UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Monday 4 June 2001 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

*Each question is divided into Part **(i)** and Part **(ii)**, which may or may not be related. Candidates may attempt either or both Parts of any question, **but** must not attempt Parts from more than **SIX** questions.*

*The number of marks for each question is the same, with Part **(ii)** of each question carrying twice as many marks as Part **(i)**. Additional credit will be given for a substantially complete answer to either part.*

Begin each answer on a separate sheet.

*Write legibly and on only **one** side of the paper.*

At the end of the examination:

*Tie your answers in separate bundles, marked **A, B, C, . . . , L** according to the letter affixed to each question. (For example, **4C, 9C** should be in one bundle and **12E, 13E** in another bundle.)*

Attach a completed cover sheet to each bundle.

*Complete a master cover sheet listing **all** Parts of **all** questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Let $X = (X_n : 0 \leqslant n \leqslant N)$ be an irreducible Markov chain on the finite state space S with transition matrix $P = (p_{ij})$ and invariant distribution π. What does it mean to say that X is reversible in equilibrium?

Show that X is reversible in equilibrium if and only if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

(ii) A finite connected graph G has vertex set V and edge set E, and has neither loops nor multiple edges. A particle performs a random walk on V, moving at each step to a randomly chosen neighbour of the current position, each such neighbour being picked with equal probability, independently of all previous moves. Show that the unique invariant distribution is given by $\pi_v = d_v/(2|E|)$ where d_v is the degree of vertex v.

A rook performs a random walk on a chessboard; at each step, it is equally likely to make any of the moves which are legal for a rook. What is the mean recurrence time of a corner square. (You should give a clear statement of any general theorem used.)

[*A chessboard is an* 8×8 *square grid. A legal move is one of any length parallel to the axes.*]

2H Principles of Dynamics

(i) Show that Newton's equations in Cartesian coordinates, for a system of N particles at positions $\mathbf{x}_i(t), i = 1, 2 \ldots N$, in a potential $V(\mathbf{x}, t)$, imply Lagrange's equations in a generalised coordinate system

$$q_j = q_j(\mathbf{x}_i, t) \quad , \quad j = 1, 2 \ldots 3N;$$

that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad , \quad j = 1, 2 \ldots 3N,$$

where $L = T - V$, $T(q, \dot{q}, t)$ being the total kinetic energy and $V(q, t)$ the total potential energy.

(ii) Consider a light rod of length L, free to rotate in a vertical plane (the xz plane), but with one end P forced to move in the x-direction. The other end of the rod is attached to a heavy mass M upon which gravity acts in the negative z direction.

(a) Write down the Lagrangian for the system.

(b) Show that, if P is stationary, the rod has two equilibrium positions, one stable and the other unstable.

(c) The end at P is now forced to move with constant acceleration, $\ddot{x} = A$. Show that, once more, there is one stable equilibrium value of the angle the rod makes with the vertical, and find it.

3A Functional Analysis

(i) Define the adjoint of a bounded, linear map $u : H \to H$ on the Hilbert space H. Find the adjoint of the map

$$u : H \to H \ ; \quad x \mapsto \phi(x)a$$

where $a, b \in H$ and $\phi \in H^*$ is the linear map $x \mapsto \langle b, x\rangle$.

Now let J be an **incomplete** inner product space and $u : J \to J$ a bounded, linear map. Is it always true that there is an adjoint $u^* : J \to J$?

(ii) Let $\mathcal{H}$ be the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ on the unit disc $\mathbb{D}$ for which

$$\int\int_{\mathbb{D}} |f(z)|^2 \, dx\, dy \ < \ \infty \qquad (z = x + iy).$$

You may assume that this is a Hilbert space for the inner product:

$$\langle f, g\rangle = \int\int_{\mathbb{D}} \overline{f(z)} g(z) \ dx\, dy \ .$$

Show that the functions $u_k : z \mapsto \alpha_k z^k$ $(k = 0, 1, 2, \ldots)$ form an orthonormal sequence in $\mathcal{H}$ when the constants α_k are chosen appropriately.

Prove carefully that every function $f \in \mathcal{H}$ can be written as the sum of a convergent series $\sum_{k=0}^{\infty} f_k u_k$ in $\mathcal{H}$ with $f_k \in \mathbb{C}$.

For each smooth curve γ in the disc $\mathbb{D}$ starting from 0, prove that

$$\phi : \mathcal{H} \to \mathbb{C} \ ; \quad f \mapsto \int_{\gamma} f(z) \ dz$$

is a continuous, linear map. Show that the norm of ϕ satisfies

$$||\phi||^2 = \frac{1}{\pi} \log\left(\frac{1}{1 - |w|^2}\right) \ ,$$

where w is the endpoint of γ.

4C Groups, Rings and Fields

(i) Define the notion of a Sylow p-subgroup of a finite group G, and state a theorem concerning the number of them and the relation between them.

(ii) Show that any group of order 30 has a non-trivial normal subgroup. Is it true that every group of order 30 is commutative?

5J Electromagnetism

(i) Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a current sheet, $\mathbf{J}$, with unit normal to the sheet $\mathbf{n}$, are

$$\mathbf{n} \wedge \mathbf{B}_2 - \mathbf{n} \wedge \mathbf{B}_1 = \mu_0 \mathbf{J}.$$

State without proof the force per unit area on $\mathbf{J}$.

(ii) Conducting gas occupies the infinite slab $0 \leqslant x \leqslant a$. It carries a steady current $\mathbf{j} = (0, 0, j)$ and a magnetic field $\mathbf{B} = (0, B, 0)$ where $\mathbf{j}, \mathbf{B}$ depend only on x. The pressure is $p(x)$. The equation of hydrostatic equilibrium is $\nabla p = \mathbf{j} \wedge \mathbf{B}$. Write down the equations to be solved in this case. Show that $p + (1/2\mu_0)B^2$ is independent of x. Using the suffixes 1,2 to denote values at $x = 0, a$, respectively, verify that your results are in agreement with those of Part (i) in the case of $a \to 0$.

Suppose that

$$j(x) = \frac{\pi j_0}{2a} \sin\left(\frac{\pi x}{a}\right), \quad B_1 = 0, \quad p_2 = 0.$$

Find $B(x)$ everywhere in the slab.

6K Dynamics of Differential Equations

(i) Given a differential equation $\dot{x} = f(x)$ for $x \in \mathbb{R}^n$, explain what it means to say that the solution is given by a flow $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Define the orbit, $o(x)$, through a point x and the ω-limit set, $\omega(x)$, of x. Define also a homoclinic orbit to a fixed point x_0. Sketch a flow in $\mathbb{R}^2$ with a homoclinic orbit, and identify (without detailed justification) the ω-limit sets $\omega(x)$ for each point x in your diagram.

(ii) Consider the differential equations

$$\dot{x} = zy, \qquad \dot{y} = -zx, \qquad \dot{z} = -z^2 \,.$$

Transform the equations to polar coordinates (r, θ) in the (x, y) plane. Solve the equation for z to find $z(t)$, and hence find $\theta(t)$. Hence, or otherwise, determine (with justification) the ω-limit set for all points $(x_0, y_0, z_0) \in \mathbb{R}^3$.

7B Logic, Computation and Set Theory

(i) What is the *Halting Problem*? What is an *unsolvable* problem?

(ii) Prove that the Halting Problem is unsolvable. Is it decidable whether or not a machine halts with input zero?

8A Graph Theory

(i) Show that any graph G with minimal degree δ contains a cycle of length at least $\delta + 1$. Give examples to show that, for each possible value of δ, there is a graph with minimal degree δ but no cycle of length greater than $\delta + 1$.

(ii) Let K_N be the complete graph with N vertices labelled $v_1, v_2, \ldots, v_N$. Prove, from first principles, that there are N^{N-2} different spanning trees in K_N. In how many of these spanning trees does the vertex v_1 have degree 1?

A spanning tree in K_N is chosen at random, with each of the N^{N-2} trees being equally likely. Show that the average number of vertices of degree 1 in the random tree is approximately N/e when N is large.

Find the average degree of vertices in the random tree.

9C Number Theory

(i) Describe Euclid's algorithm.

Find, in the RSA algorithm, the deciphering key corresponding to the enciphering key $7, 527$.

(ii) Explain what is meant by a primitive root modulo an odd prime p.

Show that, if g is a primitive root modulo p, then all primitive roots modulo p are given by g^m, where $1 \leqslant m < p$ and $(m, p-1) = 1$.

Verify, by Euler's criterion, that 3 is a primitive root modulo 17. Hence find all primitive roots modulo 17.

10A Coding and Cryptography

(i) Explain briefly how and why a signature scheme is used. Describe the el Gamal scheme.

(ii) Define a cyclic code. Define the generator of a cyclic code and show that it exists. Prove a necessary and sufficient condition for a polynomial to be the generator of a cyclic code of length n.

What is the BCH code? Show that the BCH code associated with $\{\beta, \beta^2\}$, where β is a root of $X^3 + X + 1$ in an appropriate field, is Hamming's original code.

11D Stochastic Financial Models

(i) The price of the stock in the binomial model at time r, $1 \leqslant r \leqslant n$, is $S_r = S_0 \prod_{j=1}^{r} Y_j$, where $Y_1, Y_2, \ldots, Y_n$ are independent, identically-distributed random variables with $\mathbb{P}(Y_1 = u) = p = 1 - \mathbb{P}(Y_1 = d)$ and the initial price S_0 is a constant. Denote the fixed interest rate on the bank account by ρ, where $u > 1 + \rho > d > 0$, and let the discount factor $\alpha = 1/(1+\rho)$. Determine the unique value $p = \overline{p}$ for which the sequence $\{\alpha^r S_r,\ 0 \leqslant r \leqslant n\}$ is a martingale.

Explain briefly the significance of $\overline{p}$ for the pricing of contingent claims in the model.

(ii) Let T_a denote the first time that a standard Brownian motion reaches the level $a > 0$. Prove that for $t > 0$,

$$\mathbb{P}(T_a \leqslant t) = 2\left[1 - \Phi\left(a/\sqrt{t}\right)\right],$$

where Φ is the standard normal distribution function.

Suppose that A_t and B_t represent the prices at time t of two different stocks with initial prices 1 and 2, respectively; the prices evolve so that they may be represented as $A_t = e^{\sigma_1 X_t + \mu t}$ and $B_t = 2e^{\sigma_2 Y_t + \mu t}$, respectively, where $\{X_t\}_{t \geqslant 0}$ and $\{Y_t\}_{t \geqslant 0}$ are independent standard Brownian motions and σ_1, σ_2 and μ are constants. Let T denote the first time, if ever, that the prices of the two stocks are the same. Determine $\mathbb{P}(T \leqslant t)$, for $t > 0$.

12E Principles of Statistics

(i) What are the main approaches by which prior distributions are specified in Bayesian inference?

Define the risk function of a decision rule d. Given a prior distribution, define what is meant by a Bayes decision rule and explain how this is obtained from the posterior distribution.

(ii) Dashing late into King's Cross, I discover that Harry must have already boarded the Hogwarts Express. I must therefore make my own way onto platform nine and three-quarters. Unusually, there are two guards on duty, and I will ask one of them for directions. It is safe to assume that one guard is a Wizard, who will certainly be able to direct me, and the other a Muggle, who will certainly not. But which is which? Before choosing one of them to ask for directions to platform nine and three-quarters, I have just enough time to ask one of them "Are you a Wizard?", and on the basis of their answer I must make my choice of which guard to ask for directions. I know that a Wizard will answer this question truthfully, but that a Muggle will, with probability $\frac{1}{3}$, answer it untruthfully.

Failure to catch the Hogwarts Express results in a loss which I measure as 1000 galleons, there being no loss associated with catching up with Harry on the train.

Write down an exhaustive set of non-randomised decision rules for my problem and, by drawing the associated risk set, determine my minimax decision rule.

My prior probability is $\frac{2}{3}$ that the guard I ask "Are you a Wizard?" is indeed a Wizard. What is my Bayes decision rule?

13E Computational Statistics and Statistical Modelling

(i) Assume that the n-dimensional observation vector Y may be written as

$$Y = X\beta + \epsilon \quad ,$$

where X is a given $n \times p$ matrix of rank p, β is an unknown vector, and

$$\epsilon \sim N_n(0, \sigma^2 I).$$

Let $Q(\beta) = (Y - X\beta)^T(Y - X\beta)$. Find $\widehat{\beta}$, the least-squares estimator of β, and show that

$$Q(\widehat{\beta}) = Y^T(I - H)Y \quad ,$$

where H is a matrix that you should define.

(ii) Show that $\sum_i H_{ii} = p$. Show further for the special case of

$$Y_i = \beta_1 + \beta_2 x_i + \beta_3 z_i + \epsilon_i, \quad 1 \leqslant i \leqslant n,$$

where $\Sigma x_i = 0, \quad \Sigma z_i = 0$, that

$$H \quad = \quad \frac{1}{n}\mathbf{1}\mathbf{1}^T + axx^T + b(xz^T + zx^T) + czz^T \quad ;$$

here, $\mathbf{1}$ is a vector of which every element is one, and a, b, c, are constants that you should derive.

Hence show that, if $\widehat{Y} = X\widehat{\beta}$ is the vector of fitted values, then

$$\frac{1}{\sigma^2}\mathrm{var}(\widehat{Y}_i) = \frac{1}{n} + ax_i^2 + 2bx_iz_i + cz_i^2, \quad 1 \leqslant i \leqslant n.$$

14F Quantum Physics

(i) A spinless quantum mechanical particle of mass m moving in two dimensions is confined to a square box with sides of length L. Write down the energy eigenfunctions for the particle and the associated energies.

Show that, for large L, the number of states in the energy range $E \to E + dE$ is $\rho(E)dE$, where

$$\rho(E) = \frac{mL^2}{2\pi\hbar^2}.$$

(ii) If, instead, the particle is an electron with magnetic moment μ moving in an external magnetic field, H, show that

$$\begin{aligned} \rho(E) =& \frac{mL^2}{2\pi\hbar^2}, \qquad & -\mu H < E < \mu H \\ =& \frac{mL^2}{\pi\hbar^2}, \qquad & \mu H < E. \end{aligned}$$

Let there be N electrons in the box. Explain briefly how to construct the ground state of the system. Let E_F be the Fermi energy. Show that when $E_F > \mu H$,

$$N = \frac{mL^2}{\pi\hbar^2}\, E_F.$$

Show also that the magnetic moment, M, of the system in the ground state is

$$M = \frac{\mu^2 mL^2}{\pi\hbar^2}\, H,$$

and that the ground state energy is

$$\frac{1}{2}\frac{\pi\hbar^2}{mL^2}\, N^2 - \frac{1}{2} MH.$$

15J General Relativity

(i) The metric of any two-dimensional curved space, rotationally symmetric about a point P, can by suitable choice of coordinates be written locally in the form

$$ds^2 = e^{2\phi(r)}(dr^2 + r^2 d\theta^2),$$

where $r = 0$ at P, $r > 0$ away from P, and $0 \leqslant \theta < 2\pi$. Labelling the coordinates as $(x^1, x^2) = (r, \theta)$, show that the Christoffel symbols $\Gamma^1_{12}, \Gamma^2_{11}$ and Γ^2_{22} are each zero, and compute the non-zero Christoffel symbols $\Gamma^1_{11}, \Gamma^1_{22}$ and $\Gamma^2_{12} = \Gamma^2_{21}$.

The Ricci tensor R_{ab} $(a, b = 1, 2)$ is defined by

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^d_{ac}\Gamma^c_{bd},$$

where a comma denotes a partial derivative. Show that $R_{12} = 0$ and that

$$R_{11} = -\phi'' - r^{-1}\phi', \qquad R_{22} = r^2 R_{11}.$$

(ii) Suppose further that, in a neighbourhood of P, the Ricci scalar R takes the constant value -2. Find a second order differential equation, which you should denote by $(*)$, for $\phi(r)$.

This space of constant Ricci scalar can, by a suitable coordinate transformation $r \to \chi(r)$, leaving θ invariant, be written locally as

$$ds^2 = d\chi^2 + \sinh^2 \chi d\theta^2$$

By studying this coordinate transformation, or otherwise, find $\cosh\chi$ and $\sinh\chi$ in terms of r (up to a constant of integration). Deduce that

$$e^{\phi(r)} = \frac{2A}{(1 - A^2 r^2)} \quad , \quad (0 \leqslant Ar < 1),$$

where A is a positive constant and verify that your equation $(*)$ for ϕ holds.

[*Note that*

$$\int \frac{d\chi}{\sinh\chi} = \text{const.} + \frac{1}{2}\log\bigl(\cosh\chi - 1\bigr) - \frac{1}{2}\log\bigl(\cosh\chi + 1\bigr).\Bigr]$$

16J Statistical Physics and Cosmology

(i) Introducing the concept of a co-moving distance co-ordinate, explain briefly how the velocity of a galaxy in an isotropic and homogeneous universe is determined by the scale factor $a(t)$. How is the scale factor related to the Hubble constant H_0?

A homogeneous and isotropic universe has an energy density $\rho(t)c^2$ and a pressure $P(t)$. Use the relation $dE = -PdV$ to derive the "fluid equation"

$$\dot{\rho} = -3\left(\rho + \frac{P}{c^2}\right)\left(\frac{\dot{a}}{a}\right),$$

where the overdot indicates differentiation with respect to time, t. Given that $a(t)$ satisfies the "acceleration equation"

$$\ddot{a} = -\frac{4\pi G}{3}\, a\left(\rho + \frac{3P}{c^2}\right),$$

show that the quantity

$$k = c^{-2}\left(\frac{8\pi G}{3}\rho a^2 \; - \dot{a}^2\right)$$

is time-independent.

The pressure P is related to ρ by the "equation of state"

$$P = \sigma\rho c^2, \quad |\sigma| < 1\ .$$

Given that $a(t_0) = 1$, find $a(t)$ for $k = 0$, and hence show that $a(0) = 0$.

(ii) What is meant by the expression "the Hubble time"?

Assuming that $a(0) = 0$ and $a(t_0) = 1$, where t_0 is the time now (of the current cosmological era), obtain a formula for the radius R_0 of the observable universe.

Given that

$$a(t) \quad = \quad \left(\frac{t}{t_0}\right)^{\alpha}$$

for constant α, find the values of α for which R_0 is finite. Given that R_0 is finite, show that the age of the universe is less than the Hubble time. Explain briefly, and qualitatively, why this result is to be expected as long as

$$\rho + 3\frac{P}{c^2} > 0.$$

17F Symmetries and Groups in Physics

(i) Let $h : G \to G'$ be a surjective homomorphism between two groups, G and G'. If $D' : G' \to GL(\mathbb{C}^n)$ is a representation of G', show that $D(g) = D'\left(h(g)\right)$ for $g \in G$ is a representation of G and, if D' is irreducible, show that D is also irreducible. Show further that $\widetilde{D}(\widetilde{g}) = D'\left(\widetilde{h}(\widetilde{g})\right)$ is a representation of $G/\ker(h)$, where $\widetilde{h}(\widetilde{g}) = h(g)$ for $g \in G$ and $\widetilde{g} \in G/\ker(h)$ (with $g \in \widetilde{g}$). Deduce that the characters $\chi, \widetilde{\chi}, \chi'$ of $D, \widetilde{D}, D'$, respectively, satisfy

$$\chi(g) = \widetilde{\chi}(\widetilde{g}) = \chi'(h(g)) \,.$$

(ii) D_4 is the symmetry group of rotations and reflections of a square. If c is a rotation by $\pi/2$ about the centre of the square and b is a reflection in one of its symmetry axes, then $D_4 = \{e, c, c^2, c^3, b, bc, bc^2, bc^3\}$. Given that the conjugacy classes are $\{e\}$ $\{c^2\}$, $\{c, c^3\}$ $\{b, bc^2\}$ and $\{bc, bc^3\}$ derive the character table of D_4.

Let H_0 be the Hamiltonian of a particle moving in a central potential. The $SO(3)$ symmetry ensures that the energy eigenvalues of H_0 are the same for all the angular momentum eigenstates in a given irreducible $SO(3)$ representation. Therefore, the energy eigenvalues of H_0 are labelled E_{nl} with $n \in \mathbb{N}$ and $l \in \mathbb{N}_0$, $l < n$. Assume now that in a crystal the symmetry is reduced to a D_4 symmetry by an additional term H_1 of the total Hamiltonian, $H = H_0 + H_1$. Find how the H_0 eigenstates in the $SO(3)$ irreducible representation with $l = 2$ (the D-wave orbital) decompose into irreducible representations of H. You may assume that the character, $g(\theta)$, of a group element of $SO(3)$, in a representation labelled by l is given by

$$\chi(g_\theta) = 1 + 2\cos\theta + 2\cos(2\theta) + \ldots + 2\cos(l\theta),$$

where θ is a rotation angle and $l(l+1)$ is the eigenvalue of the total angular momentum, $\mathbf{L}^2$.

