m ones.

4/II/5E Numbers and Sets

(a) Prove Wilson's theorem, that $(p-1)! \equiv -1 \pmod{p}$, where p is prime.

(b) Suppose that p is an odd prime. Express $1^2.3^2.5^2.\ldots.(p-2)^2 \pmod{p}$ as a power of -1.

[*Hint:* $k \equiv -(p-k) \pmod{p}$.]

4/II/6E Numbers and Sets

State and prove the principle of inclusion-exclusion. Use it to calculate $\phi(4199)$, where ϕ is Euler's ϕ-function.

In a certain large college, a survey revealed that 90% of the fellows detest at least one of the pop stars Hairy, Dirty and Screamer. 45% detest Hairy, 28% detest Dirty and 46% detest Screamer. If 27% detest only Screamer and 6% detest all three, what proportion detest Hairy and Dirty but not Screamer?

4/II/7E **Numbers and Sets**

(a) Prove that, if p is prime and a is not a multiple of p, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) The *order* of $a \pmod{p}$ is the least positive integer d such that $a^d \equiv 1 \pmod{p}$. Suppose now that $a^x \equiv 1 \pmod{p}$; what can you say about x in terms of d? Show that $p \equiv 1 \pmod{d}$.

(c) Suppose that p is an odd prime. What is the order of $x \pmod{p}$ if $x^2 \equiv -1 \pmod{p}$? Find a condition on $p \pmod{4}$ that is equivalent to the existence of an integer x with $x^2 \equiv -1 \pmod{p}$.

4/II/8E **Numbers and Sets**

What is the Principle of Mathematical Induction? Derive it from the statement that every non-empty set of positive integers has a least element.

Prove, by induction on n, that $9^n \equiv 2^n \pmod{7}$ for all $n \geq 1$.

What is wrong with the following argument?

"Theorem: $\sum_{i=1}^{n} i = n(n+1)/2 + 126$.

Proof: Assume that $m \geq 1$ and $\sum_{i=1}^{m} i = m(m+1)/2 + 126$. Add $m+1$ to both sides to get

$$\sum_{i=1}^{m+1} i \;=\; m(m+1)/2 + m + 1 + 126 \;=\; (m+1)(m+2)/2 + 126.$$

So, by induction, the theorem is proved."

2/I/3F **Probability**

The following problem is known as Bertrand's paradox. A chord has been chosen at random in a circle of radius r. Find the probability that it is longer than the side of the equilateral triangle inscribed in the circle. Consider three different cases:

a) the middle point of the chord is distributed uniformly inside the circle,

b) the two endpoints of the chord are independent and uniformly distributed over the circumference,

c) the distance between the middle point of the chord and the centre of the circle is uniformly distributed over the interval $[0, r]$.

[*Hint: drawing diagrams may help considerably.*]

2/I/4F **Probability**

The Ruritanian authorities decided to pardon and release one out of three remaining inmates, A, B and C, kept in strict isolation in the notorious Alkazaf prison. The inmates know this, but can't guess who among them is the lucky one; the waiting is agonising. A sympathetic, but corrupted, prison guard approaches A and offers to name, in exchange for a fee, another inmate (not A) who is doomed to stay. He says: "This reduces your chances to remain here from 2/3 to 1/2: will it make you feel better?" A hesitates but then accepts the offer; the guard names B.

Assume that indeed B will not be released. Determine the conditional probability

$$P\big(A \text{ remains } \bigm| B \text{ named}\big) = \frac{P(A\&B \text{ remain})}{P(B \text{ named})}$$

and thus check the guard's claim, in three cases:

a) when the guard is completely unbiased (i.e., names any of B and C with probability $1/2$ if the pair B, C is to remain jailed),

b) if he hates B and would certainly name him if B is to remain jailed,

c) if he hates C and would certainly name him if C is to remain jailed.

2/II/9F Probability

I play tennis with my parents; the chances for me to win a game against Mum (M) are p and against Dad (D) q, where $0 < q < p < 1$. We agreed to have three games, and their order can be DMD (where I play against Dad, then Mum then again Dad) or MDM. The results of games are independent.

Calculate under each of the two orders the probabilities of the following events:

a) that I win at least one game,

b) that I win at least two games,

c) that I win at least two games in succession (i.e., games 1 and 2 or 2 and 3, or 1, 2 and 3),

d) that I win exactly two games in succession (i.e., games 1 and 2 or 2 and 3, but not 1, 2 and 3),

e) that I win exactly two games (i.e., 1 and 2 or 2 and 3 or 1 and 3, but not 1, 2 and 3).

In each case a)– e) determine which order of games maximizes the probability of the event. In case e) assume in addition that $p + q > 3pq$.

2/II/10F Probability

A random point is distributed uniformly in a unit circle $\mathcal{D}$ so that the probability that it falls within a subset $\mathcal{A} \subseteq \mathcal{D}$ is proportional to the area of $\mathcal{A}$. Let R denote the distance between the point and the centre of the circle. Find the distribution function $F_R(x) = P(R < x)$, the expected value ER and the variance $\text{Var } R = ER^2 - (ER)^2$.

Let Θ be the angle formed by the radius through the random point and the horizontal line. Prove that R and Θ are independent random variables.

Consider a coordinate system where the origin is placed at the centre of $\mathcal{D}$. Let X and Y denote the horizontal and vertical coordinates of the random point. Find the covariance $\text{Cov}(X, Y) = E(XY) - EXEY$ and determine whether X and Y are independent.

Calculate the sum of expected values $E\frac{X}{R} + iE\frac{Y}{R}$. Show that it can be written as the expected value $Ee^{i\xi}$ and determine the random variable ξ.

2/II/11F **Probability**

Dipkomsky, a desperado in the wild West, is surrounded by an enemy gang and fighting tooth and nail for his survival. He has m guns, $m > 1$, pointing in different directions and tries to use them in succession to give an impression that there are several defenders. When he turns to a subsequent gun and discovers that the gun is loaded he fires it with probability 1/2 and moves to the next one. Otherwise, i.e. when the gun is unloaded, he loads it with probability 3/4 or simply moves to the next gun with complementary probability 1/4. If he decides to load the gun he then fires it or not with probability 1/2 and after that moves to the next gun anyway.

Initially, each gun had been loaded independently with probability p. Show that if after each move this distribution is preserved, then $p = 3/7$. Calculate the expected value EN and variance $\mathrm{Var}\ N$ of the number N of loaded guns under this distribution.

[*Hint: it may be helpful to represent* N *as a sum* $\sum_{1 \le j \le m} X_j$ *of random variables taking values* 0 *and* 1.]

2/II/12F **Probability**

A taxi travels between four villages, W, X, Y, Z, situated at the corners of a rectangle. The four roads connecting the villages follow the sides of the rectangle; the distance from W to X and Y to Z is 5 miles and from W to Z and Y to X 10 miles. After delivering a customer the taxi waits until the next call then goes to pick up the new customer and takes him to his destination. The calls may come from any of the villages with probability 1/4 and each customer goes to any other village with probability 1/3. Naturally, when travelling between a pair of adjacent corners of the rectangle, the taxi takes the straight route, otherwise (when it travels from W to Y or X to Z or vice versa) it does not matter. Distances within a given village are negligible. Let D be the distance travelled to pick up and deliver a single customer. Find the probabilitites that D takes each of its possible values. Find the expected value ED and the variance $\mathrm{Var}\ D$.

3/I/3C Vector Calculus

For a real function $f(x,y)$ with $x = x(t)$ and $y = y(t)$ state the chain rule for the derivative $\frac{d}{dt}f(x(t),y(t))$.

By changing variables to u and v, where $u = \alpha(x)y$ and $v = y/x$ with a suitable function $\alpha(x)$ to be determined, find the general solution of the equation

$$x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y} = 6f \ .$$

3/I/4A Vector Calculus

Suppose that

$$u = y^2 \sin(xz) + xy^2 z\cos(xz), \quad v = 2xy\sin(xz), \quad w = x^2y^2\cos(xz).$$

Show that $u\,dx + v\,dy + w\,dz$ is an exact differential.

Show that

$$\int_{(0,0,0)}^{(\pi/2,1,1)} u\,dx + v\,dy + w\,dz = \frac{\pi}{2}.$$

3/II/9C Vector Calculus

Explain, with justification, how the nature of a critical (stationary) point of a function $f(\mathbf{x})$ can be determined by consideration of the eigenvalues of the Hessian matrix H of $f(\mathbf{x})$ if H is non-singular. What happens if H is singular?

Let $f(x,y) = (y - x^2)(y - 2x^2) + \alpha x^2$. Find the critical points of f and determine their nature in the different cases that arise according to the values of the parameter $\alpha \in \mathbb{R}$.

3/II/10A Vector Calculus

State the rule for changing variables in a double integral.

Let D be the region defined by

$$\begin{cases} 1/x \le y \le 4x & \text{when } \frac{1}{2} \le x \le 1, \\ x \le y \le 4/x & \text{when } 1 \le x \le 2. \end{cases}$$

Using the transformation $u = y/x$ and $v = xy$, show that

$$\int_D \frac{4xy^3}{x^2+y^2}\, dx dy \;=\; \frac{15}{2} \ln \frac{17}{2}.$$

3/II/11B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{r})$ in a closed region V bounded by a smooth surface S.

Let $\Omega(\mathbf{r})$ be a scalar field. By choosing $\mathbf{u} = \mathbf{c}\,\Omega$ for arbitrary constant vector $\mathbf{c}$, show that

$$\int_V \nabla\,\Omega\, dv \;=\; \int_S \Omega\, d\mathbf{S}\,. \qquad (*)$$

Let V be the bounded region enclosed by the surface S which consists of the cone $(x, y, z) = (r\cos\theta, r\sin\theta, r/\sqrt{3})$ with $0 \le r \le \sqrt{3}$ and the plane $z = 1$, where r, θ, z are cylindrical polar coordinates. Verify that $(*)$ holds for the scalar field $\Omega = (a - z)$ where a is a constant.

3/II/12B Vector Calculus

In $\mathbb{R}^3$ show that, within a closed surface S, there is at most one solution of Poisson's equation, $\nabla^2\phi = \rho$, satisfying the boundary condition on S

$$\alpha\,\frac{\partial\phi}{\partial n} + \phi = \gamma\ ,$$

where α and γ are functions of position on S, and α is everywhere non-negative.

Show that

$$\phi(x, y) = e^{\pm lx}\, \sin ly$$

are solutions of Laplace's equation $\nabla^2\phi = 0$ on $\mathbb{R}^2$.

Find a solution $\phi(x, y)$ of Laplace's equation in the region $0 < x < \pi$, $0 < y < \pi$ that satisfies the boundary conditions

$$\begin{array}{cccc} \phi = 0 & \text{on} & 0 < x < \pi & y = 0 \\ \phi = 0 & \text{on} & 0 < x < \pi & y = \pi \\ \phi + \partial\phi/\partial n = 0 & \text{on} & x = 0 & 0 < y < \pi \\ \phi = \sin(ky) & \text{on} & x = \pi & 0 < y < \pi \end{array}$$

where k is a positive integer. Is your solution the only possible solution?

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 30 May 2002 9.00 to 12.00

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I. In Section II at most* ***five*** *answers will be taken into account and no more than* ***three*** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in three bundles, marked ***B****,* ***C*** *and* ***D*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet <u>*must*</u> ***bear your examination number and desk number.***

SECTION I

1B Algebra and Geometry

(a) State the Orbit-Stabilizer Theorem for a finite group G acting on a set X.

(b) Suppose that G is the group of rotational symmetries of a cube C. Two regular tetrahedra T and T' are inscribed in C, each using half the vertices of C. What is the order of the stabilizer in G of T?

2D Algebra and Geometry

State the Fundamental Theorem of Algebra. Define the characteristic equation for an arbitrary 3×3 matrix A whose entries are complex numbers. Explain why the matrix must have three eigenvalues, not necessarily distinct.

Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

and hence find the three eigenvalues of A. Find a set of linearly independent eigenvectors, specifying which eigenvector belongs to which eigenvalue.

3C Analysis I

Suppose $a_n \in \mathbb{R}$ for $n \geqslant 1$ and $a \in \mathbb{R}$. What does it mean to say that $a_n \to a$ as $n \to \infty$? What does it mean to say that $a_n \to \infty$ as $n \to \infty$?

Show that, if $a_n \neq 0$ for all n and $a_n \to \infty$ as $n \to \infty$, then $1/a_n \to 0$ as $n \to \infty$. Is the converse true? Give a proof or a counter example.

Show that, if $a_n \neq 0$ for all n and $a_n \to a$ with $a \neq 0$, then $1/a_n \to 1/a$ as $n \to \infty$.

4C Analysis I

Show that any bounded sequence of real numbers has a convergent subsequence.

Give an example of a sequence of real numbers with no convergent subsequence.

Give an example of an unbounded sequence of real numbers with a convergent subsequence.

SECTION II

5B Algebra and Geometry

(a) Find a subset T of the Euclidean plane $\mathbb{R}^2$ that is not fixed by any isometry (rigid motion) except the identity.

Let G be a subgroup of the group of isometries of $\mathbb{R}^2$, T a subset of $\mathbb{R}^2$ not fixed by any isometry except the identity, and let S denote the union $\bigcup_{g \in G} g(T)$. Does the group H of isometries of S contain G? Justify your answer.

(b) Find an example of such a G and T with $H \neq G$.

6B Algebra and Geometry

(a) Suppose that g is a Möbius transformation, acting on the extended complex plane. What are the possible numbers of fixed points that g can have? Justify your answer.

(b) Show that the operation c of complex conjugation, defined by $c(z) = \bar{z}$, is not a Möbius transformation.

7B Algebra and Geometry

(a) Find, with justification, the matrix, with respect to the standard basis of $\mathbb{R}^2$, of the rotation through an angle α about the origin.

(b) Find the matrix, with respect to the standard basis of $\mathbb{R}^3$, of the rotation through an angle α about the axis containing the point $(\frac{3}{5}, \frac{4}{5}, 0)$ and the origin. You may express your answer in the form of a product of matrices.

8D Algebra and Geometry

Define what is meant by a vector space V over the real numbers $\mathbb{R}$. Define subspace, proper subspace, spanning set, basis, and dimension.

Define the sum $U + W$ and intersection $U \cap W$ of two subspaces U and W of a vector space V. Why is the intersection never empty?

Let $V = \mathbb{R}^4$ and let $U = \{\mathbf{x} \in V : x_1 - x_2 + x_3 - x_4 = 0\}$, where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, and let $W = \{\mathbf{x} \in V : x_1 - x_2 - x_3 + x_4 = 0\}$. Show that $U \cap W$ has the orthogonal basis $\mathbf{b}_1$, $\mathbf{b}_2$ where $\mathbf{b}_1 = (1, 1, 0, 0)$ and $\mathbf{b}_2 = (0, 0, 1, 1)$. Extend this basis to find orthogonal bases of U, W, and $U + W$. Show that $U + W = V$ and hence verify that, in this case,

$$\dim U + \dim W \;=\; \dim(U + W) + \dim(U \cap W) \;.$$

9C Analysis I

State some version of the fundamental axiom of analysis. State the alternating series test and prove it from the fundamental axiom.

In each of the following cases state whether $\sum_{n=1}^{\infty} a_n$ converges or diverges and prove your result. You may use any test for convergence provided you state it correctly.

(i) $a_n = (-1)^n (\log(n+1))^{-1}$.

(ii) $a_{2n} = (2n)^{-2}$, $a_{2n-1} = -n^{-2}$.

(iii) $a_{3n-2} = -(2n-1)^{-1}$, $a_{3n-1} = (4n-1)^{-1}$, $a_{3n} = (4n)^{-1}$.

(iv) $a_{2^n+r} = (-1)^n (2^n + r)^{-1}$ for $0 \leqslant r \leqslant 2^n - 1$, $n \geqslant 0$.

10C Analysis I

Show that a continuous real-valued function on a closed bounded interval is bounded and attains its bounds.

Write down examples of the following functions (no proof is required).

(i) A continuous function $f_1 : (0,1) \to \mathbb{R}$ which is not bounded.

(ii) A continuous function $f_2 : (0,1) \to \mathbb{R}$ which is bounded but does not attain its bounds.

(iii) A bounded function $f_3 : [0,1] \to \mathbb{R}$ which is not continuous.

(iv) A function $f_4 : [0,1] \to \mathbb{R}$ which is not bounded on any interval $[a,b]$ with $0 \leqslant a < b \leqslant 1$.

[*Hint: Consider first how to define f_4 on the rationals.*]

11C Analysis I

State the mean value theorem and deduce it from Rolle's theorem.

Use the mean value theorem to show that, if $h : \mathbb{R} \to \mathbb{R}$ is differentiable with $h'(x) = 0$ for all x, then h is constant.

By considering the derivative of the function g given by $g(x) = e^{-ax}f(x)$, find all the solutions of the differential equation $f'(x) = af(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is differentiable and a is a fixed real number.

Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \int_0^x f(t)\,dt$$

is differentiable with $F'(x) = f(x)$.

Find the solution of the equation

$$g(x) = A + \int_0^x g(t)\,dt$$

where $g : \mathbb{R} \to \mathbb{R}$ is differentiable and A is a real number. You should explain why the solution is unique.

12C Analysis I

Prove Taylor's theorem with some form of remainder.

An infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation

$$f^{(3)}(x) = f(x)$$

and the conditions $f(0) = 1$, $f'(0) = f''(0) = 0$. If $R > 0$ and j is a positive integer, explain why we can find an M_j such that

$$|f^{(j)}(x)| \leqslant M_j$$

for all x with $|x| \leqslant R$. Explain why we can find an M such that

$$|f^{(j)}(x)| \leqslant M$$

for all x with $|x| \leqslant R$ and all $j \geqslant 0$.

Use your form of Taylor's theorem to show that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}\ .$$

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 31 May 2002 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I. In Section II at most* ***five*** *answers will be taken into account and no more than* ***three*** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in two bundles, marked ***D*** *and* ***F*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet *must* ***bear your examination number and desk number.***

SECTION I

1D Differential Equations

Solve the equation

$$\ddot{y} + \dot{y} - 2y \;=\; e^{-t}$$

subject to the conditions $y(t) = \dot{y}(t) = 0$ at $t = 0$. Solve the equation

$$\ddot{y} + \dot{y} - 2y \;=\; e^{t}$$

subject to the same conditions $y(t) = \dot{y}(t) = 0$ at $t = 0$.

2D Differential Equations

Consider the equation

$$\frac{dy}{dx} \;=\; x\left(\frac{1-y^2}{1-x^2}\right)^{1/2}, \qquad (*)$$

where the positive square root is taken, within the square $\mathcal{S}:\; 0 \leqslant x < 1,\; 0 \leqslant y \leqslant 1$. Find the solution that begins at $x = y = 0$. Sketch the corresponding solution curve, commenting on how its tangent behaves near each extremity. By inspection of the right-hand side of $(*)$, or otherwise, roughly sketch, using small line segments, the directions of flow throughout the square $\mathcal{S}$.

3F Probability

Define the *indicator function* I_A of an event A.

Let I_i be the indicator function of the event A_i, $1 \le i \le n$, and let $N = \sum_1^n I_i$ be the number of values of i such that A_i occurs. Show that $E(N) = \sum_i p_i$ where $p_i = P(A_i)$, and find $\mathrm{var}(N)$ in terms of the quantities $p_{ij} = P(A_i \cap A_j)$.

Using Chebyshev's inequality or otherwise, show that

$$P(N=0) \le \frac{\mathrm{var}(N)}{\{E(N)\}^2}.$$

4F Probability

A coin shows heads with probability p on each toss. Let π_n be the probability that the number of heads after n tosses is even. Show carefully that $\pi_{n+1} = (1-p)\pi_n + p(1-\pi_n)$, $n \ge 1$, and hence find π_n. [The number 0 is even.]

SECTION II

5D Differential Equations

Explain what is meant by an *integrating factor* for an equation of the form

$$\frac{dy}{dx} + f(x,y) \;=\; 0 \;.$$

Show that $2ye^x$ is an integrating factor for

$$\frac{dy}{dx} \;+\; \frac{2x+x^2+y^2}{2y} \;=\; 0 \;,$$

and find the solution $y = y(x)$ such that $y(0) = a$, for given $a > 0$.

Show that $2x + x^2 \geqslant -1$ for all x and hence that

$$\frac{dy}{dx} \;\leqslant\; \frac{1-y^2}{2y} \;.$$

For a solution with $a \geqslant 1$, show graphically, by considering the sign of dy/dx first for $x = 0$ and then for $x < 0$, that $dy/dx < 0$ for all $x \leqslant 0$.

Sketch the solution for the case $a = 1$, and show that property that $dy/dx \to -\infty$ both as $x \to -\infty$ and as $x \to b$ from below, where $b \approx 0.7035$ is the positive number that satisfies $b^2 = e^{-b}$.

[*Do not consider the range* $x \geqslant b$.]

6D Differential Equations

Solve the differential equation

$$\frac{dy}{dt} = ry(1-ay)$$

for the general initial condition $y = y_0$ at $t = 0$, where r, a, and y_0 are positive constants. Deduce that the equilibria at $y = a^{-1}$ and $y = 0$ are stable and unstable, respectively.

By using the approximate finite-difference formula

$$\frac{dy}{dt} = \frac{y_{n+1} - y_n}{\delta t}$$

for the derivative of y at $t = n\delta t$, where δt is a positive constant and $y_n = y(n\,\delta t)$, show that the differential equation when thus approximated becomes the difference equation

$$u_{n+1} = \lambda\,(1-u_n)\,u_n\ ,$$

where $\lambda = 1 + r\,\delta t > 1$ and where $u_n = \lambda^{-1}a(\lambda-1)\,y_n$. Find the two equilibria and, by linearizing the equation about them or otherwise, show that one is always unstable (given that $\lambda > 1$) and that the other is stable or unstable according as $\lambda < 3$ or $\lambda > 3$. Show that this last instability is oscillatory with period $2\,\delta t$. Why does this last instability have no counterpart for the differential equation? Show graphically how this instability can equilibrate to a periodic, finite-amplitude oscillation when $\lambda = 3.2$.

7D Differential Equations

The homogeneous equation

$$\ddot{y} + p(t)\dot{y} + q(t)y \;=\; 0$$

has non-constant, non-singular coefficients $p(t)$ and $q(t)$. Two solutions of the equation, $y(t) = y_1(t)$ and $y(t) = y_2(t)$, are given. The solutions are known to be such that the determinant

$$W(t) \;=\; \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}$$

is non-zero for all t. Define what is meant by linear dependence, and show that the two given solutions are linearly *independent*. Show also that

$$W(t) \;\propto\; \exp\left(-\int^t p(s)\,ds\right) \;.$$

In the corresponding inhomogeneous equation

$$\ddot{y} + p(t)\dot{y} + q(t)y \;=\; f(t)$$

the right-hand side $f(t)$ is a prescribed forcing function. Construct a particular integral of this inhomogeneous equation in the form

$$y(t) \;=\; a_1(t)\,y_1(t) + a_2(t)\,y_2(t) \;,$$

where the two functions $a_i(t)$ are to be determined such that

$$y_1(t)\,\dot{a}_1(t) + y_2(t)\,\dot{a}_2(t) \;=\; 0$$

for all t. Express your result for the functions $a_i(t)$ in terms of integrals of the functions $f(t)\,y_1(t)/W(t)$ and $f(t)\,y_2(t)/W(t)$.

Consider the case in which $p(t) = 0$ for all t and $q(t)$ is a positive constant, $q = \omega^2$ say, and in which the forcing $f(t) = \sin(\omega t)$. Show that in this case $y_1(t)$ and $y_2(t)$ can be taken as $\cos(\omega t)$ and $\sin(\omega t)$ respectively. Evaluate $f(t)\,y_1(t)/W(t)$ and $f(t)\,y_2(t)/W(t)$ and show that, as $t \to \infty$, one of the $a_i(t)$ increases in magnitude like a power of t to be determined.

8D Differential Equations

For any solution of the equations

$$\begin{aligned} \dot{x} &= \alpha x - y + y^3 \qquad (\alpha \text{ constant}) \\ \dot{y} &= -x \end{aligned}$$

show that

$$\frac{d}{dt}\left(x^2 - y^2 + \tfrac{1}{2}y^4\right) \; = \; 2\alpha x^2 \; .$$

What does this imply about the behaviour of phase-plane trajectories at large distances from the origin as $t \to \infty$, in the case $\alpha = 0$? Give brief reasoning but do not try to find explicit solutions.

Analyse the properties of the critical points and sketch the phase portrait (a) in the case $\alpha = 0$, (b) in the case $\alpha = 0.1$, and (c) in the case $\alpha = -0.1$.

9F Probability

(a) Define the *conditional probability* $P(A \mid B)$ of the event A given the event B. Let $\{B_i : 1 \le i \le n\}$ be a partition of the sample space Ω such that $P(B_i) > 0$ for all i. Show that, if $P(A) > 0$,

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_j P(A \mid B_j)P(B_j)}.$$

(b) There are n urns, the rth of which contains $r-1$ red balls and $n-r$ blue balls. You pick an urn (uniformly) at random and remove two balls without replacement. Find the probability that the first ball is blue, and the conditional probability that the second ball is blue given that the first is blue. [You may assume that $\sum_{i=1}^{n-1} i(i-1) = \frac{1}{3}n(n-1)(n-2)$.]

(c) What is meant by saying that two events A and B are independent?

(d) Two fair dice are rolled. Let A_s be the event that the sum of the numbers shown is s, and let B_i be the event that the first die shows i. For what values of s and i are the two events A_s, B_i independent?

10F Probability

There is a random number N of foreign objects in my soup, with mean μ and finite variance. Each object is a fly with probability p, and otherwise is a spider; different objects have independent types. Let F be the number of flies and S the number of spiders.

(a) Show that $G_F(s) = G_N(ps + 1 - p)$. [G_X denotes the probability generating function of a random variable X. You should present a clear statement of any general result used.]

(b) Suppose N has the Poisson distribution with parameter μ. Show that F has the Poisson distribution with parameter μp, and that F and S are independent.

(c) Let $p = \frac{1}{2}$ and suppose that F and S are independent. [You are given nothing about the distribution of N.] Show that $G_N(s) = G_N(\frac{1}{2}(1+s))^2$. By working with the function $H(s) = G_N(1-s)$ or otherwise, deduce that N has the Poisson distribution. [You may assume that $\left(1 + \frac{x}{n} + \mathrm{o}(n^{-1})\right)^n \to e^x$ as $n \to \infty$.]

11F Probability

Let X, Y, Z be independent random variables each with the uniform distribution on the interval $[0,1]$.

(a) Show that $X+Y$ has density function

$$f_{X+Y}(u) = \begin{cases} u & \text{if } 0 \le u \le 1, \\ 2-u & \text{if } 1 \le u \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that $P(Z > X+Y) = \frac{1}{6}$.

