

## ANALYSIS II

### Uniform Convergence

The general principle of uniform convergence. A uniform limit of continuous functions is continuous.

Uniform convergence and termwise integration and differentiation of series of real-valued functions.

Local uniform convergence of power series.

### Uniform continuity and integration

Continuous functions on closed bounded intervals are uniform continuous. Review of basic facts on Riemann integration (from Analysis I). Informal discussion of integration of complex-valued and  $\mathbb{R}^n$ -valued functions of one variable; proof that

$$\left\| \int_a^b f(x) dx \right\| \leq \int_a^b \|f(x)\| dx$$

## $\mathbb{R}^n$ as a normed space

Definition of a normed space. Examples, including the Euclidean norm on  $\mathbb{R}^n$  and the uniform norm on  $C[a,b]$ . Lipschitz mapping and Lipschitz equivalence norms. The Bolzano-Weierstrass theorem in  $\mathbb{R}^n$ . Completeness. Open and closed sets. Continuity for functions between normed spaces. A continuous function on a closed bounded set in  $\mathbb{R}^n$  is uniformly continuous and has closed bounded image. All norms on a finite-dimensional are Lipschitz equivalent.

## Differentiation from $\mathbb{R}^m$ to $\mathbb{R}^n$

Definition of derivative as a linear map; elementary properties, the chain rule. Partial derivatives, continuous partial derivatives imply differentiability. Higher-order derivatives; Symmetry of mixed derivatives (assumed continuous). Taylor theorem. The mean value inequality. Path-connectedness for subsets of  $\mathbb{R}^n$ ; a function having zero derivative on a path-connected open subset is constant.

## Metric Spaces

Definition and examples, \* Metrics used in Geometry\*.

Limit, continuity, balls, neighbourhoods, open and closed sets.

## The Contradiction Mapping Theorem

The contradiction mapping theorem. Applications including the inverse function theorem (proof of continuity of inverse function, statement of differentiability). Picard's solution of differential equations.

## Appropriate books

- † J.C Burkhill and H Burkhill A Second Course in Mathematical Analysis CUP 2002
- A.F Beardon Limits: A New approach to Real Analysis Springer 1997
- D.J.H Garling A Course in Mathematical Analysis (Vol 3) CUP 2014
- † W. Rudin Principles of Mathematical Analysis McGraw Hill 1976
- W.A Sutherland Introduction to Metric and Topological Spaces Clarendon
- A.J White Real Analysis: An Introduction Addison-Wesley 1968/ 1975
- T.W Korner A Companion to analysis AMS, 2004

# Complex Analysis

## Analytic functions

Complex differentiation and the Cauchy-Riemann equations.  
Examples. Conformal mappings. Informal discussion of branch points, examples of  $\log z$  and  $z^\alpha$ .

## Contour integration and Cauchy's theorem

Contour integration (for piecewise continuously differentiable curves). Statement and proof of Cauchy theorem for star domains. Cauchy's integral formula, maximum modulus theorem, Liouville theorem, fundamental theorem of algebra. Morera's theorem.

## Expansions and singularities

Uniform convergence of analytic functions; local uniform convergence. Differentiability of a power series. Taylor and Laurent expansions. Principle of isolated zeros. Residue at an isolated singularity. Classification of isolated singularities.

## The residue theorem

Winding numbers. Residue theorem. Jordan Lemma.  
Evaluation of definite integrals by contour ~~integral~~  
integration. Rouche's theorem, principle of the argument.  
Open mapping theorem.

## Appropriate books

- L.V Ahlfors Complex Analysis McGraw-Hill 1978
- + A.F Beardon Complex Analysis Wiley
- D.J.H Garling A Course in mathematical Analysis (Vol 3) CUP 2014
- + H.A Priestley Introduction to Complex Analysis OUP 2003
- I. Stewart and D.Tall Complex Analysis CUP 1983

# Complex methods

## Analytic functions

Definition of an analytic function. Cauchy-Riemann equations. Analytic functions as conformal mappings; examples. Application to the solutions of Laplace's equations in various domains. Discussion of  $\log z$  and  $z^n$ .