18H Transport Processes

(i) The diffusion equation for a spherically-symmetric concentration field $C(r,t)$ is

$$C_t = \frac{D}{r^2}\left(r^2 C_r\right)_r , \tag{1}$$

where r is the radial coordinate. Find and sketch the similarity solution to (1) which satisfies $C \to 0$ as $r \to \infty$ and $\int_0^\infty 4\pi r^2 C(r,t)dr = M =$ constant, assuming it to be of the form

$$C = \frac{M}{(Dt)^a}F(\eta), \quad \eta = \frac{r}{(Dt)^b},$$

where a and b are numbers to be found.

(ii) A two-dimensional piece of heat-conducting material occupies the region $a \leqslant r \leqslant b,\ -\pi/2 \leqslant \theta \leqslant \pi/2$ (in plane polar coordinates). The surfaces $r = a,\ \theta = -\pi/2,\ \theta = \pi/2$ are maintained at a constant temperature T_1; at the surface $r = b$ the boundary condition on temperature $T(r,\theta)$ is

$$T_r + \beta T = 0,$$

where $\beta > 0$ is a constant. Show that the temperature, which satisfies the steady heat conduction equation

$$T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta} = 0,$$

is given by a Fourier series of the form

$$\frac{T}{T_1} = K + \sum_{n=0}^{\infty}\cos\left(\alpha_n\theta\right)\left[A_n\left(\frac{r}{a}\right)^{2n+1} + B_n\left(\frac{a}{r}\right)^{2n+1}\right],$$

where $K,\ \alpha_n,\ A_n,\ B_n$ are to be found.

In the limits $a/b \ll 1$ and $\beta b \ll 1$, show that

$$\int_{-\pi/2}^{\pi/2} T_r r d\theta \approx -\pi\beta b T_1,$$

given that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Explain how, in these limits, you could have obtained this result much more simply.

19L Theoretical Geophysics

(i) From the surface of a flat Earth, an explosive source emits P-waves downward into a horizontal homogeneous elastic layer of uniform thickness h and P-wave speed α_1 overlying a lower layer of infinite depth and P-wave speed α_2, where $\alpha_2 > \alpha_1$. A line of seismometers on the surface records the travel time t as a function of distance x from the source for the various arrivals along different ray paths.

Sketch the ray paths associated with the direct, reflected and head waves arriving at a given position. Calculate the travel times $t(x)$ of the direct and reflected waves, and sketch the corresponding travel-time curves. Hence explain how to estimate α_1 and h from the recorded arrival times. Explain briefly why head waves are only observed beyond a minimum distance x_c from the source and why they have a travel-time curve of the form $t = t_c + (x - x_c)/\alpha_2$ for $x > x_c$.
[*You need not calculate x_c or t_c.*]

(ii) A plane SH-wave in a homogeneous elastic solid has displacement proportional to $\exp[i(kx+mz-\omega t)]$. Express the slowness vector $\mathbf{s}$ in terms of the wavevector $\mathbf{k} = (k, 0, m)$ and ω. Deduce an equation for m in terms of k, ω and the S-wave speed β.

A homogeneous elastic layer of uniform thickness h, S-wave speed β_1 and shear modulus μ_1 has a stress-free surface $z = 0$ and overlies a lower layer of infinite depth, S-wave speed β_2 ($> \beta_1$) and shear modulus μ_2. Find the vertical structure of Love waves with displacement proportional to $\exp[i(kx - \omega t)]$, and show that the horizontal phase speed c obeys

$$\tan\left[\left(\frac{1}{\beta_1^2} - \frac{1}{c^2}\right)^{1/2} \omega h\right] = \frac{\mu_2}{\mu_1}\left(\frac{1/c^2 - 1/\beta_2^2}{1/\beta_1^2 - 1/c^2}\right)^{1/2} .$$

By sketching both sides of the equation as a function of c in $\beta_1 \leqslant c \leqslant \beta_2$ show that at least one mode exists for every value of ω.

20K Numerical Analysis

(i) Let A be a symmetric $n \times n$ matrix such that

$$A_{k,k} > \sum_{\substack{l=1 \\ l \neq k}}^{n} |A_{k,l}| \qquad 1 \leqslant k \leqslant n.$$

Prove that it is positive definite.

(ii) Prove that both Jacobi and Gauss-Seidel methods for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, where the matrix A obeys the conditions of (i), converge.

[*You may quote the Householder-John theorem without proof.*]

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Tuesday 5 June 2001 9.00 to 12.00

PAPER 2

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It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) The fire alarm in Mill Lane is set off at random times. The probability of an alarm during the time-interval $(u, u+h)$ is $\lambda(u)h + o(h)$ where the 'intensity function' $\lambda(u)$ may vary with the time u. Let $N(t)$ be the number of alarms by time t, and set $N(0) = 0$. Show, subject to reasonable extra assumptions to be stated clearly, that $p_i(t) = P(N(t) = i)$ satisfies

$$p_0'(t) = -\lambda(t)p_0(t), \quad p_i'(t) = \lambda(t)\{p_{i-1}(t) - p_i(t)\}, \quad i \geqslant 1.$$

Deduce that $N(t)$ has the Poisson distribution with parameter $\Lambda(t) = \int_0^t \lambda(u)du$.

(ii) The fire alarm in Clarkson Road is different. The number $M(t)$ of alarms by time t is such that

$$P(M(t+h) = m+1 \mid M(t) = m) = \lambda_m h + o(h) \ ,$$

where $\lambda_m = \alpha m + \beta$, $m \geqslant 0$, and $\alpha, \beta > 0$. Show, subject to suitable extra conditions, that $p_m(t) = P(M(t) = m)$ satisfies a set of differential-difference equations to be specified. Deduce without solving these equations in their entirety that $M(t)$ has mean $\beta(e^{\alpha t} - 1)/\alpha$, and find the variance of $M(t)$.

2H Principles of Dynamics

(i) An axially symmetric top rotates freely about a fixed point O on its axis. The principal moments of inertia are A, A, C and the centre of gravity G is a distance h from O.

Define the three Euler angles θ, ϕ and ψ, specifying the orientation of the top. Use Lagrange's equations to show that there are three conserved quantities in the motion. Interpret them physically.

(ii) Initially the top is spinning with angular speed n about OG, with OG vertical, before it is slightly disturbed.

Show that, in the subsequent motion, θ stays close to zero if $C^2n^2 > 4mghA$, but if this condition fails then θ attains a maximum value given approximately by

$$\cos\theta \approx \frac{C^2n^2}{2mghA} - 1.$$

Why is this only an approximation?

3A Functional Analysis

(i) State the Stone-Weierstrass theorem for complex-valued functions. Use it to show that the trigonometric polynomials are dense in the space $C(\mathbb{T})$ of continuous, complex-valued functions on the unit circle $\mathbb{T}$ with the uniform norm.

Show further that, for $f \in C(\mathbb{T})$, the nth Fourier coefficient

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta$$

tends to 0 as $|n|$ tends to infinity.

(ii) **(a)** Let X be a normed space with the property that the series $\sum_{n=1}^{\infty} x_n$ converges whenever (x_n) is a sequence in X with $\sum_{n=1}^{\infty} ||x_n||$ convergent. Show that X is a Banach space.

(b) Let K be a compact metric space and L a closed subset of K. Let $R : C(K) \to C(L)$ be the map sending $f \in C(K)$ to its restriction $R(f) = f|L$ to L. Show that R is a bounded, linear map and that its image is a subalgebra of $C(L)$ separating the points of L.

Show further that, for each function g in the image of R, there is a function $f \in C(K)$ with $R(f) = g$ and $||f||_\infty = ||g||_\infty$. Deduce that every continuous, complex-valued function on L can be extended to a continuous function on all of K.

4C Groups, Rings and Fields

(i) Show that the ring $k = \mathbf{F}_2[X]/(X^2 + X + 1)$ is a field. How many elements does it have?

(ii) Let k be as in (i). By considering what happens to a chosen basis of the vector space k^2, or otherwise, find the order of the groups $GL_2(k)$ and $SL_2(k)$.

By considering the set of lines in k^2, or otherwise, show that $SL_2(k)$ is a subgroup of the symmetric group S_5, and identify this subgroup.

5J Electromagnetism

(i) Write down the expression for the electrostatic potential $\phi(\mathbf{r})$ due to a distribution of charge $\rho(\mathbf{r})$ contained in a volume V. Perform the multipole expansion of $\phi(\mathbf{r})$ taken only as far as the dipole term.

(ii) If the volume V is the sphere $|\mathbf{r}| \leqslant a$ and the charge distribution is given by

$$\rho(\mathbf{r}) = \begin{cases} r^2 \cos\theta & r \leqslant a \\ 0 & r > a \end{cases},$$

where r, θ are spherical polar coordinates, calculate the charge and dipole moment. Hence deduce ϕ as far as the dipole term.

Obtain an exact solution for $r > a$ by solving the boundary value problem using trial solutions of the forms

$$\phi = \frac{A\cos\theta}{r^2} \quad \text{for} \quad r > a,$$

and

$$\phi = Br\cos\theta + Cr^4\cos\theta \quad \text{for} \quad r < a.$$

Show that the solution obtained from the multipole expansion is in fact exact for $r > a$.

[*You may use without proof the result*

$$\nabla^2(r^k \cos\theta) = (k+2)(k-1)r^{k-2}\cos\theta, \qquad k \in \mathbb{N}.]$$

6K Dynamics of Differential Equations

(i) Define a Liapounov function for a flow ϕ on $\mathbb{R}^n$. Explain what it means for a fixed point of the flow to be Liapounov stable. State and prove Liapounov's first stability theorem.

(ii) Consider the damped pendulum

$$\ddot{\theta} + k\dot{\theta} + \sin\theta = 0,$$

where $k > 0$. Show that there are just two fixed points (considering the phase space as an infinite cylinder), and that one of these is the origin and is Liapounov stable. Show further that the origin is asymptotically stable, and that the the ω-limit set of each point in the phase space is one or other of the two fixed points, justifying your answer carefully.

[*You should state carefully any theorems you use in your answer, but you need not prove them.*]

7C Geometry of Surfaces

(i) Give the definition of the curvature $\kappa(t)$ of a plane curve $\gamma : [a,b] \longrightarrow \mathbf{R}^2$. Show that, if $\gamma : [a,b] \longrightarrow \mathbf{R}^2$ is a simple closed curve, then

$$\int_a^b \kappa(t)\, \|\dot{\gamma}(t)\|\, dt = 2\pi.$$

(ii) Give the definition of a geodesic on a parametrized surface in $\mathbf{R}^3$. Derive the differential equations characterizing geodesics. Show that a great circle on the unit sphere is a geodesic.

8A Graph Theory

(i) Prove that any graph G drawn on a compact surface S with negative Euler characteristic $E(S)$ has a vertex colouring that uses at most

$$h = \lfloor \tfrac{1}{2}(7 + \sqrt{49 - 24E(S)}) \rfloor$$

colours.

Briefly discuss whether the result is still true when $E(S) \geqslant 0$.

(ii) Prove that a graph G is k edge-connected if and only if the removal of no set of less than k edges from G disconnects G.

[*If you use any form of Menger's theorem, you must prove it.*]

Let G be a minimal example of a graph that requires $k+1$ colours for a vertex colouring. Show that G must be k edge-connected.

9A Coding and Cryptography

(i) Give brief answers to the following questions.

(a) What is a stream cypher?

(b) Explain briefly why a one-time pad is safe if used only once but becomes unsafe if used many times.

(c) What is a feedback register of length d? What is a linear feedback register of length d?

(d) A cypher stream is given by a linear feedback register of known length d. Show that, given plain text and cyphered text of length $2d$, we can find the complete cypher stream.

(e) State and prove a similar result for a general feedback register.

(ii) Describe the construction of a Reed-Muller code. Establish its information rate and its weight.

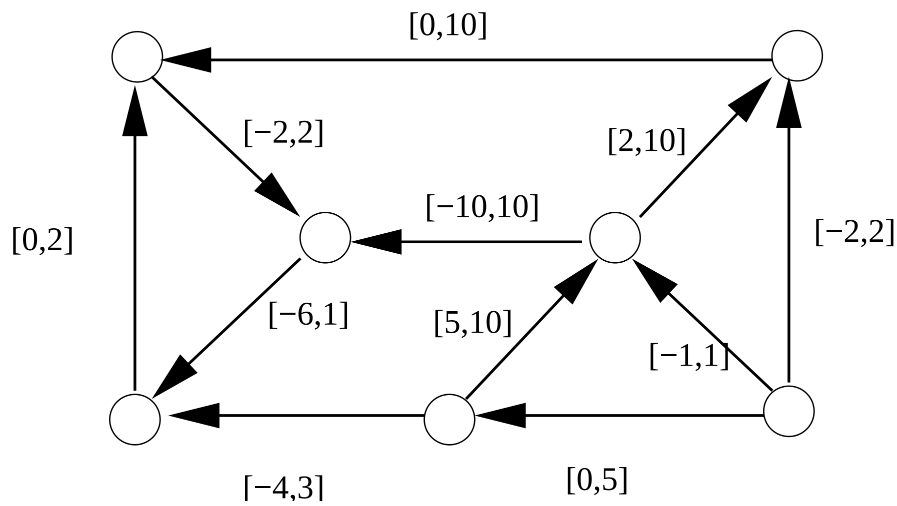
10E Algorithms and Networks

(i) Let G be a directed network with nodes N and arcs A. Let $S \subset N$ be a subset of the nodes, x be a flow on G, and y be the divergence of x. Describe carefully what is meant by a *cut* $Q = [S, N \backslash S]$. Define the *arc-cut incidence* e_Q, and the *flux of x across Q*. Define also the *divergence $y(S)$ of S*. Show that $y(S) = x.e_Q$.

Now suppose that capacity constraints are specified on each of the arcs. Define the *upper cut capacity $c^+(Q)$ of Q*. State the feasible distribution problem for a specified divergence b, and show that the problem only has a solution if $b(N) = 0$ and $b(S) \leqslant c^+(Q)$ for all cuts $Q = [S, N \backslash S]$.

(ii) Describe an algorithm to find a feasible distribution given a specified divergence b and capacity constraints on each arc. Explain what happens when no feasible distribution exists.

Illustrate the algorithm by either finding a feasible circulation, or demonstrating that one does not exist, in the network given below. Arcs are labelled with capacity constraint intervals.



11E Principles of Statistics

(i) Let $X_1, \ldots, X_n$ be independent, identically-distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a minimal sufficient statistic for μ.

Let $T_1 = n^{-1} \sum_{i=1}^n X_i$ and $T_2 = \sqrt{n^{-1} \sum_{i=1}^n X_i^2}$. Write down the distribution of X_i/μ, and hence show that $Z = T_1/T_2$ is ancillary. Explain briefly why the Conditionality Principle would lead to inference about μ being drawn from the conditional distribution of T_2 given Z.

What is the maximum likelihood estimator of μ?

(ii) Describe briefly the Bayesian approach to predictive inference.

Let $Z_1, \ldots, Z_n$ be independent, identically-distributed $N(\mu, \sigma^2)$ random variables, with μ, σ^2 both unknown. Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2 based on $Z_1, \ldots, Z_n$, and state, without proof, their joint distribution.

Suppose that it is required to construct a prediction interval $I_{1-\alpha} \equiv I_{1-\alpha}(Z_1, \ldots, Z_n)$ for a future, independent, random variable Z_0 with the same $N(\mu, \sigma^2)$ distribution, such that

$$P(Z_0 \in I_{1-\alpha}) = 1 - \alpha,$$

with the probability over the *joint* distribution of $Z_0, Z_1, \ldots, Z_n$. Let

$$I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2) = \left[\bar{Z}_n - z_{\alpha/2}\sigma\sqrt{1+1/n},\ \bar{Z}_n + z_{\alpha/2}\sigma\sqrt{1+1/n} \right],$$

where $\bar{Z}_n = n^{-1} \sum_{i=1}^n Z_i$, and $\Phi(z_\beta) = 1 - \beta$, with Φ the distribution function of $N(0,1)$.

Show that $P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2)) = 1 - \alpha$.

By considering the distribution of $(Z_0 - \bar{Z}_n)/\left(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}}\right)$, or otherwise, show that

$$P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) < 1 - \alpha,$$

and show how to construct an interval $I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)$ with

$$P(Z_0 \in I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) = 1 - \alpha.$$

[*Hint: if Y has the t-distribution with m degrees of freedom and $t_\beta^{(m)}$ is defined by $P(Y < t_\beta^{(m)}) = 1 - \beta$ then $t_\beta > z_\beta$ for $\beta < \frac{1}{2}$.*]

12E Computational Statistics and Statistical Modelling

(i) Suppose that $Y_1, \ldots, Y_n$ are independent random variables, and that Y_i has probability density function

$$f(y_i|\theta_i, \phi) = \exp[(y_i\theta_i - b(\theta_i))/\phi + c(y_i, \phi)].$$

Assume that $E(Y_i) = \mu_i$, and that $g(\mu_i) = \beta^T x_i$, where $g(\cdot)$ is a known 'link' function, $x_1, \ldots, x_n$ are known covariates, and β is an unknown vector. Show that

$$\mathbb{E}(Y_i) = b'(\theta_i),\ \text{var}(Y_i) = \phi b''(\theta_i) = V_i,\ \text{say},$$

and hence

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} \frac{(y_i - \mu_i)x_i}{g'(\mu_i)V_i},\ \text{where } l = l(\beta, \phi) \text{ is the log-likelihood.}$$

(ii) The table below shows the number of train miles (in millions) and the number of collisions involving British Rail passenger trains between 1970 and 1984. Give a detailed interpretation of the R output that is shown under this table:

	year	collisions	miles
1	1970	3	281
2	1971	6	276
3	1972	4	268
4	1973	7	269
5	1974	6	281
6	1975	2	271
7	1976	2	265
8	1977	4	264
9	1978	1	267
10	1979	7	265
11	1980	3	267
12	1981	5	260
13	1982	6	231
14	1983	1	249

```
Call:

glm(formula = collisions ~ year + log(miles), family = poisson)

Coefficients:
                 Estimate  Std. Error  z value  Pr(>|z|)
     (Intercept) 127.14453   121.37796    1.048     0.295
     year         -0.05398     0.05175   -1.043     0.297
     log(miles)   -3.41654     4.18616   -0.816     0.414

(Dispersion parameter for poisson family taken to be 1)

     Null deviance:  15.937 on 13 degrees of freedom

Residual deviance:  14.843 on 11 degrees of freedom

Number of Fisher Scoring iterations:  4
```

13F Foundations of Quantum Mechanics

(i) Hermitian operators $\hat{x}$, $\hat{p}$, satisfy $[\hat{x},\hat{p}] = i\hbar$. The eigenvectors $|p\rangle$, satisfy $\hat{p}|p\rangle = p|p\rangle$ and $\langle p'|p\rangle = \delta(p'-p)$. By differentiating with respect to b verify that

$$e^{-ib\hat{x}/\hbar}\hat{p}\,e^{ib\hat{x}/\hbar} = \hat{p}+b$$

and hence show that

$$e^{ib\hat{x}/\hbar}|p\rangle = |p+b\rangle.$$

Show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\,\langle p|\psi\rangle$$

and

$$\langle p|\hat{p}|\psi\rangle = p\,\langle p|\psi\rangle\,.$$

(ii) A quantum system has Hamiltonian $H = H_0 + H_1$, where H_1 is a small perturbation. The eigenvalues of H_0 are ϵ_n. Give (without derivation) the formulae for the first order and second order perturbations in the energy level of a non-degenerate state. Suppose that the rth energy level of H_0 has j degenerate states. Explain how to determine the eigenvalues of H corresponding to these states to first order in H_1.