(c) You are provided with three rods of respective lengths X, Y, Z. Show that the probability that these rods may be used to form the sides of a triangle is $\frac{1}{2}$.

(d) Find the density function $f_{X+Y+Z}(s)$ of $X+Y+Z$ for $0 \leqslant s \leqslant 1$. Let W be uniformly distributed on $[0,1]$, and independent of X, Y, Z. Show that the probability that rods of lengths W, X, Y, Z may be used to form the sides of a quadrilateral is $\frac{5}{6}$.

12F Probability

(a) Explain what is meant by the term ‘branching process’.

(b) Let X_n be the size of the nth generation of a branching process in which each family size has probability generating function G, and assume that $X_0 = 1$. Show that the probability generating function G_n of X_n satisfies $G_{n+1}(s) = G_n(G(s))$ for $n \geq 1$.

(c) Show that $G(s) = 1 - \alpha(1-s)^{\beta}$ is the probability generating function of a non-negative integer-valued random variable when $\alpha, \beta \in (0,1)$, and find G_n explicitly when G is thus given.

(d) Find the probability that $X_n = 0$, and show that it converges as $n \to \infty$ to $1-\alpha^{1/(1-\beta)}$. Explain carefully why this implies that the probability of ultimate extinction equals $1 - \alpha^{1/(1-\beta)}$.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 4 June 2002 1.30 - 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I. In Section II at most* ***five*** *answers will be taken into account and no more than* ***three*** *answers on each course will be taken into account.*

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Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in three separate bundles, marked ***A***, ***B*** *and* ***E*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet <u>*must*</u> ***bear your examination number and desk number.***

SECTION I

1A Algebra and Geometry

Given two real non-zero 2×2 matrices A and B, with $AB = 0$, show that A maps $\mathbb{R}^2$ onto a line. Is it always true that $BA = 0$? Show that there is always a non-zero matrix C with $CA = 0 = AC$. Justify your answers.

2B Algebra and Geometry

(a) What does it mean for a group to be cyclic? Give an example of a finite abelian group that is not cyclic, and justify your assertion.

(b) Suppose that G is a finite group of rotations of $\mathbb{R}^2$ about the origin. Is G necessarily cyclic? Justify your answer.

3A Vector Calculus

Determine whether each of the following is the exact differential of a function, and if so, find such a function:

(a) $(\cosh\theta + \sinh\theta\cos\phi)d\theta + (\cosh\theta\sin\phi + \cos\phi)d\phi$,

(b) $3x^2(y^2+1)dx + 2(yx^3 - z^2)dy - 4yzdz$.

4A Vector Calculus

State the divergence theorem.

Consider the integral
$$I = \int_S r^n \mathbf{r} \cdot d\mathbf{S} ,$$
where $n > 0$ and S is the sphere of radius R centred at the origin. Evaluate I directly, and by means of the divergence theorem.

SECTION II

5E Algebra and Geometry

Prove, using the standard formula connecting δ_{ij} and ϵ_{ijk}, that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \, .$$

Define, in terms of the dot and cross product, the triple scalar product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ and show that it is invariant under cyclic permutation of the vectors.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a *not necessarily orthonormal* basis for $\mathbb{R}^3$, and define

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \, , \quad \hat{\mathbf{e}}_2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \, , \quad \hat{\mathbf{e}}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \, .$$

By calculating $[\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3]$, show that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ is also a basis for $\mathbb{R}^3$.

The vectors $\hat{\hat{\mathbf{e}}}_1, \hat{\hat{\mathbf{e}}}_2, \hat{\hat{\mathbf{e}}}_3$ are constructed from $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in the same way that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are constructed from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Show that

$$\hat{\hat{\mathbf{e}}}_1 = \mathbf{e}_1, \ \hat{\hat{\mathbf{e}}}_2 = \mathbf{e}_2, \ \hat{\hat{\mathbf{e}}}_3 = \mathbf{e}_3,$$

Show that a vector $\mathbf{V}$ has components $\mathbf{V} \cdot \hat{\mathbf{e}}_1, \ \mathbf{V} \cdot \hat{\mathbf{e}}_2, \ \mathbf{V} \cdot \hat{\mathbf{e}}_3$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. What are the components of the vector $\mathbf{V}$ with respect to the basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$?

6E Algebra and Geometry

(a) Give the general solution for $\mathbf{x}$ and $\mathbf{y}$ of the equations

$$\mathbf{x} + \mathbf{y} = 2\mathbf{a} \, , \qquad \mathbf{x} \cdot \mathbf{y} = c \qquad (c < \mathbf{a} \cdot \mathbf{a}) \, .$$

Show in particular that $\mathbf{x}$ and $\mathbf{y}$ must lie at opposite ends of a diameter of a sphere whose centre and radius should be specified.

(b) If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair are also perpendicular to each other. Show also that the sum of the lengths squared of two opposite edges is the same for each pair.

7A Algebra and Geometry

Explain why the number of solutions $\mathbf{x} \in \mathbb{R}^3$ of the simultaneous linear equations $A\mathbf{x} = \mathbf{b}$ is $0, 1$ or infinite, where A is a real 3×3 matrix and $\mathbf{b} \in \mathbb{R}^3$. Let α be the mapping which A represents. State necessary and sufficient conditions on $\mathbf{b}$ and α for each of these possibilities to hold.

Let A and B be 3×3 matrices representing linear mappings α and β. Give necessary and sufficient conditions on α and β for the existence of a 3×3 matrix X with $AX = B$. When is X unique?

Find X when

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

8B Algebra and Geometry

Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the vertices of a regular tetrahedron T in $\mathbb{R}^3$ and that $\mathbf{a} = (1,1,1)$, $\mathbf{b} = (-1,-1,1)$, $\mathbf{c} = (-1,1,-1)$, $\mathbf{d} = (1,x,y)$.

(a) Find x and y.

(b) Find a matrix M that is a rotation leaving T invariant such that $M\mathbf{a} = \mathbf{b}$ and $M\mathbf{b} = \mathbf{a}$.

9A Vector Calculus

Two independent variables x_1 and x_2 are related to a third variable t by

$$x_1 = a + \alpha t\ , \quad x_2 = b + \beta t\ ,$$

where a, b, α and β are constants. Let f be a smooth function of x_1 and x_2, and let $F(t) = f(x_1, x_2)$. Show, by using the Taylor series for $F(t)$ about $t = 0$, that

$$f(x_1,x_2) = f(a,b) + (x_1 - a)\frac{\partial f}{\partial x_1} + (x_2 - b)\frac{\partial f}{\partial x_2}$$
$$+\frac{1}{2}\left((x_1-a)^2\frac{\partial^2 f}{\partial x_1^2} + 2(x_1-a)(x_2-b)\frac{\partial^2 f}{\partial x_1 \partial x_2} + (x_2-b)^2\frac{\partial^2 f}{\partial x_2^2}\right) + \ldots,$$

where all derivatives are evaluated at $x_1 = a,\ x_2 = b$.

Hence show that a stationary point (a, b) of $f(x_1, x_2)$ is a local minimum if

$$H_{11} > 0, \quad \det H_{ij} > 0\ ,$$

where $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ is the Hessian matrix evaluated at (a, b).

Find two local minima of

$$f(x_1, x_2) = x_1^4 - x_1^2 + 2x_1 x_2 + x_2^2\ .$$

10A Vector Calculus

The domain S in the (x, y) plane is bounded by $y = x$, $y = ax\,(0 \leqslant a \leqslant 1)$ and $xy^2 = 1\,(x, y \geqslant 0)$. Find a transformation

$$u = f(x, y), \quad v = g(x, y)\ ,$$

such that S is transformed into a rectangle in the (u, v) plane.

Evaluate

$$\int_D \frac{y^2 z^2}{x}\, dx\, dy\, dz\ ,$$

where D is the region bounded by

$$y = x\,, \qquad y = zx, \qquad xy^2 = 1 \qquad (x, y \geqslant 0)$$

and the planes

$$z = 0\,, \qquad z = 1\,.$$

11A Vector Calculus

Prove that

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}.$$

S is an open orientable surface in $\mathbb{R}^3$ with unit normal $\mathbf{n}$, and $\mathbf{v}(\mathbf{x})$ is any continuously differentiable vector field such that $\mathbf{n} \cdot \mathbf{v} = 0$ on S. Let $\mathbf{m}$ be a continuously differentiable unit vector field which coincides with $\mathbf{n}$ on S. By applying Stokes' theorem to $\mathbf{m} \times \mathbf{v}$, show that

$$\int_S \left(\delta_{ij} - n_i n_j\right) \frac{\partial v_i}{\partial x_j}\, dS = \oint_C \mathbf{u} \cdot \mathbf{v}\, ds\ ,$$

where s denotes arc-length along the boundary C of S, and $\mathbf{u}$ is such that $\mathbf{u}ds = d\mathbf{s} \times \mathbf{n}$. Verify this result by taking $\mathbf{v} = \mathbf{r}$, and S to be the disc $|\mathbf{r}| \leqslant R$ in the $z = 0$ plane.

12A Vector Calculus

(a) Show, using Cartesian coordinates, that $\psi = 1/r$ satisfies Laplace's equation, $\nabla^2\psi = 0$, on $\mathbb{R}^3\backslash\{0\}$.

(b) ϕ and ψ are smooth functions defined in a 3-dimensional domain V bounded by a smooth surface S. Show that

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi) dV = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\mathbf{S}.$$

(c) Let $\psi = 1/|\mathbf{r} - \mathbf{r}_0|$, and let V_ε be a domain bounded by a smooth outer surface S and an inner surface S_ε, where S_ε is a sphere of radius ε, centre $\mathbf{r}_0$. The function ϕ satisfies

$$\nabla^2\phi = -\rho(\mathbf{r}).$$

Use parts (a) and (b) to show, taking the limit $\varepsilon \to 0$, that ϕ at $\mathbf{r}_0$ is given by

$$4\pi\phi(\mathbf{r}_0) = \int_V \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} dV + \int_S \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \frac{\partial\phi}{\partial n} - \phi(\mathbf{r}) \frac{\partial}{\partial n} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) dS,$$

where V is the domain bounded by S.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 3 June 2002 9.00 to 12.00

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You may attempt ***all four*** *questions in Section I. In Section II at most* ***five*** *answers will be taken into account and no more than* ***three*** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in four bundles, marked ***C***, ***B***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions attempted by you.

Every cover sheet *must* ***bear your examination number and desk number.***

SECTION I

1C Numbers and Sets

What does it mean to say that a function $f : A \to B$ is injective? What does it mean to say that a function $g : A \to B$ is surjective?

Consider the functions $f : A \to B$, $g : B \to C$ and their composition $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$. Prove the following results.

(i) If f and g are surjective, then so is $g \circ f$.

(ii) If f and g are injective, then so is $g \circ f$.

(iii) If $g \circ f$ is injective, then so is f.

(iv) If $g \circ f$ is surjective, then so is g.

Give an example where $g \circ f$ is injective and surjective but f is not surjective and g is not injective.

2C Numbers and Sets

If $f,\, g : \mathbb{R} \to \mathbb{R}$ are infinitely differentiable, Leibniz's rule states that, if $n \geqslant 1$,

$$\frac{d^n}{dx^n}\big(f(x)g(x)\big) = \sum_{r=0}^{n} \binom{n}{r} f^{(n-r)}(x)g^{(r)}(x).$$

Prove this result by induction. (You should prove any results on binomial coefficients that you need.)

3E Dynamics

The position x of the leading edge of an avalanche moving down a mountain side making a positive angle α to the horizontal satisfies the equation

$$\frac{d}{dt}\left(x\frac{dx}{dt}\right) = gx\sin\alpha,$$

where g is the acceleration due to gravity.

By multiplying the equation by $x\frac{dx}{dt}$, obtain the first integral

$$x^2\dot{x}^2 = \frac{2g}{3}x^3\sin\alpha + c,$$

where c is an arbitrary constant of integration and the dot denotes differentiation with respect to time.

Sketch the positive quadrant of the $(x,\dot{x})$ phase plane. Show that all solutions approach the trajectory

$$\dot{x} = \left(\frac{2g\sin\alpha}{3}\right)^{\frac{1}{2}} x^{\frac{1}{2}}.$$

Hence show that, independent of initial conditions, the avalanche ultimately has acceleration $\frac{1}{3}g\sin\alpha$.

4E Dynamics

An inertial reference frame S and another reference frame S' have a common origin O. S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to S. Assuming the result that

$$\left(\frac{d\mathbf{a}}{dt}\right)_S = \left(\frac{d\mathbf{a}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}$$

for an arbitrary vector $\mathbf{a}(t)$, show that

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}} = \left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{x}}{dt}\right)_{\mathcal{S}'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}),$$

where $\mathbf{x}$ is the position vector of a point P measured from the origin.

A system of electrically charged particles, all with equal masses m and charges e, moves under the influence of mutual central forces $\mathbf{F}_{ij}$ of the form

$$\mathbf{F}_{ij} = (\mathbf{x}_i - \mathbf{x}_j) f(|\mathbf{x}_i - \mathbf{x}_j|).$$

In addition each particle experiences a Lorentz force due to a constant weak magnetic field $\mathbf{B}$ given by

$$e\frac{d\mathbf{x}_i}{dt} \times \mathbf{B}.$$

Transform the equations of motion to the rotating frame $\mathcal{S}'$. Show that if the angular velocity is chosen to satisfy

$$\boldsymbol{\omega} = -\frac{e}{2m}\mathbf{B},$$

and if terms of second order in $\mathbf{B}$ are neglected, then the equations of motion in the rotating frame are identical to those in the non-rotating frame in the absence of the magnetic field $\mathbf{B}$.

SECTION II

5F Numbers and Sets

What is meant by saying that a set is countable?

Prove that the union of countably many countable sets is itself countable.

Let $\{J_i : i \in I\}$ be a collection of disjoint intervals of the real line, each having strictly positive length. Prove that the index set I is countable.

6F Numbers and Sets

(a) Let S be a finite set, and let $\mathbb{P}(S)$ be the power set of S, that is, the set of all subsets of S. Let $f : \mathbb{P}(S) \to \mathbb{R}$ be additive in the sense that $f(A \cup B) = f(A) + f(B)$ whenever $A \cap B = \varnothing$. Show that, for $A_1, A_2, \ldots, A_n \in \mathbb{P}(S)$,

$$f\left(\bigcup_i A_i\right) = \sum_i f(A_i) - \sum_{i<j} f(A_i \cap A_j) + \sum_{i<j<k} f(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} f\left(\bigcap_i A_i\right).$$

(b) Let $A_1, A_2, \ldots, A_n$ be finite sets. Deduce from part (a) the inclusion–exclusion formula for the size (or cardinality) of $\bigcup_i A_i$.

(c) A *derangement* of the set $S = \{1, 2, \ldots, n\}$ is a permutation π (that is, a bijection from S to itself) in which no member of the set is fixed (that is, $\pi(i) \neq i$ for all i). Using the inclusion–exclusion formula, show that the number d_n of derangements satisfies $d_n/n! \to e^{-1}$ as $n \to \infty$.

7B Numbers and Sets

(a) Suppose that p is an odd prime. Find $1^p + 2^p + \ldots + (p-1)^p$ modulo p.

(b) Find $(p-1)!$ modulo $(1 + 2 + \ldots + (p-1))$, when p is an odd prime.

8B Numbers and Sets

Suppose that a, b are coprime positive integers. Write down an integer $d > 0$ such that $a^d \equiv 1$ modulo b. The least such d is the *order* of a modulo b. Show that if the order of a modulo b is y, and $a^x \equiv 1$ modulo b, then y divides x.

Let $n \geqslant 2$ and $F_n = 2^{2^n} + 1$. Suppose that p is a prime factor of F_n. Find the order of 2 modulo p, and show that $p \equiv 1$ modulo 2^{n+1}.

9E Dynamics

Write down the equations of motion for a system of n gravitating point particles with masses m_i and position vectors $\mathbf{x}_i = \mathbf{x}_i(t)$, $i = 1, 2, \ldots, n$.

Assume that $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Obtain a system of equations for the vectors $\mathbf{a}_i$ which does not involve the time variable t.

Show that the constant vectors $\mathbf{a}_i$ must be located at stationary points of the function

$$\sum_i \frac{1}{9} m_i \, \mathbf{a}_i \cdot \mathbf{a}_i \; + \; \frac{1}{2} \sum_j \sum_{i \neq j} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|} \; .$$

Show that for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other point?

10E Dynamics

Derive the equation

$$\frac{d^2 u}{d\theta^2} + u = \frac{f(u)}{m h^2 u^2} \; ,$$

for the orbit $r^{-1} = u(\theta)$ of a particle of mass m and angular momentum hm moving under a central force $f(u)$ directed towards a fixed point O. Give an interpretation of h in terms of the area swept out by a radius vector.

If the orbits are found to be circles passing through O, then deduce that the force varies inversely as the fifth power of the distance, $f = cu^5$, where c is a constant. Is the force attractive or repulsive?

Show that, for fixed mass, the radius R of the circle varies inversely as the angular momentum of the particle, and hence that the time taken to traverse a complete circle is proportional to R^3.

[*You may assume, if you wish, the expressions for radial and transverse acceleration in the forms* $\ddot{r} - r\dot{\theta}^2$, $2\dot{r}\dot{\theta} + r\ddot{\theta}$.]

11E Dynamics

An electron of mass m moving with velocity $\dot{\mathbf{x}}$ in the vicinity of the North Pole experiences a force

$$\mathbf{F} = a\dot{\mathbf{x}} \times \frac{\mathbf{x}}{|\mathbf{x}|^3} \ ,$$

where a is a constant and the position vector $\mathbf{x}$ of the particle is with respect to an origin located at the North Pole. Write down the equation of motion of the electron, neglecting gravity. By taking the dot product of the equation with $\dot{\mathbf{x}}$ show that the speed of the electron is constant. By taking the cross product of the equation with $\mathbf{x}$ show that

$$m\mathbf{x} \times \dot{\mathbf{x}} - a\frac{\mathbf{x}}{|\mathbf{x}|} = \mathbf{L} \ ,$$

where $\mathbf{L}$ is a constant vector. By taking the dot product of this equation with $\mathbf{x}$, show that the electron moves on a cone centred on the North Pole.

12E Dynamics

Calculate the moment of inertia of a uniform rod of length $2l$ and mass M about an axis through its centre and perpendicular to its length. Assuming it moves in a plane, give an expression for the kinetic energy of the rod in terms of the speed of the centre and the angle that it makes with a fixed direction.

Two such rods are freely hinged together at one end and the other two ends slide on a perfectly smooth horizontal floor. The rods are initially at rest and lie in a vertical plane, each making an angle α to the horizontal. The rods subsequently move under gravity. Calculate the speed with which the hinge strikes the ground.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

List of Courses

1/I/1B **Algebra and Geometry**

(a) State the Orbit-Stabilizer Theorem for a finite group G acting on a set X.

(b) Suppose that G is the group of rotational symmetries of a cube C. Two regular tetrahedra T and T' are inscribed in C, each using half the vertices of C. What is the order of the stabilizer in G of T?

1/I/2D **Algebra and Geometry**

State the Fundamental Theorem of Algebra. Define the characteristic equation for an arbitrary 3×3 matrix A whose entries are complex numbers. Explain why the matrix must have three eigenvalues, not necessarily distinct.

Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

and hence find the three eigenvalues of A. Find a set of linearly independent eigenvectors, specifying which eigenvector belongs to which eigenvalue.

1/II/5B **Algebra and Geometry**

(a) Find a subset T of the Euclidean plane $\mathbb{R}^2$ that is not fixed by any isometry (rigid motion) except the identity.

Let G be a subgroup of the group of isometries of $\mathbb{R}^2$, T a subset of $\mathbb{R}^2$ not fixed by any isometry except the identity, and let S denote the union $\bigcup_{g \in G} g(T)$. Does the group H of isometries of S contain G? Justify your answer.

(b) Find an example of such a G and T with $H \neq G$.

1/II/6B **Algebra and Geometry**

(a) Suppose that g is a Möbius transformation, acting on the extended complex plane. What are the possible numbers of fixed points that g can have? Justify your answer.

(b) Show that the operation c of complex conjugation, defined by $c(z) = \bar{z}$, is not a Möbius transformation.

1/II/7B Algebra and Geometry

(a) Find, with justification, the matrix, with respect to the standard basis of $\mathbb{R}^2$, of the rotation through an angle α about the origin.

(b) Find the matrix, with respect to the standard basis of $\mathbb{R}^3$, of the rotation through an angle α about the axis containing the point $(\frac{3}{5}, \frac{4}{5}, 0)$ and the origin. You may express your answer in the form of a product of matrices.

1/II/8D Algebra and Geometry

Define what is meant by a vector space V over the real numbers $\mathbb{R}$. Define subspace, proper subspace, spanning set, basis, and dimension.

Define the sum $U + W$ and intersection $U \cap W$ of two subspaces U and W of a vector space V. Why is the intersection never empty?

Let $V = \mathbb{R}^4$ and let $U = \{\mathbf{x} \in V : x_1 - x_2 + x_3 - x_4 = 0\}$, where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, and let $W = \{\mathbf{x} \in V : x_1 - x_2 - x_3 + x_4 = 0\}$. Show that $U \cap W$ has the orthogonal basis $\mathbf{b}_1$, $\mathbf{b}_2$ where $\mathbf{b}_1 = (1, 1, 0, 0)$ and $\mathbf{b}_2 = (0, 0, 1, 1)$. Extend this basis to find orthogonal bases of U, W, and $U + W$. Show that $U + W = V$ and hence verify that, in this case,

$$\dim U + \dim W \;=\; \dim(U + W) + \dim(U \cap W)\ .$$

3/I/1A Algebra and Geometry

Given two real non-zero 2×2 matrices A and B, with $AB = 0$, show that A maps $\mathbb{R}^2$ onto a line. Is it always true that $BA = 0$? Show that there is always a non-zero matrix C with $CA = 0 = AC$. Justify your answers.

3/I/2B Algebra and Geometry

(a) What does it mean for a group to be cyclic? Give an example of a finite abelian group that is not cyclic, and justify your assertion.

(b) Suppose that G is a finite group of rotations of $\mathbb{R}^2$ about the origin. Is G necessarily cyclic? Justify your answer.

3/II/5E **Algebra and Geometry**

Prove, using the standard formula connecting δ_{ij} and ϵ_{ijk}, that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \,.$$

Define, in terms of the dot and cross product, the triple scalar product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ and show that it is invariant under cyclic permutation of the vectors.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a *not necessarily orthonormal* basis for $\mathbb{R}^3$, and define

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,, \quad \hat{\mathbf{e}}_2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,, \quad \hat{\mathbf{e}}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \,.$$

By calculating $[\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3]$, show that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ is also a basis for $\mathbb{R}^3$.

The vectors $\hat{\hat{\mathbf{e}}}_1, \hat{\hat{\mathbf{e}}}_2, \hat{\hat{\mathbf{e}}}_3$ are constructed from $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in the same way that $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are constructed from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Show that

$$\hat{\hat{\mathbf{e}}}_1 = \mathbf{e}_1, \;\; \hat{\hat{\mathbf{e}}}_2 = \mathbf{e}_2, \;\; \hat{\hat{\mathbf{e}}}_3 = \mathbf{e}_3,$$

Show that a vector $\mathbf{V}$ has components $\mathbf{V} \cdot \hat{\mathbf{e}}_1,\ \mathbf{V} \cdot \hat{\mathbf{e}}_2,\ \mathbf{V} \cdot \hat{\mathbf{e}}_3$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. What are the components of the vector $\mathbf{V}$ with respect to the basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$?

3/II/6E **Algebra and Geometry**

(a) Give the general solution for $\mathbf{x}$ and $\mathbf{y}$ of the equations

$$\mathbf{x} + \mathbf{y} = 2\mathbf{a} \,, \qquad \mathbf{x} \cdot \mathbf{y} = c \qquad (c < \mathbf{a} \cdot \mathbf{a}) \,.$$

Show in particular that $\mathbf{x}$ and $\mathbf{y}$ must lie at opposite ends of a diameter of a sphere whose centre and radius should be specified.

(b) If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair are also perpendicular to each other. Show also that the sum of the lengths squared of two opposite edges is the same for each pair.

3/II/7A **Algebra and Geometry**

Explain why the number of solutions $\mathbf{x} \in \mathbb{R}^3$ of the simultaneous linear equations $A\mathbf{x} = \mathbf{b}$ is $0, 1$ or infinite, where A is a real 3×3 matrix and $\mathbf{b} \in \mathbb{R}^3$. Let α be the mapping which A represents. State necessary and sufficient conditions on $\mathbf{b}$ and α for each of these possibilities to hold.

Let A and B be 3×3 matrices representing linear mappings α and β. Give necessary and sufficient conditions on α and β for the existence of a 3×3 matrix X with $AX = B$. When is X unique?

Find X when

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

3/II/8B **Algebra and Geometry**

Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the vertices of a regular tetrahedron T in $\mathbb{R}^3$ and that $\mathbf{a} = (1,1,1)$, $\mathbf{b} = (-1,-1,1)$, $\mathbf{c} = (-1,1,-1)$, $\mathbf{d} = (1,x,y)$.

(a) Find x and y.

(b) Find a matrix M that is a rotation leaving T invariant such that $M\mathbf{a} = \mathbf{b}$ and $M\mathbf{b} = \mathbf{a}$.

1/I/3C **Analysis I**

Suppose $a_n \in \mathbb{R}$ for $n \geqslant 1$ and $a \in \mathbb{R}$. What does it mean to say that $a_n \to a$ as $n \to \infty$? What does it mean to say that $a_n \to \infty$ as $n \to \infty$?

Show that, if $a_n \neq 0$ for all n and $a_n \to \infty$ as $n \to \infty$, then $1/a_n \to 0$ as $n \to \infty$. Is the converse true? Give a proof or a counter example.

Show that, if $a_n \neq 0$ for all n and $a_n \to a$ with $a \neq 0$, then $1/a_n \to 1/a$ as $n \to \infty$.

1/I/4C **Analysis I**

Show that any bounded sequence of real numbers has a convergent subsequence.

Give an example of a sequence of real numbers with no convergent subsequence.

Give an example of an unbounded sequence of real numbers with a convergent subsequence.

1/II/9C **Analysis I**

State some version of the fundamental axiom of analysis. State the alternating series test and prove it from the fundamental axiom.

In each of the following cases state whether $\sum_{n=1}^{\infty} a_n$ converges or diverges and prove your result. You may use any test for convergence provided you state it correctly.

(i) $a_n = (-1)^n(\log(n+1))^{-1}$.

(ii) $a_{2n} = (2n)^{-2}$, $a_{2n-1} = -n^{-2}$.