## Contour integration and Cauchy theorem

[Proofs of theorems in this section will not be examined]

Contours, Contour integrals. Cauchy's theorem and Cauchy's integral formula. Taylor and Laurent Series. Zeros, poles and essential singularities.

## Residue calculus

Residue theorem, calculus of residues, Jordan's lemma. Evaluation of definite integrals by contour integration.

## Fourier and Laplace transforms

Laplace transform: definition and basic properties; inversion theorem (proof not required); convolution theorem. Examples of inversion of Fourier and Laplace transform by contour integration. Applications to differential equations.

Appropriate Books

Daniel Zill (First Course in  
Complex Analysis)

## Appropriate books

- M. J. Ablowitz and A S Fokas *Complex Variables: Introduction and applications* CUP 2003
- G. B. Arfken, H. J. Weber & F. E. Harris *Mathematical Methods for Physicists* Elsevier 2013
- G. J. O. Jameson *A First Course in Complex Functions* CRC 1970
- T Needham *Visual complex analysis* Clarendon 1998
- + H. A. Priestley *Introduction to Complex Analysis* Clarendon 1990
- + J. C. F. Riley , M. P Hobson and S .J Bence *Mathematical methods for Physics and Engineering : a Comprehensive guide* • CUP 2002

# Electromagnetism

## Electrostatics

Currents and the conservation of charge. Lorentz force law and Maxwell's equation. Gauss law. Application to spherically symmetric and cylindrically symmetric charge distributions. Point, line and surface charges. Electrostatic potentials; general charge distribution, dipoles. Electrostatic energy. Conductors.

## Magneto statics

Magnetic fields due to steady currents. Ampere's law. Simple examples. Vector potentials and the Biot-Savart law for general current distributions. Magnetic dipoles. ~~Lorentz~~ Lorentz force on current distributions and force between current-carrying wires.

## Electrodynamics

Faraday's Law of induction for fixed and moving circuits. Ohm's Law. Plane electromagnetic waves in vacuum, polarization. Electromagnetic energy and Poynting vector.

# Electromagnetism and relativity

Review of Special relativity; tensors and index notation.

Charge conservation, 4-vector potential, gauge transformation

Electromagnetic tensor. Lorentz transformations of electric  
and magnetic fields. Maxwell's equations in relativistic  
form. Lorentz force law.

## A appropriate books

- D.J Griffiths Introduction to Electrodynamics Pearson 2013  
E-M Purcell and D-J Morin Electricity and magnetism CUP 2013  
A Zangwill Modern Electromagnetism CUP 2013  
J-D Jackson Classical Electrodynamics Wiley 1975  
P Lorrain and D Corson Electromagnetism, Principles and Applications Freeman 1990.  
R. Feynman, R Leighton and M Sands The Feynmann Lectures on Physics, Vol 1 & 2  
Basic Books : 2011.

# FLUID DYNAMICS

## Parallel viscous flow

Plane Couette flow, dynamics viscosity. Momentum equation and boundary conditions. Steady flows including Poiseuille flow in a channel. Unsteady flows, kinematics viscosity, brief description of viscous boundary layers (skin depth).

## Kinematics

Material time derivatives. Conservation of mass and the kinematics boundary condition. Incompressibility; stream function for two-dimensional flow. Streamlines and path lines.

## Dynamics

Statement of Navier-Stokes momentum equation. Reynolds number, stagnation-point flow; discussion of viscous boundary layer and pressure field. Conservation of momentum; Euler momentum equation. Bernoulli's equation.

Vorticity, vorticity equation, vortex line sketching, irrotational flow remains irrotational.

## Potential flow

Velocity potential; Laplace's equation, examples of solutions in spherical and cylindrical geometry by separation of variables. Translating Sphere. Lift on a cylinder with circulation.

Expression for pressure in time-dependent potential flow with potential forces. Oscillations manometer and of a bubble.

## Geophysical flows

Linear water waves: dispersion relation, deep and shallow water, standing waves in a container, Rayleigh-Taylor instability.

Euler equations in a rotating frame. Steady geostrophic flow, pressure as streamfunction. Motion in a shallow layer, hydrostatic assumption, modified continuity equation. Conservation of potential vorticity, Rossby radius of deformation.