In a particular quantum system an orthonormal basis of states is given by $|n_1,n_2\rangle$, where n_i are integers. The Hamiltonian is given by

$$H = \sum_{n_1,n_2}(n_1^2+n_2^2)\,|n_1,n_2\rangle\langle n_1,n_2| + \sum_{n_1,n_2,n'_1,n'_2}\lambda_{|n_1-n'_1|,|n_2-n'_2|}\,|n_1,n_2\rangle\langle n'_1,n'_2|\,,$$

where $\lambda_{r,s} = \lambda_{s,r}$, $\lambda_{0,0} = 0$ and $\lambda_{r,s} = 0$ unless r and s are both even.

Obtain an expression for the ground state energy to second order in the perturbation, $\lambda_{r,s}$. Find the energy eigenvalues of the first excited state to first order in the perturbation. Determine a matrix (which depends on two independent parameters) whose eigenvalues give the first order energy shift of the second excited state.

14F Quantum Physics

(i) Each particle in a system of N identical fermions has a set of energy levels, E_i, with degeneracy g_i, where $1 \leq i < \infty$. Explain why, in thermal equilibrium, the average number of particles with energy E_i is

$$N_i = \frac{g_i}{e^{\beta(E_i-\mu)}+1}.$$

The physical significance of the parameters β and μ should be made clear.

(ii) A simple model of a crystal consists of a linear array of sites with separation a. At the nth site an electron may occupy either of two states with probability amplitudes b_n and c_n, respectively. The time-dependent Schrödinger equation governing the amplitudes gives

$$i\hbar\dot{b}_n = E_0 b_n - A(b_{n+1} + b_{n-1} + c_{n+1} + c_{n-1}),$$

$$i\hbar\dot{c}_n = E_1 c_n - A(b_{n+1} + b_{n-1} + c_{n+1} + c_{n-1}),$$

where $A > 0$.

By examining solutions of the form

$$\begin{pmatrix} b_n \\ c_n \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} e^{i(kna - Et/\hbar)},$$

show that the energies of the electron fall into two bands given by

$$E = \frac{1}{2}(E_0 + E_1 - 4A\cos ka) \pm \frac{1}{2}\sqrt{(E_0 - E_1)^2 + 16A^2\cos^2 ka}.$$

Describe briefly how the energy band structure for electrons in real crystalline materials can be used to explain why they are insulators, conductors or semiconductors.

15J General Relativity

(i) Show that the geodesic equation follows from a variational principle with Lagrangian

$$L \quad = \quad g_{ab}\dot{x}^a\dot{x}^b$$

where the path of the particle is $x^a(\lambda)$, and λ is an affine parameter along that path.

(ii) The Schwarzschild metric is given by

$$ds^2 = dr^2\left(1 - \frac{2M}{r}\right)^{-1} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \left(1 - \frac{2M}{r}\right)dt^2.$$

Consider a photon which moves within the equatorial plane $\theta = \frac{\pi}{2}$. Using the above Lagrangian, or otherwise, show that

$$\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right) = E, \quad \text{and} \quad r^2\left(\frac{d\phi}{d\lambda}\right) = h,$$

for constants E and h. Deduce that

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right). \qquad (*)$$

Assume further that the photon approaches from infinity. Show that the impact parameter b is given by

$$b = \frac{h}{E} \quad .$$

By considering the equation $(*)$, or otherwise

(a) show that, if $b^2 > 27M^2$, the photon is deflected but not captured by the black hole;

(b) show that, if $b^2 < 27M^2$, the photon is captured;

(c) describe, with justification, the qualitative form of the photon's orbit in the case $b^2 = 27M^2$.

16L Theoretical Geophysics

(i) In a reference frame rotating with constant angular velocity $\mathbf{\Omega}$ the equations of motion for an inviscid, incompressible fluid of density ρ in a gravitational field $\mathbf{g} = -\nabla\Phi$ are

$$\rho\frac{D\mathbf{u}}{Dt} + 2\rho\mathbf{\Omega} \wedge \mathbf{u} = -\nabla p + \rho\mathbf{g} \ , \qquad \nabla \cdot \mathbf{u} = 0 \ .$$

Define the Rossby number and explain what is meant by geostrophic flow.

Derive the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\mathbf{\Omega}) \cdot \nabla\mathbf{u} + \frac{\nabla\rho \wedge \nabla p}{\rho^2}.$$

[*Recall that* $\mathbf{u} \cdot \nabla\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}^2) - \mathbf{u} \wedge (\nabla \wedge \mathbf{u})$.]

Give a physical interpretation for the term $(\boldsymbol{\omega} + 2\mathbf{\Omega}) \cdot \nabla\mathbf{u}$.

(ii) Consider the rotating fluid of part (i), but now let ρ be constant and absorb the effects of gravity into a modified pressure $P = p - \rho\mathbf{g} \cdot \mathbf{x}$. State the *linearized* equations of motion and the *linearized* vorticity equation for small-amplitude motions (inertial waves).

Use the linearized equations of motion to show that

$$\nabla^2 P = 2\rho\mathbf{\Omega} \cdot \boldsymbol{\omega} \ .$$

Calculate the time derivative of the curl of the linearized vorticity equation. Hence show that

$$\frac{\partial^2}{\partial t^2}(\nabla^2\mathbf{u}) = -(2\mathbf{\Omega} \cdot \nabla)^2\mathbf{u} \ .$$

Deduce the dispersion relation for waves proportional to $\exp[i(\mathbf{k} \cdot \mathbf{x} - nt)]$. Show that $|n| \leq 2\Omega$. Show further that if $n = 2\Omega$ then $P = 0$.

17H Mathematical Methods

(i) A certain physical quantity $q(x)$ can be represented by the series $\sum\limits_{n=0}^{\infty} c_n x^n$ in $0 \leqslant x < x_0$, but the series diverges for $x > x_0$. Describe the Euler transformation to a new series which may enable $q(x)$ to be computed for $x > x_0$. Give the first four terms of the new series.

Describe briefly the disadvantages of the method.

(ii) The series $\sum\limits_{1}^{\infty} c_r$ has partial sums $S_n = \sum\limits_{1}^{n} c_r$. Describe Shanks' method to approximate S_n by

$$S_n = A + BC^n \ , \qquad (*)$$

giving expressions for A, B and C.

Denote by B_N and C_N the values of B and C respectively derived from these expressions using S_{N-1}, S_N and S_{N+1} for some fixed N. Now let $A^{(n)}$ be the value of A obtained from $(*)$ with $B = B_N, C = C_N$. Show that, if $|C_N| < 1$,

$$\sum_{1}^{\infty} c_r = \lim_{n \to \infty} A^{(n)} \ .$$

If, in fact, the partial sums satisfy

$$S_n = a + \alpha c^n + \beta d^n \ ,$$

with $1 > |c| > |d|$, show that

$$A^{(n)} = A + \gamma d^n + o(d^n) \ ,$$

where γ is to be found.

18K Nonlinear Waves

(i) Establish two conservation laws for the MKdV equation

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

State sufficient boundary conditions that u should satisfy for the conservation laws to be valid.

(ii) The equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\Big(\rho V\Big) = 0$$

models traffic flow on a single-lane road, where $\rho(x,t)$ represents the density of cars, and V is a given function of ρ. By considering the rate of change of the integral

$$\int_a^b \rho \, dx,$$

show that V represents the velocity of the cars.

Suppose now that $V = 1 - \rho$ (in suitable units), and that $0 \leqslant \rho \leqslant 1$ everywhere. Assume that a queue is building up at a traffic light at $x = 1$, so that, when the light turns green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0 & \text{for} \quad x < 0 \quad \text{and} \quad x > 1 \\ x & \text{for} \quad 0 \leqslant x < 1. \end{cases}$$

For this problem, find and sketch the characteristics in the (x,t) plane, for $t > 0$, paying particular attention to those emerging from the point $(1,0)$. Show that a shock forms at $t = \frac{1}{2}$. Find the density of cars $\rho(x,t)$ for $0 < t < \frac{1}{2}$, and all x.

19K Numerical Analysis

(i) Define m-step BDF (backward differential formula) methods for the numerical solution of ordinary differential equations and derive explicitly their coefficients.

(ii) Prove that the linear stability domain of the two-step BDF method includes the interval $(-\infty, 0)$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Wednesday 6 June 2001 9.00 to 12.00

PAPER 3

Before you begin read these instructions carefully.

Each question is divided into Part ***(i)*** *and Part* ***(ii)****, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* ***SIX*** *questions.*

The number of marks for each question is the same, with Part ***(ii)*** *of each question carrying twice as many marks as Part* ***(i)****. Additional credit will be given for a substantially complete answer to either part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***10E, 12E*** *should be in one bundle and* ***1D, 11D*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *Parts of* ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Explain what is meant by the *transition semigroup* $\{P_t\}$ of a Markov chain X in continuous time. If the state space is finite, show under assumptions to be stated clearly, that $P_t' = GP_t$ for some matrix G. Show that a distribution π satisfies $\pi G = 0$ if and only if $\pi P_t = \pi$ for all $t \geqslant 0$, and explain the importance of such π.

(ii) Let X be a continuous-time Markov chain on the state space $S = \{1, 2\}$ with generator

$$G = \begin{pmatrix} -\beta & \beta \\ \gamma & -\gamma \end{pmatrix}, \qquad \text{where} \quad \beta, \gamma > 0.$$

Show that the transition semigroup $P_t = \exp(tG)$ is given by

$$(\beta + \gamma)P_t = \begin{pmatrix} \gamma + \beta h(t) & \beta(1 - h(t)) \\ \gamma(1 - h(t)) & \beta + \gamma h(t) \end{pmatrix},$$

where $h(t) = e^{-t(\beta+\gamma)}$.

For $0 < \alpha < 1$, let

$$H(\alpha) = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}.$$

For a continuous-time chain X, let M be a matrix with (i, j) entry $P(X(1) = j \mid X(0) = i)$, for $i, j \in S$. Show that there is a chain X with $M = H(\alpha)$ if and only if $\alpha > \frac{1}{2}$.

2H Principles of Dynamics

(i) (a) Write down Hamilton's equations for a dynamical system. Under what condition is the Hamiltonian a constant of the motion? What is the condition for one of the momenta to be a constant of the motion?

(b) Explain the notion of an adiabatic invariant. Give an expression, in terms of Hamiltonian variables, for one such invariant.

(ii) A mass m is attached to one end of a straight spring with potential energy $\frac{1}{2}kr^2$, where k is a constant and r is the length. The other end is fixed at a point O. Neglecting gravity, consider a general motion of the mass in a plane containing O. Show that the Hamiltonian is given by

$$H = \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}kr^2, \tag{1}$$

where θ is the angle made by the spring relative to a fixed direction, and p_θ, p_r are the generalised momenta. Show that p_θ and the energy E are constants of the motion, using Hamilton's equations.

If the mass moves in a non-circular orbit, and the spring constant k is slowly varied, the orbit gradually changes. Write down the appropriate adiabatic invariant $J\left(E, p_\theta, k, m\right)$. Show that J is proportional to

$$\sqrt{mk}\left(r_+ - r_-\right)^2,$$

where

$$r_\pm^2 = \frac{E}{k} \pm \sqrt{\left(\frac{E}{k}\right)^2 - \frac{p_\theta^2}{mk}}.$$

Consider an orbit for which p_θ is zero. Show that, as k is slowly varied, the energy $E \propto k^\alpha$, for a constant α which should be found.

[*You may assume without proof that*

$$\int_{r_-}^{r_+} dr \sqrt{\left(1 - \frac{r^2}{r_+^2}\right)\left(1 - \frac{r_-^2}{r^2}\right)} = \frac{\pi}{4r_+}\left(r_+ - r_-\right)^2.\Big]$$

3A Functional Analysis

(i) Define the notion of a measurable function between measurable spaces. Show that a continuous function $\mathbb{R}^2 \to \mathbb{R}$ is measurable with respect to the Borel σ-fields on $\mathbb{R}^2$ and $\mathbb{R}$.

By using this, or otherwise, show that, when $f, g : X \to \mathbb{R}$ are measurable with respect to some σ-field $\mathcal{F}$ on X and the Borel σ-field on $\mathbb{R}$, then $f + g$ is also measurable.

(ii) State the Monotone Convergence Theorem for $[0, \infty]$-valued functions. Prove the Dominated Convergence Theorem.

[*You may assume the Monotone Convergence Theorem but any other results about integration that you use will need to be stated carefully and proved.*]

Let X be the real Banach space of continuous real-valued functions on $[0, 1]$ with the uniform norm. Fix $u \in X$ and define

$$T : X \to \mathbb{R} \; ; \quad f \mapsto \int_0^1 f(t)u(t)\, dt \; .$$

Show that T is a bounded, linear map with norm

$$||T|| = \int_0^1 |u(t)|\, dt \; .$$

Is it true, for every choice of u, that there is function $f \in X$ with $||f|| = 1$ and $||T(f)|| = ||T||$?

4C Groups, Rings and Fields

(i) Let G be the cyclic subgroup of $GL_2(\mathbf{C})$ generated by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$,

acting on the polynomial ring $\mathbf{C}[X, Y]$. Determine the ring of invariants $\mathbf{C}[X, Y]^G$.

(ii) Determine $\mathbf{C}[X, Y]^G$ when G is the cyclic group generated by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

[*Hint: consider the eigenvectors.*]

5J Electromagnetism

(i) Develop the theory of electromagnetic waves starting from Maxwell equations in vacuum. You should relate the wave-speed c to ϵ_0 and μ_0 and establish the existence of plane, plane-polarized waves in which $\mathbf{E}$ takes the form

$$\mathbf{E} = (E_0 \cos(kz - \omega t), 0, 0) \ .$$

You should give the form of the magnetic field $\mathbf{B}$ in this case.

(ii) Starting from Maxwell's equation, establish Poynting's theorem.

$$-\mathbf{j} \cdot \mathbf{E} \ = \ \frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} \ ,$$

where $W = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$ and $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B}$. Give physical interpretations of W, S and the theorem.

Compute the averages over space and time of W and $\mathbf{S}$ for the plane wave described in (i) and relate them. Comment on the result.

6K Dynamics of Differential Equations

(i) Define a hyperbolic fixed point x_0 of a flow determined by a differential equation $\dot{x} = f(x)$ where $x \in R^n$ and f is C^1 (i.e. differentiable). State the Hartman-Grobman Theorem for flow near a hyperbolic fixed point. For nonlinear flows in R^2 with a hyperbolic fixed point x_0, does the theorem necessarily allow us to distinguish, on the basis of the linearized flow near x_0 between (a) a stable focus and a stable node; and (b) a saddle and a stable node? Justify your answers briefly.

(ii) Show that the system:

$$\begin{aligned}\dot{x} &= -(\mu+1) + (\mu-3)x - y + 6x^2 + 12xy + 5y^2 - 2x^3 - 6x^2y - 5xy^2, \\ \dot{y} &= 2 - 2x + (\mu-5)y + 4xy + 6y^2 - 2x^2y - 6xy^2 - 5y^3\end{aligned}$$

has a fixed point $(x_0, 0)$ on the x-axis. Show that there is a bifurcation at $\mu = 0$ and determine the stability of the fixed point for $\mu > 0$ and for $\mu < 0$.

Make a linear change of variables of the form $u = x - x_0 + \alpha y$, $v = x - x_0 + \beta y$, where α and β are constants to be determined, to bring the system into the form:

$$\begin{aligned}\dot{u} &= v + u[\mu - (u^2 + v^2)] \\ \dot{v} &= -u + v[\mu - (u^2 + v^2)]\end{aligned}$$

and hence determine whether the periodic orbit produced in the bifurcation is stable or unstable, and whether it exists in $\mu < 0$ or $\mu > 0$.

7C Geometry of Surfaces

(i) Give the definition of the surface area of a parametrized surface in $\mathbf{R}^3$ and show that it does not depend on the parametrization.

(ii) Let $\varphi(u) > 0$ be a differentiable function of u. Consider the surface of revolution:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto f(u,v) = \begin{pmatrix} \varphi(u)\cos(v) \\ \varphi(u)\sin(v) \\ u \end{pmatrix}.$$

Find a formula for each of the following:

(a) The first fundamental form.

(b) The unit normal.

(c) The second fundamental form.

(d) The Gaussian curvature.

8B Logic, Computation and Set Theory

(i) Write down a set of axioms for the theory of dense linear order with a bottom element but no top element.

(ii) Prove that this theory has, up to isomorphism, precisely one countable model.

9C Number Theory

(i) State the law of quadratic reciprocity.

Prove that 5 is a quadratic residue modulo primes $p \equiv \pm 1 \pmod{10}$ and a quadratic non-residue modulo primes $p \equiv \pm 3 \pmod{10}$.

Determine whether 5 is a quadratic residue or non-residue modulo 119 and modulo 539.

(ii) Explain what is meant by the continued fraction of a real number θ. Define the convergents to θ and write down the recurrence relations satisfied by their numerators and denominators.

Use the continued fraction method to find two solutions in positive integers x, y of the equation $x^2 - 15y^2 = 1$.

10E Algorithms and Networks

(i) Let P be the problem

$$\text{minimize} \quad f(x) \qquad \text{subject to} \quad h(x) = b, \quad x \in X.$$

Explain carefully what it means for the problem P to be *Strong Lagrangian.*

Outline the main steps in a proof that a quadratic programming problem

$$\text{minimize} \quad \frac{1}{2} x^T Q x + c^T x \qquad \text{subject to} \quad Ax \geqslant b\,,$$

where Q is a symmetric positive semi-definite matrix, is Strong Lagrangian.

[*You should carefully state the results you need, but should not prove them.*]

(ii) Consider the quadratic programming problem:

$$\text{minimize} \quad x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 - 4x_2$$

$$\text{subject to} \quad 3x_1 + 2x_2 \leqslant 6, \quad x_1 + x_2 \geqslant 1\,.$$

State necessary and sufficient conditions for (x_1, x_2) to be optimal, and use the active-set algorithm (explaining your steps briefly) to solve the problem starting with initial condition $(2, 0)$. Demonstrate that the solution you have found is optimal by showing that it satisfies the necessary and sufficient conditions stated previously.

11D Stochastic Financial Models

(i) Suppose that Z is a random variable having the normal distribution with $\mathbb{E}Z = \beta$ and $\operatorname{Var} Z = \tau^2$.

For positive constants a, c show that

$$\mathbb{E}\left(ae^Z - c\right)_+ = ae^{\left(\beta+\tau^2/2\right)}\Phi\left(\frac{\log(a/c)+\beta}{\tau}+\tau\right) - c\Phi\left(\frac{\log(a/c)+\beta}{\tau}\right),$$

where Φ is the standard normal distribution function.

In the context of the Black-Scholes model, derive the formula for the price at time 0 of a European call option on the stock at strike price c and maturity time t_0 when the interest rate is ρ and the volatility of the stock is σ.

(ii) Let p denote the price of the call option in the Black-Scholes model specified in (i). Show that $\frac{\partial p}{\partial \rho} > 0$ and sketch carefully the dependence of p on the volatility σ (when the other parameters in the model are held fixed).

Now suppose that it is observed that the interest rate lies in the range $0 < \rho < \rho_0$ and when it changes it is linked to the volatility by the formula $\sigma = \ln(\rho_0/\rho)$. Consider a call option at strike price $c = S_0$, where S_0 is the stock price at time 0. There is a small increase $\Delta\rho$ in the interest rate; will the price of the option increase or decrease (assuming that the stock price is unaffected)? Justify your answer carefully.

[*You may assume that the function* $\phi(x)/\Phi(x)$ *is decreasing in* x, $-\infty < x < \infty$, *and increases to* $+\infty$ *as* $x \to -\infty$, *where* Φ *is the standard-normal distribution function and* $\phi = \Phi'$.]

12E Principles of Statistics

(i) Explain what is meant by a *uniformly most powerful unbiased test* of a null hypothesis against an alternative.

Let $Y_1, \ldots, Y_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, with σ^2 known. Explain how to construct a uniformly most powerful unbiased size α test of the null hypothesis that $\mu = 0$ against the alternative that $\mu \neq 0$.

(ii) Outline briefly the Bayesian approach to hypothesis testing based on *Bayes factors*.

Let the distribution of $Y_1, \ldots, Y_n$ be as in (i) above, and suppose we wish to test, as in (i), $\mu = 0$ against the alternative $\mu \neq 0$. Suppose we assume a $N(0, \tau^2)$ prior for μ under the alternative. Find the form of the Bayes factor B, and show that, for fixed n, $B \to \infty$ as $\tau \to \infty$.