(iii) $a_{3n-2} = -(2n-1)^{-1}$, $a_{3n-1} = (4n-1)^{-1}$, $a_{3n} = (4n)^{-1}$.

(iv) $a_{2^n+r} = (-1)^n(2^n+r)^{-1}$ for $0 \leqslant r \leqslant 2^n - 1$, $n \geqslant 0$.

1/II/10C Analysis I

Show that a continuous real-valued function on a closed bounded interval is bounded and attains its bounds.

Write down examples of the following functions (no proof is required).

(i) A continuous function $f_1 : (0,1) \to \mathbb{R}$ which is not bounded.

(ii) A continuous function $f_2 : (0,1) \to \mathbb{R}$ which is bounded but does not attain its bounds.

(iii) A bounded function $f_3 : [0,1] \to \mathbb{R}$ which is not continuous.

(iv) A function $f_4 : [0,1] \to \mathbb{R}$ which is not bounded on any interval $[a,b]$ with $0 \leqslant a < b \leqslant 1$.

[*Hint: Consider first how to define* f_4 *on the rationals.*]

1/II/11C Analysis I

State the mean value theorem and deduce it from Rolle's theorem.

Use the mean value theorem to show that, if $h : \mathbb{R} \to \mathbb{R}$ is differentiable with $h'(x) = 0$ for all x, then h is constant.

By considering the derivative of the function g given by $g(x) = e^{-ax} f(x)$, find all the solutions of the differential equation $f'(x) = af(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is differentiable and a is a fixed real number.

Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \int_0^x f(t)\,dt$$

is differentiable with $F'(x) = f(x)$.

Find the solution of the equation

$$g(x) = A + \int_0^x g(t)\,dt$$

where $g : \mathbb{R} \to \mathbb{R}$ is differentiable and A is a real number. You should explain why the solution is unique.

1/II/12C **Analysis I**

Prove Taylor's theorem with some form of remainder.

An infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation

$$f^{(3)}(x) = f(x)$$

and the conditions $f(0) = 1$, $f'(0) = f''(0) = 0$. If $R > 0$ and j is a positive integer, explain why we can find an M_j such that

$$|f^{(j)}(x)| \leqslant M_j$$

for all x with $|x| \leqslant R$. Explain why we can find an M such that

$$|f^{(j)}(x)| \leqslant M$$

for all x with $|x| \leqslant R$ and all $j \geqslant 0$.

Use your form of Taylor's theorem to show that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \ .$$

2/I/1D **Differential Equations**

Solve the equation

$$\ddot{y} + \dot{y} - 2y \ = \ e^{-t}$$

subject to the conditions $y(t) = \dot{y}(t) = 0$ at $t = 0$. Solve the equation

$$\ddot{y} + \dot{y} - 2y \ = \ e^{t}$$

subject to the same conditions $y(t) = \dot{y}(t) = 0$ at $t = 0$.

2/I/2D **Differential Equations**

Consider the equation

$$\frac{dy}{dx} \ = \ x\left(\frac{1-y^2}{1-x^2}\right)^{1/2}, \qquad (*)$$

where the positive square root is taken, within the square $\mathcal{S}: \ 0 \leqslant x < 1, \ 0 \leqslant y \leqslant 1$. Find the solution that begins at $x = y = 0$. Sketch the corresponding solution curve, commenting on how its tangent behaves near each extremity. By inspection of the right-hand side of $(*)$, or otherwise, roughly sketch, using small line segments, the directions of flow throughout the square $\mathcal{S}$.

2/II/5D Differential Equations

Explain what is meant by an *integrating factor* for an equation of the form

$$\frac{dy}{dx} + f(x,y) \;=\; 0\;.$$

Show that $2ye^x$ is an integrating factor for

$$\frac{dy}{dx} \;+\; \frac{2x + x^2 + y^2}{2y} \;=\; 0\;,$$

and find the solution $y = y(x)$ such that $y(0) = a$, for given $a > 0$.

Show that $2x + x^2 \geqslant -1$ for all x and hence that

$$\frac{dy}{dx} \;\leqslant\; \frac{1 - y^2}{2y}\;.$$

For a solution with $a \geqslant 1$, show graphically, by considering the sign of dy/dx first for $x = 0$ and then for $x < 0$, that $dy/dx < 0$ for all $x \leqslant 0$.

Sketch the solution for the case $a = 1$, and show that property that $dy/dx \to -\infty$ both as $x \to -\infty$ and as $x \to b$ from below, where $b \approx 0.7035$ is the positive number that satisfies $b^2 = e^{-b}$.

[*Do not consider the range $x \geqslant b$.*]

2/II/6D **Differential Equations**

Solve the differential equation

$$\frac{dy}{dt} = ry(1-ay)$$

for the general initial condition $y = y_0$ at $t = 0$, where r, a, and y_0 are positive constants. Deduce that the equilibria at $y = a^{-1}$ and $y = 0$ are stable and unstable, respectively.

By using the approximate finite-difference formula

$$\frac{dy}{dt} = \frac{y_{n+1} - y_n}{\delta t}$$

for the derivative of y at $t = n\delta t$, where δt is a positive constant and $y_n = y(n\,\delta t)$, show that the differential equation when thus approximated becomes the difference equation

$$u_{n+1} = \lambda\,(1-u_n)\,u_n\ ,$$

where $\lambda = 1 + r\,\delta t > 1$ and where $u_n = \lambda^{-1}a(\lambda-1)\,y_n$. Find the two equilibria and, by linearizing the equation about them or otherwise, show that one is always unstable (given that $\lambda > 1$) and that the other is stable or unstable according as $\lambda < 3$ or $\lambda > 3$. Show that this last instability is oscillatory with period $2\,\delta t$. Why does this last instability have no counterpart for the differential equation? Show graphically how this instability can equilibrate to a periodic, finite-amplitude oscillation when $\lambda = 3.2$.

2/II/7D Differential Equations

The homogeneous equation

$$\ddot{y} + p(t)\dot{y} + q(t)y \;=\; 0$$

has non-constant, non-singular coefficients $p(t)$ and $q(t)$. Two solutions of the equation, $y(t) = y_1(t)$ and $y(t) = y_2(t)$, are given. The solutions are known to be such that the determinant

$$W(t) \;=\; \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}$$

is non-zero for all t. Define what is meant by linear dependence, and show that the two given solutions are linearly *independent*. Show also that

$$W(t) \;\propto\; \exp\left(-\int^t p(s)\,ds\right) \;.$$

In the corresponding inhomogeneous equation

$$\ddot{y} + p(t)\dot{y} + q(t)y \;=\; f(t)$$

the right-hand side $f(t)$ is a prescribed forcing function. Construct a particular integral of this inhomogeneous equation in the form

$$y(t) \;=\; a_1(t)\,y_1(t) + a_2(t)\,y_2(t) \;,$$

where the two functions $a_i(t)$ are to be determined such that

$$y_1(t)\,\dot{a}_1(t) + y_2(t)\,\dot{a}_2(t) \;=\; 0$$

for all t. Express your result for the functions $a_i(t)$ in terms of integrals of the functions $f(t)\,y_1(t)/W(t)$ and $f(t)\,y_2(t)/W(t)$.

Consider the case in which $p(t) = 0$ for all t and $q(t)$ is a positive constant, $q = \omega^2$ say, and in which the forcing $f(t) = \sin(\omega t)$. Show that in this case $y_1(t)$ and $y_2(t)$ can be taken as $\cos(\omega t)$ and $\sin(\omega t)$ respectively. Evaluate $f(t)\,y_1(t)/W(t)$ and $f(t)\,y_2(t)/W(t)$ and show that, as $t \to \infty$, one of the $a_i(t)$ increases in magnitude like a power of t to be determined.

2/II/8D Differential Equations

For any solution of the equations

$$\begin{aligned} \dot{x} &= \alpha x - y + y^3 \qquad (\alpha \text{ constant}) \\ \dot{y} &= -x \end{aligned}$$

show that

$$\frac{d}{dt}\left(x^2 - y^2 + \tfrac{1}{2}y^4\right) \;=\; 2\alpha x^2 \;.$$

What does this imply about the behaviour of phase-plane trajectories at large distances from the origin as $t \to \infty$, in the case $\alpha = 0$? Give brief reasoning but do not try to find explicit solutions.

Analyse the properties of the critical points and sketch the phase portrait (a) in the case $\alpha = 0$, (b) in the case $\alpha = 0.1$, and (c) in the case $\alpha = -0.1$.

4/I/3E **Dynamics**

The position x of the leading edge of an avalanche moving down a mountain side making a positive angle α to the horizontal satisfies the equation

$$\frac{d}{dt}\left(x\frac{dx}{dt}\right) = gx\sin\alpha,$$

where g is the acceleration due to gravity.

By multiplying the equation by $x\frac{dx}{dt}$, obtain the first integral

$$x^2\dot{x}^2 = \frac{2g}{3}x^3\sin\alpha + c,$$

where c is an arbitrary constant of integration and the dot denotes differentiation with respect to time.

Sketch the positive quadrant of the $(x, \dot{x})$ phase plane. Show that all solutions approach the trajectory

$$\dot{x} = \left(\frac{2g\sin\alpha}{3}\right)^{\frac{1}{2}} x^{\frac{1}{2}}.$$

Hence show that, independent of initial conditions, the avalanche ultimately has acceleration $\frac{1}{3}g\sin\alpha$.

4/I/4E **Dynamics**

An inertial reference frame S and another reference frame S' have a common origin O. S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to S. Assuming the result that

$$\left(\frac{d\mathbf{a}}{dt}\right)_S = \left(\frac{d\mathbf{a}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}$$

for an arbitrary vector $\mathbf{a}(t)$, show that

$$\left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}} = \left(\frac{d^2\mathbf{x}}{dt^2}\right)_{\mathcal{S}'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{x}}{dt}\right)_{\mathcal{S}'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}),$$

where $\mathbf{x}$ is the position vector of a point P measured from the origin.

A system of electrically charged particles, all with equal masses m and charges e, moves under the influence of mutual central forces $\mathbf{F}_{ij}$ of the form

$$\mathbf{F}_{ij} = (\mathbf{x}_i - \mathbf{x}_j) f(|\mathbf{x}_i - \mathbf{x}_j|).$$

In addition each particle experiences a Lorentz force due to a constant weak magnetic field $\mathbf{B}$ given by

$$e\frac{d\mathbf{x}_i}{dt} \times \mathbf{B}.$$

Transform the equations of motion to the rotating frame $\mathcal{S}'$. Show that if the angular velocity is chosen to satisfy

$$\boldsymbol{\omega} = -\frac{e}{2m}\mathbf{B},$$

and if terms of second order in $\mathbf{B}$ are neglected, then the equations of motion in the rotating frame are identical to those in the non-rotating frame in the absence of the magnetic field $\mathbf{B}$.

4/II/9E **Dynamics**

Write down the equations of motion for a system of n gravitating point particles with masses m_i and position vectors $\mathbf{x}_i = \mathbf{x}_i(t)$, $i = 1, 2, \ldots, n$.

Assume that $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Obtain a system of equations for the vectors $\mathbf{a}_i$ which does not involve the time variable t.

Show that the constant vectors $\mathbf{a}_i$ must be located at stationary points of the function

$$\sum_i \frac{1}{9} m_i \, \mathbf{a}_i \cdot \mathbf{a}_i \; + \; \frac{1}{2} \sum_j \sum_{i \neq j} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|} \; .$$

Show that for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other point?

4/II/10E **Dynamics**

Derive the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{f(u)}{mh^2u^2} \ ,$$

for the orbit $r^{-1} = u(\theta)$ of a particle of mass m and angular momentum hm moving under a central force $f(u)$ directed towards a fixed point O. Give an interpretation of h in terms of the area swept out by a radius vector.

If the orbits are found to be circles passing through O, then deduce that the force varies inversely as the fifth power of the distance, $f = cu^5$, where c is a constant. Is the force attractive or repulsive?

Show that, for fixed mass, the radius R of the circle varies inversely as the angular momentum of the particle, and hence that the time taken to traverse a complete circle is proportional to R^3.

[*You may assume, if you wish, the expressions for radial and transverse acceleration in the forms* $\ddot{r} - r\dot{\theta}^2,\ 2\dot{r}\dot{\theta} + r\ddot{\theta}$.]

4/II/11E **Dynamics**

An electron of mass m moving with velocity $\dot{\mathbf{x}}$ in the vicinity of the North Pole experiences a force

$$\mathbf{F} = a\dot{\mathbf{x}} \times \frac{\mathbf{x}}{|\mathbf{x}|^3} \ ,$$

where a is a constant and the position vector $\mathbf{x}$ of the particle is with respect to an origin located at the North Pole. Write down the equation of motion of the electron, neglecting gravity. By taking the dot product of the equation with $\dot{\mathbf{x}}$ show that the speed of the electron is constant. By taking the cross product of the equation with $\mathbf{x}$ show that

$$m\mathbf{x} \times \dot{\mathbf{x}} - a\frac{\mathbf{x}}{|\mathbf{x}|} = \mathbf{L} \ ,$$

where $\mathbf{L}$ is a constant vector. By taking the dot product of this equation with $\mathbf{x}$, show that the electron moves on a cone centred on the North Pole.

4/II/12E **Dynamics**

Calculate the moment of inertia of a uniform rod of length $2l$ and mass M about an axis through its centre and perpendicular to its length. Assuming it moves in a plane, give an expression for the kinetic energy of the rod in terms of the speed of the centre and the angle that it makes with a fixed direction.

Two such rods are freely hinged together at one end and the other two ends slide on a perfectly smooth horizontal floor. The rods are initially at rest and lie in a vertical plane, each making an angle α to the horizontal. The rods subsequently move under gravity. Calculate the speed with which the hinge strikes the ground.

4/I/1C Numbers and Sets

What does it mean to say that a function $f : A \to B$ is injective? What does it mean to say that a function $g : A \to B$ is surjective?

Consider the functions $f : A \to B$, $g : B \to C$ and their composition $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$. Prove the following results.

(i) If f and g are surjective, then so is $g \circ f$.

(ii) If f and g are injective, then so is $g \circ f$.

(iii) If $g \circ f$ is injective, then so is f.

(iv) If $g \circ f$ is surjective, then so is g.

Give an example where $g \circ f$ is injective and surjective but f is not surjective and g is not injective.

4/I/2C Numbers and Sets

If $f,\, g : \mathbb{R} \to \mathbb{R}$ are infinitely differentiable, Leibniz's rule states that, if $n \geqslant 1$,

$$\frac{d^n}{dx^n}\big(f(x)g(x)\big) = \sum_{r=0}^{n} \binom{n}{r} f^{(n-r)}(x)g^{(r)}(x).$$

Prove this result by induction. (You should prove any results on binomial coefficients that you need.)

4/II/5F Numbers and Sets

What is meant by saying that a set is countable?

Prove that the union of countably many countable sets is itself countable.

Let $\{J_i : i \in I\}$ be a collection of disjoint intervals of the real line, each having strictly positive length. Prove that the index set I is countable.

4/II/6F Numbers and Sets

(a) Let S be a finite set, and let $\mathbb{P}(S)$ be the power set of S, that is, the set of all subsets of S. Let $f : \mathbb{P}(S) \to \mathbb{R}$ be additive in the sense that $f(A \cup B) = f(A) + f(B)$ whenever $A \cap B = \varnothing$. Show that, for $A_1, A_2, \ldots, A_n \in \mathbb{P}(S)$,

$$f\left(\bigcup_i A_i\right) = \sum_i f(A_i) - \sum_{i<j} f(A_i \cap A_j) + \sum_{i<j<k} f(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} f\left(\bigcap_i A_i\right).$$

(b) Let $A_1, A_2, \ldots, A_n$ be finite sets. Deduce from part (a) the inclusion–exclusion formula for the size (or cardinality) of $\bigcup_i A_i$.

(c) A *derangement* of the set $S = \{1, 2, \ldots, n\}$ is a permutation π (that is, a bijection from S to itself) in which no member of the set is fixed (that is, $\pi(i) \neq i$ for all i). Using the inclusion–exclusion formula, show that the number d_n of derangements satisfies $d_n/n! \to e^{-1}$ as $n \to \infty$.

4/II/7B Numbers and Sets

(a) Suppose that p is an odd prime. Find $1^p + 2^p + \ldots + (p-1)^p$ modulo p.

(b) Find $(p-1)!$ modulo $(1 + 2 + \ldots + (p-1))$, when p is an odd prime.

4/II/8B Numbers and Sets

Suppose that a, b are coprime positive integers. Write down an integer $d > 0$ such that $a^d \equiv 1$ modulo b. The least such d is the *order* of a modulo b. Show that if the order of a modulo b is y, and $a^x \equiv 1$ modulo b, then y divides x.

Let $n \geqslant 2$ and $F_n = 2^{2^n} + 1$. Suppose that p is a prime factor of F_n. Find the order of 2 modulo p, and show that $p \equiv 1$ modulo 2^{n+1}.

2/I/3F Probability

Define the *indicator function* I_A of an event A.

Let I_i be the indicator function of the event A_i, $1 \leq i \leq n$, and let $N = \sum_1^n I_i$ be the number of values of i such that A_i occurs. Show that $E(N) = \sum_i p_i$ where $p_i = P(A_i)$, and find $\operatorname{var}(N)$ in terms of the quantities $p_{ij} = P(A_i \cap A_j)$.

Using Chebyshev's inequality or otherwise, show that

$$P(N=0) \leq \frac{\operatorname{var}(N)}{\{E(N)\}^2}.$$

2/I/4F Probability

A coin shows heads with probability p on each toss. Let π_n be the probability that the number of heads after n tosses is even. Show carefully that $\pi_{n+1} = (1-p)\pi_n + p(1-\pi_n)$, $n \geq 1$, and hence find π_n. [The number 0 is even.]

2/II/9F Probability

(a) Define the *conditional probability* $P(A \mid B)$ of the event A given the event B. Let $\{B_i : 1 \leq i \leq n\}$ be a partition of the sample space Ω such that $P(B_i) > 0$ for all i. Show that, if $P(A) > 0$,

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_j P(A \mid B_j)P(B_j)}.$$

(b) There are n urns, the rth of which contains $r-1$ red balls and $n-r$ blue balls. You pick an urn (uniformly) at random and remove two balls without replacement. Find the probability that the first ball is blue, and the conditional probability that the second ball is blue given that the first is blue. [You may assume that $\sum_{i=1}^{n-1} i(i-1) = \frac{1}{3}n(n-1)(n-2)$.]

(c) What is meant by saying that two events A and B are independent?

(d) Two fair dice are rolled. Let A_s be the event that the sum of the numbers shown is s, and let B_i be the event that the first die shows i. For what values of s and i are the two events A_s, B_i independent?

2/II/10F Probability

There is a random number N of foreign objects in my soup, with mean μ and finite variance. Each object is a fly with probability p, and otherwise is a spider; different objects have independent types. Let F be the number of flies and S the number of spiders.

(a) Show that $G_F(s) = G_N(ps + 1 - p)$. [G_X denotes the probability generating function of a random variable X. You should present a clear statement of any general result used.]

(b) Suppose N has the Poisson distribution with parameter μ. Show that F has the Poisson distribution with parameter μp, and that F and S are independent.

(c) Let $p = \frac{1}{2}$ and suppose that F and S are independent. [You are given nothing about the distribution of N.] Show that $G_N(s) = G_N(\frac{1}{2}(1+s))^2$. By working with the function $H(s) = G_N(1-s)$ or otherwise, deduce that N has the Poisson distribution. [You may assume that $\left(1 + \frac{x}{n} + \mathrm{o}(n^{-1})\right)^n \to e^x$ as $n \to \infty$.]

2/II/11F Probability

Let X, Y, Z be independent random variables each with the uniform distribution on the interval $[0,1]$.

(a) Show that $X+Y$ has density function

$$f_{X+Y}(u) = \begin{cases} u & \text{if } 0 \le u \le 1, \\ 2-u & \text{if } 1 \le u \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that $P(Z > X+Y) = \frac{1}{6}$.

(c) You are provided with three rods of respective lengths X, Y, Z. Show that the probability that these rods may be used to form the sides of a triangle is $\frac{1}{2}$.

(d) Find the density function $f_{X+Y+Z}(s)$ of $X+Y+Z$ for $0 \leqslant s \leqslant 1$. Let W be uniformly distributed on $[0,1]$, and independent of X, Y, Z. Show that the probability that rods of lengths W, X, Y, Z may be used to form the sides of a quadrilateral is $\frac{5}{6}$.

2/II/12F Probability

(a) Explain what is meant by the term 'branching process'.

(b) Let X_n be the size of the nth generation of a branching process in which each family size has probability generating function G, and assume that $X_0 = 1$. Show that the probability generating function G_n of X_n satisfies $G_{n+1}(s) = G_n(G(s))$ for $n \ge 1$.

(c) Show that $G(s) = 1 - \alpha(1-s)^\beta$ is the probability generating function of a non-negative integer-valued random variable when $\alpha, \beta \in (0,1)$, and find G_n explicitly when G is thus given.

(d) Find the probability that $X_n = 0$, and show that it converges as $n \to \infty$ to $1 - \alpha^{1/(1-\beta)}$. Explain carefully why this implies that the probability of ultimate extinction equals $1 - \alpha^{1/(1-\beta)}$.

3/I/3A **Vector Calculus**

Determine whether each of the following is the exact differential of a function, and if so, find such a function:

(a) $(\cosh\theta + \sinh\theta\cos\phi)d\theta + (\cosh\theta\sin\phi + \cos\phi)d\phi$,

(b) $3x^2(y^2+1)dx + 2(yx^3 - z^2)dy - 4yzdz$.

3/I/4A **Vector Calculus**

State the divergence theorem.

Consider the integral

$$I = \int_S r^n \mathbf{r} \cdot d\mathbf{S} \ ,$$

where $n > 0$ and S is the sphere of radius R centred at the origin. Evaluate I directly, and by means of the divergence theorem.

3/II/9A **Vector Calculus**

Two independent variables x_1 and x_2 are related to a third variable t by

$$x_1 = a + \alpha t \ , \quad x_2 = b + \beta t \ ,$$

where a, b, α and β are constants. Let f be a smooth function of x_1 and x_2, and let $F(t) = f(x_1, x_2)$. Show, by using the Taylor series for $F(t)$ about $t = 0$, that

$$f(x_1,x_2) = f(a,b) + (x_1 - a)\frac{\partial f}{\partial x_1} + (x_2 - b)\frac{\partial f}{\partial x_2}$$
$$+\frac{1}{2}\left((x_1-a)^2\frac{\partial^2 f}{\partial x_1^2} + 2(x_1-a)(x_2-b)\frac{\partial^2 f}{\partial x_1 \partial x_2} + (x_2-b)^2\frac{\partial^2 f}{\partial x_2^2}\right) + \ldots,$$

where all derivatives are evaluated at $x_1 = a, \ x_2 = b$.

Hence show that a stationary point (a, b) of $f(x_1, x_2)$ is a local minimum if

$$H_{11} > 0, \quad \det H_{ij} > 0 \ ,$$

where $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ is the Hessian matrix evaluated at (a, b).

Find two local minima of

$$f(x_1, x_2) = x_1^4 - x_1^2 + 2x_1x_2 + x_2^2 \ .$$

3/II/10A **Vector Calculus**

The domain S in the (x,y) plane is bounded by $y=x$, $y=ax\,(0\leqslant a\leqslant 1)$ and $xy^2=1\,(x,y\geqslant 0)$. Find a transformation

$$u=f(x,y),\quad v=g(x,y)\ ,$$

such that S is transformed into a rectangle in the (u,v) plane.

Evaluate
$$\int_D \frac{y^2z^2}{x}\,dx\,dy\,dz\ ,$$
where D is the region bounded by

$$y=x\,,\qquad y=zx,\qquad xy^2=1\qquad (x,y\geqslant 0)$$

and the planes
$$z=0\,,\qquad z=1\,.$$

3/II/11A **Vector Calculus**

Prove that

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}\nabla\cdot\mathbf{b}-\mathbf{b}\nabla\cdot\mathbf{a}+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}.$$

S is an open orientable surface in $\mathbb{R}^3$ with unit normal $\mathbf{n}$, and $\mathbf{v}(\mathbf{x})$ is any continuously differentiable vector field such that $\mathbf{n}\cdot\mathbf{v}=0$ on S. Let $\mathbf{m}$ be a continuously differentiable unit vector field which coincides with $\mathbf{n}$ on S. By applying Stokes' theorem to $\mathbf{m}\times\mathbf{v}$, show that
$$\int_S\left(\delta_{ij}-n_in_j\right)\frac{\partial v_i}{\partial x_j}\,dS=\oint_C \mathbf{u}\cdot\mathbf{v}\,ds\ ,$$
where s denotes arc-length along the boundary C of S, and $\mathbf{u}$ is such that $\mathbf{u}ds=d\mathbf{s}\times\mathbf{n}$. Verify this result by taking $\mathbf{v}=\mathbf{r}$, and S to be the disc $|\mathbf{r}|\leqslant R$ in the $z=0$ plane.

3/II/12A **Vector Calculus**

(a) Show, using Cartesian coordinates, that $\psi = 1/r$ satisfies Laplace's equation, $\nabla^2\psi = 0$, on $\mathbb{R}^3\backslash\{0\}$.

(b) ϕ and ψ are smooth functions defined in a 3-dimensional domain V bounded by a smooth surface S. Show that

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}.$$

(c) Let $\psi = 1/|\mathbf{r}-\mathbf{r}_0|$, and let V_ε be a domain bounded by a smooth outer surface S and an inner surface S_ε, where S_ε is a sphere of radius ε, centre $\mathbf{r}_0$. The function ϕ satisfies

$$\nabla^2\phi = -\rho(\mathbf{r}).$$

Use parts (a) and (b) to show, taking the limit $\varepsilon \to 0$, that ϕ at $\mathbf{r}_0$ is given by

$$4\pi\phi(\mathbf{r}_0) = \int_V \frac{\rho(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}dV + \int_S \left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\frac{\partial\phi}{\partial n} - \phi(\mathbf{r})\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\right) dS,$$

where V is the domain bounded by S.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 29 May 2003 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked **A**, **B**, **C**, **D** *and* **F** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Algebra and Geometry

(a) Write the permutation

$$(123)(234)$$

as a product of disjoint cycles. Determine its order. Compute its sign.

(b) Elements x and y of a group G are *conjugate* if there exists a $g \in G$ such that $gxg^{-1} = y$.

Show that if permutations x and y are conjugate, then they have the same sign and the same order. Is the converse true? (Justify your answer with a proof or counter-example.)

2D Algebra and Geometry

Find the characteristic equation, the eigenvectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and the corresponding eigenvalues $\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}, \lambda_{\mathbf{d}}$ of the matrix

$$A = \begin{pmatrix} i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & i \end{pmatrix} .$$

Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ spans the complex vector space $\mathbb{C}^4$.