## Appropriate books

- D. J. Acheson Elementary Fluid dynamics Oxford University Press 1990
- G.K Batchelor An Introduction to Fluid dynamics CUP 2000
- G.M Homsy et al. Multi-Media Fluid mechanics CUP 2008
- M van Dyke An Album of Fluid motion Parabolic Press
- M.C Worster Understanding Fluid Flow CUP 2009

# Geometry

Euclidean and spherical geometry; length, lines and groups of isometries; Möbius maps and stereographic projection.

Triangulations of the sphere and the torus.\* Informal discussion of abstract smooth surfaces, orientability and statement of the classification of compact smooth surfaces.\*

Riemannian metrics on open subsets of the plane. The hyperbolic plane. Poincaré models and their metrics. The isometry group. Hyperbolic triangles and the Gauss-Bonnet theorem. The hyperboloid model.

Embedded surfaces in  $\mathbb{R}^3$ . The first fundamental form. Length and area. Examples.

~~Light~~ Length and energy. Geodesics for general Riemannian metrics as stationary points of the energy. First variation of the energy and geodesics as solutions of the corresponding Euler-Lagrange equations. Geodesic polar coordinates (informal proof of existence). Surfaces of revolution.

The second fundamental form and Gaussian curvature.

For metrics of the form  $du^2 + G(u,v)dv^2$ , expression of the curvature as  $-(\sqrt{G})_{uu}/\sqrt{G}$ . Abstract smooth surfaces and isometries, with examples. Euler numbers and Statement of Gauss-Bonnet theorem, examples and applications.

## Appropriate books

- P. M. H. Wilson Curved Spaces (CUP January 2008)
- M. Do Carmo Differential Geometry of Curves and Surfaces. Pearson, Englewood Cliffs, N.J 1976
- A Pressley Elementary Differential Geometry. Springer Undergraduate Series. Springer-Verlag London, 2001
- E. Rees Notes on Geometry Springer, 1983
- M. Reid and B. Szendroi Geometry and Topology CUP 2005

# Groups, Rings and Modules

## Groups

ParIA prerequisites. Normal subgroups, quotient groups and isomorphism theorems. Permutation groups. Groups acting on sets, permutation representations. Conjugacy classes. Centralizers and normalizers. The centre of a group. Elementary properties of finite  $p$ -groups. Examples of finite linear groups and groups arising from geometry. Simplicity of  $A_n$ . Sylow subgroups and Sylow theorems. Applications, groups of small order.

## Rings

Definition and examples of rings (commutative, with 1).  
Ideals, homomorphisms, quotient rings, isomorphism theorems  
Prime and maximal ideals. Fields. The characteristic of a field.  
Field of fractions of an integral domain.

Factorization in rings, units, primes and irreducibles.  
Unique factorization in principal ideal domains and in polynomial rings. Gauss Lemma and Eisenstein irreducibility criterion.

Rings in  $\mathbb{Z}[\alpha]$  of algebraic integers as subsets of  $\mathbb{C}$  and quotients of  $\mathbb{Z}[x]$ . Examples of Euclidean domains and uniqueness and non-uniqueness of factorization. Factorization in the ring of Gaussian integers, representation of integers as sum of two squares.

Ideals in polynomial rings. Hilbert basis theorem

## Modules

Definitions, examples of vector spaces, abelian group and vector spaces with an endomorphism. Submodules, homomorphisms, quotient modules over Euclidean domains, applications, applications to abelian groups and Jordan normal form.

## Appropriate Books

- P. M. Cohn      Classic Algebra      Wiley 2000
- P. J. Cameron      Introduction to Algebra      OUP
- J. B. Fraleigh      A First Course in Abstract Algebra      Addison Wesley 2003  
(Pearson)
- B. Hartley and T.O. Hawkes      Rings, modules and Linear Algebra: a further course in Algebra      CRC 1970
- I. Herstein      Topics in Algebra      John Wiley and Sons, 1975
- P. M. Neumann, G. A. Stoy and E.C. Thompson      Groups and Geometry. OUP 1994
- M. Artin      Algebra (Pearson 1991)

# Linear Algebra

Definition of a vector space (over  $\mathbb{R}$  or  $\mathbb{C}$ ), subspaces, the space spanned by a subset. Linear independence, bases, dimension, Direct sums and complementary subspaces. Quotient spaces

Linear maps, isomorphisms. Relation between rank and nullity. The space of linear maps from  $U$  to  $V$ , Representation by matrices. Change of Basis. Row Rank and Column Rank

Determinant and trace of a square matrix. Determinant of the product of two matrices and of the inverse matrix. Determinant of an endomorphism. The adjugate matrix.

