UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

List of Courses

1/I/5C **Linear Mathematics**

Determine for which values of $x \in \mathbb{C}$ the matrix

$$M = \begin{pmatrix} x & 1 & 1 \\ 1-x & 0 & -1 \\ 2 & 2x & 1 \end{pmatrix}$$

is invertible. Determine the rank of M as a function of x. Find the adjugate and hence the inverse of M for general x.

1/II/14C **Linear Mathematics**

(a) Find a matrix M over $\mathbb{C}$ with both minimal polynomial and characteristic polynomial equal to $(x-2)^3(x+1)^2$. Furthermore find two matrices M_1 and M_2 over $\mathbb{C}$ which have the same characteristic polynomial, $(x-3)^5(x-1)^2$, and the same minimal polynomial, $(x-3)^2(x-1)^2$, but which are not conjugate to one another. Is it possible to find a third such matrix, M_3, neither conjugate to M_1 nor to M_2? Justify your answer.

(b) Suppose A is an $n \times n$ matrix over $\mathbb{R}$ which has minimal polynomial of the form $(x-\lambda_1)(x-\lambda_2)$ for distinct roots $\lambda_1 \neq \lambda_2$ in $\mathbb{R}$. Show that the vector space $V = \mathbb{R}^n$ on which A defines an endomorphism $\alpha : V \to V$ decomposes as a direct sum into $V = \ker(\alpha - \lambda_1 \iota) \oplus \ker(\alpha - \lambda_2 \iota)$, where ι is the identity.

[*Hint: Express* $v \in V$ *in terms of* $(\alpha - \lambda_1\iota)(v)$ *and* $(\alpha - \lambda_2\iota)(v)$.]

Now suppose that A has minimal polynomial $(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m)$ for distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. By induction or otherwise show that

$$V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota) \oplus \ldots \oplus \ker(\alpha - \lambda_m\iota).$$

Use this last statement to prove that an arbitrary matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if all roots of its minimal polynomial lie in $\mathbb{R}$ and have multiplicity 1.

2/I/6C **Linear Mathematics**

Show that right multiplication by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$ defines a linear transformation $\rho_A : M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$. Find the matrix representing ρ_A with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_{2\times 2}(\mathbb{C})$. Prove that the characteristic polynomial of ρ_A is equal to the square of the characteristic polynomial of A, and that A and ρ_A have the same minimal polynomial.

2/II/15C Linear Mathematics

Define the dual V^* of a vector space V. Given a basis $\{v_1, \ldots, v_n\}$ of V define its dual and show it is a basis of V^*. For a linear transformation $\alpha : V \to W$ define the dual $\alpha^* : W^* \to V^*$.

Explain (with proof) how the matrix representing $\alpha : V \to W$ with respect to given bases of V and W relates to the matrix representing $\alpha^* : W^* \to V^*$ with respect to the corresponding dual bases of V^* and W^*.

Prove that α and α^* have the same rank.

Suppose that α is an invertible endomorphism. Prove that $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

3/I/7C Linear Mathematics

Determine the dimension of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \\ 5 \\ -1 \end{pmatrix}.$$

Write down a 5×5 matrix M which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ whose image is W and which contains $(1,1,1,1,1)^T$ in its kernel. What is the dimension of the space of all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1,1,1,1,1)^T$ in the kernel, and image contained in W?

3/II/17C Linear Mathematics

Let V be a vector space over $\mathbb{R}$. Let $\alpha : V \to V$ be a nilpotent endomorphism of V, i.e. $\alpha^m = 0$ for some positive integer m. Prove that α can be represented by a strictly upper-triangular matrix (with zeros along the diagonal). [*You may wish to consider the subspaces* $\ker(\alpha^j)$ *for* $j = 1, \ldots, m$.]

Show that if α is nilpotent, then $\alpha^n = 0$ where n is the dimension of V. Give an example of a 4×4 matrix M such that $M^4 = 0$ but $M^3 \neq 0$.

Let A be a nilpotent matrix and I the identity matrix. Prove that $I + A$ has all eigenvalues equal to 1. Is the same true of $(I+A)(I+B)$ if A and B are nilpotent? Justify your answer.

4/I/6C Linear Mathematics

Find the Jordan normal form J of the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} ,$$

and determine both the characteristic and the minimal polynomial of M.

Find a basis of $\mathbb{C}^4$ such that J (the Jordan normal form of M) is the matrix representing the endomorphism $M : \mathbb{C}^4 \to \mathbb{C}^4$ in this basis. Give a change of basis matrix P such that $P^{-1}MP = J$.

4/II/15C Linear Mathematics

Let A and B be $n \times n$ matrices over $\mathbb{C}$. Show that AB and BA have the same characteristic polynomial. [*Hint: Look at* $\det(CBC - xC)$ *for* $C = A + yI$, *where* x *and* y *are scalar variables.*]

Show by example that AB and BA need not have the same minimal polynomial.

Suppose that AB is diagonalizable, and let $p(x)$ be its minimal polynomial. Show that the minimal polynomial of BA must divide $xp(x)$. Using this and the first part of the question prove that $(AB)^2$ and $(BA)^2$ are conjugate.

1/I/4B **Geometry**

Write down the Riemannian metric on the disc model Δ of the hyperbolic plane. What are the geodesics passing through the origin? Show that the hyperbolic circle of radius ρ centred on the origin is just the Euclidean circle centred on the origin with Euclidean radius $\tanh(\rho/2)$.

Write down an isometry between the upper half-plane model H of the hyperbolic plane and the disc model Δ, under which $i \in H$ corresponds to $0 \in \Delta$. Show that the hyperbolic circle of radius ρ centred on i in H is a Euclidean circle with centre $i \cosh \rho$ and of radius $\sinh \rho$.

1/II/13B **Geometry**

Describe geometrically the stereographic projection map ϕ from the unit sphere S^2 to the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, and find a formula for ϕ. Show that any rotation of S^2 about the z-axis corresponds to a Möbius transformation of $\mathbb{C}_\infty$. You are given that the rotation of S^2 defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

corresponds under ϕ to a Möbius transformation of $\mathbb{C}_\infty$; deduce that any rotation of S^2 about the x-axis also corresponds to a Möbius transformation.

Suppose now that $u, v \in \mathbb{C}$ correspond under ϕ to distinct points $P, Q \in S^2$, and let d denote the angular distance from P to Q on S^2. Show that $-\tan^2(d/2)$ is the cross-ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in some order (which you should specify). [*You may assume that the cross-ratio is invariant under Möbius transformations.*]

3/I/4B **Geometry**

State and prove the Gauss–Bonnet theorem for the area of a spherical triangle.

Suppose $\mathbf{D}$ is a regular dodecahedron, with centre the origin. Explain how each face of $\mathbf{D}$ gives rise to a spherical pentagon on the 2-sphere S^2. For each such spherical pentagon, calculate its angles and area.

3/II/14B **Geometry**

Describe the hyperbolic lines in the upper half-plane model H of the hyperbolic plane. The group $G = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ acts on H via Möbius transformations, which you may assume are isometries of H. Show that G acts transitively on the hyperbolic lines. Find explicit formulae for the reflection in the hyperbolic line L in the cases (i) L is a vertical line $x = a$, and (ii) L is the unit semi-circle with centre the origin. Verify that the composite T of a reflection of type (ii) followed afterwards by one of type (i) is given by $T(z) = -z^{-1} + 2a$.

Suppose now that L_1 and L_2 are distinct hyperbolic lines in the hyperbolic plane, with R_1, R_2 denoting the corresponding reflections. By considering different models of the hyperbolic plane, or otherwise, show that

(a) R_1R_2 has infinite order if L_1 and L_2 are parallel or ultraparallel, and

(b) R_1R_2 has finite order if and only if L_1 and L_2 meet at an angle which is a rational multiple of π.

1/I/1A Analysis II

Define uniform continuity for functions defined on a (bounded or unbounded) interval in $\mathbb{R}$.

Is it true that a real function defined and uniformly continuous on $[0,1]$ is necessarily bounded?

Is it true that a real function defined and with a bounded derivative on $[1,\infty)$ is necessarily uniformly continuous there?

Which of the following functions are uniformly continuous on $[1,\infty)$:

(i) x^2;

(ii) $\sin(x^2)$;

(iii) $\dfrac{\sin x}{x}$?

Justify your answers.

1/II/10A Analysis II

Show that each of the functions below is a metric on the set of functions $x(t) \in C[a,b]$:

$$d_1(x,y) = \sup_{t\in[a,b]} |x(t)-y(t)|,$$

$$d_2(x,y) = \left\{ \int_a^b |x(t)-y(t)|^2 dt \right\}^{1/2}.$$

Is the space complete in the d_1 metric? Justify your answer.

Show that the set of functions

$$x_n(t) = \begin{cases} 0, & -1 \leqslant t < 0 \\ nt, & 0 \leqslant t < 1/n \\ 1, & 1/n \leqslant t \leqslant 1 \end{cases}$$

is a Cauchy sequence with respect to the d_2 metric on $C[-1,1]$, yet does not tend to a limit in the d_2 metric in this space. Hence, deduce that this space is not complete in the d_2 metric.

2/I/1A **Analysis II**

State and prove the contraction mapping theorem.

Let $A = \{x, y, z\}$, let d be the discrete metric on A, and let d' be the metric given by: d' is symmetric and

$$d'(x,y) = 2,\ d'(x,z) = 2,\ d'(y,z) = 1,$$

$$d'(x,x) = d'(y,y) = d'(z,z) = 0.$$

Verify that d' is a metric, and that it is Lipschitz equivalent to d.

Define an appropriate function $f : A \to A$ such that f is a contraction in the d' metric, but not in the d metric.

2/II/10A **Analysis II**

Define total boundedness for metric spaces.

Prove that a metric space has the Bolzano–Weierstrass property if and only if it is complete and totally bounded.

3/I/1A **Analysis II**

Define what is meant by a norm on a real vector space.

(a) Prove that two norms on a vector space (not necessarily finite-dimensional) give rise to equivalent metrics if and only if they are Lipschitz equivalent.

(b) Prove that if the vector space V has an inner product, then for all $x, y \in V$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

in the induced norm.

Hence show that the norm on $\mathbb{R}^2$ defined by $\|x\| = \max(|x_1|, |x_2|)$, where $x = (x_1, x_2) \in \mathbb{R}^2$, cannot be induced by an inner product.

3/II/11A Analysis II

Prove that if all the partial derivatives of $f : \mathbb{R}^p \to \mathbb{R}$ (with $p \geqslant 2$) exist in an open set containing $(0, 0, \ldots, 0)$ and are continuous at this point, then f is differentiable at $(0, 0, \ldots, 0)$.

Let

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$f(x, y) = g(x) + g(y).$$

At which points of the plane is the partial derivative f_x continuous?

At which points is the function $f(x, y)$ differentiable? Justify your answers.

4/I/1A Analysis II

Let f be a mapping of a metric space (X, d) into itself such that $d(f(x), f(y)) < d(x, y)$ for all distinct x, y in X.

Show that $f(x)$ and $d(x, f(x))$ are continuous functions of x.

Now suppose that (X, d) is compact and let

$$h = \inf_{x \in X} d(x, f(x)).$$

Show that we cannot have $h > 0$.

[*You may assume that a continuous function on a compact metric space is bounded and attains its bounds.*]

Deduce that f possesses a fixed point, and that it is unique.

4/II/10A Analysis II

Let $\{f_n\}$ be a pointwise convergent sequence of real-valued functions on a closed interval $[a, b]$. Prove that, if for every $x \in [a, b]$, the sequence $\{f_n(x)\}$ is monotonic in n, and if all the functions f_n, $n = 1, 2, \ldots$, and $f = \lim f_n$ are continuous, then $f_n \to f$ uniformly on $[a, b]$.

By considering a suitable sequence of functions $\{f_n\}$ on $[0, 1)$, show that if the interval is not closed but all other conditions hold, the conclusion of the theorem may fail.

1/I/7E Complex Methods

State the Cauchy integral formula.

Assuming that the function $f(z)$ is analytic in the disc $|z| < 1$, prove that, for every $0 < r < 1$, it is true that

$$\frac{d^n f(0)}{dz^n} = \frac{n!}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^{n+1}} d\xi, \qquad n = 0, 1, \ldots .$$

[*Taylor's theorem may be used if clearly stated.*]

1/II/16E Complex Methods

Let the function F be integrable for all real arguments x, such that

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty \ ,$$

and assume that the series

$$f(\tau) = \sum_{n=-\infty}^{\infty} F(2n\pi + \tau)$$

converges uniformly for all $0 \leqslant \tau \leqslant 2\pi$.

Prove the Poisson summation formula

$$f(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{F}(n) e^{in\tau},$$

where $\hat{F}$ is the Fourier transform of F. [*Hint: You may show that*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-imx} f(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-imx} F(x) dx$$

or, alternatively, prove that f is periodic and express its Fourier expansion coefficients explicitly in terms of $\hat{F}$.]

Letting $F(x) = e^{-|x|}$, use the Poisson summation formula to evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \ .$$

2/I/7E **Complex Methods**

A complex function is defined for every $z \in V$, where V is a non-empty open subset of $\mathbb{C}$, and it possesses a derivative at every $z \in V$. Commencing from a formal definition of derivative, deduce the Cauchy–Riemann equations.

2/II/16E **Complex Methods**

Let R be a rational function such that $\lim_{z\to\infty}\{zR(z)\} = 0$. Assuming that R has no real poles, use the residue calculus to evaluate

$$\int_{-\infty}^{\infty} R(x)dx.$$

Given that $n \geqslant 1$ is an integer, evaluate

$$\int_0^{\infty} \frac{dx}{1+x^{2n}}.$$

4/I/8F **Complex Methods**

Consider a conformal mapping of the form

$$f(z) = \frac{a+bz}{c+dz}, \quad z \in \mathbb{C}\ ,$$

where $a, b, c, d \in \mathbb{C}$, and $ad \neq bc$. You may assume $b \neq 0$. Show that any such $f(z)$ which maps the unit circle onto itself is necessarily of the form

$$f(z) = e^{i\psi}\frac{a+z}{1+\bar{a}z}.$$

[*Hint: Show that it is always possible to choose* $b = 1$.]

4/II/17F **Complex Methods**

State Jordan's Lemma.

Consider the integral

$$I = \oint_C dz \frac{z\ \sin(xz)}{(a^2+z^2)\sin \pi z},$$

for real x and a. The rectangular contour C runs from $+\infty + i\epsilon$ to $-\infty + i\epsilon$, to $-\infty - i\epsilon$, to $+\infty - i\epsilon$ and back to $+\infty + i\epsilon$, where ϵ is infinitesimal and positive. Perform the integral in two ways to show that

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{n \sin nx}{a^2+n^2} = -\pi\frac{\sinh ax}{\sinh a\pi},$$

for $|x| < \pi$.

1/I/2H **Methods**

The even function $f(x)$ has the Fourier cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

in the interval $-\pi \leqslant x \leqslant \pi$. Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2.$$

Find the Fourier cosine series of x^2 in the same interval, and show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

1/II/11H **Methods**

Use the substitution $y = x^p$ to find the general solution of

$$\mathcal{L}_x y \equiv \frac{d^2 y}{dx^2} - \frac{2}{x^2} y = 0.$$

Find the Green's function $G(x, \xi)$, $0 < \xi < \infty$, which satisfies

$$\mathcal{L}_x G(x, \xi) = \delta(x - \xi)$$

for $x > 0$, subject to the boundary conditions $G(x, \xi) \to 0$ as $x \to 0$ and as $x \to \infty$, for each fixed ξ.

Hence, find the solution of the equation

$$\mathcal{L}_x y = \begin{cases} 1, & 0 \leqslant x < 1, \\ 0, & x > 1, \end{cases}$$

subject to the same boundary conditions.

Verify that both forms of your solution satisfy the appropriate equation and boundary conditions, and match at $x = 1$.

2/I/2G **Methods**

Show that the symmetric and antisymmetric parts of a second-rank tensor are themselves tensors, and that the decomposition of a tensor into symmetric and antisymmetric parts is unique.

For the tensor A having components

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix},$$

find the scalar a, vector $\mathbf{p}$ and symmetric traceless tensor B such that

$$A\mathbf{x} = a\mathbf{x} + \mathbf{p} \wedge \mathbf{x} + B\mathbf{x}$$

for every vector $\mathbf{x}$.

2/II/11G **Methods**

Explain what is meant by an *isotropic* tensor.

Show that the fourth-rank tensor

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \qquad (*)$$

is isotropic for arbitrary scalars α, β and γ.

Assuming that the most general isotropic tensor of rank 4 has the form $(*)$, or otherwise, evaluate

$$B_{ijkl} = \int_{r<a} x_i x_j \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV,$$

where $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$.

3/I/2G Methods

Laplace's equation in the plane is given in terms of plane polar coordinates r and θ in the form

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

In each of the cases

$$\text{(i)} \quad 0 \leqslant r \leqslant 1, \quad \text{and} \quad \text{(ii)} \quad 1 \leqslant r < \infty,$$

find the general solution of Laplace's equation which is single-valued and finite.

Solve also Laplace's equation in the annulus $a \leqslant r \leqslant b$ with the boundary conditions

$$\phi = 1 \quad \text{on} \quad r = a \quad \text{for} \quad \text{all} \quad \theta,$$

$$\phi = 2 \quad \text{on} \quad r = b \quad \text{for} \quad \text{all} \quad \theta.$$

3/II/12H Methods

Find the Fourier sine series representation on the interval $0 \leqslant x \leqslant l$ of the function

$$f(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ 1, & a \leqslant x \leqslant b, \\ 0, & b < x \leqslant l. \end{cases}$$

The motion of a struck string is governed by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\,\frac{\partial^2 y}{\partial x^2}, \quad \text{for} \quad 0 \leqslant x \leqslant l \quad \text{and} \quad t \geqslant 0,$$

subject to boundary conditions $y = 0$ at $x = 0$ and $x = l$ for $t \geqslant 0$, and to the initial conditions $y = 0$ and $\dfrac{\partial y}{\partial t} = \delta(x - \dfrac{1}{4}l)$ at $t = 0$.

Obtain the solution $y(x,t)$ for this motion. Evaluate $y(x,t)$ for $t = \frac{1}{2}l/c$, and sketch it clearly.

4/I/2H Methods

The Legendre polynomial $P_n(x)$ satisfies

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad n = 0,1,\ldots, \ -1 \leqslant x \leqslant 1.$$

Show that $R_n(x) = P_n'(x)$ obeys an equation which can be recast in Sturm–Liouville form and has the eigenvalue $(n-1)(n+2)$. What is the orthogonality relation for $R_n(x), R_m(x)$ for $n \neq m$?

4/II/11H **Methods**

A curve $y(x)$ in the xy-plane connects the points $(\pm a, 0)$ and has a fixed length l, $2a < l < \pi a$. Find an expression for the area A of the surface of the revolution obtained by rotating $y(x)$ about the x-axis.

Show that the area A has a stationary value for

$$y = \frac{1}{k}(\cosh kx - \cosh ka),$$

where k is a constant such that

$$lk = 2\sinh ka.$$

Show that the latter equation admits a unique positive solution for k.

1/I/9F **Quantum Mechanics**

A quantum mechanical particle of mass m and energy E encounters a potential step,

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geqslant 0. \end{cases}$$

Calculate the probability P that the particle is reflected in the case $E > V_0$.

If V_0 is positive, what is the limiting value of P when E tends to V_0? If V_0 is negative, what is the limiting value of P as V_0 tends to $-\infty$ for fixed E?

1/II/18F **Quantum Mechanics**

Consider a quantum-mechanical particle of mass m moving in a potential well,

$$V(x) = \begin{cases} 0, & -a < x < a, \\ \infty, & \text{elsewhere.} \end{cases}$$

(a) Verify that the set of normalised energy eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{1}{a}}\ \sin\left(\frac{n\pi(x+a)}{2a}\right), \qquad n = 1, 2, \ldots,$$

and evaluate the corresponding energy eigenvalues E_n.

(b) At time $t = 0$ the wavefunction for the particle is only nonzero in the positive half of the well,

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}}\ \sin\left(\frac{\pi x}{a}\right), & 0 < x < a, \\ 0, & \text{elsewhere.} \end{cases}$$

Evaluate the expectation value of the energy, first using

$$\langle E \rangle = \int_{-a}^{a} \psi H \psi dx,$$

and secondly using

$$\langle E \rangle = \sum_n |a_n|^2 E_n,$$

where the a_n are the expansion coefficients in

$$\psi(x) = \sum_n a_n \psi_n(x).$$

Hence, show that

$$1 = \frac{1}{2} + \frac{8}{\pi^2} \sum_{p=0}^{\infty} \frac{(2p+1)^2}{[(2p+1)^2 - 4]^2}\ .$$

2/I/9F Quantum Mechanics

Consider a solution $\psi(x,t)$ of the time-dependent Schrödinger equation for a particle of mass m in a potential $V(x)$. The expectation value of an operator $\mathcal{O}$ is defined as

$$\langle\ \mathcal{O}\ \rangle = \int dx\ \psi^*(x,t)\ \mathcal{O}\ \psi(x,t).$$

Show that

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m},$$

where

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x},$$

and that

$$\frac{d}{dt}\langle p\rangle = \left\langle -\frac{\partial V}{\partial x}(x)\right\rangle.$$

[*You may assume that* $\psi(x,t)$ *vanishes as* $x \to \pm\infty$.]

2/II/18F Quantum Mechanics

(a) Write down the angular momentum operators L_1, L_2, L_3 in terms of x_i and

$$p_i = -i\hbar\frac{\partial}{\partial x_i},\ i = 1,2,3.$$

Verify the commutation relation

$$[L_1, L_2] = i\hbar L_3.$$

Show that this result and its cyclic permutations imply

$$[L_3, L_1 \pm iL_2] = \pm\hbar\ (L_1 \pm iL_2),$$
$$[\mathbf{L}^2, L_1 \pm iL_2] = 0.$$

(b) Consider a wavefunction of the form $\psi = (x_3^2 + ar^2)f(r)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. Show that for a particular value of a, ψ is an eigenfunction of both $\mathbf{L}^2$ and L_3. What are the corresponding eigenvalues?

3/II/20F **Quantum Mechanics**

A quantum system has a complete set of orthonormalised energy eigenfunctions $\psi_n(x)$ with corresponding energy eigenvalues $E_n,\ n = 1, 2, 3, \ldots$.

(a) If the time-dependent wavefunction $\psi(x,t)$ is, at $t = 0$,

$$\psi(x,0) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

determine $\psi(x,t)$ for all $t > 0$.

(b) A linear operator $\mathcal{S}$ acts on the energy eigenfunctions as follows:

$$\begin{aligned} \mathcal{S}\psi_1 &= 7\psi_1 + 24\psi_2, \\ \mathcal{S}\psi_2 &= 24\psi_1 - 7\psi_2, \\ \mathcal{S}\psi_n &= 0, \quad n \geqslant 3. \end{aligned}$$

Find the eigenvalues of $\mathcal{S}$. At time $t = 0$, $\mathcal{S}$ is measured and its lowest eigenvalue is found. At time $t > 0$, $\mathcal{S}$ is measured again. Show that the probability for obtaining the lowest eigenvalue again is

$$\frac{1}{625}\left(337 + 288\cos(\omega t)\right),$$

where $\omega = (E_1 - E_2)/\hbar$.

3/I/10F **Special Relativity**

A particle of rest mass m and four-momentum $P = (E/c, \mathbf{p})$ is detected by an observer with four-velocity U, satisfying $U \cdot U = c^2$, where the product of two four-vectors $P = (p^0, \mathbf{p})$ and $Q = (q^0, \mathbf{q})$ is $P \cdot Q = p^0 q^0 - \mathbf{p} \cdot \mathbf{q}$.

Show that the speed of the detected particle in the observer's rest frame is

$$v = c\,\sqrt{1 - \frac{P \cdot P c^2}{(P \cdot U)^2}}.$$

4/I/9F **Special Relativity**

What is Einstein's principle of relativity?

Show that a spherical shell expanding at the speed of light, $\mathbf{x}^2 = c^2 t^2$, in one coordinate system $(t, \mathbf{x})$, is still spherical in a second coordinate system $(t', \mathbf{x}')$ defined by

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{u}{c}x\right), \\ x' &= \gamma(x - ut), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$. Why do these equations define a Lorentz transformation?

4/II/18F **Special Relativity**

A particle of mass M is at rest at $x = 0$, in coordinates (t, x). At time $t = 0$ it decays into two particles A and B of equal mass $m < M/2$. Assume that particle A moves in the *negative* x direction.

(a) Using relativistic energy and momentum conservation compute the energy, momentum and velocity of both particles A and B.

(b) After a proper time τ, measured in its own rest frame, particle A decays. Show that the spacetime coordinates of this event are

$$\begin{aligned} t &= \frac{M}{2m}\tau, \\ x &= -\frac{MV}{2m}\tau, \end{aligned}$$

where $V = c\sqrt{1 - 4(m/M)^2}$.

1/I/6G **Fluid Dynamics**

Determine the pressure at a depth z below the surface of a static fluid of density ρ subject to gravity g. A rigid body having volume V is fully submerged in such a fluid. Calculate the buoyancy force on the body.

An iceberg of uniform density ρ_I is observed to float with volume V_I protruding above a large static expanse of seawater of density ρ_w. What is the total volume of the iceberg?

1/II/15G **Fluid Dynamics**

A fluid motion has velocity potential $\phi(x, y, t)$ given by

$$\phi = \epsilon y \cos (x - t)$$

where ϵ is a constant. Find the corresponding velocity field $\mathbf{u}\,(x, y, t)$. Determine $\nabla \cdot \mathbf{u}$.

The *time-average* of a quantity $\psi\,(x, y, t)$ is defined as $\frac{1}{2\pi}\int_0^{2\pi} \psi\,(x, y, t)\,dt$.

Show that the time-average of this velocity field at every point (x, y) is zero.

Write down an expression for the fluid acceleration and find the time-average acceleration at (x, y).

Suppose now that $|\epsilon| \ll 1$. The material particle at $(0, 0)$ at time $t = 0$ is marked with dye. Write down equations for its subsequent motion and verify that its position (x, y) at time $t > 0$ is given (correct to terms of order ϵ^2) as

$$\begin{aligned} x &= \epsilon^2(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t), \\ y &= \epsilon \sin t\ . \end{aligned}$$

Deduce the time-average velocity of the dyed particle correct to this order.

3/I/8G **Fluid Dynamics**

Inviscid incompressible fluid occupies the region $y > 0$, which is bounded by a rigid barrier along $y = 0$. At time $t = 0$, a line vortex of strength κ is placed at position (a, b). By considering the flow due to an image vortex at $(a, -b)$, or otherwise, determine the velocity potential in the fluid.

Derive the position of the original vortex at time $t > 0$.

3/II/18G Fluid Dynamics

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

A circular cylinder of radius a is immersed in unbounded inviscid fluid of uniform density ρ. The cylinder moves in a prescribed direction perpendicular to its axis, with speed U. Use cylindrical polar coordinates, with the direction $\theta = 0$ parallel to the direction of the motion, to find the velocity potential in the fluid.

If U depends on time t and gravity is negligible, determine the pressure field in the fluid at time t. Deduce the fluid force per unit length on the cylinder.

[In cylindrical polar coordinates, $\nabla\phi = \dfrac{\partial\phi}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$.]

4/I/7G Fluid Dynamics

Starting from the Euler equation, derive the *vorticity equation* for the motion of an inviscid incompressible fluid under a conservative body force, and give a physical interpretation of each term in the equation. Deduce that in a flow field of the form $\mathbf{u} = (u(x,y,t), v(x,y,t), 0)$ the vorticity of a material particle is conserved.

Find the vorticity for such a flow in terms of the stream function ψ. Deduce that if the flow is steady, there must be a function f such that

$$\nabla^2\psi = f(\psi)\ .$$

4/II/16G Fluid Dynamics

A long straight canal has rectangular cross-section with a horizontal bottom and width $w(x)$ that varies slowly with distance x downstream. Far upstream, w has a constant value W, the water depth has a constant value H, and the velocity has a constant value U. Assuming that the water velocity is steady and uniform across the channel, use mass conservation and Bernoulli's theorem, which should be stated carefully, to show that the water depth $h(x)$ satisfies

$$\left(\frac{W}{w}\right)^2 = \left(1 + \frac{2}{F}\right)\left(\frac{h}{H}\right)^2 - \frac{2}{F}\left(\frac{h}{H}\right)^3 \quad \text{where} \quad F = \frac{U^2}{gH}\ .$$

Deduce that for a given value of F, a flow of this kind can exist only if $w(x)$ is everywhere greater than or equal to a critical value w_c, which is to be determined in terms of F.

Suppose that $w(x) > w_c$ everywhere. At locations where the channel width exceeds W, determine graphically, or otherwise, under what circumstances the water depth exceeds H.

2/I/5E Numerical Analysis

Find an LU factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 & 2 \\ -4 & 3 & -4 & -2 \\ 4 & -2 & 3 & 6 \\ -6 & 5 & -8 & 1 \end{pmatrix},$$

and use it to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 4 \\ 11 \end{pmatrix}.$$

2/II/14E Numerical Analysis

(a) Let B be an $n \times n$ positive-definite, symmetric matrix. Define the Cholesky factorization of B and prove that it is unique.

(b) Let A be an $m \times n$ matrix, $m \geqslant n$, such that $\operatorname{rank} A = n$. Prove the uniqueness of the "skinny QR factorization"

$$A = QR,$$

where the matrix Q is $m \times n$ with orthonormal columns, while R is an $n \times n$ upper-triangular matrix with positive diagonal elements.

[*Hint: Show that you may choose R as a matrix that features in the Cholesky factorization of $B = A^T A$.*]

3/I/6E Numerical Analysis

Given $f \in C^{n+1}[a,b]$, let the nth-degree polynomial p interpolate the values $f(x_i)$, $i = 0, 1, \ldots, n$, where $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct. Given $x \in [a,b]$, find the error $f(x) - p(x)$ in terms of a derivative of f.

3/II/16E **Numerical Analysis**

Let the monic polynomials p_n, $n \geqslant 0$, be orthogonal with respect to the weight function $w(x) > 0$, $a < x < b$, where the degree of each p_n is exactly n.

(a) Prove that each p_n, $n \geqslant 1$, has n distinct zeros in the interval (a, b).

(b) Suppose that the p_n satisfy the three-term recurrence relation

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n^2 p_{n-2}(x), \quad n \geqslant 2,$$

where $p_0(x) \equiv 1$, $p_1(x) = x - a_1$. Set

$$A_n = \begin{pmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}, \quad n \geqslant 2.$$

Prove that $p_n(x) = \det(xI - A_n)$, $n \geqslant 2$, and deduce that all the eigenvalues of A_n reside in (a, b).

1/I/3D **Statistics**

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a two-dimensional sufficient statistic for μ, quoting carefully, without proof, any result you use.

What is the maximum likelihood estimator of μ?

1/II/12D **Statistics**

What is a *simple hypothesis*? Define the terms *size* and *power* for a test of one simple hypothesis against another.

State, without proof, the Neyman–Pearson lemma.

Let X be a **single** random variable, with distribution F. Consider testing the null hypothesis $H_0 : F$ is standard normal, $N(0,1)$, against the alternative hypothesis $H_1 : F$ is double exponential, with density $\frac{1}{4}e^{-|x|/2}, x \in \mathbb{R}$.

Find the test of size $\alpha, \alpha < \frac{1}{4}$, which maximises power, and show that the power is $e^{-t/2}$, where $\Phi(t) = 1 - \alpha/2$ and Φ is the distribution function of $N(0,1)$.

[*Hint: if* $X \sim N(0,1)$, $P(|X| > 1) = 0.3174$.]

2/I/3D **Statistics**

Suppose the **single** random variable X has a uniform distribution on the interval $[0, \theta]$ and it is required to estimate θ with the loss function

$$L(\theta, a) = c(\theta - a)^2,$$

where $c > 0$.

Find the posterior distribution for θ and the optimal Bayes point estimate with respect to the prior distribution with density $p(\theta) = \theta e^{-\theta}$, $\theta > 0$.

2/II/12D Statistics

What is meant by a *generalized likelihood ratio test*? Explain in detail how to perform such a test.

Let $X_1, \ldots, X_n$ be independent random variables, and let X_i have a Poisson distribution with unknown mean λ_i, $i = 1, \ldots, n$.

Find the form of the generalized likelihood ratio statistic for testing $H_0 : \lambda_1 = \ldots = \lambda_n$, and show that it may be approximated by

$$\frac{1}{\bar{X}} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

If, for $n = 7$, you found that the value of this statistic was 27.3, would you accept H_0? Justify your answer.

4/I/3D Statistics

Consider the linear regression model

$$Y_i = \beta x_i + \epsilon_i,$$

$i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are given constants, and $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, with σ^2 unknown.

Find the least squares estimator $\widehat{\beta}$ of β. State, without proof, the distribution of $\widehat{\beta}$ and describe how you would test $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$, where β_0 is given.

4/II/12D Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, where μ and σ^2 are unknown.

Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2, based on $X_1, \ldots, X_n$. Show that $\widehat{\mu}$ and $\widehat{\sigma}^2$ are independent, and derive their distributions.

Suppose now it is intended to construct a "prediction interval" $I(X_1, \ldots, X_n)$ for a future, independent, $N(\mu, \sigma^2)$ random variable X_0. We require

$$P\Big\{X_0 \in I(X_1, \ldots, X_n)\Big\} = 1 - \alpha,$$

with the probability over the *joint* distribution of $X_0, X_1, \ldots, X_n$.

Let

$$I_\gamma(X_1, \ldots, X_n) = \left(\widehat{\mu} - \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}},\ \widehat{\mu} + \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}}\right).$$

By considering the distribution of $(X_0 - \widehat{\mu})/(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}})$, find the value of γ for which $P\{X_0 \in I_\gamma(X_1, \ldots, X_n)\} = 1 - \alpha$.

3/I/5D Optimization

Let $a_1, \ldots, a_n$ be given constants, not all equal.

Use the Lagrangian sufficiency theorem, which you should state clearly, without proof, to minimize $\sum_{i=1}^{n} x_i^2$ subject to the two constraints $\sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} a_i x_i = 0$.

3/II/15D Optimization

Consider the following linear programming problem,

$$\begin{array}{ll} \text{minimize} & (3-p)x_1 + px_2 \\ \text{subject to} & \begin{array}{rcl} 2x_1 + x_2 & \geqslant & 8 \\ x_1 + 3x_2 & \geqslant & 9 \\ x_1 & \leqslant & 6 \\ x_1,\ x_2 & \geqslant & 0. \end{array} \end{array}$$

Formulate the problem in a suitable way for solution by the two-phase simplex method.

Using the two-phase simplex method, show that if $2 \leqslant p \leqslant \frac{9}{4}$ then the optimal solution has objective function value $9 - p$, while if $\frac{9}{4} < p \leqslant 3$ the optimal objective function value is $18 - 5p$.

4/I/5D Optimization

Explain what is meant by a two-person zero-sum game with payoff matrix $A = [a_{ij}]$. Write down a set of sufficient conditions for a pair of strategies to be optimal for such a game.

A fair coin is tossed and the result is shown to player I, who must then decide to 'pass' or 'bet'. If he passes, he must pay player II £1. If he bets, player II, who does not know the result of the coin toss, may either 'fold' or 'call the bet'. If player II folds, she pays player I £1. If she calls the bet and the toss was a head, she pays player I £2; if she calls the bet and the toss was a tail, player I must pay her £2.

Formulate this as a two-person zero-sum game and find optimal strategies for the two players. Show that the game has value $\frac{1}{3}$.

[*Hint: Player I has four possible moves and player II two.*]

4/II/14D **Optimization**

Dumbledore Publishers must decide how many copies of the best-selling "History of Hogwarts" to print in the next two months to meet demand. It is known that the demands will be for 40 thousand and 60 thousand copies in the first and second months respectively, and these demands must be met on time. At the beginning of the first month, a supply of 10 thousand copies is available, from existing stock. During each month, Dumbledore can produce up to 40 thousand copies, at a cost of 400 galleons per thousand copies. By having employees work overtime, up to 150 thousand additional copies can be printed each month, at a cost of 450 galleons per thousand copies. At the end of each month, after production and the current month's demand has been satisfied, a holding cost of 20 galleons per thousand copies is incurred.

Formulate a transportation problem, with 5 supply points and 3 demand points, to minimize the sum of production and holding costs during the two month period, and solve it.

[*You may assume that copies produced during a month can be used to meet demand in that month.*]

1/I/8B **Quadratic Mathematics**

Let $q(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Define what is meant by the *discriminant* d of q, and show that q is positive-definite if and only if $a > 0 > d$. Define what it means for the form q to be *reduced*. For any integer $d < 0$, we define the class number $h(d)$ to be the number of positive-definite reduced binary quadratic forms (with integer coefficients) with discriminant d. Show that $h(d)$ is always finite (for negative d). Find $h(-39)$, and exhibit the corresponding reduced forms.

1/II/17B **Quadratic Mathematics**

Let ϕ be a symmetric bilinear form on a finite dimensional vector space V over a field k of characteristic $\neq 2$. Prove that the form ϕ may be diagonalized, and interpret the rank r of ϕ in terms of the resulting diagonal form.

For ϕ a symmetric bilinear form on a real vector space V of finite dimension n, define the *signature* σ of ϕ, proving that it is well-defined. A subspace U of V is called *null* if $\phi|_U \equiv 0$; show that V has a null subspace of dimension $n - \frac{1}{2}(r + |\sigma|)$, but no null subspace of higher dimension.

Consider now the quadratic form q on $\mathbb{R}^5$ given by

$$2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1).$$

Write down the matrix A for the corresponding symmetric bilinear form, and calculate $\det A$. Hence, or otherwise, find the rank and signature of q.

2/I/8B **Quadratic Mathematics**

Let V be a finite-dimensional vector space over a field k. Describe a bijective correspondence between the set of bilinear forms on V, and the set of linear maps of V to its dual space V^*. If ϕ_1, ϕ_2 are non-degenerate bilinear forms on V, prove that there exists an isomorphism $\alpha : V \to V$ such that $\phi_2(u, v) = \phi_1(u, \alpha v)$ for all $u, v \in V$. If furthermore both ϕ_1, ϕ_2 are symmetric, show that α is self-adjoint (i.e. equals its adjoint) with respect to ϕ_1.

2/II/17B Quadratic Mathematics

Suppose p is an odd prime and a an integer coprime to p. Define the Legendre symbol $(\frac{a}{p})$, and state (without proof) Euler's criterion for its calculation.

For j any positive integer, we denote by r_j the (unique) integer with $|r_j| \leq (p-1)/2$ and $r_j \equiv aj \bmod p$. Let l be the number of integers $1 \leq j \leq (p-1)/2$ for which r_j is negative. Prove that

$$\left(\frac{a}{p}\right) = (-1)^l.$$

Hence determine the odd primes for which 2 is a quadratic residue.

Suppose that $p_1, \ldots, p_m$ are primes congruent to 7 modulo 8, and let

$$N \ = \ 8(p_1 \ldots p_m)^2 - 1.$$

Show that 2 is a quadratic residue for any prime dividing N. Prove that N is divisible by some prime $p \equiv 7 \bmod 8$. Hence deduce that there are infinitely many primes congruent to 7 modulo 8.

3/I/9B Quadratic Mathematics

Let A be the Hermitian matrix

$$\begin{pmatrix} 1 & i & 2i \\ -i & 3 & -i \\ -2i & i & 5 \end{pmatrix}.$$

Explaining carefully the method you use, find a diagonal matrix D with **rational** entries, and an invertible (complex) matrix T such that $T^*DT = A$, where T^* here denotes the conjugated transpose of T.

Explain briefly why we cannot find T, D as above with T unitary.

[*You may assume that if a monic polynomial* $t^3 + a_2t^2 + a_1t + a_0$ *with integer coefficients has all its roots rational, then all its roots are in fact integers.*]

3/II/19B **Quadratic Mathematics**

Let J_1 denote the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose that T is a 2×2 upper-triangular real matrix with strictly positive diagonal entries and that ${J_1}^{-1}TJ_1T^{-1}$ is orthogonal. Verify that $J_1T = TJ_1$.

Prove that any real invertible matrix A has a decomposition $A = BC$, where B is an orthogonal matrix and C is an upper-triangular matrix with strictly positive diagonal entries.

Let A now denote a $2n \times 2n$ real matrix, and $A = BC$ be the decomposition of the previous paragraph. Let K denote the $2n \times 2n$ matrix with n copies of J_1 on the diagonal, and zeros elsewhere, and suppose that $KA = AK$. Prove that $K^{-1}CKC^{-1}$ is orthogonal. From this, deduce that the entries of $K^{-1}CKC^{-1}$ are zero, apart from n orthogonal 2×2 blocks $E_1, \ldots, E_n$ along the diagonal. Show that each E_i has the form ${J_1}^{-1}C_iJ_1{C_i}^{-1}$, for some 2×2 upper-triangular matrix C_i with strictly positive diagonal entries. Deduce that $KC = CK$ and $KB = BK$.

[*Hint: The invertible* $2n \times 2n$ *matrices* S *with* 2×2 *blocks* $S_1, \ldots, S_n$ *along the diagonal, but with all other entries below the diagonal zero, form a group under matrix multiplication.*]

2/I/4B **Further Analysis**

Define the terms *connected* and *path connected* for a topological space. If a topological space X is path connected, prove that it is connected.

Consider the following subsets of $\mathbb{R}^2$:

$$I = \{(x, 0): \ 0 \leq x \leq 1\}, \quad A = \{(0, y): \ \tfrac{1}{2} \leq y \leq 1\}, \text{ and}$$

$$J_n = \{(n^{-1}, \ y): \ 0 \leq y \leq 1\} \quad \text{for } n \geq 1.$$

Let

$$X = A \cup I \cup \bigcup_{n \geq 1} J_n$$

with the subspace (metric) topology. Prove that X is connected.

[*You may assume that any interval in* $\mathbb{R}$ *(with the usual topology) is connected.*]

2/II/13A **Further Analysis**

State Liouville's Theorem. Prove it by considering

$$\int_{|z|=R} \frac{f(z)\ dz}{(z-a)(z-b)}$$

and letting $R \to \infty$.

Prove that, if $g(z)$ is a function analytic on all of $\mathbb{C}$ with real and imaginary parts $u(z)$ and $v(z)$, then either of the conditions:

$$\text{(i) } u + v \geqslant 0 \text{ for all } z; \quad \text{or} \quad \text{(ii) } uv \geqslant 0 \text{ for all } z,$$

implies that $g(z)$ is constant.

3/I/3B **Further Analysis**

State a version of Rouché's Theorem. Find the number of solutions (counted with multiplicity) of the equation

$$z^4 = a(z-1)(z^2-1) + \tfrac{1}{2}$$

inside the open disc $\{z : |z| < \sqrt{2}\}$, for the cases $a = \frac{1}{3}, 12$ and 5.

[*Hint: For the case* $a = 5$, *you may find it helpful to consider the function* $(z^2-1)(z-2)(z-3)$.]

3/II/13B Further Analysis

If X and Y are topological spaces, describe the open sets in the *product topology* on $X \times Y$. If the topologies on X and Y are induced from metrics, prove that the same is true for the product.

What does it mean to say that a topological space is *compact*? If the topologies on X and Y are compact, prove that the same is true for the product.

4/I/4A Further Analysis

Let $f(z)$ be analytic in the disc $|z| < R$. Assume the formula

$$f'(z_0) = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{f(z)\,dz}{(z-z_0)^2}, \quad 0 \leqslant |z_0| < r < R.$$

By combining this formula with a complex conjugate version of Cauchy's Theorem, namely

$$0 = \int\limits_{|z|=r} \overline{f(z)}\,d\bar{z},$$

prove that

$$f'(0) = \frac{1}{\pi r} \int_0^{2\pi} u(\theta) e^{-i\theta} d\theta,$$

where $u(\theta)$ is the real part of $f(re^{i\theta})$.

4/II/13B Further Analysis

Let $\Delta^* = \{z : 0 < |z| < r\}$ be a punctured disc, and f an analytic function on Δ^*. What does it mean to say that f has the origin as (i) a removable singularity, (ii) a pole, and (iii) an essential singularity? State criteria for (i), (ii), (iii) to occur, in terms of the Laurent series for f at 0.

Suppose now that the origin is an essential singularity for f. Given any $w \in \mathbb{C}$, show that there exists a sequence (z_n) of points in Δ^* such that $z_n \to 0$ and $f(z_n) \to w$. [*You may assume the fact that an isolated singularity is removable if the function is bounded in some open neighbourhood of the singularity.*]

State the Open Mapping Theorem. Prove that if f is analytic and injective on Δ^*, then the origin cannot be an essential singularity. By applying this to the function $g(1/z)$, or otherwise, deduce that if g is an injective analytic function on $\mathbb{C}$, then g is linear of the form $az + b$, for some non-zero complex number a. [*Here, you may assume that g injective implies that its derivative g' is nowhere vanishing.*]

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 6 June 2001 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions in Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, …, H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1A Analysis II

Define uniform continuity for functions defined on a (bounded or unbounded) interval in $\mathbb{R}$.

Is it true that a real function defined and uniformly continuous on $[0,1]$ is necessarily bounded?

Is it true that a real function defined and with a bounded derivative on $[1,\infty)$ is necessarily uniformly continuous there?

Which of the following functions are uniformly continuous on $[1,\infty)$:

(i) x^2;

(ii) $\sin(x^2)$;

(iii) $\dfrac{\sin x}{x}$?

Justify your answers.

2H Methods

The even function $f(x)$ has the Fourier cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

in the interval $-\pi \leqslant x \leqslant \pi$. Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 .$$

Find the Fourier cosine series of x^2 in the same interval, and show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

3D Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a two-dimensional sufficient statistic for μ, quoting carefully, without proof, any result you use.

What is the maximum likelihood estimator of μ?

4B Geometry

Write down the Riemannian metric on the disc model Δ of the hyperbolic plane. What are the geodesics passing through the origin? Show that the hyperbolic circle of radius ρ centred on the origin is just the Euclidean circle centred on the origin with Euclidean radius $\tanh(\rho/2)$.

Write down an isometry between the upper half-plane model H of the hyperbolic plane and the disc model Δ, under which $i \in H$ corresponds to $0 \in \Delta$. Show that the hyperbolic circle of radius ρ centred on i in H is a Euclidean circle with centre $i \cosh \rho$ and of radius $\sinh \rho$.

5C Linear Mathematics

Determine for which values of $x \in \mathbb{C}$ the matrix

$$M = \begin{pmatrix} x & 1 & 1 \\ 1-x & 0 & -1 \\ 2 & 2x & 1 \end{pmatrix}$$

is invertible. Determine the rank of M as a function of x. Find the adjugate and hence the inverse of M for general x.