13F Foundations of Quantum Mechanics

(i) Write the Hamiltonian for the harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 ,$$

in terms of creation and annihilation operators, defined by

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - i\frac{p}{m\omega}\right), \qquad a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x + i\frac{p}{m\omega}\right).$$

Obtain an expression for $[a^\dagger, a]$ by using the usual commutation relation between p and x. Deduce the quantized energy levels for this system.

(ii) Define the number operator, N, in terms of creation and annihilation operators, $a^\dagger$ and a. The normalized eigenvector of N with eigenvalue n is $|n\rangle$. Show that $n \geq 0$.

Determine $a|n\rangle$ and $a^\dagger|n\rangle$ in the basis defined by $\{|n\rangle\}$.

Show that

$$a^{\dagger m} a^m |n\rangle = \begin{cases} \frac{n!}{(n-m)!}|n\rangle, & m \leq n\,, \\ 0\,, & m > n\,. \end{cases}$$

Verify the relation

$$|0\rangle\langle 0| = \sum_{m=0} \frac{1}{m!}(-1)^m a^{\dagger m} a^m\,,$$

by considering the action of both sides of the equation on an arbitrary basis vector.

14J Statistical Physics and Cosmology

(i) A spherically symmetric star has pressure $P(r)$ and mass density $\rho(r)$, where r is distance from the star's centre. Stating without proof any theorems you may need, show that mechanical equilibrium implies the Newtonian pressure support equation

$$P' = -\frac{Gm\rho}{r^2} ,$$

where $m(r)$ is the mass within radius r and $P' = dP/dr$.

Write down an integral expression for the total gravitational potential energy, E_{gr}. Use this to derive the "virial theorem"

$$E_{gr} = -3\langle P\rangle V ,$$

when $\langle P\rangle$ is the average pressure.

(ii) Given that the total kinetic energy, E_{kin}, of a spherically symmetric star is related to its average pressure by the formula

$$E_{kin} = \alpha\langle P\rangle V \qquad (*)$$

for constant α, use the virial theorem (stated in part (i)) to determine the condition on α needed for gravitational binding. State the relation between pressure P and "internal energy" U for an ideal gas of non-relativistic particles. What is the corresponding relation for ultra-relativistic particles? Hence show that the formula $(*)$ applies in these cases, and determine the values of α.

Why does your result imply a maximum mass for any star, whatever the source of its pressure? What is the maximum mass, approximately, for stars supported by

(a) thermal pressure,

(b) electron degeneracy pressure (White Dwarf),

(c) neutron degeneracy pressure (Neutron Star).

A White Dwarf can accrete matter from a companion star until its mass exceeds the Chandrasekar limit. Explain briefly the process by which it then evolves into a neutron star.

15F Symmetries and Groups in Physics

(i) The pions form an isospin triplet with $\pi^+ = |1,1\rangle$, $\pi^0 = |1,0\rangle$ and $\pi^- = |1,-1\rangle$, whilst the nucleons form an isospin doublet with $p = |\frac{1}{2},\frac{1}{2}\rangle$ and $n = |\frac{1}{2},-\frac{1}{2}\rangle$. Consider the isospin representation of two-particle states spanned by the basis

$$T = \{|\pi^+ p\rangle,\ |\pi^+ n\rangle,\ |\pi^0 p\rangle,\ |\pi^0 n\rangle,\ |\pi^- p\rangle,\ |\pi^- n\rangle\}\,.$$

State which irreducible representations are contained in this representation and explain why $|\pi^+ p\rangle$ is an isospin eigenstate.

Using

$$I_-\,|j,m\rangle = \sqrt{(j-m+1)(j+m)}\,|j,m-1\rangle,$$

where I_- is the isospin ladder operator, write the isospin eigenstates in terms of the basis, T.

(ii) The Lie algebra $su(2)$ of generators of $SU(2)$ is spanned by the operators $\{J_+, J_-, J_3\}$ satisfying the commutator algebra $[J_+, J_-] = 2J_3$ and $[J_3, J_\pm] = \pm J_\pm$. Let Ψ_j be an eigenvector of J_3: $J_3\,(\Psi_j) = j\Psi_j$ such that $J_+\Psi_j = 0$. The vector space $V_j = \mathrm{span}\{J_-^n\Psi_j : n \in \mathbb{N}_0\}$ together with the action of an arbitrary $su(2)$ operator A on V_j defined by

$$J_-\left(J_-^n\Psi_j\right) = J_-^{n+1}\Psi_j\ , \qquad A\left(J_-^n\Psi_j\right) = [A, J_-]\left(J_-^{n-1}\Psi_j\right) + J_-\left(A\left(J_-^{n-1}\Psi_j\right)\right)\ ,$$

forms a representation (not necessarily reducible) of $su(2)$. Show that if $J_-^n\Psi_j$ is non-trivial then it is an eigenvector of J_3 and find its eigenvalue. Given that $[J_+, J_-^n] = \alpha_n J_-^{n-1}J_3 + \beta_n J_-^{n-1}$ show that α_n and β_n satisfy

$$\alpha_n = \alpha_{n-1} + 2, \qquad \beta_n = \beta_{n-1} - \alpha_{n-1}.$$

By solving these equations evaluate $[J_+, J_-^n]$. Show that $J_+J_-^{2j+1}\Psi_j = 0$. Hence show that $J_-^{2j+1}\Psi_j$ is contained in a proper sub-representation of V_j.

16H Transport Processes

(i) Incompressible fluid of kinematic viscosity ν occupies a parallel-sided channel $0 \leqslant y \leqslant h_0$, $-\infty < x < \infty$. The wall $y = 0$ is moving parallel to itself, in the x-direction, with velocity $\mathrm{Re}\left\{Ue^{i\omega t}\right\}$, where t is time and U, ω are real constants. The fluid velocity $u(y, t)$ satisfies the equation

$$u_t = \nu u_{yy};$$

write down the boundary conditions satisfied by u.

Assuming that

$$u = \mathrm{Re}\left\{a \sinh[b(1-\eta)]e^{i\omega t}\right\},$$

where $\eta = y/h_0$, find the complex constants a, b. Calculate the velocity (in real, not complex, form) in the limit $h_0(\omega/\nu)^{1/2} \to 0$.

(ii) Incompressible fluid of viscosity μ fills the narrow gap between the rigid plane $y = 0$, which moves parallel to itself in the x-direction with constant speed U, and the rigid wavy wall $y = h(x)$, which is at rest. The length-scale, L, over which h varies is much larger than a typical value, h_0, of h.

Assume that inertia is negligible, and therefore that the governing equations for the velocity field (u, v) and the pressure p are

$$u_x + v_y = 0,\ p_x = \mu\left(u_{xx} + u_{yy}\right), p_y = \mu\left(v_{xx} + v_{yy}\right).$$

Use scaling arguments to show that these equations reduce approximately to

$$p_x = \mu u_{yy}, \quad p_y = 0.$$

Hence calculate the velocity $u(x, y)$, the flow rate

$$Q = \int_0^h u dy,$$

and the viscous shear stress exerted by the fluid on the plane wall,

$$\tau = -\mu u_y|_{y=0}$$

in terms of p_x, h, U and μ.

Now assume that $h = h_0(1 + \epsilon \sin kx)$, where $\epsilon \ll 1$ and $kh_0 \ll 1$, and that p is periodic in x with wavelength $2\pi/k$. Show that

$$Q = \frac{h_0 U}{2}\left(1 - \frac{3}{2}\epsilon^2 + O\left(\epsilon^4\right)\right)$$

and calculate τ correct to $O\left(\epsilon^2\right)$. Does increasing the amplitude ϵ of the corrugation cause an increase or a decrease in the force required to move the plane $y = 0$ at the chosen speed U?

17H Mathematical Methods

(i) The function $y(x)$ satisfies the differential equation

$$y'' + by' + cy = 0\ , \quad 0 < x < 1\ ,$$

where b and c are constants, with boundary conditions $y(0) = 0,\ y'(0) = 1$. By integrating this equation or otherwise, show that y must also satisfy the integral equation

$$y(x) = g(x) + \int_0^1 K(x,t)y(t)dt\ ,$$

and find the functions $g(x)$ and $K(x,t)$.

(ii) Solve the integral equation

$$\varphi(x) = 1 + \lambda^2 \int_0^x (x-t)\varphi(t)dt\ , \quad x > 0\ , \quad \lambda \text{ real}\ ,$$

by finding an ordinary differential equation satisfied by $\varphi(x)$ together with boundary conditions.

Now solve the integral equation by the method of successive approximations and show that the two solutions are the same.

18K Nonlinear Waves

(i) The so-called breather solution of the sine-Gordon equation is

$$\phi(x,t) = 4\tan^{-1}\left(\frac{(1-\lambda^2)^{\frac{1}{2}}}{\lambda}\frac{\sin\lambda t}{\cosh(1-\lambda^2)^{\frac{1}{2}}x}\right), \quad 0 < \lambda < 1.$$

Describe qualitatively the behaviour of $\phi(x,t)$, for $\lambda \ll 1$, when $|x| \gg \ln(2/\lambda)$, when $|x| \ll 1$, and when $\cosh x \approx \frac{1}{\lambda}|\sin\lambda t|$. Explain how this solution can be interpreted in terms of motion of a kink and an antikink. Estimate the greatest separation of the kink and antikink.

(ii) The field $\psi(x,t)$ obeys the nonlinear wave equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + \frac{dU}{d\psi} = 0,$$

where the potential U has the form

$$U(\psi) = \frac{1}{2}(\psi - \psi^3)^2.$$

Show that $\psi = 0$ and $\psi = 1$ are stable constant solutions.

Find a steady wave solution $\psi = f(x - vt)$ satisfying the boundary conditions $\psi \to 0$ as $x \to -\infty$, $\psi \to 1$ as $x \to \infty$. What constraint is there on the velocity v?

19K Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is discretized by the finite-difference method

$$u_m^{n+1} - \frac{1}{2}(\mu - \alpha)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}(\mu + \alpha)(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

where $u_m^n \approx u(m\Delta x, n\Delta t), \mu = \Delta t/(\Delta x)^2$ and α is a constant. Derive the order of magnitude (as a power of Δx) of the local error for different choices of α.

(ii) Investigate the stability of the above finite-difference method for different values of α by the Fourier technique.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Thursday 7 June 2001 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than ***FOUR*** *questions.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***3A, 9A*** *should be in one bundle and* ***4C, 10C*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

Write an essay on the convergence to equilibrium of a discrete-time Markov chain on a countable state-space. You should include a discussion of the existence of invariant distributions, and of the limit theorem in the non-null recurrent case.

2H Principles of Dynamics

(i) Consider a particle of charge q and mass m, moving in a stationary magnetic field $\mathbf{B}$. Show that Lagrange's equations applied to the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}),$$

where $\mathbf{A}$ is the vector potential such that $\mathbf{B} = \text{curl } \mathbf{A}$, lead to the correct Lorentz force law. Compute the canonical momentum $\mathbf{p}$, and show that the Hamiltonian is $H = \frac{1}{2}m\dot{\mathbf{r}}^2$.

(ii) Expressing the velocity components $\dot{r}_i$ in terms of the canonical momenta and co-ordinates for the above system, derive the following formulae for Poisson brackets:

(a) $\{FG, H\} = F\{G, H\} + \{F, H\}G$, for any functions F, G, H;

(b) $\{m\dot{r}_i, m\dot{r}_j\} = q\epsilon_{ijk}B_k$;

(c) $\{m\dot{r}_i, r_j\} = -\delta_{ij}$;

(d) $\{m\dot{r}_i, f(r_j)\} = -\frac{\partial}{\partial r_i}f(r_j)$.

Now consider a particle moving in the field of a magnetic monopole,

$$B_i = g\frac{r_i}{r^3}.$$

Show that $\{H, \mathbf{J}\} = 0$, where $\mathbf{J} = m\mathbf{r} \wedge \dot{\mathbf{r}} - gq\hat{\mathbf{r}}$. Explain why this means that $\mathbf{J}$ is conserved.

Show that, if $g = 0$, conservation of $\mathbf{J}$ implies that the particle moves in a plane perpendicular to $\mathbf{J}$. What type of surface does the particle move on if $g \neq 0$?

3A Functional Analysis

Write an account of the classical sequence spaces: ℓ_p $(1 \leqslant p \leqslant \infty)$ and c_0. You should define them, prove that they are Banach spaces, and discuss their properties, including their dual spaces. Show that ℓ_∞ is inseparable but that c_0 and ℓ_p for $1 \leqslant p < \infty$ are separable.

Prove that, if $T : X \to Y$ is an isomorphism between two Banach spaces, then

$$T^* : Y^* \to X^* \ ; \quad f \mapsto f \circ T$$

is an isomorphism between their duals.

Hence, or otherwise, show that no two of the spaces $c_0, \ell_1, \ell_2, \ell_\infty$ are isomorphic.

4C Groups, Rings and Fields

Show that the ring $\mathbf{Z}[\omega]$ is Euclidean, where $\omega = \exp(2\pi i/3)$.

Show that a prime number $p \neq 3$ is reducible in $\mathbf{Z}[\omega]$ if and only if $p \equiv 1 (\text{mod } 3)$.

Which prime numbers p can be written in the form $p = a^2 + ab + b^2$ with $a, b \in \mathbf{Z}$ (and why)?

5J Electromagnetism

Write down the form of Ohm's Law that applies to a conductor if at a point $\mathbf{r}$ it is moving with velocity $\mathbf{v}(\mathbf{r})$.

Use two of Maxwell's equations to prove that

$$\int_C (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot d\mathbf{r} \quad = \quad -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad ,$$

where $C(t)$ is a moving closed loop, $\mathbf{v}$ is the velocity at the point $\mathbf{r}$ on C, and S is a surface spanning C. The time derivative on the right hand side accounts for changes in both C and $\mathbf{B}$. Explain briefly the physical importance of this result.

Find and sketch the magnetic field $\mathbf{B}$ described in the vector potential

$$\mathbf{A}(r, \theta, z) \quad = \quad (0, \tfrac{1}{2}\, brz, 0)$$

in cylindrical polar coordinates (r, θ, z) , where $b > 0$ is constant.

A conducting circular loop of radius a and resistance R lies in the plane $z = h(t)$ with its centre on the z-axis.

Find the magnitude and direction of the current induced in the loop as $h(t)$ changes with time, neglecting self-inductance.

At time $t = 0$ the loop is at rest at $z = 0$. For time $t > 0$ the loop moves with constant velocity $dh/dt = v > 0$. Ignoring the inertia of the loop, use energy considerations to find the force $F(t)$ necessary to maintain this motion.

[*In cylindrical polar coordinates*

$$\text{curl } \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right). \Bigg]$$

6K Dynamics of Differential Equations

Write a short essay about periodic orbits in flows in two dimensions. Your essay should include criteria for the existence and non-existence of periodic orbits, and should mention (with sketches) at least two bifurcations that create or destroy periodic orbits in flows as a parameter is altered (though a detailed analysis of any bifurcation is not required).

7C Geometry of Surfaces

Write an essay on the Gauss-Bonnet theorem. Make sure that your essay contains a precise statement of the theorem, in its local form, and a discussion of some of its applications, including the global Gauss-Bonnet theorem.

8B Logic, Computation and Set Theory

What is a wellfounded relation, and how does wellfoundedness underpin wellfounded induction?

A formula $\phi(x, y)$ with two free variables *defines an* $\in$*-automorphism* if for all x there is a unique y, the function f, defined by $y = f(x)$ if and only if $\phi(x, y)$, is a permutation of the universe, and we have $(\forall xy)(x \in y \leftrightarrow f(x) \in f(y))$.

Use wellfounded induction over $\in$ to prove that all formulæ defining $\in$-automorphisms are equivalent to $x = y$.

9A Graph Theory

Write an essay on extremal graph theory. Your essay should include proofs of at least two major results and a discussion of variations on these results or their proofs.

10C Number Theory

Attempt **one** of the following:

(i) Discuss pseudoprimes and primality testing. Find all bases for which 57 is a Fermat pseudoprime. Determine whether 57 is also an Euler pseudoprime for these bases.

(ii) Write a brief account of various methods for factoring large numbers. Use Fermat factorization to find the factors of 10033. Would Pollard's $p - 1$ method also be practical in this instance?

(iii) Show that $\sum 1/p_n$ is divergent, where p_n denotes the n*-th* prime.

Write a brief account of basic properties of the Riemann zeta-function.

State the prime number theorem. Show that it implies that for all sufficiently large positive integers n there is a prime p satisfying $n < p \leqslant 2n$.

11E Algorithms and Networks

State the optimal distribution problem. Carefully describe the simplex-on-a-graph algorithm for solving optimal distribution problems when the flow in each arc in the network is constrained to lie in the interval $[0, \infty)$. Explain how the algorithm can be initialised if there is no obvious feasible solution with which to begin. Describe the adjustments that are needed for the algorithm to cope with more general capacity constraints $x(j) \in [c^-(j), c^+(j)]$ for each arc j (where $c^{\pm}(j)$ may be finite or infinite).

12D Stochastic Financial Models

Write an essay on the mean-variance approach to portfolio selection in a one-period model. Your essay should contrast the solution in the case when all the assets are risky with that for the case when there is a riskless asset.

13E Principles of Statistics

Write an account, with appropriate examples, of **one** of the following:

(a) Inference in multi-parameter exponential families;

(b) Asymptotic properties of maximum-likelihood estimators and their use in hypothesis testing;

(c) Bootstrap inference.

14E Computational Statistics and Statistical Modelling

(i) Assume that independent observations $Y_1, \ldots, Y_n$ are such that

$$Y_i \sim \text{Binomial}(t_i, \pi_i), \ \log \frac{\pi_i}{1-\pi_i} = \beta^T x_i \quad \text{for } 1 \leqslant i \leqslant n \quad ,$$

where $x_1, \ldots, x_n$ are given covariates. Discuss carefully how to estimate β, and how to test that the model fits.

(ii) Carmichael *et al.* (1989) collected data on the numbers of 5-year old children with "dmft", i.e. with 5 or more decayed, missing or filled teeth, classified by social class, and by whether or not their tap water was fluoridated or non-fluoridated. The numbers of such children with dmft, and the total numbers, are given in the table below:

dmft

Social Class	Fluoridated	Non-fluoridated
I	12/117	12/56
II	26/170	48/146
III	11/52	29/64
Unclassified	24/118	49/104

A (slightly edited) version of the *R* output is given below. Explain carefully what model is being fitted, whether it does actually fit, and what the parameter estimates and Std. Errors are telling you. (You may assume that the factors SClass (social class) and Fl (with/without) have been correctly set up.)

```
Call:

     glm(formula = Yes/Total ~ SClass + Fl, family = binomial,
          weights = Total)

Coefficients:

                Estimate  Std.  Error  z value
     (Intercept) -2.2716        0.2396   -9.480
     SClassII     0.5099        0.2628    1.940
     SClassIII    0.9857        0.3021    3.262
     SClassUnc    1.0020        0.2684    3.734
     Flwithout    1.0813        0.1694    6.383

(Dispersion parameter for binomial family taken to be 1)

      Null deviance:  68.53785 on 7 degrees of freedom

Residual deviance:  0.64225 on 3 degrees of freedom

Number of Fisher Scoring iterations:  3
```

Here 'Yes' is the vector of numbers with dmft, taking values $12, 12, \ldots, 24, 49$, 'Total' is the vector of Total in each category, taking values $117, 56, \ldots, 118, 104$, and SClass, Fl are the factors corresponding to Social class and Fluoride status, defined in the obvious way.

15F Foundations of Quantum Mechanics

(i) The two states of a spin-$\frac{1}{2}$ particle corresponding to spin pointing along the z axis are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$. Explain why the states

$$|\uparrow, \theta\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle, \qquad |\downarrow, \theta\rangle = -\sin\frac{\theta}{2}\,|\uparrow\rangle + \cos\frac{\theta}{2}\,|\downarrow\rangle$$

correspond to the spins being aligned along a direction at an angle θ to the z direction.

The spin-0 state of two spin-$\frac{1}{2}$ particles is

$$|0\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\Big).$$

Show that this is independent of the direction chosen to define $|\uparrow\rangle_{1,2}$, $|\downarrow\rangle_{1,2}$. If the spin of particle 1 along some direction is measured to be $\frac{1}{2}\hbar$ show that the spin of particle 2 along the same direction is determined, giving its value.