Consider the four subspaces of $\mathbb{C}^4$ defined parametrically by

$$\mathbf{z} = s\mathbf{a}, \quad \mathbf{z} = s\mathbf{b}, \quad \mathbf{z} = s\mathbf{c}, \quad \mathbf{z} = s\mathbf{d} \qquad (\mathbf{z} \in \mathbb{C}^4,\ s \in \mathbb{C}) .$$

Show that multiplication by A maps three of these subspaces onto themselves, and the remaining subspace into a smaller subspace to be specified.

3B Analysis

Define what it means for a function of a real variable to be *differentiable* at $x \in \mathbb{R}$.

Prove that if a function is differentiable at $x \in \mathbb{R}$, then it is continuous there.

Show directly from the definition that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at 0 with derivative 0.

Show that the derivative $f'(x)$ is not continuous at 0.

4C Analysis

Explain what is meant by the *radius of convergence* of a power series.

Find the radius of convergence R of each of the following power series:

$$\text{(i)} \quad \sum_{n=1}^{\infty} n^{-2} z^n, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \left(n + \frac{1}{2^n}\right) z^n .$$

In each case, determine whether the series converges on the circle $|z| = R$.

SECTION II

5B Algebra and Geometry

(a) In the standard basis of $\mathbb{R}^2$, write down the matrix for a rotation through an angle θ about the origin.

(b) Let A be a real 3×3 matrix such that $\det A = 1$ and $AA^{\mathrm{T}} = I$, where A^{T} is the transpose of A.

(i) Suppose that A has an eigenvector $\mathbf{v}$ with eigenvalue 1. Show that A is a rotation through an angle θ about the line through the origin in the direction of $\mathbf{v}$, where $\cos\theta = \frac{1}{2}(\mathrm{trace} A - 1)$.

(ii) Show that A must have an eigenvector $\mathbf{v}$ with eigenvalue 1.

6A Algebra and Geometry

Let α be a linear map

$$\alpha : \mathbb{R}^3 \to \mathbb{R}^3 .$$

Define the kernel K and image I of α.

Let $\mathbf{y} \in \mathbb{R}^3$. Show that the equation $\alpha\mathbf{x} = \mathbf{y}$ has a solution $\mathbf{x} \in \mathbb{R}^3$ if and only if $\mathbf{y} \in I$.

Let α have the matrix

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t & -2b \\ 1 & t & 0 \end{pmatrix}$$

with respect to the standard basis, where $b \in \mathbb{R}$ and t is a real variable. Find K and I for α. Hence, or by evaluating the determinant, show that if $0 < b < 2$ and $\mathbf{y} \in I$ then the equation $\alpha\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x} \in \mathbb{R}^3$ for all values of t.

7B Algebra and Geometry

(i) State the orbit-stabilizer theorem for a group G acting on a set X.

(ii) Denote the group of *all* symmetries of the cube by G. Using the orbit-stabilizer theorem, show that G has 48 elements.

Does G have any non-trivial normal subgroups?

Let L denote the line between two diagonally opposite vertices of the cube, and let

$$H = \{g \in G \mid gL = L\}$$

be the subgroup of symmetries that preserve the line. Show that H is isomorphic to the direct product $S_3 \times C_2$, where S_3 is the symmetric group on 3 letters and C_2 is the cyclic group of order 2.

8D Algebra and Geometry

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy–Schwarz inequality, and deduce the triangle inequality.

By means of a sketch, give a geometric interpretation of the scalar product $\mathbf{x} \cdot \mathbf{y}$ in the case $n = 3$, relating the value of $\mathbf{x} \cdot \mathbf{y}$ to the angle α between the directions of $\mathbf{x}$ and $\mathbf{y}$.

What is a unit vector? Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u}\,.$$

Show that

(i) for any fixed, linearly independent $\mathbf{u}$ and $\mathbf{v}$, the minimum of S over $\mathbf{w}$ is attained when $\mathbf{w} = \lambda(\mathbf{u} + \mathbf{v})$ for some $\lambda \in \mathbb{R}$;

(ii) $\lambda \leqslant -\frac{1}{2}$ in all cases;

(iii) $\lambda = -1$ and $S = -3/2$ in the case where $\mathbf{u} \cdot \mathbf{v} = \cos(2\pi/3)$.

9F Analysis

Prove the Axiom of Archimedes.

Let x be a real number in $[0, 1]$, and let m, n be positive integers. Show that the limit

$$\lim_{m\to\infty} \left[\lim_{n\to\infty} \cos^{2n}(m!\pi x) \right]$$

exists, and that its value depends on whether x is rational or irrational.

[*You may assume standard properties of the cosine function provided they are clearly stated.*]

10F Analysis

State without proof the *Integral Comparison Test* for the convergence of a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms.

Determine for which positive real numbers α the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges.

In each of the following cases determine whether the series is convergent or divergent:

(i) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n}$,

(ii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n \log\, n)\ (\log \log\, n)^2}$,

(iii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{(1+1/n)} \log n}$.

11B Analysis

Let $f : [a,b] \to \mathbb{R}$ be continuous. Define the *integral* $\int_a^b f(x)dx$. (You are not asked to prove existence.)

Suppose that m, M are real numbers such that $m \leqslant f(x) \leqslant M$ for all $x \in [a,b]$. Stating clearly any properties of the integral that you require, show that

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)\,.$$

The function $g : [a,b] \to \mathbb{R}$ is continuous and non-negative. Show that

$$m \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M \int_a^b g(x)dx\,.$$

Now let f be continuous on $[0,1]$. By suitable choice of g show that

$$\lim_{n\to\infty} \int_0^{1/\sqrt{n}} nf(x)e^{-nx}dx = f(0)\ ,$$

and by making an appropriate change of variable, or otherwise, show that

$$\lim_{n\to\infty} \int_0^1 nf(x)e^{-nx}dx = f(0)\,.$$

12C Analysis

State carefully the formula for integration by parts for functions of a real variable.

Let $f : (-1,1) \to \mathbb{R}$ be infinitely differentiable. Prove that for all $n \geqslant 1$ and all $t \in (-1,1)$,

$$f(t) = f(0) + f'(0)t + \frac{1}{2!}f''(0)t^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(0)t^{n-1} + \frac{1}{(n-1)!}\int_0^t f^{(n)}(x)(t-x)^{n-1}\,dx.$$

By considering the function $f(x) = \log(1-x)$ at $x = 1/2$, or otherwise, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

converges to $\log 2$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 30 May 2003 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked **D** *and* **F** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Differential Equations

Consider the equation

$$\frac{dy}{dx} = 1 - y^2 \,. \tag{$*$}$$

Using small line segments, sketch the flow directions in $x \geqslant 0$, $-2 \leqslant y \leqslant 2$ implied by the right-hand side of $(*)$. Find the general solution

(i) in $|y| < 1$,

(ii) in $|y| > 1$.

Sketch a solution curve in each of the three regions $y > 1$, $|y| < 1$, and $y < -1$.

2D Differential Equations

Consider the differential equation

$$\frac{dx}{dt} + Kx = 0 \,,$$

where K is a positive constant. By using the approximate finite-difference formula

$$\frac{dx_n}{dt} = \frac{x_{n+1} - x_{n-1}}{2\delta t} \,,$$

where δt is a positive constant, and where x_n denotes the function $x(t)$ evaluated at $t = n\delta t$ for integer n, convert the differential equation to a difference equation for x_n.

Solve both the differential equation and the difference equation for general initial conditions. Identify those solutions of the difference equation that agree with solutions of the differential equation over a finite interval $0 \leqslant t \leqslant T$ in the limit $\delta t \to 0$, and demonstrate the agreement. Demonstrate that the remaining solutions of the difference equation cannot agree with the solution of the differential equation in the same limit.

[*You may use the fact that, for bounded* $|u|$, $\lim_{\epsilon \to 0} (1 + \epsilon u)^{1/\epsilon} = e^u$.]

3F Probability

(a) Define the *probability generating function* of a random variable. Calculate the probability generating function of a binomial random variable with parameters n and p, and use it to find the mean and variance of the random variable.

(b) X is a binomial random variable with parameters n and p, Y is a binomial random variable with parameters m and p, and X and Y are independent. Find the distribution of $X + Y$; that is, determine $P\{X + Y = k\}$ for all possible values of k.

4F Probability

The random variable X is uniformly distributed on the interval $[0,1]$. Find the distribution function and the probability density function of Y, where

$$Y = \frac{3X}{1-X}\,.$$

SECTION II

5D Differential Equations

(a) Show that if $\mu(x,y)$ is an integrating factor for an equation of the form

$$f(x,y)\,dy \;+\; g(x,y)\,dx \;=\; 0$$

then $\partial(\mu f)/\partial x \;=\; \partial(\mu g)/\partial y$.

Consider the equation

$$\cot x\;dy \;-\; \tan y\;dx \;=\; 0$$

in the domain $0 \leqslant x \leqslant \frac{1}{2}\pi$, $0 \leqslant y \leqslant \frac{1}{2}\pi$. Using small line segments, sketch the flow directions in that domain. Show that $\sin x\,\cos y$ is an integrating factor for the equation. Find the general solution of the equation, and sketch the family of solutions that occupies the larger domain $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$, $-\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi$.

(b) The following example illustrates that the concept of integrating factor extends to higher-order equations. Multiply the equation

$$\left[y\,\frac{d^2y}{dx^2} \;+\; \left(\frac{dy}{dx}\right)^2\right]\cos^2 x \;=\; 1$$

by $\sec^2 x$, and show that the result takes the form $\dfrac{d}{dx}h(x,y)=0$, for some function $h(x,y)$ to be determined. Find a particular solution $y=y(x)$ such that $y(0)=0$ with dy/dx finite at $x=0$, and sketch its graph in $0 \leqslant x < \frac{1}{2}\pi$.

6D Differential Equations

Define the *Wronskian* $W(x)$ associated with solutions of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y \;=\; 0$$

and show that

$$W(x) \;\propto\; \exp\left(-\int^x p(\xi)\,d\xi\right) \;.$$

Evaluate the expression on the right when $p(x) = -2/x$.

Given that $p(x) = -2/x$ and that $q(x) = -1$, show that solutions in the form of power series,

$$y \;=\; x^\lambda \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0)\,,$$

can be found if and only if $\lambda = 0$ or 3. By constructing and solving the appropriate recurrence relations, find the coefficients a_n for each power series.

You may assume that the equation is satisfied by $y = \cosh x - x\sinh x$ and by $y = \sinh x - x\cosh x$. Verify that these two solutions agree with the two power series found previously, and that they give the $W(x)$ found previously, up to multiplicative constants.

[*Hint:* $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dots$, $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dots$.]

7D Differential Equations

Consider the linear system

$$\dot{\mathbf{x}}(t) - A\mathbf{x}(t) \;=\; \mathbf{z}(t)$$

where the n-vector $\mathbf{z}(t)$ and the $n \times n$ matrix A are given; A has constant real entries, and has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and n linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$. Find the complementary function. Given a particular integral $\mathbf{x}_{\mathrm{p}}(t)$, write down the general solution. In the case $n = 2$ show that the complementary function is purely oscillatory, with no growth or decay, if and only if

$$\operatorname{trace} A = 0 \qquad \text{and} \qquad \det A > 0 \;.$$

Consider the same case $n = 2$ with $\operatorname{trace} A = 0$ and $\det A > 0$ and with

$$\mathbf{z}(t) = \mathbf{a}_1 \exp(i\omega_1 t) + \mathbf{a}_2 \exp(i\omega_2 t)\,,$$

where $\omega_1,\ \omega_2$ are given real constants. Find a particular integral when

(i) $i\omega_1 \neq \lambda_1$ and $i\omega_2 \neq \lambda_2\,;$

(ii) $i\omega_1 \neq \lambda_1$ but $i\omega_2 = \lambda_2\,.$

In the case

$$A \;=\; \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$$

with $\mathbf{z}(t) = \begin{pmatrix} 2 \\ 3i - 1 \end{pmatrix} \exp(3it)$, find the solution subject to the initial condition $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$.

8D Differential Equations

For all solutions of

$$\begin{aligned}\dot{x} &= \tfrac{1}{2}\alpha x + y - 2y^3, \\ \dot{y} &= -x\end{aligned}$$

show that $dK/dt = \alpha x^2$ where

$$K = K(x,y) = x^2 + y^2 - y^4 \ .$$

In the case $\alpha = 0$, analyse the properties of the critical points and sketch the phase portrait, including the special contours for which $K(x,y) = \frac{1}{4}$. Comment on the asymptotic behaviour, as $t \to \infty$, of solution trajectories that pass near each critical point, indicating whether or not any such solution trajectories approach from, or recede to, infinity.

Briefly discuss how the picture changes when α is made small and positive, using your result for dK/dt to describe, in qualitative terms, how solution trajectories cross K-contours.

9F Probability

State the inclusion-exclusion formula for the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs.

After a party the n guests take coats randomly from a pile of their n coats. Calculate the probability that no-one goes home with the correct coat.

Let $p(m,n)$ be the probability that exactly m guests go home with the correct coats. By relating $p(m,n)$ to $p(0,n-m)$, or otherwise, determine $p(m,n)$ and deduce that

$$\lim_{n\to\infty} p(m,n) = \frac{1}{em!} \ .$$

10F Probability

The random variables X and Y each take values in $\{0,1\}$, and their joint distribution $p(x,y) = P\{X = x,\ Y = y\}$ is given by

$$p(0,0) = a, \qquad p(0,1) = b, \qquad p(1,0) = c, \qquad p(1,1) = d\,.$$

Find necessary and sufficient conditions for X and Y to be

(i) uncorrelated;

(ii) independent.

Are the conditions established in (i) and (ii) equivalent?

11F Probability

A laboratory keeps a population of aphids. The probability of an aphid passing a day uneventfully is $q < 1$. Given that a day is not uneventful, there is probability r that the aphid will have one offspring, probability s that it will have two offspring and probability t that it will die, where $r + s + t = 1$. Offspring are ready to reproduce the next day. The fates of different aphids are independent, as are the events of different days. The laboratory starts out with one aphid.

Let X_1 be the number of aphids at the end of the first day. What is the expected value of X_1? Determine an expression for the probability generating function of X_1.

Show that the probability of extinction does not depend on q, and that if $2r+3s \leqslant 1$ then the aphids will certainly die out. Find the probability of extinction if $r = 1/5,\ s = 2/5$ and $t = 2/5$.

[*Standard results on branching processes may be used without proof, provided that they are clearly stated.*]

12F Probability

Planet Zog is a ball with centre O. Three spaceships A, B and C land at random on its surface, their positions being independent and each uniformly distributed on its surface. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Spaceships A and B can communicate directly by radio if $\angle AOB < \pi/2$, and similarly for spaceships B and C and spaceships A and C. Given angle $\angle AOB = \gamma < \pi/2$, calculate the probability that C can communicate directly with *either* A or B. Given angle $\angle AOB = \gamma > \pi/2$, calculate the probability that C can communicate directly with *both* A and B. Hence, or otherwise, show that the probability that all three spaceships can keep in in touch (with, for example, A communicating with B via C if necessary) is $(\pi + 2)/(4\pi)$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 3 June 2003 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

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At the end of the examination:

Tie up your answers in bundles, marked **A**, **B** *and* **E** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

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SECTION I

1A Algebra and Geometry

The mapping α of $\mathbb{R}^3$ into itself is a reflection in the plane $x_2 = x_3$. Find the matrix A of α with respect to any basis of your choice, which should be specified.

The mapping β of $\mathbb{R}^3$ into itself is a rotation about the line $x_1 = x_2 = x_3$ through $2\pi/3$, followed by a dilatation by a factor of 2. Find the matrix B of β with respect to a choice of basis that should again be specified.

Show explicitly that

$$B^3 = 8A^2$$

and explain why this must hold, irrespective of your choices of bases.

2B Algebra and Geometry

Show that if a group G contains a normal subgroup of order 3, and a normal subgroup of order 5, then G contains an element of order 15.

Give an example of a group of order 10 with no element of order 10.

3A Vector Calculus

Sketch the curve $y^2 = x^2 + 1$. By finding a parametric representation, or otherwise, determine the points on the curve where the radius of curvature is least, and compute its value there.

[*Hint: you may use the fact that the radius of curvature of a parametrized curve* $(x(t), y(t))$ *is* $(\dot{x}^2 + \dot{y}^2)^{3/2}/|\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.]

4A Vector Calculus

Suppose V is a region in $\mathbb{R}^3$, bounded by a piecewise smooth closed surface S, and $\phi(\mathbf{x})$ is a scalar field satisfying

$$\begin{aligned} \boldsymbol{\nabla}^2\phi &= 0 \quad \text{in } V, \\ \text{and} \quad \phi &= f(\mathbf{x}) \quad \text{on } S\,. \end{aligned}$$

Prove that ϕ is determined uniquely in V.

How does the situation change if the normal derivative of ϕ rather than ϕ itself is specified on S?

SECTION II

5E Algebra and Geometry

(a) Show, using vector methods, that the distances from the centroid of a tetrahedron to the centres of opposite pairs of edges are equal. If the three distances are u, v, w and if a, b, c, d are the distances from the centroid to the vertices, show that

$$u^2 + v^2 + w^2 \;=\; \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2).$$

[*The centroid of k points in $\mathbb{R}^3$ with position vectors $\mathbf{x}_i$ is the point with position vector* $\dfrac{1}{k}\sum \mathbf{x}_i$.]

(b) Show that

$$|\mathbf{x} - \mathbf{a}|^2 \cos^2 \alpha = [(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n}]^2,$$

with $\mathbf{n}^2 = 1$, is the equation of a right circular double cone whose vertex has position vector $\mathbf{a}$, axis of symmetry $\mathbf{n}$ and opening angle α.

Two such double cones, with vertices $\mathbf{a}_1$ and $\mathbf{a}_2$, have parallel axes and the same opening angle. Show that if $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 \neq \mathbf{0}$, then the intersection of the cones lies in a plane with unit normal

$$\mathbf{N} = \frac{\mathbf{b}\cos^2\alpha - \mathbf{n}(\mathbf{n} \cdot \mathbf{b})}{\sqrt{\mathbf{b}^2 \cos^4 \alpha + (\mathbf{b} \cdot \mathbf{n})^2 (1 - 2\cos^2 \alpha)}}\,.$$

6E Algebra and Geometry

Derive an expression for the triple scalar product $(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3$ in terms of the determinant of the matrix E whose rows are given by the components of the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Use the geometrical interpretation of the cross product to show that $\mathbf{e}_a$, $a = 1, 2, 3$, will be a *not necessarily orthogonal* basis for $\mathbb{R}^3$ as long as $\det E \neq 0$.

The rows of another matrix $\hat{E}$ are given by the components of three other vectors $\hat{\mathbf{e}}_b$, $b = 1, 2, 3$. By considering the matrix $E\hat{E}^{\mathrm{T}}$, where ${}^{\mathrm{T}}$ denotes the transpose, show that there is a unique choice of $\hat{E}$ such that $\hat{\mathbf{e}}_b$ is also a basis and

$$\mathbf{e}_a \cdot \hat{\mathbf{e}}_b = \delta_{ab}.$$

Show that the new basis is given by

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3} \qquad \text{etc.}$$

Show that if either $\mathbf{e}_a$ or $\hat{\mathbf{e}}_b$ is an orthonormal basis then E is a rotation matrix.

7B Algebra and Geometry

Let G be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $X = \{\alpha, \beta, \gamma\}$ be a set of three distinct points in $\mathbb{C} \cup \{\infty\}$.

(i) Show that there exists a $g \in G$ sending α to 0, β to 1, and γ to ∞.

(ii) Hence show that if $H = \{g \in G \mid gX = X\}$, then H is isomorphic to S_3, the symmetric group on 3 letters.

8B Algebra and Geometry

(a) Determine the characteristic polynomial and the eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

Is it diagonalizable?

(b) Show that an $n \times n$ matrix A with characteristic polynomial $f(t) = (t - \mu)^n$ is diagonalizable if and only if $A = \mu I$.

9A Vector Calculus

Let C be the closed curve that is the boundary of the triangle T with vertices at the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Specify a direction along C and consider the integral

$$\oint_C \mathbf{A} \cdot d\mathbf{x} ,$$

where $\mathbf{A} = (z^2 - y^2,\ x^2 - z^2,\ y^2 - x^2)$. Explain why the contribution to the integral is the same from each edge of C, and evaluate the integral.

State Stokes's theorem and use it to evaluate the surface integral

$$\int_T (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S} ,$$

the components of the normal to T being positive.

Show that $d\mathbf{S}$ in the above surface integral can be written in the form $(1,1,1)\,dy\,dz$. Use this to verify your result by a direct calculation of the surface integral.

10A Vector Calculus

Write down an expression for the Jacobian J of a transformation

$$(x,y,z) \to (u,v,w) .$$

Use it to show that

$$\int_D f\ dx\,dy\,dz = \int_\Delta \phi\ |J|\ du\,dv\,dw$$

where D is mapped one-to-one onto Δ, and

$$\phi(u,v,w) \ = \ f\big(x(u,v,w),\ y(u,v,w),\ z(u,v,w)\big) .$$

Find a transformation that maps the ellipsoid D,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1 ,$$

onto a sphere. Hence evaluate

$$\int_D x^2\,dx\,dy\,dz .$$

11A Vector Calculus

(a) Prove the identity

$$\nabla(\mathbf{F}\cdot\mathbf{G}) \;=\; (\mathbf{F}\cdot\nabla)\mathbf{G} \;+\; (\mathbf{G}\cdot\nabla)\mathbf{F} \;+\; \mathbf{F}\times(\nabla\times\mathbf{G}) \;+\; \mathbf{G}\times(\nabla\times\mathbf{F})\,.$$

(b) If $\mathbf{E}$ is an irrotational vector field ($\nabla\times\mathbf{E}=\mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E}=-\nabla\phi$.

Show that

$$(2xy^2ze^{-x^2z},\; -2ye^{-x^2z},\; x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

12A Vector Calculus

State the divergence theorem. By applying this to $f(\mathbf{x})\mathbf{k}$, where $f(\mathbf{x})$ is a scalar field in a closed region V in $\mathbb{R}^3$ bounded by a piecewise smooth surface S, and $\mathbf{k}$ an arbitrary constant vector, show that

$$\int_V \nabla f\, dV = \int_S f\, d\mathbf{S}\,. \qquad (*)$$

A vector field $\mathbf{G}$ satisfies

$$\nabla\cdot\mathbf{G}=\rho(\mathbf{x})$$

$$\text{with}\quad \rho(\mathbf{x})=\begin{cases}\rho_0 & |\mathbf{x}|\leqslant a\\ 0 & |\mathbf{x}|>a\,.\end{cases}$$

By applying the divergence theorem to $\displaystyle\int_V \nabla\cdot\mathbf{G}\,dV$, prove Gauss's law

$$\int_S \mathbf{G}\cdot d\mathbf{S} = \int_V \rho(\mathbf{x})\,dV,$$

where S is the piecewise smooth surface bounding the volume V.

Consider the spherically symmetric solution

$$\mathbf{G}(\mathbf{x}) = G(r)\,\frac{\mathbf{x}}{r}\ ,$$

where $r=|\mathbf{x}|$. By using Gauss's law with S a sphere of radius r, centre $\mathbf{0}$, in the two cases $0<r\leqslant a$ and $r>a$, show that

$$\mathbf{G}(\mathbf{x})=\begin{cases}\dfrac{\rho_0}{3}\,\mathbf{x} & r\leqslant a\\[2mm] \dfrac{\rho_0}{3}\left(\dfrac{a}{r}\right)^3\mathbf{x} & r>a\,.\end{cases}$$

The scalar field $f(\mathbf{x})$ satisfies $\mathbf{G}=\nabla f$. Assuming that $f\to 0$ as $r\to\infty$, and that f is continuous at $r=a$, find f everywhere.

By using a symmetry argument, explain why $(*)$ is clearly satisfied for this f if S is any sphere centred at the origin.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 2 June 2003 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. In Section I, you may attempt **all four** *questions. In Section II, at most* **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

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At the end of the examination:

Tie up your answers in bundles, marked **C** *and* **E** *according to the code letter affixed to each question. Attach a blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet. Do not tie up questions from Section I and Section II in separate bundles.*

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SECTION I

1C Numbers and Sets

(i) Prove by induction or otherwise that for every $n \geqslant 1$,

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2 .$$

(ii) Show that the sum of the first n positive cubes is divisible by 4 if and only if $n \equiv 0$ or 3 (mod 4).

2C Numbers and Sets

What is an *equivalence relation*? For each of the following pairs $(X, \sim)$, determine whether or not $\sim$ is an equivalence relation on X:

(i) $X = \mathbb{R}$, $x \sim y$ iff $x - y$ is an even integer;

(ii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{R}$;

(iii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{Z}$;

(iv) $X = \mathbb{Z} \setminus \{0\}$, $x \sim y$ iff $x^2 - y^2$ is ± 1 times a perfect square.

3E Dynamics

Because of an accident on launching, a rocket of unladen mass M lies horizontally on the ground. It initially contains fuel of mass m_0, which ignites and is emitted horizontally at a constant rate and at uniform speed u relative to the rocket. The rocket is initially at rest. If the coefficient of friction between the rocket and the ground is μ, and the fuel is completely burnt in a total time T, show that the final speed of the rocket is

$$u \log\left(\frac{M + m_0}{M}\right) - \mu g T.$$

4E Dynamics

Write down an expression for the total momentum $\mathbf{P}$ and angular momentum $\mathbf{L}$ with respect to an origin O of a system of n point particles of masses m_i, position vectors (with respect to O) $\mathbf{x}_i$, and velocities $\mathbf{v}_i$, $i = 1, \ldots, n$.

Show that with respect to a new origin O' the total momentum $\mathbf{P}'$ and total angular momentum $\mathbf{L}'$ are given by

$$\mathbf{P}' = \mathbf{P}, \qquad \mathbf{L}' = \mathbf{L} - \mathbf{b} \times \mathbf{P},$$

and hence

$$\mathbf{L}' \cdot \mathbf{P}' = \mathbf{L} \cdot \mathbf{P},$$

where $\mathbf{b}$ is the constant vector displacement of O' with respect to O. How does $\mathbf{L} \times \mathbf{P}$ change under change of origin?

Hence show that **either**

(1) the total momentum vanishes and the total angular momentum is independent of origin, **or**

(2) by choosing $\mathbf{b}$ in a way that should be specified, the total angular momentum with respect to O' can be made parallel to the total momentum.

SECTION II

5C Numbers and Sets

Define what is meant by the term *countable*. Show directly from your definition that if X is countable, then so is any subset of X.