6G Fluid Dynamics

Determine the pressure at a depth z below the surface of a static fluid of density ρ subject to gravity g. A rigid body having volume V is fully submerged in such a fluid. Calculate the buoyancy force on the body.

An iceberg of uniform density ρ_I is observed to float with volume V_I protruding above a large static expanse of seawater of density ρ_w. What is the total volume of the iceberg?

7E Complex Methods

State the Cauchy integral formula.

Assuming that the function $f(z)$ is analytic in the disc $|z| < 1$, prove that, for every $0 < r < 1$, it is true that

$$\frac{d^n f(0)}{dz^n} = \frac{n!}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi^{n+1}} d\xi, \qquad n = 0, 1, \ldots .$$

[*Taylor's theorem may be used if clearly stated.*]

8B Quadratic Mathematics

Let $q(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Define what is meant by the *discriminant* d of q, and show that q is positive-definite if and only if $a > 0 > d$. Define what it means for the form q to be *reduced*. For any integer $d < 0$, we define the class number $h(d)$ to be the number of positive-definite reduced binary quadratic forms (with integer coefficients) with discriminant d. Show that $h(d)$ is always finite (for negative d). Find $h(-39)$, and exhibit the corresponding reduced forms.

9F Quantum Mechanics

A quantum mechanical particle of mass m and energy E encounters a potential step,

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geqslant 0. \end{cases}$$

Calculate the probability P that the particle is reflected in the case $E > V_0$.

If V_0 is positive, what is the limiting value of P when E tends to V_0? If V_0 is negative, what is the limiting value of P as V_0 tends to $-\infty$ for fixed E?

SECTION II

10A Analysis II

Show that each of the functions below is a metric on the set of functions $x(t) \in C[a,b]$:

$$d_1(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|,$$

$$d_2(x,y) = \left\{ \int_a^b |x(t) - y(t)|^2 dt \right\}^{1/2}.$$

Is the space complete in the d_1 metric? Justify your answer.

Show that the set of functions

$$x_n(t) = \begin{cases} 0, & -1 \leqslant t < 0 \\ nt, & 0 \leqslant t < 1/n \\ 1, & 1/n \leqslant t \leqslant 1 \end{cases}$$

is a Cauchy sequence with respect to the d_2 metric on $C[-1,1]$, yet does not tend to a limit in the d_2 metric in this space. Hence, deduce that this space is not complete in the d_2 metric.

11H Methods

Use the substitution $y = x^p$ to find the general solution of

$$\mathcal{L}_x y \equiv \frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0.$$

Find the Green's function $G(x,\xi)$, $0 < \xi < \infty$, which satisfies

$$\mathcal{L}_x G(x,\xi) = \delta(x-\xi)$$

for $x > 0$, subject to the boundary conditions $G(x,\xi) \to 0$ as $x \to 0$ and as $x \to \infty$, for each fixed ξ.

Hence, find the solution of the equation

$$\mathcal{L}_x y = \begin{cases} 1, & 0 \leqslant x < 1, \\ 0, & x > 1, \end{cases}$$

subject to the same boundary conditions.

Verify that both forms of your solution satisfy the appropriate equation and boundary conditions, and match at $x = 1$.

12D Statistics

What is a *simple hypothesis*? Define the terms *size* and *power* for a test of one simple hypothesis against another.

State, without proof, the Neyman–Pearson lemma.

Let X be a **single** random variable, with distribution F. Consider testing the null hypothesis $H_0 : F$ is standard normal, $N(0,1)$, against the alternative hypothesis $H_1 : F$ is double exponential, with density $\frac{1}{4}e^{-|x|/2}, x \in \mathbb{R}$.

Find the test of size $\alpha, \alpha < \frac{1}{4}$, which maximises power, and show that the power is $e^{-t/2}$, where $\Phi(t) = 1 - \alpha/2$ and Φ is the distribution function of $N(0,1)$.

[*Hint: if* $X \sim N(0,1)$, $P(|X| > 1) = 0.3174$.]

13B Geometry

Describe geometrically the stereographic projection map ϕ from the unit sphere S^2 to the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, and find a formula for ϕ. Show that any rotation of S^2 about the z-axis corresponds to a Möbius transformation of $\mathbb{C}_\infty$. You are given that the rotation of S^2 defined by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

corresponds under ϕ to a Möbius transformation of $\mathbb{C}_\infty$; deduce that any rotation of S^2 about the x-axis also corresponds to a Möbius transformation.

Suppose now that $u, v \in \mathbb{C}$ correspond under ϕ to distinct points $P, Q \in S^2$, and let d denote the angular distance from P to Q on S^2. Show that $-\tan^2(d/2)$ is the cross-ratio of the points $u, v, -1/\bar{u}, -1/\bar{v}$, taken in some order (which you should specify). [*You may assume that the cross-ratio is invariant under Möbius transformations.*]

14C Linear Mathematics

(a) Find a matrix M over $\mathbb{C}$ with both minimal polynomial and characteristic polynomial equal to $(x-2)^3(x+1)^2$. Furthermore find two matrices M_1 and M_2 over $\mathbb{C}$ which have the same characteristic polynomial, $(x-3)^5(x-1)^2$, and the same minimal polynomial, $(x-3)^2(x-1)^2$, but which are not conjugate to one another. Is it possible to find a third such matrix, M_3, neither conjugate to M_1 nor to M_2? Justify your answer.

(b) Suppose A is an $n \times n$ matrix over $\mathbb{R}$ which has minimal polynomial of the form $(x-\lambda_1)(x-\lambda_2)$ for distinct roots $\lambda_1 \neq \lambda_2$ in $\mathbb{R}$. Show that the vector space $V = \mathbb{R}^n$ on which A defines an endomorphism $\alpha : V \to V$ decomposes as a direct sum into $V = \ker(\alpha - \lambda_1 \iota) \oplus \ker(\alpha - \lambda_2 \iota)$, where ι is the identity.

[*Hint: Express* $v \in V$ *in terms of* $(\alpha - \lambda_1\iota)(v)$ *and* $(\alpha - \lambda_2\iota)(v)$.]

Now suppose that A has minimal polynomial $(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m)$ for distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. By induction or otherwise show that

$$V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota) \oplus \ldots \oplus \ker(\alpha - \lambda_m\iota).$$

Use this last statement to prove that an arbitrary matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if all roots of its minimal polynomial lie in $\mathbb{R}$ and have multiplicity 1.

15G Fluid Dynamics

A fluid motion has velocity potential $\phi(x, y, t)$ given by

$$\phi = \epsilon y \cos(x - t)$$

where ϵ is a constant. Find the corresponding velocity field $\mathbf{u}(x, y, t)$. Determine $\nabla \cdot \mathbf{u}$.

The *time-average* of a quantity $\psi(x, y, t)$ is defined as $\frac{1}{2\pi}\int_0^{2\pi} \psi(x, y, t)\, dt$.

Show that the time-average of this velocity field at every point (x, y) is zero.

Write down an expression for the fluid acceleration and find the time-average acceleration at (x, y).

Suppose now that $|\epsilon| \ll 1$. The material particle at $(0, 0)$ at time $t = 0$ is marked with dye. Write down equations for its subsequent motion and verify that its position (x, y) at time $t > 0$ is given (correct to terms of order ϵ^2) as

$$x = \epsilon^2(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t),$$
$$y = \epsilon \sin t\ .$$

Deduce the time-average velocity of the dyed particle correct to this order.

16E Complex Methods

Let the function F be integrable for all real arguments x, such that

$$\int_{-\infty}^{\infty} |F(x)| dx < \infty \ ,$$

and assume that the series

$$f(\tau) = \sum_{n=-\infty}^{\infty} F(2n\pi + \tau)$$

converges uniformly for all $0 \leqslant \tau \leqslant 2\pi$.

Prove the Poisson summation formula

$$f(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{F}(n) e^{in\tau},$$

where $\hat{F}$ is the Fourier transform of F. [*Hint: You may show that*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-imx} f(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-imx} F(x) dx$$

or, alternatively, prove that f is periodic and express its Fourier expansion coefficients explicitly in terms of $\hat{F}$.]

Letting $F(x) = e^{-|x|}$, use the Poisson summation formula to evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \ .$$

17B Quadratic Mathematics

Let ϕ be a symmetric bilinear form on a finite dimensional vector space V over a field k of characteristic $\neq 2$. Prove that the form ϕ may be diagonalized, and interpret the rank r of ϕ in terms of the resulting diagonal form.

For ϕ a symmetric bilinear form on a real vector space V of finite dimension n, define the *signature* σ of ϕ, proving that it is well-defined. A subspace U of V is called *null* if $\phi|_U \equiv 0$; show that V has a null subspace of dimension $n - \frac{1}{2}(r + |\sigma|)$, but no null subspace of higher dimension.

Consider now the quadratic form q on $\mathbb{R}^5$ given by

$$2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1).$$

Write down the matrix A for the corresponding symmetric bilinear form, and calculate $\det A$. Hence, or otherwise, find the rank and signature of q.

18F Quantum Mechanics

Consider a quantum-mechanical particle of mass m moving in a potential well,

$$V(x) = \begin{cases} 0, & -a < x < a, \\ \infty, & \text{elsewhere.} \end{cases}$$

(a) Verify that the set of normalised energy eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{1}{a}}\ \sin\left(\frac{n\pi(x+a)}{2a}\right), \qquad n = 1, 2, \ldots,$$

and evaluate the corresponding energy eigenvalues E_n.

(b) At time $t = 0$ the wavefunction for the particle is only nonzero in the positive half of the well,

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}}\ \sin\left(\frac{\pi x}{a}\right), & 0 < x < a, \\ 0, & \text{elsewhere.} \end{cases}$$

Evaluate the expectation value of the energy, first using

$$\langle E \rangle = \int_{-a}^{a} \psi H \psi dx,$$

and secondly using

$$\langle E \rangle = \sum_n |a_n|^2 E_n,$$

where the a_n are the expansion coefficients in

$$\psi(x) = \sum_n a_n \psi_n(x).$$

Hence, show that

$$1 = \frac{1}{2} + \frac{8}{\pi^2} \sum_{p=0}^{\infty} \frac{(2p+1)^2}{[(2p+1)^2 - 4]^2}\ .$$

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 7 June 2001 9 to 12

PAPER 2

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It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1A Analysis II

State and prove the contraction mapping theorem.

Let $A = \{x, y, z\}$, let d be the discrete metric on A, and let d' be the metric given by: d' is symmetric and

$$d'(x,y) = 2,\ d'(x,z) = 2,\ d'(y,z) = 1,$$

$$d'(x,x) = d'(y,y) = d'(z,z) = 0.$$

Verify that d' is a metric, and that it is Lipschitz equivalent to d.

Define an appropriate function $f : A \to A$ such that f is a contraction in the d' metric, but not in the d metric.

2G Methods

Show that the symmetric and antisymmetric parts of a second-rank tensor are themselves tensors, and that the decomposition of a tensor into symmetric and antisymmetric parts is unique.

For the tensor A having components

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix},$$

find the scalar a, vector $\mathbf{p}$ and symmetric traceless tensor B such that

$$A\mathbf{x} = a\mathbf{x} + \mathbf{p} \wedge \mathbf{x} + B\mathbf{x}$$

for every vector $\mathbf{x}$.

3D Statistics

Suppose the **single** random variable X has a uniform distribution on the interval $[0, \theta]$ and it is required to estimate θ with the loss function

$$L(\theta, a) = c(\theta - a)^2,$$

where $c > 0$.

Find the posterior distribution for θ and the optimal Bayes point estimate with respect to the prior distribution with density $p(\theta) = \theta e^{-\theta},\ \theta > 0$.

4B Further Analysis

Define the terms *connected* and *path connected* for a topological space. If a topological space X is path connected, prove that it is connected.

Consider the following subsets of $\mathbb{R}^2$:

$$I = \{(x,0) : \ 0 \le x \le 1\}, \quad A = \{(0,y) : \ \tfrac{1}{2} \le y \le 1\}, \text{ and}$$

$$J_n = \{(n^{-1}, \ y) : \ 0 \le y \le 1\} \quad \text{for } n \ge 1.$$

Let

$$X = A \cup I \cup \bigcup_{n \ge 1} J_n$$

with the subspace (metric) topology. Prove that X is connected.

[*You may assume that any interval in* $\mathbb{R}$ *(with the usual topology) is connected.*]

5E Numerical Analysis

Find an LU factorization of the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 & 2 \\ -4 & 3 & -4 & -2 \\ 4 & -2 & 3 & 6 \\ -6 & 5 & -8 & 1 \end{pmatrix},$$

and use it to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 4 \\ 11 \end{pmatrix}.$$

6C Linear Mathematics

Show that right multiplication by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$ defines a linear transformation $\rho_A : M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$. Find the matrix representing ρ_A with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_{2\times 2}(\mathbb{C})$. Prove that the characteristic polynomial of ρ_A is equal to the square of the characteristic polynomial of A, and that A and ρ_A have the same minimal polynomial.

7E Complex Methods

A complex function is defined for every $z \in V$, where V is a non-empty open subset of $\mathbb{C}$, and it possesses a derivative at every $z \in V$. Commencing from a formal definition of derivative, deduce the Cauchy–Riemann equations.

8B Quadratic Mathematics

Let V be a finite-dimensional vector space over a field k. Describe a bijective correspondence between the set of bilinear forms on V, and the set of linear maps of V to its dual space V^*. If ϕ_1, ϕ_2 are non-degenerate bilinear forms on V, prove that there exists an isomorphism $\alpha : V \to V$ such that $\phi_2(u, v) = \phi_1(u, \alpha v)$ for all $u, v \in V$. If furthermore both ϕ_1, ϕ_2 are symmetric, show that α is self-adjoint (i.e. equals its adjoint) with respect to ϕ_1.

9F Quantum Mechanics

Consider a solution $\psi(x,t)$ of the time-dependent Schrödinger equation for a particle of mass m in a potential $V(x)$. The expectation value of an operator $\mathcal{O}$ is defined as

$$\langle\ \mathcal{O}\ \rangle = \int dx\ \psi^*(x,t)\ \mathcal{O}\ \psi(x,t).$$

Show that

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m},$$

where

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x},$$

and that

$$\frac{d}{dt}\langle p\rangle = \left\langle -\frac{\partial V}{\partial x}(x)\right\rangle.$$

[*You may assume that $\psi(x,t)$ vanishes as $x \to \pm\infty$.*]

SECTION II

10A Analysis II

Define total boundedness for metric spaces.

Prove that a metric space has the Bolzano–Weierstrass property if and only if it is complete and totally bounded.

11G Methods

Explain what is meant by an *isotropic* tensor.

Show that the fourth-rank tensor

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \qquad (*)$$

is isotropic for arbitrary scalars α, β and γ.

Assuming that the most general isotropic tensor of rank 4 has the form $(*)$, or otherwise, evaluate

$$B_{ijkl} = \int_{r<a} x_i x_j \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) dV,$$

where $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$.

12D Statistics

What is meant by a *generalized likelihood ratio test*? Explain in detail how to perform such a test.

Let $X_1, \ldots, X_n$ be independent random variables, and let X_i have a Poisson distribution with unknown mean λ_i, $i = 1, \ldots, n$.

Find the form of the generalized likelihood ratio statistic for testing $H_0 : \lambda_1 = \ldots = \lambda_n$, and show that it may be approximated by

$$\frac{1}{\bar{X}} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

If, for $n = 7$, you found that the value of this statistic was 27.3, would you accept H_0? Justify your answer.

13A Further Analysis

State Liouville's Theorem. Prove it by considering

$$\int_{|z|=R} \frac{f(z)\, dz}{(z-a)(z-b)}$$

and letting $R \to \infty$.

Prove that, if $g(z)$ is a function analytic on all of $\mathbb{C}$ with real and imaginary parts $u(z)$ and $v(z)$, then either of the conditions:

$$\text{(i) } u + v \geqslant 0 \text{ for all } z; \quad \text{or} \quad \text{(ii) } uv \geqslant 0 \text{ for all } z,$$

implies that $g(z)$ is constant.

14E Numerical Analysis

(a) Let B be an $n \times n$ positive-definite, symmetric matrix. Define the Cholesky factorization of B and prove that it is unique.

(b) Let A be an $m \times n$ matrix, $m \geqslant n$, such that $\operatorname{rank} A = n$. Prove the uniqueness of the "skinny QR factorization"

$$A = QR,$$

where the matrix Q is $m \times n$ with orthonormal columns, while R is an $n \times n$ upper-triangular matrix with positive diagonal elements.

[*Hint: Show that you may choose R as a matrix that features in the Cholesky factorization of $B = A^T A$.*]

15C Linear Mathematics

Define the dual V^* of a vector space V. Given a basis $\{v_1, \ldots, v_n\}$ of V define its dual and show it is a basis of V^*. For a linear transformation $\alpha : V \to W$ define the dual $\alpha^* : W^* \to V^*$.

Explain (with proof) how the matrix representing $\alpha : V \to W$ with respect to given bases of V and W relates to the matrix representing $\alpha^* : W^* \to V^*$ with respect to the corresponding dual bases of V^* and W^*.

Prove that α and α^* have the same rank.

Suppose that α is an invertible endomorphism. Prove that $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

16E Complex Methods

Let R be a rational function such that $\lim_{z\to\infty}\{zR(z)\} = 0$. Assuming that R has no real poles, use the residue calculus to evaluate

$$\int_{-\infty}^{\infty} R(x)dx.$$

Given that $n \geqslant 1$ is an integer, evaluate

$$\int_0^{\infty} \frac{dx}{1+x^{2n}}.$$

17B Quadratic Mathematics

Suppose p is an odd prime and a an integer coprime to p. Define the Legendre symbol $(\frac{a}{p})$, and state (without proof) Euler's criterion for its calculation.

For j any positive integer, we denote by r_j the (unique) integer with $|r_j| \leq (p-1)/2$ and $r_j \equiv aj \bmod p$. Let l be the number of integers $1 \leq j \leq (p-1)/2$ for which r_j is negative. Prove that

$$\left(\frac{a}{p}\right) = (-1)^l.$$

Hence determine the odd primes for which 2 is a quadratic residue.

Suppose that $p_1, \ldots, p_m$ are primes congruent to 7 modulo 8, and let

$$N \;=\; 8(p_1 \ldots p_m)^2 - 1.$$

Show that 2 is a quadratic residue for any prime dividing N. Prove that N is divisible by some prime $p \equiv 7 \bmod 8$. Hence deduce that there are infinitely many primes congruent to 7 modulo 8.

18F Quantum Mechanics

(a) Write down the angular momentum operators L_1, L_2, L_3 in terms of x_i and

$$p_i = -i\hbar\frac{\partial}{\partial x_i},\ i = 1,2,3.$$

Verify the commutation relation

$$[L_1, L_2] = i\hbar L_3.$$

Show that this result and its cyclic permutations imply

$$[L_3, L_1 \pm iL_2] = \pm\hbar\ (L_1 \pm iL_2),$$
$$[\mathbf{L}^2, L_1 \pm iL_2] = 0.$$

(b) Consider a wavefunction of the form $\psi = (x_3^2 + ar^2)f(r)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. Show that for a particular value of a, ψ is an eigenfunction of both $\mathbf{L}^2$ and L_3. What are the corresponding eigenvalues?

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 8 June 2001 9 to 12

PAPER 3

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SECTION I

1A Analysis II

Define what is meant by a norm on a real vector space.

(a) Prove that two norms on a vector space (not necessarily finite-dimensional) give rise to equivalent metrics if and only if they are Lipschitz equivalent.

(b) Prove that if the vector space V has an inner product, then for all $x, y \in V$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

in the induced norm.

Hence show that the norm on $\mathbb{R}^2$ defined by $\|x\| = \max(|x_1|, |x_2|)$, where $x = (x_1, x_2) \in \mathbb{R}^2$, cannot be induced by an inner product.

2G Methods

Laplace's equation in the plane is given in terms of plane polar coordinates r and θ in the form

$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

In each of the cases

$$\text{(i)} \quad 0 \leqslant r \leqslant 1, \quad \text{and} \quad \text{(ii)} \quad 1 \leqslant r < \infty,$$

find the general solution of Laplace's equation which is single-valued and finite.

Solve also Laplace's equation in the annulus $a \leqslant r \leqslant b$ with the boundary conditions

$$\phi = 1 \quad \text{on} \quad r = a \quad \text{for} \quad \text{all} \quad \theta,$$

$$\phi = 2 \quad \text{on} \quad r = b \quad \text{for} \quad \text{all} \quad \theta.$$

3B Further Analysis

State a version of Rouché's Theorem. Find the number of solutions (counted with multiplicity) of the equation

$$z^4 = a(z-1)(z^2-1) + \tfrac{1}{2}$$

inside the open disc $\{z : |z| < \sqrt{2}\}$, for the cases $a = \frac{1}{3}, 12$ and 5.

[*Hint: For the case $a = 5$, you may find it helpful to consider the function $(z^2-1)(z-2)(z-3)$.*]

4B Geometry

State and prove the Gauss–Bonnet theorem for the area of a spherical triangle.

Suppose **D** is a regular dodecahedron, with centre the origin. Explain how each face of **D** gives rise to a spherical pentagon on the 2-sphere S^2. For each such spherical pentagon, calculate its angles and area.

5D Optimization

Let $a_1, \ldots, a_n$ be given constants, not all equal.

Use the Lagrangian sufficiency theorem, which you should state clearly, without proof, to minimize $\sum_{i=1}^{n} x_i^2$ subject to the two constraints $\sum_{i=1}^{n} x_i = 1, \sum_{i=1}^{n} a_i x_i = 0$.

6E Numerical Analysis

Given $f \in C^{n+1}[a,b]$, let the nth-degree polynomial p interpolate the values $f(x_i)$, $i = 0, 1, \ldots, n$, where $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct. Given $x \in [a,b]$, find the error $f(x) - p(x)$ in terms of a derivative of f.

7C Linear Mathematics

Determine the dimension of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \\ 5 \\ -1 \end{pmatrix}.$$

Write down a 5×5 matrix M which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ whose image is W and which contains $(1,1,1,1,1)^T$ in its kernel. What is the dimension of the space of all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1,1,1,1,1)^T$ in the kernel, and image contained in W?

8G Fluid Dynamics

Inviscid incompressible fluid occupies the region $y > 0$, which is bounded by a rigid barrier along $y = 0$. At time $t = 0$, a line vortex of strength κ is placed at position (a,b). By considering the flow due to an image vortex at $(a,-b)$, or otherwise, determine the velocity potential in the fluid.

Derive the position of the original vortex at time $t > 0$.

9B Quadratic Mathematics

Let A be the Hermitian matrix

$$\begin{pmatrix} 1 & i & 2i \\ -i & 3 & -i \\ -2i & i & 5 \end{pmatrix}.$$

Explaining carefully the method you use, find a diagonal matrix D with **rational** entries, and an invertible (complex) matrix T such that $T^*DT = A$, where T^* here denotes the conjugated transpose of T.

Explain briefly why we cannot find T, D as above with T unitary.

[*You may assume that if a monic polynomial* $t^3 + a_2t^2 + a_1t + a_0$ *with integer coefficients has all its roots rational, then all its roots are in fact integers.*]

10F Special Relativity

A particle of rest mass m and four-momentum $P = (E/c, \mathbf{p})$ is detected by an observer with four-velocity U, satisfying $U \cdot U = c^2$, where the product of two four-vectors $P = (p^0, \mathbf{p})$ and $Q = (q^0, \mathbf{q})$ is $P \cdot Q = p^0q^0 - \mathbf{p} \cdot \mathbf{q}$.

Show that the speed of the detected particle in the observer's rest frame is

$$v = c\,\sqrt{1 - \frac{P \cdot P c^2}{(P \cdot U)^2}}.$$

SECTION II

11A Analysis II

Prove that if all the partial derivatives of $f : \mathbb{R}^p \to \mathbb{R}$ (with $p \geqslant 2$) exist in an open set containing $(0, 0, \ldots, 0)$ and are continuous at this point, then f is differentiable at $(0, 0, \ldots, 0)$.

Let

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$f(x, y) = g(x) + g(y).$$

At which points of the plane is the partial derivative f_x continuous?

At which points is the function $f(x, y)$ differentiable? Justify your answers.

12H Methods

Find the Fourier sine series representation on the interval $0 \leqslant x \leqslant l$ of the function

$$f(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ 1, & a \leqslant x \leqslant b, \\ 0, & b < x \leqslant l. \end{cases}$$

The motion of a struck string is governed by the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{for} \quad 0 \leqslant x \leqslant l \quad \text{and} \quad t \geqslant 0,$$

subject to boundary conditions $y = 0$ at $x = 0$ and $x = l$ for $t \geqslant 0$, and to the initial conditions $y = 0$ and $\dfrac{\partial y}{\partial t} = \delta(x - \dfrac{1}{4}l)$ at $t = 0$.

Obtain the solution $y(x, t)$ for this motion. Evaluate $y(x, t)$ for $t = \frac{1}{2}l/c$, and sketch it clearly.

13B Further Analysis

If X and Y are topological spaces, describe the open sets in the *product topology* on $X \times Y$. If the topologies on X and Y are induced from metrics, prove that the same is true for the product.

What does it mean to say that a topological space is *compact*? If the topologies on X and Y are compact, prove that the same is true for the product.

14B Geometry

Describe the hyperbolic lines in the upper half-plane model H of the hyperbolic plane. The group $G = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ acts on H via Möbius transformations, which you may assume are isometries of H. Show that G acts transitively on the hyperbolic lines. Find explicit formulae for the reflection in the hyperbolic line L in the cases (i) L is a vertical line $x = a$, and (ii) L is the unit semi-circle with centre the origin. Verify that the composite T of a reflection of type (ii) followed afterwards by one of type (i) is given by $T(z) = -z^{-1} + 2a$.

Suppose now that L_1 and L_2 are distinct hyperbolic lines in the hyperbolic plane, with R_1, R_2 denoting the corresponding reflections. By considering different models of the hyperbolic plane, or otherwise, show that

(a) R_1R_2 has infinite order if L_1 and L_2 are parallel or ultraparallel, and

(b) R_1R_2 has finite order if and only if L_1 and L_2 meet at an angle which is a rational multiple of π.

15D Optimization

Consider the following linear programming problem,

$$\begin{array}{lrcl} \text{minimize} & (3-p)x_1 + px_2 & & \\ \text{subject to} & 2x_1 + x_2 & \geqslant & 8 \\ & x_1 + 3x_2 & \geqslant & 9 \\ & x_1 & \leqslant & 6 \\ & x_1,\ x_2 & \geqslant & 0. \end{array}$$

Formulate the problem in a suitable way for solution by the two-phase simplex method.

Using the two-phase simplex method, show that if $2 \leqslant p \leqslant \frac{9}{4}$ then the optimal solution has objective function value $9 - p$, while if $\frac{9}{4} < p \leqslant 3$ the optimal objective function value is $18 - 5p$.

16E Numerical Analysis

Let the monic polynomials p_n, $n \geqslant 0$, be orthogonal with respect to the weight function $w(x) > 0$, $a < x < b$, where the degree of each p_n is exactly n.

(a) Prove that each p_n, $n \geqslant 1$, has n distinct zeros in the interval (a, b).

(b) Suppose that the p_n satisfy the three-term recurrence relation

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n^2 p_{n-2}(x), \quad n \geqslant 2,$$

where $p_0(x) \equiv 1$, $p_1(x) = x - a_1$. Set

$$A_n = \begin{pmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ b_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}, \quad n \geqslant 2.$$

Prove that $p_n(x) = \det(xI - A_n)$, $n \geqslant 2$, and deduce that all the eigenvalues of A_n reside in (a, b).

17C Linear Mathematics

Let V be a vector space over $\mathbb{R}$. Let $\alpha : V \to V$ be a nilpotent endomorphism of V, i.e. $\alpha^m = 0$ for some positive integer m. Prove that α can be represented by a strictly upper-triangular matrix (with zeros along the diagonal). [*You may wish to consider the subspaces* $\ker(\alpha^j)$ *for* $j = 1, \ldots, m$.]

Show that if α is nilpotent, then $\alpha^n = 0$ where n is the dimension of V. Give an example of a 4×4 matrix M such that $M^4 = 0$ but $M^3 \neq 0$.

Let A be a nilpotent matrix and I the identity matrix. Prove that $I + A$ has all eigenvalues equal to 1. Is the same true of $(I+A)(I+B)$ if A and B are nilpotent? Justify your answer.

18G Fluid Dynamics

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

A circular cylinder of radius a is immersed in unbounded inviscid fluid of uniform density ρ. The cylinder moves in a prescribed direction perpendicular to its axis, with speed U. Use cylindrical polar coordinates, with the direction $\theta = 0$ parallel to the direction of the motion, to find the velocity potential in the fluid.

If U depends on time t and gravity is negligible, determine the pressure field in the fluid at time t. Deduce the fluid force per unit length on the cylinder.

[In cylindrical polar coordinates, $\nabla\phi = \dfrac{\partial\phi}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$.]

19B Quadratic Mathematics

Let J_1 denote the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose that T is a 2×2 upper-triangular real matrix with strictly positive diagonal entries and that ${J_1}^{-1}TJ_1T^{-1}$ is orthogonal. Verify that $J_1T = TJ_1$.

Prove that any real invertible matrix A has a decomposition $A = BC$, where B is an orthogonal matrix and C is an upper-triangular matrix with strictly positive diagonal entries.

Let A now denote a $2n \times 2n$ real matrix, and $A = BC$ be the decomposition of the previous paragraph. Let K denote the $2n \times 2n$ matrix with n copies of J_1 on the diagonal, and zeros elsewhere, and suppose that $KA = AK$. Prove that $K^{-1}CKC^{-1}$ is orthogonal. From this, deduce that the entries of $K^{-1}CKC^{-1}$ are zero, apart from n orthogonal 2×2 blocks $E_1, \ldots, E_n$ along the diagonal. Show that each E_i has the form ${J_1}^{-1}C_iJ_1{C_i}^{-1}$, for some 2×2 upper-triangular matrix C_i with strictly positive diagonal entries. Deduce that $KC = CK$ and $KB = BK$.

[*Hint: The invertible* $2n \times 2n$ *matrices* S *with* 2×2 *blocks* $S_1, \ldots, S_n$ *along the diagonal, but with all other entries below the diagonal zero, form a group under matrix multiplication.*]

20F Quantum Mechanics

A quantum system has a complete set of orthonormalised energy eigenfunctions $\psi_n(x)$ with corresponding energy eigenvalues $E_n,\ n = 1, 2, 3, \ldots$.

(a) If the time-dependent wavefunction $\psi(x,t)$ is, at $t = 0$,

$$\psi(x,0) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

determine $\psi(x,t)$ for all $t > 0$.

(b) A linear operator $\mathcal{S}$ acts on the energy eigenfunctions as follows:

$$\begin{aligned} \mathcal{S}\psi_1 &= 7\psi_1 + 24\psi_2, \\ \mathcal{S}\psi_2 &= 24\psi_1 - 7\psi_2, \\ \mathcal{S}\psi_n &= 0, \quad n \geqslant 3. \end{aligned}$$

Find the eigenvalues of $\mathcal{S}$. At time $t = 0,\ \mathcal{S}$ is measured and its lowest eigenvalue is found. At time $t > 0,\ \mathcal{S}$ is measured again. Show that the probability for obtaining the lowest eigenvalue again is

$$\frac{1}{625}\bigg(337 + 288\cos(\omega t)\bigg),$$

where $\omega = (E_1 - E_2)/\hbar$.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 8 June 2001 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions in Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, …, H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1A Analysis II

Let f be a mapping of a metric space (X, d) into itself such that $d(f(x), f(y)) < d(x, y)$ for all distinct x, y in X.

Show that $f(x)$ and $d(x, f(x))$ are continuous functions of x.

Now suppose that (X, d) is compact and let

$$h = \inf_{x \in X} d(x, f(x)).$$

Show that we cannot have $h > 0$.

[*You may assume that a continuous function on a compact metric space is bounded and attains its bounds.*]

Deduce that f possesses a fixed point, and that it is unique.

2H Methods

The Legendre polynomial $P_n(x)$ satisfies

$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad n = 0, 1, \ldots, \; -1 \leqslant x \leqslant 1.$$

Show that $R_n(x) = P_n'(x)$ obeys an equation which can be recast in Sturm–Liouville form and has the eigenvalue $(n-1)(n+2)$. What is the orthogonality relation for $R_n(x), R_m(x)$ for $n \neq m$?

3D Statistics

Consider the linear regression model

$$Y_i = \beta x_i + \epsilon_i,$$

$i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are given constants, and $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, with σ^2 unknown.

Find the least squares estimator $\widehat{\beta}$ of β. State, without proof, the distribution of $\widehat{\beta}$ and describe how you would test $H_0 : \beta = \beta_0$ against $H_1 : \beta \neq \beta_0$, where β_0 is given.

4A Further Analysis

Let $f(z)$ be analytic in the disc $|z| < R$. Assume the formula

$$f'(z_0) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)\,dz}{(z-z_0)^2}, \quad 0 \leqslant |z_0| < r < R.$$

By combining this formula with a complex conjugate version of Cauchy's Theorem, namely

$$0 = \int_{|z|=r} \overline{f(z)}\,d\bar{z},$$

prove that

$$f'(0) = \frac{1}{\pi r} \int_0^{2\pi} u(\theta) e^{-i\theta} d\theta,$$

where $u(\theta)$ is the real part of $f(re^{i\theta})$.

5D Optimization

Explain what is meant by a two-person zero-sum game with payoff matrix $A = [a_{ij}]$. Write down a set of sufficient conditions for a pair of strategies to be optimal for such a game.

A fair coin is tossed and the result is shown to player I, who must then decide to 'pass' or 'bet'. If he passes, he must pay player II £1. If he bets, player II, who does not know the result of the coin toss, may either 'fold' or 'call the bet'. If player II folds, she pays player I £1. If she calls the bet and the toss was a head, she pays player I £2; if she calls the bet and the toss was a tail, player I must pay her £2.

Formulate this as a two-person zero-sum game and find optimal strategies for the two players. Show that the game has value $\frac{1}{3}$.

[*Hint: Player I has four possible moves and player II two.*]

6C Linear Mathematics

Find the Jordan normal form J of the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

and determine both the characteristic and the minimal polynomial of M.

Find a basis of $\mathbb{C}^4$ such that J (the Jordan normal form of M) is the matrix representing the endomorphism $M : \mathbb{C}^4 \to \mathbb{C}^4$ in this basis. Give a change of basis matrix P such that $P^{-1}MP = J$.

7G Fluid Dynamics

Starting from the Euler equation, derive the *vorticity equation* for the motion of an inviscid incompressible fluid under a conservative body force, and give a physical interpretation of each term in the equation. Deduce that in a flow field of the form $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$ the vorticity of a material particle is conserved.

Find the vorticity for such a flow in terms of the stream function ψ. Deduce that if the flow is steady, there must be a function f such that

$$\nabla^2\psi = f(\psi) \ .$$

8F Complex Methods

Consider a conformal mapping of the form

$$f(z) = \frac{a + bz}{c + dz}, \quad z \in \mathbb{C} \ ,$$

where $a, b, c, d \in \mathbb{C}$, and $ad \neq bc$. You may assume $b \neq 0$. Show that any such $f(z)$ which maps the unit circle onto itself is necessarily of the form

$$f(z) = e^{i\psi} \frac{a + z}{1 + \bar{a}z} \cdot$$

[*Hint: Show that it is always possible to choose* $b = 1$.]

9F Special Relativity

What is Einstein's principle of relativity?

Show that a spherical shell expanding at the speed of light, $\mathbf{x}^2 = c^2t^2$, in one coordinate system $(t, \mathbf{x})$, is still spherical in a second coordinate system $(t', \mathbf{x}')$ defined by

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{u}{c}x\right), \\ x' &= \gamma(x - ut), \\ y' &= y, \\ z' &= z, \end{aligned}$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$. Why do these equations define a Lorentz transformation?

SECTION II

10A Analysis II

Let $\{f_n\}$ be a pointwise convergent sequence of real-valued functions on a closed interval $[a,b]$. Prove that, if for every $x \in [a,b]$, the sequence $\{f_n(x)\}$ is monotonic in n, and if all the functions $f_n,\ n = 1,2,\ldots$, and $f = \lim f_n$ are continuous, then $f_n \to f$ uniformly on $[a,b]$.

By considering a suitable sequence of functions $\{f_n\}$ on $[0,1)$, show that if the interval is not closed but all other conditions hold, the conclusion of the theorem may fail.

11H Methods

A curve $y(x)$ in the xy-plane connects the points $(\pm a, 0)$ and has a fixed length $l,\ 2a < l < \pi a$. Find an expression for the area A of the surface of the revolution obtained by rotating $y(x)$ about the x-axis.

Show that the area A has a stationary value for

$$y = \frac{1}{k}(\cosh kx - \cosh ka),$$

where k is a constant such that

$$lk = 2\sinh ka.$$

Show that the latter equation admits a unique positive solution for k.

12D Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, where μ and σ^2 are unknown.

Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2, based on $X_1, \ldots, X_n$. Show that $\widehat{\mu}$ and $\widehat{\sigma}^2$ are independent, and derive their distributions.

Suppose now it is intended to construct a "prediction interval" $I(X_1, \ldots, X_n)$ for a future, independent, $N(\mu, \sigma^2)$ random variable X_0. We require

$$P\Big\{X_0 \in I(X_1, \ldots, X_n)\Big\} = 1 - \alpha,$$

with the probability over the *joint* distribution of $X_0, X_1, \ldots, X_n$.

Let

$$I_\gamma(X_1, \ldots, X_n) = \left(\widehat{\mu} - \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}},\ \widehat{\mu} + \gamma\widehat{\sigma}\sqrt{1 + \frac{1}{n}}\right).$$

By considering the distribution of $(X_0 - \widehat{\mu})/(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}})$, find the value of γ for which $P\{X_0 \in I_\gamma(X_1, \ldots, X_n)\} = 1 - \alpha$.

13B Further Analysis

Let $\Delta^* = \{z : 0 < |z| < r\}$ be a punctured disc, and f an analytic function on Δ^*. What does it mean to say that f has the origin as (i) a removable singularity, (ii) a pole, and (iii) an essential singularity? State criteria for (i), (ii), (iii) to occur, in terms of the Laurent series for f at 0.

Suppose now that the origin is an essential singularity for f. Given any $w \in \mathbb{C}$, show that there exists a sequence (z_n) of points in Δ^* such that $z_n \to 0$ and $f(z_n) \to w$. [*You may assume the fact that an isolated singularity is removable if the function is bounded in some open neighbourhood of the singularity.*]

State the Open Mapping Theorem. Prove that if f is analytic and injective on Δ^*, then the origin cannot be an essential singularity. By applying this to the function $g(1/z)$, or otherwise, deduce that if g is an injective analytic function on $\mathbb{C}$, then g is linear of the form $az + b$, for some non-zero complex number a. [*Here, you may assume that g injective implies that its derivative g' is nowhere vanishing.*]

14D Optimization

Dumbledore Publishers must decide how many copies of the best-selling "History of Hogwarts" to print in the next two months to meet demand. It is known that the demands will be for 40 thousand and 60 thousand copies in the first and second months respectively, and these demands must be met on time. At the beginning of the first month, a supply of 10 thousand copies is available, from existing stock. During each month, Dumbledore can produce up to 40 thousand copies, at a cost of 400 galleons per thousand copies. By having employees work overtime, up to 150 thousand additional copies can be printed each month, at a cost of 450 galleons per thousand copies. At the end of each month, after production and the current month's demand has been satisfied, a holding cost of 20 galleons per thousand copies is incurred.

Formulate a transportation problem, with 5 supply points and 3 demand points, to minimize the sum of production and holding costs during the two month period, and solve it.

[*You may assume that copies produced during a month can be used to meet demand in that month.*]

15C Linear Mathematics

Let A and B be $n \times n$ matrices over $\mathbb{C}$. Show that AB and BA have the same characteristic polynomial. [*Hint: Look at* $\det(CBC - xC)$ *for* $C = A + yI$*, where* x *and* y *are scalar variables.*]

Show by example that AB and BA need not have the same minimal polynomial.

Suppose that AB is diagonalizable, and let $p(x)$ be its minimal polynomial. Show that the minimal polynomial of BA must divide $xp(x)$. Using this and the first part of the question prove that $(AB)^2$ and $(BA)^2$ are conjugate.

16G Fluid Dynamics

A long straight canal has rectangular cross-section with a horizontal bottom and width $w(x)$ that varies slowly with distance x downstream. Far upstream, w has a constant value W, the water depth has a constant value H, and the velocity has a constant value U. Assuming that the water velocity is steady and uniform across the channel, use mass conservation and Bernoulli's theorem, which should be stated carefully, to show that the water depth $h(x)$ satisfies

$$\left(\frac{W}{w}\right)^2 = \left(1+\frac{2}{F}\right)\left(\frac{h}{H}\right)^2 - \frac{2}{F}\left(\frac{h}{H}\right)^3 \quad \text{where} \quad F = \frac{U^2}{gH} \ .$$

Deduce that for a given value of F, a flow of this kind can exist only if $w(x)$ is everywhere greater than or equal to a critical value w_c, which is to be determined in terms of F.

Suppose that $w(x) > w_c$ everywhere. At locations where the channel width exceeds W, determine graphically, or otherwise, under what circumstances the water depth exceeds H.

17F Complex Methods

State Jordan's Lemma.

Consider the integral

$$I = \oint_C dz \frac{z \ \sin(xz)}{(a^2+z^2)\sin \pi z},$$

for real x and a. The rectangular contour C runs from $+\infty + i\epsilon$ to $-\infty + i\epsilon$, to $-\infty - i\epsilon$, to $+\infty - i\epsilon$ and back to $+\infty + i\epsilon$, where ϵ is infinitesimal and positive. Perform the integral in two ways to show that

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{n \sin nx}{a^2+n^2} = -\pi \frac{\sinh ax}{\sinh a\pi},$$

for $|x| < \pi$.

18F Special Relativity

A particle of mass M is at rest at $x = 0$, in coordinates (t, x). At time $t = 0$ it decays into two particles A and B of equal mass $m < M/2$. Assume that particle A moves in the *negative* x direction.

(a) Using relativistic energy and momentum conservation compute the energy, momentum and velocity of both particles A and B.

(b) After a proper time τ, measured in its own rest frame, particle A decays. Show that the spacetime coordinates of this event are

$$\begin{aligned} t &= \frac{M}{2m}\tau, \\ x &= -\frac{MV}{2m}\tau, \end{aligned}$$

where $V = c\sqrt{1 - 4(m/M)^2}$.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

List of Courses

1/I/5G **Linear Mathematics**

Define $f : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$f(a,b,c) = (a + 3b - c, 2b + c, -4b - c).$$

Find the characteristic polynomial and the minimal polynomial of f. Is f diagonalisable? Are f and f^2 linearly independent endomorphisms of $\mathbb{C}^3$? Justify your answers.

1/II/14G **Linear Mathematics**

Let α be an endomorphism of a vector space V of finite dimension n .

(a) What is the dimension of the vector space of linear endomorphisms of V? Show that there exists a non-trivial polynomial $p(X)$ such that $p(\alpha) = 0$. Define what is meant by the minimal polynomial m_α of α.

(b) Show that the eigenvalues of α are precisely the roots of the minimal polynomial of α.

(c) Let W be a subspace of V such that $\alpha(W) \subseteq W$ and let β be the restriction of α to W. Show that m_β divides m_α.

(d) Give an example of an endomorphism α and a subspace W as in (c) not equal to V for which $m_\alpha = m_\beta$, and $\deg(m_\alpha) > 1$.

2/I/6G **Linear Mathematics**

Let A be a complex 4×4 matrix such that $A^3 = A^2$. What are the possible minimal polynomials of A? If A is not diagonalisable and $A^2 \neq 0$, list all possible Jordan normal forms of A.

2/II/15G **Linear Mathematics**

(a) A complex $n \times n$ matrix is said to be unipotent if $U - I$ is nilpotent, where I is the identity matrix. Show that U is unipotent if and only if 1 is the only eigenvalue of U.

(b) Let T be an invertible complex matrix. By considering the Jordan normal form of T show that there exists an invertible matrix P such that

$$PTP^{-1} = D_0 + N,$$

where D_0 is an invertible diagonal matrix, N is an upper triangular matrix with zeros in the diagonal and $D_0 N = N D_0$.

(c) Set $D = P^{-1} D_0 P$ and show that $U = D^{-1} T$ is unipotent.

(d) Conclude that any invertible matrix T can be written as $T = DU$ where D is diagonalisable, U is unipotent and $DU = UD$.

3/I/7F Linear Mathematics

Which of the following statements are true, and which false? Give brief justifications for your answers.

(a) If U and W are subspaces of a vector space V, then $U \cap W$ is always a subspace of V.

(b) If U and W are distinct subspaces of a vector space V, then $U \cup W$ is never a subspace of V.

(c) If U, W and X are subspaces of a vector space V, then $U \cap (W + X) = (U \cap W) + (U \cap X)$.

(d) If U is a subspace of a finite-dimensional space V, then there exists a subspace W such that $U \cap W = \{0\}$ and $U + W = V$.

3/II/17F Linear Mathematics

Define the *determinant* of an $n \times n$ matrix A, and prove from your definition that if A' is obtained from A by an elementary row operation (i.e. by adding a scalar multiple of the ith row of A to the jth row, for some $j \neq i$), then $\det A' = \det A$.

Prove also that if X is a $2n \times 2n$ matrix of the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix} ,$$

where O denotes the $n \times n$ zero matrix, then $\det X = \det A \det C$. Explain briefly how the $2n \times 2n$ matrix

$$\begin{pmatrix} B & I \\ O & A \end{pmatrix}$$

can be transformed into the matrix

$$\begin{pmatrix} B & I \\ -AB & O \end{pmatrix}$$

by a sequence of elementary row operations. Hence or otherwise prove that $\det AB = \det A \det B$.

4/I/6F Linear Mathematics

Define the *rank* and *nullity* of a linear map between finite-dimensional vector spaces. State the rank–nullity formula.

Let $\alpha \colon U \to V$ and $\beta \colon V \to W$ be linear maps. Prove that

$$\operatorname{rank}(\alpha) + \operatorname{rank}(\beta) - \dim V \leqslant \operatorname{rank}(\beta\alpha) \leqslant \min\{\operatorname{rank}(\alpha), \operatorname{rank}(\beta)\} .$$

4/II/15F Linear Mathematics

Define the *dual space* V^* of a finite-dimensional real vector space V, and explain what is meant by the basis of V^* dual to a given basis of V. Explain also what is meant by the statement that the second dual V^{**} is naturally isomorphic to V.

Let V_n denote the space of real polynomials of degree at most n. Show that, for any real number x, the function e_x mapping p to $p(x)$ is an element of V_n^*. Show also that, if $x_1, x_2, \ldots, x_{n+1}$ are distinct real numbers, then $\{e_{x_1}, e_{x_2}, \ldots, e_{x_{n+1}}\}$ is a basis of V_n^*, and find the basis of V_n dual to it.