[*The Pauli matrices are given by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \Bigg]$$

(ii) Starting from the commutation relation for angular momentum in the form

$$[J_3, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_3,$$

obtain the possible values of j, m, where $m\hbar$ are the eigenvalues of J_3 and $j(j+1)\hbar^2$ are the eigenvalues of $\mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$. Show that the corresponding normalized eigenvectors, $|j, m\rangle$, satisfy

$$J_\pm|j, m\rangle = \hbar\left((j \mp m)(j \pm m + 1)\right)^{1/2}|j, m{\pm}1\rangle,$$

and that

$$\frac{1}{n!}J_-^{\,n}|j, j\rangle = \hbar^n\left(\frac{(2j)!}{n!(2j-n)!}\right)^{1/2}|j, j{-}n\rangle, \qquad n \le 2j.$$

The state $|w\rangle$ is defined by

$$|w\rangle = e^{wJ_-/\hbar}|j, j\rangle,$$

for any complex w. By expanding the exponential show that $\langle w|w\rangle = (1+|w|^2)^{2j}$. Verify that

$$e^{-wJ_-/\hbar}J_3\,e^{wJ_-/\hbar} = J_3 - wJ_-,$$

and hence show that

$$J_3|w\rangle = \hbar\left(j - w\frac{\partial}{\partial w}\right)|w\rangle.$$

If $H = \alpha J_3$ verify that $|e^{i\alpha t}\rangle\,e^{-ij\alpha t}$ is a solution of the time-dependent Schrödinger equation.

16F Quantum Physics

A harmonic oscillator of frequency ω is in thermal equilibrium with a heat bath at temperature T. Show that the mean number of quanta n in the oscillator is

$$n = \frac{1}{e^{\hbar\omega/kT} - 1}.$$

Use this result to show that the density of photons of frequency ω for cavity radiation at temperature T is

$$n(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}.$$

By considering this system in thermal equilibrium with a set of distinguishable atoms, derive formulae for the Einstein A and B coefficients.

Give a brief description of the operation of a laser.

17J General Relativity

Discuss how Einstein's theory of gravitation reduces to Newton's in the limit of weak fields. Your answer should include discussion of:

(a) the field equations;

(b) the motion of a point particle;

(c) the motion of a pressureless fluid.

[*The metric in a weak gravitational field, with Newtonian potential ϕ, may be taken as*

$$ds^2 = dx^2 + dy^2 + dz^2 - (1 + 2\phi)dt^2.$$

The Riemann tensor is

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc}.$$]

18J Statistical Physics and Cosmology

(i) Given that $g(p)dp$ is the number of eigenstates of a gas particle with momentum between p and $p + dp$, write down the Bose-Einstein distribution $\bar{n}(p)$ for the average number of particles with momentum between p and $p + dp$, as a function of temperature T and chemical potential μ.

Given that $\mu = 0$ and $g(p) = 8\pi \frac{Vp^2}{h^3}$ for a gas of photons, obtain a formula for the energy density ρ_T at temperature T in the form

$$\rho_T \quad = \quad \int_0^\infty \epsilon_T(\nu) d\nu,$$

where $\epsilon_T(\nu)$ is a function of the photon frequency ν that you should determine. Hence show that the value ν_{peak} of ν at the maximum of $\epsilon_T(\nu)$ is proportional to T.

A thermally isolated photon gas undergoes a slow change of its volume V. Why is $\bar{n}(p)$ unaffected by this change? Use this fact to show that VT^3 remains constant.

(ii) According to the "Hot Big Bang" theory, the Universe evolved by expansion from an earlier state in which it was filled with a gas of electrons, protons and photons (with $n_e = n_p$) at thermal equilibrium at a temperature T such that

$$2m_e c^2 \gg \; kT \; \gg B \quad ,$$

where m_e is the electron mass and B is the binding energy of a hydrogen atom. Why must the composition have been different when $kT \gg 2m_e c^2$? Why must it change as the temperature falls to $kT \ll B$? Why does this lead to a thermal decoupling of radiation from matter?

The baryon number of the Universe can be taken to be the number of protons, either as free particles or as hydrogen atom nuclei. Let n_b be the baryon number density and n_γ the photon number density. Why is the ratio $\eta = n_b/n_\gamma$ unchanged by the expansion of the universe? Given that $\eta \ll 1$, obtain an estimate of the temperature T_D at which decoupling occurs, as a function of η and B. How does this decoupling lead to the concept of a "surface of last scattering" and a prediction of a Cosmic Microwave Background Radiation (CMBR)?

19H Transport Processes

Fluid flows in the x-direction past the infinite plane $y = 0$ with uniform but time-dependent velocity $U(t) = U_0 G\left(t/t_0\right)$, where G is a positive function with timescale t_0. A long region of the plane, $0 < x < L$, is heated and has temperature $T_0\left(1 + \gamma\left(x/L\right)^n\right)$, where $T_0,\ \gamma,\ n$ are constants $[\gamma = O(1)]$; the remainder of the plane is insulating ($T_y = 0$). The fluid temperature far from the heated region is T_0. A thermal boundary layer is formed over the heated region. The full advection–diffusion equation for temperature $T(x, y, t)$ is

$$T_t + U(t)T_x = D\left(T_{yy} + T_{xx}\right), \tag{1}$$

where D is the thermal diffusivity. By considering the steady case ($G \equiv 1$), derive a scale for the thickness of the boundary layer, and explain why the term T_{xx} in (1) can be neglected if $U_0 L/D \gg 1$.

Neglecting T_{xx}, use the change of variables

$$\tau = \frac{t}{t_0}, \quad \xi = \frac{x}{L}, \quad \eta = y\left[\frac{U(t)}{Dx}\right]^{1/2}, \quad \frac{T - T_0}{T_0} = \gamma\left(\frac{x}{L}\right)^n f(\xi, \eta, \tau)$$

to transform the governing equation to

$$f_{\eta\eta} + \frac{1}{2}\eta f_\eta - nf = \xi f_\xi + \frac{L\xi}{t_0 U_0}\left(\frac{G_\tau}{2G^2}\eta f_\eta + \frac{1}{G}f_\tau\right). \tag{2}$$

Write down the boundary conditions to be satisfied by f in the region $0 < \xi < 1$.

In the case in which U is slowly-varying, so $\epsilon = \frac{L}{t_0 U_0} \ll 1$, consider a solution for f in the form

$$f = f_0(\eta) + \epsilon f_1(\xi, \eta, \tau) + O\left(\epsilon^2\right).$$

Explain why f_0 is independent of ξ and τ.

Henceforth take $n = \frac{1}{2}$. Calculate $f_0(\eta)$ and show that

$$f_1 = \frac{G_\tau \xi}{G^2} g(\eta) \quad ,$$

where g satisfies the ordinary differential equation

$$g'' + \frac{1}{2}\eta g' - \frac{3}{2}g = \frac{-\eta}{4}\int_\eta^\infty e^{-u^2/4} du.$$

State the boundary conditions on $g(\eta)$.

The heat transfer per unit length of the heated region is $-DT_y|_{y=0}$. Use the above results to show that the total rate of heat transfer is

$$T_0\left[DLU(t)\right]^{1/2}\frac{\gamma}{2}\left\{\sqrt{\pi} - \frac{\epsilon G_\tau}{G^2}g'(0) + O\left(\epsilon^2\right)\right\}.$$

20L Theoretical Geophysics

Write down expressions for the phase speed c and group velocity c_g in one dimension for general waves of the form $A\exp[i(kx-\omega t)]$ with dispersion relation $\omega(k)$. Briefly indicate the physical significance of c and c_g for a wavetrain of finite size.

The dispersion relation for internal gravity waves with wavenumber $\mathbf{k}=(k,0,m)$ in an incompressible stratified fluid with constant buoyancy frequency N is

$$\omega=\frac{\pm Nk}{(k^2+m^2)^{1/2}}.$$

Calculate the group velocity $\mathbf{c}_g$ and show that it is perpendicular to $\mathbf{k}$. Show further that the horizontal components of $\mathbf{k}/\omega$ and $\mathbf{c}_g$ have the same sign and that the vertical components have the opposite sign.

The vertical velocity w of small-amplitude internal gravity waves is governed by

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right)+N^2\nabla_h^2 w=0\ , \qquad (*)$$

where ∇_h^2 is the horizontal part of the Laplacian and N is constant.

Find separable solutions to $(*)$ of the form $w(x,z,t)=X(x-Ut)Z(z)$ corresponding to waves with constant horizontal phase speed $U>0$. Comment on the nature of these solutions for $0<k<N/U$ and for $k>N/U$.

A semi-infinite stratified fluid occupies the region $z>h(x,t)$ above a moving lower boundary $z=h(x,t)$. Construct the solution to $(*)$ for the case $h=\epsilon\sin[k(x-Ut)]$, where ϵ and k are constants and $\epsilon\ll 1$.

Sketch the orientation of the wavecrests, the propagation direction and the group velocity for the case $0<k<N/U$.

21H Mathematical Methods

The equation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \ ,$$

where $\mathbf{A}$ is a real square matrix and $\mathbf{x}$ a column vector, has a simple eigenvalue $\lambda = \mu$ with corresponding right-eigenvector $\mathbf{x} = \mathbf{X}$. Show how to find expressions for the perturbed eigenvalue and right-eigenvector solutions of

$$\mathbf{A}\mathbf{x} + \epsilon \mathbf{b}(\mathbf{x}) = \lambda \mathbf{x} \ , \quad |\epsilon| \ll 1 \ ,$$

to first order in ϵ, where $\mathbf{b}$ is a vector function of $\mathbf{x}$. State clearly any assumptions you make.

If $\mathbf{A}$ is $(n \times n)$ and has a complete set of right-eigenvectors $\mathbf{X}^{(j)}, \ j = 1, 2, \ldots n$, which span $\mathbb{R}^n$ and correspond to separate eigenvalues $\mu^{(j)}, \ j = 1, 2, \ldots n$, find an expression for the first-order perturbation to $\mathbf{X}^{(1)}$ in terms of the $\left\{ \mathbf{X}^{(j)} \right\}$ and the corresponding left-eigenvectors of $\mathbf{A}$.

Find the normalised eigenfunctions and eigenvalues of the equation

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \ 0 < x < 1 \ ,$$

with $y(0) = y(1) = 0$. Let these be the zeroth order approximations to the eigenfunctions of

$$\frac{d^2 y}{dx^2} + \lambda y + \epsilon b(y) = 0, \ 0 < x < 1 \ ,$$

with $y(0) = y(1) = 0$ and where b is a function of y. Show that the first-order perturbations of the eigenvalues are given by

$$\lambda_n^{(1)} = -\epsilon \sqrt{2} \int_0^1 \sin(n\pi x) \ \ b\left(\sqrt{2} \sin n\pi x\right) dx \ .$$

22K Numerical Analysis

Write an essay on the computation of eigenvalues and eigenvectors of matrices.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Monday 4 June 2001 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

*Tie your answers in separate bundles, marked **A, B, C, . . . , L** according to the letter affixed to each question. (For example, **1D, 13D** should be in one bundle and **8B, 9B** in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Let $X = (X_n : 0 \leqslant n \leqslant N)$ be an irreducible Markov chain on the finite state space S with transition matrix $P = (p_{ij})$ and invariant distribution π. What does it mean to say that X is reversible in equilibrium?

Show that X is reversible in equilibrium if and only if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

(ii) A finite connected graph G has vertex set V and edge set E, and has neither loops nor multiple edges. A particle performs a random walk on V, moving at each step to a randomly chosen neighbour of the current position, each such neighbour being picked with equal probability, independently of all previous moves. Show that the unique invariant distribution is given by $\pi_v = d_v/(2|E|)$ where d_v is the degree of vertex v.

A rook performs a random walk on a chessboard; at each step, it is equally likely to make any of the moves which are legal for a rook. What is the mean recurrence time of a corner square. (You should give a clear statement of any general theorem used.)

[*A chessboard is an* 8×8 *square grid. A legal move is one of any length parallel to the axes.*]

2H Principles of Dynamics

(i) Show that Newton's equations in Cartesian coordinates, for a system of N particles at positions $\mathbf{x}_i(t), i = 1, 2 \ldots N$, in a potential $V(\mathbf{x}, t)$, imply Lagrange's equations in a generalised coordinate system

$$q_j = q_j(\mathbf{x}_i, t) \quad , \quad j = 1, 2 \ldots 3N;$$

that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad , \quad j = 1, 2 \ldots 3N,$$

where $L = T - V$, $T(q, \dot{q}, t)$ being the total kinetic energy and $V(q, t)$ the total potential energy.

(ii) Consider a light rod of length L, free to rotate in a vertical plane (the xz plane), but with one end P forced to move in the x-direction. The other end of the rod is attached to a heavy mass M upon which gravity acts in the negative z direction.

(a) Write down the Lagrangian for the system.

(b) Show that, if P is stationary, the rod has two equilibrium positions, one stable and the other unstable.

(c) The end at P is now forced to move with constant acceleration, $\ddot{x} = A$. Show that, once more, there is one stable equilibrium value of the angle the rod makes with the vertical, and find it.

3C Groups, Rings and Fields

(i) Define the notion of a Sylow p-subgroup of a finite group G, and state a theorem concerning the number of them and the relation between them.

(ii) Show that any group of order 30 has a non-trivial normal subgroup. Is it true that every group of order 30 is commutative?

4J Electromagnetism

(i) Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a current sheet, $\mathbf{J}$, with unit normal to the sheet $\mathbf{n}$, are

$$\mathbf{n} \wedge \mathbf{B}_2 - \mathbf{n} \wedge \mathbf{B}_1 = \mu_0 \mathbf{J}.$$

State without proof the force per unit area on $\mathbf{J}$.

(ii) Conducting gas occupies the infinite slab $0 \leqslant x \leqslant a$. It carries a steady current $\mathbf{j} = (0, 0, j)$ and a magnetic field $\mathbf{B} = (0, B, 0)$ where $\mathbf{j}, \mathbf{B}$ depend only on x. The pressure is $p(x)$. The equation of hydrostatic equilibrium is $\nabla p = \mathbf{j} \wedge \mathbf{B}$. Write down the equations to be solved in this case. Show that $p + (1/2\mu_0)B^2$ is independent of x. Using the suffixes 1,2 to denote values at $x = 0, a$, respectively, verify that your results are in agreement with those of Part (i) in the case of $a \to 0$.

Suppose that

$$j(x) = \frac{\pi j_0}{2a} \sin\left(\frac{\pi x}{a}\right), \quad B_1 = 0, \quad p_2 = 0.$$

Find $B(x)$ everywhere in the slab.

5A Combinatorics

Let $\mathcal{A} \subset [n]^{(r)}$ where $r \leqslant n/2$. Prove that, if $\mathcal{A}$ is 1-intersecting, then $|\mathcal{A}| \leqslant \binom{n-1}{r-1}$. State an upper bound on $|\mathcal{A}|$ that is valid if $\mathcal{A}$ is t-intersecting and n is large compared to r and t.

Let $\mathcal{B} \subset \mathcal{P}([n])$ be maximal 1-intersecting; that is, $\mathcal{B}$ is 1-intersecting but if $\mathcal{B} \subset \mathcal{C} \subset \mathcal{P}([n])$ and $\mathcal{B} \neq \mathcal{C}$ then $\mathcal{C}$ is not 1-intersecting. Show that $|\mathcal{B}| = 2^{n-1}$.

Let $\mathcal{B} \subset \mathcal{P}([n])$ be 2-intersecting. Show that $|\mathcal{B}| \geqslant 2^{n-2}$ is possible. Can the inequality be strict?

6C Representation Theory

Compute the character table of A_5 (begin by listing the conjugacy classes and their orders).

[*It is not enough to write down the result; you must justify your answer.*]

7C Galois Theory

Prove that the Galois group G of the polynomial $X^6 + 3$ over $\mathbf{Q}$ is of order 6. By explicitly describing the elements of G, show that they have orders $1, 2$ or 3. Hence deduce that G is isomorphic to S_3.

Why does it follow that $X^6 + 3$ is reducible over the finite field $\mathbf{F}_p$, for all primes p?

8B Differentiable Manifolds

Define an immersion and an embedding of one manifold in another. State a necessary and sufficient condition for an immersion to be an embedding and prove its necessity.

Assuming the existence of "bump functions" on Euclidean spaces, state and prove a version of Whitney's embedding theorem.

Deduce that $\mathbb{RP}^n$ embeds in $\mathbb{R}^{(n+1)^2}$.

9B Number Fields

Let $K = \mathbf{Q}(\alpha)$ be a number field, where $\alpha \in \mathcal{O}_K$. Let f be the (normalized) minimal polynomial of α over $\mathbf{Q}$. Show that the discriminant $\operatorname{disc}(f)$ of f is equal to $(\mathcal{O}_K : \mathbf{Z}[\alpha])^2 D_K$.

Show that $f(x) = x^3 + 5x^2 - 19$ is irreducible over $\mathbf{Q}$. Determine $\operatorname{disc}(f)$ and the ring of algebraic integers $\mathcal{O}_K$ of $K = \mathbf{Q}(\alpha)$, where $\alpha \in \mathbf{C}$ is a root of f.

10A Hilbert Spaces

State and prove the Riesz representation theorem for bounded linear functionals on a Hilbert space H.

[*You may assume, without proof, that* $H = E \oplus E^{\perp}$, *for every closed subspace* E *of* H.]

Prove that, for every $T \in \mathcal{B}(H)$, there is a unique $T^* \in \mathcal{B}(H)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for every $x, y \in H$. Prove that $\|T^*T\| = \|T\|^2$ for every $T \in \mathcal{B}(H)$.

Define a *normal* operator $T \in \mathcal{B}(H)$. Prove that T is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in H$. Deduce that every point in the spectrum of a normal operator T is an approximate eigenvalue of T.

[*You may assume, without proof, any general criterion for the invertibility of a bounded linear operator on* H.]

11B Riemann Surfaces

Recall that an *automorphism* of a Riemann surface is a bijective analytic map onto itself, and that the inverse map is then guaranteed to be analytic.

Let Δ denote the disc $\{z \in \mathbb{C} \,\big|\, |z| < 1\}$, and let $\Delta^* = \Delta - \{0\}$.

(a) Prove that an automorphism $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ is a Euclidian rotation.

[*Hint: Apply the maximum modulus principle to the functions* $\phi(z)/z$ *and* $\phi^{-1}(z)/z$.]

(b) Prove that a holomorphic map $\phi : \Delta^* \to \Delta$ extends to the entire disc, and use this to conclude that any automorphism of Δ^* is a Euclidean rotation.

[*You may use the result stated in part (a).*]

(c) Define an analytic map between Riemann surfaces. Show that a continuous map between Riemann surfaces, known to be analytic everywhere except perhaps at a single point P, is, in fact, analytic everywhere.

12B Logic, Computation and Set Theory

(i) What is the *Halting Problem*? What is an *unsolvable* problem?

(ii) Prove that the Halting Problem is unsolvable. Is it decidable whether or not a machine halts with input zero?

13D Probability and Measure

State and prove Hölder's Inequality.

[*Jensen's inequality, and other standard results, may be assumed.*]

Let (X_n) be a sequence of random variables bounded in L_p for some $p > 1$. Prove that (X_n) is uniformly integrable.

Suppose that $X \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $p \in (1, \infty)$. Show that $X \in L_r(\Omega, \mathcal{F}, \mathbb{P})$ for all $1 \leqslant r < p$ and that $||X||_r$ is an increasing function of r on $[1, p]$.

Show further that $\lim_{r \to 1^+} ||X||_r = ||X||_1$.

14E Information Theory

Let $p_1, \ldots, p_n$ be a probability distribution, with $p^* = \max_i[p_i]$. Prove that

$$(i) - \sum_i p_i \log p_i \geqslant -p^* \log p^* - (1-p^*)\log(1-p^*);$$

$$(ii) - \sum_i p_i \log p_i \geqslant \log(1/p^*); \text{ and}$$

$$(iii) - \sum_i p_i \log p_i \geqslant 2(1-p^*).$$

All logarithms are to base 2.

[*Hint: To prove (iii), it is convenient to use (i) for* $p^* \geqslant \frac{1}{2}$ *and (ii) for* $p^* \leqslant \frac{1}{2}$.]