Eigenvalues and eigenvectors. Diagonals and triangular forms. Characteristic and minimal polynomials. Cayley-Hamilton theorem over  $\mathbb{C}$ . Algebraic and geometric multiplicity of eigenvalues. Statement and illustration of Jordan normal form.

Dual of a finite-dimensional vector space, dual bases and maps. Matrix representation, rank and determinant of dual map

Bilinear forms. Matrix representation, change of basis.

Symmetric forms and their link with quadratic forms.

Diagonalisation of quadratic forms. Law of inertia, classification by rank and signature. Complex Hermitian forms

Inner product spaces, orthonormal sets, orthogonal projection,  $V = W \oplus W^\perp$ . Gram-Schmidt orthogonalisation. Adjoints. Diagonalisation of Hermitian matrices. Orthogonality of eigenvectors and properties of eigenvectors.

## Appropriate Books

- C.W Curtis Linear Algebra: an introductory approach Springer 1984  
P.R Halmos Finite-dimensional vector spaces Springer 1974  
K Hoffman and R Kunze Linear Algebra (Pearson 1971)

# MAR KOV CHAINS

## Discrete-time chains

Definition and basic properties, the transition matrix.  
Calculation of n-step transition probabilities. Communicating classes, closed classes, absorption, irreducibility. Calculation of hitting probabilities and mean hitting times; survival probability of birth and death chains. Stopping times and statement of the Strong Markov property.

Recurrence and transience; equivalence of ~~transient~~ transience and summability of n-step transition probabilities, equivalence of recurrence and certainty of returns.

Recurrence as a class property, relation with closed classes.  
Simple random walks in dimensions one, two and three

Invariant distributions, statement of existence and uniqueness. Mean return time, positive recurrence; equivalence of positive recurrence and the existence of an invariant distribution. Convergence to equilibrium, positive recurrent, aperiodic chains and proof by coupling. \*Long-run proportion of time spent in given state.\*

## Appropriate Books

- G.-R Grimmett and D. R Stirzaker Probability and Random Processes (OUP 2001)
- G.-R Grimmett and D Welsh Probability, An Introduction OUP, 2nd edition, 2014
- J. R Norris Markov Chains CUP 1997

## METHODS

### Self-adjoint ODEs (slowly)

Periodic functions. Fourier series: definition and simple properties; Parseval's theorem. Equations of second order. Self-adjoint differential operators. The Sturm-Liouville equation; eigenfunctions and eigenvalues; reality of eigenvalues and orthogonality of eigenfunctions; eigenfunctions expansion (Fourier series as prototype), approximation in mean square, statement of completeness.

## PDEs on bounded domains: separation of variables

Physical basis of Laplace's equation, the wave equation and the diffusion equation. General method of separation of variables in Cartesian, cylindrical and spherical coordinates. Legendre's equation: derivation, solutions including explicit forms of  $P_0, P_1$  and  $P_2$ , orthogonality. Bessel's equations of integer order as an example of a self-adjoint eigenvalue problem with non-trivial weight.

Examples including potentials on rectangular and circular domains on a spherical domain (easy axisymmetric case only), waves on a finite string and heat flow down a semi-infinite rod.

## Inhomogeneous ODEs: Green functions

Properties of the Dirac delta function. Initial value problems and forced problems with two fixed end points; Solution using Green's functions. Eigenfunction expansions of the delta function and Green's function.

## Fourier transforms

Fourier transforms: definition and simple properties; inversion and convolution theorems. The discrete fourier transform. Examples of application of to linear systems.

Relationship of transfer function to Green function for initial value problems.

## PDEs on unbounded domains

Classification of PDE's in two independent variables.

Well posedness.( not a very poised position huh!).

Solution by the method of characteristics. Green's functions for PDEs in 1,2 and 3 independent variables; fundamental solutions of the wave equation, Laplace's equation and the diffusion equation. The method of images. Application of the forced wave equation, Poission's equation and forced diffusion equation. Transient Solutions of diffusion problems; the error function ( erf(x) ).