Deduce that, for any $(n+1)$ distinct points $x_1, \ldots, x_{n+1}$ of the interval $[-1, 1]$, there exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n+1} \lambda_i p(x_i)$$

for all $p \in V_n$. For $n = 4$ and $(x_1, x_2, x_3, x_4, x_5) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$, find the corresponding scalars λ_i.

1/I/4E Geometry

Show that any finite group of orientation-preserving isometries of the Euclidean plane is cyclic.

Show that any finite group of orientation-preserving isometries of the hyperbolic plane is cyclic.

[*You may assume that given any non-empty finite set E in the hyperbolic plane, or the Euclidean plane, there is a unique smallest closed disc that contains E. You may also use any general fact about the hyperbolic plane without proof providing that it is stated carefully.*]

1/II/13E Geometry

Let $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$, and let $\mathbb{H}$ have the hyperbolic metric ρ derived from the line element $|dz|/y$. Let Γ be the group of Möbius maps of the form $z \mapsto (az+b)/(cz+d)$, where a, b, c and d are real and $ad - bc = 1$. Show that every g in Γ is an isometry of the metric space $(\mathbb{H}, \rho)$. For z and w in $\mathbb{H}$, let

$$h(z,w) = \frac{|z-w|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}.$$

Show that for every g in Γ, $h\big(g(z), g(w)\big) = h(z,w)$. By considering $z = iy$, where $y > 1$, and $w = i$, or otherwise, show that for all z and w in $\mathbb{H}$,

$$\cosh \rho(z,w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\mathrm{Im}(w)}.$$

By considering points i, iy, where $y > 1$ and $s + it$, where $s^2 + t^2 = 1$, or otherwise, derive Pythagoras' Theorem for hyperbolic geometry in the form $\cosh a \cosh b = \cosh c$, where a, b and c are the lengths of sides of a right-angled triangle whose hypotenuse has length c.

3/I/4E Geometry

State Euler's formula for a graph $\mathcal{G}$ with F faces, E edges and V vertices on the surface of a sphere.

Suppose that every face in $\mathcal{G}$ has at least three edges, and that at least three edges meet at every vertex of $\mathcal{G}$. Let F_n be the number of faces of $\mathcal{G}$ that have exactly n edges ($n \geqslant 3$), and let V_m be the number of vertices at which exactly m edges meet ($m \geqslant 3$). By expressing $6F - \sum_n nF_n$ in terms of the V_j, or otherwise, show that every convex polyhedron has at least four faces each of which is a triangle, a quadrilateral or a pentagon.

3/II/14E **Geometry**

Show that every isometry of Euclidean space $\mathbb{R}^3$ is a composition of reflections in planes.

What is the smallest integer N such that every isometry f of $\mathbb{R}^3$ with $f(0) = 0$ can be expressed as the composition of at most N reflections? Give an example of an isometry that needs this number of reflections and justify your answer.

Describe (geometrically) all twelve orientation-reversing isometries of a regular tetrahedron.

1/I/1E **Analysis II**

Suppose that for each $n = 1, 2, \ldots$, the function $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(a) If $f_n \to f$ pointwise on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

(b) If $f_n \to f$ uniformly on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

In each case, give a proof or a counter-example (with justification).

1/II/10E **Analysis II**

Suppose that (X, d) is a metric space that has the Bolzano-Weierstrass property (that is, any sequence has a convergent subsequence). Let (Y, d') be any metric space, and suppose that f is a continuous map of X onto Y. Show that (Y, d') also has the Bolzano-Weierstrass property.

Show also that if f is a bijection of X onto Y, then $f^{-1} : Y \to X$ is continuous.

By considering the map $x \mapsto e^{ix}$ defined on the real interval $[-\pi/2, \pi/2]$, or otherwise, show that there exists a continuous choice of $\arg z$ for the complex number z lying in the right half-plane $\{x + iy : x > 0\}$.

2/I/1E **Analysis II**

Define what is meant by (i) a complete metric space, and (ii) a totally bounded metric space.

Give an example of a metric space that is complete but not totally bounded. Give an example of a metric space that is totally bounded but not complete.

Give an example of a continuous function that maps a complete metric space onto a metric space that is not complete. Give an example of a continuous function that maps a totally bounded metric space onto a metric space that is not totally bounded.

[*You need not justify your examples.*]

2/II/10E **Analysis II**

(a) Let f be a map of a complete metric space (X, d) into itself, and suppose that there exists some k in $(0,1)$, and some positive integer N, such that $d\big(f^N(x), f^N(y)\big) \leqslant k\,d(x,y)$ for all distinct x and y in X, where f^m is the mth iterate of f. Show that f has a unique fixed point in X.

(b) Let f be a map of a compact metric space (X, d) into itself such that $d\big(f(x), f(y)\big) < d(x,y)$ for all distinct x and y in X. By considering the function $d\big(f(x), x\big)$, or otherwise, show that f has a unique fixed point in X.

(c) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|f(x) - f(y)| < |x - y|$ for every distinct x and y in $\mathbb{R}^n$. Suppose that for some x, the orbit $O(x) = \{x, f(x), f^2(x), \ldots\}$ is bounded. Show that f maps the closure of $O(x)$ into itself, and deduce that f has a unique fixed point in $\mathbb{R}^n$.

[*The Contraction Mapping Theorem may be used without proof providing that it is correctly stated.*]

3/I/1E **Analysis II**

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f = (u, v)$, where u and v are defined by $u(0) = v(0) = 0$ and, for $t \neq 0$, $u(t) = t^2 \sin(1/t)$ and $v(t) = t^2 \cos(1/t)$. Show that f is differentiable on $\mathbb{R}$.

Show that for any real non-zero a, $||f'(a) - f'(0)|| > 1$, where we regard $f'(a)$ as the vector $\big(u'(a), v'(a)\big)$ in $\mathbb{R}^2$.

3/II/11E **Analysis II**

Show that if a, b and c are non-negative numbers, and $a \leqslant b+c$, then

$$\frac{a}{1+a} \leqslant \frac{b}{1+b} + \frac{c}{1+c}.$$

Deduce that if (X,d) is a metric space, then $d(x,y)/[1+d(x,y)]$ is a metric on X.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K_n = \{z \in D : |z| \leqslant (n-1)/n\}$. Let $\mathcal{F}$ be the class of continuous complex-valued functions on D and, for f and g in $\mathcal{F}$, define

$$\sigma(f,g) = \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{||f-g||_n}{1+||f-g||_n},$$

where $||f-g||_n = \sup\{|f(z)-g(z)| : z \in K_n\}$. Show that the series for $\sigma(f,g)$ converges, and that σ is a metric on $\mathcal{F}$.

For $|z| < 1$, let $s_k(z) = 1 + z + z^2 + \cdots + z^k$ and $s(z) = 1 + z + z^2 + \cdots$. Show that for $n \geqslant 2$, $||s_k - s||_n = n(1-\frac{1}{n})^{k+1}$. By considering the sums for $2 \leqslant n \leqslant N$ and $n > N$ separately, show that for each N,

$$\sigma(s_k, s) \leqslant \sum_{n=2}^{N} ||s_k - s||_n + 2^{-N},$$

and deduce that $\sigma(s_k, s) \to 0$ as $k \to \infty$.

4/I/1E **Analysis II**

(a) Let (X,d) be a metric space containing the point x_0, and let

$$U = \{x \in X : d(x,x_0) < 1\}, \quad K = \{x \in X : d(x,x_0) \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

(b) Let X be a normed space with norm $||\cdot||$, and let

$$U = \{x \in X : ||x|| < 1\}, \quad K = \{x \in X : ||x|| \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

4/II/10E **Analysis II**

(a) Let V be a finite-dimensional real vector space, and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on V. Show that a function $f : V \to \mathbb{R}$ is differentiable at a point a in V with respect to $||\cdot||_1$ if and only if it is differentiable at a with respect to $||\cdot||_2$, and that if this is so then the derivative $f'(a)$ of f is independent of the norm used. [*You may assume that all norms on a finite-dimensional vector space are equivalent.*]

(b) Let V_1, V_2 and V_3 be finite-dimensional normed real vector spaces with V_j having norm $||\cdot||_j$, $j = 1, 2, 3$, and let $f : V_1 \times V_2 \to V_3$ be a continuous bilinear mapping. Show that f is differentiable at any point (a, b) in $V_1 \times V_2$, and that $f'(a, b)(h, k) = f(h, b) + f(a, k)$. [*You may assume that* $\left(||u||_1^2 + ||v||_2^2\right)^{1/2}$ *is a norm on* $V_1 \times V_2$, *and that* $\{(x, y) \in V_1 \times V_2 : ||x||_1 = 1, ||y||_2 = 1\}$ *is compact.*]

1/I/7B Complex Methods

Using contour integration around a rectangle with vertices

$$-x,\ x,\ x+iy,\ -x+iy,$$

prove that, for all real y,

$$\int_{-\infty}^{+\infty} e^{-(x+iy)^2}\,dx = \int_{-\infty}^{+\infty} e^{-x^2}\,dx\,.$$

Hence derive that the function $f(x)=e^{-x^2/2}$ is an eigenfunction of the Fourier transform

$$\widehat{f}(y) = \int_{-\infty}^{+\infty} f(x)e^{-ixy}dx,$$

i.e. $\widehat{f}$ is a constant multiple of f.

1/II/16B Complex Methods

(a) Show that if f is an analytic function at z_0 and $f'(z_0)\neq 0$, then f is conformal at z_0, i.e. it preserves angles between paths passing through z_0.

(b) Let D be the disc given by $|z+i|<\sqrt{2}$, and let H be the half-plane given by $y>0$, where $z=x+iy$. Construct a map of the domain $D\cap H$ onto H, and hence find a conformal mapping of $D\cap H$ onto the disc $\{z : |z|<1\}$. [*Hint: You may find it helpful to consider a mapping of the form* $(az+b)/(cz+d)$, *where* $ad-bc\neq 0$.]

2/I/7B Complex Methods

Suppose that f is analytic, and that $|f(z)|^2$ is constant in an open disk D. Use the Cauchy–Riemann equations to show that $f(z)$ is constant in D.

2/II/16B **Complex Methods**

A function $f(z)$ has an isolated singularity at a, with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

(a) Define $\operatorname{res}(f, a)$, the residue of f at the point a.

(b) Prove that if a is a pole of order $k+1$, then

$$\operatorname{res}(f, a) = \lim_{z \to a} \frac{h^{(k)}(z)}{k!}, \quad \text{where} \quad h(z) = (z-a)^{k+1} f(z).$$

(c) Using the residue theorem and the formula above show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{k+1}} = \pi \frac{(2k)!}{(k!)^2} 4^{-k}, \quad k \geq 1.$$

4/I/8B **Complex Methods**

Let f be a function such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$. Prove that

$$\int_{-\infty}^{+\infty} f(x+k)\overline{f(x+l)}\, dx = 0 \quad \text{for all integers } k \text{ and } l \text{ with } k \neq l,$$

if and only if

$$\int_{-\infty}^{+\infty} |\widehat{f}(t)|^2 e^{-imt} dt = 0 \quad \text{for all integers } m \neq 0,$$

where $\widehat{f}$ is the Fourier transform of f.

4/II/17B **Complex Methods**

(a) Using the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \frac{\pi}{2}$, show that, if f is continuous for large $|z|$, and if $f(z) \to 0$ as $z \to \infty$, then

$$\lim_{R \to \infty} \int_{\Gamma_R} f(z) e^{i\lambda z} dz = 0 \quad \text{for} \quad \lambda > 0,$$

where $\Gamma_R = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

(b) By integrating an appropriate function $f(z)$ along the contour formed by the semicircles Γ_R and Γ_r in the upper half-plane with the segments of the real axis $[-R, -r]$ and $[r, R]$, show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

1/I/2A **Methods**

Find the Fourier sine series for $f(x) = x$, on $0 \leqslant x < L$. To which value does the series converge at $x = \frac{3}{2}L$?

Now consider the corresponding cosine series for $f(x) = x$, on $0 \leqslant x < L$. Sketch the cosine series between $x = -2L$ and $x = 2L$. To which value does the series converge at $x = \frac{3}{2}L$? [*You do not need to determine the cosine series explicitly.*]

1/II/11A **Methods**

The potential $\Phi(r, \vartheta)$, satisfies Laplace's equation everywhere except on a sphere of unit radius and $\Phi \to 0$ as $r \to \infty$. The potential is continuous at $r = 1$, but the derivative of the potential satisfies

$$\lim_{r \to 1^+} \frac{\partial \Phi}{\partial r} - \lim_{r \to 1^-} \frac{\partial \Phi}{\partial r} = V \cos^2 \vartheta,$$

where V is a constant. Use the method of separation of variables to find Φ for both $r > 1$ and $r < 1$.

[*The Laplacian in spherical polar coordinates for axisymmetric systems is*

$$\nabla^2 \equiv \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin \vartheta}\left(\frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta}\right).$$

You may assume that the equation

$$\left((1 - x^2) y'\right)' + \lambda y = 0$$

has polynomial solutions of degree n, *which are regular at* $x = \pm 1$, *if and only if* $\lambda = n(n+1)$.]

2/I/2C **Methods**

Write down the transformation law for the components of a second-rank tensor A_{ij} explaining the meaning of the symbols that you use.

A tensor is said to have *cubic symmetry* if its components are unchanged by rotations of $\pi/2$ about each of the three co-ordinate axes. Find the most general second-rank tensor having cubic symmetry.

2/II/11C **Methods**

If $\mathbf{B}$ is a vector, and

$$T_{ij} = \alpha B_i B_j + \beta B_k B_k \delta_{ij} \ ,$$

show for arbitrary scalars α and β that T_{ij} is a symmetric second-rank tensor.

Find the eigenvalues and eigenvectors of T_{ij}.

Suppose now that $\mathbf{B}$ depends upon position $\mathbf{x}$ and that $\nabla \cdot \mathbf{B} = 0$. Find constants α and β such that

$$\frac{\partial}{\partial x_j} T_{ij} = [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i \ .$$

Hence or otherwise show that if $\mathbf{B}$ vanishes everywhere on a surface S that encloses a volume V then

$$\int_V (\nabla \times \mathbf{B}) \times \mathbf{B} \ dV = 0 \ .$$

3/I/2A **Methods**

Write down the wave equation for the displacement $y(x,t)$ of a stretched string with constant mass density and tension. Obtain the general solution in the form

$$y(x,t) = f(x+ct) + g(x-ct),$$

where c is the wave velocity. For a solution in the region $0 \leqslant x < \infty$, with $y(0,t) = 0$ and $y \to 0$ as $x \to \infty$, show that

$$E = \int_0^\infty \left[\frac{1}{2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} c^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx,$$

is constant in time. Express E in terms of the general solution in this case.

3/II/12A Methods

Consider the real Sturm-Liouville problem

$$\mathcal{L}y(x) = -(p(x)y')' + q(x)y = \lambda r(x)y,$$

with the boundary conditions $y(a) = y(b) = 0$, where p, q and r are continuous and positive on $[a, b]$. Show that, with suitable choices of inner product and normalisation, the eigenfunctions $y_n(x)$, $n = 1, 2, 3 \dots$, form an orthonormal set.

Hence show that the corresponding Green's function $G(x, \xi)$ satisfying

$$(\mathcal{L} - \mu r(x))G(x, \xi) = \delta(x - \xi),$$

where μ is not an eigenvalue, is

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)}{\lambda_n - \mu},$$

where λ_n is the eigenvalue corresponding to y_n.

Find the Green's function in the case where

$$\mathcal{L}y \equiv y'',$$

with boundary conditions $y(0) = y(\pi) = 0$, and deduce, by suitable choice of μ, that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

4/I/2A Methods

Use the method of Lagrange multipliers to find the largest volume of a rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4/II/11A Methods

A function $y(x)$ is chosen to make the integral

$$I = \int_a^b f\left(x, y, y', y''\right) dx$$

stationary, subject to given values of $y(a), y'(a), y(b)$ and $y'(b)$. Derive an analogue of the Euler–Lagrange equation for $y(x)$.

Solve this equation for the case where

$$f = x^4 y''^2 + 4y^2 y',$$

in the interval $[0, 1]$ and

$$x^2 y(x) \to 0, \quad xy(x) \to 1$$

as $x \to 0$, whilst

$$y(1) = 2, \quad y'(1) = 0.$$

1/I/9D **Quantum Mechanics**

Consider a quantum mechanical particle of mass m moving in one dimension, in a potential well

$$V(x) = \begin{cases} \infty, & x < 0, \\ 0, & 0 < x < a, \\ V_0, & x > a. \end{cases}$$

Sketch the ground state energy eigenfunction $\chi(x)$ and show that its energy is $E = \frac{\hbar^2 k^2}{2m}$, where k satisfies

$$\tan ka = -\frac{k}{\sqrt{\frac{2mV_0}{\hbar^2} - k^2}}.$$

[*Hint: You may assume that* $\chi(0) = 0$*.*]

1/II/18D **Quantum Mechanics**

A quantum mechanical particle of mass M moves in one dimension in the presence of a negative delta function potential

$$V = -\frac{\hbar^2}{2M\Delta}\delta(x),$$

where Δ is a parameter with dimensions of length.

(a) Write down the time-independent Schrödinger equation for energy eigenstates $\chi(x)$, with energy E. By integrating this equation across $x = 0$, show that the gradient of the wavefunction jumps across $x = 0$ according to

$$\lim_{\epsilon \to 0} \left(\frac{d\chi}{dx}(\epsilon) - \frac{d\chi}{dx}(-\epsilon) \right) = -\frac{1}{\Delta}\chi(0).$$

[*You may assume that* χ *is continuous across* $x = 0$*.*]

(b) Show that there exists a negative energy solution and calculate its energy.

(c) Consider a double delta function potential

$$V(x) = -\frac{\hbar^2}{2M\Delta}[\delta(x+a) + \delta(x-a)].$$

For sufficiently small Δ, this potential yields a negative energy solution of odd parity, i.e. $\chi(-x) = -\chi(x)$. Show that its energy is given by

$$E = -\frac{\hbar^2}{2M}\lambda^2, \quad \text{where} \quad \tanh \lambda a = \frac{\lambda\Delta}{1 - \lambda\Delta}.$$

[*You may again assume* χ *is continuous across* $x = \pm a$*.*]

2/I/9D **Quantum Mechanics**

From the expressions

$$L_x = yP_z - zP_y, \quad L_y = zP_x - xP_z, \quad L_z = xP_y - yP_x,$$

show that

$$(x + iy)z$$

is an eigenfunction of $\mathbf{L}^2$ and L_z, and compute the corresponding eigenvalues.

2/II/18D **Quantum Mechanics**

Consider a quantum mechanical particle moving in an upside-down harmonic oscillator potential. Its wavefunction $\Psi(x,t)$ evolves according to the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{2}x^2\Psi. \tag{1}$$

(a) Verify that

$$\Psi(x,t) = A(t)e^{-B(t)x^2} \tag{2}$$

is a solution of equation (1), provided that

$$\frac{dA}{dt} = -i\hbar\, AB,$$

and

$$\frac{dB}{dt} = -\frac{i}{2\hbar} - 2i\hbar B^2. \tag{3}$$

(b) Verify that $B = \frac{1}{2\hbar}\tan(\phi - it)$ provides a solution to (3), where ϕ is an arbitrary real constant.

(c) The expectation value of an operator $\mathcal{O}$ at time t is

$$\langle\mathcal{O}\rangle(t) \equiv \int_{-\infty}^{\infty} dx \Psi^*(x,t)\mathcal{O}\Psi(x,t),$$

where $\Psi(x,t)$ is the normalised wave function. Show that for $\Psi(x,t)$ given by (2),

$$\langle x^2\rangle = \frac{1}{4Re(B)}, \quad \langle p^2\rangle = 4\hbar^2|B|^2\langle x^2\rangle.$$

Hence show that as $t\to\infty$,

$$\langle x^2\rangle \approx \langle p^2\rangle \approx \frac{\hbar}{4\sin 2\phi}e^{2t}.$$

[*Hint: You may use*

$$\frac{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}x^2}{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}} = \frac{1}{2C}.]$$

3/II/20D **Quantum Mechanics**

A quantum mechanical system has two states χ_0 and χ_1, which are normalised energy eigenstates of a Hamiltonian H_3, with

$$H_3\chi_0 = -\chi_0, \quad H_3\chi_1 = +\chi_1.$$

A general time-dependent state may be written

$$\Psi(t) = a_0(t)\chi_0 + a_1(t)\chi_1, \tag{1}$$

where $a_0(t)$ and $a_1(t)$ are complex numbers obeying $|a_0(t)|^2 + |a_1(t)|^2 = 1$.

(a) Write down the time-dependent Schrödinger equation for $\Psi(t)$, and show that if the Hamiltonian is H_3, then

$$i\hbar\frac{da_0}{dt} = -a_0, \qquad i\hbar\frac{da_1}{dt} = +a_1.$$

For the general state given in equation (1) above, write down the probability to observe the system, at time t, in a state $\alpha\chi_0 + \beta\chi_1$, properly normalised so that $|\alpha|^2 + |\beta|^2 = 1$.

(b) Now consider starting the system in the state χ_0 at time $t = 0$, and evolving it with a different Hamiltonian H_1, which acts on the states χ_0 and χ_1 as follows:

$$H_1\chi_0 = \chi_1, \qquad H_1\chi_1 = \chi_0.$$

By solving the time-dependent Schrödinger equation for the Hamiltonian H_1, find $a_0(t)$ and $a_1(t)$ in this case. Hence determine the shortest time $T > 0$ such that $\Psi(T)$ is an eigenstate of H_3 with eigenvalue $+1$.

(c) Now consider taking the state $\Psi(T)$ from part (b), and evolving it for further length of time T, with Hamiltonian H_2, which acts on the states χ_0 and χ_1 as follows:

$$H_2\chi_0 = -i\chi_1, \qquad H_2\chi_1 = i\chi_0.$$

What is the final state of the system? Is this state observationally distinguishable from the original state χ_0?

3/I/10D Special Relativity

Write down the formulae for a Lorentz transformation with velocity v taking one set of co-ordinates (t, x) to another (t', x').

Imagine you observe a train travelling past Cambridge station at a relativistic speed u. Someone standing still on the train throws a ball in the direction the train is moving, with speed v. How fast do you observe the ball to be moving? Justify your answer.

4/I/9D Special Relativity

A particle with mass M is observed to be at rest. It decays into a particle of mass $m < M$, and a massless particle. Calculate the energies and momenta of both final particles.

4/II/18D Special Relativity

A javelin of length 2m is thrown horizontally and lengthwise into a shed of length 1.5m at a speed of $0.8c$, where c is the speed of light.

(a) What is the length of the javelin in the rest frame of the shed?

(b) What is the length of the shed in the rest frame of the javelin?

(c) Draw a space-time diagram in the rest frame coordinates (ct, x) of the shed, showing the world lines of both ends of the javelin, and of the front and back of the shed. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the shed.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the shed, and (B) the leading end of the javelin hitting the back of the shed. Give the corresponding (ct, x) and (ct', x') coordinates for both (A) and (B). Are these two events space-like, null or time-like separated? How does the javelin fit inside the shed, even though it is initially longer than the shed in its own rest frame?

1/I/6C **Fluid Dynamics**

A fluid flow has velocity given in Cartesian co-ordinates as $\mathbf{u} = (kty, 0, 0)$ where k is a constant and t is time. Show that the flow is incompressible. Find a stream function and determine an equation for the streamlines at time t.

At $t = 0$ the points along the straight line segment $x = 0,\ 0 \leqslant y \leqslant a,\ z = 0$ are marked with dye. Show that at any later time the marked points continue to form a segment of a straight line. Determine the length of this line segment at time t and the angle that it makes with the x-axis.

1/II/15C **Fluid Dynamics**

State the unsteady form of Bernoulli's theorem.

A spherical bubble having radius R_0 at time $t = 0$ is located with its centre at the origin in unbounded fluid. The fluid is inviscid, has constant density ρ and is everywhere at rest at $t = 0$. The pressure at large distances from the bubble has the constant value p_∞, and the pressure inside the bubble has the constant value $p_\infty - \triangle p$. In consequence the bubble starts to collapse so that its radius at time t is $R(t)$. Find the velocity everywhere in the fluid in terms of $R(t)$ at time t and, assuming that surface tension is negligible, show that R satisfies the equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = -\frac{\triangle p}{\rho}\ .$$

Find the total kinetic energy of the fluid in terms of $R(t)$ at time t. Hence or otherwise obtain a first integral of the above equation.

3/I/8C **Fluid Dynamics**

State and prove Kelvin's circulation theorem.

Consider a planar flow in the unbounded region outside a cylinder for which the vorticity vanishes everywhere at time $t = 0$. What may be deduced about the circulation around closed loops in the fluid at time t:

(i) that do not enclose the cylinder;

(ii) that enclose the cylinder?

Give a brief justification for your answer in each case.

3/II/18C **Fluid Dynamics**

Use Euler's equation to derive Bernoulli's theorem for the steady flow of an inviscid fluid of uniform density ρ in the absence of body forces.

Such a fluid flows steadily through a long cylindrical elastic tube having circular cross-section. The variable z measures distance downstream along the axis of the tube. The tube wall has thickness $h(z)$, so that if the external radius of the tube is $r(z)$, its internal radius is $r(z)-h(z)$, where $h(z) \geqslant 0$ is a given slowly-varying function that tends to zero as $z \to \pm\infty$. The elastic tube wall exerts a pressure $p(z)$ on the fluid given as

$$p(z) = p_0 + k[r(z) - R],$$

where $p_0,\ k$ and R are positive constants. Far upstream, r has the constant value R, the fluid pressure has the constant value p_0, and the fluid velocity u has the constant value V. Assume that gravity is negligible and that $h(z)$ varies sufficiently slowly that the velocity may be taken as uniform across the tube at each value of z. Use mass conservation and Bernoulli's theorem to show that $u(z)$ satisfies

$$\frac{h}{R} = 1 - \left(\frac{V}{u}\right)^{1/2} + \frac{1}{4}\lambda\left[1 - \left(\frac{u}{V}\right)^2\right], \quad \text{where} \quad \lambda = \frac{2\rho V^2}{kR} \ .$$

Sketch a graph of h/R against u/V. Show that if $h(z)$ exceeds a critical value $h_c(\lambda)$, no such flow is possible and find $h_c(\lambda)$.

Show that if $h < h_c(\lambda)$ everywhere, then for given h the equation has two positive solutions for u. Explain how the given value of λ determines which solution should be chosen.

4/I/7C **Fluid Dynamics**

If $\mathbf{u}$ is given in Cartesian co-ordinates as $\mathbf{u} = (-\Omega y, \Omega x, 0)$, with Ω a constant, verify that

$$\mathbf{u}\cdot \nabla \mathbf{u} = \nabla(-\tfrac{1}{2}\mathbf{u}^2) \ .$$

When incompressible fluid is placed in a stationary cylindrical container of radius a with its axis vertical, the depth of the fluid is h. Assuming that the free surface does not reach the bottom of the container, use cylindrical polar co-ordinates to find the equation of the free surface when the fluid and the container rotate steadily about this axis with angular velocity Ω.

Deduce the angular velocity at which the free surface first touches the bottom of the container.

4/II/16C **Fluid Dynamics**

Use Euler's equation to show that in a planar flow of an inviscid fluid the vorticity $\boldsymbol{\omega}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = 0\ .$$

Write down the velocity field associated with a point vortex of strength κ in unbounded fluid.

Consider now the flow generated in unbounded fluid by two point vortices of strengths κ_1 and κ_2 at $\mathbf{x}_1(t) = (x_1, y_1)$ and $\mathbf{x}_2(t) = (x_2, y_2)$, respectively. Show that in the subsequent motion the quantity

$$\mathbf{q} = \kappa_1\mathbf{x}_1 + \kappa_2\mathbf{x}_2$$

remains constant. Show also that the separation of the vortices, $|\mathbf{x}_2 - \mathbf{x}_1|$, remains constant.

Suppose finally that $\kappa_1 = \kappa_2$ and that the vortices are placed at time $t = 0$ at positions $(a, 0)$ and $(-a, 0)$. What are the positions of the vortices at time t?

2/I/5B **Numerical Analysis**

Applying the Gram–Schmidt orthogonalization, compute a "skinny" QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 6 \\ 1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix},$$

i.e. find a 4×3 matrix Q with orthonormal columns and an upper triangular 3×3 matrix R such that $A = QR$.

2/II/14B **Numerical Analysis**

Let $f \in C[a, b]$ and let $n+1$ distinct points $x_0, \ldots, x_n \in [a, b]$ be given.

(a) Define the divided difference $f[x_0, \ldots, x_n]$ of order n in terms of interpolating polynomials. Prove that it is a symmetric function of the variables x_i, $i = 0, \ldots, n$.

(b) Prove the recurrence relation

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

(c) Hence or otherwise deduce that, for any $i \neq j$, we have

$$f[x_0, \ldots, x_n] = \frac{f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] - f[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]}{x_j - x_i}.$$

(d) From the formulas above, show that, for any $i = 1, \ldots, n-1$,

$$f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] = \gamma f[x_0, \ldots, x_{n-1}] + (1 - \gamma) f[x_1, \ldots, x_n],$$

where $\gamma = \dfrac{x_i - x_0}{x_n - x_0}$.

3/I/6B **Numerical Analysis**

For numerical integration, a quadrature formula

$$\int_a^b f(x)\, dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

is applied which is exact on $\mathcal{P}_n$, i.e., for all polynomials of degree n.

Prove that such a formula is exact for all $f \in \mathcal{P}_{2n+1}$ if and only if x_i, $i = 0, \ldots, n$, are the zeros of an orthogonal polynomial $p_{n+1} \in \mathcal{P}_{n+1}$ which satisfies $\int_a^b p_{n+1}(x) r(x)\, dx = 0$ for all $r \in \mathcal{P}_n$. [*You may assume that p_{n+1} has $(n+1)$ distinct zeros.*]

3/II/16B **Numerical Analysis**

(a) Consider a system of linear equations $Ax = b$ with a non-singular square $n \times n$ matrix A. To determine its solution $x = x^*$ we apply the iterative method

$$x^{k+1} = Hx^k + v.$$

Here $v \in \mathbb{R}^n$, while the matrix $H \in \mathbb{R}^{n\times n}$ is such that $x^* = Hx^* + v$ implies $Ax^* = b$. The initial vector $x^0 \in \mathbb{R}^n$ is arbitrary. Prove that, if the matrix H possesses n linearly independent eigenvectors $w_1, \ldots, w_n$ whose corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy $\max_i |\lambda_i| < 1$, then the method converges for any choice of x^0, i.e. $x^k \to x^*$ as $k \to \infty$.

(b) Describe the Jacobi iteration method for solving $Ax = b$. Show directly from the definition of the method that, if the matrix A is strictly diagonally dominant by rows, i.e.

$$|a_{ii}|^{-1} \sum_{j=1, j\neq i}^{n} |a_{ij}| \leq \gamma < 1, \quad i = 1, \ldots, n,$$

then the method converges.

1/I/3H **Statistics**

State the factorization criterion for sufficient statistics and give its proof in the discrete case.

Let $X_1, \ldots, X_n$ form a random sample from a Poisson distribution for which the value of the mean θ is unknown. Find a one-dimensional sufficient statistic for θ.

1/II/12H **Statistics**

Suppose we ask 50 men and 150 women whether they are early risers, late risers, or risers with no preference. The data are given in the following table.

	Early risers	*Late risers*	*No preference*	*Totals*
Men	17	22	11	50
Women	43	78	29	150
Totals	60	100	40	200

Derive carefully a (generalized) likelihood ratio test of independence of classification. What is the result of applying this test at the 0.01 level?

[

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_5^2	χ_6^2
99%*percentile*	*6.63*	*9.21*	*11.34*	*15.09*	*16.81*

]

2/I/3H **Statistics**

Explain what is meant by a uniformly most powerful test, its power function and size.

Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables with common density $\rho e^{-\rho y}$, $y \geq 0$. Obtain the uniformly most powerful test of $\rho = \rho_0$ against alternatives $\rho < \rho_0$ and determine the power function of the test.

2/II/12H Statistics

For ten steel ingots from a production process the following measures of hardness were obtained:

73.2, 74.3, 75.4, 73.8, 74.4, 76.7, 76.1, 73.0, 74.6, 74.1.

On the assumption that the variation is well described by a normal density function obtain an estimate of the process mean.

The manufacturer claims that he is supplying steel with mean hardness 75. Derive carefully a (generalized) likelihood ratio test of this claim. Knowing that for the data above

$$S_{XX} = \sum_{j=1}^{n} (X_i - \bar{X})^2 = 12.824,$$

what is the result of the test at the 5% significance level?

[

Distribution	t_9	t_{10}
95% percentile	*1.83*	*1.81*
97.5% percentile	*2.26*	*2.23*

]

4/I/3H Statistics

From each of 100 concrete mixes six sample blocks were taken and subjected to strength tests, the number out of the six blocks failing the test being recorded in the following table:

No. x failing strength tests	0	1	2	3	4	5	6
No. of mixes with x failures	53	32	12	2	1	0	0

On the assumption that the probability of failure is the same for each block, obtain an unbiased estimate of this probability and explain how to find a 95% confidence interval for it.

4/II/12H Statistics

Explain what is meant by a prior distribution, a posterior distribution, and a Bayes estimator. Relate the Bayes estimator to the posterior distribution for both quadratic and absolute error loss functions.

Suppose $X_1, \ldots, X_n$ are independent identically distributed random variables from a distribution uniform on $(\theta - 1, \theta + 1)$, and that the prior for θ is uniform on $(20, 50)$.

Calculate the posterior distribution for θ, given $\mathbf{x} = (x_1, \ldots, x_n)$, and find the point estimate for θ under both quadratic and absolute error loss function.

3/I/5H **Optimization**

Consider a two-person zero-sum game with a payoff matrix

$$\begin{pmatrix} 3 & b \\ 5 & 2 \end{pmatrix},$$

where $0 < b < \infty$. Here, the (i, j) entry of the matrix indicates the payoff to player one if he chooses move i and player two move j. Suppose player one chooses moves 1 and 2 with probabilities p and $1 - p$, $0 \le p \le 1$. Write down the maximization problem for the optimal strategy and solve it for each value of b.

3/II/15H **Optimization**

Consider the following linear programming problem

$$\begin{array}{ll} \text{maximise} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 - x_2 \ge 1, \\ & 4x_1 - x_2 \ge 10, \\ & x_2 \le 6, \\ & x_i \ge 0,\ i = 1, 2. \end{array} \qquad (1)$$

Write down the Phase One problem for (1) and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve (1), i.e., find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

4/I/5H **Optimization**

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow, maximal flow, cut, capacity.

4/II/14H **Optimization**

A gambler at a horse race has an amount $b > 0$ to bet. The gambler assesses p_i, the probability that horse i will win, and knows that $s_i \geq 0$ has been bet on horse i by others, for $i = 1, 2, \ldots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gambler's optimal strategy is to choose $(x_1, x_2, \ldots, x_n)$ so that it maximizes

$$\sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0, \tag{1}$$

where x_i is the amount the gambler bets on horse i. Show that the optimal solution to (1) also solves the following problem:

$$\text{minimize } \sum_{i=1}^{n} \frac{p_i s_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0.$$

Assume that $p_1/s_1 \geq p_2/s_2 \geq \ldots \geq p_n/s_n$. Applying the Lagrangian sufficiency theorem, prove that the optimal solution to (1) satisfies

$$\frac{p_1 s_1}{(s_1 + x_1)^2} = \ldots = \frac{p_k s_k}{(s_k + x_k)^2}, \quad x_{k+1} = \ldots = x_n = 0,$$

with maximal possible $k \in \{1, 2, \ldots, n\}$.

[*You may use the fact that for all $\lambda < 0$, the minimum of the function $x \mapsto \frac{ps}{s+x} - \lambda x$ on the non-negative axis $0 \leq x < \infty$ is attained at*

$$x(\lambda) = \left(\sqrt{\frac{ps}{-\lambda}} - s\right)^+ \equiv \max\left(\sqrt{\frac{ps}{-\lambda}} - s, 0\right).]$$

Deduce that if b is small enough, the gambler's optimal strategy is to bet on the horses for which the ratio p_i/s_i is maximal. What is his expected gain in this case?

1/I/8F Quadratic Mathematics

Define the *rank* and *signature* of a symmetric bilinear form ϕ on a finite-dimensional real vector space. (If your definitions involve a matrix representation of ϕ, you should explain why they are independent of the choice of representing matrix.)

Let V be the space of all $n \times n$ real matrices (where $n \geqslant 2$), and let ϕ be the bilinear form on V defined by

$$\phi(A, B) = \operatorname{tr} AB - \operatorname{tr} A \operatorname{tr} B \ .$$

Find the rank and signature of ϕ.

[*Hint: You may find it helpful to consider the subspace of symmetric matrices having trace zero, and a suitable complement for this subspace.*]

1/II/17F Quadratic Mathematics

Let A and B be $n \times n$ real symmetric matrices, such that the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite. Show that it is possible to find an invertible matrix P such that $P^T AP = I$ and $P^T BP$ is diagonal. Show also that the diagonal entries of the matrix $P^T BP$ may be calculated directly from A and B, without finding the matrix P. If

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,$$

find the diagonal entries of $P^T BP$.

2/I/8F Quadratic Mathematics

Explain what is meant by a *sesquilinear form* on a complex vector space V. If ϕ and ψ are two such forms, and $\phi(v, v) = \psi(v, v)$ for all $v \in V$, prove that $\phi(v, w) = \psi(v, w)$ for all $v, w \in V$. Deduce that if $\alpha \colon V \to V$ is a linear map satisfying $\phi(\alpha(v), \alpha(v)) = \phi(v, v)$ for all $v \in V$, then $\phi(\alpha(v), \alpha(w)) = \phi(v, w)$ for all $v, w \in V$.

2/II/17F **Quadratic Mathematics**

Define the *adjoint* α^* of an endomorphism α of a complex inner-product space V. Show that if W is a subspace of V, then $\alpha(W) \subseteq W$ if and only if $\alpha^*(W^\perp) \subseteq W^\perp$.

An endomorphism of a complex inner-product space is said to be *normal* if it commutes with its adjoint. Prove the following facts about a normal endomorphism α of a finite-dimensional space V.

(i) α and α^* have the same kernel.

(ii) α and α^* have the same eigenvectors, with complex conjugate eigenvalues.

(iii) If $E_\lambda = \{x \in V : \alpha(x) = \lambda x\}$, then $\alpha(E_\lambda^\perp) \subseteq E_\lambda^\perp$.

(iv) There is an orthonormal basis of V consisting of eigenvectors of α.

Deduce that an endomorphism α is normal if and only if it can be written as a product $\beta\gamma$, where β is Hermitian, γ is unitary and β and γ commute with each other. [*Hint: Given α, define β and γ in terms of their effect on the basis constructed in (iv).*]

3/I/9F **Quadratic Mathematics**

Explain what is meant by a *quadratic residue* modulo an odd prime p, and show that a is a quadratic residue modulo p if and only if $a^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}$. Hence characterize the odd primes p for which -1 is a quadratic residue.

State the law of quadratic reciprocity, and use it to determine whether 73 is a quadratic residue (mod 127).

3/II/19F **Quadratic Mathematics**

Explain what is meant by saying that a positive definite integral quadratic form $f(x,y) = ax^2 + bxy + cy^2$ is *reduced*, and show that every positive definite form is equivalent to a reduced form.

State a criterion for a prime number p to be representable by some form of discriminant d, and deduce that p is representable by a form of discriminant -32 if and only if $p \equiv 1, 2$ or $3 \pmod 8$. Find the reduced forms of discriminant -32, and hence or otherwise show that a prime p is representable by the form $3x^2 + 2xy + 3y^2$ if and only if $p \equiv 3 \pmod 8$.

[*Standard results on when -1 and 2 are squares* (mod p) *may be assumed.*]

2/I/4G **Further Analysis**

Let the function $f = u + iv$ be analytic in the complex plane $\mathbb{C}$ with u,v real-valued. Prove that, if uv is bounded above everywhere on $\mathbb{C}$, then f is constant.

2/II/13G **Further Analysis**

(a) Given a topology $\mathcal{T}$ on X, a collection $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every non-empty set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$. Prove that a collection $\mathcal{B}$ is a basis for some topology if it satisfies:

(i) the union of all sets in $\mathcal{B}$ is X;

(ii) if $x \in B_1 \cap B_2$ for two sets B_1 and B_2 in $\mathcal{B}$, then there is a set $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

(b) On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consider the dictionary order given by

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. Given points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\mathbf{z} \in \mathbb{R}^2 : \ \mathbf{x} < \mathbf{z} < \mathbf{y}\}.$$

Show that the sets $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ form a basis of a topology.

(c) Show that this topology on $\mathbb{R}^2$ does not have a countable basis.

3/I/3G **Further Analysis**

Let $f : X \to Y$ be a continuous map between topological spaces. Let

$$G_f = \{(x, f(x)) : \ x \in X\}.$$

(a) Show that if Y is Hausdorff, then G_f is closed in $X \times Y$.

(b) Show that if X is compact, then G_f is also compact.

3/II/13G **Further Analysis**

(a) Let f and g be two analytic functions on a domain D and let $\gamma \subset D$ be a simple closed curve homotopic in D to a point. If $|g(z)| < |f(z)|$ for every z in γ, prove that γ encloses the same number of zeros of f as of $f + g$.

(b) Let g be an analytic function on the disk $|z| < 1 + \epsilon$, for some $\epsilon > 0$. Suppose that g maps the closed unit disk into the open unit disk (both centred at 0). Prove that g has exactly one fixed point in the open unit disk.

(c) Prove that, if $|a| < 1$, then

$$z^m \left(\frac{z - a}{1 - \bar{a} z} \right)^n - a$$

has $m + n$ zeros in $|z| < 1$.

4/I/4G **Further Analysis**

(a) Let X be a topological space and suppose $X = C \cup D$, where C and D are disjoint nonempty open subsets of X. Show that, if Y is a connected subset of X, then Y is entirely contained in either C or D.

(b) Let X be a topological space and let $\{A_n\}$ be a sequence of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$, for $n = 1, 2, 3, \ldots$. Show that $\bigcup_{n \geqslant 1} A_n$ is connected.

4/II/13G **Further Analysis**

A function f is said to be analytic at ∞ if there exists a real number $r > 0$ such that f is analytic for $|z| > r$ and $\lim_{z \to 0} f(1/z)$ is finite (i.e. $f(1/z)$ has a removable singularity at $z = 0$). f is said to have a pole at ∞ if $f(1/z)$ has a pole at $z = 0$. Suppose that f is a meromorphic function on the extended plane $\mathbb{C}_\infty$, that is, f is analytic at each point of $\mathbb{C}_\infty$ except for poles.

(a) Show that if f has a pole at $z = \infty$, then there exists $r > 0$ such that $f(z)$ has no poles for $r < |z| < \infty$.

(b) Show that the number of poles of f is finite.

(c) By considering the Laurent expansions around the poles show that f is in fact a rational function, i.e. of the form p/q, where p and q are polynomials.

(d) Deduce that the only bijective meromorphic maps of $\mathbb{C}_\infty$ onto itself are the Möbius maps.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 5 June 2002 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, …, H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1E Analysis II

Suppose that for each $n = 1, 2, \ldots$, the function $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(a) If $f_n \to f$ pointwise on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

(b) If $f_n \to f$ uniformly on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

In each case, give a proof or a counter-example (with justification).

2A Methods

Find the Fourier sine series for $f(x) = x$, on $0 \leqslant x < L$. To which value does the series converge at $x = \frac{3}{2}L$?

Now consider the corresponding cosine series for $f(x) = x$, on $0 \leqslant x < L$. Sketch the cosine series between $x = -2L$ and $x = 2L$. To which value does the series converge at $x = \frac{3}{2}L$? [*You do not need to determine the cosine series explicitly.*]

3H Statistics

State the factorization criterion for sufficient statistics and give its proof in the discrete case.

Let $X_1, \ldots, X_n$ form a random sample from a Poisson distribution for which the value of the mean θ is unknown. Find a one-dimensional sufficient statistic for θ.

4E Geometry

Show that any finite group of orientation-preserving isometries of the Euclidean plane is cyclic.

Show that any finite group of orientation-preserving isometries of the hyperbolic plane is cyclic.

[*You may assume that given any non-empty finite set E in the hyperbolic plane, or the Euclidean plane, there is a unique smallest closed disc that contains E. You may also use any general fact about the hyperbolic plane without proof providing that it is stated carefully.*]

5G Linear Mathematics

Define $f : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$f(a,b,c) = (a + 3b - c, 2b + c, -4b - c).$$

Find the characteristic polynomial and the minimal polynomial of f. Is f diagonalisable? Are f and f^2 linearly independent endomorphisms of $\mathbb{C}^3$? Justify your answers.

6C Fluid Dynamics

A fluid flow has velocity given in Cartesian co-ordinates as $\mathbf{u} = (kty, 0, 0)$ where k is a constant and t is time. Show that the flow is incompressible. Find a stream function and determine an equation for the streamlines at time t.

At $t = 0$ the points along the straight line segment $x = 0$, $0 \leqslant y \leqslant a$, $z = 0$ are marked with dye. Show that at any later time the marked points continue to form a segment of a straight line. Determine the length of this line segment at time t and the angle that it makes with the x-axis.

7B Complex Methods

Using contour integration around a rectangle with vertices

$$-x,\ x,\ x + iy,\ -x + iy,$$

prove that, for all real y,

$$\int_{-\infty}^{+\infty} e^{-(x+iy)^2}\, dx = \int_{-\infty}^{+\infty} e^{-x^2}\, dx\,.$$

Hence derive that the function $f(x) = e^{-x^2/2}$ is an eigenfunction of the Fourier transform

$$\widehat{f}(y) = \int_{-\infty}^{+\infty} f(x) e^{-ixy} dx,$$

i.e. $\widehat{f}$ is a constant multiple of f.

8F Quadratic Mathematics

Define the *rank* and *signature* of a symmetric bilinear form ϕ on a finite-dimensional real vector space. (If your definitions involve a matrix representation of ϕ, you should explain why they are independent of the choice of representing matrix.)

Let V be the space of all $n \times n$ real matrices (where $n \geqslant 2$), and let ϕ be the bilinear form on V defined by

$$\phi(A, B) = \operatorname{tr} AB - \operatorname{tr} A \operatorname{tr} B \ .$$

Find the rank and signature of ϕ.

[*Hint: You may find it helpful to consider the subspace of symmetric matrices having trace zero, and a suitable complement for this subspace.*]

9D Quantum Mechanics

Consider a quantum mechanical particle of mass m moving in one dimension, in a potential well

$$V(x) = \begin{cases} \infty, & x < 0, \\ 0, & 0 < x < a, \\ V_0, & x > a. \end{cases}$$

Sketch the ground state energy eigenfunction $\chi(x)$ and show that its energy is $E = \frac{\hbar^2 k^2}{2m}$, where k satisfies

$$\tan ka = -\frac{k}{\sqrt{\frac{2mV_0}{\hbar^2} - k^2}}.$$

[*Hint: You may assume that* $\chi(0) = 0$.]