Show that $\mathbb{N} \times \mathbb{N}$ is countable. Hence or otherwise, show that a countable union of countable sets is countable. Show also that for any $n \geqslant 1$, $\mathbb{N}^n$ is countable.

A function $f : \mathbb{Z} \to \mathbb{N}$ is *periodic* if there exists a positive integer m such that, for every $x \in \mathbb{Z}$, $f(x+m) = f(x)$. Show that the set of periodic functions $f : \mathbb{Z} \to \mathbb{N}$ is countable.

6C Numbers and Sets

(i) Prove Wilson's theorem: if p is prime then $(p-1)! \equiv -1 \pmod{p}$.

Deduce that if $p \equiv 1 \pmod{4}$ then

$$\left(\left(\frac{p-1}{2}\right)!\right)^2 \equiv -1 \pmod{p}.$$

(ii) Suppose that p is a prime of the form $4k+3$. Show that if $x^4 \equiv 1 \pmod{p}$ then $x^2 \equiv 1 \pmod{p}$.

(iii) Deduce that if p is an odd prime, then the congruence

$$x^2 \equiv -1 \pmod{p}$$

has exactly two solutions (modulo p) if $p \equiv 1 \pmod{4}$, and none otherwise.

7C Numbers and Sets

Let m, n be integers. Explain what is their *greatest common divisor* (m,n). Show from your definition that, for any integer k, $(m,n) = (m+kn, n)$.

State Bezout's theorem, and use it to show that if p is prime and p divides mn, then p divides at least one of m and n.

The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$ is defined by $x_0 = 0$, $x_1 = 1$ and $x_{n+1} = x_n + x_{n-1}$ for $n \geqslant 1$. Prove:

(i) $(x_{n+1}, x_n) = 1$ and $(x_{n+2}, x_n) = 1$ for every $n \geqslant 0$;

(ii) $x_{n+3} \equiv x_n \pmod{2}$ and $x_{n+8} \equiv x_n \pmod{3}$ for every $n \geqslant 0$;

(iii) if $n \equiv 0 \pmod{5}$ then $x_n \equiv 0 \pmod{5}$.

8C Numbers and Sets

Let X be a finite set with n elements. How many functions are there from X to X? How many relations are there on X?

Show that the number of relations R on X such that, for each $y \in X$, there exists at least one $x \in X$ with xRy, is $(2^n - 1)^n$.

Using the inclusion–exclusion principle or otherwise, deduce that the number of such relations R for which, in addition, for each $x \in X$, there exists at least one $y \in X$ with xRy, is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^n .$$

9E Dynamics

Write down the equation of motion for a point particle with mass m, charge e, and position vector $\mathbf{x}(t)$ moving in a time-dependent magnetic field $\mathbf{B}(\mathbf{x}, t)$ with vanishing electric field, and show that the kinetic energy of the particle is constant. If the magnetic field is constant in direction, show that the component of velocity in the direction of $\mathbf{B}$ is constant. Show that, in general, the angular momentum of the particle is not conserved.

Suppose that the magnetic field is independent of time and space and takes the form $\mathbf{B} = (0, 0, B)$ and that $\dot{A}$ is the rate of change of area swept out by a radius vector joining the origin to the projection of the particle's path on the (x, y) plane. Obtain the equation

$$\frac{d}{dt}\left(m\dot{A} + \frac{eBr^2}{4} \right) = 0 \,, \qquad (*)$$

where (r, θ) are plane polar coordinates. Hence obtain an equation replacing the equation of conservation of angular momentum.

Show further, using energy conservation and $(*)$, that the equations of motion in plane polar coordinates may be reduced to the first order non-linear system

$$\dot{r} = \sqrt{v^2 - \left(\frac{2c}{mr} - \frac{erB}{2m} \right)^2} \,,$$

$$\dot{\theta} = \frac{2c}{mr^2} - \frac{eB}{2m} \,,$$

where v and c are constants.

10E Dynamics

Write down the equations of motion for a system of n gravitating particles with masses m_i, and position vectors $\mathbf{x}_i$, $i = 1, 2, \ldots, n$.

The particles undergo a motion for which $\mathbf{x}_i(t) = a(t)\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Show that the equations of motion will be satisfied as long as the function $a(t)$ satisfies

$$\ddot{a} = -\frac{\Lambda}{a^2} \,, \qquad (*)$$

where Λ is a constant and the vectors $\mathbf{a}_i$ satisfy

$$\Lambda m_i \mathbf{a}_i = \mathbf{G}_i = \sum_{j \neq i} \frac{G m_i m_j \big(\mathbf{a}_i - \mathbf{a}_j\big)}{|\mathbf{a}_i - \mathbf{a}_j|^3} \,. \qquad (**)$$

Show that $(*)$ has as first integral

$$\frac{\dot{a}^2}{2} - \frac{\Lambda}{a} = \frac{k}{2} \,,$$

where k is another constant. Show that

$$\mathbf{G}_i = \boldsymbol{\nabla}_i W \,,$$

where $\boldsymbol{\nabla}_i$ is the gradient operator with respect to $\mathbf{a}_i$ and

$$W = -\sum_i \sum_{j<i} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|} \,.$$

Using Euler's theorem for homogeneous functions (see below), or otherwise, deduce that

$$\sum_i \mathbf{a}_i \cdot \mathbf{G}_i = -W \,.$$

Hence show that all solutions of $(**)$ satisfy

$$\Lambda I = -W$$

where

$$I = \sum_i m_i \mathbf{a}_i^2 \,.$$

Deduce that Λ must be positive and that the total kinetic energy plus potential energy of the system of particles is equal to $\frac{k}{2} I$.

[*Euler's theorem states that if*

$$f(\lambda x, \lambda y, \lambda z, \ldots) = \lambda^p f(x, y, z, \ldots) \,,$$

then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + \ldots = pf \,.]$$

11E Dynamics

State the parallel axis theorem and use it to calculate the moment of inertia of a uniform hemisphere of mass m and radius a about an axis through its centre of mass and parallel to the base.

[*You may assume that the centre of mass is located at a distance* $\frac{3}{8}a$ *from the flat face of the hemisphere, and that the moment of inertia of a full sphere about its centre is* $\frac{2}{5}Ma^2$*, with* $M = 2m$.]

The hemisphere initially rests on a rough horizontal plane with its base vertical. It is then released from rest and subsequently rolls on the plane without slipping. Let θ be the angle that the base makes with the horizontal at time t. Express the instantaneous speed of the centre of mass in terms of b and the rate of change of θ, where b is the instantaneous distance from the centre of mass to the point of contact with the plane. Hence write down expressions for the kinetic energy and potential energy of the hemisphere and deduce that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{(28 - 15\cos\theta)a}\,.$$

12E Dynamics

Let (r, θ) be plane polar coordinates and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ unit vectors in the direction of increasing r and θ respectively. Show that the velocity of a particle moving in the plane with polar coordinates $\big(r(t),\ \theta(t)\big)$ is given by

$$\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\,,$$

and that the unit normal $\mathbf{n}$ to the particle path is parallel to

$$r\dot{\theta}\mathbf{e}_r - \dot{r}\mathbf{e}_\theta\,.$$

Deduce that the perpendicular distance p from the origin to the tangent of the curve $r = r(\theta)$ is given by

$$\frac{r^2}{p^2} = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2.$$

The particle, whose mass is m, moves under the influence of a central force with potential $V(r)$. Use the conservation of energy E and angular momentum h to obtain the equation

$$\frac{1}{p^2} = \frac{2m\big(E - V(r)\big)}{h^2}\,.$$

Hence express θ as a function of r as the integral

$$\theta = \int \frac{hr^{-2}dr}{\sqrt{2m\big(E - V_{\text{eff}}(r)\big)}}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{h^2}{2mr^2}\,.$$

Evaluate the integral and describe the orbit when $V(r) = \dfrac{c}{r^2}$, with c a positive constant.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2003

List of Courses

1/I/1B **Algebra and Geometry**

(a) Write the permutation

$$(123)(234)$$

as a product of disjoint cycles. Determine its order. Compute its sign.

(b) Elements x and y of a group G are *conjugate* if there exists a $g \in G$ such that $gxg^{-1} = y$.

Show that if permutations x and y are conjugate, then they have the same sign and the same order. Is the converse true? (Justify your answer with a proof or counter-example.)

1/I/2D **Algebra and Geometry**

Find the characteristic equation, the eigenvectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and the corresponding eigenvalues $\lambda_{\mathbf{a}}, \lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}, \lambda_{\mathbf{d}}$ of the matrix

$$A = \begin{pmatrix} i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & i \end{pmatrix} .$$

Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ spans the complex vector space $\mathbb{C}^4$.

Consider the four subspaces of $\mathbb{C}^4$ defined parametrically by

$$\mathbf{z} = s\mathbf{a}, \quad \mathbf{z} = s\mathbf{b}, \quad \mathbf{z} = s\mathbf{c}, \quad \mathbf{z} = s\mathbf{d} \qquad (\mathbf{z} \in \mathbb{C}^4,\ s \in \mathbb{C})\,.$$

Show that multiplication by A maps three of these subspaces onto themselves, and the remaining subspace into a smaller subspace to be specified.

1/II/5B **Algebra and Geometry**

(a) In the standard basis of $\mathbb{R}^2$, write down the matrix for a rotation through an angle θ about the origin.

(b) Let A be a real 3×3 matrix such that $\det A = 1$ and $AA^{\mathrm{T}} = I$, where A^{T} is the transpose of A.

(i) Suppose that A has an eigenvector $\mathbf{v}$ with eigenvalue 1. Show that A is a rotation through an angle θ about the line through the origin in the direction of $\mathbf{v}$, where $\cos\theta = \frac{1}{2}(\mathrm{trace}A - 1)$.

(ii) Show that A must have an eigenvector $\mathbf{v}$ with eigenvalue 1.

1/II/6A **Algebra and Geometry**

Let α be a linear map
$$\alpha : \mathbb{R}^3 \to \mathbb{R}^3 .$$
Define the kernel K and image I of α.

Let $\mathbf{y} \in \mathbb{R}^3$. Show that the equation $\alpha\mathbf{x} = \mathbf{y}$ has a solution $\mathbf{x} \in \mathbb{R}^3$ if and only if $\mathbf{y} \in I$.

Let α have the matrix
$$\begin{pmatrix} 1 & 1 & t \\ 0 & t & -2b \\ 1 & t & 0 \end{pmatrix}$$
with respect to the standard basis, where $b \in \mathbb{R}$ and t is a real variable. Find K and I for α. Hence, or by evaluating the determinant, show that if $0 < b < 2$ and $\mathbf{y} \in I$ then the equation $\alpha\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x} \in \mathbb{R}^3$ for all values of t.

1/II/7B **Algebra and Geometry**

(i) State the orbit-stabilizer theorem for a group G acting on a set X.

(ii) Denote the group of *all* symmetries of the cube by G. Using the orbit-stabilizer theorem, show that G has 48 elements.

Does G have any non-trivial normal subgroups?

Let L denote the line between two diagonally opposite vertices of the cube, and let
$$H = \{g \in G \mid gL = L\}$$
be the subgroup of symmetries that preserve the line. Show that H is isomorphic to the direct product $S_3 \times C_2$, where S_3 is the symmetric group on 3 letters and C_2 is the cyclic group of order 2.

1/II/8D **Algebra and Geometry**

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be non-zero vectors in $\mathbb{R}^n$. What is meant by saying that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent? What is the dimension of the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}$ and $\mathbf{y}$ if they are (1) linearly independent, (2) linearly dependent?

Define the scalar product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Define the corresponding norm $\|\mathbf{x}\|$ of $\mathbf{x} \in \mathbb{R}^n$. State and prove the Cauchy–Schwarz inequality, and deduce the triangle inequality.

By means of a sketch, give a geometric interpretation of the scalar product $\mathbf{x} \cdot \mathbf{y}$ in the case $n = 3$, relating the value of $\mathbf{x} \cdot \mathbf{y}$ to the angle α between the directions of $\mathbf{x}$ and $\mathbf{y}$.

What is a unit vector? Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be unit vectors in $\mathbb{R}^3$. Let

$$S = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u} \,.$$

Show that

(i) for any fixed, linearly independent $\mathbf{u}$ and $\mathbf{v}$, the minimum of S over $\mathbf{w}$ is attained when $\mathbf{w} = \lambda(\mathbf{u} + \mathbf{v})$ for some $\lambda \in \mathbb{R}$;

(ii) $\lambda \leqslant -\frac{1}{2}$ in all cases;

(iii) $\lambda = -1$ and $S = -3/2$ in the case where $\mathbf{u} \cdot \mathbf{v} = \cos(2\pi/3)$.

3/I/1A **Algebra and Geometry**

The mapping α of $\mathbb{R}^3$ into itself is a reflection in the plane $x_2 = x_3$. Find the matrix A of α with respect to any basis of your choice, which should be specified.

The mapping β of $\mathbb{R}^3$ into itself is a rotation about the line $x_1 = x_2 = x_3$ through $2\pi/3$, followed by a dilatation by a factor of 2. Find the matrix B of β with respect to a choice of basis that should again be specified.

Show explicitly that

$$B^3 = 8A^2$$

and explain why this must hold, irrespective of your choices of bases.

3/I/2B **Algebra and Geometry**

Show that if a group G contains a normal subgroup of order 3, and a normal subgroup of order 5, then G contains an element of order 15.

Give an example of a group of order 10 with no element of order 10.

3/II/5E **Algebra and Geometry**

(a) Show, using vector methods, that the distances from the centroid of a tetrahedron to the centres of opposite pairs of edges are equal. If the three distances are u, v, w and if a, b, c, d are the distances from the centroid to the vertices, show that

$$u^2 + v^2 + w^2 = \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2).$$

[*The centroid of k points in $\mathbb{R}^3$ with position vectors $\mathbf{x}_i$ is the point with position vector* $\dfrac{1}{k}\sum \mathbf{x}_i$.]

(b) Show that

$$|\mathbf{x} - \mathbf{a}|^2 \cos^2 \alpha = [(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n}]^2,$$

with $\mathbf{n}^2 = 1$, is the equation of a right circular double cone whose vertex has position vector $\mathbf{a}$, axis of symmetry $\mathbf{n}$ and opening angle α.

Two such double cones, with vertices $\mathbf{a}_1$ and $\mathbf{a}_2$, have parallel axes and the same opening angle. Show that if $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 \neq \mathbf{0}$, then the intersection of the cones lies in a plane with unit normal

$$\mathbf{N} = \frac{\mathbf{b}\cos^2\alpha - \mathbf{n}(\mathbf{n}\cdot\mathbf{b})}{\sqrt{\mathbf{b}^2\cos^4\alpha + (\mathbf{b}\cdot\mathbf{n})^2(1 - 2\cos^2\alpha)}}\,.$$

3/II/6E **Algebra and Geometry**

Derive an expression for the triple scalar product $\big(\mathbf{e}_1 \times \mathbf{e}_2\big) \cdot \mathbf{e}_3$ in terms of the determinant of the matrix E whose rows are given by the components of the three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Use the geometrical interpretation of the cross product to show that $\mathbf{e}_a$, $a = 1, 2, 3$, will be a *not necessarily orthogonal* basis for $\mathbb{R}^3$ as long as $\det E \neq 0$.

The rows of another matrix $\hat{E}$ are given by the components of three other vectors $\hat{\mathbf{e}}_b$, $b = 1, 2, 3$. By considering the matrix $E\hat{E}^{\mathrm{T}}$, where ${}^{\mathrm{T}}$ denotes the transpose, show that there is a unique choice of $\hat{E}$ such that $\hat{\mathbf{e}}_b$ is also a basis and

$$\mathbf{e}_a \cdot \hat{\mathbf{e}}_b = \delta_{ab}.$$

Show that the new basis is given by

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\big(\mathbf{e}_1 \times \mathbf{e}_2\big) \cdot \mathbf{e}_3} \qquad \text{etc.}$$

Show that if either $\mathbf{e}_a$ or $\hat{\mathbf{e}}_b$ is an orthonormal basis then E is a rotation matrix.

3/II/7B **Algebra and Geometry**

Let G be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $X = \{\alpha, \beta, \gamma\}$ be a set of three distinct points in $\mathbb{C} \cup \{\infty\}$.

(i) Show that there exists a $g \in G$ sending α to 0, β to 1, and γ to ∞.

(ii) Hence show that if $H = \{g \in G \mid gX = X\}$, then H is isomorphic to S_3, the symmetric group on 3 letters.

3/II/8B **Algebra and Geometry**

(a) Determine the characteristic polynomial and the eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

Is it diagonalizable?

(b) Show that an $n \times n$ matrix A with characteristic polynomial $f(t) = (t-\mu)^n$ is diagonalizable if and only if $A = \mu I$.

1/I/3B Analysis

Define what it means for a function of a real variable to be *differentiable* at $x \in \mathbb{R}$.

Prove that if a function is differentiable at $x \in \mathbb{R}$, then it is continuous there.

Show directly from the definition that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at 0 with derivative 0.

Show that the derivative $f'(x)$ is not continuous at 0.

1/I/4C Analysis

Explain what is meant by the *radius of convergence* of a power series.

Find the radius of convergence R of each of the following power series:

$$\text{(i)} \quad \sum_{n=1}^{\infty} n^{-2} z^n, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \left(n + \frac{1}{2^n}\right) z^n .$$

In each case, determine whether the series converges on the circle $|z| = R$.

1/II/9F Analysis

Prove the Axiom of Archimedes.

Let x be a real number in $[0,1]$, and let m, n be positive integers. Show that the limit

$$\lim_{m\to\infty} \left[\lim_{n\to\infty} \cos^{2n}(m!\pi x) \right]$$

exists, and that its value depends on whether x is rational or irrational.

[*You may assume standard properties of the cosine function provided they are clearly stated.*]

1/II/10F **Analysis**

State without proof the *Integral Comparison Test* for the convergence of a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms.

Determine for which positive real numbers α the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges.

In each of the following cases determine whether the series is convergent or divergent:

(i) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n}$,

(ii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n \log\, n)\ (\log \log\, n)^2}$,

(iii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{(1+1/n)} \log n}$.

1/II/11B **Analysis**

Let $f : [a, b] \to \mathbb{R}$ be continuous. Define the *integral* $\int_a^b f(x)dx$. (You are not asked to prove existence.)

Suppose that m, M are real numbers such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. Stating clearly any properties of the integral that you require, show that

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)\,.$$

The function $g : [a, b] \to \mathbb{R}$ is continuous and non-negative. Show that

$$m \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M \int_a^b g(x)dx\,.$$

Now let f be continuous on $[0, 1]$. By suitable choice of g show that

$$\lim_{n\to\infty} \int_0^{1/\sqrt{n}} nf(x)e^{-nx}dx = f(0)\ ,$$

and by making an appropriate change of variable, or otherwise, show that

$$\lim_{n\to\infty} \int_0^{1} nf(x)e^{-nx}dx = f(0)\,.$$

1/II/12C **Analysis**

State carefully the formula for integration by parts for functions of a real variable.

Let $f : (-1,1) \to \mathbb{R}$ be infinitely differentiable. Prove that for all $n \geqslant 1$ and all $t \in (-1,1)$,

$$f(t) = f(0) + f'(0)t + \frac{1}{2!}f''(0)t^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(0)t^{n-1} + \frac{1}{(n-1)!}\int_0^t f^{(n)}(x)(t-x)^{n-1}\,dx.$$

By considering the function $f(x) = \log(1-x)$ at $x = 1/2$, or otherwise, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

converges to $\log 2$.

2/I/1D **Differential Equations**

Consider the equation

$$\frac{dy}{dx} \;=\; 1 - y^2 \; . \qquad (*)$$

Using small line segments, sketch the flow directions in $x \geqslant 0$, $-2 \leqslant y \leqslant 2$ implied by the right-hand side of $(*)$. Find the general solution

(i) in $|y| < 1$,

(ii) in $|y| > 1$.

Sketch a solution curve in each of the three regions $y > 1$, $|y| < 1$, and $y < -1$.

2/I/2D **Differential Equations**

Consider the differential equation

$$\frac{dx}{dt} + Kx = 0 \; ,$$

where K is a positive constant. By using the approximate finite-difference formula

$$\frac{dx_n}{dt} \;=\; \frac{x_{n+1} - x_{n-1}}{2\delta t} \; ,$$

where δt is a positive constant, and where x_n denotes the function $x(t)$ evaluated at $t = n\delta t$ for integer n, convert the differential equation to a difference equation for x_n.

Solve both the differential equation and the difference equation for general initial conditions. Identify those solutions of the difference equation that agree with solutions of the differential equation over a finite interval $0 \leqslant t \leqslant T$ in the limit $\delta t \to 0$, and demonstrate the agreement. Demonstrate that the remaining solutions of the difference equation cannot agree with the solution of the differential equation in the same limit.

[*You may use the fact that, for bounded* $|u|$, $\lim_{\epsilon \to 0} (1 + \epsilon u)^{1/\epsilon} \;=\; e^u$.]

2/II/5D **Differential Equations**

(a) Show that if $\mu(x,y)$ is an integrating factor for an equation of the form

$$f(x,y)\,dy \;+\; g(x,y)\,dx \;=\; 0$$

then $\partial(\mu f)/\partial x \;=\; \partial(\mu g)/\partial y\,.$

Consider the equation

$$\cot x\;dy \;-\; \tan y\;dx \;=\; 0$$

in the domain $0 \leqslant x \leqslant \frac{1}{2}\pi\,,\;\; 0 \leqslant y \leqslant \frac{1}{2}\pi\,.$ Using small line segments, sketch the flow directions in that domain. Show that $\sin x\,\cos y$ is an integrating factor for the equation. Find the general solution of the equation, and sketch the family of solutions that occupies the larger domain $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi\,,\;\; -\frac{1}{2}\pi \leqslant y \leqslant \frac{1}{2}\pi\,.$

(b) The following example illustrates that the concept of integrating factor extends to higher-order equations. Multiply the equation

$$\left[y\,\frac{d^2y}{dx^2} \;+\; \left(\frac{dy}{dx}\right)^2 \right] \cos^2 x \;=\; 1$$

by $\sec^2 x$, and show that the result takes the form $\dfrac{d}{dx}h(x,y) = 0$, for some function $h(x,y)$ to be determined. Find a particular solution $y = y(x)$ such that $y(0) = 0$ with dy/dx finite at $x = 0\,,$ and sketch its graph in $0 \leqslant x < \frac{1}{2}\pi\,.$

2/II/6D **Differential Equations**

Define the *Wronskian* $W(x)$ associated with solutions of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y \ = \ 0$$

and show that

$$W(x) \ \propto \ \exp\left(-\int^x p(\xi)\,d\xi\right) \ .$$

Evaluate the expression on the right when $p(x) = -2/x$.

Given that $p(x) = -2/x$ and that $q(x) = -1$, show that solutions in the form of power series,

$$y \ = \ x^\lambda \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0)\,,$$

can be found if and only if $\lambda = 0$ or 3. By constructing and solving the appropriate recurrence relations, find the coefficients a_n for each power series.

You may assume that the equation is satisfied by $y = \cosh x - x\sinh x$ and by $y = \sinh x - x\cosh x$. Verify that these two solutions agree with the two power series found previously, and that they give the $W(x)$ found previously, up to multiplicative constants.

[*Hint:* $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\,,\ \ \sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\,.$]

2/II/7D Differential Equations

Consider the linear system

$$\dot{\mathbf{x}}(t) - A\mathbf{x}(t) = \mathbf{z}(t)$$

where the n-vector $\mathbf{z}(t)$ and the $n \times n$ matrix A are given; A has constant real entries, and has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and n linearly independent eigenvectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$. Find the complementary function. Given a particular integral $\mathbf{x}_{\mathrm{p}}(t)$, write down the general solution. In the case $n = 2$ show that the complementary function is purely oscillatory, with no growth or decay, if and only if

$$\operatorname{trace} A = 0 \qquad \text{and} \qquad \det A > 0 \ .$$

Consider the same case $n = 2$ with $\operatorname{trace} A = 0$ and $\det A > 0$ and with

$$\mathbf{z}(t) = \mathbf{a}_1 \exp(i\omega_1 t) + \mathbf{a}_2 \exp(i\omega_2 t) \,,$$

where $\omega_1, \ \omega_2$ are given real constants. Find a particular integral when

(i) $i\omega_1 \neq \lambda_1$ and $i\omega_2 \neq \lambda_2$;

(ii) $i\omega_1 \neq \lambda_1$ but $i\omega_2 = \lambda_2$.

In the case

$$A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$$

with $\mathbf{z}(t) = \begin{pmatrix} 2 \\ 3i - 1 \end{pmatrix} \exp(3it)$, find the solution subject to the initial condition $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at $t = 0$.

2/II/8D Differential Equations

For all solutions of

$$\begin{aligned} \dot{x} &= \tfrac{1}{2}\alpha x + y - 2y^3, \\ \dot{y} &= -x \end{aligned}$$

show that $dK/dt = \alpha x^2$ where

$$K = K(x, y) = x^2 + y^2 - y^4 \ .$$

In the case $\alpha = 0$, analyse the properties of the critical points and sketch the phase portrait, including the special contours for which $K(x, y) = \frac{1}{4}$. Comment on the asymptotic behaviour, as $t \to \infty$, of solution trajectories that pass near each critical point, indicating whether or not any such solution trajectories approach from, or recede to, infinity.

Briefly discuss how the picture changes when α is made small and positive, using your result for dK/dt to describe, in qualitative terms, how solution trajectories cross K-contours.

4/I/3E **Dynamics**

Because of an accident on launching, a rocket of unladen mass M lies horizontally on the ground. It initially contains fuel of mass m_0, which ignites and is emitted horizontally at a constant rate and at uniform speed u relative to the rocket. The rocket is initially at rest. If the coefficient of friction between the rocket and the ground is μ, and the fuel is completely burnt in a total time T, show that the final speed of the rocket is

$$u \log\left(\frac{M+m_0}{M}\right) - \mu g T.$$

4/I/4E **Dynamics**

Write down an expression for the total momentum $\mathbf{P}$ and angular momentum $\mathbf{L}$ with respect to an origin O of a system of n point particles of masses m_i, position vectors (with respect to O) $\mathbf{x}_i$, and velocities $\mathbf{v}_i$, $i = 1, \ldots, n$.

Show that with respect to a new origin O' the total momentum $\mathbf{P}'$ and total angular momentum $\mathbf{L}'$ are given by

$$\mathbf{P}' = \mathbf{P}, \qquad \mathbf{L}' = \mathbf{L} - \mathbf{b} \times \mathbf{P},$$

and hence

$$\mathbf{L}' \cdot \mathbf{P}' = \mathbf{L} \cdot \mathbf{P},$$

where $\mathbf{b}$ is the constant vector displacement of O' with respect to O. How does $\mathbf{L} \times \mathbf{P}$ change under change of origin?