## Appropriate Books

6. B Arfken, H-J Weber & F.E Harris Mathematical methods for physicists. Elsevier 2013
- M.L Boas Mathematical methods in the physical sciences Wiley 2005
- J Matthews and R L Walker Mathematical Methods ~~and~~ of physics o Benjamin Cummings 1970
- K. F Riley, M. P Hobson and S. J Bence Mathematical methods for Physics and engineering: a comprehensive guide . CUP 2002
- Erwin Kreyszig Advanced Engineering Mathematics Wiley

# Metric and topological spaces

## Metrics

Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets

## Topology

Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity.  $\ell^1$  and Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness, Subspace, quotient and product topologies

## Connectedness

Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected. Path-connectedness. Path-connected spaces. are connected but not conversely. Connected open sets in Euclidean space are path-connected.

## Compactness

Definition using open covers. Examples: finite sets and  $[0, 1]$ . Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness.

## Appropriate books

- + W.A Sutherland Introduction to metric and Topological Spaces. Clarendon  
D.J.H Garling A Course in Mathematical Analysis(Vol 2) CUP 2013  
A.J White Real analysis: an introduction Addison-Wesley 1968  
B Mendelson Introduction to Topology Dover 1990.

# Numerical Analysis

## Polynomial approximation

Interpolation by polynomials • Divided differences of functions and relations to derivatives • Orthogonal polynomials and their recurrence relations • Least square approximation by polynomial • Gaussian quadrature formulae • Peano kernel theorem and applications.

## Computation of ordinary differential equations

Euler's method and proof of convergence • Multi-step methods, including order, the root condition and the concept of convergence • Runge-Kutta schemes • Stiff equations and A-Stability

## Systems of equation and least square calculations

S-D Conte and C de Boor Elementary Numerical  
At Analysis: an algorithmic approach Mc-Graw-Hill 1990  
is a book notable for these content!

LU triangular factorization of matrices. Relation to Gaussian elimination. Column pivoting. Factorizations of symmetric and band matrices. The Newton-Raphson method for system of non-linear algebraic equation. QR factorization of rectangular matrices by Gram-Schmidt, Givens and ~~Holder~~ house-holder techniques. Application to linear least square calculations.

## Appropriate books

- S.D Conte and C. de Boor Elementary Numerical Analysis: an algorithmic approach McGraw-Hill 1980
- G.H Golub and C. Van Loan Matrix Computations Johns Hopkins University Press
- A Tsytler A First Course in the Numerical Analysis of Differential Equation CUP 2009
- E Suli and D.F Meyers An Introduction to numerical analysis CUP 2003
- A Ralston and P.Rabinowitz A First in Course in numerical course Dover 2001.
- M.J.D Powell Approximation Theory and Methods CUP 1981
- P.J Davis Interpolation and Approximation Dover 1975

# Optimisation

## Elements of convex optimisation

Convex sets and functions in  $\mathbb{R}^n$ , global and constrained optimality. Algorithms for unconstrained convex optimisation: gradient descent, Newton's algorithm.

Introduction to convex optimisation on a convex set, the barrier method. Examples.

## Lagrangian methods & duality

General information of constrained problems; the Lagrangian sufficiency theorem. Interpretation of Lagrange multipliers as shadow prices. The dual linear problem, duality theorem in a standardized case, complementary slackness, dual variables and their interpretation as shadow prices.

Relationship of the primal simplex algorithm to dual problem. Examples.

## Linear programming in the nondegenerate case

nondegenerate

Convexity of feasible region; sufficiency of extreme points.  
Standardization of problems, slack variables, equivalence  
of extreme points and basic solutions. The primal  
Simplex algorithm and the tableau. Examples.

## Applications of Linear Programming

Two person zero-sum games. Network flows; the max-flow min-cut theorem; the Ford-Fulkerson algorithm,  
the rational case. Network flows with costs, the  
transportation algorithm, relationship of dual variables  
with nodes. Examples, conditions for optimality in more  
general networks. The formulation of simple practical  
and combinatorial problems as linear programming or  
network problems.