SECTION II

10E Analysis II

Suppose that (X, d) is a metric space that has the Bolzano-Weierstrass property (that is, any sequence has a convergent subsequence). Let (Y, d') be any metric space, and suppose that f is a continuous map of X onto Y. Show that (Y, d') also has the Bolzano-Weierstrass property.

Show also that if f is a bijection of X onto Y, then $f^{-1} : Y \to X$ is continuous.

By considering the map $x \mapsto e^{ix}$ defined on the real interval $[-\pi/2, \pi/2]$, or otherwise, show that there exists a continuous choice of $\arg z$ for the complex number z lying in the right half-plane $\{x + iy : x > 0\}$.

11A Methods

The potential $\Phi(r, \vartheta)$, satisfies Laplace's equation everywhere except on a sphere of unit radius and $\Phi \to 0$ as $r \to \infty$. The potential is continuous at $r = 1$, but the derivative of the potential satisfies

$$\lim_{r\to 1^+} \frac{\partial \Phi}{\partial r} - \lim_{r\to 1^-} \frac{\partial \Phi}{\partial r} = V \cos^2 \vartheta,$$

where V is a constant. Use the method of separation of variables to find Φ for both $r > 1$ and $r < 1$.

[*The Laplacian in spherical polar coordinates for axisymmetric systems is*

$$\nabla^2 \equiv \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\left(\frac{\partial}{\partial \vartheta}\sin\vartheta \frac{\partial}{\partial \vartheta}\right).$$

You may assume that the equation

$$\left((1 - x^2)y'\right)' + \lambda y = 0$$

has polynomial solutions of degree n, which are regular at $x = \pm 1$, if and only if $\lambda = n(n+1)$.]

12H Statistics

Suppose we ask 50 men and 150 women whether they are early risers, late risers, or risers with no preference. The data are given in the following table.

	Early risers	*Late risers*	*No preference*	*Totals*
Men	17	22	11	50
Women	43	78	29	150
Totals	60	100	40	200

Derive carefully a (generalized) likelihood ratio test of independence of classification. What is the result of applying this test at the 0.01 level?

[

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_5^2	χ_6^2
99%*percentile*	*6.63*	*9.21*	*11.34*	*15.09*	*16.81*

]

13E Geometry

Let $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$, and let $\mathbb{H}$ have the hyperbolic metric ρ derived from the line element $|dz|/y$. Let Γ be the group of Möbius maps of the form $z \mapsto (az+b)/(cz+d)$, where a, b, c and d are real and $ad - bc = 1$. Show that every g in Γ is an isometry of the metric space $(\mathbb{H}, \rho)$. For z and w in $\mathbb{H}$, let

$$h(z,w) = \frac{|z-w|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}.$$

Show that for every g in Γ, $h\big(g(z), g(w)\big) = h(z,w)$. By considering $z = iy$, where $y > 1$, and $w = i$, or otherwise, show that for all z and w in $\mathbb{H}$,

$$\cosh \rho(z,w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\mathrm{Im}(w)}.$$

By considering points i, iy, where $y > 1$ and $s + it$, where $s^2 + t^2 = 1$, or otherwise, derive Pythagoras' Theorem for hyperbolic geometry in the form $\cosh a \cosh b = \cosh c$, where a, b and c are the lengths of sides of a right-angled triangle whose hypotenuse has length c.

14G Linear Mathematics

Let α be an endomorphism of a vector space V of finite dimension n .

(a) What is the dimension of the vector space of linear endomorphisms of V? Show that there exists a non-trivial polynomial $p(X)$ such that $p(\alpha) = 0$. Define what is meant by the minimal polynomial m_α of α.

(b) Show that the eigenvalues of α are precisely the roots of the minimal polynomial of α.

(c) Let W be a subspace of V such that $\alpha(W) \subseteq W$ and let β be the restriction of α to W. Show that m_β divides m_α.

(d) Give an example of an endomorphism α and a subspace W as in (c) not equal to V for which $m_\alpha = m_\beta$, and $\deg(m_\alpha) > 1$.

15C Fluid Dynamics

State the unsteady form of Bernoulli's theorem.

A spherical bubble having radius R_0 at time $t = 0$ is located with its centre at the origin in unbounded fluid. The fluid is inviscid, has constant density ρ and is everywhere at rest at $t = 0$. The pressure at large distances from the bubble has the constant value p_∞, and the pressure inside the bubble has the constant value $p_\infty - \triangle p$. In consequence the bubble starts to collapse so that its radius at time t is $R(t)$. Find the velocity everywhere in the fluid in terms of $R(t)$ at time t and, assuming that surface tension is negligible, show that R satisfies the equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = -\frac{\triangle p}{\rho} .$$

Find the total kinetic energy of the fluid in terms of $R(t)$ at time t. Hence or otherwise obtain a first integral of the above equation.

16B Complex Methods

(a) Show that if f is an analytic function at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0, i.e. it preserves angles between paths passing through z_0.

(b) Let D be the disc given by $|z + i| < \sqrt{2}$, and let H be the half-plane given by $y > 0$, where $z = x + iy$. Construct a map of the domain $D \cap H$ onto H, and hence find a conformal mapping of $D \cap H$ onto the disc $\{z : |z| < 1\}$. [*Hint: You may find it helpful to consider a mapping of the form* $(az + b)/(cz + d)$, *where* $ad - bc \neq 0$.]

17F Quadratic Mathematics

Let A and B be $n \times n$ real symmetric matrices, such that the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite. Show that it is possible to find an invertible matrix P such that $P^T AP = I$ and $P^T BP$ is diagonal. Show also that the diagonal entries of the matrix $P^T BP$ may be calculated directly from A and B, without finding the matrix P. If

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

find the diagonal entries of $P^T BP$.

18D Quantum Mechanics

A quantum mechanical particle of mass M moves in one dimension in the presence of a negative delta function potential

$$V = -\frac{\hbar^2}{2M\Delta}\delta(x),$$

where Δ is a parameter with dimensions of length.

(a) Write down the time-independent Schrödinger equation for energy eigenstates $\chi(x)$, with energy E. By integrating this equation across $x = 0$, show that the gradient of the wavefunction jumps across $x = 0$ according to

$$\lim_{\epsilon \to 0} \left(\frac{d\chi}{dx}(\epsilon) - \frac{d\chi}{dx}(-\epsilon) \right) \;=\; -\frac{1}{\Delta}\chi(0).$$

[*You may assume that* χ *is continuous across* $x = 0$.]

(b) Show that there exists a negative energy solution and calculate its energy.

(c) Consider a double delta function potential

$$V(x) = -\frac{\hbar^2}{2M\Delta}[\delta(x+a) + \delta(x-a)].$$

For sufficiently small Δ, this potential yields a negative energy solution of odd parity, i.e. $\chi(-x) = -\chi(x)$. Show that its energy is given by

$$E = -\frac{\hbar^2}{2M}\lambda^2, \quad \text{where} \quad \tanh \lambda a = \frac{\lambda\Delta}{1 - \lambda\Delta}.$$

[*You may again assume* χ *is continuous across* $x = \pm a$.]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 6 June 2002 9 to 12

PAPER 2

Before you begin read these instructions carefully.

*Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

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*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Answers must be tied up in separate bundles, marked **A, B, . . . , H** according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1E Analysis II

Define what is meant by (i) a complete metric space, and (ii) a totally bounded metric space.

Give an example of a metric space that is complete but not totally bounded. Give an example of a metric space that is totally bounded but not complete.

Give an example of a continuous function that maps a complete metric space onto a metric space that is not complete. Give an example of a continuous function that maps a totally bounded metric space onto a metric space that is not totally bounded.

[*You need not justify your examples.*]

2C Methods

Write down the transformation law for the components of a second-rank tensor A_{ij} explaining the meaning of the symbols that you use.

A tensor is said to have *cubic symmetry* if its components are unchanged by rotations of $\pi/2$ about each of the three co-ordinate axes. Find the most general second-rank tensor having cubic symmetry.

3H Statistics

Explain what is meant by a uniformly most powerful test, its power function and size.

Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables with common density $\rho e^{-\rho y}$, $y \geq 0$. Obtain the uniformly most powerful test of $\rho = \rho_0$ against alternatives $\rho < \rho_0$ and determine the power function of the test.

4G Further Analysis

Let the function $f = u + iv$ be analytic in the complex plane $\mathbb{C}$ with u,v real-valued. Prove that, if uv is bounded above everywhere on $\mathbb{C}$, then f is constant.

5B Numerical Analysis

Applying the Gram–Schmidt orthogonalization, compute a "skinny" QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 6 \\ 1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix},$$

i.e. find a 4×3 matrix Q with orthonormal columns and an upper triangular 3×3 matrix R such that $A = QR$.

6G Linear Mathematics

Let A be a complex 4×4 matrix such that $A^3 = A^2$. What are the possible minimal polynomials of A? If A is not diagonalisable and $A^2 \neq 0$, list all possible Jordan normal forms of A.

7B Complex Methods

Suppose that f is analytic, and that $|f(z)|^2$ is constant in an open disk D. Use the Cauchy–Riemann equations to show that $f(z)$ is constant in D.

8F Quadratic Mathematics

Explain what is meant by a *sesquilinear form* on a complex vector space V. If ϕ and ψ are two such forms, and $\phi(v,v) = \psi(v,v)$ for all $v \in V$, prove that $\phi(v,w) = \psi(v,w)$ for all $v,w \in V$. Deduce that if $\alpha \colon V \to V$ is a linear map satisfying $\phi(\alpha(v),\alpha(v)) = \phi(v,v)$ for all $v \in V$, then $\phi(\alpha(v),\alpha(w)) = \phi(v,w)$ for all $v,w \in V$.

9D Quantum Mechanics

From the expressions

$$L_x = yP_z - zP_y, \quad L_y = zP_x - xP_z, \quad L_z = xP_y - yP_x,$$

show that

$$(x+iy)z$$

is an eigenfunction of $\mathbf{L}^2$ and L_z, and compute the corresponding eigenvalues.

SECTION II

10E Analysis II

(a) Let f be a map of a complete metric space (X,d) into itself, and suppose that there exists some k in $(0,1)$, and some positive integer N, such that $d\big(f^N(x), f^N(y)\big) \leqslant k\,d(x,y)$ for all distinct x and y in X, where f^m is the mth iterate of f. Show that f has a unique fixed point in X.

(b) Let f be a map of a compact metric space (X,d) into itself such that $d\big(f(x), f(y)\big) < d(x,y)$ for all distinct x and y in X. By considering the function $d\big(f(x), x\big)$, or otherwise, show that f has a unique fixed point in X.

(c) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|f(x) - f(y)| < |x - y|$ for every distinct x and y in $\mathbb{R}^n$. Suppose that for some x, the orbit $O(x) = \{x, f(x), f^2(x), \ldots\}$ is bounded. Show that f maps the closure of $O(x)$ into itself, and deduce that f has a unique fixed point in $\mathbb{R}^n$.

[*The Contraction Mapping Theorem may be used without proof providing that it is correctly stated.*]

11C Methods

If $\mathbf{B}$ is a vector, and

$$T_{ij} = \alpha B_i B_j + \beta B_k B_k \delta_{ij} \ ,$$

show for arbitrary scalars α and β that T_{ij} is a symmetric second-rank tensor.

Find the eigenvalues and eigenvectors of T_{ij}.

Suppose now that $\mathbf{B}$ depends upon position $\mathbf{x}$ and that $\nabla \cdot \mathbf{B} = 0$. Find constants α and β such that

$$\frac{\partial}{\partial x_j} T_{ij} = [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i \ .$$

Hence or otherwise show that if $\mathbf{B}$ vanishes everywhere on a surface S that encloses a volume V then

$$\int_V (\nabla \times \mathbf{B}) \times \mathbf{B} \; dV = 0 \ .$$

12H Statistics

For ten steel ingots from a production process the following measures of hardness were obtained:

73.2, 74.3, 75.4, 73.8, 74.4, 76.7, 76.1, 73.0, 74.6, 74.1.

On the assumption that the variation is well described by a normal density function obtain an estimate of the process mean.

The manufacturer claims that he is supplying steel with mean hardness 75. Derive carefully a (generalized) likelihood ratio test of this claim. Knowing that for the data above

$$S_{XX} = \sum_{j=1}^{n} (X_i - \bar{X})^2 = 12.824,$$

what is the result of the test at the 5% significance level?

[

Distribution	t_9	t_{10}
95% percentile	*1.83*	*1.81*
97.5% percentile	*2.26*	*2.23*

]

13G Further Analysis

(a) Given a topology $\mathcal{T}$ on X, a collection $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every non-empty set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$. Prove that a collection $\mathcal{B}$ is a basis for some topology if it satisfies:

(i) the union of all sets in $\mathcal{B}$ is X;

(ii) if $x \in B_1 \cap B_2$ for two sets B_1 and B_2 in $\mathcal{B}$, then there is a set $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

(b) On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consider the dictionary order given by

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. Given points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\mathbf{z} \in \mathbb{R}^2 : \ \mathbf{x} < \mathbf{z} < \mathbf{y}\}.$$

Show that the sets $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ form a basis of a topology.

(c) Show that this topology on $\mathbb{R}^2$ does not have a countable basis.

14B Numerical Analysis

Let $f \in C[a,b]$ and let $n+1$ distinct points $x_0, \ldots, x_n \in [a,b]$ be given.

(a) Define the divided difference $f[x_0, \ldots, x_n]$ of order n in terms of interpolating polynomials. Prove that it is a symmetric function of the variables x_i, $i = 0, \ldots, n$.

(b) Prove the recurrence relation

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

(c) Hence or otherwise deduce that, for any $i \neq j$, we have

$$f[x_0, \ldots, x_n] = \frac{f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] - f[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]}{x_j - x_i}.$$

(d) From the formulas above, show that, for any $i = 1, \ldots, n-1$,

$$f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] = \gamma f[x_0, \ldots, x_{n-1}] + (1-\gamma) f[x_1, \ldots, x_n],$$

where $\gamma = \dfrac{x_i - x_0}{x_n - x_0}$.

15G Linear Mathematics

(a) A complex $n \times n$ matrix is said to be unipotent if $U - I$ is nilpotent, where I is the identity matrix. Show that U is unipotent if and only if 1 is the only eigenvalue of U.

(b) Let T be an invertible complex matrix. By considering the Jordan normal form of T show that there exists an invertible matrix P such that

$$PTP^{-1} = D_0 + N,$$

where D_0 is an invertible diagonal matrix, N is an upper triangular matrix with zeros in the diagonal and $D_0 N = N D_0$.

(c) Set $D = P^{-1} D_0 P$ and show that $U = D^{-1} T$ is unipotent.

(d) Conclude that any invertible matrix T can be written as $T = DU$ where D is diagonalisable, U is unipotent and $DU = UD$.

16B Complex Methods

A function $f(z)$ has an isolated singularity at a, with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

(a) Define $\operatorname{res}(f, a)$, the residue of f at the point a.

(b) Prove that if a is a pole of order $k+1$, then

$$\operatorname{res}(f, a) = \lim_{z \to a} \frac{h^{(k)}(z)}{k!}, \quad \text{where} \quad h(z) = (z-a)^{k+1} f(z).$$

(c) Using the residue theorem and the formula above show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{k+1}} = \pi \frac{(2k)!}{(k!)^2} 4^{-k}, \quad k \geq 1.$$

17F Quadratic Mathematics

Define the *adjoint* α^* of an endomorphism α of a complex inner-product space V. Show that if W is a subspace of V, then $\alpha(W) \subseteq W$ if and only if $\alpha^*(W^\perp) \subseteq W^\perp$.

An endomorphism of a complex inner-product space is said to be *normal* if it commutes with its adjoint. Prove the following facts about a normal endomorphism α of a finite-dimensional space V.

(i) α and α^* have the same kernel.

(ii) α and α^* have the same eigenvectors, with complex conjugate eigenvalues.

(iii) If $E_\lambda = \{x \in V : \alpha(x) = \lambda x\}$, then $\alpha(E_\lambda^\perp) \subseteq E_\lambda^\perp$.

(iv) There is an orthonormal basis of V consisting of eigenvectors of α.

Deduce that an endomorphism α is normal if and only if it can be written as a product $\beta\gamma$, where β is Hermitian, γ is unitary and β and γ commute with each other. [*Hint: Given α, define β and γ in terms of their effect on the basis constructed in (iv).*]

18D Quantum Mechanics

Consider a quantum mechanical particle moving in an upside-down harmonic oscillator potential. Its wavefunction $\Psi(x,t)$ evolves according to the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{2}x^2\Psi. \tag{1}$$

(a) Verify that

$$\Psi(x,t) = A(t)e^{-B(t)x^2} \tag{2}$$

is a solution of equation (1), provided that

$$\frac{dA}{dt} = -i\hbar\, AB,$$

and

$$\frac{dB}{dt} = -\frac{i}{2\hbar} - 2i\hbar B^2. \tag{3}$$

(b) Verify that $B = \frac{1}{2\hbar}\tan(\phi - it)$ provides a solution to (3), where ϕ is an arbitrary real constant.

(c) The expectation value of an operator $\mathcal{O}$ at time t is

$$\langle\mathcal{O}\rangle(t) \equiv \int_{-\infty}^{\infty} dx\Psi^*(x,t)\mathcal{O}\Psi(x,t),$$

where $\Psi(x,t)$ is the normalised wave function. Show that for $\Psi(x,t)$ given by (2),

$$\langle x^2\rangle = \frac{1}{4Re(B)}, \quad \langle p^2\rangle = 4\hbar^2|B|^2\langle x^2\rangle.$$

Hence show that as $t \to \infty$,

$$\langle x^2\rangle \approx \langle p^2\rangle \approx \frac{\hbar}{4\sin 2\phi}e^{2t}.$$

[*Hint: You may use*

$$\frac{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}x^2}{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}} = \frac{1}{2C}.]$$

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 7 June 2002 9 to 12

PAPER 3

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SECTION I

1E Analysis II

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f = (u, v)$, where u and v are defined by $u(0) = v(0) = 0$ and, for $t \neq 0$, $u(t) = t^2 \sin(1/t)$ and $v(t) = t^2 \cos(1/t)$. Show that f is differentiable on $\mathbb{R}$.

Show that for any real non-zero a, $||f'(a) - f'(0)|| > 1$, where we regard $f'(a)$ as the vector $\big(u'(a), v'(a)\big)$ in $\mathbb{R}^2$.

2A Methods

Write down the wave equation for the displacement $y(x,t)$ of a stretched string with constant mass density and tension. Obtain the general solution in the form

$$y(x,t) = f(x+ct) + g(x-ct),$$

where c is the wave velocity. For a solution in the region $0 \leqslant x < \infty$, with $y(0,t) = 0$ and $y \to 0$ as $x \to \infty$, show that

$$E = \int_0^\infty \left[\frac{1}{2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} c^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx,$$

is constant in time. Express E in terms of the general solution in this case.

3G Further Analysis

Let $f : X \to Y$ be a continuous map between topological spaces. Let

$$G_f = \{(x, f(x)) : \ x \in X\}.$$

(a) Show that if Y is Hausdorff, then G_f is closed in $X \times Y$.

(b) Show that if X is compact, then G_f is also compact.

4E Geometry

State Euler's formula for a graph $\mathcal{G}$ with F faces, E edges and V vertices on the surface of a sphere.

Suppose that every face in $\mathcal{G}$ has at least three edges, and that at least three edges meet at every vertex of $\mathcal{G}$. Let F_n be the number of faces of $\mathcal{G}$ that have exactly n edges ($n \geqslant 3$), and let V_m be the number of vertices at which exactly m edges meet ($m \geqslant 3$). By expressing $6F - \sum_n nF_n$ in terms of the V_j, or otherwise, show that every convex polyhedron has at least four faces each of which is a triangle, a quadrilateral or a pentagon.

5H Optimization

Consider a two-person zero-sum game with a payoff matrix

$$\begin{pmatrix} 3 & b \\ 5 & 2 \end{pmatrix},$$

where $0 < b < \infty$. Here, the (i,j) entry of the matrix indicates the payoff to player one if he chooses move i and player two move j. Suppose player one chooses moves 1 and 2 with probabilities p and $1-p$, $0 \leq p \leq 1$. Write down the maximization problem for the optimal strategy and solve it for each value of b.

6B Numerical Analysis

For numerical integration, a quadrature formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

is applied which is exact on $\mathcal{P}_n$, i.e., for all polynomials of degree n.

Prove that such a formula is exact for all $f \in \mathcal{P}_{2n+1}$ if and only if x_i, $i = 0, \ldots, n$, are the zeros of an orthogonal polynomial $p_{n+1} \in \mathcal{P}_{n+1}$ which satisfies $\int_a^b p_{n+1}(x) r(x)\,dx = 0$ for all $r \in \mathcal{P}_n$. [*You may assume that p_{n+1} has $(n+1)$ distinct zeros.*]

7F Linear Mathematics

Which of the following statements are true, and which false? Give brief justifications for your answers.

(a) If U and W are subspaces of a vector space V, then $U \cap W$ is always a subspace of V.

(b) If U and W are distinct subspaces of a vector space V, then $U \cup W$ is never a subspace of V.

(c) If U, W and X are subspaces of a vector space V, then $U \cap (W + X) = (U \cap W) + (U \cap X)$.

(d) If U is a subspace of a finite-dimensional space V, then there exists a subspace W such that $U \cap W = \{0\}$ and $U + W = V$.

8C Fluid Dynamics

State and prove Kelvin's circulation theorem.

Consider a planar flow in the unbounded region outside a cylinder for which the vorticity vanishes everywhere at time $t = 0$. What may be deduced about the circulation around closed loops in the fluid at time t:

(i) that do not enclose the cylinder;

(ii) that enclose the cylinder?

Give a brief justification for your answer in each case.

9F Quadratic Mathematics

Explain what is meant by a *quadratic residue* modulo an odd prime p, and show that a is a quadratic residue modulo p if and only if $a^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}$. Hence characterize the odd primes p for which -1 is a quadratic residue.

State the law of quadratic reciprocity, and use it to determine whether 73 is a quadratic residue (mod 127).

10D Special Relativity

Write down the formulae for a Lorentz transformation with velocity v taking one set of co-ordinates (t, x) to another (t', x').

Imagine you observe a train travelling past Cambridge station at a relativistic speed u. Someone standing still on the train throws a ball in the direction the train is moving, with speed v. How fast do you observe the ball to be moving? Justify your answer.

SECTION II

11E Analysis II

Show that if a, b and c are non-negative numbers, and $a \leqslant b+c$, then

$$\frac{a}{1+a} \leqslant \frac{b}{1+b} + \frac{c}{1+c}.$$

Deduce that if (X,d) is a metric space, then $d(x,y)/[1+d(x,y)]$ is a metric on X.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K_n = \{z \in D : |z| \leqslant (n-1)/n\}$. Let $\mathcal{F}$ be the class of continuous complex-valued functions on D and, for f and g in $\mathcal{F}$, define

$$\sigma(f,g) = \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{||f-g||_n}{1+||f-g||_n},$$

where $||f-g||_n = \sup\{|f(z)-g(z)| : z \in K_n\}$. Show that the series for $\sigma(f,g)$ converges, and that σ is a metric on $\mathcal{F}$.

For $|z| < 1$, let $s_k(z) = 1 + z + z^2 + \cdots + z^k$ and $s(z) = 1 + z + z^2 + \cdots$. Show that for $n \geqslant 2$, $||s_k - s||_n = n(1-\frac{1}{n})^{k+1}$. By considering the sums for $2 \leqslant n \leqslant N$ and $n > N$ separately, show that for each N,

$$\sigma(s_k, s) \leqslant \sum_{n=2}^{N} ||s_k - s||_n + 2^{-N},$$

and deduce that $\sigma(s_k, s) \to 0$ as $k \to \infty$.

12A Methods

Consider the real Sturm-Liouville problem

$$\mathcal{L}y(x) = -(p(x)y')' + q(x)y = \lambda r(x)y,$$

with the boundary conditions $y(a) = y(b) = 0$, where p, q and r are continuous and positive on $[a, b]$. Show that, with suitable choices of inner product and normalisation, the eigenfunctions $y_n(x)$, $n = 1, 2, 3 \ldots$, form an orthonormal set.

Hence show that the corresponding Green's function $G(x, \xi)$ satisfying

$$(\mathcal{L} - \mu r(x))G(x, \xi) = \delta(x - \xi),$$

where μ is not an eigenvalue, is

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)}{\lambda_n - \mu},$$

where λ_n is the eigenvalue corresponding to y_n.

Find the Green's function in the case where

$$\mathcal{L}y \equiv y'',$$

with boundary conditions $y(0) = y(\pi) = 0$, and deduce, by suitable choice of μ, that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

13G Further Analysis

(a) Let f and g be two analytic functions on a domain D and let $\gamma \subset D$ be a simple closed curve homotopic in D to a point. If $|g(z)| < |f(z)|$ for every z in γ, prove that γ encloses the same number of zeros of f as of $f + g$.

(b) Let g be an analytic function on the disk $|z| < 1 + \epsilon$, for some $\epsilon > 0$. Suppose that g maps the closed unit disk into the open unit disk (both centred at 0). Prove that g has exactly one fixed point in the open unit disk.

(c) Prove that, if $|a| < 1$, then

$$z^m \left(\frac{z - a}{1 - \bar{a}z} \right)^n - a$$

has $m + n$ zeros in $|z| < 1$.

14E Geometry

Show that every isometry of Euclidean space $\mathbb{R}^3$ is a composition of reflections in planes.

What is the smallest integer N such that every isometry f of $\mathbb{R}^3$ with $f(0) = 0$ can be expressed as the composition of at most N reflections? Give an example of an isometry that needs this number of reflections and justify your answer.

Describe (geometrically) all twelve orientation-reversing isometries of a regular tetrahedron.

15H Optimization

Consider the following linear programming problem

$$\begin{array}{ll} \text{maximise} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 - x_2 \geq 1, \\ & 4x_1 - x_2 \geq 10, \\ & x_2 \leq 6, \\ & x_i \geq 0,\ i = 1, 2. \end{array} \tag{1}$$

Write down the Phase One problem for (1) and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve (1), i.e., find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

16B Numerical Analysis

(a) Consider a system of linear equations $Ax = b$ with a non-singular square $n \times n$ matrix A. To determine its solution $x = x^*$ we apply the iterative method

$$x^{k+1} = Hx^k + v.$$

Here $v \in \mathbb{R}^n$, while the matrix $H \in \mathbb{R}^{n \times n}$ is such that $x^* = Hx^* + v$ implies $Ax^* = b$. The initial vector $x^0 \in \mathbb{R}^n$ is arbitrary. Prove that, if the matrix H possesses n linearly independent eigenvectors $w_1, \ldots, w_n$ whose corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy $\max_i |\lambda_i| < 1$, then the method converges for any choice of x^0, i.e. $x^k \to x^*$ as $k \to \infty$.

(b) Describe the Jacobi iteration method for solving $Ax = b$. Show directly from the definition of the method that, if the matrix A is strictly diagonally dominant by rows, i.e.

$$|a_{ii}|^{-1} \sum_{j=1, j \neq i}^{n} |a_{ij}| \leq \gamma < 1, \quad i = 1, \ldots, n,$$

then the method converges.

17F Linear Mathematics

Define the *determinant* of an $n \times n$ matrix A, and prove from your definition that if A' is obtained from A by an elementary row operation (i.e. by adding a scalar multiple of the ith row of A to the jth row, for some $j \neq i$), then $\det A' = \det A$.

Prove also that if X is a $2n \times 2n$ matrix of the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where O denotes the $n \times n$ zero matrix, then $\det X = \det A \det C$. Explain briefly how the $2n \times 2n$ matrix

$$\begin{pmatrix} B & I \\ O & A \end{pmatrix}$$

can be transformed into the matrix

$$\begin{pmatrix} B & I \\ -AB & O \end{pmatrix}$$

by a sequence of elementary row operations. Hence or otherwise prove that $\det AB = \det A \det B$.

18C Fluid Dynamics

Use Euler's equation to derive Bernoulli's theorem for the steady flow of an inviscid fluid of uniform density ρ in the absence of body forces.

Such a fluid flows steadily through a long cylindrical elastic tube having circular cross-section. The variable z measures distance downstream along the axis of the tube. The tube wall has thickness $h(z)$, so that if the external radius of the tube is $r(z)$, its internal radius is $r(z) - h(z)$, where $h(z) \geqslant 0$ is a given slowly-varying function that tends to zero as $z \to \pm\infty$. The elastic tube wall exerts a pressure $p(z)$ on the fluid given as

$$p(z) = p_0 + k[r(z) - R],$$

where p_0, k and R are positive constants. Far upstream, r has the constant value R, the fluid pressure has the constant value p_0, and the fluid velocity u has the constant value V. Assume that gravity is negligible and that $h(z)$ varies sufficiently slowly that the velocity may be taken as uniform across the tube at each value of z. Use mass conservation and Bernoulli's theorem to show that $u(z)$ satisfies

$$\frac{h}{R} = 1 - \left(\frac{V}{u}\right)^{1/2} + \frac{1}{4}\lambda\left[1 - \left(\frac{u}{V}\right)^2\right], \quad \text{where} \quad \lambda = \frac{2\rho V^2}{kR}\ .$$

Sketch a graph of h/R against u/V. Show that if $h(z)$ exceeds a critical value $h_c(\lambda)$, no such flow is possible and find $h_c(\lambda)$.

Show that if $h < h_c(\lambda)$ everywhere, then for given h the equation has two positive solutions for u. Explain how the given value of λ determines which solution should be chosen.

19F Quadratic Mathematics

Explain what is meant by saying that a positive definite integral quadratic form $f(x,y) = ax^2 + bxy + cy^2$ is *reduced*, and show that every positive definite form is equivalent to a reduced form.

State a criterion for a prime number p to be representable by some form of discriminant d, and deduce that p is representable by a form of discriminant -32 if and only if $p \equiv 1, 2$ or $3 \pmod 8$. Find the reduced forms of discriminant -32, and hence or otherwise show that a prime p is representable by the form $3x^2 + 2xy + 3y^2$ if and only if $p \equiv 3 \pmod 8$.

[*Standard results on when* -1 *and* 2 *are squares* $\pmod p$ *may be assumed.*]

20D Quantum Mechanics

A quantum mechanical system has two states χ_0 and χ_1, which are normalised energy eigenstates of a Hamiltonian H_3, with

$$H_3\chi_0 = -\chi_0, \quad H_3\chi_1 = +\chi_1.$$

A general time-dependent state may be written

$$\Psi(t) = a_0(t)\chi_0 + a_1(t)\chi_1, \tag{1}$$

where $a_0(t)$ and $a_1(t)$ are complex numbers obeying $|a_0(t)|^2 + |a_1(t)|^2 = 1$.

(a) Write down the time-dependent Schrödinger equation for $\Psi(t)$, and show that if the Hamiltonian is H_3, then

$$i\hbar\frac{da_0}{dt} = -a_0, \qquad i\hbar\frac{da_1}{dt} = +a_1.$$

For the general state given in equation (1) above, write down the probability to observe the system, at time t, in a state $\alpha\chi_0 + \beta\chi_1$, properly normalised so that $|\alpha|^2 + |\beta|^2 = 1$.

(b) Now consider starting the system in the state χ_0 at time $t = 0$, and evolving it with a different Hamiltonian H_1, which acts on the states χ_0 and χ_1 as follows:

$$H_1\chi_0 = \chi_1, \qquad H_1\chi_1 = \chi_0.$$

By solving the time-dependent Schrödinger equation for the Hamiltonian H_1, find $a_0(t)$ and $a_1(t)$ in this case. Hence determine the shortest time $T > 0$ such that $\Psi(T)$ is an eigenstate of H_3 with eigenvalue $+1$.

(c) Now consider taking the state $\Psi(T)$ from part (b), and evolving it for further length of time T, with Hamiltonian H_2, which acts on the states χ_0 and χ_1 as follows:

$$H_2\chi_0 = -i\chi_1, \qquad H_2\chi_1 = i\chi_0.$$

What is the final state of the system? Is this state observationally distinguishable from the original state χ_0?

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 7 June 2002 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, ..., H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1E Analysis II

(a) Let (X,d) be a metric space containing the point x_0, and let

$$U = \{x \in X : d(x,x_0) < 1\}, \quad K = \{x \in X : d(x,x_0) \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

(b) Let X be a normed space with norm $||\cdot||$, and let

$$U = \{x \in X : ||x|| < 1\}, \quad K = \{x \in X : ||x|| \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

2A Methods

Use the method of Lagrange multipliers to find the largest volume of a rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

3H Statistics

From each of 100 concrete mixes six sample blocks were taken and subjected to strength tests, the number out of the six blocks failing the test being recorded in the following table:

No. x failing strength tests	0	1	2	3	4	5	6
No. of mixes with x failures	53	32	12	2	1	0	0

On the assumption that the probability of failure is the same for each block, obtain an unbiased estimate of this probability and explain how to find a 95% confidence interval for it.

4G Further Analysis

(a) Let X be a topological space and suppose $X = C \cup D$, where C and D are disjoint nonempty open subsets of X. Show that, if Y is a connected subset of X, then Y is entirely contained in either C or D.

(b) Let X be a topological space and let $\{A_n\}$ be a sequence of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$, for $n = 1, 2, 3, \ldots$. Show that $\bigcup_{n \geqslant 1} A_n$ is connected.

5H Optimization

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow, maximal flow, cut, capacity.

6F Linear Mathematics

Define the *rank* and *nullity* of a linear map between finite-dimensional vector spaces. State the rank–nullity formula.

Let $\alpha \colon U \to V$ and $\beta \colon V \to W$ be linear maps. Prove that

$$\operatorname{rank}(\alpha) + \operatorname{rank}(\beta) - \dim V \leqslant \operatorname{rank}(\beta\alpha) \leqslant \min\{\operatorname{rank}(\alpha), \operatorname{rank}(\beta)\} \ .$$

7C Fluid Dynamics

If $\mathbf{u}$ is given in Cartesian co-ordinates as $\mathbf{u} = (-\Omega y, \Omega x, 0)$, with Ω a constant, verify that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(-\tfrac{1}{2}\mathbf{u}^2) \ .$$

When incompressible fluid is placed in a stationary cylindrical container of radius a with its axis vertical, the depth of the fluid is h. Assuming that the free surface does not reach the bottom of the container, use cylindrical polar co-ordinates to find the equation of the free surface when the fluid and the container rotate steadily about this axis with angular velocity Ω.

Deduce the angular velocity at which the free surface first touches the bottom of the container.

8B Complex Methods

Let f be a function such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$. Prove that

$$\int_{-\infty}^{+\infty} f(x+k)\overline{f(x+l)}\, dx = 0 \quad \text{for all integers } k \text{ and } l \text{ with } k \neq l,$$

if and only if

$$\int_{-\infty}^{+\infty} |\widehat{f}(t)|^2 e^{-imt} dt = 0 \quad \text{for all integers } m \neq 0,$$

where $\widehat{f}$ is the Fourier transform of f.

9D Special Relativity

A particle with mass M is observed to be at rest. It decays into a particle of mass $m < M$, and a massless particle. Calculate the energies and momenta of both final particles.

SECTION II

10E Analysis II

(a) Let V be a finite-dimensional real vector space, and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on V. Show that a function $f : V \to \mathbb{R}$ is differentiable at a point a in V with respect to $||\cdot||_1$ if and only if it is differentiable at a with respect to $||\cdot||_2$, and that if this is so then the derivative $f'(a)$ of f is independent of the norm used. [*You may assume that all norms on a finite-dimensional vector space are equivalent.*]

(b) Let V_1, V_2 and V_3 be finite-dimensional normed real vector spaces with V_j having norm $||\cdot||_j$, $j = 1, 2, 3$, and let $f : V_1 \times V_2 \to V_3$ be a continuous bilinear mapping. Show that f is differentiable at any point (a, b) in $V_1 \times V_2$, and that $f'(a, b)(h, k) = f(h, b) + f(a, k)$. [*You may assume that* $\left(||u||_1^2 + ||v||_2^2\right)^{1/2}$ *is a norm on* $V_1 \times V_2$, *and that* $\{(x, y) \in V_1 \times V_2 : ||x||_1 = 1, ||y||_2 = 1\}$ *is compact.*]

11A Methods

A function $y(x)$ is chosen to make the integral

$$I = \int_a^b f\left(x, y, y', y''\right) dx$$

stationary, subject to given values of $y(a), y'(a), y(b)$ and $y'(b)$. Derive an analogue of the Euler–Lagrange equation for $y(x)$.

Solve this equation for the case where

$$f = x^4 y''^2 + 4y^2 y',$$

in the interval $[0, 1]$ and

$$x^2 y(x) \to 0, \quad xy(x) \to 1$$

as $x \to 0$, whilst

$$y(1) = 2, \quad y'(1) = 0.$$

12H Statistics

Explain what is meant by a prior distribution, a posterior distribution, and a Bayes estimator. Relate the Bayes estimator to the posterior distribution for both quadratic and absolute error loss functions.

Suppose $X_1, \ldots, X_n$ are independent identically distributed random variables from a distribution uniform on $(\theta - 1, \theta + 1)$, and that the prior for θ is uniform on $(20, 50)$.

Calculate the posterior distribution for θ, given $\mathbf{x} = (x_1, \ldots, x_n)$, and find the point estimate for θ under both quadratic and absolute error loss function.

13G Further Analysis

A function f is said to be analytic at ∞ if there exists a real number $r > 0$ such that f is analytic for $|z| > r$ and $\lim_{z\to 0} f(1/z)$ is finite (i.e. $f(1/z)$ has a removable singularity at $z = 0$). f is said to have a pole at ∞ if $f(1/z)$ has a pole at $z = 0$. Suppose that f is a meromorphic function on the extended plane $\mathbb{C}_\infty$, that is, f is analytic at each point of $\mathbb{C}_\infty$ except for poles.

(a) Show that if f has a pole at $z = \infty$, then there exists $r > 0$ such that $f(z)$ has no poles for $r < |z| < \infty$.

(b) Show that the number of poles of f is finite.

(c) By considering the Laurent expansions around the poles show that f is in fact a rational function, i.e. of the form p/q, where p and q are polynomials.

(d) Deduce that the only bijective meromorphic maps of $\mathbb{C}_\infty$ onto itself are the Möbius maps.

14H Optimization

A gambler at a horse race has an amount $b > 0$ to bet. The gambler assesses p_i, the probability that horse i will win, and knows that $s_i \geq 0$ has been bet on horse i by others, for $i = 1, 2, \ldots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gambler's optimal strategy is to choose $(x_1, x_2, \ldots, x_n)$ so that it maximizes

$$\sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0, \tag{1}$$

where x_i is the amount the gambler bets on horse i. Show that the optimal solution to (1) also solves the following problem:

$$\text{minimize } \sum_{i=1}^{n} \frac{p_i s_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0.$$

Assume that $p_1/s_1 \geq p_2/s_2 \geq \ldots \geq p_n/s_n$. Applying the Lagrangian sufficiency theorem, prove that the optimal solution to (1) satisfies

$$\frac{p_1 s_1}{(s_1 + x_1)^2} = \ldots = \frac{p_k s_k}{(s_k + x_k)^2}, \quad x_{k+1} = \ldots = x_n = 0,$$

with maximal possible $k \in \{1, 2, \ldots, n\}$.

[*You may use the fact that for all $\lambda < 0$, the minimum of the function $x \mapsto \frac{ps}{s+x} - \lambda x$ on the non-negative axis $0 \leq x < \infty$ is attained at*

$$x(\lambda) = \left(\sqrt{\frac{ps}{-\lambda}} - s\right)^{+} \equiv \max\left(\sqrt{\frac{ps}{-\lambda}} - s, 0\right).]$$

Deduce that if b is small enough, the gambler's optimal strategy is to bet on the horses for which the ratio p_i/s_i is maximal. What is his expected gain in this case?

15F Linear Mathematics

Define the *dual space* V^* of a finite-dimensional real vector space V, and explain what is meant by the basis of V^* dual to a given basis of V. Explain also what is meant by the statement that the second dual V^{**} is naturally isomorphic to V.

Let V_n denote the space of real polynomials of degree at most n. Show that, for any real number x, the function e_x mapping p to $p(x)$ is an element of V_n^*. Show also that, if $x_1, x_2, \ldots, x_{n+1}$ are distinct real numbers, then $\{e_{x_1}, e_{x_2}, \ldots, e_{x_{n+1}}\}$ is a basis of V_n^*, and find the basis of V_n dual to it.

Deduce that, for any $(n+1)$ distinct points $x_1, \ldots, x_{n+1}$ of the interval $[-1, 1]$, there exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n+1} \lambda_i p(x_i)$$

for all $p \in V_n$. For $n = 4$ and $(x_1, x_2, x_3, x_4, x_5) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$, find the corresponding scalars λ_i.

16C Fluid Dynamics

Use Euler's equation to show that in a planar flow of an inviscid fluid the vorticity $\boldsymbol{\omega}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = 0 \ .$$

Write down the velocity field associated with a point vortex of strength κ in unbounded fluid.

Consider now the flow generated in unbounded fluid by two point vortices of strengths κ_1 and κ_2 at $\mathbf{x}_1(t) = (x_1, y_1)$ and $\mathbf{x}_2(t) = (x_2, y_2)$, respectively. Show that in the subsequent motion the quantity

$$\mathbf{q} = \kappa_1 \mathbf{x}_1 + \kappa_2 \mathbf{x}_2$$

remains constant. Show also that the separation of the vortices, $|\mathbf{x}_2 - \mathbf{x}_1|$, remains constant.

Suppose finally that $\kappa_1 = \kappa_2$ and that the vortices are placed at time $t = 0$ at positions $(a, 0)$ and $(-a, 0)$. What are the positions of the vortices at time t?

17B Complex Methods

(a) Using the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \frac{\pi}{2}$, show that, if f is continuous for large $|z|$, and if $f(z) \to 0$ as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z) e^{i\lambda z} dz = 0 \quad \text{for} \quad \lambda > 0,$$

where $\Gamma_R = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

(b) By integrating an appropriate function $f(z)$ along the contour formed by the semicircles Γ_R and Γ_r in the upper half-plane with the segments of the real axis $[-R, -r]$ and $[r, R]$, show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

18D Special Relativity

A javelin of length 2m is thrown horizontally and lengthwise into a shed of length 1.5m at a speed of $0.8c$, where c is the speed of light.

(a) What is the length of the javelin in the rest frame of the shed?

(b) What is the length of the shed in the rest frame of the javelin?

(c) Draw a space-time diagram in the rest frame coordinates (ct, x) of the shed, showing the world lines of both ends of the javelin, and of the front and back of the shed. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the shed.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the shed, and (B) the leading end of the javelin hitting the back of the shed. Give the corresponding (ct, x) and (ct', x') coordinates for both (A) and (B). Are these two events space-like, null or time-like separated? How does the javelin fit inside the shed, even though it is initially longer than the shed in its own rest frame?

END OF PAPER

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2003

List of Courses

1/I/1F **Analysis II**

Let E be a subset of $\mathbb{R}^n$. Prove that the following conditions on E are equivalent:

(i) E is closed and bounded.

(ii) E has the Bolzano–Weierstrass property (i.e., every sequence in E has a subsequence convergent to a point of E).

(iii) Every continuous real-valued function on E is bounded.

[*The Bolzano–Weierstrass property for bounded closed intervals in* $\mathbb{R}^1$ *may be assumed.*]

1/II/10F **Analysis II**

Explain briefly what is meant by a *metric space*, and by a *Cauchy sequence* in a metric space.

A function $d : X \times X \to \mathbb{R}$ is called a pseudometric on X if it satisfies all the conditions for a metric except the requirement that $d(x, y) = 0$ implies $x = y$. If d is a pseudometric on X, show that the binary relation R on X defined by $x\,R\,y \Leftrightarrow d(x, y) = 0$ is an equivalence relation, and that the function d induces a metric on the set X/R of equivalence classes.

Now let (X, d) be a metric space. If (x_n) and (y_n) are Cauchy sequences in X, show that the sequence whose nth term is $d(x_n, y_n)$ is a Cauchy sequence of real numbers. Deduce that the function $\overline{d}$ defined by

$$\overline{d}((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$$

is a pseudometric on the set C of all Cauchy sequences in X. Show also that there is an isometric embedding (that is, a distance-preserving mapping) $X \to C/R$, where R is the equivalence relation on C induced by the pseudometric $\overline{d}$ as in the previous paragraph. Under what conditions on X is $X \to C/R$ bijective? Justify your answer.

2/I/1F **Analysis II**

Explain what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ to be *differentiable* at a point (a, b). Show that if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in a neighbourhood of (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Let

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad ((x, y) \neq (0, 0))$$

and $f(0, 0) = 0$. Do the partial derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$? Justify your answers.

2/II/10F Analysis II

Let V be the space of $n \times n$ real matrices. Show that the function

$$N(A) = \sup\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| = 1\}$$

(where $\| - \|$ denotes the usual Euclidean norm on $\mathbb{R}^n$) defines a norm on V. Show also that this norm satisfies $N(AB) \leqslant N(A)N(B)$ for all A and B, and that if $N(A) < \epsilon$ then all entries of A have absolute value less than ϵ. Deduce that any function $f\colon V \to \mathbb{R}$ such that $f(A)$ is a polynomial in the entries of A is continuously differentiable.

Now let $d : V \to \mathbb{R}$ be the mapping sending a matrix to its determinant. By considering $d(I+H)$ as a polynomial in the entries of H, show that the derivative $d'(I)$ is the function $H \mapsto \operatorname{tr} H$. Deduce that, for any A, $d'(A)$ is the mapping $H \mapsto \operatorname{tr}((\operatorname{adj} A)H)$, where $\operatorname{adj} A$ is the adjugate of A, i.e. the matrix of its cofactors.

[*Hint: consider first the case when A is invertible. You may assume the results that the set U of invertible matrices is open in V and that its closure is the whole of V, and the identity* $(\operatorname{adj} A)A = \det A.I$.]

3/I/1F Analysis II

Let V be the vector space of continuous real-valued functions on $[-1,1]$. Show that the function

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx$$

defines a norm on V.

Let $f_n(x) = x^n$. Show that (f_n) is a Cauchy sequence in V. Is (f_n) convergent? Justify your answer.

3/II/11F Analysis II

State and prove the Contraction Mapping Theorem.