Hence show that **either**

(1) the total momentum vanishes and the total angular momentum is independent of origin, **or**

(2) by choosing $\mathbf{b}$ in a way that should be specified, the total angular momentum with respect to O' can be made parallel to the total momentum.

4/II/9E **Dynamics**

Write down the equation of motion for a point particle with mass m, charge e, and position vector $\mathbf{x}(t)$ moving in a time-dependent magnetic field $\mathbf{B}(\mathbf{x}, t)$ with vanishing electric field, and show that the kinetic energy of the particle is constant. If the magnetic field is constant in direction, show that the component of velocity in the direction of $\mathbf{B}$ is constant. Show that, in general, the angular momentum of the particle is not conserved.

Suppose that the magnetic field is independent of time and space and takes the form $\mathbf{B} = (0, 0, B)$ and that $\dot{A}$ is the rate of change of area swept out by a radius vector joining the origin to the projection of the particle's path on the (x, y) plane. Obtain the equation

$$\frac{d}{dt}\left(m\dot{A} + \frac{eBr^2}{4}\right) = 0\,, \qquad (*)$$

where (r, θ) are plane polar coordinates. Hence obtain an equation replacing the equation of conservation of angular momentum.

Show further, using energy conservation and $(*)$, that the equations of motion in plane polar coordinates may be reduced to the first order non-linear system

$$\dot{r} = \sqrt{v^2 - \left(\frac{2c}{mr} - \frac{erB}{2m}\right)^2}\,,$$

$$\dot{\theta} = \frac{2c}{mr^2} - \frac{eB}{2m}\,,$$

where v and c are constants.

4/II/10E **Dynamics**

Write down the equations of motion for a system of n gravitating particles with masses m_i, and position vectors $\mathbf{x}_i$, $i = 1, 2, \ldots, n$.

The particles undergo a motion for which $\mathbf{x}_i(t) = a(t)\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are independent of time t. Show that the equations of motion will be satisfied as long as the function $a(t)$ satisfies

$$\ddot{a} = -\frac{\Lambda}{a^2} \; , \qquad (*)$$

where Λ is a constant and the vectors $\mathbf{a}_i$ satisfy

$$\Lambda m_i \mathbf{a}_i = \mathbf{G}_i = \sum_{j \neq i} \frac{G m_i m_j \big(\mathbf{a}_i - \mathbf{a}_j\big)}{|\mathbf{a}_i - \mathbf{a}_j|^3} \; . \qquad (**)$$

Show that $(*)$ has as first integral

$$\frac{\dot{a}^2}{2} - \frac{\Lambda}{a} = \frac{k}{2} \; ,$$

where k is another constant. Show that

$$\mathbf{G}_i = \boldsymbol{\nabla}_i W \; ,$$

where $\boldsymbol{\nabla}_i$ is the gradient operator with respect to $\mathbf{a}_i$ and

$$W = -\sum_i \sum_{j<i} \frac{G m_i m_j}{|\mathbf{a}_i - \mathbf{a}_j|} \; .$$

Using Euler's theorem for homogeneous functions (see below), or otherwise, deduce that

$$\sum_i \mathbf{a}_i \cdot \mathbf{G}_i = -W \; .$$

Hence show that all solutions of $(**)$ satisfy

$$\Lambda I = -W$$

where

$$I = \sum_i m_i \mathbf{a}_i^2 \; .$$

Deduce that Λ must be positive and that the total kinetic energy plus potential energy of the system of particles is equal to $\frac{k}{2} I$.

[*Euler's theorem states that if*

$$f(\lambda x, \lambda y, \lambda z, \ldots) = \lambda^p f(x, y, z, \ldots) \; ,$$

then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} + \ldots = p f \; .]$$

4/II/11E **Dynamics**

State the parallel axis theorem and use it to calculate the moment of inertia of a uniform hemisphere of mass m and radius a about an axis through its centre of mass and parallel to the base.

[*You may assume that the centre of mass is located at a distance* $\frac{3}{8}a$ *from the flat face of the hemisphere, and that the moment of inertia of a full sphere about its centre is* $\frac{2}{5}Ma^2$, *with* $M = 2m$.]

The hemisphere initially rests on a rough horizontal plane with its base vertical. It is then released from rest and subsequently rolls on the plane without slipping. Let θ be the angle that the base makes with the horizontal at time t. Express the instantaneous speed of the centre of mass in terms of b and the rate of change of θ, where b is the instantaneous distance from the centre of mass to the point of contact with the plane. Hence write down expressions for the kinetic energy and potential energy of the hemisphere and deduce that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{15g\cos\theta}{(28 - 15\cos\theta)a}\,.$$

4/II/12E **Dynamics**

Let (r,θ) be plane polar coordinates and $\mathbf{e}_r$ and $\mathbf{e}_\theta$ unit vectors in the direction of increasing r and θ respectively. Show that the velocity of a particle moving in the plane with polar coordinates $\big(r(t),\ \theta(t)\big)$ is given by

$$\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\,,$$

and that the unit normal $\mathbf{n}$ to the particle path is parallel to

$$r\dot{\theta}\mathbf{e}_r - \dot{r}\mathbf{e}_\theta\,.$$

Deduce that the perpendicular distance p from the origin to the tangent of the curve $r = r(\theta)$ is given by

$$\frac{r^2}{p^2} = 1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2.$$

The particle, whose mass is m, moves under the influence of a central force with potential $V(r)$. Use the conservation of energy E and angular momentum h to obtain the equation

$$\frac{1}{p^2} = \frac{2m\big(E - V(r)\big)}{h^2}\,.$$

Hence express θ as a function of r as the integral

$$\theta = \int \frac{hr^{-2}dr}{\sqrt{2m\big(E - V_{\text{eff}}(r)\big)}}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{h^2}{2mr^2}\,.$$

Evaluate the integral and describe the orbit when $V(r) = \dfrac{c}{r^2}$, with c a positive constant.

4/I/1C **Numbers and Sets**

(i) Prove by induction or otherwise that for every $n \geqslant 1$,

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2 .$$

(ii) Show that the sum of the first n positive cubes is divisible by 4 if and only if $n \equiv 0$ or 3 (mod 4).

4/I/2C **Numbers and Sets**

What is an *equivalence relation*? For each of the following pairs $(X, \sim)$, determine whether or not $\sim$ is an equivalence relation on X:

(i) $X = \mathbb{R}$, $x \sim y$ iff $x - y$ is an even integer;

(ii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{R}$;

(iii) $X = \mathbb{C} \setminus \{0\}$, $x \sim y$ iff $x\bar{y} \in \mathbb{Z}$;

(iv) $X = \mathbb{Z} \setminus \{0\}$, $x \sim y$ iff $x^2 - y^2$ is ± 1 times a perfect square.

4/II/5C **Numbers and Sets**

Define what is meant by the term *countable*. Show directly from your definition that if X is countable, then so is any subset of X.

Show that $\mathbb{N} \times \mathbb{N}$ is countable. Hence or otherwise, show that a countable union of countable sets is countable. Show also that for any $n \geqslant 1$, $\mathbb{N}^n$ is countable.

A function $f : \mathbb{Z} \to \mathbb{N}$ is *periodic* if there exists a positive integer m such that, for every $x \in \mathbb{Z}$, $f(x + m) = f(x)$. Show that the set of periodic functions $f : \mathbb{Z} \to \mathbb{N}$ is countable.

4/II/6C Numbers and Sets

(i) Prove Wilson's theorem: if p is prime then $(p-1)! \equiv -1 \pmod p$.

Deduce that if $p \equiv 1 \pmod 4$ then

$$\left(\left(\frac{p-1}{2}\right)!\right)^2 \equiv -1 \pmod p.$$

(ii) Suppose that p is a prime of the form $4k+3$. Show that if $x^4 \equiv 1 \pmod p$ then $x^2 \equiv 1 \pmod p$.

(iii) Deduce that if p is an odd prime, then the congruence

$$x^2 \equiv -1 \pmod p$$

has exactly two solutions (modulo p) if $p \equiv 1 \pmod 4$, and none otherwise.

4/II/7C Numbers and Sets

Let m, n be integers. Explain what is their *greatest common divisor* (m,n). Show from your definition that, for any integer k, $(m,n) = (m+kn,n)$.

State Bezout's theorem, and use it to show that if p is prime and p divides mn, then p divides at least one of m and n.

The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, … is defined by $x_0 = 0$, $x_1 = 1$ and $x_{n+1} = x_n + x_{n-1}$ for $n \geqslant 1$. Prove:

(i) $(x_{n+1}, x_n) = 1$ and $(x_{n+2}, x_n) = 1$ for every $n \geqslant 0$;

(ii) $x_{n+3} \equiv x_n \pmod 2$ and $x_{n+8} \equiv x_n \pmod 3$ for every $n \geqslant 0$;

(iii) if $n \equiv 0 \pmod 5$ then $x_n \equiv 0 \pmod 5$.

4/II/8C Numbers and Sets

Let X be a finite set with n elements. How many functions are there from X to X? How many relations are there on X?

Show that the number of relations R on X such that, for each $y \in X$, there exists at least one $x \in X$ with xRy, is $(2^n - 1)^n$.

Using the inclusion–exclusion principle or otherwise, deduce that the number of such relations R for which, in addition, for each $x \in X$, there exists at least one $y \in X$ with xRy, is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^n.$$

2/I/3F **Probability**

(a) Define the *probability generating function* of a random variable. Calculate the probability generating function of a binomial random variable with parameters n and p, and use it to find the mean and variance of the random variable.

(b) X is a binomial random variable with parameters n and p, Y is a binomial random variable with parameters m and p, and X and Y are independent. Find the distribution of $X+Y$; that is, determine $P\{X+Y=k\}$ for all possible values of k.

2/I/4F **Probability**

The random variable X is uniformly distributed on the interval $[0,1]$. Find the distribution function and the probability density function of Y, where

$$Y=\frac{3X}{1-X}\,.$$

2/II/9F **Probability**

State the inclusion-exclusion formula for the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs.

After a party the n guests take coats randomly from a pile of their n coats. Calculate the probability that no-one goes home with the correct coat.

Let $p(m,n)$ be the probability that exactly m guests go home with the correct coats. By relating $p(m,n)$ to $p(0,n-m)$, or otherwise, determine $p(m,n)$ and deduce that

$$\lim_{n\to\infty} p(m,n)=\frac{1}{em!}\,.$$

2/II/10F **Probability**

The random variables X and Y each take values in $\{0,1\}$, and their joint distribution $p(x,y)=P\{X=x,\ Y=y\}$ is given by

$$p(0,0)=a, \qquad p(0,1)=b, \qquad p(1,0)=c, \qquad p(1,1)=d\,.$$

Find necessary and sufficient conditions for X and Y to be

(i) uncorrelated;

(ii) independent.

Are the conditions established in (i) and (ii) equivalent?

2/II/11F Probability

A laboratory keeps a population of aphids. The probability of an aphid passing a day uneventfully is $q < 1$. Given that a day is not uneventful, there is probability r that the aphid will have one offspring, probability s that it will have two offspring and probability t that it will die, where $r + s + t = 1$. Offspring are ready to reproduce the next day. The fates of different aphids are independent, as are the events of different days. The laboratory starts out with one aphid.

Let X_1 be the number of aphids at the end of the first day. What is the expected value of X_1? Determine an expression for the probability generating function of X_1.

Show that the probability of extinction does not depend on q, and that if $2r+3s \leqslant 1$ then the aphids will certainly die out. Find the probability of extinction if $r = 1/5,\ s = 2/5$ and $t = 2/5$.

[*Standard results on branching processes may be used without proof, provided that they are clearly stated.*]

2/II/12F Probability

Planet Zog is a ball with centre O. Three spaceships A, B and C land at random on its surface, their positions being independent and each uniformly distributed on its surface. Calculate the probability density function of the angle $\angle AOB$ formed by the lines OA and OB.

Spaceships A and B can communicate directly by radio if $\angle AOB < \pi/2$, and similarly for spaceships B and C and spaceships A and C. Given angle $\angle AOB = \gamma < \pi/2$, calculate the probability that C can communicate directly with *either* A or B. Given angle $\angle AOB = \gamma > \pi/2$, calculate the probability that C can communicate directly with *both* A and B. Hence, or otherwise, show that the probability that all three spaceships can keep in in touch (with, for example, A communicating with B via C if necessary) is $(\pi + 2)/(4\pi)$.

3/I/3A Vector Calculus

Sketch the curve $y^2 = x^2 + 1$. By finding a parametric representation, or otherwise, determine the points on the curve where the radius of curvature is least, and compute its value there.

[*Hint: you may use the fact that the radius of curvature of a parametrized curve* $(x(t), y(t))$ *is* $(\dot{x}^2 + \dot{y}^2)^{3/2}/|\dot{x}\ddot{y} - \ddot{x}\dot{y}|$.]

3/I/4A Vector Calculus

Suppose V is a region in $\mathbb{R}^3$, bounded by a piecewise smooth closed surface S, and $\phi(\mathbf{x})$ is a scalar field satisfying

$$\begin{aligned} \boldsymbol{\nabla}^2\phi &= 0 \quad \text{in } V, \\ \text{and} \quad \phi &= f(\mathbf{x}) \quad \text{on } S\,. \end{aligned}$$

Prove that ϕ is determined uniquely in V.

How does the situation change if the normal derivative of ϕ rather than ϕ itself is specified on S?

3/II/9A Vector Calculus

Let C be the closed curve that is the boundary of the triangle T with vertices at the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Specify a direction along C and consider the integral

$$\oint_C \mathbf{A} \cdot d\mathbf{x}\ ,$$

where $\mathbf{A} = (z^2 - y^2,\ x^2 - z^2,\ y^2 - x^2)$. Explain why the contribution to the integral is the same from each edge of C, and evaluate the integral.

State Stokes's theorem and use it to evaluate the surface integral

$$\int_T (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S}\ ,$$

the components of the normal to T being positive.

Show that $d\mathbf{S}$ in the above surface integral can be written in the form $(1,1,1)\,dy\,dz$. Use this to verify your result by a direct calculation of the surface integral.

3/II/10A **Vector Calculus**

Write down an expression for the Jacobian J of a transformation

$$(x, y, z) \to (u, v, w)\,.$$

Use it to show that

$$\int_D f\ dx\,dy\,dz = \int_\Delta \phi\ |J|\ du\,dv\,dw$$

where D is mapped one-to-one onto Δ, and

$$\phi(u, v, w) \ = \ f\big(x(u, v, w),\ y(u, v, w),\ z(u, v, w)\big)\,.$$

Find a transformation that maps the ellipsoid D,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\,,$$

onto a sphere. Hence evaluate

$$\int_D x^2\,dx\,dy\,dz\,.$$

3/II/11A **Vector Calculus**

(a) Prove the identity

$$\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G}) \ = \ (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G} \ + \ (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} \ + \ \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G}) \ + \ \mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F})\,.$$

(b) If $\mathbf{E}$ is an irrotational vector field ($\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$ everywhere), prove that there exists a scalar potential $\phi(\mathbf{x})$ such that $\mathbf{E} = -\boldsymbol{\nabla}\phi$.

Show that

$$(2xy^2ze^{-x^2z},\ -2ye^{-x^2z},\ x^2y^2e^{-x^2z})$$

is irrotational, and determine the corresponding potential ϕ.

3/II/12A **Vector Calculus**

State the divergence theorem. By applying this to $f(\mathbf{x})\mathbf{k}$, where $f(\mathbf{x})$ is a scalar field in a closed region V in $\mathbb{R}^3$ bounded by a piecewise smooth surface S, and $\mathbf{k}$ an arbitrary constant vector, show that

$$\int_V \boldsymbol{\nabla} f \, dV = \int_S f \, d\mathbf{S} \,. \qquad (*)$$

A vector field $\mathbf{G}$ satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \rho(\mathbf{x})$$
$$\text{with} \quad \rho(\mathbf{x}) = \begin{cases} \rho_0 & |\mathbf{x}| \leqslant a \\ 0 & |\mathbf{x}| > a \,. \end{cases}$$

By applying the divergence theorem to $\displaystyle\int_V \boldsymbol{\nabla} \cdot \mathbf{G} \, dV$, prove Gauss's law

$$\int_S \mathbf{G} \cdot d\mathbf{S} = \int_V \rho(\mathbf{x}) \, dV,$$

where S is the piecewise smooth surface bounding the volume V.

Consider the spherically symmetric solution

$$\mathbf{G}(\mathbf{x}) = G(r) \, \frac{\mathbf{x}}{r} \ ,$$

where $r = |\mathbf{x}|$. By using Gauss's law with S a sphere of radius r, centre $\mathbf{0}$, in the two cases $0 < r \leqslant a$ and $r > a$, show that

$$\mathbf{G}(\mathbf{x}) = \begin{cases} \dfrac{\rho_0}{3} \mathbf{x} & r \leqslant a \\ \dfrac{\rho_0}{3} \left(\dfrac{a}{r}\right)^3 \mathbf{x} & r > a \,. \end{cases}$$

The scalar field $f(\mathbf{x})$ satisfies $\mathbf{G} = \boldsymbol{\nabla} f$. Assuming that $f \to 0$ as $r \to \infty$, and that f is continuous at $r = a$, find f everywhere.

By using a symmetry argument, explain why $(*)$ is clearly satisfied for this f if S is any sphere centred at the origin.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 27th May 2004 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **A**, **B**, **C**, **D**, **E** *and* **F** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Algebra and Geometry

The linear map $H : \mathbb{R}^3 \to \mathbb{R}^3$ represents reflection in the plane through the origin with normal $\mathbf{n}$, where $|\mathbf{n}| = 1$, and $\mathbf{n} = (n_1, n_2, n_3)$ referred to the standard basis. The map is given by $\mathbf{x} \mapsto \mathbf{x}' = \mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is a (3×3) matrix.

Show that

$$M_{ij} = \delta_{ij} - 2n_i n_j \,.$$

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors such that $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ is an orthonormal set. Show that

$$\mathbf{Mn} = -\mathbf{n}, \quad \mathbf{Mu} = \mathbf{u}, \quad \mathbf{Mv} = \mathbf{v}\,,$$

and find the matrix $\mathbf{N}$ which gives the mapping relative to the basis $(\mathbf{u}, \mathbf{v}, \mathbf{n})$.

2C Algebra and Geometry

Show that

$$\sum_{i=1}^{n} a_i b_i \leqslant \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2}$$

for any real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. Using this inequality, show that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of unit length in $\mathbb{R}^n$ then $|\mathbf{a} \cdot \mathbf{b}| \leqslant 1$.

3D Analysis

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

State the Least Upper Bound Axiom for the real numbers.

Starting from the Least Upper Bound Axiom, show that if (a_n) is a bounded monotonic sequence of real numbers, then it converges.

4E Analysis

Let $f(x) = (1+x)^{1/2}$ for $x \in (-1,1)$. Show by induction or otherwise that for every integer $r \geq 1$,

$$f^{(r)}(x) = (-1)^{r-1} \frac{(2r-2)!}{2^{2r-1}(r-1)!}(1+x)^{\frac{1}{2}-r}.$$

Evaluate the series
$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-2)!}{8^r r!(r-1)!}.$$

[You may use Taylor's Theorem in the form

$$f(x) = f(0) + \sum_{r=1}^{n} \frac{f^{(r)}(0)}{r!} x^r + \int_0^x \frac{(x-t)^n f^{(n+1)}(t)}{n!}\, dt$$

without proof.]

SECTION II

5B Algebra and Geometry

The vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfies the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,,$$

where $\mathbf{A}$ is a (3×3) matrix and $\mathbf{b}$ is a (3×1) column vector. State the conditions under which this equation has (a) a unique solution, (b) an infinity of solutions, (c) no solution for $\mathbf{x}$.

Find all possible solutions for the unknowns x, y and z which satisfy the following equations:

$$\begin{aligned} x + y + z &= 1 \\ x + y + \lambda z &= 2 \\ x + 2y + \lambda z &= 4\,, \end{aligned}$$

in the cases (a) $\lambda = 0$, and (b) $\lambda = 1$.

6A Algebra and Geometry

Express the product $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ in suffix notation and thence prove that the result is zero.

Silver Beard the space pirate believed people relied so much on space-age navigation techniques that he could safely write down the location of his treasure using the ancient art of vector algebra. Spikey the space jockey thought he could follow the instructions, by moving by the sequence of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ one stage at a time. The vectors (expressed in 1000 parsec units) were defined as follows:

1. Start at the centre of the galaxy, which has coordinates $(0, 0, 0)$.

2. Vector $\mathbf{a}$ has length $\sqrt{3}$, is normal to the plane $x + y + z = 1$ and is directed into the positive quadrant.

3. Vector $\mathbf{b}$ is given by $\mathbf{b} = (\mathbf{a} \cdot \mathbf{m})\mathbf{a} \times \mathbf{m}$, where $\mathbf{m} = (2, 0, 1)$.

4. Vector $\mathbf{c}$ has length $2\sqrt{2}$, is normal to $\mathbf{a}$ and $\mathbf{b}$, and moves you closer to the x axis.

5. Vector $\mathbf{d} = (1, -2, 2)$.

6. Vector $\mathbf{e}$ has length $\mathbf{a} \cdot \mathbf{b}$. Spikey was initially a little confused with this one, but then realised the orientation of the vector did not matter.

7. Vector $\mathbf{f}$ has unknown length but is parallel to $\mathbf{m}$ and takes you to the treasure located somewhere on the plane $2x - y + 4z = 10$.

Determine the location of the way-points Spikey will use and thence the location of the treasure.

7A Algebra and Geometry

Simplify the fraction

$$\zeta = \cfrac{1}{\bar{z} + \cfrac{1}{z + \cfrac{1}{\bar{z}}}},$$

where $\bar{z}$ is the complex conjugate of z. Determine the geometric form that satisfies

$$\mathrm{Re}(\zeta) = \mathrm{Re}\left(\frac{z + \frac{1}{4}}{|z|^2}\right).$$

Find solutions to

$$\mathrm{Im}(\log z) = \frac{\pi}{3}$$

and

$$z^2 = x^2 - y^2 + 2ix,$$

where $z = x + iy$ is a complex variable. Sketch these solutions in the complex plane and describe the region they enclose. Derive complex equations for the circumscribed and inscribed circles for the region. [The circumscribed circle is the circle that passes through the vertices of the region and the inscribed circle is the largest circle that fits within the region.]

8C Algebra and Geometry

(i) The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfy $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3 \neq 0$. Are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(ii) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$ have the property that every subset comprising $(n-1)$ of the vectors is linearly independent. Are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(iii) For each value of t in the range $0 \leqslant t < 1$, give a construction of a linearly independent set of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfying

$$\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij} + t(1 - \delta_{ij}),$$

where δ_{ij} is the Kronecker delta.

9D Analysis

i) State Rolle's theorem.

Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions which are differentiable on (a,b).

ii) Prove that for some $c \in (a,b)$,

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

iii) Suppose that $f(a) = g(a) = 0$, and that $\lim_{x\to a+} \frac{f'(x)}{g'(x)}$ exists and is equal to L. Prove that $\lim_{x\to a+} \frac{f(x)}{g(x)}$ exists and is also equal to L.

[You may assume there exists a $\delta > 0$ such that, for all $x \in (a, a+\delta)$, $g'(x) \neq 0$ and $g(x) \neq 0$.]

iv) Evaluate $\lim_{x\to 0} \frac{\log\cos x}{x^2}$.

10E Analysis

Define, for an integer $n \geq 0$,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

Show that for every $n \geq 2$, $nI_n = (n-1)I_{n-2}$, and deduce that

$$I_{2n} = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}.$$

Show that $0 < I_n < I_{n-1}$, and that

$$\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1.$$

Hence prove that

$$\lim_{n\to\infty} \frac{2^{4n+1}(n!)^4}{(2n+1)(2n)!^2} = \pi.$$

11F Analysis

Let f be defined on $\mathbb{R}$, and assume that there exists at least one point $x_0 \in \mathbb{R}$ at which f is continuous. Suppose also that, for every $x, y \in \mathbb{R}$, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Show that f is continuous on $\mathbb{R}$.

Show that there exists a constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Suppose that g is a continuous function defined on $\mathbb{R}$ and that, for every $x, y \in \mathbb{R}$, g satisfies the equation

$$g(x+y) \;=\; g(x)g(y).$$

Show that if g is not identically zero, then g is everywhere positive. Find the general form of g.

12F Analysis

(i) Show that if $a_n > 0$, $b_n > 0$ and

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$$

for all $n \geqslant 1$, and if $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(ii) Let

$$c_n = \binom{2n}{n} 4^{-n}.$$

By considering $\log c_n$, or otherwise, show that $c_n \to 0$ as $n \to \infty$.

[*Hint*: $\log(1-x) \leqslant -x$ for $x \in (0,1)$.]

(iii) Determine the convergence or otherwise of

$$\sum_{n=1}^{\infty} \binom{2n}{n} x^n$$

for (a) $x = \frac{1}{4}$, (b) $x = -\frac{1}{4}$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday 28th May 2004 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **B** *and* **F** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

By writing $y(x) = mx$ where m is a constant, solve the differential equation

$$\frac{dy}{dx} = \frac{x - 2y}{2x + y}$$

and find the possible values of m.

Describe the isoclines of this differential equation and sketch the flow vectors. Use these to sketch at least two characteristically different solution curves.

Now, by making the substitution $y(x) = xu(x)$ or otherwise, find the solution of the differential equation which satisfies $y(0) = 1$.

2B Differential Equations

Find two linearly independent solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2 y = 0\,.$$

Find also the solution of

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2 y = e^{-px}$$

that satisfies

$$y = 0,\ \frac{dy}{dx} = 0 \quad \text{at } x = 0\,.$$

3F Probability

Define the covariance, $\operatorname{cov}(X, Y)$, of two random variables X and Y.

Prove, or give a counterexample to, each of the following statements.

(a) For any random variables X, Y, Z

$$\operatorname{cov}(X + Y, Z) = \operatorname{cov}(X, Z) + \operatorname{cov}(Y, Z).$$

(b) If X and Y are identically distributed, not necessarily independent, random variables then

$$\operatorname{cov}(X + Y, X - Y) = 0.$$

4F Probability

The random variable X has probability density function

$$f(x) = \begin{cases} cx(1-x) & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

Determine c, and the mean and variance of X.