Random variables X and Y with values x and y from finite 'alphabets' I and J represent the input and output of a transmission channel, with the conditional probability $p(x \mid y) = \mathbb{P}(X = x \mid Y = y)$. Let $h(p(\cdot \mid y))$ denote the entropy of the conditional distribution $p(\cdot \mid y)$, $y \in J$, and $h(X \mid Y)$ denote the conditional entropy of X given Y. Define the ideal observer decoding rule as a map $f : J \to I$ such that $p(f(y) \mid y) = \max_{x \in I} p(x \mid y)$ for all $y \in J$. Show that under this rule the error probability

$$\pi_{\text{er}}(y) = \sum_{\substack{x \in I \\ x \neq f(y)}} p(x \mid y)$$

satisfies $\pi_{\text{er}}(y) \leqslant \frac{1}{2} h(p(\cdot \mid y))$, and the expected value satisfies

$$\mathbb{E}\pi_{\text{er}}(Y) \leqslant \frac{1}{2} h(X \mid Y).$$

15E Principles of Statistics

(i) What are the main approaches by which prior distributions are specified in Bayesian inference?

Define the risk function of a decision rule d. Given a prior distribution, define what is meant by a Bayes decision rule and explain how this is obtained from the posterior distribution.

(ii) Dashing late into King's Cross, I discover that Harry must have already boarded the Hogwarts Express. I must therefore make my own way onto platform nine and three-quarters. Unusually, there are two guards on duty, and I will ask one of them for directions. It is safe to assume that one guard is a Wizard, who will certainly be able to direct me, and the other a Muggle, who will certainly not. But which is which? Before choosing one of them to ask for directions to platform nine and three-quarters, I have just enough time to ask one of them "Are you a Wizard?", and on the basis of their answer I must make my choice of which guard to ask for directions. I know that a Wizard will answer this question truthfully, but that a Muggle will, with probability $\frac{1}{3}$, answer it untruthfully.

Failure to catch the Hogwarts Express results in a loss which I measure as 1000 galleons, there being no loss associated with catching up with Harry on the train.

Write down an exhaustive set of non-randomised decision rules for my problem and, by drawing the associated risk set, determine my minimax decision rule.

My prior probability is $\frac{2}{3}$ that the guard I ask "Are you a Wizard?" is indeed a Wizard. What is my Bayes decision rule?

16D Stochastic Financial Models

(i) The price of the stock in the binomial model at time r, $1 \leqslant r \leqslant n$, is $S_r = S_0 \prod_{j=1}^{r} Y_j$, where $Y_1, Y_2, \ldots, Y_n$ are independent, identically-distributed random variables with $\mathbb{P}(Y_1 = u) = p = 1 - \mathbb{P}(Y_1 = d)$ and the initial price S_0 is a constant. Denote the fixed interest rate on the bank account by ρ, where $u > 1 + \rho > d > 0$, and let the discount factor $\alpha = 1/(1+\rho)$. Determine the unique value $p = \overline{p}$ for which the sequence $\{\alpha^r S_r,\ 0 \leqslant r \leqslant n\}$ is a martingale.

Explain briefly the significance of $\overline{p}$ for the pricing of contingent claims in the model.

(ii) Let T_a denote the first time that a standard Brownian motion reaches the level $a > 0$. Prove that for $t > 0$,

$$\mathbb{P}(T_a \leqslant t) = 2\left[1 - \Phi\left(a/\sqrt{t}\right)\right],$$

where Φ is the standard normal distribution function.

Suppose that A_t and B_t represent the prices at time t of two different stocks with initial prices 1 and 2, respectively; the prices evolve so that they may be represented as $A_t = e^{\sigma_1 X_t + \mu t}$ and $B_t = 2e^{\sigma_2 Y_t + \mu t}$, respectively, where $\{X_t\}_{t \geqslant 0}$ and $\{Y_t\}_{t \geqslant 0}$ are independent standard Brownian motions and σ_1, σ_2 and μ are constants. Let T denote the first time, if ever, that the prices of the two stocks are the same. Determine $\mathbb{P}(T \leqslant t)$, for $t > 0$.

17K Dynamical Systems

Define topological conjugacy and C^1-conjugacy.

Let a, b be real numbers with $a > b > 0$ and let F_a, F_b be the maps of $(0, \infty)$ to itself given by $F_a(x) = ax, F_b(x) = bx$. For which pairs a, b are F_a and F_b topologically conjugate? Would the answer be the same for C^1-conjugacy? Justify your statements.

18A Partial Differential Equations

(a) Solve the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2$$

together with the boundary condition on the x-axis:

$$u(x,0) = f(x) \ ,$$

where f is a smooth function. You should discuss the domain on which the solution is smooth. For which functions f can the solution be extended to give a smooth solution on the upper half plane $\{y > 0\}$?

(b) Solve the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

together with the boundary condition on the unit circle:

$$u(x,y) = x \qquad \text{when} \qquad x^2 + y^2 = 1.$$

19L Methods of Mathematical Physics

State and prove the convolution theorem for Laplace transforms.

Use the convolution theorem to prove that the Beta function

$$B(p,q) = \int_0^1 (1-\tau)^{p-1}\tau^{q-1} d\tau$$

may be written in terms of the Gamma function as

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \ .$$

20K Numerical Analysis

(i) Let A be a symmetric $n \times n$ matrix such that

$$A_{k,k} > \sum_{\substack{l=1 \\ l \neq k}}^{n} |A_{k,l}| \qquad 1 \leqslant k \leqslant n.$$

Prove that it is positive definite.

(ii) Prove that both Jacobi and Gauss-Seidel methods for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, where the matrix A obeys the conditions of (i), converge.

[*You may quote the Householder-John theorem without proof.*]

21F Electrodynamics

Explain the multipole expansion in electrostatics, and devise formulae for the total charge, dipole moments and quadrupole moments given by a static charge distribution $\rho(\mathbf{r})$.

A nucleus is modelled as a uniform distribution of charge inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

The total charge of the nucleus is Q. What are the dipole moments and quadrupole moments of this distribution?

Describe qualitatively what happens if the nucleus starts to oscillate.

22F Statistical Physics

Write down the first law of thermodynamics in differential form for an infinitesimal reversible change in terms of the increments dE, dS and dV, where E, S and V are to be defined. Briefly give an interpretation of each term and deduce that

$$P = -\left(\frac{\partial E}{\partial V}\right)_S , \qquad T = \left(\frac{\partial E}{\partial S}\right)_V .$$

Define the specific heat at constant volume C_V and show that for an adiabatic change

$$C_V \, dT + \left(\left(\frac{\partial E}{\partial V}\right)_T + P\right) dV = 0 .$$

Derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V ,$$

where T is temperature and hence show that

$$\left(\frac{\partial E}{\partial V}\right)_T = -P + T\left(\frac{\partial P}{\partial T}\right)_V .$$

An imperfect gas of volume V obeys the van der Waals equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT ,$$

where a and b are non-negative constants. Show that

$$\left(\frac{\partial C_V}{\partial V}\right)_T = 0 ,$$

and deduce that C_V is a function of T only. It can further be shown that in this case C_V is independent of T. Hence show that

$$T(V - b)^{R/C_V}$$

is constant on adiabatic curves.

23J Applications of Quantum Mechanics

A steady beam of particles, having wavenumber k and moving in the z direction, scatters on a spherically-symmetric potential. Write down the asymptotic form of the wave function at large r.

The incoming wave is written as a partial-wave series

$$\sum_{\ell=0}^{\infty} \chi_\ell(kr) P_\ell(\cos\theta).$$

Show that for large r

$$\chi_\ell(kr) \sim \frac{\ell+\frac{1}{2}}{ikr}\left(e^{ikr} - (-1)^\ell e^{-ikr}\right)$$

and calculate $\chi_0(kr)$ and $\chi_1(kr)$ for all r.

Write down the second-order differential equation satisfied by the $\chi_\ell(kr)$. Construct a second linearly-independent solution for each ℓ that is singular at $r=0$ and, when it is suitably normalised, has large-r behaviour

$$\frac{\ell+\frac{1}{2}}{ikr}\left(e^{ikr} + (-1)^\ell e^{-ikr}\right).$$

24J General Relativity

(i) The metric of any two-dimensional curved space, rotationally symmetric about a point P, can by suitable choice of coordinates be written locally in the form

$$ds^2 = e^{2\phi(r)}(dr^2 + r^2 d\theta^2),$$

where $r = 0$ at P, $r > 0$ away from P, and $0 \leqslant \theta < 2\pi$. Labelling the coordinates as $(x^1, x^2) = (r, \theta)$, show that the Christoffel symbols $\Gamma^1_{12}, \Gamma^2_{11}$ and Γ^2_{22} are each zero, and compute the non-zero Christoffel symbols $\Gamma^1_{11}, \Gamma^1_{22}$ and $\Gamma^2_{12} = \Gamma^2_{21}$.

The Ricci tensor R_{ab} $(a, b = 1, 2)$ is defined by

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^d_{ac}\Gamma^c_{bd},$$

where a comma denotes a partial derivative. Show that $R_{12} = 0$ and that

$$R_{11} = -\phi'' - r^{-1}\phi', \qquad R_{22} = r^2 R_{11}.$$

(ii) Suppose further that, in a neighbourhood of P, the Ricci scalar R takes the constant value -2. Find a second order differential equation, which you should denote by $(*)$, for $\phi(r)$.

This space of constant Ricci scalar can, by a suitable coordinate transformation $r \to \chi(r)$, leaving θ invariant, be written locally as

$$ds^2 = d\chi^2 + \sinh^2 \chi d\theta^2$$

By studying this coordinate transformation, or otherwise, find $\cosh\chi$ and $\sinh\chi$ in terms of r (up to a constant of integration). Deduce that

$$e^{\phi(r)} = \frac{2A}{(1 - A^2 r^2)} \quad , \quad (0 \leqslant Ar < 1),$$

where A is a positive constant and verify that your equation $(*)$ for ϕ holds.

[*Note that*

$$\int \frac{d\chi}{\sinh\chi} = \text{const.} + \frac{1}{2}\log\bigl(\cosh\chi - 1\bigr) - \frac{1}{2}\log\bigl(\cosh\chi + 1\bigr).\Bigr]$$

25H Fluid Dynamics II

The energy equation for the motion of a viscous, incompressible fluid states that

$$\frac{d}{dt}\int_{V(t)} \frac{1}{2}\rho u^2 dV = \int_{S(t)} u_i \sigma_{ij} n_j dS - 2\mu \int_{V(t)} e_{ij} e_{ij} dV.$$

Interpret each term in this equation and explain the meaning of the symbols used.

For steady rectilinear flow in a (not necessarily circular) pipe having rigid stationary walls, deduce a relation between the viscous dissipation per unit length of the pipe, the pressure gradient G, and the volume flux Q.

Starting from the Navier-Stokes equations, calculate the velocity field for steady rectilinear flow in a circular pipe of radius a. Using the relationship derived above, or otherwise, find in terms of G the viscous dissipation per unit length for this flow.

[*In cylindrical polar coordinates,*

$$\nabla^2 w(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) .]$$

26L Waves in Fluid and Solid Media

Derive Riemann's equations for finite amplitude, one-dimensional sound waves in a perfect gas with ratio of specific heats γ.

At time $t = 0$ the gas is at rest and has uniform density ρ_0, pressure p_0 and sound speed c_0. A piston initially at $x = 0$ starts moving backwards at time $t = 0$ with displacement $x = -a\sin\omega t$, where a and ω are positive constants. Explain briefly how to find the resulting disturbance using a graphical construction in the xt-plane, and show that prior to any shock forming $c = c_0 + \frac{1}{2}(\gamma - 1)u$.

For small amplitude a, show that the excess pressure $\Delta p = p - p_0$ and the excess sound speed $\Delta c = c - c_0$ are related by

$$\frac{\Delta p}{p_0} = \frac{2\gamma}{\gamma - 1}\frac{\Delta c}{c_0} + \frac{\gamma(\gamma+1)}{(\gamma-1)^2}\left(\frac{\Delta c}{c_0}\right)^2 + O\left(\left(\frac{\Delta c}{c_0}\right)^3\right).$$

Deduce that the time-averaged pressure on the face of the piston exceeds p_0 by

$$\frac{1}{8}\rho_0 a^2\omega^2(\gamma+1) + O(a^3).$$

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Tuesday 5 June 2001 9.00 to 12.00

PAPER 2

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***2A, 5A*** *should be in one bundle and* ***1H, 24H*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1H Principles of Dynamics

(i) An axially symmetric top rotates freely about a fixed point O on its axis. The principal moments of inertia are A, A, C and the centre of gravity G is a distance h from O.

Define the three Euler angles θ, ϕ and ψ, specifying the orientation of the top. Use Lagrange's equations to show that there are three conserved quantities in the motion. Interpret them physically.

(ii) Initially the top is spinning with angular speed n about OG, with OG vertical, before it is slightly disturbed.

Show that, in the subsequent motion, θ stays close to zero if $C^2n^2 > 4mghA$, but if this condition fails then θ attains a maximum value given approximately by

$$\cos\theta \approx \frac{C^2n^2}{2mghA} - 1.$$

Why is this only an approximation?

2A Functional Analysis

(i) State the Stone-Weierstrass theorem for complex-valued functions. Use it to show that the trigonometric polynomials are dense in the space $C(\mathbb{T})$ of continuous, complex-valued functions on the unit circle $\mathbb{T}$ with the uniform norm.

Show further that, for $f \in C(\mathbb{T})$, the nth Fourier coefficient

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta$$

tends to 0 as $|n|$ tends to infinity.

(ii) **(a)** Let X be a normed space with the property that the series $\sum_{n=1}^{\infty} x_n$ converges whenever (x_n) is a sequence in X with $\sum_{n=1}^{\infty} ||x_n||$ convergent. Show that X is a Banach space.

(b) Let K be a compact metric space and L a closed subset of K. Let $R : C(K) \to C(L)$ be the map sending $f \in C(K)$ to its restriction $R(f) = f|L$ to L. Show that R is a bounded, linear map and that its image is a subalgebra of $C(L)$ separating the points of L.

Show further that, for each function g in the image of R, there is a function $f \in C(K)$ with $R(f) = g$ and $||f||_\infty = ||g||_\infty$. Deduce that every continuous, complex-valued function on L can be extended to a continuous function on all of K.

3C Groups, Rings and Fields

(i) Show that the ring $k = \mathbf{F}_2[X]/(X^2 + X + 1)$ is a field. How many elements does it have?

(ii) Let k be as in (i). By considering what happens to a chosen basis of the vector space k^2, or otherwise, find the order of the groups $GL_2(k)$ and $SL_2(k)$.

By considering the set of lines in k^2, or otherwise, show that $SL_2(k)$ is a subgroup of the symmetric group S_5, and identify this subgroup.

4K Dynamics of Differential Equations

(i) Define a Liapounov function for a flow ϕ on $\mathbb{R}^n$. Explain what it means for a fixed point of the flow to be Liapounov stable. State and prove Liapounov's first stability theorem.

(ii) Consider the damped pendulum

$$\ddot{\theta} + k\dot{\theta} + \sin\theta = 0,$$

where $k > 0$. Show that there are just two fixed points (considering the phase space as an infinite cylinder), and that one of these is the origin and is Liapounov stable. Show further that the origin is asymptotically stable, and that the the ω-limit set of each point in the phase space is one or other of the two fixed points, justifying your answer carefully.

[*You should state carefully any theorems you use in your answer, but you need not prove them.*]

5A Combinatorics

As usual, $R_k^{(r)}(m)$ denotes the smallest integer n such that every k-colouring of $[n]^{(r)}$ yields a monochromatic m-subset $M \in [n]^{(m)}$. Prove that $R_2^{(2)}(m) > 2^{m/2}$ for $m \geqslant 3$.

Let $\mathcal{P}([n])$ have the colex order, and for $a, b \in \mathcal{P}([n])$ let $\delta(a,b) = \max a \triangle b$; thus $a < b$ means $\delta(a,b) \in b$. Show that if $a < b < c$ then $\delta(a,b) \neq \delta(b,c)$, and that $\delta(a,c) = \max\{\delta(a,b), \delta(b,c)\}$.

Given a red-blue colouring of $[n]^{(2)}$, the 4-colouring

$$\chi : \mathcal{P}([n])^{(3)} \to \{\text{red}, \text{blue}\} \times \{0, 1\}$$

is defined as follows:

$$\chi(\{a,b,c\}) = \begin{cases} (\text{red}, 0) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is red } \text{ and } \delta(a,b) < \delta(b,c) \\ (\text{red}, 1) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is red } \text{ and } \delta(a,b) > \delta(b,c) \\ (\text{blue}, 0) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is blue and } \delta(a,b) < \delta(b,c) \\ (\text{blue}, 1) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is blue and } \delta(a,b) > \delta(b,c) \end{cases}$$

where $a < b < c$. Show that if $M = \{a_0, a_1, \ldots, a_m\} \in \mathcal{P}([n])^{(m+1)}$ is monochromatic then $\{\delta_1, \ldots, \delta_m\} \in [n]^{(m)}$ is monochromatic, where $\delta_i = \delta(a_{i-1}, a_i)$ and $a_0 < a_1 < \cdots < a_m$.

Deduce that $R_4^{(3)}(m+1) > 2^{2^{m/2}}$ for $m \geqslant 3$.

6C Representation Theory

(i) Let G be a group, and X and Y finite G-sets. Define the permutation representation $\mathbf{C}[X]$ and compute its character. Show that

$$\dim \operatorname{Hom}_G(\mathbf{C}[X], \mathbf{C}[Y])$$

is equal to the number of G-orbits in $X \times Y$.

(ii) Let $G = S_n$ $(n \geqslant 4)$, $X = \{1, \ldots, n\}$, and

$$Z = \{\, \{i, j\} \subseteq X \mid i \neq j \}$$

be the set of 2-element subsets of X. Decompose $\mathbf{C}[Z]$ into irreducibles, and determine the dimension of each irreducible constituent.

7B Differentiable Manifolds

State Stokes' Theorem.

Prove that, if M^m is a compact connected manifold and $\Phi : U \to \mathbb{R}^m$ is a surjective chart on M, then for any $\omega \in \Omega^m(M)$ there is $\eta \in \Omega^{m-1}(M)$ such that $\operatorname{supp}(\omega + d\eta) \subseteq \Phi^{-1}(\mathbf{B}^m)$, where $\mathbf{B}^m$ is the unit ball in $\mathbb{R}^m$.

[*You may assume that, if* $\omega \in \Omega^m(\mathbb{R}^m)$ *with* $\operatorname{supp}(\omega) \subseteq \mathbf{B}^m$ *and* $\int_{\mathbb{R}^m} \omega = 0$, *then* $\exists\, \eta \in \Omega^{m-1}(\mathbb{R}^m)$ *with* $\operatorname{supp}(\eta) \subseteq \mathbf{B}^m$ *such that* $d\eta = \omega$.]

By considering the m-form

$$\omega = x_1 dx_2 \wedge ... \wedge dx_{m+1} + \cdots + x_{m+1} dx_1 \wedge ... \wedge dx_m$$

on $\mathbb{R}^{m+1}$, or otherwise, deduce that $H^m(S^m) \cong \mathbb{R}$.

8C Algebraic Topology

Show that the fundamental group of the 2-torus $S^1 \times S^1$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

Show that an injective continuous map from the circle S^1 to itself induces multiplication by ± 1 on the fundamental group.

Show that there is no retraction from the solid torus $S^1 \times D^2$ to its boundary.

9B Number Fields

Determine the ideal class group of $\mathbf{Q}(\sqrt{-11})$.

Find all solutions of the diophantine equation

$$y^2 + 11 = x^3 \qquad (x, y \in \mathbf{Z})\ .$$

[*Minkowski's bound is* $n! n^{-n} (4/\pi)^{r_2} |D_k|^{1/2}$.]

10B Algebraic Curves

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map given by $f(X_0 : X_1 : X_2) = (X_1X_2 : X_0X_2 : X_0X_1)$. Determine whether f is defined at the following points: $(1:1:1), (0:1:1), (0:0:1)$.

Let $C \subset \mathbb{P}^2$ be the curve defined by $X_1^2 X_2 - X_0^3 = 0$. Define a bijective morphism $\alpha : \mathbb{P}^1 \to C$. Prove that α is not an isomorphism.

11B Logic, Computation and Set Theory

Let U be an arbitrary set, and $\mathcal{P}(U)$ the power set of U. For X a subset of $\mathcal{P}(U)$, the *dual* $X^\vee$ of X is the set $\{y \subseteq U : (\forall x \in X)(y \cap x \neq \emptyset)\}$.

(i) Show that $X \subseteq Y \to Y^\vee \subseteq X^\vee$.