Let (X,d) be a bounded metric space, and let F denote the set of all continuous maps $X \to X$. Let $\rho\colon F \times F \to \mathbb{R}$ be the function

$$\rho(f,g) = \sup\{d(f(x),g(x)) : x \in X\}\ .$$

Show that ρ is a metric on F, and that (F,ρ) is complete if (X,d) is complete. [*You may assume that a uniform limit of continuous functions is continuous.*]

Now suppose that (X,d) is complete. Let $C \subseteq F$ be the set of contraction mappings, and let $\theta\colon C \to X$ be the function which sends a contraction mapping to its unique fixed point. Show that θ is continuous. [*Hint: fix $f \in C$ and consider $d(\theta(g), f(\theta(g)))$, where $g \in C$ is close to f.*]

4/I/1F Analysis II

Explain what it means for a sequence of functions (f_n) to converge uniformly to a function f on an interval. If (f_n) is a sequence of continuous functions converging uniformly to f on a finite interval $[a, b]$, show that

$$\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx \qquad \text{as } n \to \infty\ .$$

Let $f_n(x) = x \exp(-x/n)/n^2$, $x \geqslant 0$. Does $f_n \to 0$ uniformly on $[0, \infty)$? Does $\int_0^\infty f_n(x)\,dx \to 0$? Justify your answers.

4/II/10F Analysis II

Let $(f_n)_{n \geqslant 1}$ be a sequence of continuous complex-valued functions defined on a set $E \subseteq \mathbb{C}$, and converging uniformly on E to a function f. Prove that f is continuous on E.

State the Weierstrass M-test for uniform convergence of a series $\sum_{n=1}^{\infty} u_n(z)$ of complex-valued functions on a set E.

Now let $f(z) = \sum_{n=1}^{\infty} u_n(z)$, where

$$u_n(z) = n^{-2} \sec\left(\pi z/2n\right)\ .$$

Prove carefully that f is continuous on $\mathbb{C} \setminus \mathbb{Z}$.

[*You may assume the inequality* $|\cos z| \geqslant |\cos(\operatorname{Re} z)|$.]

1/I/7B Complex Methods

Let $u(x, y)$ and $v(x, y)$ be a pair of conjugate harmonic functions in a domain D. Prove that

$$U(x, y) = e^{-2uv} \cos(u^2 - v^2) \qquad \text{and} \qquad V(x, y) = e^{-2uv} \sin(u^2 - v^2)$$

also form a pair of conjugate harmonic functions in D.

1/II/16B Complex Methods

Sketch the region A which is the intersection of the discs

$$D_0 = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad D_1 = \{z \in \mathbb{C} : |z - (1 + i)| < 1\}\,.$$

Find a conformal mapping that maps A onto the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Also find a conformal mapping that maps A onto D_0.

[*Hint: You may find it useful to consider maps of the form* $w(z) = \frac{az+b}{cz+d}$.]

2/I/7B Complex Methods

(a) Using the residue theorem, evaluate

$$\int_{|z|=1} \left(z - \frac{1}{z}\right)^{2n} \frac{dz}{z}\,.$$

(b) Deduce that

$$\int_0^{2\pi} \sin^{2n} t \, dt = \frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}\,.$$

2/II/16B **Complex Methods**

(a) Show that if f satisfies the equation

$$f''(x) - x^2 f(x) = \mu f(x), \quad x \in \mathbb{R}, \tag{$*$}$$

where μ is a constant, then its Fourier transform $\widehat{f}$ satisfies the same equation, i.e.

$$\widehat{f}''(\lambda) - \lambda^2 \widehat{f}(\lambda) = \mu \widehat{f}(\lambda)\,.$$

(b) Prove that, for each $n \geq 0$, there is a polynomial $p_n(x)$ of degree n, unique up to multiplication by a constant, such that

$$f_n(x) = p_n(x) e^{-x^2/2}$$

is a solution of $(*)$ for some $\mu = \mu_n$.

(c) Using the fact that $g(x) = e^{-x^2/2}$ satisfies $\widehat{g} = cg$ for some constant c, show that the Fourier transform of f_n has the form

$$\widehat{f_n}(\lambda) = q_n(\lambda) e^{-\lambda^2/2}$$

where q_n is also a polynomial of degree n.

(d) Deduce that the f_n are eigenfunctions of the Fourier transform operator, i.e. $\widehat{f_n}(x) = c_n f_n(x)$ for some constants c_n.

4/I/8B **Complex Methods**

Find the Laurent series centred on 0 for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in each of the domains

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$.

4/II/17B **Complex Methods**

Let

$$f(z) = \frac{z^m}{1+z^n}, \quad n > m+1, \quad m, n \in \mathbb{N},$$

and let C_R be the boundary of the domain

$$D_R = \{z = re^{i\theta} : \ 0 < r < R, \quad 0 < \theta < \tfrac{2\pi}{n}\}, \quad R > 1.$$

(a) Using the residue theorem, determine

$$\int_{C_R} f(z)\, dz.$$

(b) Show that the integral of $f(z)$ along the circular part γ_R of C_R tends to 0 as $R \to \infty$.

(c) Deduce that

$$\int_0^\infty \frac{x^m}{1+x^n}\, dx = \frac{\pi}{n \sin \frac{\pi(m+1)}{n}}\,.$$

1/I/6C **Fluid Dynamics**

An unsteady fluid flow has velocity field given in Cartesian coordinates (x, y, z) by $\mathbf{u} = (1, xt, 0)$, where t denotes time. Dye is released into the fluid from the origin continuously. Find the position at time t of the dye particle that was released at time s and hence show that the dye streak lies along the curve

$$y = \tfrac{1}{2}tx^2 - \tfrac{1}{6}x^3.$$

1/II/15C **Fluid Dynamics**

Starting from the Euler equations for incompressible, inviscid flow

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p, \qquad \nabla \cdot \mathbf{u} = 0,$$

derive the vorticity equation governing the evolution of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

Consider the flow

$$\mathbf{u} = \beta(-x, -y, 2z) + \Omega(t)(-y, x, 0),$$

in Cartesian coordinates (x, y, z), where t is time and β is a constant. Compute the vorticity and show that it evolves in time according to

$$\boldsymbol{\omega} = \omega_0 \mathrm{e}^{2\beta t}\mathbf{k},$$

where ω_0 is the initial magnitude of the vorticity and $\mathbf{k}$ is a unit vector in the z-direction.

Show that the material curve $C(t)$ that takes the form

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 1$$

at $t = 0$ is given later by

$$x^2 + y^2 = a^2(t) \quad \text{and} \quad z = \frac{1}{a^2(t)},$$

where the function $a(t)$ is to be determined.

Calculate the circulation of $\mathbf{u}$ around C and state how this illustrates Kelvin's circulation theorem.

3/I/8C **Fluid Dynamics**

Show that the velocity field

$$\mathbf{u} = \mathbf{U} + \frac{\mathbf{\Gamma} \times \mathbf{r}}{2\pi r^2},$$

where $\mathbf{U} = (U, 0, 0)$, $\mathbf{\Gamma} = (0, 0, \Gamma)$ and $\mathbf{r} = (x, y, 0)$ in Cartesian coordinates (x, y, z), represents the combination of a uniform flow and the flow due to a line vortex. Define and evaluate the circulation of the vortex.

Show that

$$\oint_{C_R} (\mathbf{u} \cdot \mathbf{n})\mathbf{u}\, dl \to \tfrac{1}{2}\mathbf{\Gamma} \times \mathbf{U} \qquad \text{as} \qquad R \to \infty,$$

where C_R is a circle $x^2 + y^2 = R^2$, $z =$ const. Explain how this result is related to the lift force on a two-dimensional aerofoil or other obstacle.

3/II/18C **Fluid Dynamics**

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid in the absence of gravity.

Water of density ρ is driven through a tube of length L and internal radius a by the pressure exerted by a spherical, water-filled balloon of radius $R(t)$ attached to one end of the tube. The balloon maintains the pressure of the water entering the tube at $2\gamma/R$ in excess of atmospheric pressure, where γ is a constant. It may be assumed that the water exits the tube at atmospheric pressure. Show that

$$R^3 \ddot{R} + 2R^2 \dot{R}^2 = -\frac{\gamma a^2}{2\rho L}. \qquad (\dagger)$$

Solve equation ($\dagger$), by multiplying through by $2R\dot{R}$ or otherwise, to obtain

$$t = R_0^2 \left(\frac{2\rho L}{\gamma a^2}\right)^{1/2} \left[\frac{\pi}{4} - \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right],$$

where $\theta = \sin^{-1}(R/R_0)$ and R_0 is the initial radius of the balloon. Hence find the time when $R = 0$.

4/I/7C **Fluid Dynamics**

Inviscid fluid issues vertically downwards at speed u_0 from a circular tube of radius a. The fluid falls onto a horizontal plate a distance H below the end of the tube, where it spreads out axisymmetrically.

Show that while the fluid is falling freely it has speed

$$u = u_0 \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{1/2},$$

and occupies a circular jet of radius

$$R = a \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{-1/4},$$

where z is the height above the plate and g is the acceleration due to gravity.

Show further that along the plate, at radial distances $r \gg a$ (i.e. far from the falling jet), where the fluid is flowing almost horizontally, it does so as a film of height $h(r)$, where

$$\frac{a^4}{4r^2h^2} = 1 + \frac{2g}{u_0^2}(H - h).$$

4/II/16C **Fluid Dynamics**

Define the terms *irrotational flow* and *incompressible flow.* The two-dimensional flow of an incompressible fluid is given in terms of a streamfunction $\psi(x, y)$ as

$$\mathbf{u} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$$

in Cartesian coordinates (x, y). Show that the line integral

$$\int_{\mathbf{x_1}}^{\mathbf{x_2}} \mathbf{u} \cdot \mathbf{n} \, dl = \psi(\mathbf{x_2}) - \psi(\mathbf{x_1})$$

along any path joining the points $\mathbf{x_1}$ and $\mathbf{x_2}$, where $\mathbf{n}$ is the unit normal to the path. Describe how this result is related to the concept of mass conservation.

Inviscid, incompressible fluid is contained in the semi-infinite channel $x > 0$, $0 < y < 1$, which has rigid walls at $x = 0$ and at $y = 0$, 1, apart from a small opening at the origin through which the fluid is withdrawn with volume flux m per unit distance in the third dimension. Show that the streamfunction for irrotational flow in the channel can be chosen (up to an additive constant) to satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

and boundary conditions

$$\begin{aligned} &\psi = 0 \qquad \text{on } y = 0,\ x > 0,\\ &\psi = -m \qquad \text{on } x = 0,\ 0 < y < 1,\\ &\psi = -m \qquad \text{on } y = 1,\ x > 0,\\ &\psi \to -my \qquad \text{as } x \to \infty, \end{aligned}$$

if it is assumed that the flow at infinity is uniform. Solve the boundary-value problem above using separation of variables to obtain

$$\psi = -my + \frac{2m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi y \, e^{-n\pi x}.$$

2/I/4E Further Analysis

Let τ_1 be the collection of all subsets $A \subset \mathbb{N}$ such that $A = \emptyset$ or $\mathbb{N} \setminus A$ is finite. Let τ_2 be the collection of all subsets of $\mathbb{N}$ of the form $I_n = \{n, n+1, n+2, \ldots\}$, together with the empty set. Prove that τ_1 and τ_2 are both topologies on $\mathbb{N}$.

Show that a function f from the topological space $(\mathbb{N}, \tau_1)$ to the topological space $(\mathbb{N}, \tau_2)$ is continuous if and only if one of the following alternatives holds:

(i) $f(n) \to \infty$ as $n \to \infty$;

(ii) there exists $N \in \mathbb{N}$ such that $f(n) = N$ for all but finitely many n and $f(n) \leqslant N$ for all n.

2/II/13E Further Analysis

(a) Let $f\colon [1, \infty) \to \mathbb{C}$ be defined by $f(t) = t^{-1} e^{2\pi i t}$ and let X be the image of f. Prove that $X \cup \{0\}$ is compact and path-connected. [*Hint: you may find it helpful to set* $s = t^{-1}$.]

(b) Let $g\colon [1, \infty) \to \mathbb{C}$ be defined by $g(t) = (1 + t^{-1}) e^{2\pi i t}$, let Y be the image of g and let $\overline{D}$ be the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Prove that $Y \cup \overline{D}$ is connected. Explain briefly why it is not path-connected.

3/I/3E Further Analysis

(a) Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leqslant 1 + |z|^{1/2}$ for every z. Prove that f is constant.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\mathrm{Re}\,(f(z)) \geqslant 0$ for every z. Prove that f is constant.

3/II/13E Further Analysis

(a) State Taylor's Theorem.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ be defined whenever $|z - z_0| < r$. Suppose that $z_k \to z_0$ as $k \to \infty$, that no z_k equals z_0 and that $f(z_k) = g(z_k)$ for every k. Prove that $a_n = b_n$ for every $n \geqslant 0$.

(c) Let D be a domain, let $z_0 \in D$ and let (z_k) be a sequence of points in D that converges to z_0, but such that no z_k equals z_0. Let $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ be analytic functions such that $f(z_k) = g(z_k)$ for every k. Prove that $f(z) = g(z)$ for every $z \in D$.

(d) Let D be the domain $\mathbb{C} \setminus \{0\}$. Give an example of an analytic function $f\colon D \to \mathbb{C}$ such that $f(n^{-1}) = 0$ for every positive integer n but f is not identically 0.

(e) Show that any function with the property described in (d) must have an essential singularity at the origin.

4/I/4E **Further Analysis**

(a) State and prove Morera's Theorem.

(b) Let D be a domain and for each $n \in \mathbb{N}$ let $f_n : D \to \mathbb{C}$ be an analytic function. Suppose that $f : D \to \mathbb{C}$ is another function and that $f_n \to f$ uniformly on D. Prove that f is analytic.

4/II/13E **Further Analysis**

(a) State the residue theorem and use it to deduce the principle of the argument, in a form that involves winding numbers.

(b) Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\mathrm{Im}\,(p(z)) = 0$. Calculate $\mathrm{Re}\,(p(z))$ for each such z. [*It will be helpful to set* $z = e^{i\theta}$. *You may use the addition formulae* $\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ *and* $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$.]

(c) Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed path $\theta \mapsto e^{i\theta}$. Use your answer to (b) to give a rough sketch of the path $p \circ \gamma$, paying particular attention to where it crosses the real axis.

(d) Hence, or otherwise, determine for every real t the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = t$. (You need not give rigorous justifications for your calculations.)

1/I/4F Geometry

Describe the geodesics (that is, hyperbolic straight lines) in **either** the disc model **or** the half-plane model of the hyperbolic plane. Explain what is meant by the statements that two hyperbolic lines are parallel, and that they are ultraparallel.

Show that two hyperbolic lines l and l' have a unique common perpendicular if and only if they are ultraparallel.

[*You may assume standard results about the group of isometries of whichever model of the hyperbolic plane you use.*]

1/II/13F Geometry

Write down the Riemannian metric in the half-plane model of the hyperbolic plane. Show that Möbius transformations mapping the upper half-plane to itself are isometries of this model.

Calculate the hyperbolic distance from ib to ic, where b and c are positive real numbers. Assuming that the hyperbolic circle with centre ib and radius r is a Euclidean circle, find its Euclidean centre and radius.

Suppose that a and b are positive real numbers for which the points ib and $a + ib$ of the upper half-plane are such that the hyperbolic distance between them coincides with the Euclidean distance. Obtain an expression for b as a function of a. Hence show that, for any b with $0 < b < 1$, there is a unique positive value of a such that the hyperbolic distance between ib and $a + ib$ coincides with the Euclidean distance.

3/I/4F Geometry

Show that any isometry of Euclidean 3-space which fixes the origin can be written as a composite of at most three reflections in planes through the origin, and give (with justification) an example of an isometry for which three reflections are necessary.

3/II/14F Geometry

State and prove the Gauss–Bonnet formula for the area of a spherical triangle. Deduce a formula for the area of a spherical n-gon with angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. For what range of values of α does there exist a (convex) regular spherical n-gon with angle α?

Let Δ be a spherical triangle with angles π/p, π/q and π/r where p, q, r are integers, and let G be the group of isometries of the sphere generated by reflections in the three sides of Δ. List the possible values of (p, q, r), and in each case calculate the order of the corresponding group G. If $(p, q, r) = (2, 3, 5)$, show how to construct a regular dodecahedron whose group of symmetries is G.

[*You may assume that the images of* Δ *under the elements of* G *form a tessellation of the sphere.*]

1/I/5E Linear Mathematics

Let V be the subset of $\mathbb{R}^5$ consisting of all quintuples $(a_1, a_2, a_3, a_4, a_5)$ such that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0 \ .$$

Prove that V is a subspace of $\mathbb{R}^5$. Solve the above equations for a_1 and a_2 in terms of a_3, a_4 and a_5. Hence, exhibit a basis for V, explaining carefully why the vectors you give form a basis.

1/II/14E Linear Mathematics

(a) Let U, U' be subspaces of a finite-dimensional vector space V. Prove that $\dim(U + U') = \dim U + \dim U' - \dim(U \cap U')$.

(b) Let V and W be finite-dimensional vector spaces and let α and β be linear maps from V to W. Prove that

$$\operatorname{rank}(\alpha + \beta) \leqslant \operatorname{rank}\alpha + \operatorname{rank}\beta \ .$$

(c) Deduce from this result that

$$\operatorname{rank}(\alpha + \beta) \geqslant |\operatorname{rank}\alpha - \operatorname{rank}\beta| \ .$$

(d) Let $V = W = \mathbb{R}^n$ and suppose that $1 \leqslant r \leqslant s \leqslant n$. Exhibit linear maps $\alpha, \beta \colon V \to W$ such that $\operatorname{rank}\alpha = r$, $\operatorname{rank}\beta = s$ and $\operatorname{rank}(\alpha + \beta) = s - r$. Suppose that $r + s \geqslant n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\operatorname{rank}\alpha = r$, $\operatorname{rank}\beta = s$ and $\operatorname{rank}(\alpha + \beta) = n$.

2/I/6E Linear Mathematics

Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. For each i let $\mathbf{v}_i$ be the vector $(1, a_i, a_i^2, \ldots, a_i^{n-1})$. Let A be the $n \times n$ matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and let $\mathbf{c}$ be a column vector of size n. Prove that $A\mathbf{c} = \mathbf{0}$ if and only if $\mathbf{c} = \mathbf{0}$. Deduce that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$.

[*You may use general facts about matrices if you state them clearly.*]

2/II/15E Linear Mathematics

(a) Let $A = (a_{ij})$ be an $m \times n$ matrix and for each $k \leqslant n$ let A_k be the $m \times k$ matrix formed by the first k columns of A. Suppose that $n > m$. Explain why the nullity of A is non-zero. Prove that if k is minimal such that A_k has non-zero nullity, then the nullity of A_k is 1.

(b) Suppose that no column of A consists entirely of zeros. Deduce from (a) that there exist scalars $b_1, \ldots, b_k$ (where k is defined as in (a)) such that $\sum_{j=1}^{k} a_{ij} b_j = 0$ for every $i \leqslant m$, but whenever $\lambda_1, \ldots, \lambda_k$ are distinct real numbers there is some $i \leqslant m$ such that $\sum_{j=1}^{k} a_{ij} \lambda_j b_j \neq 0$.

(c) Now let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be bases for the same real m-dimensional vector space. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real numbers such that for every j the vectors $\mathbf{v}_1 + \lambda_j \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda_j \mathbf{w}_m$ are linearly dependent. For each j, let $a_{1j}, \ldots, a_{mj}$ be scalars, not all zero, such that $\sum_{i=1}^{m} a_{ij} (\mathbf{v}_i + \lambda_j \mathbf{w}_i) = \mathbf{0}$. By applying the result of (b) to the matrix (a_{ij}), deduce that $n \leqslant m$.

(d) It follows that the vectors $\mathbf{v}_1 + \lambda \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda \mathbf{w}_m$ are linearly dependent for at most m values of λ. Explain briefly how this result can also be proved using determinants.

3/I/7G Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U and let β be another endomorphism of U that commutes with α. If λ is an eigenvalue of α, show that β maps the kernel of $\alpha - \lambda \iota$ into itself, where ι is the identity map. Suppose now that α is diagonalizable with n distinct real eigenvalues where $n = \dim U$. Prove that if there exists an endomorphism β of U such that $\alpha = \beta^2$, then $\lambda \geqslant 0$ for all eigenvalues λ of α.

3/II/17G Linear Mathematics

Define the *determinant* $\det(A)$ of an $n \times n$ complex matrix A. Let $A_1, \ldots, A_n$ be the columns of A, let σ be a permutation of $\{1, \ldots, n\}$ and let A^σ be the matrix whose columns are $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$. Prove from your definition of determinant that $\det(A^\sigma) = \epsilon(\sigma) \det(A)$, where $\epsilon(\sigma)$ is the sign of the permutation σ. Prove also that $\det(A) = \det(A^t)$.

Define the *adjugate* matrix $\operatorname{adj}(A)$ and prove from your definitions that $A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A)\, I$, where I is the identity matrix. Hence or otherwise, prove that if $\det(A) \neq 0$, then A is invertible.

Let C and D be real $n \times n$ matrices such that the complex matrix $C + iD$ is invertible. By considering $\det(C + \lambda D)$ as a function of λ or otherwise, prove that there exists a real number λ such that $C + \lambda D$ is invertible. [*You may assume that if a matrix A is invertible, then* $\det(A) \neq 0$.]

Deduce that if two real matrices A and B are such that there exists an invertible complex matrix P with $P^{-1} A P = B$, then there exists an invertible **real** matrix Q such that $Q^{-1} A Q = B$.

4/I/6G **Linear Mathematics**

Let α be an endomorphism of a finite-dimensional real vector space U such that $\alpha^2 = \alpha$. Show that U can be written as the direct sum of the kernel of α and the image of α. Hence or otherwise, find the characteristic polynomial of α in terms of the dimension of U and the rank of α. Is α diagonalizable? Justify your answer.

4/II/15G **Linear Mathematics**

Let $\alpha \in L(U, V)$ be a linear map between finite-dimensional vector spaces. Let

$$M^l(\alpha) = \{\beta \in L(V, U) : \beta\,\alpha = 0\} \qquad \text{and}$$

$$M^r(\alpha) = \{\beta \in L(V, U) : \alpha\,\beta = 0\}\ .$$

(a) Prove that $M^l(\alpha)$ and $M^r(\alpha)$ are subspaces of $L(V, U)$ of dimensions

$$\dim M^l(\alpha) = (\dim V - \operatorname{rank}\alpha)\dim U \qquad \text{and}$$

$$\dim M^r(\alpha) = \dim\ker(\alpha)\dim V\ .$$

[*You may use the result that there exist bases in U and V so that α is represented by*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and r is the rank of α.]

(b) Let $\Phi\colon L(U, V) \to L(V^*, U^*)$ be given by $\Phi(\alpha) = \alpha^*$, where α^* is the dual map induced by α. Prove that Φ is an isomorphism. [*You may assume that Φ is linear, and you may use the result that a finite-dimensional vector space and its dual have the same dimension.*]

(c) Prove that

$$\Phi(M^l(\alpha)) = M^r(\alpha^*) \quad \text{and} \quad \Phi(M^r(\alpha)) = M^l(\alpha^*).$$

[*You may use the results that $(\beta\,\alpha)^* = \alpha^*\,\beta^*$ and that β^{**} can be identified with β under the canonical isomorphism between a vector space and its double dual.*]

(d) Conclude that $\operatorname{rank}(\alpha) = \operatorname{rank}(\alpha^*)$.

1/I/2D **Methods**

Fermat's principle of optics states that the path of a light ray connecting two points will be such that the travel time t is a minimum. If the speed of light varies continuously in a medium and is a function $c(y)$ of the distance from the boundary $y = 0$, show that the path of a light ray is given by the solution to

$$c(y)y'' + c'(y)(1 + y'^2) = 0\,,$$

where $y' = \frac{dy}{dx}$, etc. Show that the path of a light ray in a medium where the speed of light c is a constant is a straight line. Also find the path from $(0, 0)$ to $(1, 0)$ if $c(y) = y$, and sketch it.

1/II/11D **Methods**

(a) Determine the Green's function $G(x, \xi)$ for the operator $\frac{d^2}{dx^2} + k^2$ on $[0, \pi]$ with Dirichlet boundary conditions by solving the boundary value problem

$$\frac{d^2G}{dx^2} + k^2 G = \delta(x - \xi)\ , \quad G(0) = 0,\ G(\pi) = 0$$

when k is not an integer.

(b) Use the method of Green's functions to solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2 y = f(x)\ , \quad y(0) = a,\ y(\pi) = b$$

when k is not an integer.

2/I/2C **Methods**

Explain briefly why the second-rank tensor

$$\int_S x_i x_j \, dS(\mathbf{x})$$

is isotropic, where S is the surface of the unit sphere centred on the origin.

A second-rank tensor is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j) \, dS(\mathbf{x}) \,,$$

where S is the surface of the unit sphere centred on the origin. Calculate $T(\mathbf{y})$ in the form

$$T_{ij} = \lambda \delta_{ij} + \mu y_i y_j \,,$$

where λ and μ are to be determined.

By considering the action of T on $\mathbf{y}$ and on vectors perpendicular to $\mathbf{y}$, determine the eigenvalues and associated eigenvectors of T.

2/II/11C **Methods**

State the transformation law for an nth-rank tensor $T_{ij \cdots k}$.

Show that the fourth-rank tensor

$$c_{ijkl} = \alpha \, \delta_{ij} \, \delta_{kl} + \beta \, \delta_{ik} \, \delta_{jl} + \gamma \, \delta_{il} \, \delta_{jk}$$

is isotropic for arbitrary scalars α, β and γ.

The stress σ_{ij} and strain e_{ij} in a linear elastic medium are related by

$$\sigma_{ij} = c_{ijkl} \, e_{kl}.$$

Given that e_{ij} is symmetric and that the medium is isotropic, show that the stress-strain relationship can be written in the form

$$\sigma_{ij} = \lambda \, e_{kk} \, \delta_{ij} + 2\mu \, e_{ij}.$$

Show that e_{ij} can be written in the form $e_{ij} = p\delta_{ij} + d_{ij}$, where d_{ij} is a traceless tensor and p is a scalar to be determined. Show also that necessary and sufficient conditions for the stored elastic energy density $E = \frac{1}{2}\sigma_{ij} \, e_{ij}$ to be non-negative for any deformation of the solid are that

$$\mu \geq 0 \qquad \text{and} \qquad \lambda \geq -\tfrac{2}{3}\mu.$$

3/I/2D **Methods**

Consider the path between two arbitrary points on a cone of interior angle 2α. Show that the arc-length of the path $r(\theta)$ is given by

$$\int (r^2 + r'^2 \mathrm{cosec}^2\,\alpha)^{1/2}\, d\theta \ ,$$

where $r' = \frac{dr}{d\theta}$. By minimizing the total arc-length between the points, determine the equation for the shortest path connecting them.

3/II/12D **Methods**

The transverse displacement $y(x,t)$ of a stretched string clamped at its ends $x = 0, l$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2k \frac{\partial y}{\partial t} \ , \quad y(x,0) = 0, \ \frac{\partial y}{\partial t}(x,0) = \delta(x-a)\,,$$

where $c > 0$ is the wave velocity, and $k > 0$ is the damping coefficient. The initial conditions correspond to a sharp blow at $x = a$ at time $t = 0$.

(a) Show that the subsequent motion of the string is given by

$$y(x,t) = \frac{1}{\sqrt{\alpha_n^2 - k^2}} \sum_n 2e^{-kt} \sin \frac{\alpha_n a}{c} \, \sin \frac{\alpha_n x}{c} \, \sin/(\sqrt{\alpha_n^2 - k^2} \ \ t)$$

where $\alpha_n = \pi c n / l$.

(b) Describe what happens in the limits of small and large damping. What critical parameter separates the two cases?

4/I/2D **Methods**

Consider the wave equation in a spherically symmetric coordinate system

$$\frac{\partial^2 u(r,t)}{\partial t^2} = c^2 \Delta u(r,t)\,,$$

where $\Delta u = \frac{1}{r} \frac{\partial^2}{\partial r^2}(ru)$ is the spherically symmetric Laplacian operator.

(a) Show that the general solution to the equation above is

$$u(r,t) = \tfrac{1}{r}[f(r+ct) + g(r-ct)]\,,$$

where $f(x), g(x)$ are arbitrary functions.

(b) Using separation of variables, determine the wave field $u(r,t)$ in response to a pulsating source at the origin $u(0,t) = A \sin \omega t$.

4/II/11D **Methods**

The velocity potential $\phi(r,\theta)$ for inviscid flow in two dimensions satisfies the Laplace equation

$$\Delta\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0\ .$$

(a) Using separation of variables, derive the general solution to the equation above that is single-valued and finite in each of the domains (i) $0 \leqslant r \leqslant a$; (ii) $a \leqslant r < \infty$.

(b) Assuming ϕ is single-valued, solve the Laplace equation subject to the boundary conditions $\frac{\partial\phi}{\partial r} = 0$ at $r = a$, and $\frac{\partial\phi}{\partial r} \to U\cos\theta$ as $r \to \infty$. Sketch the lines of constant potential.

2/I/5B **Numerical Analysis**

Let

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} \gamma \\ 0 \\ 0 \\ \gamma a \end{pmatrix}, \quad \gamma = 1 - a^4 \neq 0.$$

Find the LU factorization of the matrix A and use it to solve the system $Ax = b$.

2/II/14B **Numerical Analysis**

Let

$$f''(0) \approx a_0 f(-1) + a_1 f(0) + a_2 f(1) = \mu(f)$$

be an approximation of the second derivative which is exact for $f \in \mathcal{P}_2$, the set of polynomials of degree ≤ 2, and let

$$e(f) = f''(0) - \mu(f)$$

be its error.

(a) Determine the coefficients a_0, a_1, a_2.

(b) Using the Peano kernel theorem prove that, for $f \in C^3[-1,1]$, the set of three-times continuously differentiable functions, the error satisfies the inequality

$$|e(f)| \leq \frac{1}{3} \max_{x \in [-1,1]} |f'''(x)|.$$

3/I/6B **Numerical Analysis**

Given $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$, let

$$\ell_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - x_k}{x_i - x_k}$$

be the fundamental Lagrange polynomials of degree n, let

$$\omega(x) = \prod_{i=0}^{n} (x - x_i),$$

and let p be any polynomial of degree $\leq n$.

(a) Prove that $\sum_{i=0}^{n} p(x_i)\ell_i(x) \equiv p(x)$.

(b) Hence or otherwise derive the formula

$$\frac{p(x)}{\omega(x)} = \sum_{i=0}^{n} \frac{A_i}{x - x_i}, \quad A_i = \frac{p(x_i)}{\omega'(x_i)}\,,$$

which is the decomposition of $p(x)/\omega(x)$ into partial fractions.

3/II/16B **Numerical Analysis**

The functions $H_0, H_1, \ldots$ are generated by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\,.$$

(a) Show that H_n is a polynomial of degree n, and that the H_n are orthogonal with respect to the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2}\, dx\,.$$

(b) By induction or otherwise, prove that the H_n satisfy the three-term recurrence relation

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)\,.$$

[*Hint: you may need to prove the equality* $H_n'(x) = 2n H_{n-1}(x)$ *as well.*]

3/I/5H **Optimization**

Two players A and B play a zero-sum game with the pay-off matrix

	B_1	B_2	B_3
A_1	4	-2	-5
A_2	-2	4	3
A_3	-3	6	2
A_4	3	-8	-6

Here, the (i, j) entry of the matrix indicates the pay-off to player A if he chooses move A_i and player B chooses move B_j. Show that the game can be reduced to a zero-sum game with 2×2 pay-off matrix.

Determine the value of the game and the optimal strategy for player A.

3/II/15H **Optimization**

Explain what is meant by a transportation problem where the total demand equals the total supply. Write the Lagrangian and describe an algorithm for solving such a problem. Starting from the north-west initial assignment, solve the problem with three sources and three destinations described by the table

5	9	1	36
3	10	6	84
7	2	5	40
14	68	78	

where the figures in the 3×3 box denote the transportation costs (per unit), the right-hand column denotes supplies, and the bottom row demands.

4/I/5H **Optimization**

State and prove the Lagrangian sufficiency theorem for a general optimization problem with constraints.

4/II/14H **Optimization**

Use the two-phase simplex method to solve the problem

$$\begin{array}{lrcrcrcr}
\text{minimize} & 5x_1 & - & 12x_2 & + & 13x_3 & & \\
\\
\text{subject to} & 4x_1 & + & 5x_2 & & & \leq & 9, \\
& 6x_1 & + & 4x_2 & + & x_3 & \geq & 12, \\
& 3x_1 & + & 2x_2 & - & x_3 & \leq & 3, \\
& \multicolumn{7}{l}{x_i \geq 0, \qquad i = 1, 2, 3.}
\end{array}$$

1/I/8G **Quadratic Mathematics**

Let U and V be finite-dimensional vector spaces. Suppose that b and c are bilinear forms on $U \times V$ and that b is non-degenerate. Show that there exist linear endomorphisms S of U and T of V such that $c(x,y) = b(S(x),y) = b(x,T(y))$ for all $(x,y) \in U \times V$.

1/II/17G **Quadratic Mathematics**

(a) Suppose p is an odd prime and a an integer coprime to p. Define the *Legendre symbol* $(\frac{a}{p})$ and state Euler's criterion.

(b) Compute $(\frac{-1}{p})$ and prove that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

whenever a and b are coprime to p.

(c) Let n be any integer such that $1 \leqslant n \leqslant p-2$. Let m be the unique integer such that $1 \leqslant m \leqslant p-2$ and $mn \equiv 1 \pmod{p}$. Prove that

$$\left(\frac{n(n+1)}{p}\right) = \left(\frac{1+m}{p}\right) .$$

(d) Find

$$\sum_{n=1}^{p-2} \left(\frac{n(n+1)}{p}\right) .$$

2/I/8G **Quadratic Mathematics**

Let U be a finite-dimensional real vector space and b a positive definite symmetric bilinear form on $U \times U$. Let $\psi\colon U \to U$ be a linear map such that $b(\psi(x),y)+b(x,\psi(y)) = 0$ for all x and y in U. Prove that if ψ is invertible, then the dimension of U must be even. By considering the restriction of ψ to its image or otherwise, prove that the rank of ψ is always even.

2/II/17G **Quadratic Mathematics**

Let S be the set of all 2×2 complex matrices A which are *hermitian*, that is, $A^* = A$, where $A^* = \overline{A}^t$.

(a) Show that S is a real 4-dimensional vector space. Consider the real symmetric bilinear form b on this space defined by

$$b(A, B) = \tfrac{1}{2}\left(\operatorname{tr}(A\,B) - \operatorname{tr}(A)\operatorname{tr}(B)\right) .$$

Prove that $b(A, A) = -\det A$ and $b(A, I) = -\frac{1}{2}\operatorname{tr}(A)$, where I denotes the identity matrix.

(b) Consider the three matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Prove that the basis I, A_1, A_2, A_3 of S diagonalizes b. Hence or otherwise find the rank and signature of b.

(c) Let Q be the set of all 2×2 complex matrices C which satisfy $C + C^* = \operatorname{tr}(C)\, I$. Show that Q is a real 4-dimensional vector space. Given $C \in Q$, put

$$\Phi(C) = \frac{1-i}{2}\operatorname{tr}(C)\, I + i\, C.$$

Show that Φ takes values in S and is a linear isomorphism between Q and S.

(d) Define a real symmetric bilinear form on Q by setting $c(C, D) = -\frac{1}{2}\operatorname{tr}(C\,D)$, $C, D \in Q$. Show that $b(\Phi(C), \Phi(D)) = c(C, D)$ for all $C, D \in Q$. Find the rank and signature of the symmetric bilinear form c defined on Q.

3/I/9G **Quadratic Mathematics**

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Explain what is meant by the *discriminant* d of f. State a necessary and sufficient condition for some form of discriminant d to represent an odd prime number p. Using this result or otherwise, find the primes p which can be represented by the form $x^2 + 3y^2$.

3/II/19G **Quadratic Mathematics**

Let U be a finite-dimensional real vector space endowed with a positive definite inner product. A linear map $\tau : U \to U$ is said to be an *orthogonal projection* if τ is self-adjoint and $\tau^2 = \tau$.

(a) Prove that for every orthogonal projection τ there is an orthogonal decomposition

$$U = \ker(\tau) \oplus \operatorname{im}(\tau).$$

(b) Let $\phi\colon U \to U$ be a linear map. Show that if $\phi^2 = \phi$ and $\phi\,\phi^* = \phi^*\,\phi$, where ϕ^* is the adjoint of ϕ, then ϕ is an orthogonal projection. [*You may find it useful to prove first that if $\phi\,\phi^* = \phi^*\,\phi$, then ϕ and ϕ^* have the same kernel.*]

(c) Show that given a subspace W of U there exists a unique orthogonal projection τ such that $\operatorname{im}(\tau) = W$. If W_1 and W_2 are two subspaces with corresponding orthogonal projections τ_1 and τ_2, show that $\tau_2 \circ \tau_1 = 0$ if and only if W_1 is orthogonal to W_2.

(d) Let $\phi\colon U \to U$ be a linear map satisfying $\phi^2 = \phi$. Prove that one can define a positive definite inner product on U such that ϕ becomes an orthogonal projection.

1/I/9A **Quantum Mechanics**

A particle of mass m is confined inside a one-dimensional box of length a. Determine the possible energy eigenvalues.

1/II/18A **Quantum Mechanics**

What is the significance of the expectation value

$$\langle Q \rangle = \int \psi^*(x)\ Q\ \psi(x) dx$$

of an observable Q in the normalized state $\psi(x)$? Let Q and P be two observables. By considering the norm of $(Q + i\lambda P)\psi$ for real values of λ, show that

$$\langle Q^2 \rangle \langle P^2 \rangle \geqslant \tfrac{1}{4} |\langle [Q, P] \rangle|^2 .$$

The uncertainty ΔQ of Q in the state $\psi(x)$ is defined as

$$(\Delta Q)^2 = \langle (Q - \langle Q \rangle)^2 \rangle .$$

Deduce the generalized uncertainty relation,

$$\Delta Q \Delta P \geqslant \tfrac{1}{2} |\langle [Q, P] \rangle| .$$

A particle of mass m moves in one dimension under the influence of the potential $\frac{1}{2} m\omega^2 x^2$. By considering the commutator $[x, p]$, show that the expectation value of the Hamiltonian satisfies

$$\langle H \rangle \geqslant \tfrac{1}{2} \hbar\omega .$$

2/I/9A **Quantum Mechanics**

What is meant by the statement than an operator is *hermitian*?

A particle of mass m moves in the real potential $V(x)$ in one dimension. Show that the Hamiltonian of the system is hermitian.

Show that

$$\begin{aligned} \frac{d}{dt} \langle x \rangle &= \frac{1}{m} \langle p \rangle , \\ \frac{d}{dt} \langle p \rangle &= \langle -V'(x) \rangle , \end{aligned}$$

where p is the momentum operator and $\langle A \rangle$ denotes the expectation value of the operator A.

2/II/18A Quantum Mechanics

A particle of mass m and energy E moving in one dimension is incident from the left on a potential barrier $V(x)$ given by

$$V(x) = \begin{cases} V_0 & 0 \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

with $V_0 > 0$.

In the limit $V_0 \to \infty, a \to 0$ with $V_0 a = U$ held fixed, show that the transmission probability is

$$T = \left(1 + \frac{mU^2}{2E\hbar^2}\right)^{-1}.$$

3/II/20A Quantum Mechanics

The radial wavefunction for the hydrogen atom satisfies the equation

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + \frac{\hbar^2}{2mr^2}\ell(\ell+1)R(r) - \frac{e^2}{4\pi\epsilon_0 r}R(r) = ER(r)\,.$$

Explain the origin of each term in this equation.

The wavefunctions for the ground state and first radially excited state, both with $\ell = 0$, can be written as

$$R_1(r) = N_1 \exp(-\alpha r)$$
$$R_2(r) = N_2(r+b)\exp(-\beta r)$$

respectively, where N_1 and N_2 are normalization constants. Determine α, β, b and the corresponding energy eigenvalues E_1 and E_2.

A hydrogen atom is in the first radially excited state. It makes the transition to the ground state, emitting a photon. What is the frequency of the emitted photon?

3/I/10A **Special Relativity**

What are the momentum and energy of a photon of wavelength λ?

A photon of wavelength λ is incident on an electron. After the collision, the photon has wavelength λ'. Show that

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta),$$

where θ is the scattering angle and m is the electron rest mass.

4/I/9A **Special Relativity**

Prove that the two-dimensional Lorentz transformation can be written in the form

$$\begin{aligned} x' &= x\cosh\phi - ct\sinh\phi \\ ct' &= -x\sinh\phi + ct\cosh\phi, \end{aligned}$$

where $\tanh\phi = v/c$. Hence, show that

$$\begin{aligned} x' + ct' &= e^{-\phi}(x + ct) \\ x' - ct' &= e^{\phi}(x - ct)\,. \end{aligned}$$

Given that frame S' has speed v with respect to S and S'' has speed v' with respect to S', use this formalism to find the speed v'' of S'' with respect to S.

[*Hint: rotation through a hyperbolic angle ϕ, followed by rotation through ϕ', is equivalent to rotation through $\phi + \phi'$.*]

4/II/18A **Special Relativity**

A pion of rest mass M decays at rest into a muon of rest mass $m < M$ and a neutrino of zero rest mass. What is the speed u of the muon?

In the pion rest frame S, the muon moves in the y-direction. A moving observer, in a frame S' with axes parallel to those in the pion rest frame, wishes to take measurements of the decay along the x-axis, and notes that the pion has speed v with respect to the x-axis. Write down the four-dimensional Lorentz transformation relating S' to S and determine the momentum of the muon in S'. Hence show that in S' the direction of motion of the muon makes an angle θ with respect to the y-axis, where

$$\tan\theta = \frac{M^2 + m^2}{M^2 - m^2}\frac{v}{(c^2 - v^2)^{1/2}}\,.$$

1/I/3H Statistics

Derive the least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ for the coefficients of the simple linear regression model

$$Y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are given constants, $\bar{x} = n^{-1}\sum_{i=1}^n x_i$, and ε_i are independent with $\mathrm{E}\,\varepsilon_i = 0$, $\mathrm{Var}\,\varepsilon_i = \sigma^2$, $i = 1, \ldots, n$.

A manufacturer of optical equipment has the following data on the unit cost (in pounds) of certain custom-made lenses and the number of units made in each order:

No. of units, x_i	1	3	5	10	12
Cost per unit, y_i	58	55	40	37	22

Assuming that the conditions underlying simple linear regression analysis are met, estimate the regression coefficients and use the estimated regression equation to predict the unit cost in an order for 8 of these lenses.

[*Hint: for the data above,* $S_{xy} = \sum_{i=1}^n (x_i - \bar{x}) y_i = -257.4$.]

1/II/12H Statistics

Suppose that six observations $X_1, \ldots, X_6$ are selected at random from a normal distribution for which both the mean μ_X and the variance σ_X^2 are unknown, and it is found that $S_{XX} = \sum_{i=1}^6 (x_i - \bar{x})^2 = 30$, where $\bar{x} = \frac{1}{6}\sum_{i=1}^6 x_i$. Suppose also that 21 observations $Y_1, \ldots, Y_{21}$ are selected at random from another normal distribution for which both the mean μ_Y and the variance σ_Y^2 are unknown, and it is found that $S_{YY} = 40$. Derive carefully the likelihood ratio test of the hypothesis H_0: $\sigma_X^2 = \sigma_Y^2$ against H_1: $\sigma_X^2 > \sigma_Y^2$ and apply it to the data above at the 0.05 level.

[*Hint:*

Distribution	χ_5^2	χ_6^2	χ_{20}^2	χ_{21}^2	$F_{5,20}$	$F_{6,21}$
95% percentile	11.07	12.59	31.41	32.68	2.71	2.57

]

2/I/3H Statistics

Let $X_1, \ldots, X_n$ be a random sample from the $N(\theta, \sigma^2)$ distribution, and suppose that the prior distribution for θ is $N(\mu, \tau^2)$, where σ^2, μ, τ^2 are known. Determine the posterior distribution for θ, given $X_1, \ldots, X_n$, and the best point estimate of θ under both quadratic and absolute error loss.

2/II/12H **Statistics**

An examination was given to 500 high-school students in each of two large cities, and their grades were recorded as low, medium, or high. The results are given in the table below.

	Low	*Medium*	*High*
City A	103	145	252
City B	140	136	224

Derive carefully the test of homogeneity and test the hypothesis that the distributions of scores among students in the two cities are the same.

[*Hint:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_5^2	χ_6^2
99% percentile	6.63	9.21	11.34	15.09	16.81
95% percentile	3.84	5.99	7.81	11.07	12.59

]

4/I/3H **Statistics**

The following table contains a distribution obtained in 320 tosses of 6 coins and the corresponding expected frequencies calculated with the formula for the binomial distribution for $p = 0.5$ and $n = 6$.

No. heads	0	1	2	3	4	5	6
Observed frequencies	3	21	85	110	62	32	7
Expected frequencies	5	30	75	100	75	30	5

Conduct a goodness-of-fit test at the 0.05 level for the null hypothesis that the coins are all fair.

[*Hint:*

Distribution	χ_5^2	χ_6^2	χ_7^2
95% percentile	11.07	12.59	14.07

]

4/II/12H **Statistics**

State and prove the Rao–Blackwell theorem.

Suppose that $X_1, \ldots, X_n$ are independent random variables uniformly distributed over $(\theta, 3\theta)$. Find a two-dimensional sufficient statistic $T(X)$ for θ. Show that an unbiased estimator of θ is $\hat{\theta} = X_1/2$.

Find an unbiased estimator of θ which is a function of $T(X)$ and whose mean square error is no more than that of $\hat{\theta}$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday 3 June 2003 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B, . . . , H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Analysis II

Let E be a subset of $\mathbb{R}^n$. Prove that the following conditions on E are equivalent:

(i) E is closed and bounded.

(ii) E has the Bolzano–Weierstrass property (i.e., every sequence in E has a subsequence convergent to a point of E).

(iii) Every continuous real-valued function on E is bounded.

[*The Bolzano–Weierstrass property for bounded closed intervals in* $\mathbb{R}^1$ *may be assumed.*]

2D Methods

Fermat's principle of optics states that the path of a light ray connecting two points will be such that the travel time t is a minimum. If the speed of light varies continuously in a medium and is a function $c(y)$ of the distance from the boundary $y = 0$, show that the path of a light ray is given by the solution to

$$c(y)y'' + c'(y)(1 + y'^2) = 0\,,$$

where $y' = \frac{dy}{dx}$, etc. Show that the path of a light ray in a medium where the speed of light c is a constant is a straight line. Also find the path from $(0,0)$ to $(1,0)$ if $c(y) = y$, and sketch it.

3H Statistics

Derive the least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ for the coefficients of the simple linear regression model

$$Y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are given constants, $\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$, and ε_i are independent with $\mathrm{E}\,\varepsilon_i = 0$, $\mathrm{Var}\,\varepsilon_i = \sigma^2$, $i = 1, \ldots, n$.