SECTION II

5B Differential Equations

Construct a series solution $y = y_1(x)$ valid in the neighbourhood of $x = 0$, for the differential equation

$$\frac{d^2y}{dx^2} + 4x^3\frac{dy}{dx} + x^2y = 0\,,$$

satisfying

$$y_1 = 1,\ \frac{dy_1}{dx} = 0 \quad \text{at } x = 0\,.$$

Find also a second solution $y = y_2(x)$ which satisfies

$$y_2 = 0,\ \frac{dy_2}{dx} = 1 \quad \text{at } x = 0\,.$$

Obtain an expression for the Wronskian of these two solutions and show that

$$y_2(x) = y_1(x)\int_0^x \frac{e^{-\xi^4}}{y_1^2(\xi)}\,d\xi\,.$$

6B Differential Equations

Two solutions of the recurrence relation

$$x_{n+2} + b(n)x_{n+1} + c(n)x_n = 0$$

are given as p_n and q_n, and their Wronskian is defined to be

$$W_n = p_nq_{n+1} - p_{n+1}q_n\,.$$

Show that

$$W_{n+1} = W_1\prod_{m=1}^{n} c(m)\,. \qquad (*)$$

Suppose that $b(n) = \alpha$, where α is a real constant lying in the range $[-2, 2]$, and that $c(n) = 1$. Show that two solutions are $x_n = e^{in\theta}$ and $x_n = e^{-in\theta}$, where $\cos\theta = -\alpha/2$. Evaluate the Wronskian of these two solutions and verify $(*)$.

7B Differential Equations

Show how a second-order differential equation $\ddot{x} = f(x, \dot{x})$ may be transformed into a pair of coupled first-order equations. Explain what is meant by a *critical point* on the phase diagram for a pair of first-order equations. Hence find the critical points of the following equations. Describe their stability type, sketching their behaviour near the critical points on a phase diagram.

(i) $\ddot{x} + \cos x = 0$

(ii) $\ddot{x} + x(x^2 + \lambda x + 1) = 0, \quad \text{for } \lambda = 1,\ 5/2\,.$

Sketch the phase portraits of these equations marking clearly the direction of flow.

8B Differential Equations

Construct the general solution of the system of equations

$$\dot{x} + 4x + 3y = 0$$
$$\dot{y} + 4y - 3x = 0$$

in the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{x} = \sum_{j=1}^{2} a_j \mathbf{x}^{(j)} e^{\lambda_j t}$$

and find the eigenvectors $\mathbf{x}^{(j)}$ and eigenvalues λ_j.

Explain what is meant by resonance in a forced system of linear differential equations.

Consider the forced system

$$\dot{x} + 4x + 3y = \sum_{j=1}^{2} p_j e^{\lambda_j t}$$
$$\dot{y} + 4y - 3x = \sum_{j=1}^{2} q_j e^{\lambda_j t}\,.$$

Find conditions on p_j and q_j $(j = 1, 2)$ such that there is no resonant response to the forcing.

9F Probability

Let X be a positive-integer valued random variable. Define its *probability generating function* p_X. Show that if X and Y are independent positive-integer valued random variables, then $p_{X+Y} = p_X p_Y$.

A non-standard pair of dice is a pair of six-sided unbiased dice whose faces are numbered with strictly positive integers in a non-standard way (for example, $(2,2,2,3,5,7)$ and $(1,1,5,6,7,8)$). Show that there exists a non-standard pair of dice A and B such that when thrown

$$P\{\text{total shown by } A \text{ and } B \text{ is } n\} = P\{\text{total shown by pair of ordinary dice is } n\}$$

for all $2 \leqslant n \leqslant 12$.

[*Hint:* $(x+x^2+x^3+x^4+x^5+x^6) = x(1+x)(1+x^2+x^4) = x(1+x+x^2)(1+x^3)$.]

10F Probability

Define the *conditional probability* $P(A \mid B)$ of the event A given the event B.

A bag contains four coins, each of which when tossed is equally likely to land on either of its two faces. One of the coins shows a head on each of its two sides, while each of the other three coins shows a head on only one side. A coin is chosen at random, and tossed three times in succession. If heads turn up each time, what is the probability that if the coin is tossed once more it will turn up heads again? Describe the sample space you use and explain carefully your calculations.

11F Probability

The random variables X_1 and X_2 are independent, and each has an exponential distribution with parameter λ. Find the joint density function of

$$Y_1 = X_1 + X_2\,, \quad Y_2 = X_1/X_2\,,$$

and show that Y_1 and Y_2 are independent. What is the density of Y_2?

12F Probability

Let $A_1, A_2, \ldots, A_r$ be events such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Show that the number N of events that occur satisfies

$$P(N=0) \quad = \quad 1 - \sum_{i=1}^{r} P(A_i)\,.$$

Planet Zog is a sphere with centre O. A number N of spaceships land at random on its surface, their positions being independent, each uniformly distributed over the surface. A spaceship at A is in direct radio contact with another point B on the surface if $\angle AOB < \frac{\pi}{2}$. Calculate the probability that every point on the surface of the planet is in direct radio contact with at least one of the N spaceships.

[*Hint*: The intersection of the surface of a sphere with a plane through the centre of the sphere is called a *great circle*. You may find it helpful to use the fact that N random great circles partition the surface of a sphere into $N(N-1)+2$ disjoint regions with probability one.]

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 1st June 2004 1.30 to 4.30

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **C** *and* **D** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

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Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Algebra and Geometry

State Lagrange's Theorem.

Show that there are precisely two non-isomorphic groups of order 10. [You may assume that a group whose elements are all of order 1 or 2 has order 2^k.]

2D Algebra and Geometry

Define the Möbius group, and describe how it acts on $\mathbb{C} \cup \{\infty\}$.

Show that the subgroup of the Möbius group consisting of transformations which fix 0 and ∞ is isomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Now show that the subgroup of the Möbius group consisting of transformations which fix 0 and 1 is also isomorphic to $\mathbb{C}^*$.

3C Vector Calculus

If $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields, show that

(i) $\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - \mathbf{G}(\boldsymbol{\nabla} \cdot \mathbf{F}) + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}\,,$

(ii) $\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} + \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times \mathbf{F})\,.$

4C Vector Calculus

Define the curvature, κ, of a curve in $\mathbb{R}^3$.

The curve C is parametrised by

$$\mathbf{x}(t) = \left(\frac{1}{2}e^t \cos t,\ \frac{1}{2}e^t \sin t,\ \frac{1}{\sqrt{2}}e^t\right) \quad \text{for } -\infty < t < \infty\,.$$

Obtain a parametrisation of the curve in terms of its arc length, s, measured from the origin. Hence obtain its curvature, $\kappa(s)$, as a function of s.

SECTION II

5D Algebra and Geometry

Let $G = \langle g, h \mid h^2 = 1, g^6 = 1, hgh^{-1} = g^{-1} \rangle$ be the dihedral group of order 12.

i) List all the subgroups of G of order 2. Which of them are normal?

ii) Now list all the remaining proper subgroups of G. [There are 6+3 of them.]

iii) For each proper normal subgroup N of G, describe the quotient group G/N.

iv) Show that G is not isomorphic to the alternating group A_4.

6D Algebra and Geometry

State the conditions on a matrix A that ensure it represents a rotation of $\mathbb{R}^3$ with respect to the standard basis.

Check that the matrix

$$A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$

represents a rotation. Find its axis of rotation $\mathbf{n}$.

Let Π be the plane perpendicular to the axis $\mathbf{n}$. The matrix A induces a rotation of Π by an angle θ. Find $\cos\theta$.

7D Algebra and Geometry

Let A be a real symmetric matrix. Show that all the eigenvalues of A are real, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Give an example of a non-zero *complex* (2×2) symmetric matrix whose only eigenvalues are zero. Is it diagonalisable?

8D Algebra and Geometry

Compute the characteristic polynomial of

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4-s & 2s-2 \\ 0 & -2s+2 & 4s-1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A for all values of s.

For which values of s is A diagonalisable?

9C Vector Calculus

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ state if the following implications are true or false. (No justification is required.)

(i) f is differentiable $\Rightarrow$ f is continuous.

(ii) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist $\Rightarrow$ f is continuous.

(iii) directional derivatives $\dfrac{\partial f}{\partial \mathbf{n}}$ exist for all unit vectors $\mathbf{n} \in \mathbb{R}^2 \Rightarrow f$ is differentiable.

(iv) f is differentiable $\Rightarrow \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous.

(v) all second order partial derivatives of f exist $\Rightarrow \dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$.

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that f is continuous at $(0,0)$ and find the partial derivatives $\dfrac{\partial f}{\partial x}(0,y)$ and $\dfrac{\partial f}{\partial y}(x,0)$. Then show that f is differentiable at $(0,0)$ and find its derivative. Investigate whether the second order partial derivatives $\dfrac{\partial^2 f}{\partial x\, \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\, \partial x}(0,0)$ are the same. Are the second order partial derivatives of f at $(0,0)$ continuous? Justify your answer.

10C Vector Calculus

Explain what is meant by an exact differential. The three-dimensional vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = \left(e^x z^3 + 3x^2(e^y - e^z),\, e^y(x^3 - z^3),\, 3z^2(e^x - e^y) - e^z x^3\right).$$

Find the most general function that has $\mathbf{F} \cdot \mathbf{dx}$ as its differential.

Hence show that the line integral

$$\int_{P_1}^{P_2} \mathbf{F} \cdot \mathbf{dx}$$

along any path in $\mathbb{R}^3$ between points $P_1 = (0, a, 0)$ and $P_2 = (b, b, b)$ vanishes for any values of a and b.

The two-dimensional vector field $\mathbf{G}$ is defined at all points in $\mathbb{R}^2$ except $(0,0)$ by

$$\mathbf{G} = \left(\frac{-y}{x^2+y^2}\,,\; \frac{x}{x^2+y^2}\right).$$

($\mathbf{G}$ is not defined at $(0,0)$.) Show that

$$\oint_C \mathbf{G} \cdot \mathbf{dx} = 2\pi$$

for any closed curve C in $\mathbb{R}^2$ that goes around $(0,0)$ anticlockwise precisely once without passing through $(0,0)$.

11C Vector Calculus

Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0,0,0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2},0,0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(\frac{-1}{4},0,0)$. The eccentrically shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1. Is there a point in the interior of S_3 where a test particle would remain stably at rest? Justify your answer.

12C Vector Calculus

State (without proof) the divergence theorem for a vector field $\mathbf{F}$ with continuous first-order partial derivatives throughout a volume V enclosed by a bounded oriented piecewise-smooth non-self-intersecting surface S.

By calculating the relevant volume and surface integrals explicitly, verify the divergence theorem for the vector field

$$\mathbf{F} = \left(x^3 + 2xy^2,\, y^3 + 2yz^2,\, z^3 + 2zx^2\right),$$

defined within a sphere of radius R centred at the origin.

Suppose that functions ϕ, ψ are continuous and that their first and second partial derivatives are all also continuous in a region V bounded by a smooth surface S.

Show that

$$(1) \qquad \int_V (\phi\nabla^2\psi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi)\,d\tau = \int_S \phi\boldsymbol{\nabla}\psi\cdot\mathbf{dS}\,.$$

$$(2) \qquad \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d\tau = \int_S \phi\boldsymbol{\nabla}\psi\cdot\mathbf{dS} - \int_S \psi\boldsymbol{\nabla}\phi\cdot\mathbf{dS}\,.$$

Hence show that if $\rho(\mathbf{x})$ is a continuous function on V and $g(\mathbf{x})$ a continuous function on S and ϕ_1 and ϕ_2 are two continuous functions such that

$$\nabla^2\phi_1(\mathbf{x}) = \nabla^2\phi_2(\mathbf{x}) = \rho(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ in } V\text{, and}$$
$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ on } S,$$

then $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x}$ in V.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday 31st May 2004 9 to 12

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I.

In Section I, you may attempt **all four** *questions.*

In Section II, at most **five** *answers will be taken into account and no more than* **three** *answers on each course will be taken into account.*

Additional credit will be awarded for substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked **A** *and* **E** *according to the code letter affixed to each question. Include in the same bundle questions from Sections I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

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You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

(a) Use Euclid's algorithm to find positive integers m, n such that $79m - 100n = 1$.

(b) Determine all integer solutions of the congruence

$$237x \equiv 21 \pmod{300}.$$

(c) Find the set of all integers x satisfying the simultaneous congruences

$$x \equiv 8 \pmod{79}$$
$$x \equiv 11 \pmod{100}.$$

2E Numbers and Sets

Prove by induction the following statements:

i) For every integer $n \geq 1$,

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n).$$

ii) For every integer $n \geq 1$, $n^3 + 5n$ is divisible by 6.

3A Dynamics

A lecturer driving his car of mass m_1 along the flat at speed U_1 accidentally collides with a stationary vehicle of mass m_2. As both vehicles are old and very solidly built, neither suffers damage in the collision: they simply bounce elastically off each other in a straight line. Determine how both vehicles are moving after the collision if neither driver applied their brakes. State any assumptions made and consider all possible values of the mass ratio $R = m_1/m_2$. You may neglect friction and other such losses.

An undergraduate drives into a rigid rock wall at speed V. The undergraduate's car of mass M is modern and has a crumple zone of length L at its front. As this zone crumples upon impact, it exerts a net force $F = (L-y)^{-1/2}$ on the car, where y is the amount the zone has crumpled. Determine the value of y at the point the car stops moving forwards as a function of V, where $V < 2L^{\frac{1}{4}}/M^{\frac{1}{2}}$.

4A Dynamics

A small spherical bubble of radius a containing carbon dioxide rises in water due to a buoyancy force $\rho g V$, where ρ is the density of water, g is gravitational attraction and V is the volume of the bubble. The drag on a bubble moving at speed u is $6\pi\mu a u$, where μ is the dynamic viscosity of water, and an accelerating bubble acts like a particle of mass $\alpha\rho V$, for some constant α. Find the location at time t of a bubble released from rest at $t = 0$ and show the bubble approaches a steady rise speed

$$U = \frac{2}{9}\frac{\rho g}{\mu}a^2. \qquad (*)$$

Under some circumstances the carbon dioxide gradually dissolves in the water, which leads to the bubble radius varying as $a^2 = a_0^2 - \beta t$, where a_0 is the bubble radius at $t = 0$ and β is a constant. Under the assumption that the bubble rises at speed given by $(*)$, determine the height to which it rises before it disappears.

SECTION II

5E Numbers and Sets

Show that the set of all subsets of $\mathbb{N}$ is uncountable, and that the set of all finite subsets of $\mathbb{N}$ is countable.

Let X be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$, and let $Y \subset X$ be the set

$$Y = \{f \in X \mid \text{for all but finitely many } n \in \mathbb{N},\ f(n) = n\}.$$

Show that X is uncountable, but that Y is countable.

6E Numbers and Sets

Prove Fermat's Theorem: if p is prime and $(x, p) = 1$ then $x^{p-1} \equiv 1 \pmod{p}$.

Let n and x be positive integers with $(x, n) = 1$. Show that if $n = mp$ where p is prime and $(m, p) = 1$, then

$$x^{n-1} \equiv 1 \pmod{p} \quad \text{if and only if} \quad x^{m-1} \equiv 1 \pmod{p}.$$

Now assume that n is a product of distinct primes. Show that $x^{n-1} \equiv 1 \pmod{n}$ if and only if, for every prime divisor p of n,

$$x^{(n/p)-1} \equiv 1 \pmod{p}.$$

Deduce that if every prime divisor p of n satisfies $(p-1)|(n-1)$, then for every x with $(x, n) = 1$, the congruence

$$x^{n-1} \equiv 1 \pmod{n}$$

holds.

7E Numbers and Sets

Polynomials $P_r(X)$ for $r \geq 0$ are defined by

$$P_0(X) = 1$$
$$P_r(X) = \frac{X(X-1)\cdots(X-r+1)}{r!} = \prod_{i=1}^{r} \frac{X-i+1}{i} \qquad \text{for } r \geq 1.$$

Show that $P_r(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, and that if $r \geq 1$ then $P_r(X) - P_r(X-1) = P_{r-1}(X-1)$.

Prove that if F is any polynomial of degree d with rational coefficients, then there are unique rational numbers $c_r(F)$ $(0 \leq r \leq d)$ for which

$$F(X) = \sum_{r=0}^{d} c_r(F) P_r(X).$$

Let $\Delta F(X) = F(X+1) - F(X)$. Show that

$$\Delta F(X) = \sum_{r=0}^{d-1} c_{r+1}(F) P_r(X).$$

Show also that, if F and G are polynomials such that $\Delta F = \Delta G$, then $F - G$ is a constant.

By induction on the degree of F, or otherwise, show that if $F(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$, then $c_r(F) \in \mathbb{Z}$ for all r.

8E Numbers and Sets

Let X be a finite set, $X_1, \ldots, X_m$ subsets of X and $Y = X \setminus \bigcup X_i$. Let g_i be the characteristic function of X_i, so that

$$g_i(x) = \begin{cases} 1 & \text{if } x \in X_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $f\colon X \to \mathbb{R}$ be any function. By considering the expression

$$\sum_{x \in X} f(x) \prod_{i=1}^{m} (1 - g_i(x)) \,,$$

or otherwise, prove the Inclusion–Exclusion Principle in the form

$$\sum_{x \in Y} f(x) = \sum_{r \geq 0} (-1)^r \sum_{i_1 < \cdots < i_r} \Bigg(\sum_{x \in X_{i_1} \cap \cdots \cap X_{i_r}} f(x) \Bigg).$$

Let $n > 1$ be an integer. For an integer m dividing n let

$$X_m = \{0 \leq x < n \mid x \equiv 0 \pmod{m}\}.$$

By considering the sets X_p for prime divisors p of n, show that

$$\phi(n) = n \prod_{p \mid n} \Big(1 - \frac{1}{p}\Big)$$

(where ϕ is Euler's function) and

$$\sum_{\substack{0 < x < n \\ (x,n)=1}} x = \frac{n^2}{2} \prod_{p \mid n} \Big(1 - \frac{1}{p}\Big).$$

9A Dynamics

A horizontal table oscillates with a displacement $\mathbf{A}\sin\omega t$, where $\mathbf{A} = (A_x, 0, A_z)$ is the amplitude vector and ω the angular frequency in an inertial frame of reference with the z axis vertically upwards, normal to the table. A block sitting on the table has mass m and linear friction that results in a force $\mathbf{F} = -\lambda\mathbf{u}$, where λ is a constant and $\mathbf{u}$ is the velocity difference between the block and the table. Derive the equations of motion for this block in the frame of reference of the table using axes (ξ, η, ζ) on the table parallel to the axes (x, y, z) in the inertial frame.

For the case where $A_z = 0$, show that at late time the block will approach the steady orbit

$$\xi = \xi_0 - A_x \sin\theta\cos(\omega t - \theta),$$

where

$$\sin^2\theta = \frac{m^2\omega^2}{\lambda^2 + m^2\omega^2}$$

and ξ_0 is a constant.

Given that there are no attractive forces between block and table, show that the block will only remain in contact with the table if $\omega^2 A_z < g$.

10A Dynamics

A small probe of mass m is in low orbit about a planet of mass M. If there is no drag on the probe then its orbit is governed by

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^3}\,\mathbf{r},$$

where $\mathbf{r}$ is the location of the probe relative to the centre of the planet and G is the gravitational constant. Show that the basic orbital trajectory is elliptical. Determine the orbital period for the probe if it is in a circular orbit at a distance r_0 from the centre of the planet.

Data returned by the probe shows that the planet has a very extensive but diffuse atmosphere. This atmosphere induces a drag on the probe that may be approximated by the linear law $\mathbf{D} = -A\dot{\mathbf{r}}$, where $\mathbf{D}$ is the drag force and A is a constant. Show that the angular momentum of the probe about the planet decays exponentially.

11A Dynamics

A particle of mass m and charge q moves through a magnetic field $\mathbf{B}$. There is no electric field or external force so that the particle obeys

$$m\ddot{\mathbf{r}} = q\,\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}$ is the location of the particle. Prove that the kinetic energy of the particle is preserved.

Consider an axisymmetric magnetic field described by $\mathbf{B} = (0, 0, B(r))$ in cylindrical polar coordinates $\mathbf{r} = (r, \theta, z)$. Determine the angular velocity of a circular orbit centred on $\mathbf{r} = \mathbf{0}$.

For a general orbit when $B(r) = B_0/r$, show that the angular momentum about the z-axis varies as $L = L_0 - qB_0(r - r_0)$, where L_0 is the angular momentum at radius r_0. Determine and sketch the relationship between $\dot{r}^2$ and r. [Hint: Use conservation of energy.] What is the escape velocity for the particle?

12A Dynamics

A circular cylinder of radius a, length L and mass m is rolling along a surface. Show that its moment of inertia is given by $\frac{1}{2}ma^2$.

At $t = 0$ the cylinder is at the bottom of a slope making an angle α to the horizontal, and is rolling with velocity V and angular velocity V/a. Assuming slippage does not occur, determine the position of the cylinder as a function of time. What is the maximum height that the cylinder reaches?

The frictional force between the cylinder and surface is given by $\mu mg \cos\alpha$, where μ is the friction coefficient. Show that the cylinder begins to slip rather than roll if $\tan\alpha > 3\mu$. Determine as a function of time the location, speed and angular velocity of the cylinder on the slope if this condition is satisfied. Show that slipping continues as the cylinder ascends and descends the slope. Find also the maximum height the cylinder reaches, and its speed and angular velocity when it returns to the bottom of the slope.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2004

List of Courses

Algebra and Geometry
Analysis
Differential Equations
Dynamics
Numbers and Sets
Probability
Vector Calculus

1/I/1B **Algebra and Geometry**

The linear map $H : \mathbb{R}^3 \to \mathbb{R}^3$ represents reflection in the plane through the origin with normal $\mathbf{n}$, where $|\mathbf{n}| = 1$, and $\mathbf{n} = (n_1, n_2, n_3)$ referred to the standard basis. The map is given by $\mathbf{x} \mapsto \mathbf{x}' = \mathbf{M}\mathbf{x}$, where $\mathbf{M}$ is a (3×3) matrix.

Show that

$$M_{ij} = \delta_{ij} - 2n_i n_j \,.$$

Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors such that $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ is an orthonormal set. Show that

$$\mathbf{M}\mathbf{n} = -\mathbf{n}, \quad \mathbf{M}\mathbf{u} = \mathbf{u}, \quad \mathbf{M}\mathbf{v} = \mathbf{v}\,,$$

and find the matrix $\mathbf{N}$ which gives the mapping relative to the basis $(\mathbf{u}, \mathbf{v}, \mathbf{n})$.

1/I/2C **Algebra and Geometry**

Show that

$$\sum_{i=1}^{n} a_i b_i \leqslant \Big(\sum_{i=1}^{n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2}$$

for any real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. Using this inequality, show that if $\mathbf{a}$ and $\mathbf{b}$ are vectors of unit length in $\mathbb{R}^n$ then $|\mathbf{a} \cdot \mathbf{b}| \leqslant 1$.

1/II/5B **Algebra and Geometry**

The vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfies the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,,$$

where $\mathbf{A}$ is a (3×3) matrix and $\mathbf{b}$ is a (3×1) column vector. State the conditions under which this equation has (a) a unique solution, (b) an infinity of solutions, (c) no solution for $\mathbf{x}$.

Find all possible solutions for the unknowns x, y and z which satisfy the following equations:

$$\begin{aligned} x + y + z &= 1 \\ x + y + \lambda z &= 2 \\ x + 2y + \lambda z &= 4\,, \end{aligned}$$

in the cases (a) $\lambda = 0$, and (b) $\lambda = 1$.

1/II/6A **Algebra and Geometry**

Express the product $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ in suffix notation and thence prove that the result is zero.

Silver Beard the space pirate believed people relied so much on space-age navigation techniques that he could safely write down the location of his treasure using the ancient art of vector algebra. Spikey the space jockey thought he could follow the instructions, by moving by the sequence of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$ one stage at a time. The vectors (expressed in 1000 parsec units) were defined as follows:

1. Start at the centre of the galaxy, which has coordinates $(0, 0, 0)$.

2. Vector $\mathbf{a}$ has length $\sqrt{3}$, is normal to the plane $x + y + z = 1$ and is directed into the positive quadrant.

3. Vector $\mathbf{b}$ is given by $\mathbf{b} = (\mathbf{a} \cdot \mathbf{m})\mathbf{a} \times \mathbf{m}$, where $\mathbf{m} = (2, 0, 1)$.

4. Vector $\mathbf{c}$ has length $2\sqrt{2}$, is normal to $\mathbf{a}$ and $\mathbf{b}$, and moves you closer to the x axis.

5. Vector $\mathbf{d} = (1, -2, 2)$.

6. Vector $\mathbf{e}$ has length $\mathbf{a} \cdot \mathbf{b}$. Spikey was initially a little confused with this one, but then realised the orientation of the vector did not matter.

7. Vector $\mathbf{f}$ has unknown length but is parallel to $\mathbf{m}$ and takes you to the treasure located somewhere on the plane $2x - y + 4z = 10$.

Determine the location of the way-points Spikey will use and thence the location of the treasure.

1/II/7A Algebra and Geometry

Simplify the fraction

$$\zeta = \cfrac{1}{\bar{z} + \cfrac{1}{z + \cfrac{1}{\bar{z}}}},$$

where $\bar{z}$ is the complex conjugate of z. Determine the geometric form that satisfies

$$\mathrm{Re}(\zeta) = \mathrm{Re}\left(\frac{z + \frac{1}{4}}{|z|^2}\right).$$

Find solutions to

$$\mathrm{Im}(\log z) = \frac{\pi}{3}$$

and

$$z^2 = x^2 - y^2 + 2ix,$$

where $z = x + iy$ is a complex variable. Sketch these solutions in the complex plane and describe the region they enclose. Derive complex equations for the circumscribed and inscribed circles for the region. [The circumscribed circle is the circle that passes through the vertices of the region and the inscribed circle is the largest circle that fits within the region.]

1/II/8C Algebra and Geometry

(i) The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfy $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3 \neq 0$. Are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(ii) The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$ have the property that every subset comprising $(n-1)$ of the vectors is linearly independent. Are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ necessarily linearly independent? Justify your answer by a proof or a counterexample.

(iii) For each value of t in the range $0 \leqslant t < 1$, give a construction of a linearly independent set of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$ satisfying

$$\mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij} + t(1 - \delta_{ij}),$$

where δ_{ij} is the Kronecker delta.

3/I/1D Algebra and Geometry

State Lagrange's Theorem.

Show that there are precisely two non-isomorphic groups of order 10. [You may assume that a group whose elements are all of order 1 or 2 has order 2^k.]

3/I/2D **Algebra and Geometry**

Define the Möbius group, and describe how it acts on $\mathbb{C} \cup \{\infty\}$.

Show that the subgroup of the Möbius group consisting of transformations which fix 0 and ∞ is isomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Now show that the subgroup of the Möbius group consisting of transformations which fix 0 and 1 is also isomorphic to $\mathbb{C}^*$.