Show that for $\{X_i : i \in I\}$ a family of subsets of $\mathcal{P}(U)$

$$\left(\bigcup\{X_i : i \in I\}\right)^\vee = \bigcap\{X_i^\vee : i \in I\}.$$

(ii) Consider $S = \{X \subseteq \mathcal{P}(U) : X \subseteq X^\vee\}$. Show that $S, \subseteq$ is a chain-complete poset.

State Zorn's lemma and use it to deduce that there exists X with $X = X^\vee$.

Show that if $X = X^\vee$ then the following hold:

X is closed under superset; for all $U' \subseteq U$, X contains either U' or $U \setminus U'$.

12D Probability and Measure

(a) Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}\left((0,1)\right)$ be the Borel σ-field and let $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. What is the distribution of the random variable Z, where $Z(\omega) = 2\omega - 1$?

Let $\omega = \sum_{n=1}^{\infty} 2^{-n} R_n(\omega)$ be the binary expansion of the point $\omega \in \Omega$ and set $U(\omega) = \sum\limits_{n\ odd} 2^{-n} Q_n(\omega)$, where $Q_n(\omega) = 2R_n(\omega) - 1$. Find a random variable V independent of U such that U and V are identically distributed and $U + \frac{1}{2}V$ is uniformly distributed on $(-1,1)$.

(b) Now suppose that on some probability triple X and Y are independent, identically-distributed random variables such that $X + \frac{1}{2}Y$ is uniformly distributed on $(-1,1)$.

Let ϕ be the characteristic function of X. Calculate $\phi(t)/\phi(t/4)$. Show that the distribution of X must be the same as the distribution of the random variable U in (a).

13D Applied Probability

Let M be a Poisson random measure on $E = \mathbb{R} \times [0, \pi)$ with constant intensity λ. For $(x, \theta) \in E$, denote by $l(x, \theta)$ the line in $\mathbb{R}^2$ obtained by rotating the line $\{(x, y) : y \in \mathbb{R}\}$ through an angle θ about the origin.

Consider the line process $L = M \circ\ l^{-1}$.

(i) What is the distribution of the number of lines intersecting the disc $\{z \in \mathbb{R}^2 : |z| \leqslant a\}$?

(ii) What is the distribution of the distance from the origin to the nearest line?

(iii) What is the distribution of the distance from the origin to the kth nearest line?

14E Information Theory

A subset $\mathcal{C}$ of the Hamming space $\{0,1\}^n$ of cardinality $|\mathcal{C}| = r$ and with the minimal (Hamming) distance $\min\,[d(x,x') : x,x' \in \mathcal{C}, x \neq x'] = \delta$ is called an $[n,r,\delta]$-code (not necessarily linear). An $[n,r,\delta]$-code is called *maximal* if it is not contained in any $[n,r+1,\delta]$-code. Prove that an $[n,r,\delta]$-code is maximal if and only if for any $y \in \{0,1\}^n$ there exists $x \in \mathcal{C}$ such that $d(x,y) < \delta$. Conclude that if there are δ or more changes made in a codeword then the new word is closer to some other codeword than to the original one.

Suppose that a maximal $[n,r,\delta]$-code is used for transmitting information via a binary memoryless channel with the error probability p, and the receiver uses the maximum likelihood decoder. Prove that the probability of erroneous decoding, $\pi_{\rm err}^{\rm ml}$, obeys the bounds

$$1 - b(n,\delta-1) \leqslant \pi_{\rm err}^{\rm ml} \leqslant 1 - b(n,[(\delta-1)/2]),$$

where

$$b(n,m) = \sum_{0 \leqslant k \leqslant m} \binom{n}{k} p^k (1-p)^{n-k}$$

is a partial binomial sum and $[\,\cdot\,]$ is the integer part.

15D Optimization and Control

A street trader wishes to dispose of k counterfeit Swiss watches. If he offers one for sale at price u he will sell it with probability ae^{-u}. Here a is known and less than 1. Subsequent to each attempted sale (successful or not) there is a probability $1-\beta$ that he will be arrested and can make no more sales. His aim is to choose the prices at which he offers the watches so as to maximize the expected values of his sales up until the time he is arrested or has sold all k watches.

Let $V(k)$ be the maximum expected amount he can obtain when he has k watches remaining and has not yet been arrested. Explain why $V(k)$ is the solution to

$$V(k) = \max_{u>0} \left\{ ae^{-u}[u + \beta V(k-1)] + (1-ae^{-u})\beta V(k) \right\} .$$

Denote the optimal price by u_k and show that

$$u_k = 1 + \beta V(k) - \beta V(k-1)$$

and that

$$V(k) = ae^{-u_k}/(1-\beta)\,.$$

Show inductively that $V(k)$ is a nondecreasing and concave function of k.

16E Principles of Statistics

(i) Let $X_1, \ldots, X_n$ be independent, identically-distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a minimal sufficient statistic for μ.

Let $T_1 = n^{-1} \sum_{i=1}^{n} X_i$ and $T_2 = \sqrt{n^{-1} \sum_{i=1}^{n} X_i^2}$. Write down the distribution of X_i/μ, and hence show that $Z = T_1/T_2$ is ancillary. Explain briefly why the Conditionality Principle would lead to inference about μ being drawn from the conditional distribution of T_2 given Z.

What is the maximum likelihood estimator of μ?

(ii) Describe briefly the Bayesian approach to predictive inference.

Let $Z_1, \ldots, Z_n$ be independent, identically-distributed $N(\mu, \sigma^2)$ random variables, with μ, σ^2 both unknown. Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2 based on $Z_1, \ldots, Z_n$, and state, without proof, their joint distribution.

Suppose that it is required to construct a prediction interval $I_{1-\alpha} \equiv I_{1-\alpha}(Z_1, \ldots, Z_n)$ for a future, independent, random variable Z_0 with the same $N(\mu, \sigma^2)$ distribution, such that

$$P(Z_0 \in I_{1-\alpha}) = 1 - \alpha,$$

with the probability over the *joint* distribution of $Z_0, Z_1, \ldots, Z_n$. Let

$$I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2) = \left[\bar{Z}_n - z_{\alpha/2}\sigma\sqrt{1+1/n},\ \bar{Z}_n + z_{\alpha/2}\sigma\sqrt{1+1/n}\right],$$

where $\bar{Z}_n = n^{-1} \sum_{i=1}^{n} Z_i$, and $\Phi(z_\beta) = 1 - \beta$, with Φ the distribution function of $N(0,1)$.

Show that $P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2)) = 1 - \alpha$.

By considering the distribution of $(Z_0 - \bar{Z}_n)/\left(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}}\right)$, or otherwise, show that

$$P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) < 1 - \alpha,$$

and show how to construct an interval $I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)$ with

$$P(Z_0 \in I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) = 1 - \alpha.$$

[*Hint: if Y has the t-distribution with m degrees of freedom and $t_\beta^{(m)}$ is defined by $P(Y < t_\beta^{(m)}) = 1 - \beta$ then $t_\beta > z_\beta$ for $\beta < \frac{1}{2}$.*]

17A Partial Differential Equations

Define the Schwartz space $\mathcal{S}(\mathbb{R})$ and the corresponding space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

Use the Fourier transform to give an integral formula for the solution of the equation

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} + u = f \qquad (*)$$

for $f \in \mathcal{S}(\mathbb{R})$. Prove that your solution lies in $\mathcal{S}(\mathbb{R})$. Is your formula the unique solution to $(*)$ in the Schwartz space?

Deduce from this formula an integral expression for the fundamental solution of the operator $P = -\frac{d^2}{dx^2} + \frac{d}{dx} + 1$.

Let K be the function:

$$K(x) = \begin{cases} \frac{1}{\sqrt{5}} e^{-(\sqrt{5}-1)x/2} & \text{for } x \geqslant 0, \\ \frac{1}{\sqrt{5}} e^{(\sqrt{5}+1)x/2} & \text{for } x \leqslant 0. \end{cases}$$

Using the definition of distributional derivatives verify that this function is a fundamental solution for P.

18L Methods of Mathematical Physics

The Bessel function $J_\nu(z)$ is defined, for $|\arg z| < \pi/2$, by

$$J_\nu(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} \mathrm{e}^{(t-t^{-1})z/2} t^{-\nu-1} dt \ ,$$

where the path of integration is the Hankel contour and $t^{-\nu-1}$ is the principal branch.

Use the method of steepest descent to show that, as $z \to +\infty$,

$$J_\nu(z) \sim (2/\pi z)^{\frac{1}{2}} \cos(z - \pi\nu/2 - \pi/4) \ .$$

You should give a rough sketch of the steepest descent paths.

19K Numerical Analysis

(i) Define m-step BDF (backward differential formula) methods for the numerical solution of ordinary differential equations and derive explicitly their coefficients.

(ii) Prove that the linear stability domain of the two-step BDF method includes the interval $(-\infty, 0)$.

20F Electrodynamics

In a superconductor, there are superconducting charge carriers with number density n, mass m and charge q. Starting from the quantum mechanical wavefunction $\Psi = Re^{i\Phi}$ (with real R and Φ), construct a formula for the electric current and explain carefully why your result is gauge invariant.

Now show that inside a superconductor a static magnetic field obeys the equation

$$\nabla^2 \, \mathbf{B} = \frac{\mu_0 \, n \, q^2}{m} \mathbf{B}.$$

A superconductor occupies the region $z > 0$, while for $z < 0$ there is a vacuum with a constant magnetic field in the x direction. Show that the magnetic field cannot penetrate deep into the superconductor.

21F Foundations of Quantum Mechanics

(i) Hermitian operators $\hat{x}$, $\hat{p}$, satisfy $[\hat{x},\hat{p}] = i\hbar$. The eigenvectors $|p\rangle$, satisfy $\hat{p}|p\rangle = p|p\rangle$ and $\langle p'|p\rangle = \delta(p'-p)$. By differentiating with respect to b verify that

$$e^{-ib\hat{x}/\hbar}\hat{p}\, e^{ib\hat{x}/\hbar} = \hat{p} + b$$

and hence show that

$$e^{ib\hat{x}/\hbar}|p\rangle = |p+b\rangle.$$

Show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\,\langle p|\psi\rangle$$

and

$$\langle p|\hat{p}|\psi\rangle = p\,\langle p|\psi\rangle\,.$$

(ii) A quantum system has Hamiltonian $H = H_0 + H_1$, where H_1 is a small perturbation. The eigenvalues of H_0 are ϵ_n. Give (without derivation) the formulae for the first order and second order perturbations in the energy level of a non-degenerate state. Suppose that the rth energy level of H_0 has j degenerate states. Explain how to determine the eigenvalues of H corresponding to these states to first order in H_1.

In a particular quantum system an orthonormal basis of states is given by $|n_1,n_2\rangle$, where n_i are integers. The Hamiltonian is given by

$$H = \sum_{n_1,n_2}(n_1^2+n_2^2)\,|n_1,n_2\rangle\langle n_1,n_2| + \sum_{n_1,n_2,n'_1,n'_2}\lambda_{|n_1-n'_1|,|n_2-n'_2|}\,|n_1,n_2\rangle\langle n'_1,n'_2|\,,$$

where $\lambda_{r,s} = \lambda_{s,r}$, $\lambda_{0,0} = 0$ and $\lambda_{r,s} = 0$ unless r and s are both even.

Obtain an expression for the ground state energy to second order in the perturbation, $\lambda_{r,s}$. Find the energy eigenvalues of the first excited state to first order in the perturbation. Determine a matrix (which depends on two independent parameters) whose eigenvalues give the first order energy shift of the second excited state.

22J Applications of Quantum Mechanics

A particle of charge e moves freely within a cubical box of side a. Its initial wavefunction is

$$(2/a)^{-\frac{3}{2}}\sin(\pi x/a)\sin(\pi y/a)\sin(\pi z/a).$$

A uniform electric field $\mathcal{E}$ in the x direction is switched on for a time T. Derive from first principles the probability, correct to order $\mathcal{E}^2$, that after the field has been switched off the wave function will be found to be

$$(2/a)^{-\frac{3}{2}}\sin(2\pi x/a)\sin(\pi y/a)\sin(\pi z/a).$$

23J General Relativity

(i) Show that the geodesic equation follows from a variational principle with Lagrangian

$$L \quad = \quad g_{ab}\dot{x}^a\dot{x}^b$$

where the path of the particle is $x^a(\lambda)$, and λ is an affine parameter along that path.

(ii) The Schwarzschild metric is given by

$$ds^2 = dr^2\left(1-\frac{2M}{r}\right)^{-1} + r^2(d\theta^2+\sin^2\theta d\phi^2) - \left(1-\frac{2M}{r}\right)dt^2.$$

Consider a photon which moves within the equatorial plane $\theta = \frac{\pi}{2}$. Using the above Lagrangian, or otherwise, show that

$$\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right) = E, \quad \text{and} \quad r^2\left(\frac{d\phi}{d\lambda}\right) = h,$$

for constants E and h. Deduce that

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1-\frac{2M}{r}\right). \qquad (*)$$

Assume further that the photon approaches from infinity. Show that the impact parameter b is given by

$$b = \frac{h}{E} \quad .$$

By considering the equation $(*)$, or otherwise

(a) show that, if $b^2 > 27M^2$, the photon is deflected but not captured by the black hole;

(b) show that, if $b^2 < 27M^2$, the photon is captured;

(c) describe, with justification, the qualitative form of the photon's orbit in the case $b^2 = 27M^2$.

24H Fluid Dynamics II

Explain what is meant by a Stokes flow and show that, in such a flow, in the absence of body forces, $\partial\sigma_{ij}/\partial x_j = 0$, where σ_{ij} is the stress tensor.

State and prove the *reciprocal theorem* for Stokes flow.

When a rigid sphere of radius a translates with velocity $\mathbf{U}$ through unbounded fluid at rest at infinity, it may be shown that the traction per unit area, $\boldsymbol{\sigma}\cdot\mathbf{n}$, exerted by the sphere on the fluid, has the uniform value $3\mu\mathbf{U}/2a$ over the sphere surface. Find the drag on the sphere.

Suppose that the same sphere is free of external forces and is placed with its centre at the origin in an unbounded Stokes flow given in the absence of the sphere as $\mathbf{u}_s(\mathbf{x})$. By applying the reciprocal theorem to the perturbation to the flow generated by the presence of the sphere, and assuming this to tend to zero sufficiently rapidly at infinity, show that the instantaneous velocity of the centre of the sphere is

$$\mathbf{V} = \frac{1}{4\pi a^2}\int_{r=a} \mathbf{u}_s(\mathbf{x})dS.$$

25L Waves in Fluid and Solid Media

A semi-infinite elastic medium with shear modulus μ_1 and shear-wave speed c_1 lies in $y < 0$. Above it there is a layer $0 \leq y \leqslant h$ of a second elastic medium with shear modulus μ_2 and shear-wave speed c_2 ($< c_1$). The top boundary $y = h$ is stress-free. Consider a monochromatic shear wave propagating at speed c with wavenumber k in the x-direction and with displacements only in the z-direction.

Obtain the dispersion relation

$$\tan kh\theta = \frac{\mu_1 c_2}{\mu_2 c_1}\,\frac{1}{\theta}\left(\frac{c_1^2}{c_2^2} - 1 - \theta^2\right)^{1/2}, \qquad \text{where} \quad \theta = \sqrt{\frac{c^2}{c_2^2} - 1}.$$

Deduce that the modes have a cut-off frequency $\pi n c_1 c_2/h\sqrt{c_1^2 - c_2^2}$ where they propagate at speed $c = c_1$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Wednesday 6 June 2001 9.00 to 12.00

PAPER 3

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

*Tie your answers in separate bundles, marked **A, B, C, . . . , L** according to the letter affixed to each question. (For example, **5C, 7C** should be in one bundle and **1D, 13D** in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Explain what is meant by the *transition semigroup* $\{P_t\}$ of a Markov chain X in continuous time. If the state space is finite, show under assumptions to be stated clearly, that $P_t' = GP_t$ for some matrix G. Show that a distribution π satisfies $\pi G = 0$ if and only if $\pi P_t = \pi$ for all $t \geqslant 0$, and explain the importance of such π.

(ii) Let X be a continuous-time Markov chain on the state space $S = \{1, 2\}$ with generator

$$G = \begin{pmatrix} -\beta & \beta \\ \gamma & -\gamma \end{pmatrix}, \qquad \text{where} \quad \beta, \gamma > 0.$$

Show that the transition semigroup $P_t = \exp(tG)$ is given by

$$(\beta + \gamma)P_t = \begin{pmatrix} \gamma + \beta h(t) & \beta(1 - h(t)) \\ \gamma(1 - h(t)) & \beta + \gamma h(t) \end{pmatrix},$$

where $h(t) = e^{-t(\beta+\gamma)}$.

For $0 < \alpha < 1$, let

$$H(\alpha) = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}.$$

For a continuous-time chain X, let M be a matrix with (i, j) entry $P(X(1) = j \mid X(0) = i)$, for $i, j \in S$. Show that there is a chain X with $M = H(\alpha)$ if and only if $\alpha > \frac{1}{2}$.

2A Functional Analysis

(i) Define the notion of a measurable function between measurable spaces. Show that a continuous function $\mathbb{R}^2 \to \mathbb{R}$ is measurable with respect to the Borel σ-fields on $\mathbb{R}^2$ and $\mathbb{R}$.

By using this, or otherwise, show that, when $f, g : X \to \mathbb{R}$ are measurable with respect to some σ-field $\mathcal{F}$ on X and the Borel σ-field on $\mathbb{R}$, then $f + g$ is also measurable.

(ii) State the Monotone Convergence Theorem for $[0, \infty]$-valued functions. Prove the Dominated Convergence Theorem.

[*You may assume the Monotone Convergence Theorem but any other results about integration that you use will need to be stated carefully and proved.*]

Let X be the real Banach space of continuous real-valued functions on $[0, 1]$ with the uniform norm. Fix $u \in X$ and define

$$T : X \to \mathbb{R} \; ; \quad f \mapsto \int_0^1 f(t)u(t) \, dt \; .$$

Show that T is a bounded, linear map with norm

$$||T|| = \int_0^1 |u(t)| \, dt \; .$$

Is it true, for every choice of u, that there is function $f \in X$ with $||f|| = 1$ and $||T(f)|| = ||T||$?

3J Electromagnetism

(i) Develop the theory of electromagnetic waves starting from Maxwell equations in vacuum. You should relate the wave-speed c to ϵ_0 and μ_0 and establish the existence of plane, plane-polarized waves in which $\mathbf{E}$ takes the form

$$\mathbf{E} = (E_0 \cos(kz - \omega t), 0, 0) \; .$$

You should give the form of the magnetic field $\mathbf{B}$ in this case.

(ii) Starting from Maxwell's equation, establish Poynting's theorem.

$$-\mathbf{j} \cdot \mathbf{E} \quad = \quad \frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} \; ,$$

where $W = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$ and $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B}$. Give physical interpretations of W, S and the theorem.

Compute the averages over space and time of W and $\mathbf{S}$ for the plane wave described in (i) and relate them. Comment on the result.

4K Dynamics of Differential Equations

(i) Define a hyperbolic fixed point x_0 of a flow determined by a differential equation $\dot{x} = f(x)$ where $x \in R^n$ and f is C^1 (i.e. differentiable). State the Hartman-Grobman Theorem for flow near a hyperbolic fixed point. For nonlinear flows in R^2 with a hyperbolic fixed point x_0, does the theorem necessarily allow us to distinguish, on the basis of the linearized flow near x_0 between (a) a stable focus and a stable node; and (b) a saddle and a stable node? Justify your answers briefly.

(ii) Show that the system:

$$\begin{aligned}\dot{x} &= -(\mu+1) + (\mu-3)x - y + 6x^2 + 12xy + 5y^2 - 2x^3 - 6x^2y - 5xy^2,\\ \dot{y} &= 2 - 2x + (\mu-5)y + 4xy + 6y^2 - 2x^2y - 6xy^2 - 5y^3\end{aligned}$$

has a fixed point $(x_0, 0)$ on the x-axis. Show that there is a bifurcation at $\mu = 0$ and determine the stability of the fixed point for $\mu > 0$ and for $\mu < 0$.

Make a linear change of variables of the form $u = x - x_0 + \alpha y$, $v = x - x_0 + \beta y$, where α and β are constants to be determined, to bring the system into the form:

$$\begin{aligned}\dot{u} &= v + u[\mu - (u^2 + v^2)]\\ \dot{v} &= -u + v[\mu - (u^2 + v^2)]\end{aligned}$$

and hence determine whether the periodic orbit produced in the bifurcation is stable or unstable, and whether it exists in $\mu < 0$ or $\mu > 0$.