A manufacturer of optical equipment has the following data on the unit cost (in pounds) of certain custom-made lenses and the number of units made in each order:

No. of units, x_i	1	3	5	10	12
Cost per unit, y_i	58	55	40	37	22

Assuming that the conditions underlying simple linear regression analysis are met, estimate the regression coefficients and use the estimated regression equation to predict the unit cost in an order for 8 of these lenses.

[*Hint: for the data above,* $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})y_i = -257.4$.]

4F Geometry

Describe the geodesics (that is, hyperbolic straight lines) in **either** the disc model **or** the half-plane model of the hyperbolic plane. Explain what is meant by the statements that two hyperbolic lines are parallel, and that they are ultraparallel.

Show that two hyperbolic lines l and l' have a unique common perpendicular if and only if they are ultraparallel.

[*You may assume standard results about the group of isometries of whichever model of the hyperbolic plane you use.*]

5E Linear Mathematics

Let V be the subset of $\mathbb{R}^5$ consisting of all quintuples $(a_1, a_2, a_3, a_4, a_5)$ such that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0 \ .$$

Prove that V is a subspace of $\mathbb{R}^5$. Solve the above equations for a_1 and a_2 in terms of a_3, a_4 and a_5. Hence, exhibit a basis for V, explaining carefully why the vectors you give form a basis.

6C Fluid Dynamics

An unsteady fluid flow has velocity field given in Cartesian coordinates (x, y, z) by $\mathbf{u} = (1, xt, 0)$, where t denotes time. Dye is released into the fluid from the origin continuously. Find the position at time t of the dye particle that was released at time s and hence show that the dye streak lies along the curve

$$y = \tfrac{1}{2}tx^2 - \tfrac{1}{6}x^3.$$

7B Complex Methods

Let $u(x, y)$ and $v(x, y)$ be a pair of conjugate harmonic functions in a domain D. Prove that

$$U(x, y) = e^{-2uv}\cos(u^2 - v^2) \qquad \text{and} \qquad V(x, y) = e^{-2uv}\sin(u^2 - v^2)$$

also form a pair of conjugate harmonic functions in D.

8G Quadratic Mathematics

Let U and V be finite-dimensional vector spaces. Suppose that b and c are bilinear forms on $U \times V$ and that b is non-degenerate. Show that there exist linear endomorphisms S of U and T of V such that $c(x,y) = b(S(x),y) = b(x,T(y))$ for all $(x,y) \in U \times V$.

9A Quantum Mechanics

A particle of mass m is confined inside a one-dimensional box of length a. Determine the possible energy eigenvalues.

SECTION II

10F Analysis II

Explain briefly what is meant by a *metric space*, and by a *Cauchy sequence* in a metric space.

A function $d : X \times X \to \mathbb{R}$ is called a pseudometric on X if it satisfies all the conditions for a metric except the requirement that $d(x, y) = 0$ implies $x = y$. If d is a pseudometric on X, show that the binary relation R on X defined by $x \, R \, y \Leftrightarrow d(x, y) = 0$ is an equivalence relation, and that the function d induces a metric on the set X/R of equivalence classes.

Now let (X, d) be a metric space. If (x_n) and (y_n) are Cauchy sequences in X, show that the sequence whose nth term is $d(x_n, y_n)$ is a Cauchy sequence of real numbers. Deduce that the function $\overline{d}$ defined by

$$\overline{d}((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$$

is a pseudometric on the set C of all Cauchy sequences in X. Show also that there is an isometric embedding (that is, a distance-preserving mapping) $X \to C/R$, where R is the equivalence relation on C induced by the pseudometric $\overline{d}$ as in the previous paragraph. Under what conditions on X is $X \to C/R$ bijective? Justify your answer.

11D Methods

(a) Determine the Green's function $G(x, \xi)$ for the operator $\frac{d^2}{dx^2} + k^2$ on $[0, \pi]$ with Dirichlet boundary conditions by solving the boundary value problem

$$\frac{d^2G}{dx^2} + k^2 G = \delta(x - \xi) \ , \quad G(0) = 0, \ G(\pi) = 0$$

when k is not an integer.

(b) Use the method of Green's functions to solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2 y = f(x) \ , \quad y(0) = a, \ y(\pi) = b$$

when k is not an integer.

12H Statistics

Suppose that six observations $X_1, \ldots, X_6$ are selected at random from a normal distribution for which both the mean μ_X and the variance σ_X^2 are unknown, and it is found that $S_{XX} = \sum_{i=1}^{6}(x_i - \bar{x})^2 = 30$, where $\bar{x} = \frac{1}{6}\sum_{i=1}^{6} x_i$. Suppose also that 21 observations $Y_1, \ldots, Y_{21}$ are selected at random from another normal distribution for which both the mean μ_Y and the variance σ_Y^2 are unknown, and it is found that $S_{YY} = 40$. Derive carefully the likelihood ratio test of the hypothesis H_0: $\sigma_X^2 = \sigma_Y^2$ against H_1: $\sigma_X^2 > \sigma_Y^2$ and apply it to the data above at the 0.05 level.

[*Hint:*

Distribution	χ_5^2	χ_6^2	χ_{20}^2	χ_{21}^2	$F_{5,20}$	$F_{6,21}$
95% percentile	11.07	12.59	31.41	32.68	2.71	2.57

]

13F Geometry

Write down the Riemannian metric in the half-plane model of the hyperbolic plane. Show that Möbius transformations mapping the upper half-plane to itself are isometries of this model.

Calculate the hyperbolic distance from ib to ic, where b and c are positive real numbers. Assuming that the hyperbolic circle with centre ib and radius r is a Euclidean circle, find its Euclidean centre and radius.

Suppose that a and b are positive real numbers for which the points ib and $a + ib$ of the upper half-plane are such that the hyperbolic distance between them coincides with the Euclidean distance. Obtain an expression for b as a function of a. Hence show that, for any b with $0 < b < 1$, there is a unique positive value of a such that the hyperbolic distance between ib and $a + ib$ coincides with the Euclidean distance.

14E Linear Mathematics

(a) Let U, U' be subspaces of a finite-dimensional vector space V. Prove that $\dim(U + U') = \dim U + \dim U' - \dim(U \cap U')$.

(b) Let V and W be finite-dimensional vector spaces and let α and β be linear maps from V to W. Prove that

$$\operatorname{rank}(\alpha + \beta) \leqslant \operatorname{rank}\alpha + \operatorname{rank}\beta\ .$$

(c) Deduce from this result that

$$\operatorname{rank}(\alpha + \beta) \geqslant |\operatorname{rank}\alpha - \operatorname{rank}\beta|\ .$$

(d) Let $V = W = \mathbb{R}^n$ and suppose that $1 \leqslant r \leqslant s \leqslant n$. Exhibit linear maps $\alpha, \beta \colon V \to W$ such that $\operatorname{rank}\alpha = r$, $\operatorname{rank}\beta = s$ and $\operatorname{rank}(\alpha + \beta) = s - r$. Suppose that $r + s \geqslant n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\operatorname{rank}\alpha = r$, $\operatorname{rank}\beta = s$ and $\operatorname{rank}(\alpha + \beta) = n$.

15C Fluid Dynamics

Starting from the Euler equations for incompressible, inviscid flow

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p, \qquad \nabla \cdot \mathbf{u} = 0,$$

derive the vorticity equation governing the evolution of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

Consider the flow

$$\mathbf{u} = \beta(-x, -y, 2z) + \Omega(t)(-y, x, 0),$$

in Cartesian coordinates (x, y, z), where t is time and β is a constant. Compute the vorticity and show that it evolves in time according to

$$\boldsymbol{\omega} = \omega_0 \mathrm{e}^{2\beta t} \mathbf{k},$$

where ω_0 is the initial magnitude of the vorticity and $\mathbf{k}$ is a unit vector in the z-direction.

Show that the material curve $C(t)$ that takes the form

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 1$$

at $t = 0$ is given later by

$$x^2 + y^2 = a^2(t) \quad \text{and} \quad z = \frac{1}{a^2(t)},$$

where the function $a(t)$ is to be determined.

Calculate the circulation of $\mathbf{u}$ around C and state how this illustrates Kelvin's circulation theorem.

16B Complex Methods

Sketch the region A which is the intersection of the discs

$$D_0 = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad D_1 = \{z \in \mathbb{C} : |z - (1+i)| < 1\}.$$

Find a conformal mapping that maps A onto the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Also find a conformal mapping that maps A onto D_0.

[*Hint: You may find it useful to consider maps of the form* $w(z) = \frac{az+b}{cz+d}$.]

17G Quadratic Mathematics

(a) Suppose p is an odd prime and a an integer coprime to p. Define the *Legendre symbol* $(\frac{a}{p})$ and state Euler's criterion.

(b) Compute $(\frac{-1}{p})$ and prove that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

whenever a and b are coprime to p.

(c) Let n be any integer such that $1 \leqslant n \leqslant p-2$. Let m be the unique integer such that $1 \leqslant m \leqslant p-2$ and $mn \equiv 1 \pmod{p}$. Prove that

$$\left(\frac{n(n+1)}{p}\right) = \left(\frac{1+m}{p}\right).$$

(d) Find

$$\sum_{n=1}^{p-2} \left(\frac{n(n+1)}{p}\right).$$

18A Quantum Mechanics

What is the significance of the expectation value

$$\langle Q \rangle = \int \psi^*(x)\ Q\ \psi(x) dx$$

of an observable Q in the normalized state $\psi(x)$? Let Q and P be two observables. By considering the norm of $(Q + i\lambda P)\psi$ for real values of λ, show that

$$\langle Q^2 \rangle \langle P^2 \rangle \geqslant \tfrac{1}{4} |\langle [Q,P] \rangle|^2 .$$

The uncertainty ΔQ of Q in the state $\psi(x)$ is defined as

$$(\Delta Q)^2 = \langle (Q - \langle Q \rangle)^2 \rangle .$$

Deduce the generalized uncertainty relation,

$$\Delta Q \Delta P \geqslant \tfrac{1}{2} |\langle [Q,P] \rangle| .$$

A particle of mass m moves in one dimension under the influence of the potential $\frac{1}{2} m \omega^2 x^2$. By considering the commutator $[x,p]$, show that the expectation value of the Hamiltonian satisfies

$$\langle H \rangle \geqslant \tfrac{1}{2} \hbar \omega .$$

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 4 June 2003 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

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Attach a completed blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Analysis II

Explain what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ to be *differentiable* at a point (a, b). Show that if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist in a neighbourhood of (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Let

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad ((x, y) \neq (0, 0))$$

and $f(0, 0) = 0$. Do the partial derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$? Justify your answers.

2C Methods

Explain briefly why the second-rank tensor

$$\int_S x_i x_j \, dS(\mathbf{x})$$

is isotropic, where S is the surface of the unit sphere centred on the origin.

A second-rank tensor is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j) \, dS(\mathbf{x}) ,$$

where S is the surface of the unit sphere centred on the origin. Calculate $T(\mathbf{y})$ in the form

$$T_{ij} = \lambda \delta_{ij} + \mu y_i y_j ,$$

where λ and μ are to be determined.

By considering the action of T on $\mathbf{y}$ and on vectors perpendicular to $\mathbf{y}$, determine the eigenvalues and associated eigenvectors of T.

3H Statistics

Let $X_1, \ldots, X_n$ be a random sample from the $N(\theta, \sigma^2)$ distribution, and suppose that the prior distribution for θ is $N(\mu, \tau^2)$, where σ^2, μ, τ^2 are known. Determine the posterior distribution for θ, given $X_1, \ldots, X_n$, and the best point estimate of θ under both quadratic and absolute error loss.

4E Further Analysis

Let τ_1 be the collection of all subsets $A \subset \mathbb{N}$ such that $A = \emptyset$ or $\mathbb{N} \setminus A$ is finite. Let τ_2 be the collection of all subsets of $\mathbb{N}$ of the form $I_n = \{n, n+1, n+2, \ldots\}$, together with the empty set. Prove that τ_1 and τ_2 are both topologies on $\mathbb{N}$.

Show that a function f from the topological space $(\mathbb{N}, \tau_1)$ to the topological space $(\mathbb{N}, \tau_2)$ is continuous if and only if one of the following alternatives holds:

(i) $f(n) \to \infty$ as $n \to \infty$;

(ii) there exists $N \in \mathbb{N}$ such that $f(n) = N$ for all but finitely many n and $f(n) \leqslant N$ for all n.

5B Numerical Analysis

Let

$$A = \begin{pmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} \gamma \\ 0 \\ 0 \\ \gamma a \end{pmatrix}, \quad \gamma = 1 - a^4 \neq 0.$$

Find the LU factorization of the matrix A and use it to solve the system $Ax = b$.

6E Linear Mathematics

Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. For each i let $\mathbf{v}_i$ be the vector $(1, a_i, a_i^2, \ldots, a_i^{n-1})$. Let A be the $n \times n$ matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and let $\mathbf{c}$ be a column vector of size n. Prove that $A\mathbf{c} = \mathbf{0}$ if and only if $\mathbf{c} = \mathbf{0}$. Deduce that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$.

[*You may use general facts about matrices if you state them clearly.*]

7B Complex Methods

(a) Using the residue theorem, evaluate

$$\int_{|z|=1} \left(z - \frac{1}{z}\right)^{2n} \frac{dz}{z}\,.$$

(b) Deduce that

$$\int_0^{2\pi} \sin^{2n} t \, dt = \frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}\,.$$

8G Quadratic Mathematics

Let U be a finite-dimensional real vector space and b a positive definite symmetric bilinear form on $U \times U$. Let $\psi : U \to U$ be a linear map such that $b(\psi(x), y) + b(x, \psi(y)) = 0$ for all x and y in U. Prove that if ψ is invertible, then the dimension of U must be even. By considering the restriction of ψ to its image or otherwise, prove that the rank of ψ is always even.

9A Quantum Mechanics

What is meant by the statement than an operator is *hermitian*?

A particle of mass m moves in the real potential $V(x)$ in one dimension. Show that the Hamiltonian of the system is hermitian.

Show that

$$\frac{d}{dt}\langle x \rangle = \frac{1}{m}\langle p \rangle ,$$
$$\frac{d}{dt}\langle p \rangle = \langle -V'(x) \rangle ,$$

where p is the momentum operator and $\langle A \rangle$ denotes the expectation value of the operator A.

SECTION II

10F Analysis II

Let V be the space of $n \times n$ real matrices. Show that the function

$$N(A) = \sup\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| = 1\}$$

(where $\| - \|$ denotes the usual Euclidean norm on $\mathbb{R}^n$) defines a norm on V. Show also that this norm satisfies $N(AB) \leqslant N(A)N(B)$ for all A and B, and that if $N(A) < \epsilon$ then all entries of A have absolute value less than ϵ. Deduce that any function $f : V \to \mathbb{R}$ such that $f(A)$ is a polynomial in the entries of A is continuously differentiable.

Now let $d : V \to \mathbb{R}$ be the mapping sending a matrix to its determinant. By considering $d(I + H)$ as a polynomial in the entries of H, show that the derivative $d'(I)$ is the function $H \mapsto \operatorname{tr} H$. Deduce that, for any A, $d'(A)$ is the mapping $H \mapsto \operatorname{tr}((\operatorname{adj} A)H)$, where $\operatorname{adj} A$ is the adjugate of A, i.e. the matrix of its cofactors.

[*Hint: consider first the case when A is invertible. You may assume the results that the set U of invertible matrices is open in V and that its closure is the whole of V, and the identity* $(\operatorname{adj} A)A = \det A.I$.]

11C Methods

State the transformation law for an nth-rank tensor $T_{ij\cdots k}$.

Show that the fourth-rank tensor

$$c_{ijkl} = \alpha\,\delta_{ij}\,\delta_{kl} + \beta\,\delta_{ik}\,\delta_{jl} + \gamma\,\delta_{il}\,\delta_{jk}$$

is isotropic for arbitrary scalars α, β and γ.

The stress σ_{ij} and strain e_{ij} in a linear elastic medium are related by

$$\sigma_{ij} = c_{ijkl}\,e_{kl}.$$

Given that e_{ij} is symmetric and that the medium is isotropic, show that the stress-strain relationship can be written in the form

$$\sigma_{ij} = \lambda\,e_{kk}\,\delta_{ij} + 2\mu\,e_{ij}.$$

Show that e_{ij} can be written in the form $e_{ij} = p\delta_{ij} + d_{ij}$, where d_{ij} is a traceless tensor and p is a scalar to be determined. Show also that necessary and sufficient conditions for the stored elastic energy density $E = \frac{1}{2}\sigma_{ij}\,e_{ij}$ to be non-negative for any deformation of the solid are that

$$\mu \geq 0 \qquad \text{and} \qquad \lambda \geq -\tfrac{2}{3}\mu.$$

12H Statistics

An examination was given to 500 high-school students in each of two large cities, and their grades were recorded as low, medium, or high. The results are given in the table below.

	Low	*Medium*	*High*
City A	103	145	252
City B	140	136	224

Derive carefully the test of homogeneity and test the hypothesis that the distributions of scores among students in the two cities are the same.

[*Hint:*

Distribution	χ^2_1	χ^2_2	χ^2_3	χ^2_5	χ^2_6
99% percentile	6.63	9.21	11.34	15.09	16.81
95% percentile	3.84	5.99	7.81	11.07	12.59

]

13E Further Analysis

(a) Let $f\colon [1,\infty) \to \mathbb{C}$ be defined by $f(t) = t^{-1}e^{2\pi it}$ and let X be the image of f. Prove that $X \cup \{0\}$ is compact and path-connected. [*Hint: you may find it helpful to set* $s = t^{-1}$.]

(b) Let $g\colon [1,\infty) \to \mathbb{C}$ be defined by $g(t) = (1+t^{-1})e^{2\pi it}$, let Y be the image of g and let $\overline{D}$ be the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. Prove that $Y \cup \overline{D}$ is connected. Explain briefly why it is not path-connected.

14B Numerical Analysis

Let

$$f''(0) \approx a_0 f(-1) + a_1 f(0) + a_2 f(1) = \mu(f)$$

be an approximation of the second derivative which is exact for $f \in \mathcal{P}_2$, the set of polynomials of degree ≤ 2, and let

$$e(f) = f''(0) - \mu(f)$$

be its error.

(a) Determine the coefficients a_0, a_1, a_2.

(b) Using the Peano kernel theorem prove that, for $f \in C^3[-1,1]$, the set of three-times continuously differentiable functions, the error satisfies the inequality

$$|e(f)| \leq \frac{1}{3} \max_{x \in [-1,1]} |f'''(x)|\,.$$

15E Linear Mathematics

(a) Let $A = (a_{ij})$ be an $m \times n$ matrix and for each $k \leqslant n$ let A_k be the $m \times k$ matrix formed by the first k columns of A. Suppose that $n > m$. Explain why the nullity of A is non-zero. Prove that if k is minimal such that A_k has non-zero nullity, then the nullity of A_k is 1.

(b) Suppose that no column of A consists entirely of zeros. Deduce from (a) that there exist scalars $b_1, \ldots, b_k$ (where k is defined as in (a)) such that $\sum_{j=1}^{k} a_{ij} b_j = 0$ for every $i \leqslant m$, but whenever $\lambda_1, \ldots, \lambda_k$ are distinct real numbers there is some $i \leqslant m$ such that $\sum_{j=1}^{k} a_{ij} \lambda_j b_j \neq 0$.

(c) Now let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be bases for the same real m-dimensional vector space. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real numbers such that for every j the vectors $\mathbf{v}_1 + \lambda_j \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda_j \mathbf{w}_m$ are linearly dependent. For each j, let $a_{1j}, \ldots, a_{mj}$ be scalars, not all zero, such that $\sum_{i=1}^{m} a_{ij}(\mathbf{v}_i + \lambda_j \mathbf{w}_i) = \mathbf{0}$. By applying the result of (b) to the matrix (a_{ij}), deduce that $n \leqslant m$.

(d) It follows that the vectors $\mathbf{v}_1 + \lambda \mathbf{w}_1, \ldots, \mathbf{v}_m + \lambda \mathbf{w}_m$ are linearly dependent for at most m values of λ. Explain briefly how this result can also be proved using determinants.

16B Complex Methods

(a) Show that if f satisfies the equation

$$f''(x) - x^2 f(x) = \mu f(x), \quad x \in \mathbb{R}, \qquad (*)$$

where μ is a constant, then its Fourier transform $\widehat{f}$ satisfies the same equation, i.e.

$$\widehat{f}''(\lambda) - \lambda^2 \widehat{f}(\lambda) = \mu \widehat{f}(\lambda)\,.$$

(b) Prove that, for each $n \geq 0$, there is a polynomial $p_n(x)$ of degree n, unique up to multiplication by a constant, such that

$$f_n(x) = p_n(x) e^{-x^2/2}$$

is a solution of $(*)$ for some $\mu = \mu_n$.

(c) Using the fact that $g(x) = e^{-x^2/2}$ satisfies $\widehat{g} = cg$ for some constant c, show that the Fourier transform of f_n has the form

$$\widehat{f_n}(\lambda) = q_n(\lambda) e^{-\lambda^2/2}$$

where q_n is also a polynomial of degree n.

(d) Deduce that the f_n are eigenfunctions of the Fourier transform operator, i.e. $\widehat{f_n}(x) = c_n f_n(x)$ for some constants c_n.

17G Quadratic Mathematics

Let S be the set of all 2×2 complex matrices A which are *hermitian*, that is, $A^* = A$, where $A^* = \overline{A}^t$.

(a) Show that S is a real 4-dimensional vector space. Consider the real symmetric bilinear form b on this space defined by

$$b(A, B) = \tfrac{1}{2}\left(\operatorname{tr}(A\,B) - \operatorname{tr}(A)\operatorname{tr}(B)\right) .$$

Prove that $b(A, A) = -\det A$ and $b(A, I) = -\frac{1}{2}\operatorname{tr}(A)$, where I denotes the identity matrix.

(b) Consider the three matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Prove that the basis I, A_1, A_2, A_3 of S diagonalizes b. Hence or otherwise find the rank and signature of b.

(c) Let Q be the set of all 2×2 complex matrices C which satisfy $C + C^* = \operatorname{tr}(C)\, I$. Show that Q is a real 4-dimensional vector space. Given $C \in Q$, put

$$\Phi(C) = \frac{1-i}{2}\operatorname{tr}(C)\, I + i\, C.$$

Show that Φ takes values in S and is a linear isomorphism between Q and S.

(d) Define a real symmetric bilinear form on Q by setting $c(C, D) = -\frac{1}{2}\operatorname{tr}(C\,D)$, $C, D \in Q$. Show that $b(\Phi(C), \Phi(D)) = c(C, D)$ for all $C, D \in Q$. Find the rank and signature of the symmetric bilinear form c defined on Q.

18A Quantum Mechanics

A particle of mass m and energy E moving in one dimension is incident from the left on a potential barrier $V(x)$ given by

$$V(x) = \begin{cases} V_0 & 0 \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

with $V_0 > 0$.

In the limit $V_0 \to \infty, a \to 0$ with $V_0 a = U$ held fixed, show that the transmission probability is

$$T = \left(1 + \frac{mU^2}{2E\hbar^2}\right)^{-1} .$$

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 5 June 2003 9 to 12

PAPER 3

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SECTION I

1F Analysis II

Let V be the vector space of continuous real-valued functions on $[-1,1]$. Show that the function

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx$$

defines a norm on V.

Let $f_n(x) = x^n$. Show that (f_n) is a Cauchy sequence in V. Is (f_n) convergent? Justify your answer.

2D Methods

Consider the path between two arbitrary points on a cone of interior angle 2α. Show that the arc-length of the path $r(\theta)$ is given by

$$\int (r^2 + r'^2 \operatorname{cosec}^2 \alpha)^{1/2}\, d\theta\ ,$$

where $r' = \frac{dr}{d\theta}$. By minimizing the total arc-length between the points, determine the equation for the shortest path connecting them.

3E Further Analysis

(a) Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leqslant 1 + |z|^{1/2}$ for every z. Prove that f is constant.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\operatorname{Re}(f(z)) \geqslant 0$ for every z. Prove that f is constant.

4F Geometry

Show that any isometry of Euclidean 3-space which fixes the origin can be written as a composite of at most three reflections in planes through the origin, and give (with justification) an example of an isometry for which three reflections are necessary.

5H Optimization

Two players A and B play a zero-sum game with the pay-off matrix

	B_1	B_2	B_3
A_1	4	-2	-5
A_2	-2	4	3
A_3	-3	6	2
A_4	3	-8	-6

Here, the (i, j) entry of the matrix indicates the pay-off to player A if he chooses move A_i and player B chooses move B_j. Show that the game can be reduced to a zero-sum game with 2×2 pay-off matrix.

Determine the value of the game and the optimal strategy for player A.

6B Numerical Analysis

Given $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$, let

$$\ell_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - x_k}{x_i - x_k}$$

be the fundamental Lagrange polynomials of degree n, let

$$\omega(x) = \prod_{i=0}^{n} (x - x_i),$$

and let p be any polynomial of degree $\leq n$.

(a) Prove that $\sum_{i=0}^{n} p(x_i)\ell_i(x) \equiv p(x)$.

(b) Hence or otherwise derive the formula

$$\frac{p(x)}{\omega(x)} = \sum_{i=0}^{n} \frac{A_i}{x - x_i}, \quad A_i = \frac{p(x_i)}{\omega'(x_i)},$$

which is the decomposition of $p(x)/\omega(x)$ into partial fractions.

7G Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U and let β be another endomorphism of U that commutes with α. If λ is an eigenvalue of α, show that β maps the kernel of $\alpha - \lambda\,\iota$ into itself, where ι is the identity map. Suppose now that α is diagonalizable with n distinct real eigenvalues where $n = \dim U$. Prove that if there exists an endomorphism β of U such that $\alpha = \beta^2$, then $\lambda \geqslant 0$ for all eigenvalues λ of α.

8C Fluid Dynamics

Show that the velocity field

$$\mathbf{u} = \mathbf{U} + \frac{\boldsymbol{\Gamma} \times \mathbf{r}}{2\pi r^2},$$

where $\mathbf{U} = (U, 0, 0)$, $\boldsymbol{\Gamma} = (0, 0, \Gamma)$ and $\mathbf{r} = (x, y, 0)$ in Cartesian coordinates (x, y, z), represents the combination of a uniform flow and the flow due to a line vortex. Define and evaluate the circulation of the vortex.

Show that

$$\oint_{C_R} (\mathbf{u} \cdot \mathbf{n})\mathbf{u}\, dl \to \tfrac{1}{2}\boldsymbol{\Gamma} \times \mathbf{U} \qquad \text{as} \qquad R \to \infty,$$

where C_R is a circle $x^2 + y^2 = R^2$, $z = \text{const}$. Explain how this result is related to the lift force on a two-dimensional aerofoil or other obstacle.

9G Quadratic Mathematics

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Explain what is meant by the *discriminant* d of f. State a necessary and sufficient condition for some form of discriminant d to represent an odd prime number p. Using this result or otherwise, find the primes p which can be represented by the form $x^2 + 3y^2$.

10A Special Relativity

What are the momentum and energy of a photon of wavelength λ?

A photon of wavelength λ is incident on an electron. After the collision, the photon has wavelength λ'. Show that

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta),$$

where θ is the scattering angle and m is the electron rest mass.

SECTION II

11F Analysis II

State and prove the Contraction Mapping Theorem.

Let (X, d) be a bounded metric space, and let F denote the set of all continuous maps $X \to X$. Let $\rho \colon F \times F \to \mathbb{R}$ be the function

$$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\} \ .$$

Show that ρ is a metric on F, and that (F, ρ) is complete if (X, d) is complete. [*You may assume that a uniform limit of continuous functions is continuous.*]

Now suppose that (X, d) is complete. Let $C \subseteq F$ be the set of contraction mappings, and let $\theta \colon C \to X$ be the function which sends a contraction mapping to its unique fixed point. Show that θ is continuous. [*Hint: fix $f \in C$ and consider $d(\theta(g), f(\theta(g)))$, where $g \in C$ is close to f.*]

12D Methods

The transverse displacement $y(x, t)$ of a stretched string clamped at its ends $x = 0, l$ satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 2k \frac{\partial y}{\partial t} \ , \quad y(x, 0) = 0, \ \frac{\partial y}{\partial t}(x, 0) = \delta(x - a) \, ,$$

where $c > 0$ is the wave velocity, and $k > 0$ is the damping coefficient. The initial conditions correspond to a sharp blow at $x = a$ at time $t = 0$.

(a) Show that the subsequent motion of the string is given by

$$y(x, t) = \frac{1}{\sqrt{\alpha_n^2 - k^2}} \sum_n 2e^{-kt} \sin \frac{\alpha_n a}{c} \ \sin \frac{\alpha_n x}{c} \ \sin /(\sqrt{\alpha_n^2 - k^2} \ \ t)$$

where $\alpha_n = \pi c n / l$.

(b) Describe what happens in the limits of small and large damping. What critical parameter separates the two cases?

13E Further Analysis

(a) State Taylor's Theorem.

(b) Let $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z)=\sum_{n=0}^{\infty} b_n(z-z_0)^n$ be defined whenever $|z-z_0|<r$. Suppose that $z_k \to z_0$ as $k\to\infty$, that no z_k equals z_0 and that $f(z_k)=g(z_k)$ for every k. Prove that $a_n=b_n$ for every $n \geqslant 0$.

(c) Let D be a domain, let $z_0 \in D$ and let (z_k) be a sequence of points in D that converges to z_0, but such that no z_k equals z_0. Let $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ be analytic functions such that $f(z_k)=g(z_k)$ for every k. Prove that $f(z)=g(z)$ for every $z \in D$.

(d) Let D be the domain $\mathbb{C}\setminus\{0\}$. Give an example of an analytic function $f\colon D \to \mathbb{C}$ such that $f(n^{-1})=0$ for every positive integer n but f is not identically 0.

(e) Show that any function with the property described in (d) must have an essential singularity at the origin.

14F Geometry

State and prove the Gauss–Bonnet formula for the area of a spherical triangle. Deduce a formula for the area of a spherical n-gon with angles $\alpha_1,\alpha_2,\ldots,\alpha_n$. For what range of values of α does there exist a (convex) regular spherical n-gon with angle α?

Let Δ be a spherical triangle with angles π/p, π/q and π/r where p,q,r are integers, and let G be the group of isometries of the sphere generated by reflections in the three sides of Δ. List the possible values of (p,q,r), and in each case calculate the order of the corresponding group G. If $(p,q,r)=(2,3,5)$, show how to construct a regular dodecahedron whose group of symmetries is G.

[*You may assume that the images of Δ under the elements of G form a tessellation of the sphere.*]

15H Optimization

Explain what is meant by a transportation problem where the total demand equals the total supply. Write the Lagrangian and describe an algorithm for solving such a problem. Starting from the north-west initial assignment, solve the problem with three sources and three destinations described by the table

5	9	1	36
3	10	6	84
7	2	5	40
14	68	78	

where the figures in the 3×3 box denote the transportation costs (per unit), the right-hand column denotes supplies, and the bottom row demands.

16B Numerical Analysis

The functions $H_0, H_1, \ldots$ are generated by the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} .$$

(a) Show that H_n is a polynomial of degree n, and that the H_n are orthogonal with respect to the scalar product

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2}\, dx .$$

(b) By induction or otherwise, prove that the H_n satisfy the three-term recurrence relation

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) .$$

[*Hint: you may need to prove the equality* $H_n'(x) = 2n H_{n-1}(x)$ *as well.*]

17G Linear Mathematics

Define the *determinant* $\det(A)$ of an $n \times n$ complex matrix A. Let $A_1, \ldots, A_n$ be the columns of A, let σ be a permutation of $\{1, \ldots, n\}$ and let A^σ be the matrix whose columns are $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$. Prove from your definition of determinant that $\det(A^\sigma) = \epsilon(\sigma) \det(A)$, where $\epsilon(\sigma)$ is the sign of the permutation σ. Prove also that $\det(A) = \det(A^t)$.

Define the *adjugate* matrix $\operatorname{adj}(A)$ and prove from your definitions that $A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A)\, I$, where I is the identity matrix. Hence or otherwise, prove that if $\det(A) \neq 0$, then A is invertible.

Let C and D be real $n \times n$ matrices such that the complex matrix $C + iD$ is invertible. By considering $\det(C + \lambda D)$ as a function of λ or otherwise, prove that there exists a real number λ such that $C + \lambda D$ is invertible. [*You may assume that if a matrix* A *is invertible, then* $\det(A) \neq 0$.]

Deduce that if two real matrices A and B are such that there exists an invertible complex matrix P with $P^{-1} A P = B$, then there exists an invertible **real** matrix Q such that $Q^{-1} A Q = B$.

18C Fluid Dynamics

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid in the absence of gravity.

Water of density ρ is driven through a tube of length L and internal radius a by the pressure exerted by a spherical, water-filled balloon of radius $R(t)$ attached to one end of the tube. The balloon maintains the pressure of the water entering the tube at $2\gamma/R$ in excess of atmospheric pressure, where γ is a constant. It may be assumed that the water exits the tube at atmospheric pressure. Show that

$$R^3\ddot{R} + 2R^2\dot{R}^2 = -\frac{\gamma a^2}{2\rho L}. \qquad (\dagger)$$

Solve equation (†), by multiplying through by $2R\dot{R}$ or otherwise, to obtain

$$t = R_0^2\left(\frac{2\rho L}{\gamma a^2}\right)^{1/2}\left[\frac{\pi}{4} - \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right],$$

where $\theta = \sin^{-1}(R/R_0)$ and R_0 is the initial radius of the balloon. Hence find the time when $R = 0$.

19G Quadratic Mathematics

Let U be a finite-dimensional real vector space endowed with a positive definite inner product. A linear map $\tau : U \to U$ is said to be an *orthogonal projection* if τ is self-adjoint and $\tau^2 = \tau$.

(a) Prove that for every orthogonal projection τ there is an orthogonal decomposition

$$U = \ker(\tau) \oplus \operatorname{im}(\tau).$$

(b) Let $\phi\colon U \to U$ be a linear map. Show that if $\phi^2 = \phi$ and $\phi\,\phi^* = \phi^*\,\phi$, where ϕ^* is the adjoint of ϕ, then ϕ is an orthogonal projection. [*You may find it useful to prove first that if $\phi\,\phi^* = \phi^*\,\phi$, then ϕ and ϕ^* have the same kernel.*]

(c) Show that given a subspace W of U there exists a unique orthogonal projection τ such that $\operatorname{im}(\tau) = W$. If W_1 and W_2 are two subspaces with corresponding orthogonal projections τ_1 and τ_2, show that $\tau_2 \circ \tau_1 = 0$ if and only if W_1 is orthogonal to W_2.

(d) Let $\phi\colon U \to U$ be a linear map satisfying $\phi^2 = \phi$. Prove that one can define a positive definite inner product on U such that ϕ becomes an orthogonal projection.

20A Quantum Mechanics

The radial wavefunction for the hydrogen atom satisfies the equation

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right)+\frac{\hbar^2}{2mr^2}\ell(\ell+1)R(r)-\frac{e^2}{4\pi\epsilon_0 r}R(r)=ER(r)\,.$$

Explain the origin of each term in this equation.

The wavefunctions for the ground state and first radially excited state, both with $\ell = 0$, can be written as

$$R_1(r) = N_1 \exp(-\alpha r)$$
$$R_2(r) = N_2(r+b)\exp(-\beta r)$$

respectively, where N_1 and N_2 are normalization constants. Determine α, β, b and the corresponding energy eigenvalues E_1 and E_2.

A hydrogen atom is in the first radially excited state. It makes the transition to the ground state, emitting a photon. What is the frequency of the emitted photon?

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 6 June 2003 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** ..., **H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed blue cover sheet to each bundle; write the code letter in the box marked 'SECTION' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Analysis II

Explain what it means for a sequence of functions (f_n) to converge uniformly to a function f on an interval. If (f_n) is a sequence of continuous functions converging uniformly to f on a finite interval $[a,b]$, show that

$$\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx \qquad \text{as } n\to\infty\ .$$

Let $f_n(x) = x\exp(-x/n)/n^2,\ x \geqslant 0$. Does $f_n \to 0$ uniformly on $[0,\infty)$? Does $\int_0^\infty f_n(x)\,dx \to 0$? Justify your answers.

2D Methods

Consider the wave equation in a spherically symmetric coordinate system

$$\frac{\partial^2 u(r,t)}{\partial t^2} = c^2 \Delta u(r,t)\,,$$

where $\Delta u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$ is the spherically symmetric Laplacian operator.

(a) Show that the general solution to the equation above is

$$u(r,t) = \tfrac{1}{r}\left[f(r+ct) + g(r-ct)\right],$$

where $f(x), g(x)$ are arbitrary functions.

(b) Using separation of variables, determine the wave field $u(r,t)$ in response to a pulsating source at the origin $u(0,t) = A\,\sin\omega t$.

3H Statistics

The following table contains a distribution obtained in 320 tosses of 6 coins and the corresponding expected frequencies calculated with the formula for the binomial distribution for $p = 0.5$ and $n = 6$.

No. heads	0	1	2	3	4	5	6
Observed frequencies	3	21	85	110	62	32	7
Expected frequencies	5	30	75	100	75	30	5

Conduct a goodness-of-fit test at the 0.05 level for the null hypothesis that the coins are all fair.

[*Hint:*

Distribution	χ^2_5	χ^2_6	χ^2_7
95% percentile	11.07	12.59	14.07

]

4E Further Analysis

(a) State and prove Morera's Theorem.

(b) Let D be a domain and for each $n \in \mathbb{N}$ let $f_n : D \to \mathbb{C}$ be an analytic function. Suppose that $f : D \to \mathbb{C}$ is another function and that $f_n \to f$ uniformly on D. Prove that f is analytic.

5H Optimization

State and prove the Lagrangian sufficiency theorem for a general optimization problem with constraints.

6G Linear Mathematics

Let α be an endomorphism of a finite-dimensional real vector space U such that $\alpha^2 = \alpha$. Show that U can be written as the direct sum of the kernel of α and the image of α. Hence or otherwise, find the characteristic polynomial of α in terms of the dimension of U and the rank of α. Is α diagonalizable? Justify your answer.

7C Fluid Dynamics

Inviscid fluid issues vertically downwards at speed u_0 from a circular tube of radius a. The fluid falls onto a horizontal plate a distance H below the end of the tube, where it spreads out axisymmetrically.

Show that while the fluid is falling freely it has speed

$$u = u_0 \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{1/2},$$

and occupies a circular jet of radius

$$R = a \left[1 + \frac{2g}{u_0^2}(H - z)\right]^{-1/4},$$

where z is the height above the plate and g is the acceleration due to gravity.

Show further that along the plate, at radial distances $r \gg a$ (i.e. far from the falling jet), where the fluid is flowing almost horizontally, it does so as a film of height $h(r)$, where

$$\frac{a^4}{4r^2h^2} = 1 + \frac{2g}{u_0^2}(H - h).$$

8B Complex Methods

Find the Laurent series centred on 0 for the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in each of the domains

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$.

9A Special Relativity

Prove that the two-dimensional Lorentz transformation can be written in the form

$$\begin{aligned} x' &= x\cosh\phi - ct\sinh\phi \\ ct' &= -x\sinh\phi + ct\cosh\phi, \end{aligned}$$

where $\tanh\phi = v/c$. Hence, show that

$$\begin{aligned} x' + ct' &= e^{-\phi}(x+ct) \\ x' - ct' &= e^{\phi}(x-ct)\,. \end{aligned}$$

Given that frame S' has speed v with respect to S and S'' has speed v' with respect to S', use this formalism to find the speed v'' of S'' with respect to S.

[*Hint: rotation through a hyperbolic angle* ϕ*, followed by rotation through* ϕ'*, is equivalent to rotation through* $\phi + \phi'$*.*]

SECTION II

10F Analysis II

Let $(f_n)_{n\geqslant 1}$ be a sequence of continuous complex-valued functions defined on a set $E \subseteq \mathbb{C}$, and converging uniformly on E to a function f. Prove that f is continuous on E.

State the Weierstrass M-test for uniform convergence of a series $\sum_{n=1}^{\infty} u_n(z)$ of complex-valued functions on a set E.

Now let $f(z) = \sum_{n=1}^{\infty} u_n(z)$, where

$$u_n(z) = n^{-2}\sec\left(\pi z/2n\right) .$$

Prove carefully that f is continuous on $\mathbb{C} \setminus \mathbb{Z}$.

[*You may assume the inequality* $|\cos z| \geqslant |\cos(\operatorname{Re} z)|$.]

11D Methods

The velocity potential $\phi(r,\theta)$ for inviscid flow in two dimensions satisfies the Laplace equation

$$\Delta\phi = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\phi(r,\theta) = 0 .$$

(a) Using separation of variables, derive the general solution to the equation above that is single-valued and finite in each of the domains (i) $0 \leqslant r \leqslant a$; (ii) $a \leqslant r < \infty$.

(b) Assuming ϕ is single-valued, solve the Laplace equation subject to the boundary conditions $\frac{\partial\phi}{\partial r} = 0$ at $r = a$, and $\frac{\partial\phi}{\partial r} \to U\cos\theta$ as $r \to \infty$. Sketch the lines of constant potential.

12H Statistics

State and prove the Rao–Blackwell theorem.

Suppose that $X_1, \ldots, X_n$ are independent random variables uniformly distributed over $(\theta, 3\theta)$. Find a two-dimensional sufficient statistic $T(X)$ for θ. Show that an unbiased estimator of θ is $\hat{\theta} = X_1/2$.

Find an unbiased estimator of θ which is a function of $T(X)$ and whose mean square error is no more than that of $\hat{\theta}$.

13E Further Analysis

(a) State the residue theorem and use it to deduce the principle of the argument, in a form that involves winding numbers.

(b) Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\mathrm{Im}\,(p(z)) = 0$. Calculate $\mathrm{Re}\,(p(z))$ for each such z. [*It will be helpful to set* $z = e^{i\theta}$. *You may use the addition formulae* $\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ *and* $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$.]

(c) Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed path $\theta \mapsto e^{i\theta}$. Use your answer to (b) to give a rough sketch of the path $p \circ \gamma$, paying particular attention to where it crosses the real axis.

(d) Hence, or otherwise, determine for every real t the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = t$. (You need not give rigorous justifications for your calculations.)

14H Optimization

Use the two-phase simplex method to solve the problem

$$
\begin{array}{llcrcrcl}
\text{minimize} & 5x_1 & - & 12x_2 & + & 13x_3 & & \\
\\
\text{subject to} & 4x_1 & + & 5x_2 & & & \leq & 9, \\
 & 6x_1 & + & 4x_2 & + & x_3 & \geq & 12, \\
 & 3x_1 & + & 2x_2 & - & x_3 & \leq & 3, \\
\\
 & \multicolumn{7}{l}{x_i \geq 0, \qquad i = 1, 2, 3.}
\end{array}
$$

15G Linear Mathematics

Let $\alpha \in L(U, V)$ be a linear map between finite-dimensional vector spaces. Let

$$M^l(\alpha) = \{\beta \in L(V, U) : \ \beta\,\alpha = 0\} \qquad \text{and}$$

$$M^r(\alpha) = \{\beta \in L(V, U) : \ \alpha\,\beta = 0\} \ .$$

(a) Prove that $M^l(\alpha)$ and $M^r(\alpha)$ are subspaces of $L(V, U)$ of dimensions

$$\dim M^l(\alpha) = (\dim V - \operatorname{rank} \alpha) \dim U \qquad \text{and}$$

$$\dim M^r(\alpha) = \dim \ker(\alpha) \dim V \ .$$

[*You may use the result that there exist bases in U and V so that α is represented by*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix and r is the rank of α.]

(b) Let $\Phi\colon L(U, V) \to L(V^*, U^*)$ be given by $\Phi(\alpha) = \alpha^*$, where α^* is the dual map induced by α. Prove that Φ is an isomorphism. [*You may assume that Φ is linear, and you may use the result that a finite-dimensional vector space and its dual have the same dimension.*]

(c) Prove that

$$\Phi(M^l(\alpha)) = M^r(\alpha^*) \quad \text{and} \quad \Phi(M^r(\alpha)) = M^l(\alpha^*).$$

[*You may use the results that $(\beta\,\alpha)^* = \alpha^*\,\beta^*$ and that β^{**} can be identified with β under the canonical isomorphism between a vector space and its double dual.*]

(d) Conclude that $\operatorname{rank}(\alpha) = \operatorname{rank}(\alpha^*)$.

16C Fluid Dynamics

Define the terms *irrotational flow* and *incompressible flow*. The two-dimensional flow of an incompressible fluid is given in terms of a streamfunction $\psi(x, y)$ as

$$\mathbf{u} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$$

in Cartesian coordinates (x, y). Show that the line integral

$$\int_{\mathbf{x_1}}^{\mathbf{x_2}} \mathbf{u} \cdot \mathbf{n} \, dl = \psi(\mathbf{x_2}) - \psi(\mathbf{x_1})$$

along any path joining the points $\mathbf{x_1}$ and $\mathbf{x_2}$, where $\mathbf{n}$ is the unit normal to the path. Describe how this result is related to the concept of mass conservation.

Inviscid, incompressible fluid is contained in the semi-infinite channel $x > 0$, $0 < y < 1$, which has rigid walls at $x = 0$ and at $y = 0$, 1, apart from a small opening at the origin through which the fluid is withdrawn with volume flux m per unit distance in the third dimension. Show that the streamfunction for irrotational flow in the channel can be chosen (up to an additive constant) to satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

and boundary conditions

$$\begin{aligned}
&\psi = 0 && \text{on } y = 0,\ x > 0, \\
&\psi = -m && \text{on } x = 0,\ 0 < y < 1, \\
&\psi = -m && \text{on } y = 1,\ x > 0, \\
&\psi \to -my && \text{as } x \to \infty,
\end{aligned}$$

if it is assumed that the flow at infinity is uniform. Solve the boundary-value problem above using separation of variables to obtain

$$\psi = -my + \frac{2m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \, \sin n\pi y \; e^{-n\pi x}.$$

17B Complex Methods

Let

$$f(z) = \frac{z^m}{1+z^n}, \quad n > m+1, \quad m, n \in \mathbb{N},$$

and let C_R be the boundary of the domain

$$D_R = \{z = re^{i\theta} : \ 0 < r < R, \quad 0 < \theta < \tfrac{2\pi}{n}\}, \quad R > 1.$$

(a) Using the residue theorem, determine

$$\int_{C_R} f(z)\, dz.$$

(b) Show that the integral of $f(z)$ along the circular part γ_R of C_R tends to 0 as $R \to \infty$.

(c) Deduce that

$$\int_0^\infty \frac{x^m}{1+x^n}\, dx = \frac{\pi}{n \sin \frac{\pi(m+1)}{n}}\,.$$

18A Special Relativity

A pion of rest mass M decays at rest into a muon of rest mass $m < M$ and a neutrino of zero rest mass. What is the speed u of the muon?