3/II/5D **Algebra and Geometry**

Let $G = \langle g, h \mid h^2 = 1, g^6 = 1, hgh^{-1} = g^{-1} \rangle$ be the dihedral group of order 12.

i) List all the subgroups of G of order 2. Which of them are normal?

ii) Now list all the remaining proper subgroups of G. [There are 6+3 of them.]

iii) For each proper normal subgroup N of G, describe the quotient group G/N.

iv) Show that G is not isomorphic to the alternating group A_4.

3/II/6D **Algebra and Geometry**

State the conditions on a matrix A that ensure it represents a rotation of $\mathbb{R}^3$ with respect to the standard basis.

Check that the matrix

$$A = \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$

represents a rotation. Find its axis of rotation $\mathbf{n}$.

Let Π be the plane perpendicular to the axis $\mathbf{n}$. The matrix A induces a rotation of Π by an angle θ. Find $\cos\theta$.

3/II/7D Algebra and Geometry

Let A be a real symmetric matrix. Show that all the eigenvalues of A are real, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Give an example of a non-zero *complex* (2×2) symmetric matrix whose only eigenvalues are zero. Is it diagonalisable?

3/II/8D Algebra and Geometry

Compute the characteristic polynomial of

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4-s & 2s-2 \\ 0 & -2s+2 & 4s-1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A for all values of s.

For which values of s is A diagonalisable?

1/I/3D **Analysis**

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

State the Least Upper Bound Axiom for the real numbers.

Starting from the Least Upper Bound Axiom, show that if (a_n) is a bounded monotonic sequence of real numbers, then it converges.

1/I/4E **Analysis**

Let $f(x) = (1+x)^{1/2}$ for $x \in (-1,1)$. Show by induction or otherwise that for every integer $r \geq 1$,

$$f^{(r)}(x) = (-1)^{r-1} \frac{(2r-2)!}{2^{2r-1}(r-1)!}(1+x)^{\frac{1}{2}-r}.$$

Evaluate the series

$$\sum_{r=1}^{\infty} (-1)^{r-1} \frac{(2r-2)!}{8^r r!(r-1)!}.$$

[You may use Taylor's Theorem in the form

$$f(x) = f(0) + \sum_{r=1}^{n} \frac{f^{(r)}(0)}{r!} x^r + \int_0^x \frac{(x-t)^n f^{(n+1)}(t)}{n!}\, dt$$

without proof.]

1/II/9D **Analysis**

i) State Rolle's theorem.

Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions which are differentiable on (a,b).

ii) Prove that for some $c \in (a,b)$,

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

iii) Suppose that $f(a) = g(a) = 0$, and that $\lim_{x \to a+} \frac{f'(x)}{g'(x)}$ exists and is equal to L. Prove that $\lim_{x \to a+} \frac{f(x)}{g(x)}$ exists and is also equal to L.

[You may assume there exists a $\delta > 0$ such that, for all $x \in (a, a+\delta)$, $g'(x) \neq 0$ and $g(x) \neq 0$.]

iv) Evaluate $\lim_{x \to 0} \frac{\log \cos x}{x^2}$.

1/II/10E Analysis

Define, for an integer $n \geq 0$,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

Show that for every $n \geq 2$, $nI_n = (n-1)I_{n-2}$, and deduce that

$$I_{2n} = \frac{(2n)!}{(2^n n!)^2} \frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}.$$

Show that $0 < I_n < I_{n-1}$, and that

$$\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1.$$

Hence prove that

$$\lim_{n \to \infty} \frac{2^{4n+1}(n!)^4}{(2n+1)(2n)!^2} = \pi.$$

1/II/11F Analysis

Let f be defined on $\mathbb{R}$, and assume that there exists at least one point $x_0 \in \mathbb{R}$ at which f is continuous. Suppose also that, for every $x, y \in \mathbb{R}$, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Show that f is continuous on $\mathbb{R}$.

Show that there exists a constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Suppose that g is a continuous function defined on $\mathbb{R}$ and that, for every $x, y \in \mathbb{R}$, g satisfies the equation

$$g(x+y) \;=\; g(x)g(y).$$

Show that if g is not identically zero, then g is everywhere positive. Find the general form of g.

1/II/12F **Analysis**

(i) Show that if $a_n > 0$, $b_n > 0$ and

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$$

for all $n \geqslant 1$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Let

$$c_n = \binom{2n}{n} 4^{-n}.$$

By considering $\log c_n$, or otherwise, show that $c_n \to 0$ as $n \to \infty$.

[*Hint*: $\log(1-x) \leqslant -x$ for $x \in (0,1)$.]

(iii) Determine the convergence or otherwise of

$$\sum_{n=1}^{\infty} \binom{2n}{n} x^n$$

for (a) $x = \frac{1}{4}$, (b) $x = -\frac{1}{4}$.

2/I/1B **Differential Equations**

By writing $y(x) = mx$ where m is a constant, solve the differential equation

$$\frac{dy}{dx} = \frac{x - 2y}{2x + y}$$

and find the possible values of m.

Describe the isoclines of this differential equation and sketch the flow vectors. Use these to sketch at least two characteristically different solution curves.

Now, by making the substitution $y(x) = xu(x)$ or otherwise, find the solution of the differential equation which satisfies $y(0) = 1$.

2/I/2B **Differential Equations**

Find two linearly independent solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2 y = 0\,.$$

Find also the solution of

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + p^2 y = e^{-px}$$

that satisfies

$$y = 0,\ \frac{dy}{dx} = 0 \quad \text{at } x = 0\,.$$

2/II/5B Differential Equations

Construct a series solution $y = y_1(x)$ valid in the neighbourhood of $x = 0$, for the differential equation

$$\frac{d^2y}{dx^2} + 4x^3\frac{dy}{dx} + x^2y = 0\,,$$

satisfying

$$y_1 = 1,\ \frac{dy_1}{dx} = 0 \quad \text{at } x = 0\,.$$

Find also a second solution $y = y_2(x)$ which satisfies

$$y_2 = 0,\ \frac{dy_2}{dx} = 1 \quad \text{at } x = 0\,.$$

Obtain an expression for the Wronskian of these two solutions and show that

$$y_2(x) = y_1(x)\int_0^x \frac{e^{-\xi^4}}{y_1^2(\xi)}\,d\xi\,.$$

2/II/6B Differential Equations

Two solutions of the recurrence relation

$$x_{n+2} + b(n)x_{n+1} + c(n)x_n = 0$$

are given as p_n and q_n, and their Wronskian is defined to be

$$W_n = p_n q_{n+1} - p_{n+1}q_n\,.$$

Show that

$$W_{n+1} = W_1\prod_{m=1}^{n} c(m)\,. \qquad (*)$$

Suppose that $b(n) = \alpha$, where α is a real constant lying in the range $[-2, 2]$, and that $c(n) = 1$. Show that two solutions are $x_n = e^{in\theta}$ and $x_n = e^{-in\theta}$, where $\cos\theta = -\alpha/2$. Evaluate the Wronskian of these two solutions and verify $(*)$.

2/II/7B Differential Equations

Show how a second-order differential equation $\ddot{x} = f(x, \dot{x})$ may be transformed into a pair of coupled first-order equations. Explain what is meant by a *critical point* on the phase diagram for a pair of first-order equations. Hence find the critical points of the following equations. Describe their stability type, sketching their behaviour near the critical points on a phase diagram.

(i) $\ddot{x} + \cos x = 0$

(ii) $\ddot{x} + x(x^2 + \lambda x + 1) = 0, \quad \text{for } \lambda = 1,\ 5/2\,.$

Sketch the phase portraits of these equations marking clearly the direction of flow.

2/II/8B Differential Equations

Construct the general solution of the system of equations

$$\begin{aligned}\dot{x} + 4x + 3y &= 0\\ \dot{y} + 4y - 3x &= 0\end{aligned}$$

in the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{x} = \sum_{j=1}^{2} a_j \mathbf{x}^{(j)} e^{\lambda_j t}$$

and find the eigenvectors $\mathbf{x}^{(j)}$ and eigenvalues λ_j.

Explain what is meant by resonance in a forced system of linear differential equations.

Consider the forced system

$$\begin{aligned}\dot{x} + 4x + 3y &= \sum_{j=1}^{2} p_j e^{\lambda_j t}\\ \dot{y} + 4y - 3x &= \sum_{j=1}^{2} q_j e^{\lambda_j t}\,.\end{aligned}$$

Find conditions on p_j and q_j $(j = 1, 2)$ such that there is no resonant response to the forcing.

4/I/3A **Dynamics**

A lecturer driving his car of mass m_1 along the flat at speed U_1 accidentally collides with a stationary vehicle of mass m_2. As both vehicles are old and very solidly built, neither suffers damage in the collision: they simply bounce elastically off each other in a straight line. Determine how both vehicles are moving after the collision if neither driver applied their brakes. State any assumptions made and consider all possible values of the mass ratio $R = m_1/m_2$. You may neglect friction and other such losses.

An undergraduate drives into a rigid rock wall at speed V. The undergraduate's car of mass M is modern and has a crumple zone of length L at its front. As this zone crumples upon impact, it exerts a net force $F = (L-y)^{-1/2}$ on the car, where y is the amount the zone has crumpled. Determine the value of y at the point the car stops moving forwards as a function of V, where $V < 2L^{\frac{1}{4}}/M^{\frac{1}{2}}$.

4/I/4A **Dynamics**

A small spherical bubble of radius a containing carbon dioxide rises in water due to a buoyancy force $\rho g V$, where ρ is the density of water, g is gravitational attraction and V is the volume of the bubble. The drag on a bubble moving at speed u is $6\pi\mu a u$, where μ is the dynamic viscosity of water, and an accelerating bubble acts like a particle of mass $\alpha\rho V$, for some constant α. Find the location at time t of a bubble released from rest at $t = 0$ and show the bubble approaches a steady rise speed

$$U = \frac{2}{9}\frac{\rho g}{\mu}a^2. \qquad (*)$$

Under some circumstances the carbon dioxide gradually dissolves in the water, which leads to the bubble radius varying as $a^2 = a_0^2 - \beta t$, where a_0 is the bubble radius at $t = 0$ and β is a constant. Under the assumption that the bubble rises at speed given by $(*)$, determine the height to which it rises before it disappears.

4/II/9A **Dynamics**

A horizontal table oscillates with a displacement $\mathbf{A}\sin\omega t$, where $\mathbf{A} = (A_x, 0, A_z)$ is the amplitude vector and ω the angular frequency in an inertial frame of reference with the z axis vertically upwards, normal to the table. A block sitting on the table has mass m and linear friction that results in a force $\mathbf{F} = -\lambda\mathbf{u}$, where λ is a constant and $\mathbf{u}$ is the velocity difference between the block and the table. Derive the equations of motion for this block in the frame of reference of the table using axes (ξ, η, ζ) on the table parallel to the axes (x, y, z) in the inertial frame.

For the case where $A_z = 0$, show that at late time the block will approach the steady orbit

$$\xi = \xi_0 - A_x \sin\theta \cos(\omega t - \theta),$$

where

$$\sin^2\theta = \frac{m^2\omega^2}{\lambda^2 + m^2\omega^2}$$

and ξ_0 is a constant.

Given that there are no attractive forces between block and table, show that the block will only remain in contact with the table if $\omega^2 A_z < g$.

4/II/10A **Dynamics**

A small probe of mass m is in low orbit about a planet of mass M. If there is no drag on the probe then its orbit is governed by

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^3}\,\mathbf{r},$$

where $\mathbf{r}$ is the location of the probe relative to the centre of the planet and G is the gravitational constant. Show that the basic orbital trajectory is elliptical. Determine the orbital period for the probe if it is in a circular orbit at a distance r_0 from the centre of the planet.

Data returned by the probe shows that the planet has a very extensive but diffuse atmosphere. This atmosphere induces a drag on the probe that may be approximated by the linear law $\mathbf{D} = -A\dot{\mathbf{r}}$, where $\mathbf{D}$ is the drag force and A is a constant. Show that the angular momentum of the probe about the planet decays exponentially.

4/II/11A Dynamics

A particle of mass m and charge q moves through a magnetic field $\mathbf{B}$. There is no electric field or external force so that the particle obeys

$$m\ddot{\mathbf{r}} = q\,\dot{\mathbf{r}} \times \mathbf{B},$$

where $\mathbf{r}$ is the location of the particle. Prove that the kinetic energy of the particle is preserved.

Consider an axisymmetric magnetic field described by $\mathbf{B} = (0, 0, B(r))$ in cylindrical polar coordinates $\mathbf{r} = (r, \theta, z)$. Determine the angular velocity of a circular orbit centred on $\mathbf{r} = \mathbf{0}$.

For a general orbit when $B(r) = B_0/r$, show that the angular momentum about the z-axis varies as $L = L_0 - qB_0(r - r_0)$, where L_0 is the angular momentum at radius r_0. Determine and sketch the relationship between $\dot{r}^2$ and r. [Hint: Use conservation of energy.] What is the escape velocity for the particle?

4/II/12A Dynamics

A circular cylinder of radius a, length L and mass m is rolling along a surface. Show that its moment of inertia is given by $\frac{1}{2}ma^2$.

At $t = 0$ the cylinder is at the bottom of a slope making an angle α to the horizontal, and is rolling with velocity V and angular velocity V/a. Assuming slippage does not occur, determine the position of the cylinder as a function of time. What is the maximum height that the cylinder reaches?

The frictional force between the cylinder and surface is given by $\mu mg \cos\alpha$, where μ is the friction coefficient. Show that the cylinder begins to slip rather than roll if $\tan\alpha > 3\mu$. Determine as a function of time the location, speed and angular velocity of the cylinder on the slope if this condition is satisfied. Show that slipping continues as the cylinder ascends and descends the slope. Find also the maximum height the cylinder reaches, and its speed and angular velocity when it returns to the bottom of the slope.

4/I/1E **Numbers and Sets**

(a) Use Euclid's algorithm to find positive integers m, n such that $79m - 100n = 1$.

(b) Determine all integer solutions of the congruence

$$237x \equiv 21 \pmod{300}.$$

(c) Find the set of all integers x satisfying the simultaneous congruences

$$x \equiv 8 \pmod{79}$$
$$x \equiv 11 \pmod{100}.$$

4/I/2E **Numbers and Sets**

Prove by induction the following statements:

i) For every integer $n \geq 1$,

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n).$$

ii) For every integer $n \geq 1$, $n^3 + 5n$ is divisible by 6.

4/II/5E **Numbers and Sets**

Show that the set of all subsets of $\mathbb{N}$ is uncountable, and that the set of all finite subsets of $\mathbb{N}$ is countable.

Let X be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$, and let $Y \subset X$ be the set

$$Y = \{f \in X \mid \text{for all but finitely many } n \in \mathbb{N},\ f(n) = n\}.$$

Show that X is uncountable, but that Y is countable.

4/II/6E **Numbers and Sets**

Prove Fermat's Theorem: if p is prime and $(x,p)=1$ then $x^{p-1}\equiv 1 \pmod p$.

Let n and x be positive integers with $(x,n)=1$. Show that if $n=mp$ where p is prime and $(m,p)=1$, then

$$x^{n-1}\equiv 1 \pmod p \quad \text{if and only if} \quad x^{m-1}\equiv 1 \pmod p.$$

Now assume that n is a product of distinct primes. Show that $x^{n-1}\equiv 1 \pmod n$ if and only if, for every prime divisor p of n,

$$x^{(n/p)-1}\equiv 1 \pmod p.$$

Deduce that if every prime divisor p of n satisfies $(p-1)|(n-1)$, then for every x with $(x,n)=1$, the congruence

$$x^{n-1}\equiv 1 \pmod n$$

holds.

4/II/7E **Numbers and Sets**

Polynomials $P_r(X)$ for $r\geq 0$ are defined by

$$\begin{aligned} &P_0(X)=1 \\ &P_r(X)=\frac{X(X-1)\cdots(X-r+1)}{r!}=\prod_{i=1}^{r}\frac{X-i+1}{i} \qquad \text{for } r\geq 1. \end{aligned}$$

Show that $P_r(n)\in\mathbb{Z}$ for every $n\in\mathbb{Z}$, and that if $r\geq 1$ then $P_r(X)-P_r(X-1)=P_{r-1}(X-1)$.

Prove that if F is any polynomial of degree d with rational coefficients, then there are unique rational numbers $c_r(F)$ $(0\leq r\leq d)$ for which

$$F(X)=\sum_{r=0}^{d}c_r(F)P_r(X).$$

Let $\Delta F(X)=F(X+1)-F(X)$. Show that

$$\Delta F(X)=\sum_{r=0}^{d-1}c_{r+1}(F)P_r(X).$$

Show also that, if F and G are polynomials such that $\Delta F=\Delta G$, then $F-G$ is a constant.

By induction on the degree of F, or otherwise, show that if $F(n)\in\mathbb{Z}$ for every $n\in\mathbb{Z}$, then $c_r(F)\in\mathbb{Z}$ for all r.

4/II/8E **Numbers and Sets**

Let X be a finite set, $X_1, \ldots, X_m$ subsets of X and $Y = X \setminus \bigcup X_i$. Let g_i be the characteristic function of X_i, so that

$$g_i(x) = \begin{cases} 1 & \text{if } x \in X_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $f\colon X \to \mathbb{R}$ be any function. By considering the expression

$$\sum_{x \in X} f(x) \prod_{i=1}^{m} (1 - g_i(x)) \,,$$

or otherwise, prove the Inclusion–Exclusion Principle in the form

$$\sum_{x \in Y} f(x) = \sum_{r \geq 0} (-1)^r \sum_{i_1 < \cdots < i_r} \Bigg(\sum_{x \in X_{i_1} \cap \cdots \cap X_{i_r}} f(x) \Bigg).$$

Let $n > 1$ be an integer. For an integer m dividing n let

$$X_m = \{0 \leq x < n \mid x \equiv 0 \pmod{m}\}.$$

By considering the sets X_p for prime divisors p of n, show that

$$\phi(n) = n \prod_{p \mid n} \Big(1 - \frac{1}{p}\Big)$$

(where ϕ is Euler's function) and

$$\sum_{\substack{0 < x < n \\ (x,n)=1}} x = \frac{n^2}{2} \prod_{p \mid n} \Big(1 - \frac{1}{p}\Big).$$

2/I/3F Probability

Define the covariance, $\operatorname{cov}(X,Y)$, of two random variables X and Y.

Prove, or give a counterexample to, each of the following statements.

(a) For any random variables X, Y, Z

$$\operatorname{cov}(X+Y,Z)=\operatorname{cov}(X,Z)+\operatorname{cov}(Y,Z).$$

(b) If X and Y are identically distributed, not necessarily independent, random variables then

$$\operatorname{cov}(X+Y,X-Y)=0.$$

2/I/4F Probability

The random variable X has probability density function

$$f(x)=\begin{cases} cx(1-x) & \text{if } 0\leqslant x\leqslant 1\\ 0 & \text{otherwise.}\end{cases}$$

Determine c, and the mean and variance of X.

2/II/9F Probability

Let X be a positive-integer valued random variable. Define its *probability generating function* p_X. Show that if X and Y are independent positive-integer valued random variables, then $p_{X+Y}=p_X p_Y$.

A non-standard pair of dice is a pair of six-sided unbiased dice whose faces are numbered with strictly positive integers in a non-standard way (for example, $(2,2,2,3,5,7)$ and $(1,1,5,6,7,8)$). Show that there exists a non-standard pair of dice A and B such that when thrown

$$P\{\text{total shown by } A \text{ and } B \text{ is } n\}=P\{\text{total shown by pair of ordinary dice is } n\}$$

for all $2\leqslant n\leqslant 12$.

[*Hint:* $(x+x^2+x^3+x^4+x^5+x^6)=x(1+x)(1+x^2+x^4)=x(1+x+x^2)(1+x^3)$.]

2/II/10F Probability

Define the *conditional probability* $P(A \mid B)$ of the event A given the event B.

A bag contains four coins, each of which when tossed is equally likely to land on either of its two faces. One of the coins shows a head on each of its two sides, while each of the other three coins shows a head on only one side. A coin is chosen at random, and tossed three times in succession. If heads turn up each time, what is the probability that if the coin is tossed once more it will turn up heads again? Describe the sample space you use and explain carefully your calculations.

2/II/11F Probability

The random variables X_1 and X_2 are independent, and each has an exponential distribution with parameter λ. Find the joint density function of

$$Y_1 = X_1 + X_2\,, \quad Y_2 = X_1/X_2\,,$$

and show that Y_1 and Y_2 are independent. What is the density of Y_2?

2/II/12F Probability

Let $A_1, A_2, \ldots, A_r$ be events such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Show that the number N of events that occur satisfies

$$P(N = 0) \quad = \quad 1 - \sum_{i=1}^{r} P(A_i)\,.$$

Planet Zog is a sphere with centre O. A number N of spaceships land at random on its surface, their positions being independent, each uniformly distributed over the surface. A spaceship at A is in direct radio contact with another point B on the surface if $\angle AOB < \frac{\pi}{2}$. Calculate the probability that every point on the surface of the planet is in direct radio contact with at least one of the N spaceships.

[*Hint*: The intersection of the surface of a sphere with a plane through the centre of the sphere is called a *great circle*. You may find it helpful to use the fact that N random great circles partition the surface of a sphere into $N(N-1)+2$ disjoint regions with probability one.]

3/I/3C **Vector Calculus**

If $\mathbf{F}$ and $\mathbf{G}$ are differentiable vector fields, show that

$$\text{(i)}\quad \boldsymbol{\nabla}\times(\mathbf{F}\times\mathbf{G})=\mathbf{F}(\boldsymbol{\nabla}\cdot\mathbf{G})-\mathbf{G}(\boldsymbol{\nabla}\cdot\mathbf{F})+(\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F}-(\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G}\,,$$

$$\text{(ii)}\quad \boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G})=(\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G}+(\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F}+\mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G})+\mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F})\,.$$

3/I/4C **Vector Calculus**

Define the curvature, κ, of a curve in $\mathbb{R}^3$.

The curve C is parametrised by

$$\mathbf{x}(t)=\left(\frac{1}{2}e^t\cos t,\ \frac{1}{2}e^t\sin t,\ \frac{1}{\sqrt{2}}e^t\right)\quad\text{for }\ -\infty<t<\infty\,.$$

Obtain a parametrisation of the curve in terms of its arc length, s, measured from the origin. Hence obtain its curvature, $\kappa(s)$, as a function of s.

3/II/9C **Vector Calculus**

For a function $f : \mathbb{R}^2 \to \mathbb{R}$ state if the following implications are true or false. (No justification is required.)

(i) f is differentiable $\Rightarrow$ f is continuous.

(ii) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist $\Rightarrow$ f is continuous.

(iii) directional derivatives $\dfrac{\partial f}{\partial \mathbf{n}}$ exist for all unit vectors $\mathbf{n} \in \mathbb{R}^2 \Rightarrow f$ is differentiable.

(iv) f is differentiable $\Rightarrow \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous.

(v) all second order partial derivatives of f exist $\Rightarrow \dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$.

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that f is continuous at $(0,0)$ and find the partial derivatives $\dfrac{\partial f}{\partial x}(0,y)$ and $\dfrac{\partial f}{\partial y}(x,0)$. Then show that f is differentiable at $(0,0)$ and find its derivative. Investigate whether the second order partial derivatives $\dfrac{\partial^2 f}{\partial x\, \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y\, \partial x}(0,0)$ are the same. Are the second order partial derivatives of f at $(0,0)$ continuous? Justify your answer.

3/II/10C **Vector Calculus**

Explain what is meant by an exact differential. The three-dimensional vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = \left(e^x z^3 + 3x^2(e^y - e^z),\, e^y(x^3 - z^3),\, 3z^2(e^x - e^y) - e^z x^3\right).$$

Find the most general function that has $\mathbf{F} \cdot \mathbf{dx}$ as its differential.

Hence show that the line integral

$$\int_{P_1}^{P_2} \mathbf{F} \cdot \mathbf{dx}$$

along any path in $\mathbb{R}^3$ between points $P_1 = (0, a, 0)$ and $P_2 = (b, b, b)$ vanishes for any values of a and b.

The two-dimensional vector field $\mathbf{G}$ is defined at all points in $\mathbb{R}^2$ except $(0,0)$ by

$$\mathbf{G} = \left(\frac{-y}{x^2+y^2}\,,\ \frac{x}{x^2+y^2}\right).$$

($\mathbf{G}$ is not defined at $(0,0)$.) Show that

$$\oint_C \mathbf{G} \cdot \mathbf{dx} = 2\pi$$

for any closed curve C in $\mathbb{R}^2$ that goes around $(0,0)$ anticlockwise precisely once without passing through $(0,0)$.

3/II/11C **Vector Calculus**

Let S_1 be the 3-dimensional sphere of radius 1 centred at $(0,0,0)$, S_2 be the sphere of radius $\frac{1}{2}$ centred at $(\frac{1}{2},0,0)$ and S_3 be the sphere of radius $\frac{1}{4}$ centred at $(\frac{-1}{4},0,0)$. The eccentrically shaped planet Zog is composed of rock of uniform density ρ occupying the region within S_1 and outside S_2 and S_3. The regions inside S_2 and S_3 are empty. Give an expression for Zog's gravitational potential at a general coordinate $\mathbf{x}$ that is outside S_1. Is there a point in the interior of S_3 where a test particle would remain stably at rest? Justify your answer.

3/II/12C **Vector Calculus**

State (without proof) the divergence theorem for a vector field $\mathbf{F}$ with continuous first-order partial derivatives throughout a volume V enclosed by a bounded oriented piecewise-smooth non-self-intersecting surface S.

By calculating the relevant volume and surface integrals explicitly, verify the divergence theorem for the vector field

$$\mathbf{F} = \left(x^3 + 2xy^2,\, y^3 + 2yz^2,\, z^3 + 2zx^2\right),$$

defined within a sphere of radius R centred at the origin.

Suppose that functions ϕ, ψ are continuous and that their first and second partial derivatives are all also continuous in a region V bounded by a smooth surface S.

Show that

$$(1) \qquad \int_V (\phi \nabla^2 \psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS}\,.$$

$$(2) \qquad \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d\tau = \int_S \phi \boldsymbol{\nabla}\psi \cdot \mathbf{dS} - \int_S \psi \boldsymbol{\nabla}\phi \cdot \mathbf{dS}\,.$$

Hence show that if $\rho(\mathbf{x})$ is a continuous function on V and $g(\mathbf{x})$ a continuous function on S and ϕ_1 and ϕ_2 are two continuous functions such that

$$\nabla^2 \phi_1(\mathbf{x}) = \nabla^2 \phi_2(\mathbf{x}) = \rho(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ in } V, \text{ and}$$
$$\phi_1(\mathbf{x}) = \phi_2(\mathbf{x}) = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \text{ on } S,$$

then $\phi_1(\mathbf{x}) = \phi_2(\mathbf{x})$ for all $\mathbf{x}$ in V.