## A appropriate books

- + M. S. Bazaraa J. J. Jarvis and H. D. Sherali Linear Programming and Network Flows . Wiley 1988
- D Luenberger Linear and Non-linear Programming ( Pearson 1984)
- S Boyd and L Vandenberghe Convex Optimization CUP 2004
- D Bertsimas J. N. Tsitsiklis Introduction to linear Optimization (Athena Scientific) (1997)

# QUANTUM MECHANICS

## Physical background

Photoelectric effect. Electrons in atoms and line spectra. Particle diffusion.

## Schrödinger equation and solution

De Broglie waves, Schrödinger equations. Superposition principle. Probability interpretation, density and currents

Stationary States. Free particle, Gaussian wave packet.

Motion in 1-dimensional particle, parity. Potential step, square well and barrier. Harmonic oscillator.

## Observables and expectation values

Position and momentum operators and expectation values.

Canonical commutation relations. Uncertainty principle

Observables and Hermitian operators. Eigenvalues and eigenfunctions. Formula for expectation value.

## Hydrogen atom

Spherically symmetric wave function for spherical well and hydrogen function.

Orbital angular momentum operators. General  
solution to hydrogen atom.

## Appropriate books

- + Feynman, Leighton and Sands Vol 3 chpt 1-3 of Feynman Lectures on physics. Addison-Wesley 1970
- + P. V. Landshoff A. J. F. Metherell and W. O Rees Essential Quantum Physics . CUP 1997
- + S Gasiorowicz Quantum Physics Wiley 2003
- + A J M Rae Quantum Mechanics . TOP Science Publishing 2002
- L I Schiff Quantum Mechanics McGraw Hill 1968

# STATISTICS (Part 1B)

## Estimation

Review of distribution and density functions, parametric families. Examples: binomial, Poisson, gamma. Sufficiency, minimal sufficiency, the Rao-Blackwell theorem. Maximum likelihood estimation. Confidence intervals. Use of prior distribution and Bayesian inference.

## Hypothesis testing

Simple examples of hypothesis testing, null and alternative hypothesis, critical region, size, power, type I and II errors, Neyman-Pearson Lemma. Significance level of outcome. Uniformly most powerful tests. Likelihood ratio, and use of generalised likelihood ratio to construct test statistics for composite hypothesis. Examples, including  $t$ -tests and  $F$ -tests. Relationship with confidence intervals. Goodness-of-fit tests and contingency tables.

## Linear Models

Derivation and joint - distribution of maximum likelihood estimators, least squares, Gauss - Markov theorem. Testing hypotheses, geometric interpretation. Examples, including simple linear regression and one-way analysis of variance (ANOVA).

## Appropriate books

- D.A Berry and B.W Lindgren *Statistics, Theory and Model* Wadsworth 1995
- (S) Casella and J.O Berger *Statistical Inference* Duxbury 2001
- M.H DeGroot and M.J Schervish *Probability and Statistics* Pearson 2001

## Variational principles

Stationary points for functions on  $\mathbb{R}^n$ . Necessary and sufficient conditions for minima and maxima. Importance of convexity. Variational problems with constraints; method of Lagrange multipliers, the Legendre Transform; need for convexity to ensure invertibility; illustrations from thermodynamics.

The idea of a functional and a functional derivatives. First variation for functionals, Euler-Lagrange equations, for both ordinary and partial differential equations. Use of Lagrange multipliers and multiplier functions

Fermat's principle; geodesics; least action principles; Lagrange's and Hamilton's equations for particles and fields. Noether theorems and first integrals, including two forms of Noether's theorem for ordinary differential equations (energy, momentum, for example). Interpretation in terms of conservation laws.

Second variation for functionals; associated eigenvalue problem.

## Appropriate books

- D.S Lemons Perfect form. Princeton University Press 1997
- C Lanczos The variational principles of Mechanics Dover 1986
- R Weinstock Calculus of Variations with Applications to Physics and engineering . Dover 1974
- I.M Gelfand and S.V. Fomin Calculus of Variations Dover 2000
- W Young and S Mandelstam Variational Principles in Dynamics and Quantum Theory. Dover 2007
- S. Hildebrandt and A Tromba Mathematics and Optimal form . Scientific American Library 1985