In the pion rest frame S, the muon moves in the y-direction. A moving observer, in a frame S' with axes parallel to those in the pion rest frame, wishes to take measurements of the decay along the x-axis, and notes that the pion has speed v with respect to the x-axis. Write down the four-dimensional Lorentz transformation relating S' to S and determine the momentum of the muon in S'. Hence show that in S' the direction of motion of the muon makes an angle θ with respect to the y-axis, where

$$\tan\theta = \frac{M^2+m^2}{M^2-m^2}\frac{v}{(c^2-v^2)^{1/2}}\,.$$

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2004

List of Courses

1/I/4G **Analysis II**

Define what it means for a sequence of functions $F_n : (0,1) \to \mathbb{R}$, where $n = 1, 2, \ldots$, to converge uniformly to a function F.

For each of the following sequences of functions on $(0,1)$, find the pointwise limit function. Which of these sequences converge uniformly? Justify your answers.

(i) $F_n(x) = \frac{1}{n} e^x$

(ii) $F_n(x) = e^{-nx^2}$

(iii) $F_n(x) = \sum_{i=0}^{n} x^i$

1/II/15G **Analysis II**

State the axioms for a norm on a vector space. Show that the usual Euclidean norm on $\mathbb{R}^n$,

$$||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}\,,$$

satisfies these axioms.

Let U be any bounded convex open subset of $\mathbb{R}^n$ that contains 0 and such that if $x \in U$ then $-x \in U$. Show that there is a norm on $\mathbb{R}^n$, satisfying the axioms, for which U is the set of points in $\mathbb{R}^n$ of norm less than 1.

2/I/3G **Analysis II**

Consider a sequence of continuous functions $F_n : [-1,1] \to \mathbb{R}$. Suppose that the functions F_n converge uniformly to some continuous function F. Show that the integrals $\int_{-1}^{1} F_n(x)dx$ converge to $\int_{-1}^{1} F(x)dx$.

Give an example to show that, even if the functions $F_n(x)$ and $F(x)$ are differentiable, the derivatives $F_n'(0)$ need not converge to $F'(0)$.

2/II/14G **Analysis II**

Let X be a non-empty complete metric space. Give an example to show that the intersection of a descending sequence of non-empty closed subsets of X, $A_1 \supset A_2 \supset \cdots$, can be empty. Show that if we also assume that

$$\lim_{n\to\infty} \text{diam } (A_n) = 0$$

then the intersection is not empty. Here the diameter $\text{diam}(A)$ is defined as the supremum of the distances between any two points of a set A.

We say that a subset A of X is *dense* if it has nonempty intersection with every nonempty open subset of X. Let $U_1, U_2, \ldots$ be any sequence of dense open subsets of X. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not empty.

[*Hint: Look for a descending sequence of subsets $A_1 \supset A_2 \supset \cdots$, with $A_i \subset U_i$, such that the previous part of this problem applies.*]

3/I/4F **Analysis II**

Let X and X' be metric spaces with metrics d and d'. If $u = (x, x')$ and $v = (y, y')$ are any two points of $X \times X'$, prove that the formula

$$D(u, v) = \max\bigl\{d(x, y), d'(x', y')\bigr\}$$

defines a metric on $X \times X'$. If $X = X'$, prove that the diagonal Δ of $X \times X$ is closed in $X \times X$.

3/II/16F **Analysis II**

State and prove the contraction mapping theorem.

Let a be a positive real number, and take $X = [\sqrt{\frac{a}{2}},\ \infty)$. Prove that the function

$$f(x) = \frac{1}{2}\Bigl(x + \frac{a}{x}\Bigr)$$

is a contraction from X to X. Find the unique fixed point of f.

4/I/3F Analysis II

Let U, V be open sets in $\mathbb{R}^n, \mathbb{R}^m$, respectively, and let $f : U \to V$ be a map. What does it mean for f to be differentiable at a point u of U?

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the map given by

$$g(x, y) = |x| + |y| \,.$$

Prove that g is differentiable at all points (a, b) with $ab \neq 0$.

4/II/13F Analysis II

State the inverse function theorem for maps $f : U \to \mathbb{R}^2$, where U is a non-empty open subset of $\mathbb{R}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x, y) = (x, x^3 + y^3 - 3xy) \,.$$

Find a non-empty open subset U of $\mathbb{R}^2$ such that f is locally invertible on U, and compute the derivative of the local inverse.

Let C be the set of all points (x, y) in $\mathbb{R}^2$ satisfying

$$x^3 + y^3 - 3xy = 0 \,.$$

Prove that f is locally invertible at all points of C except $(0, 0)$ and $(2^{2/3}, 2^{1/3})$. Deduce that, for each point (a, b) in C except $(0, 0)$ and $(2^{2/3}, 2^{1/3})$, there exist open intervals I, J containing a, b, respectively, such that for each x in I, there is a unique point y in J with (x, y) in C.

1/I/5A **Complex Methods**

Determine the poles of the following functions and calculate their residues there.

$$\text{(i)} \quad \frac{1}{z^2+z^4}, \qquad \text{(ii)} \quad \frac{e^{1/z^2}}{z-1}, \qquad \text{(iii)} \quad \frac{1}{\sin(e^z)}.$$

1/II/16A **Complex Methods**

Let p and q be two polynomials such that

$$q(z) = \prod_{l=1}^{m}(z-\alpha_l),$$

where $\alpha_1, \ldots, \alpha_m$ are distinct non-real complex numbers and $\deg p \leqslant m-1$. Using contour integration, determine

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} dx\,,$$

carefully justifying all steps.

2/I/5A **Complex Methods**

Let the functions f and g be analytic in an open, nonempty domain Ω and assume that $g \neq 0$ there. Prove that if $|f(z)| \equiv |g(z)|$ in Ω then there exists $\alpha \in \mathbb{R}$ such that $f(z) \equiv e^{i\alpha} g(z)$.

2/II/16A **Complex Methods**

Prove by using the Cauchy theorem that if f is analytic in the open disc $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ then there exists a function g, analytic in Ω, such that $g'(z) = f(z)$, $z \in \Omega$.

4/I/5A **Complex Methods**

State and prove the Parseval formula.

[*You may use without proof properties of convolution, as long as they are precisely stated.*]

4/II/15A **Complex Methods**

(i) Show that the inverse Fourier transform of the function

$$\hat{g}(s) = \begin{cases} e^s - e^{-s}, & |s| \leqslant 1, \\ 0, & |s| \geqslant 1. \end{cases}$$

is

$$g(x) = \frac{2i}{\pi} \frac{1}{1+x^2} (x \sinh 1 \cos x - \cosh 1 \sin x)$$

(ii) Determine, by using Fourier transforms, the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

given in the strip $-\infty < x < \infty$, $0 < y < 1$, together with the boundary conditions

$$u(x,0) = g(x), \quad u(x,1) \equiv 0, \qquad -\infty < x < \infty,$$

where g has been given above.

[*You may use without proof properties of Fourier transforms.*]

1/I/9C **Fluid Dynamics**

From the general mass-conservation equation, show that the velocity field $\mathbf{u}(\mathbf{x})$ of an incompressible fluid is solenoidal, i.e. that $\nabla \cdot \mathbf{u} = 0$.

Verify that the two-dimensional flow

$$\mathbf{u} = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right)$$

is solenoidal and find a streamfunction $\psi(x, y)$ such that $\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x)$.

1/II/20C **Fluid Dynamics**

A layer of water of depth h flows along a wide channel with uniform velocity $(U, 0)$, in Cartesian coordinates (x, y), with x measured downstream. The bottom of the channel is at $y = -h$, and the free surface of the water is at $y = 0$. Waves are generated on the free surface so that it has the new position $y = \eta(x, t) = a\, e^{i(\omega t - kx)}$.

Write down the equation and the full nonlinear boundary conditions for the velocity potential ϕ (for the perturbation velocity) and the motion of the free surface.

By linearizing these equations about the state of uniform flow, show that

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + g\eta = 0 \qquad \text{on} \quad y = 0,$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on} \quad y = -h,$$

where g is the acceleration due to gravity.

Hence, determine the dispersion relation for small-amplitude surface waves

$$(\omega - kU)^2 = gk \tanh kh.$$

3/I/10C **Fluid Dynamics**

State Bernoulli's equation for unsteady motion of an irrotational, incompressible, inviscid fluid subject to a conservative body force $-\nabla\chi$.

A long vertical U-tube of uniform cross section contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation

$$h\frac{d^2\zeta}{dt^2} + g\zeta = 0$$

governing the displacement ζ of the surface on one side of the U-tube, where t is time and g is the acceleration due to gravity.

3/II/21C **Fluid Dynamics**

Use separation of variables to determine the irrotational, incompressible flow

$$\mathbf{u} = U\frac{a^3}{r^3}\left(\cos\theta\,\mathbf{e}_r + \tfrac{1}{2}\sin\theta\,\mathbf{e}_\theta\right)$$

around a solid sphere of radius a translating at velocity U along the direction $\theta = 0$ in spherical polar coordinates r and θ.

Show that the total kinetic energy of the fluid is

$$K = \tfrac{1}{4}M_f U^2,$$

where M_f is the mass of fluid displaced by the sphere.

A heavy sphere of mass M is released from rest in an inviscid fluid. Determine its speed after it has fallen through a distance h in terms of M, M_f, g and h.

4/I/8C **Fluid Dynamics**

Write down the vorticity equation for the unsteady flow of an incompressible, inviscid fluid with no body forces acting.

Show that the flow field

$$\mathbf{u} = (-x,\, x\,\omega(t),\, z-1)$$

has uniform vorticity of magnitude $\omega(t) = \omega_0 e^t$ for some constant ω_0.

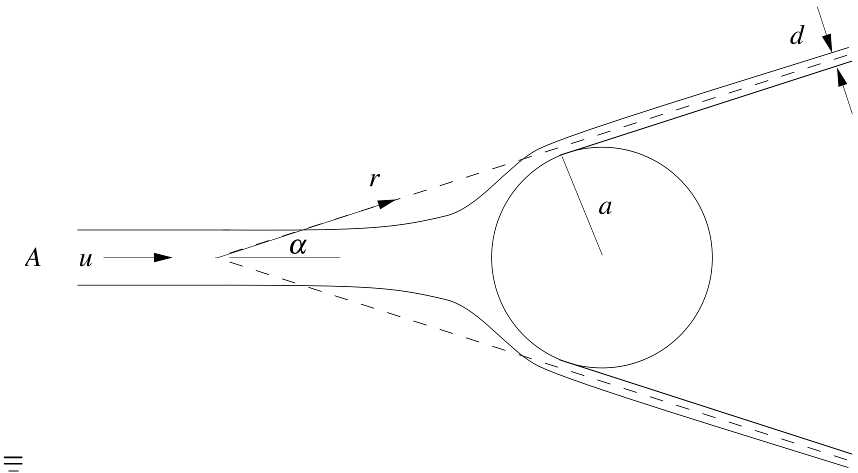
4/II/18C **Fluid Dynamics**

Use Euler's equation to derive the momentum integral

$$\int_S (pn_i + \rho n_j u_j u_i)\ dS = 0$$

for the steady flow $\mathbf{u} = (u_1, u_2, u_3)$ and pressure p of an inviscid,incompressible fluid of density ρ, where S is a closed surface with normal $\mathbf{n}$.

A cylindrical jet of water of area A and speed u impinges axisymmetrically on a stationary sphere of radius a and is deflected into a conical sheet of vertex angle α as shown. Gravity is being ignored.



Use a suitable form of Bernoulli's equation to determine the speed of the water in the conical sheet, being careful to state how the equation is being applied.

Use conservation of mass to show that the width $d(r)$ of the sheet far from the point of impact is given by

$$d = \frac{A}{2\pi r \sin\alpha},$$

where r is the distance along the sheet measured from the vertex of the cone.

Finally, use the momentum integral to determine the net force on the sphere in terms of ρ, u, A and α.

2/I/4E Further Analysis

Let τ be the topology on $\mathbb{N}$ consisting of the empty set and all sets $X \subset \mathbb{N}$ such that $\mathbb{N} \setminus X$ is finite. Let σ be the usual topology on $\mathbb{R}$, and let ρ be the topology on $\mathbb{R}$ consisting of the empty set and all sets of the form (x, ∞) for some real x.

(i) Prove that all continuous functions $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \sigma)$ are constant.

(ii) Give an example with proof of a non-constant function $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \rho)$ that is continuous.

2/II/15E Further Analysis

(i) Let X be the set of all infinite sequences $(\epsilon_1, \epsilon_2, \ldots)$ such that $\epsilon_i \in \{0, 1\}$ for all i. Let τ be the collection of all subsets $Y \subset X$ such that, for every $(\epsilon_1, \epsilon_2, \ldots) \in Y$ there exists n such that $(\eta_1, \eta_2, \ldots) \in Y$ whenever $\eta_1 = \epsilon_1, \eta_2 = \epsilon_2, \ldots, \eta_n = \epsilon_n$. Prove that τ is a topology on X.

(ii) Let a distance d be defined on X by

$$d\Big((\epsilon_1, \epsilon_2, \ldots), (\eta_1, \eta_2, \ldots)\Big) = \sum_{n=1}^{\infty} 2^{-n} |\epsilon_n - \eta_n| \, .$$

Prove that d is a metric and that the topology arising from d is the same as τ.

3/I/5E Further Analysis

Let C be the contour that goes once round the boundary of the square

$$\{z : -1 \leqslant \operatorname{Re} z \leqslant 1, -1 \leqslant \operatorname{Im} z \leqslant 1\}$$

in an anticlockwise direction. What is $\displaystyle\int_C \frac{dz}{z}$? Briefly justify your answer.

Explain why the integrals along each of the four edges of the square are equal. Deduce that $\displaystyle\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}$.

3/II/17E **Further Analysis**

(i) Explain why the formula

$$f(z)=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}$$

defines a function that is analytic on the domain $\mathbb{C}\setminus\mathbb{Z}$. [You need not give full details, but should indicate what results are used.]

Show also that $f(z+1)=f(z)$ for every z such that $f(z)$ is defined.

(ii) Write $\log z$ for $\log r+i\theta$ whenever $z=re^{i\theta}$ with $r>0$ and $-\pi<\theta\leqslant\pi$. Let g be defined by the formula

$$g(z)=f\left(\frac{1}{2\pi i}\log z\right).$$

Prove that g is analytic on $\mathbb{C}\setminus\{0,1\}$.

[*Hint: What would be the effect of redefining* $\log z$ *to be* $\log r+i\theta$ *when* $z=re^{i\theta}$, $r>0$ *and* $0\leqslant\theta<2\pi$?]

(iii) Determine the nature of the singularity of g at $z=1$.

4/I/4E **Further Analysis**

(i) Let D be the open unit disc of radius 1 about the point $3+3i$. Prove that there is an analytic function $f:D\to\mathbb{C}$ such that $f(z)^2=z$ for every $z\in D$.

(ii) Let $D'=\mathbb{C}\setminus\{z\in\mathbb{C}:\text{Im } z=0,\text{Re } z\leqslant 0\}$. Explain briefly why there is at most one extension of f to a function that is analytic on D'.

(iii) Deduce that f cannot be extended to an analytic function on $\mathbb{C}\setminus\{0\}$.

4/II/14E **Further Analysis**

(i) State and prove Rouché's theorem.

[*You may assume the principle of the argument.*]

(ii) Let $0<c<1$. Prove that the polynomial $p(z)=z^3+icz+8$ has three roots with modulus less than 3. Prove that one root α satisfies $\text{Re }\alpha>0,\text{Im }\alpha>0$; another, β, satisfies $\text{Re }\beta>0,\text{Im }\beta<0$; and the third, γ, has $\text{Re }\gamma<0$.

(iii) For sufficiently small c, prove that $\text{Im }\gamma>0$.

[*You may use results from the course if you state them precisely.*]

1/I/3G **Geometry**

Using the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,,$$

define the length of a curve and the area of a region in the upper half-plane $H = \{x + iy : y > 0\}$.

Find the hyperbolic area of the region $\{(x, y) \in H : 0 < x < 1, y > 1\}$.

1/II/14G **Geometry**

Show that for every hyperbolic line L in the hyperbolic plane H there is an isometry of H which is the identity on L but not on all of H. Call it the *reflection* R_L.

Show that every isometry of H is a composition of reflections.

3/I/3G **Geometry**

State Euler's formula for a convex polyhedron with F faces, E edges, and V vertices.

Show that any regular polyhedron whose faces are pentagons has the same number of vertices, edges and faces as the dodecahedron.

3/II/15G **Geometry**

Let a, b, c be the lengths of a right-angled triangle in spherical geometry, where c is the hypotenuse. Prove the Pythagorean theorem for spherical geometry in the form

$$\cos c = \cos a \cos b\,.$$

Now consider such a spherical triangle with the sides a, b replaced by $\lambda a, \lambda b$ for a positive number λ. Show that the above formula approaches the usual Pythagorean theorem as λ approaches zero.

1/I/6B **Methods**

Write down the general isotropic tensors of rank 2 and 3.

According to a theory of magnetostriction, the mechanical stress described by a second-rank symmetric tensor σ_{ij} is induced by the magnetic field vector B_i. The stress is linear in the magnetic field,

$$\sigma_{ij} = A_{ijk} B_k,$$

where A_{ijk} is a third-rank tensor which depends only on the material. Show that σ_{ij} can be non-zero only in anisotropic materials.

1/II/17B **Methods**

The equation governing small amplitude waves on a string can be written as

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

The end points $x = 0$ and $x = 1$ are fixed at $y = 0$. At $t = 0$, the string is held stationary in the waveform,

$$y(x, 0) = x(1 - x) \qquad \text{in} \quad 0 \le x \le 1.$$

The string is then released. Find $y(x, t)$ in the subsequent motion.

Given that the energy

$$\int_0^1 \left[\left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial x} \right)^2 \right] dx$$

is constant in time, show that

$$\sum_{\substack{n \text{ odd} \\ n \geqslant 1}} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

2/I/6B **Methods**

Write down the general form of the solution in polar coordinates (r, θ) to Laplace's equation in two dimensions.

Solve Laplace's equation for $\phi(r, \theta)$ in $0 < r < 1$ and in $1 < r < \infty$, subject to the conditions

$$\phi \to 0 \qquad \text{as} \qquad r \to 0 \text{ and } r \to \infty,$$

$$\phi|_{r=1+} = \phi|_{r=1-} \qquad \text{and} \qquad \left.\frac{\partial \phi}{\partial r}\right|_{r=1+} - \left.\frac{\partial \phi}{\partial r}\right|_{r=1-} = \cos 2\theta + \cos 4\theta.$$

2/II/17B **Methods**

Let $I_{ij}(P)$ be the moment-of-inertia tensor of a rigid body relative to the point P. If G is the centre of mass of the body and the vector GP has components X_i, show that

$$I_{ij}(P) = I_{ij}(G) + M\left(X_k X_k \delta_{ij} - X_i X_j\right),$$

where M is the mass of the body.

Consider a cube of uniform density and side $2a$, with centre at the origin. Find the inertia tensor about the centre of mass, and thence about the corner $P = (a, a, a)$.

Find the eigenvectors and eigenvalues of $I_{ij}(P)$.

3/I/6D **Methods**

Let

$$S[x] = \int_0^T \tfrac{1}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \mathrm{d}t\,, \qquad x(0) = a\,, \quad x(T) = b\,.$$

For any variation $\delta x(t)$ with $\delta x(0) = \delta x(T) = 0$, show that $\delta S = 0$ when $x = x_c$ with

$$x_c(t) = \frac{1}{\sin \omega T}\Big[a\,\sin \omega(T-t) + b\,\sin \omega t\Big]\,.$$

By using integration by parts, show that

$$S[x_c] = \left[\tfrac{1}{2} x_c \dot{x}_c\right]_0^T = \frac{\omega}{2\sin \omega T}\Big[(a^2+b^2)\cos \omega T - 2ab\Big]\,.$$

3/II/18D **Methods**

Starting from the Euler–Lagrange equations, show that the condition for the variation of the integral $\int I(y, y')\,\mathrm{d}x$ to be stationary is

$$I - y' \frac{\partial I}{\partial y'} = \text{constant}\,.$$

In a medium with speed of light $c(y)$ the ray path taken by a light signal between two points satisfies the condition that the time taken is stationary. Consider the region $0 < y < \infty$ and suppose $c(y) = e^{\lambda y}$. Derive the equation for the light ray path $y(x)$. Obtain the solution of this equation and show that the light ray between $(-a, 0)$ and $(a, 0)$ is given by

$$e^{\lambda y} = \frac{\cos \lambda x}{\cos \lambda a}\,,$$

if $\lambda a < \frac{\pi}{2}$.

Sketch the path for λa close to $\frac{\pi}{2}$ and evaluate the time taken for a light signal between these points.

[*The substitution* $u = k\,e^{\lambda y}$, *for some constant* k, *should prove useful in solving the differential equation.*]

4/I/6C **Methods**

Chebyshev polynomials $T_n(x)$ satisfy the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0 \qquad \text{on} \qquad [-1, 1], \tag{$\dagger$}$$

where n is an integer.

Recast this equation into Sturm–Liouville form and hence write down the orthogonality relationship between $T_n(x)$ and $T_m(x)$ for $n \neq m$.

By writing $x = \cos\theta$, or otherwise, show that the polynomial solutions of ($\dagger$) are proportional to $\cos\left(n \cos^{-1} x\right)$.

4/II/16C **Methods**

Obtain the Green function $G(x,\xi)$ satisfying

$$G'' + \frac{2}{x}G' + k^2 G = \delta(x-\xi),$$

where k is real, subject to the boundary conditions

$$\begin{array}{rcl} G \text{ is finite} & \text{at} & x=0, \\ G=0 & \text{at} & x=1. \end{array}$$

[*Hint: You may find the substitution $G = H/x$ helpful.*]

Use the Green function to determine that the solution of the differential equation

$$y'' + \frac{2}{x}y' + k^2 y = 1,$$

subject to the boundary conditions

$$\begin{array}{rcl} y \text{ is finite} & \text{at} & x=0, \\ y=0 & \text{at} & x=1, \end{array}$$

is

$$y = \frac{1}{k^2}\left[1 - \frac{\sin kx}{x \sin k}\right].$$

2/I/9A **Numerical Analysis**

Determine the coefficients of Gaussian quadrature for the evaluation of the integral

$$\int_0^1 f(x)x\,dx$$

that uses two function evaluations.

2/II/20A **Numerical Analysis**

Given an $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$, prove that the vector $\mathbf{x} \in \mathbb{R}^n$ is the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$ if and only if $A^T(A\mathbf{x} - \mathbf{b}) = \mathbf{0}$. Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \end{bmatrix}.$$

Determine the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$.

3/I/11A **Numerical Analysis**

The linear system

$$\begin{bmatrix} \alpha & 2 & 1 \\ 1 & \alpha & 2 \\ 2 & 1 & \alpha \end{bmatrix} \mathbf{x} = \mathbf{b},$$

where real $\alpha \neq 0$ and $\mathbf{b} \in \mathbb{R}^3$ are given, is solved by the iterative procedure

$$\mathbf{x}^{(k+1)} = -\frac{1}{\alpha} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \frac{1}{\alpha}\mathbf{b}, \qquad k \geqslant 0.$$

Determine the conditions on α that guarantee convergence.

3/II/22A **Numerical Analysis**

Given $f \in C^3[0,1]$, we approximate $f'(\frac{1}{3})$ by the linear combination

$$\mathcal{T}[f] = -\frac{5}{3}f(0) + \frac{4}{3}f(\tfrac{1}{2}) + \frac{1}{3}f(1)\,.$$

By finding the Peano kernel, determine the least constant c such that

$$\left|\mathcal{T}[f] - f'(\tfrac{1}{3})\right| \leq c\left\|f'''\right\|_\infty.$$

3/I/12G **Optimization**

Consider the two-person zero-sum game Rock, Scissors, Paper. That is, a player gets 1 point by playing Rock when the other player chooses Scissors, or by playing Scissors against Paper, or Paper against Rock; the losing player gets -1 point. Zero points are received if both players make the same move.

Suppose player one chooses Rock and Scissors (but never Paper) with probabilities p and $1-p$, $0 \leqslant p \leqslant 1$. Write down the maximization problem for player two's optimal strategy. Determine the optimal strategy for each value of p.

3/II/23G **Optimization**

Consider the following linear programming problem:

$$\text{maximize } -x_1 + 3x_2$$

$$\begin{aligned} \text{subject to} \qquad x_1 + x_2 &\geqslant 3, \\ -x_1 + 2x_2 &\geqslant 6, \\ -x_1 + x_2 &\leqslant 2, \\ x_2 &\leqslant 5, \\ x_i &\geqslant 0, \quad i = 1, 2. \end{aligned}$$

Write down the Phase One problem in this case, and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve the above maximization problem. That is, find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

4/I/10G **Optimization**

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow; maximal flow; cut; capacity.

4/II/20G **Optimization**

For any number $c \in (0,1)$, find the minimum and maximum values of

$$\sum_{i=1}^{n} x_i^c \,,$$

subject to $\sum_{i=1}^{n} x_i = 1, x_1, \ldots, x_n \geqslant 0$. Find all the points $(x_1, \ldots, x_n)$ at which the minimum and maximum are attained. Justify your answer.

1/I/8D **Quantum Mechanics**

From the time-dependent Schrödinger equation for $\psi(x,t)$, derive the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

for $\rho(x,t) = \psi^*(x,t)\psi(x,t)$ and some suitable $j(x,t)$.

Show that $\psi(x,t) = e^{i(kx-\omega t)}$ is a solution of the time-dependent Schrödinger equation with zero potential for suitable $\omega(k)$ and calculate ρ and j. What is the interpretation of this solution?

1/II/19D **Quantum Mechanics**

The angular momentum operators are $\mathbf{L} = (L_1, L_2, L_3)$. Write down their commutation relations and show that $[L_i, \mathbf{L}^2] = 0$. Let

$$L_\pm = L_1 \pm iL_2\,,$$

and show that

$$\mathbf{L}^2 = L_- L_+ + {L_3}^2 + \hbar L_3\,.$$

Verify that $\mathbf{L} f(r) = 0$, where $r^2 = x_i x_i$, for any function f. Show that

$$L_3 (x_1 + ix_2)^n f(r) = n\hbar (x_1 + ix_2)^n f(r)\,, \qquad L_+ (x_1 + ix_2)^n f(r) = 0\,,$$

for any integer n. Show that $(x_1 + ix_2)^n f(r)$ is an eigenfunction of $\mathbf{L}^2$ and determine its eigenvalue. Why must $L_-(x_1 + ix_2)^n f(r)$ be an eigenfunction of $\mathbf{L}^2$? What is its eigenvalue?

2/I/8D **Quantum Mechanics**

A quantum mechanical system is described by vectors $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$. The energy eigenvectors are

$$\psi_0 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\,, \qquad \psi_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\,,$$

with energies E_0, E_1 respectively. The system is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at time $t = 0$. What is the probability of finding it in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at a later time t?

2/II/19D **Quantum Mechanics**

Consider a Hamiltonian of the form

$$H = \frac{1}{2m}\big(p + if(x)\big)\big(p - if(x)\big), \qquad -\infty < x < \infty \,,$$

where $f(x)$ is a real function. Show that this can be written in the form $H = p^2/(2m) + V(x)$, for some real $V(x)$ to be determined. Show that there is a wave function $\psi_0(x)$, satisfying a first-order equation, such that $H\psi_0 = 0$. If f is a polynomial of degree n, show that n must be odd in order for ψ_0 to be normalisable. By considering $\int \mathrm{d}x\, \psi^* H \psi$ show that all energy eigenvalues other than that for ψ_0 must be positive.

For $f(x) = kx$, use these results to find the lowest energy and corresponding wave function for the harmonic oscillator Hamiltonian

$$H_{\text{oscillator}} = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2 \,.$$

3/I/9D **Quantum Mechanics**

Write down the expressions for the classical energy and angular momentum for an electron in a hydrogen atom. In the Bohr model the angular momentum L is quantised so that

$$L = n\hbar \,,$$

for integer n. Assuming circular orbits, show that the radius of the n'th orbit is

$$r_n = n^2 a \,,$$

and determine a. Show that the corresponding energy is then

$$E_n = -\frac{e^2}{8\pi\epsilon_0\, r_n} \,.$$

3/II/20D **Quantum Mechanics**

A one-dimensional system has the potential

$$V(x) = \begin{cases} 0 & x < 0\,, \\ \frac{\hbar^2 U}{2m} & 0 < x < L\,, \\ 0 & x > L\,. \end{cases}$$

For energy $E = \hbar^2\epsilon/(2m)$, $\epsilon < U$, the wave function has the form

$$\psi(x) = \begin{cases} a\,e^{ikx} + c\,e^{-ikx} & x < 0\,, \\ e\,\cosh Kx + f\,\sinh Kx & 0 < x < L\,, \\ d\,e^{ik(x-L)} + b\,e^{-ik(x-L)} & x > L\,. \end{cases}$$

By considering the relation between incoming and outgoing waves explain why we should expect

$$|c|^2 + |d|^2 = |a|^2 + |b|^2\,.$$

Find four linear relations between a, b, c, d, e, f. Eliminate d, e, f and show that

$$c = \frac{1}{D}\left[b + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big)\sinh KL\ a\right]\,,$$

where $D = \cosh KL - \frac{1}{2}\big(\lambda + \frac{1}{\lambda}\big)\sinh KL$ and $\lambda = K/(ik)$. By using the result for c, or otherwise, explain why the solution for d is

$$d = \frac{1}{D}\left[a + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big)\sinh KL\ b\right]\,.$$

For $b = 0$ define the transmission coefficient T and show that, for large L,

$$T \approx 16\,\frac{\epsilon(U-\epsilon)}{U^2}\,e^{-2\sqrt{U-\epsilon}\,L}\,.$$

3/I/8B **Special Relativity**

Write down the Lorentz transformation with one space dimension between two inertial frames S and S' moving relatively to one another at speed V.

A particle moves at velocity u in frame S. Find its velocity u' in frame S' and show that u' is always less than c.

4/I/7D **Special Relativity**

For a particle with energy E and momentum $(p\cos\theta, p\sin\theta, 0)$, explain why an observer moving in the x-direction with velocity v would find

$$E' = \gamma(E - p\,\cos\theta\ v)\,, \qquad p'\cos\theta' = \gamma\Big(p\,\cos\theta - E\,\frac{v}{c^2}\Big)\,, \qquad p'\sin\theta' = p\,\sin\theta\,,$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. What is the relation between E and p for a photon? Show that the same relation holds for E' and p' and that

$$\cos\theta' = \frac{\cos\theta - \dfrac{v}{c}}{1 - \dfrac{v}{c}\cos\theta}\,.$$

What happens for $v \to c$?

4/II/17D **Special Relativity**

State how the 4-momentum p_μ of a particle is related to its energy and 3-momentum. How is p_μ related to the particle mass? For two particles with 4-momenta $p_{1\mu}$ and $p_{2\mu}$ find a Lorentz-invariant expression that gives the total energy in their centre of mass frame.

A photon strikes an electron at rest. What is the minimum energy it must have in order for it to create an electron and positron, of the same mass m_e as the electron, in addition to the original electron? Express the result in units of $m_e c^2$.

[*It may be helpful to consider the minimum necessary energy in the centre of mass frame.*]

1/I/10H **Statistics**

Use the generalized likelihood-ratio test to derive Student's t-test for the equality of the means of two populations. You should explain carefully the assumptions underlying the test.

1/II/21H **Statistics**

State and prove the Rao–Blackwell Theorem.

Suppose that $X_1, X_2, \ldots, X_n$ are independent, identically-distributed random variables with distribution

$$P\left(X_1 = r\right) = p^{r-1}(1-p), \quad r = 1, 2, \ldots,$$

where p, $0 < p < 1$, is an unknown parameter. Determine a one-dimensional sufficient statistic, T, for p.

By first finding a simple unbiased estimate for p, or otherwise, determine an unbiased estimate for p which is a function of T.

2/I/10H **Statistics**

A study of 60 men and 90 women classified each individual according to eye colour to produce the figures below.

	Blue	Brown	Green
Men	20	20	20
Women	20	50	20

Explain how you would analyse these results. You should indicate carefully any underlying assumptions that you are making.

A further study took 150 individuals and classified them both by eye colour and by whether they were left or right handed to produce the following table.

	Blue	Brown	Green
Left Handed	20	20	20
Right Handed	20	50	20

How would your analysis change? You should again set out your underlying assumptions carefully.

[*You may wish to note the following percentiles of the* χ^2 *distribution.*

	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_5^2	χ_6^2
95% percentile	3.84	5.99	7.81	9.49	11.07	12.59
99% percentile	6.64	9.21	11.34	13.28	15.09	16.81

]

2/II/21H **Statistics**

Defining carefully the terminology that you use, state and prove the Neyman–Pearson Lemma.

Let X be a single observation from the distribution with density function

$$f(x \mid \theta) = \tfrac{1}{2}e^{-|x-\theta|}, \quad -\infty < x < \infty,$$

for an unknown real parameter θ. Find the best test of size α, $0 < \alpha < 1$, of the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

When $\alpha = 0.05$, for which values of θ_0 and θ_1 will the power of the best test be at least 0.95?

4/I/9H **Statistics**

Suppose that $Y_1, \ldots, Y_n$ are independent random variables, with Y_i having the normal distribution with mean βx_i and variance σ^2; here β, σ^2 are unknown and $x_1, \ldots, x_n$ are known constants.

Derive the least-squares estimate of β.

Explain carefully how to test the hypothesis $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$.

4/II/19H **Statistics**

It is required to estimate the unknown parameter θ after observing X, a single random variable with probability density function $f(x \mid \theta)$; the parameter θ has the prior distribution with density $\pi(\theta)$ and the loss function is $L(\theta, a)$. Show that the optimal Bayesian point estimate minimizes the posterior expected loss.

Suppose now that $f(x \mid \theta) = \theta e^{-\theta x}$, $x > 0$ and $\pi(\theta) = \mu e^{-\mu\theta}$, $\theta > 0$, where $\mu > 0$ is known. Determine the posterior distribution of θ given X.

Determine the optimal Bayesian point estimate of θ in the cases when

(i) $L(\theta, a) = (\theta - a)^2$, and

(ii) $L(\theta, a) = |(\theta - a)/\theta|$.

1/I/1H Linear Algebra

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_{r+1}\}$ is a linearly independent set of distinct elements of a vector space V and $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{f}_{r+1}, \ldots, \mathbf{f}_m\}$ spans V. Prove that $\mathbf{f}_{r+1}, \ldots, \mathbf{f}_m$ may be reordered, as necessary, so that $\{\mathbf{e}_1, \ldots \mathbf{e}_{r+1}, \mathbf{f}_{r+2}, \ldots, \mathbf{f}_m\}$ spans V.

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a linearly independent set of distinct elements of V and that $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ spans V. Show that $n \leqslant m$.

1/II/12H Linear Algebra

Let U and W be subspaces of the finite-dimensional vector space V. Prove that both the sum $U + W$ and the intersection $U \cap W$ are subspaces of V. Prove further that

$$\dim U + \dim W = \dim\,(U + W) + \dim\,(U \cap W)\,.$$

Let U, W be the kernels of the maps $A, B : \mathbb{R}^4 \to \mathbb{R}^2$ given by the matrices A and B respectively, where

$$A = \begin{pmatrix} 1 & 2 & -1 & -3 \\ -1 & 1 & 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & -4 \end{pmatrix}.$$

Find a basis for the intersection $U \cap W$, and extend this first to a basis of U, and then to a basis of $U + W$.

2/I/1E Linear Algebra

For each n let A_n be the $n \times n$ matrix defined by

$$(A_n)_{ij} = \begin{cases} i & i \leqslant j\,, \\ j & i > j\,. \end{cases}$$

What is $\det A_n$? Justify your answer.

[*It may be helpful to look at the cases* $n = 1, 2, 3$ *before tackling the general case.*]

2/II/12E Linear Algebra

Let Q be a quadratic form on a real vector space V of dimension n. Prove that there is a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with respect to which Q is given by the formula

$$Q\Big(\sum_{i=1}^{n} x_i \mathbf{e}_i\Big) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \,.$$

Prove that the numbers p and q are uniquely determined by the form Q. By means of an example, show that the subspaces $\langle \mathbf{e}_1, \ldots, \mathbf{e}_p \rangle$ and $\langle \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q} \rangle$ need not be uniquely determined by Q.

3/I/1E Linear Algebra

Let V be a finite-dimensional vector space over $\mathbb{R}$. What is the *dual space* of V? Prove that the dimension of the dual space is the same as that of V.

3/II/13E Linear Algebra

(i) Let V be an n-dimensional vector space over $\mathbb{C}$ and let $\alpha : V \to V$ be an endomorphism. Suppose that the characteristic polynomial of α is $\Pi_{i=1}^{k}(x - \lambda_i)^{n_i}$, where the λ_i are distinct and $n_i > 0$ for every i.

Describe all possibilities for the minimal polynomial and prove that there are no further ones.

(ii) Give an example of a matrix for which both the characteristic and the minimal polynomial are $(x-1)^3(x-3)$.

(iii) Give an example of two matrices A, B with the same rank and the same minimal and characteristic polynomials such that there is no invertible matrix P with $PAP^{-1} = B$.

4/I/1E Linear Algebra

Let V be a real n-dimensional inner-product space and let $W \subset V$ be a k-dimensional subspace. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be an orthonormal basis for W. In terms of this basis, give a formula for the orthogonal projection $\pi : V \to W$.

Let $v \in V$. Prove that πv is the closest point in W to v.

[*You may assume that the sequence* $\mathbf{e}_1, \ldots, \mathbf{e}_k$ *can be extended to an orthonormal basis* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *of* V.]

4/II/11E **Linear Algebra**

(i) Let V be an n-dimensional inner-product space over $\mathbb{C}$ and let $\alpha : V \to V$ be a Hermitian linear map. Prove that V has an orthonormal basis consisting of eigenvectors of α.

(ii) Let $\beta : V \to V$ be another Hermitian map. Prove that $\alpha\beta$ is Hermitian if and only if $\alpha\beta = \beta\alpha$.

(iii) A Hermitian map α is *positive-definite* if $\langle \alpha v, v \rangle > 0$ for every non-zero vector v. If α is a positive-definite Hermitian map, prove that there is a unique positive-definite Hermitian map β such that $\beta^2 = \alpha$.

1/I/2F **Groups, Rings and Modules**

Let G be a finite group of order n. Let H be a subgroup of G. Define the normalizer $N(H)$ of H, and prove that the number of distinct conjugates of H is equal to the index of $N(H)$ in G. If p is a prime dividing n, deduce that the number of Sylow p-subgroups of G must divide n.

[*You may assume the existence and conjugacy of Sylow subgroups.*]

Prove that any group of order 72 must have either 1 or 4 Sylow 3-subgroups.

1/II/13F **Groups, Rings and Modules**

State the structure theorem for finitely generated abelian groups. Prove that a finitely generated abelian group A is finite if and only if there exists a prime p such that $A/pA = 0$.

Show that there exist abelian groups $A \neq 0$ such that $A/pA = 0$ for all primes p. Prove directly that your example of such an A is not finitely generated.

2/I/2F **Groups, Rings and Modules**

Prove that the alternating group A_5 is simple.

2/II/13F **Groups, Rings and Modules**

Let K be a subgroup of a group G. Prove that K is normal if and only if there is a group H and a homomorphism $\phi : G \to H$ such that

$$K = \{g \in G \ : \ \phi(g) = 1\}.$$

Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d in $\mathbb{Z}$ and $ad - bc = 1$. Let p be a prime number, and take K to be the subset of G consisting of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \equiv d \equiv 1 \pmod{p}$ and $c \equiv b \equiv 0 \pmod{p}$. Prove that K is a normal subgroup of G.

3/I/2F **Groups, Rings and Modules**

Let R be the subring of all z in $\mathbb{C}$ of the form

$$z = \frac{a + b\sqrt{-3}}{2}$$

where a and b are in $\mathbb{Z}$ and $a \equiv b \pmod 2$. Prove that $N(z) = z\bar{z}$ is a non-negative element of $\mathbb{Z}$, for all z in R. Prove that the multiplicative group of units of R has order 6. Prove that $7R$ is the intersection of two prime ideals of R.

[*You may assume that R is a unique factorization domain.*]

3/II/14F **Groups, Rings and Modules**

Let L be the group $\mathbb{Z}^3$ consisting of 3-dimensional row vectors with integer components. Let M be the subgroup of L generated by the three vectors

$$u = (1,2,3),\ v = (2,3,1),\ w = (3,1,2)\,.$$

(i) What is the index of M in L?

(ii) Prove that M is not a direct summand of L.

(iii) Is the subgroup N generated by u and v a direct summand of L?

(iv) What is the structure of the quotient group L/M?

4/I/2F **Groups, Rings and Modules**

State Gauss's lemma and Eisenstein's irreducibility criterion. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$:

(i) $x^5 + 5x + 5$;

(ii) $x^3 - 4x + 1$;

(iii) $x^{p-1} + x^{p-2} + \ldots + x + 1$, where p is any prime number.

4/II/12F **Groups, Rings and Modules**

Answer the following questions, fully justifying your answer in each case.

(i) Give an example of a ring in which some non-zero prime ideal is not maximal.

(ii) Prove that $\mathbb{Z}[x]$ is not a principal ideal domain.

(iii) Does there exist a field K such that the polynomial $f(x) = 1 + x + x^3 + x^4$ is irreducible in $K[x]$?

(iv) Is the ring $\mathbb{Q}[x]/(x^3 - 1)$ an integral domain?

(v) Determine all ring homomorphisms $\phi : \mathbb{Q}[x]/(x^3 - 1) \to \mathbb{C}$.

1/I/7B **Electromagnetism**

Write down Maxwell's equations and show that they imply the conservation of charge.

In a conducting medium of conductivity σ, where $\mathbf{J} = \sigma\mathbf{E}$, show that any charge density decays in time exponentially at a rate to be determined.

1/II/18B **Electromagnetism**

Inside a volume D there is an electrostatic charge density $\rho(\mathbf{r})$, which induces an electric field $\mathbf{E}(\mathbf{r})$ with associated electrostatic potential $\phi(\mathbf{r})$. The potential vanishes on the boundary of D. The electrostatic energy is

$$W = \frac{1}{2}\int_D \rho\phi \, d^3\mathbf{r}. \tag{1}$$

Derive the alternative form

$$W = \frac{\epsilon_0}{2}\int_D E^2 \, d^3\mathbf{r}. \tag{2}$$

A capacitor consists of three identical and parallel thin metal circular plates of area A positioned in the planes $z = -H$, $z = a$ and $z = H$, with $-H < a < H$, with centres on the z axis, and at potentials 0, V and 0 respectively. Find the electrostatic energy stored, verifying that expressions (1) and (2) give the same results. Why is the energy minimal when $a = 0$?

2/I/7B **Electromagnetism**

Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a sheet carrying a surface current of density $\mathbf{s}$, with normal $\mathbf{n}$ to the sheet, are

$$\mathbf{n} \times \mathbf{B}_+ - \mathbf{n} \times \mathbf{B}_- = \mu_0\mathbf{s}.$$

Write down the force per unit area on the surface current.

2/II/18B **Electromagnetism**

The vector potential due to a steady current density $\mathbf{J}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, d^3\mathbf{r}', \qquad (*)$$

where you may assume that $\mathbf{J}$ extends only over a finite region of space. Use $(*)$ to derive the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \, d^3\mathbf{r}'.$$

A circular loop of wire of radius a carries a current I. Take Cartesian coordinates with the origin at the centre of the loop and the z-axis normal to the loop. Use the Biot–Savart law to show that on the z-axis the magnetic field is in the axial direction and of magnitude

$$B = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}.$$

3/I/7B **Electromagnetism**

A wire is bent into the shape of three sides of a rectangle and is held fixed in the $z=0$ plane, with base $x=0$ and $-\ell < y < \ell$, and with arms $y = \pm\ell$ and $0 < x < \ell$. A second wire moves smoothly along the arms: $x = X(t)$ and $-\ell < y < \ell$ with $0 < X < \ell$. The two wires have resistance R per unit length and mass M per unit length. There is a time-varying magnetic field $B(t)$ in the z-direction.

Using the law of induction, find the electromotive force around the circuit made by the two wires.

Using the Lorentz force, derive the equation

$$M\ddot{X} = -\frac{B}{R(X+2\ell)} \frac{d}{dt}\left(X\ell B\right).$$

3/II/19B **Electromagnetism**

Starting from Maxwell's equations, derive the law of energy conservation in the form

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E} = 0,$$

where $W = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ and $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$.

Evaluate W and $\mathbf{S}$ for the plane electromagnetic wave in vacuum

$$\mathbf{E} = (E_0 \cos(kz - \omega t), 0, 0) \qquad \mathbf{B} = (0, B_0 \cos(kz-\omega t), 0),$$

where the relationships between E_0, B_0, ω and k should be determined. Show that the electromagnetic energy propagates at speed $c^2 = 1/(\epsilon_0 \mu_0)$, i.e. show that $S = Wc$.

1/I/11H Markov Chains

Let $P = (P_{ij})$ be a transition matrix. What does it mean to say that P is (*a*) irreducible, (*b*) recurrent?

Suppose that P is irreducible and recurrent and that the state space contains at least two states. Define a new transition matrix $\tilde{P}$ by

$$\tilde{P}_{ij} = \begin{cases} 0 & \text{if} \quad i = j, \\ (1 - P_{ii})^{-1} P_{ij} & \text{if} \quad i \neq j. \end{cases}$$

Prove that $\tilde{P}$ is also irreducible and recurrent.

1/II/22H Markov Chains

Consider the Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Determine the communicating classes of the chain, and for each class indicate whether it is open or closed.

Suppose that the chain starts in state 2; determine the probability that it ever reaches state 6.

Suppose that the chain starts in state 3; determine the probability that it is in state 6 after exactly n transitions, $n \geqslant 1$.

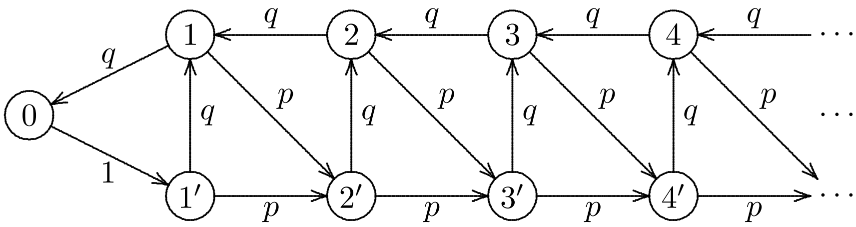
2/I/11H Markov Chains

Let $(X_r)_{r \geqslant 0}$ be an irreducible, positive-recurrent Markov chain on the state space S with transition matrix (P_{ij}) and initial distribution $P(X_0 = i) = \pi_i$, $i \in S$, where (π_i) is the unique invariant distribution. What does it mean to say that the Markov chain is reversible?

Prove that the Markov chain is reversible if and only if $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$.

2/II/22H **Markov Chains**

Consider a Markov chain on the state space $S = \{0, 1, 2, \ldots\} \cup \{1', 2', 3', \ldots\}$ with transition probabilities as illustrated in the diagram below, where $0 < q < 1$ and $p = 1 - q$.



For each value of q, $0 < q < 1$, determine whether the chain is transient, null recurrent or positive recurrent.

When the chain is positive recurrent, calculate the invariant distribution.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday 1 June 2004 9 to 12

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Additional credit will be given to substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** ..., **H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Linear Algebra

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_{r+1}\}$ is a linearly independent set of distinct elements of a vector space V and $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{f}_{r+1}, \ldots, \mathbf{f}_m\}$ spans V. Prove that $\mathbf{f}_{r+1}, \ldots, \mathbf{f}_m$ may be reordered, as necessary, so that $\{\mathbf{e}_1, \ldots \mathbf{e}_{r+1}, \mathbf{f}_{r+2}, \ldots, \mathbf{f}_m\}$ spans V.

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a linearly independent set of distinct elements of V and that $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ spans V. Show that $n \leqslant m$.

2F Groups, Rings and Modules

Let G be a finite group of order n. Let H be a subgroup of G. Define the normalizer $N(H)$ of H, and prove that the number of distinct conjugates of H is equal to the index of $N(H)$ in G. If p is a prime dividing n, deduce that the number of Sylow p-subgroups of G must divide n.

[*You may assume the existence and conjugacy of Sylow subgroups.*]

Prove that any group of order 72 must have either 1 or 4 Sylow 3-subgroups.

3G Geometry

Using the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \, ,$$

define the length of a curve and the area of a region in the upper half-plane
$H = \{x + iy : y > 0\}$.

Find the hyperbolic area of the region $\{(x, y) \in H : 0 < x < 1, y > 1\}$.

4G Analysis II

Define what it means for a sequence of functions $F_n : (0,1) \to \mathbb{R}$, where $n = 1, 2, \ldots$, to converge uniformly to a function F.

For each of the following sequences of functions on $(0,1)$, find the pointwise limit function. Which of these sequences converge uniformly? Justify your answers.

(i) $F_n(x) = \frac{1}{n} e^x$

(ii) $F_n(x) = e^{-nx^2}$

(iii) $F_n(x) = \displaystyle\sum_{i=0}^{n} x^i$

5A Complex Methods

Determine the poles of the following functions and calculate their residues there.

$$\text{(i)}\quad \frac{1}{z^2+z^4}, \qquad \text{(ii)}\quad \frac{e^{1/z^2}}{z-1}, \qquad \text{(iii)}\quad \frac{1}{\sin(e^z)}.$$

6B Methods

Write down the general isotropic tensors of rank 2 and 3.

According to a theory of magnetostriction, the mechanical stress described by a second-rank symmetric tensor σ_{ij} is induced by the magnetic field vector B_i. The stress is linear in the magnetic field,

$$\sigma_{ij} = A_{ijk} B_k,$$

where A_{ijk} is a third-rank tensor which depends only on the material. Show that σ_{ij} can be non-zero only in anisotropic materials.

7B Electromagnetism

Write down Maxwell's equations and show that they imply the conservation of charge.

In a conducting medium of conductivity σ, where $\mathbf{J} = \sigma \mathbf{E}$, show that any charge density decays in time exponentially at a rate to be determined.

8D Quantum Mechanics

From the time-dependent Schrödinger equation for $\psi(x,t)$, derive the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

for $\rho(x,t) = \psi^*(x,t)\psi(x,t)$ and some suitable $j(x,t)$.

Show that $\psi(x,t) = e^{i(kx-\omega t)}$ is a solution of the time-dependent Schrödinger equation with zero potential for suitable $\omega(k)$ and calculate ρ and j. What is the interpretation of this solution?

9C Fluid Dynamics

From the general mass-conservation equation, show that the velocity field $\mathbf{u}(\mathbf{x})$ of an incompressible fluid is solenoidal, i.e. that $\nabla \cdot \mathbf{u} = 0$.

Verify that the two-dimensional flow

$$\mathbf{u} = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right)$$

is solenoidal and find a streamfunction $\psi(x, y)$ such that $\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x)$.

10H Statistics

Use the generalized likelihood-ratio test to derive Student's t-test for the equality of the means of two populations. You should explain carefully the assumptions underlying the test.

11H Markov Chains

Let $P = (P_{ij})$ be a transition matrix. What does it mean to say that P is (a) irreducible, (b) recurrent?

Suppose that P is irreducible and recurrent and that the state space contains at least two states. Define a new transition matrix $\tilde{P}$ by

$$\tilde{P}_{ij} = \begin{cases} 0 & \text{if} \quad i = j, \\ (1 - P_{ii})^{-1} P_{ij} & \text{if} \quad i \neq j. \end{cases}$$

Prove that $\tilde{P}$ is also irreducible and recurrent.

SECTION II

12H Linear Algebra

Let U and W be subspaces of the finite-dimensional vector space V. Prove that both the sum $U+W$ and the intersection $U \cap W$ are subspaces of V. Prove further that

$$\dim U + \dim W = \dim\,(U+W) + \dim\,(U \cap W)\,.$$

Let U, W be the kernels of the maps $A, B : \mathbb{R}^4 \to \mathbb{R}^2$ given by the matrices A and B respectively, where

$$A = \begin{pmatrix} 1 & 2 & -1 & -3 \\ -1 & 1 & 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & -4 \end{pmatrix}.$$

Find a basis for the intersection $U \cap W$, and extend this first to a basis of U, and then to a basis of $U+W$.

13F Groups, Rings and Modules

State the structure theorem for finitely generated abelian groups. Prove that a finitely generated abelian group A is finite if and only if there exists a prime p such that $A/pA = 0$.

Show that there exist abelian groups $A \neq 0$ such that $A/pA = 0$ for all primes p. Prove directly that your example of such an A is not finitely generated.

14G Geometry

Show that for every hyperbolic line L in the hyperbolic plane H there is an isometry of H which is the identity on L but not on all of H. Call it the *reflection* R_L.

Show that every isometry of H is a composition of reflections.

15G Analysis II

State the axioms for a norm on a vector space. Show that the usual Euclidean norm on $\mathbb{R}^n$,

$$||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}\,,$$

satisfies these axioms.

Let U be any bounded convex open subset of $\mathbb{R}^n$ that contains 0 and such that if $x \in U$ then $-x \in U$. Show that there is a norm on $\mathbb{R}^n$, satisfying the axioms, for which U is the set of points in $\mathbb{R}^n$ of norm less than 1.

16A Complex Methods

Let p and q be two polynomials such that

$$q(z) = \prod_{l=1}^{m} (z - \alpha_l),$$

where $\alpha_1, \ldots, \alpha_m$ are distinct non-real complex numbers and $\deg p \leqslant m-1$. Using contour integration, determine

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} dx\,,$$

carefully justifying all steps.

17B Methods

The equation governing small amplitude waves on a string can be written as

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

The end points $x = 0$ and $x = 1$ are fixed at $y = 0$. At $t = 0$, the string is held stationary in the waveform,

$$y(x, 0) = x(1 - x) \qquad \text{in} \quad 0 \leq x \leq 1.$$

The string is then released. Find $y(x,t)$ in the subsequent motion.

Given that the energy

$$\int_0^1 \left[\left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial x} \right)^2 \right] dx$$

is constant in time, show that

$$\sum_{\substack{n \text{ odd} \\ n \geqslant 1}} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

18B Electromagnetism

Inside a volume D there is an electrostatic charge density $\rho(\mathbf{r})$, which induces an electric field $\mathbf{E}(\mathbf{r})$ with associated electrostatic potential $\phi(\mathbf{r})$. The potential vanishes on the boundary of D. The electrostatic energy is

$$W = \frac{1}{2}\int_D \rho\phi\, d^3\mathbf{r}. \tag{1}$$

Derive the alternative form

$$W = \frac{\epsilon_0}{2}\int_D E^2\, d^3\mathbf{r}. \tag{2}$$

A capacitor consists of three identical and parallel thin metal circular plates of area A positioned in the planes $z=-H$, $z=a$ and $z=H$, with $-H<a<H$, with centres on the z axis, and at potentials 0, V and 0 respectively. Find the electrostatic energy stored, verifying that expressions (1) and (2) give the same results. Why is the energy minimal when $a=0$?

19D Quantum Mechanics

The angular momentum operators are $\mathbf{L} = (L_1, L_2, L_3)$. Write down their commutation relations and show that $[L_i, \mathbf{L}^2] = 0$. Let

$$L_\pm = L_1 \pm iL_2\,,$$

and show that

$$\mathbf{L}^2 = L_- L_+ + {L_3}^2 + \hbar L_3\,.$$

Verify that $\mathbf{L}f(r) = 0$, where $r^2 = x_i x_i$, for any function f. Show that

$$L_3 (x_1 + ix_2)^n f(r) = n\hbar (x_1 + ix_2)^n f(r)\,, \qquad L_+ (x_1 + ix_2)^n f(r) = 0\,,$$

for any integer n. Show that $(x_1 + ix_2)^n f(r)$ is an eigenfunction of $\mathbf{L}^2$ and determine its eigenvalue. Why must $L_-(x_1 + ix_2)^n f(r)$ be an eigenfunction of $\mathbf{L}^2$? What is its eigenvalue?

20C Fluid Dynamics

A layer of water of depth h flows along a wide channel with uniform velocity $(U, 0)$, in Cartesian coordinates (x, y), with x measured downstream. The bottom of the channel is at $y = -h$, and the free surface of the water is at $y = 0$. Waves are generated on the free surface so that it has the new position $y = \eta(x, t) = a\, e^{i(\omega t - kx)}$.

Write down the equation and the full nonlinear boundary conditions for the velocity potential ϕ (for the perturbation velocity) and the motion of the free surface.

By linearizing these equations about the state of uniform flow, show that

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + g\eta = 0 \qquad \text{on} \quad y = 0,$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{on} \quad y = -h,$$

where g is the acceleration due to gravity.

Hence, determine the dispersion relation for small-amplitude surface waves

$$(\omega - kU)^2 = gk \tanh kh.$$

21H Statistics

State and prove the Rao–Blackwell Theorem.

Suppose that $X_1, X_2, \ldots, X_n$ are independent, identically-distributed random variables with distribution

$$P\left(X_1 = r\right) = p^{r-1}(1-p), \quad r = 1, 2, \ldots,$$

where p, $0 < p < 1$, is an unknown parameter. Determine a one-dimensional sufficient statistic, T, for p.

By first finding a simple unbiased estimate for p, or otherwise, determine an unbiased estimate for p which is a function of T.

22H Markov Chains

Consider the Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Determine the communicating classes of the chain, and for each class indicate whether it is open or closed.

Suppose that the chain starts in state 2; determine the probability that it ever reaches state 6.

Suppose that the chain starts in state 3; determine the probability that it is in state 6 after exactly n transitions, $n \geqslant 1$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 2 June 2004 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Additional credit will be given to substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

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You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

For each n let A_n be the $n \times n$ matrix defined by

$$(A_n)_{ij} = \begin{cases} i & i \leqslant j\,, \\ j & i > j\,. \end{cases}$$

What is $\det A_n$? Justify your answer.

[*It may be helpful to look at the cases* $n = 1, 2, 3$ *before tackling the general case.*]

2F Groups, Rings and Modules

Prove that the alternating group A_5 is simple.

3G Analysis II

Consider a sequence of continuous functions $F_n : [-1, 1] \to \mathbb{R}$. Suppose that the functions F_n converge uniformly to some continuous function F. Show that the integrals $\int_{-1}^{1} F_n(x)dx$ converge to $\int_{-1}^{1} F(x)dx$.

Give an example to show that, even if the functions $F_n(x)$ and $F(x)$ are differentiable, the derivatives $F_n'(0)$ need not converge to $F'(0)$.

4E Further Analysis

Let τ be the topology on $\mathbb{N}$ consisting of the empty set and all sets $X \subset \mathbb{N}$ such that $\mathbb{N} \setminus X$ is finite. Let σ be the usual topology on $\mathbb{R}$, and let ρ be the topology on $\mathbb{R}$ consisting of the empty set and all sets of the form (x, ∞) for some real x.

(i) Prove that all continuous functions $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \sigma)$ are constant.

(ii) Give an example with proof of a non-constant function $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \rho)$ that is continuous.

5A Complex Methods

Let the functions f and g be analytic in an open, nonempty domain Ω and assume that $g \neq 0$ there. Prove that if $|f(z)| \equiv |g(z)|$ in Ω then there exists $\alpha \in \mathbb{R}$ such that $f(z) \equiv e^{i\alpha} g(z)$.

6B Methods

Write down the general form of the solution in polar coordinates (r, θ) to Laplace's equation in two dimensions.

Solve Laplace's equation for $\phi(r, \theta)$ in $0 < r < 1$ and in $1 < r < \infty$, subject to the conditions

$$\phi \to 0 \qquad \text{as} \qquad r \to 0 \text{ and } r \to \infty,$$

$$\phi|_{r=1+} = \phi|_{r=1-} \qquad \text{and} \qquad \left.\frac{\partial \phi}{\partial r}\right|_{r=1+} - \left.\frac{\partial \phi}{\partial r}\right|_{r=1-} = \cos 2\theta + \cos 4\theta.$$

7B Electromagnetism

Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a sheet carrying a surface current of density $\mathbf{s}$, with normal $\mathbf{n}$ to the sheet, are

$$\mathbf{n} \times \mathbf{B}_+ - \mathbf{n} \times \mathbf{B}_- = \mu_0 \mathbf{s}.$$

Write down the force per unit area on the surface current.

8D Quantum Mechanics

A quantum mechanical system is described by vectors $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$. The energy eigenvectors are

$$\psi_0 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

with energies E_0, E_1 respectively. The system is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at time $t = 0$. What is the probability of finding it in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at a later time t?

9A Numerical Analysis

Determine the coefficients of Gaussian quadrature for the evaluation of the integral

$$\int_0^1 f(x) x \, dx$$

that uses two function evaluations.

10H Statistics

A study of 60 men and 90 women classified each individual according to eye colour to produce the figures below.

	Blue	Brown	Green
Men	20	20	20
Women	20	50	20

Explain how you would analyse these results. You should indicate carefully any underlying assumptions that you are making.

A further study took 150 individuals and classified them both by eye colour and by whether they were left or right handed to produce the following table.

	Blue	Brown	Green
Left Handed	20	20	20
Right Handed	20	50	20

How would your analysis change? You should again set out your underlying assumptions carefully.

[*You may wish to note the following percentiles of the* χ^2 *distribution.*

	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_5^2	χ_6^2
95% percentile	3.84	5.99	7.81	9.49	11.07	12.59
99% percentile	6.64	9.21	11.34	13.28	15.09	16.81

]

11H Markov Chains

Let $(X_r)_{r\geqslant 0}$ be an irreducible, positive-recurrent Markov chain on the state space S with transition matrix (P_{ij}) and initial distribution $P(X_0 = i) = \pi_i$, $i \in S$, where (π_i) is the unique invariant distribution. What does it mean to say that the Markov chain is reversible?

Prove that the Markov chain is reversible if and only if $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$.

SECTION II

12E Linear Algebra

Let Q be a quadratic form on a real vector space V of dimension n. Prove that there is a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with respect to which Q is given by the formula

$$Q\Big(\sum_{i=1}^{n} x_i \mathbf{e}_i\Big) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 .$$

Prove that the numbers p and q are uniquely determined by the form Q. By means of an example, show that the subspaces $\langle \mathbf{e}_1, \ldots, \mathbf{e}_p \rangle$ and $\langle \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q} \rangle$ need not be uniquely determined by Q.

13F Groups, Rings and Modules

Let K be a subgroup of a group G. Prove that K is normal if and only if there is a group H and a homomorphism $\phi : G \to H$ such that

$$K = \{ g \in G \ : \ \phi(g) = 1 \} .$$

Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d in $\mathbb{Z}$ and $ad - bc = 1$. Let p be a prime number, and take K to be the subset of G consisting of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \equiv d \equiv 1 \pmod{p}$ and $c \equiv b \equiv 0 \pmod{p}$. Prove that K is a normal subgroup of G.

14G Analysis II

Let X be a non-empty complete metric space. Give an example to show that the intersection of a descending sequence of non-empty closed subsets of X, $A_1 \supset A_2 \supset \cdots$, can be empty. Show that if we also assume that

$$\lim_{n \to \infty} \operatorname{diam}\,(A_n) = 0$$

then the intersection is not empty. Here the diameter $\operatorname{diam}(A)$ is defined as the supremum of the distances between any two points of a set A.

We say that a subset A of X is *dense* if it has nonempty intersection with every nonempty open subset of X. Let $U_1, U_2, \ldots$ be any sequence of dense open subsets of X. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not empty.

[*Hint: Look for a descending sequence of subsets* $A_1 \supset A_2 \supset \cdots$, *with* $A_i \subset U_i$, *such that the previous part of this problem applies.*]

15E Further Analysis

(i) Let X be the set of all infinite sequences $(\epsilon_1, \epsilon_2, \ldots)$ such that $\epsilon_i \in \{0,1\}$ for all i. Let τ be the collection of all subsets $Y \subset X$ such that, for every $(\epsilon_1, \epsilon_2, \ldots) \in Y$ there exists n such that $(\eta_1, \eta_2, \ldots) \in Y$ whenever $\eta_1 = \epsilon_1, \eta_2 = \epsilon_2, \ldots, \eta_n = \epsilon_n$. Prove that τ is a topology on X.

(ii) Let a distance d be defined on X by

$$d\Big((\epsilon_1, \epsilon_2, \ldots), (\eta_1, \eta_2, \ldots)\Big) = \sum_{n=1}^{\infty} 2^{-n} |\epsilon_n - \eta_n| \,.$$

Prove that d is a metric and that the topology arising from d is the same as τ.

16A Complex Methods

Prove by using the Cauchy theorem that if f is analytic in the open disc $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ then there exists a function g, analytic in Ω, such that $g'(z) = f(z)$, $z \in \Omega$.

17B Methods

Let $I_{ij}(P)$ be the moment-of-inertia tensor of a rigid body relative to the point P. If G is the centre of mass of the body and the vector GP has components X_i, show that

$$I_{ij}(P) = I_{ij}(G) + M\left(X_k X_k \delta_{ij} - X_i X_j\right),$$

where M is the mass of the body.

Consider a cube of uniform density and side $2a$, with centre at the origin. Find the inertia tensor about the centre of mass, and thence about the corner $P = (a, a, a)$.

Find the eigenvectors and eigenvalues of $I_{ij}(P)$.

18B Electromagnetism

The vector potential due to a steady current density $\mathbf{J}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}', \qquad (*)$$

where you may assume that $\mathbf{J}$ extends only over a finite region of space. Use $(*)$ to derive the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3\mathbf{r}'.$$

A circular loop of wire of radius a carries a current I. Take Cartesian coordinates with the origin at the centre of the loop and the z-axis normal to the loop. Use the Biot–Savart law to show that on the z-axis the magnetic field is in the axial direction and of magnitude

$$B = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}.$$

19D Quantum Mechanics

Consider a Hamiltonian of the form

$$H = \frac{1}{2m}\big(p + if(x)\big)\big(p - if(x)\big), \qquad -\infty < x < \infty\,,$$

where $f(x)$ is a real function. Show that this can be written in the form $H = p^2/(2m) + V(x)$, for some real $V(x)$ to be determined. Show that there is a wave function $\psi_0(x)$, satisfying a first-order equation, such that $H\psi_0 = 0$. If f is a polynomial of degree n, show that n must be odd in order for ψ_0 to be normalisable. By considering $\int \mathrm{d}x\, \psi^* H \psi$ show that all energy eigenvalues other than that for ψ_0 must be positive.

For $f(x) = kx$, use these results to find the lowest energy and corresponding wave function for the harmonic oscillator Hamiltonian

$$H_{\text{oscillator}} = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2\,.$$

20A Numerical Analysis

Given an $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$, prove that the vector $\mathbf{x} \in \mathbb{R}^n$ is the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$ if and only if $A^T(A\mathbf{x} - \mathbf{b}) = \mathbf{0}$. Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \end{bmatrix}.$$

Determine the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$.

21H Statistics

Defining carefully the terminology that you use, state and prove the Neyman–Pearson Lemma.

Let X be a single observation from the distribution with density function

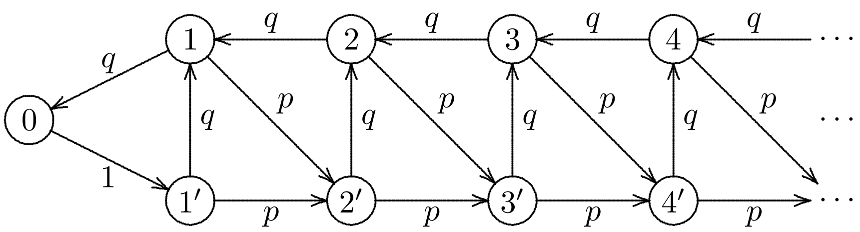
$$f(x \mid \theta) = \tfrac{1}{2} e^{-|x-\theta|}, \quad -\infty < x < \infty,$$

for an unknown real parameter θ. Find the best test of size α, $0 < \alpha < 1$, of the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

When $\alpha = 0.05$, for which values of θ_0 and θ_1 will the power of the best test be at least 0.95?

22H Markov Chains

Consider a Markov chain on the state space $S = \{0, 1, 2, \ldots\} \cup \{1', 2', 3', \ldots\}$ with transition probabilities as illustrated in the diagram below, where $0 < q < 1$ and $p = 1 - q$.



For each value of q, $0 < q < 1$, determine whether the chain is transient, null recurrent or positive recurrent.

When the chain is positive recurrent, calculate the invariant distribution.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 3 June 2004 9 to 12

PAPER 3

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SECTION I

1E Linear Algebra

Let V be a finite-dimensional vector space over $\mathbb{R}$. What is the *dual space* of V? Prove that the dimension of the dual space is the same as that of V.

2F Groups, Rings and Modules

Let R be the subring of all z in $\mathbb{C}$ of the form

$$z = \frac{a + b\sqrt{-3}}{2}$$

where a and b are in $\mathbb{Z}$ and $a \equiv b \pmod 2$. Prove that $N(z) = z\bar{z}$ is a non-negative element of $\mathbb{Z}$, for all z in R. Prove that the multiplicative group of units of R has order 6. Prove that $7R$ is the intersection of two prime ideals of R.

[*You may assume that R is a unique factorization domain.*]

3G Geometry

State Euler's formula for a convex polyhedron with F faces, E edges, and V vertices.

Show that any regular polyhedron whose faces are pentagons has the same number of vertices, edges and faces as the dodecahedron.

4F Analysis II

Let X and X' be metric spaces with metrics d and d'. If $u = (x, x')$ and $v = (y, y')$ are any two points of $X \times X'$, prove that the formula

$$D(u, v) = \max\{d(x, y), d'(x', y')\}$$

defines a metric on $X \times X'$. If $X = X'$, prove that the diagonal Δ of $X \times X$ is closed in $X \times X$.

5E Further Analysis

Let C be the contour that goes once round the boundary of the square

$$\{z : -1 \leqslant \operatorname{Re} z \leqslant 1, -1 \leqslant \operatorname{Im} z \leqslant 1\}$$

in an anticlockwise direction. What is $\displaystyle\int_C \frac{dz}{z}$? Briefly justify your answer.

Explain why the integrals along each of the four edges of the square are equal. Deduce that $\displaystyle\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}$.

6D Methods

Let

$$S[x] = \int_0^T \tfrac{1}{2}\left(\dot{x}^2 - \omega^2 x^2\right) \mathrm{d}t\,, \qquad x(0) = a\,, \quad x(T) = b\,.$$

For any variation $\delta x(t)$ with $\delta x(0) = \delta x(T) = 0$, show that $\delta S = 0$ when $x = x_c$ with

$$x_c(t) = \frac{1}{\sin \omega T}\Big[a \,\sin \omega(T-t) + b\, \sin \omega t\Big]\,.$$

By using integration by parts, show that

$$S[x_c] = \left[\tfrac{1}{2} x_c \dot{x}_c\right]_0^T = \frac{\omega}{2 \sin \omega T}\Big[(a^2+b^2)\cos \omega T - 2ab\Big]\,.$$

7B Electromagnetism

A wire is bent into the shape of three sides of a rectangle and is held fixed in the $z = 0$ plane, with base $x = 0$ and $-\ell < y < \ell$, and with arms $y = \pm\ell$ and $0 < x < \ell$. A second wire moves smoothly along the arms: $x = X(t)$ and $-\ell < y < \ell$ with $0 < X < \ell$. The two wires have resistance R per unit length and mass M per unit length. There is a time-varying magnetic field $B(t)$ in the z-direction.

Using the law of induction, find the electromotive force around the circuit made by the two wires.

Using the Lorentz force, derive the equation

$$M\ddot{X} = -\frac{B}{R(X+2\ell)} \frac{d}{dt}\,(X\ell B)\,.$$

8B Special Relativity

Write down the Lorentz transformation with one space dimension between two inertial frames S and S' moving relatively to one another at speed V.

A particle moves at velocity u in frame S. Find its velocity u' in frame S' and show that u' is always less than c.

9D Quantum Mechanics

Write down the expressions for the classical energy and angular momentum for an electron in a hydrogen atom. In the Bohr model the angular momentum L is quantised so that

$$L = n\hbar\,,$$

for integer n. Assuming circular orbits, show that the radius of the n'th orbit is

$$r_n = n^2 a\,,$$

and determine a. Show that the corresponding energy is then

$$E_n = -\frac{e^2}{8\pi\epsilon_0\, r_n}\,.$$

10C Fluid Dynamics

State Bernoulli's equation for unsteady motion of an irrotational, incompressible, inviscid fluid subject to a conservative body force $-\nabla\chi$.

A long vertical U-tube of uniform cross section contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation

$$h\frac{d^2\zeta}{dt^2} + g\zeta = 0$$

governing the displacement ζ of the surface on one side of the U-tube, where t is time and g is the acceleration due to gravity.

11A Numerical Analysis

The linear system

$$\begin{bmatrix} \alpha & 2 & 1 \\ 1 & \alpha & 2 \\ 2 & 1 & \alpha \end{bmatrix} \mathbf{x} = \mathbf{b},$$

where real $\alpha \neq 0$ and $\mathbf{b} \in \mathbb{R}^3$ are given, is solved by the iterative procedure

$$\mathbf{x}^{(k+1)} = -\frac{1}{\alpha} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{x}^{(k)} + \frac{1}{\alpha}\mathbf{b}, \qquad k \geqslant 0.$$

Determine the conditions on α that guarantee convergence.

12G Optimization

Consider the two-person zero-sum game Rock, Scissors, Paper. That is, a player gets 1 point by playing Rock when the other player chooses Scissors, or by playing Scissors against Paper, or Paper against Rock; the losing player gets -1 point. Zero points are received if both players make the same move.

Suppose player one chooses Rock and Scissors (but never Paper) with probabilities p and $1-p$, $0 \leqslant p \leqslant 1$. Write down the maximization problem for player two's optimal strategy. Determine the optimal strategy for each value of p.

SECTION II

13E Linear Algebra

(i) Let V be an n-dimensional vector space over $\mathbb{C}$ and let $\alpha : V \to V$ be an endomorphism. Suppose that the characteristic polynomial of α is $\Pi_{i=1}^{k}(x-\lambda_i)^{n_i}$, where the λ_i are distinct and $n_i > 0$ for every i.

Describe all possibilities for the minimal polynomial and prove that there are no further ones.

(ii) Give an example of a matrix for which both the characteristic and the minimal polynomial are $(x-1)^3(x-3)$.

(iii) Give an example of two matrices A, B with the same rank and the same minimal and characteristic polynomials such that there is no invertible matrix P with $PAP^{-1} = B$.

14F Groups, Rings and Modules

Let L be the group $\mathbb{Z}^3$ consisting of 3-dimensional row vectors with integer components. Let M be the subgroup of L generated by the three vectors

$$u = (1,2,3),\ v = (2,3,1),\ w = (3,1,2)\,.$$

(i) What is the index of M in L?

(ii) Prove that M is not a direct summand of L.

(iii) Is the subgroup N generated by u and v a direct summand of L?

(iv) What is the structure of the quotient group L/M?

15G Geometry

Let a, b, c be the lengths of a right-angled triangle in spherical geometry, where c is the hypotenuse. Prove the Pythagorean theorem for spherical geometry in the form

$$\cos c = \cos a \cos b\,.$$

Now consider such a spherical triangle with the sides a, b replaced by $\lambda a, \lambda b$ for a positive number λ. Show that the above formula approaches the usual Pythagorean theorem as λ approaches zero.

16F Analysis II

State and prove the contraction mapping theorem.

Let a be a positive real number, and take $X = [\sqrt{\frac{a}{2}},\ \infty)$. Prove that the function

$$f(x) = \frac{1}{2}\Big(x + \frac{a}{x}\Big)$$

is a contraction from X to X. Find the unique fixed point of f.

17E Further Analysis

(i) Explain why the formula

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

defines a function that is analytic on the domain $\mathbb{C} \setminus \mathbb{Z}$. [You need not give full details, but should indicate what results are used.]

Show also that $f(z+1) = f(z)$ for every z such that $f(z)$ is defined.

(ii) Write $\log z$ for $\log r + i\theta$ whenever $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leqslant \pi$. Let g be defined by the formula

$$g(z) = f\left(\frac{1}{2\pi i}\log z\right) .$$

Prove that g is analytic on $\mathbb{C} \setminus \{0, 1\}$.

[*Hint: What would be the effect of redefining* $\log z$ *to be* $\log r + i\theta$ *when* $z = re^{i\theta}$, $r > 0$ *and* $0 \leqslant \theta < 2\pi$?]

(iii) Determine the nature of the singularity of g at $z = 1$.

18D Methods

Starting from the Euler–Lagrange equations, show that the condition for the variation of the integral $\int I(y, y')\,\mathrm{d}x$ to be stationary is

$$I - y'\frac{\partial I}{\partial y'} = \text{constant}\,.$$

In a medium with speed of light $c(y)$ the ray path taken by a light signal between two points satisfies the condition that the time taken is stationary. Consider the region $0 < y < \infty$ and suppose $c(y) = e^{\lambda y}$. Derive the equation for the light ray path $y(x)$. Obtain the solution of this equation and show that the light ray between $(-a, 0)$ and $(a, 0)$ is given by

$$e^{\lambda y} = \frac{\cos \lambda x}{\cos \lambda a}\,,$$

if $\lambda a < \frac{\pi}{2}$.

Sketch the path for λa close to $\frac{\pi}{2}$ and evaluate the time taken for a light signal between these points.

[*The substitution* $u = k\,e^{\lambda y}$, *for some constant* k, *should prove useful in solving the differential equation.*]

19B Electromagnetism

Starting from Maxwell's equations, derive the law of energy conservation in the form

$$\frac{\partial W}{\partial t} + \nabla\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0,$$

where $W = \dfrac{\epsilon_0}{2}E^2 + \dfrac{1}{2\mu_0}B^2$ and $\mathbf{S} = \dfrac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$.

Evaluate W and $\mathbf{S}$ for the plane electromagnetic wave in vacuum

$$\mathbf{E} = (E_0\cos(kz - \omega t), 0, 0) \qquad \mathbf{B} = (0, B_0\cos(kz - \omega t), 0),$$

where the relationships between E_0, B_0, ω and k should be determined. Show that the electromagnetic energy propagates at speed $c^2 = 1/(\epsilon_0\mu_0)$, i.e. show that $S = Wc$.

20D Quantum Mechanics

A one-dimensional system has the potential

$$V(x) = \begin{cases} 0 & x < 0\,, \\ \frac{\hbar^2 U}{2m} & 0 < x < L\,, \\ 0 & x > L\,. \end{cases}$$

For energy $E = \hbar^2 \epsilon/(2m)$, $\epsilon < U$, the wave function has the form

$$\psi(x) = \begin{cases} a\, e^{ikx} + c\, e^{-ikx} & x < 0\,, \\ e \cosh Kx + f \sinh Kx & 0 < x < L\,, \\ d\, e^{ik(x-L)} + b\, e^{-ik(x-L)} & x > L\,. \end{cases}$$

By considering the relation between incoming and outgoing waves explain why we should expect

$$|c|^2 + |d|^2 = |a|^2 + |b|^2\,.$$

Find four linear relations between a, b, c, d, e, f. Eliminate d, e, f and show that

$$c = \frac{1}{D}\left[b + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big) \sinh KL\ a \right],$$

where $D = \cosh KL - \frac{1}{2}\big(\lambda + \frac{1}{\lambda}\big) \sinh KL$ and $\lambda = K/(ik)$. By using the result for c, or otherwise, explain why the solution for d is

$$d = \frac{1}{D}\left[a + \frac{1}{2}\Big(\lambda - \frac{1}{\lambda}\Big) \sinh KL\ b \right].$$

For $b = 0$ define the transmission coefficient T and show that, for large L,

$$T \approx 16\, \frac{\epsilon(U - \epsilon)}{U^2}\, e^{-2\sqrt{U-\epsilon}\, L}\,.$$

21C Fluid Dynamics

Use separation of variables to determine the irrotational, incompressible flow

$$\mathbf{u} = U\frac{a^3}{r^3}\left(\cos\theta\,\mathbf{e}_r + \tfrac{1}{2}\sin\theta\,\mathbf{e}_\theta\right)$$

around a solid sphere of radius a translating at velocity U along the direction $\theta = 0$ in spherical polar coordinates r and θ.

Show that the total kinetic energy of the fluid is

$$K = \tfrac{1}{4}M_f U^2,$$

where M_f is the mass of fluid displaced by the sphere.

A heavy sphere of mass M is released from rest in an inviscid fluid. Determine its speed after it has fallen through a distance h in terms of M, M_f, g and h.

22A Numerical Analysis

Given $f \in C^3[0,1]$, we approximate $f'(\frac{1}{3})$ by the linear combination

$$\mathcal{T}[f] = -\frac{5}{3}f(0) + \frac{4}{3}f(\tfrac{1}{2}) + \frac{1}{3}f(1)\,.$$

By finding the Peano kernel, determine the least constant c such that

$$\left|\mathcal{T}[f] - f'(\tfrac{1}{3})\right| \le c\left\|f'''\right\|_\infty.$$

23G Optimization

Consider the following linear programming problem:

$$\begin{aligned} &\text{maximize} \quad -x_1 + 3x_2 \\ &\text{subject to} \quad x_1 + x_2 \geqslant 3, \\ &\qquad -x_1 + 2x_2 \geqslant 6, \\ &\qquad -x_1 + x_2 \leqslant 2, \\ &\qquad x_2 \leqslant 5, \\ &\qquad x_i \geqslant 0, \quad i = 1, 2. \end{aligned}$$

Write down the Phase One problem in this case, and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve the above maximization problem. That is, find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 4 June 2004 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Additional credit will be given to substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B, . . . , H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Let V be a real n-dimensional inner-product space and let $W \subset V$ be a k-dimensional subspace. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be an orthonormal basis for W. In terms of this basis, give a formula for the orthogonal projection $\pi : V \to W$.

Let $v \in V$. Prove that πv is the closest point in W to v.

[*You may assume that the sequence* $\mathbf{e}_1, \ldots, \mathbf{e}_k$ *can be extended to an orthonormal basis* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *of* V.]

2F Groups, Rings and Modules

State Gauss's lemma and Eisenstein's irreducibility criterion. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$:

(i) $x^5 + 5x + 5$;

(ii) $x^3 - 4x + 1$;

(iii) $x^{p-1} + x^{p-2} + \ldots + x + 1,$ where p is any prime number.

3F Analysis II

Let U, V be open sets in $\mathbb{R}^n, \mathbb{R}^m$, respectively, and let $f : U \to V$ be a map. What does it mean for f to be differentiable at a point u of U?

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the map given by

$$g(x, y) = |x| + |y| \, .$$

Prove that g is differentiable at all points (a, b) with $ab \neq 0$.

4E Further Analysis

(i) Let D be the open unit disc of radius 1 about the point $3 + 3i$. Prove that there is an analytic function $f : D \to \mathbb{C}$ such that $f(z)^2 = z$ for every $z \in D$.

(ii) Let $D' = \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Im } z = 0, \text{Re } z \leqslant 0\}$. Explain briefly why there is at most one extension of f to a function that is analytic on D'.

(iii) Deduce that f cannot be extended to an analytic function on $\mathbb{C} \setminus \{0\}$.

5A Complex Methods

State and prove the Parseval formula.

[*You may use without proof properties of convolution, as long as they are precisely stated.*]

6C Methods

Chebyshev polynomials $T_n(x)$ satisfy the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0 \qquad \text{on} \qquad [-1,1], \tag{†}$$

where n is an integer.

Recast this equation into Sturm–Liouville form and hence write down the orthogonality relationship between $T_n(x)$ and $T_m(x)$ for $n \neq m$.

By writing $x = \cos\theta$, or otherwise, show that the polynomial solutions of (†) are proportional to $\cos\left(n\cos^{-1} x\right)$.

7D Special Relativity

For a particle with energy E and momentum $(p\cos\theta, p\sin\theta, 0)$, explain why an observer moving in the x-direction with velocity v would find

$$E' = \gamma(E - p\,\cos\theta\ v)\,, \qquad p'\cos\theta' = \gamma\Big(p\,\cos\theta - E\,\frac{v}{c^2}\Big)\,, \qquad p'\sin\theta' = p\,\sin\theta\,,$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. What is the relation between E and p for a photon? Show that the same relation holds for E' and p' and that

$$\cos\theta' = \frac{\cos\theta - \dfrac{v}{c}}{1 - \dfrac{v}{c}\cos\theta}\,.$$

What happens for $v \to c$?

8C Fluid Dynamics

Write down the vorticity equation for the unsteady flow of an incompressible, inviscid fluid with no body forces acting.

Show that the flow field

$$\mathbf{u} = (-x,\, x\,\omega(t),\, z-1)$$

has uniform vorticity of magnitude $\omega(t) = \omega_0 e^t$ for some constant ω_0.

9H Statistics

Suppose that $Y_1, \ldots, Y_n$ are independent random variables, with Y_i having the normal distribution with mean βx_i and variance σ^2; here β, σ^2 are unknown and $x_1, \ldots, x_n$ are known constants.

Derive the least-squares estimate of β.

Explain carefully how to test the hypothesis $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$.

10G Optimization

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow; maximal flow; cut; capacity.

SECTION II

11E Linear Algebra

(i) Let V be an n-dimensional inner-product space over $\mathbb{C}$ and let $\alpha : V \to V$ be a Hermitian linear map. Prove that V has an orthonormal basis consisting of eigenvectors of α.

(ii) Let $\beta : V \to V$ be another Hermitian map. Prove that $\alpha\beta$ is Hermitian if and only if $\alpha\beta = \beta\alpha$.

(iii) A Hermitian map α is *positive-definite* if $\langle \alpha v, v \rangle > 0$ for every non-zero vector v. If α is a positive-definite Hermitian map, prove that there is a unique positive-definite Hermitian map β such that $\beta^2 = \alpha$.

12F Groups, Rings and Modules

Answer the following questions, fully justifying your answer in each case.

(i) Give an example of a ring in which some non-zero prime ideal is not maximal.

(ii) Prove that $\mathbb{Z}[x]$ is not a principal ideal domain.

(iii) Does there exist a field K such that the polynomial $f(x) = 1 + x + x^3 + x^4$ is irreducible in $K[x]$?

(iv) Is the ring $\mathbb{Q}[x]/(x^3 - 1)$ an integral domain?

(v) Determine all ring homomorphisms $\phi : \mathbb{Q}[x]/(x^3 - 1) \to \mathbb{C}$.

13F Analysis II

State the inverse function theorem for maps $f : U \to \mathbb{R}^2$, where U is a non-empty open subset of $\mathbb{R}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f(x, y) = (x, x^3 + y^3 - 3xy)\,.$$

Find a non-empty open subset U of $\mathbb{R}^2$ such that f is locally invertible on U, and compute the derivative of the local inverse.

Let C be the set of all points (x, y) in $\mathbb{R}^2$ satisfying

$$x^3 + y^3 - 3xy = 0\,.$$

Prove that f is locally invertible at all points of C except $(0, 0)$ and $(2^{2/3}, 2^{1/3})$. Deduce that, for each point (a, b) in C except $(0, 0)$ and $(2^{2/3}, 2^{1/3})$, there exist open intervals I, J containing a, b, respectively, such that for each x in I, there is a unique point y in J with (x, y) in C.

14E Further Analysis

(i) State and prove Rouché's theorem.

[*You may assume the principle of the argument.*]

(ii) Let $0 < c < 1$. Prove that the polynomial $p(z) = z^3 + icz + 8$ has three roots with modulus less than 3. Prove that one root α satisfies $\mathrm{Re}\,\alpha > 0, \mathrm{Im}\,\alpha > 0$; another, β, satisfies $\mathrm{Re}\,\beta > 0, \mathrm{Im}\,\beta < 0$; and the third, γ, has $\mathrm{Re}\,\gamma < 0$.

(iii) For sufficiently small c, prove that $\mathrm{Im}\,\gamma > 0$.

[*You may use results from the course if you state them precisely.*]

15A Complex Methods

(i) Show that the inverse Fourier transform of the function

$$\hat{g}(s) = \begin{cases} e^s - e^{-s}, & |s| \leqslant 1, \\ 0, & |s| \geqslant 1. \end{cases}$$

is

$$g(x) = \frac{2i}{\pi} \frac{1}{1+x^2} (x \sinh 1 \cos x - \cosh 1 \sin x)$$

(ii) Determine, by using Fourier transforms, the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

given in the strip $-\infty < x < \infty$, $0 < y < 1$, together with the boundary conditions

$$u(x,0) = g(x), \quad u(x,1) \equiv 0, \qquad -\infty < x < \infty,$$

where g has been given above.

[*You may use without proof properties of Fourier transforms.*]

16C Methods

Obtain the Green function $G(x, \xi)$ satisfying

$$G'' + \frac{2}{x}G' + k^2 G = \delta(x - \xi),$$

where k is real, subject to the boundary conditions

$$\begin{aligned} G \text{ is finite} \quad & \text{at} \quad x = 0, \\ G = 0 \quad & \text{at} \quad x = 1. \end{aligned}$$

[*Hint: You may find the substitution* $G = H/x$ *helpful.*]

Use the Green function to determine that the solution of the differential equation

$$y'' + \frac{2}{x}y' + k^2 y = 1,$$

subject to the boundary conditions

$$\begin{aligned} y \text{ is finite} \quad & \text{at} \quad x = 0, \\ y = 0 \quad & \text{at} \quad x = 1, \end{aligned}$$

is

$$y = \frac{1}{k^2}\left[1 - \frac{\sin kx}{x \sin k}\right].$$

17D Special Relativity

State how the 4-momentum p_μ of a particle is related to its energy and 3-momentum. How is p_μ related to the particle mass? For two particles with 4-momenta $p_{1\mu}$ and $p_{2\mu}$ find a Lorentz-invariant expression that gives the total energy in their centre of mass frame.

A photon strikes an electron at rest. What is the minimum energy it must have in order for it to create an electron and positron, of the same mass m_e as the electron, in addition to the original electron? Express the result in units of $m_e c^2$.

[*It may be helpful to consider the minimum necessary energy in the centre of mass frame.*]

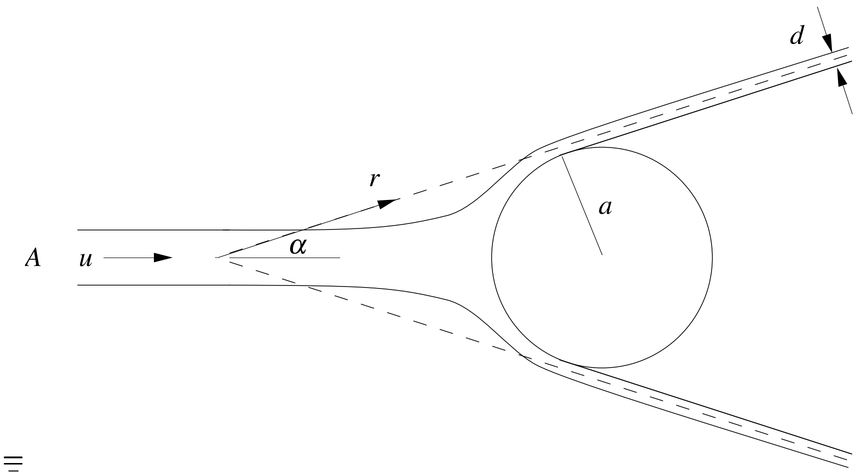
18C Fluid Dynamics

Use Euler's equation to derive the momentum integral

$$\int_S (pn_i + \rho n_j u_j u_i)\, dS = 0$$

for the steady flow $\mathbf{u} = (u_1, u_2, u_3)$ and pressure p of an inviscid,incompressible fluid of density ρ, where S is a closed surface with normal $\mathbf{n}$.

A cylindrical jet of water of area A and speed u impinges axisymmetrically on a stationary sphere of radius a and is deflected into a conical sheet of vertex angle α as shown. Gravity is being ignored.



Use a suitable form of Bernoulli's equation to determine the speed of the water in the conical sheet, being careful to state how the equation is being applied.

Use conservation of mass to show that the width $d(r)$ of the sheet far from the point of impact is given by

$$d = \frac{A}{2\pi r \sin\alpha},$$

where r is the distance along the sheet measured from the vertex of the cone.

Finally, use the momentum integral to determine the net force on the sphere in terms of ρ, u, A and α.

19H Statistics

It is required to estimate the unknown parameter θ after observing X, a single random variable with probability density function $f(x \mid \theta)$; the parameter θ has the prior distribution with density $\pi(\theta)$ and the loss function is $L(\theta, a)$. Show that the optimal Bayesian point estimate minimizes the posterior expected loss.

Suppose now that $f(x \mid \theta) = \theta e^{-\theta x}$, $x > 0$ and $\pi(\theta) = \mu e^{-\mu\theta}$, $\theta > 0$, where $\mu > 0$ is known. Determine the posterior distribution of θ given X.

Determine the optimal Bayesian point estimate of θ in the cases when

(i) $L(\theta, a) = (\theta - a)^2$, and

(ii) $L(\theta, a) = |(\theta - a)/\theta|$.

20G Optimization

For any number $c \in (0, 1)$, find the minimum and maximum values of

$$\sum_{i=1}^{n} x_i^c \, ,$$

subject to $\sum_{i=1}^{n} x_i = 1, x_1, \ldots, x_n \geqslant 0$. Find all the points $(x_1, \ldots, x_n)$ at which the minimum and maximum are attained. Justify your answer.