5C Representation Theory

Let $G = SU_2$, and V_n be the vector space of homogeneous polynomials of degree n in the variables x and y.

(i) Define the action of G on V_n, and prove that V_n is an irreducible representation of G.

(ii) Decompose $V_4 \otimes V_3$ into irreducible representations of SU_2. Briefly justify your answer.

(iii) SU_2 acts on the vector space $M_3(\mathbf{C})$ of complex 3×3 matrices via

$$\rho\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot X = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1\end{pmatrix} X \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1\end{pmatrix}^{-1}, \qquad X \in M_3(\mathbf{C}).$$

Decompose this representation into irreducible representations.

6C Galois Theory

Let $\mathbf{F}_p$ be the finite field with p elements (p a prime), and let k be a finite extension of $\mathbf{F}_p$. Define the Frobenius automorphism $\sigma : k \longrightarrow k$, verifying that it is an $\mathbf{F}_p$-automorphism of k.

Suppose $f = X^{p+1} + X^p + 1 \in \mathbf{F}_p[X]$ and that K is its splitting field over $\mathbf{F}_p$. Why are the zeros of f distinct? If α is any zero of f in K, show that $\sigma(\alpha) = -\frac{1}{\alpha+1}$. Prove that f has at most two zeros in $\mathbf{F}_p$ and that $\sigma^3 = id$. Deduce that the Galois group of f over $\mathbf{F}_p$ is a cyclic group of order three.

7C Algebraic Topology

Write down the Mayer-Vietoris sequence and describe all the maps involved.

Use the Mayer-Vietoris sequence to compute the homology of the n-sphere S^n for all n.

8A Hilbert Spaces

Let T be a bounded linear operator on a Hilbert space H. Define what it means to say that T is (i) *compact*, and (ii) *Fredholm*. What is the *index*, $\text{ind}(T)$, of a Fredholm operator T?

Let S, T be bounded linear operators on H. Prove that S and T are Fredholm if and only if both ST and TS are Fredholm. Prove also that if S is invertible and T is Fredholm then $\text{ind}(ST) = \text{ind}(TS) = \text{ind}(T)$.

Let K be a compact linear operator on H. Prove that $I + K$ is Fredholm with index zero.

9B Riemann Surfaces

Let $f : X \to Y$ be a nonconstant holomorphic map between compact connected Riemann surfaces. Define the *valency* of f at a point, and the *degree* of f.

Define the *genus* of a compact connected Riemann surface X (assuming the existence of a triangulation).

State the Riemann-Hurwitz theorem. Show that a holomorphic non-constant self-map of a compact Riemann surface of genus $g > 1$ is bijective, with holomorphic inverse. Verify that the Riemann surface in $\mathbb{C}^2$ described in the equation $w^4 = z^4 - 1$ is non-singular, and describe its topological type.

[*Note: The description can be in the form of a picture or in words. If you apply Riemann-Hurwitz, explain first how you compactify the surface.*]

10B Algebraic Curves

Let C be the projective curve (over an algebraically closed field k of characteristic zero) defined by the affine equation

$$x^5 + y^5 = 1 \ .$$

Determine the points at infinity of C and show that C is smooth.

Determine the divisors of the rational functions $x, y \in k(C)$.

Show that $\omega = dx/y^4$ is a regular differential on C.

Compute the divisor of ω. What is the genus of C?

11B Logic, Computation and Set Theory

(i) Write down a set of axioms for the theory of dense linear order with a bottom element but no top element.

(ii) Prove that this theory has, up to isomorphism, precisely one countable model.

12D Probability and Measure

State and prove Birkhoff's almost-everywhere ergodic theorem.

[*You need not prove convergence in L_p and the maximal ergodic lemma may be assumed provided that it is clearly stated.*]

Let $\Omega = [0, 1)$, $\mathcal{F} = \mathcal{B}([0, 1))$ be the Borel σ-field and let $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. Give an example of an ergodic measure-preserving map $\theta : \Omega \to \Omega$ (you need not prove it is ergodic).

Let $f(x) = x$ for $x \in [0, 1)$. Find (at least for all x outside a set of measure zero)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (f \circ \theta^{i-1})(x).$$

Briefly justify your answer.

13D Applied Probability

Consider an $M/G/1$ queue with arrival rate λ and traffic intensity less than 1. Prove that the moment-generating function of a typical busy period, $M_B(\theta)$, satisfies

$$M_B(\theta) = M_S(\theta - \lambda + \lambda\ M_B(\theta)),$$

where $M_S(\theta)$ is the moment-generating function of a typical service time.

If service times are exponentially distributed with parameter $\mu > \lambda$, show that

$$M_B(\theta) = \frac{\lambda + \mu - \theta - \{(\lambda + \mu - \theta)^2 - 4\lambda\mu\ \}^{1/2}}{2\lambda}$$

for all sufficiently small but positive values of θ.

14D Optimization and Control

A file of X Mb is to be transmitted over a communications link. At each time t the sender can choose a transmission rate, $u(t)$, within the range $[0, 1]$ Mb per second. The charge for transmitting at rate $u(t)$ at time t is $u(t)p(t)$. The function p is fully known at time 0. If it takes a total time T to transmit the file then there is a delay cost of γT^2, $\gamma > 0$. Thus u and T are to be chosen to minimize

$$\int_0^T u(t)p(t)dt + \gamma T^2\,,$$

where $u(t) \in [0, 1]$, $dx(t)/dt = -u(t)$, $x(0) = X$ and $x(T) = 0$. Quoting and applying appropriate results of Pontryagin's maximum principle show that a property of the optimal policy is that there exists p^* such that $u(t) = 1$ if $p(t) < p^*$ and $u(t) = 0$ if $p(t) > p^*$.

Show that the optimal p^* and T are related by $p^* = p(T) + 2\gamma T$.

Suppose $p(t) = t + 1/t$ and $X = 1$. For what value of γ is it optimal to transmit at a constant rate 1 between times 1/2 and 3/2?

15E Principles of Statistics

(i) Explain what is meant by a *uniformly most powerful unbiased test* of a null hypothesis against an alternative.

Let $Y_1, \ldots, Y_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, with σ^2 known. Explain how to construct a uniformly most powerful unbiased size α test of the null hypothesis that $\mu = 0$ against the alternative that $\mu \neq 0$.

(ii) Outline briefly the Bayesian approach to hypothesis testing based on *Bayes factors*.

Let the distribution of $Y_1, \ldots, Y_n$ be as in (i) above, and suppose we wish to test, as in (i), $\mu = 0$ against the alternative $\mu \neq 0$. Suppose we assume a $N(0, \tau^2)$ prior for μ under the alternative. Find the form of the Bayes factor B, and show that, for fixed n, $B \to \infty$ as $\tau \to \infty$.

16D Stochastic Financial Models

(i) Suppose that Z is a random variable having the normal distribution with $\mathbb{E}Z = \beta$ and $\mathrm{Var}\, Z = \tau^2$.

For positive constants a, c show that

$$\mathbb{E}\left(ae^Z - c\right)_+ = ae^{(\beta+\tau^2/2)}\Phi\left(\frac{\log(a/c)+\beta}{\tau} + \tau\right) - c\Phi\left(\frac{\log(a/c)+\beta}{\tau}\right),$$

where Φ is the standard normal distribution function.

In the context of the Black-Scholes model, derive the formula for the price at time 0 of a European call option on the stock at strike price c and maturity time t_0 when the interest rate is ρ and the volatility of the stock is σ.

(ii) Let p denote the price of the call option in the Black-Scholes model specified in (i). Show that $\frac{\partial p}{\partial \rho} > 0$ and sketch carefully the dependence of p on the volatility σ (when the other parameters in the model are held fixed).

Now suppose that it is observed that the interest rate lies in the range $0 < \rho < \rho_0$ and when it changes it is linked to the volatility by the formula $\sigma = \ln(\rho_0/\rho)$. Consider a call option at strike price $c = S_0$, where S_0 is the stock price at time 0. There is a small increase $\Delta\rho$ in the interest rate; will the price of the option increase or decrease (assuming that the stock price is unaffected)? Justify your answer carefully.

[*You may assume that the function $\phi(x)/\Phi(x)$ is decreasing in x, $-\infty < x < \infty$, and increases to $+\infty$ as $x \to -\infty$, where Φ is the standard-normal distribution function and $\phi = \Phi'$.*]

17K Dynamical Systems

If $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ show that $A^{n+2} = A^{n+1} + A^n$ for all $n \geqslant 0$. Show that A^5 has trace 11 and deduce that the subshift map defined by A has just two cycles of exact period 5. What are they?

18A Partial Differential Equations

Write down a formula for the solution $u = u(t, x)$, for $t > 0$ and $x \in \mathbb{R}^n$, of the initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad u(0, x) = f(x),$$

for f a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. State (without proof) a theorem which ensures that this formula is the unique solution in some class of functions (which should be explicitly described).

By writing $u = e^t v$, or otherwise, solve the initial value problem

$$\frac{\partial v}{\partial t} + v - \Delta v = 0, \qquad v(0, x) = g(x), \tag{$\dagger$}$$

for g a bounded continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and give a class of functions in which your solution is the unique one.

Hence, or otherwise, prove that for all $t > 0$:

$$\sup_{x \in \mathbb{R}^n} v(t, x) \leqslant \sup_{x \in \mathbb{R}^n} g(x)$$

and deduce that the solutions $v_1(t, x)$ and $v_2(t, x)$ of ($\dagger$) corresponding to initial values $g_1(x)$ and $g_2(x)$ satisfy, for $t > 0$,

$$\sup_{x \in \mathbb{R}^n} |v_1(t, x) - v_2(t, x)| \leqslant \sup_{x \in \mathbb{R}^n} |g_1(x) - g_2(x)|.$$

19L Methods of Mathematical Physics

Consider the integral

$$\int_0^\infty \frac{t^z e^{-at}}{1+t}\, dt\ ,$$

where t^z is the principal branch and a is a positive constant. State the region of the complex z–plane in which the integral defines a holomorphic function.

Show how the analytic continuation of this function can be obtained by means of an alternative integral representation using the Hankel contour.

Hence show that the analytic continuation is holomorphic except for simple poles at $z = -1, -2, \ldots$, and that the residue at $z = -n$ is

$$(-1)^{n-1} \sum_{r=0}^{n-1} \frac{a^r}{r!}\ .$$

20K Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is discretized by the finite-difference method

$$u_m^{n+1} - \frac{1}{2}(\mu - \alpha)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}(\mu + \alpha)(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

where $u_m^n \approx u(m\Delta x, n\Delta t), \mu = \Delta t/(\Delta x)^2$ and α is a constant. Derive the order of magnitude (as a power of Δx) of the local error for different choices of α.

(ii) Investigate the stability of the above finite-difference method for different values of α by the Fourier technique.

21F Foundations of Quantum Mechanics

(i) Write the Hamiltonian for the harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

in terms of creation and annihilation operators, defined by

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - i\frac{p}{m\omega}\right), \qquad a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x + i\frac{p}{m\omega}\right).$$

Obtain an expression for $[a^\dagger, a]$ by using the usual commutation relation between p and x. Deduce the quantized energy levels for this system.

(ii) Define the number operator, N, in terms of creation and annihilation operators, $a^\dagger$ and a. The normalized eigenvector of N with eigenvalue n is $|n\rangle$. Show that $n \geq 0$.

Determine $a|n\rangle$ and $a^\dagger|n\rangle$ in the basis defined by $\{|n\rangle\}$.

Show that

$$a^{\dagger m} a^m |n\rangle = \begin{cases} \frac{n!}{(n-m)!}|n\rangle, & m \leq n\,, \\ 0\,, & m > n\,. \end{cases}$$

Verify the relation

$$|0\rangle\langle 0| = \sum_{m=0} \frac{1}{m!}(-1)^m a^{\dagger m} a^m\,,$$

by considering the action of both sides of the equation on an arbitrary basis vector.

22F Statistical Physics

A system consists of N weakly interacting non-relativistic fermions, each of mass m, in a three-dimensional volume, V. Derive the Fermi-Dirac distribution

$$n(\epsilon) \;=\; KVg\frac{\epsilon^{1/2}}{\exp((\epsilon-\mu)/kT)+1}\ ,$$

where $n(\epsilon)d\epsilon$ is the number of particles with energy in $(\epsilon, \epsilon+d\epsilon)$ and $K = 2\pi(2m)^{3/2}/h^3$. Explain the physical significance of g.

Explain how the constant μ is determined by the number of particles N and write down expressions for N and the internal energy E in terms of $n(\epsilon)$.

Show that, in the limit $\kappa \equiv e^{-\mu/kT} \gg 1$,

$$N \;=\; \frac{V}{A\kappa}\left(1-\frac{1}{2\sqrt{2}\kappa}+O\left(\frac{1}{\kappa^2}\right)\right)\ ,$$

where $A = h^3/g(2\pi mkT)^{3/2}$.

Show also that in this limit

$$E \;=\; \frac{3}{2}NkT\left(1+\frac{1}{4\sqrt{2}\kappa}+O\left(\frac{1}{\kappa^2}\right)\right)\ .$$

Deduce that the condition $\kappa \gg 1$ implies that $AN/V \ll 1$. Discuss briefly whether or not this latter condition is satisfied in a white dwarf star and in a dilute electron gas at room temperature.

$\left[\ \textit{Note that}\ \int_0^\infty du\, e^{-u^2 a} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\right].$

23J Applications of Quantum Mechanics

Write down the commutation relations satisfied by the cartesian components of the total angular momentum operator $\mathbf{J}$.

In quantum mechanics an operator $\mathbf{V}$ is said to be a vector operator if, under rotations, its components transform in the same way as those of the coordinate operator $\mathbf{r}$. Show from first principles that this implies that its cartesian components satisfy the commutation relations

$$[J_j, V_k] = i\epsilon_{jkl}V_l\ .$$

Hence calculate the total angular momentum of the nonvanishing states $V_j|0\rangle$, where $|0\rangle$ is the vacuum state.

24H Fluid Dynamics II

A planar flow of an inviscid, incompressible fluid is everywhere in the x-direction and has velocity profile

$$u = \begin{cases} U & y > 0, \\ 0 & y < 0. \end{cases}$$

By examining linear perturbations to the vortex sheet at $y = 0$ that have the form $e^{ikx-i\omega t}$, show that

$$\omega = \frac{1}{2}kU(1 \pm i)$$

and deduce the temporal stability of the sheet to disturbances of wave number k.

Use this result to determine also the spatial growth rate and propagation speed of disturbances of frequency ω introduced at a fixed spatial position.

25L Waves in Fluid and Solid Media

Consider the equation

$$\phi_{tt} + \alpha^2 \phi_{xxxx} + \beta^2 \phi = 0, \qquad (*)$$

where α and β are real constants. Find the dispersion relation for waves of frequency ω and wavenumber k. Find the phase velocity $c(k)$ and the group velocity $c_g(k)$ and sketch graphs of these functions.

Multiplying equation $(*)$ by ϕ_t, obtain an equation of the form

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} = 0$$

where A and B are expressions involving ϕ and its derivatives. Give a physical interpretation of this equation.

Evaluate the time-averaged energy $\langle E \rangle$ and energy flux $\langle I \rangle$ of a monochromatic wave $\phi = \cos(kx - wt)$, and show that

$$\langle I \rangle = c_g \langle E \rangle.$$

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Thursday 7 June 2001 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than ***FOUR*** *questions.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, ..., L*** *according to the letter affixed to each question. (For example,* ***2C, 5C*** *should be in one bundle and* ***11D, 14D*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1A Combinatorics

Write an essay on extremal graph theory. You should give proofs of at least two major theorems and you should also include a description of alternative proofs or of further results.

2C Representation Theory

Let G be the Heisenberg group of order p^3. This is the subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, x \in \mathbf{F}_p \right\}$$

of 3×3 matrices over the finite field $\mathbf{F}_p$ (p prime). Let H be the subgroup of G of such matrices with $a = 0$.

(i) Find all one dimensional representations of G.

[*You may assume without proof that* $[G, G]$ *is equal to the set of matrices in* G *with* $a = b = 0$.]

(ii) Let $\psi : \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{C}^*$ be a non-trivial one dimensional representation of $\mathbf{F}_p$, and define a one dimensional representation ρ of H by

$$\rho \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that $V_\psi = \mathrm{Ind}_H^G(\rho)$ is irreducible.

(iii) List all the irreducible representations of G and explain why your list is complete.

3C Galois Theory

Define the concept of separability and normality for algebraic field extensions. Suppose $K = k(\alpha)$ is a simple algebraic extension of k, and that $\mathrm{Aut}(K/k)$ denotes the group of k-automorphisms of K. Prove that
$|\mathrm{Aut}(K/k)| \leqslant [K : k]$, with equality if and only if K/k is normal and separable.

[*You may assume that the splitting field of a separable polynomial $f \in k[X]$ is normal and separable over k.*]

Suppose now that G is a finite group of automorphisms of a field F, and $E = F^G$ is the fixed subfield. Prove:

(i) F/E is separable.

(ii) $G = \mathrm{Aut}(F/E)$ and $[F : E] = |G|$.

(iii) F/E is normal.

[*The Primitive Element Theorem for finite separable extensions may be used without proof.*]

4B Differentiable Manifolds

Describe the Mayer-Vietoris exact sequence for forms on a manifold M and show how to derive from it the Mayer-Vietoris exact sequence for the de Rham cohomology.

Calculate $H^*(\mathbb{RP}^n)$.

5C Algebraic Topology

Write an essay on the definition of simplicial homology groups. The essay should include a discussion of orientations, of the action of a simplicial map and a proof of $\partial^2 = 0$.

6B Number Fields

For a prime number $p > 2$, set $\zeta = e^{2\pi i/p}, K = \mathbf{Q}(\zeta)$ and $K^+ = \mathbf{Q}(\zeta + \zeta^{-1})$.

(a) Show that the (normalized) minimal polynomial of $\zeta - 1$ over $\mathbf{Q}$ is equal to

$$f(x) = \frac{(x+1)^p - 1}{x}.$$

(b) Determine the degrees $[K : \mathbf{Q}]$ and $[K^+ : \mathbf{Q}]$.

(c) Show that

$$\prod_{j=1}^{p-1} (1 - \zeta^j) = p.$$

(d) Show that $\operatorname{disc}(f) = (-1)^{\frac{p-1}{2}} p^{p-2}$.

(e) Show that K contains $\mathbf{Q}(\sqrt{p^*})$, where $p^* = (-1)^{\frac{p-1}{2}} p$.

(f) If $j, k \in \mathbf{Z}$ are not divisible by p, show that $\frac{1-\zeta^j}{1-\zeta^k}$ lies in $\mathcal{O}_K^*$.

(g) Show that the ideal $(p) = p\mathcal{O}_K$ is equal to $(1 - \zeta)^{p-1}$.

7A Hilbert Spaces

Write an essay on the use of Hermite functions in the elementary theory of the Fourier transform on $L^2(\mathbb{R})$.

[*You should assume, without proof, any results that you need concerning the approximation of functions by Hermite functions.*]

8B Riemann Surfaces

Let λ and μ be fixed, non-zero complex numbers, with $\lambda/\mu \notin \mathbb{R}$, and let $\Lambda = \mathbb{Z}\mu + \mathbb{Z}\lambda$ be the lattice they generate in $\mathbb{C}$. The series

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n} \Big[\frac{1}{(z - m\lambda - n\mu)^2} - \frac{1}{(m\lambda + n\mu)^2} \Big],$$

with the sum taken over all pairs $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ other than (0,0), is known to converge to an *elliptic function*, meaning a meromorphic function $\wp : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ satisfying $\wp(z) = \wp(z+\mu) = \wp(z+\lambda)$ for all $z \in \mathbb{C}$. ($\wp$ is called the *Weierstrass function.*)

(a) Find the three zeros of $\wp'$ modulo Λ, explaining why there are no others.

(b) Show that, for any number $a \in \mathbb{C}$, other than the three values $\wp(\lambda/2), \wp(\mu/2)$ and $\wp((\lambda+\mu)/2)$, the equation $\wp(z) = a$ has exactly two solutions, modulo Λ; whereas, for each of the specified values, it has a single solution.

[*In (a) and (b), you may use, without proof, any known results about valencies and degrees of holomorphic maps between compact Riemann surfaces, provided you state them correctly.*]

(c) Prove that every *even elliptic function* $\phi(z)$ is a rational function of $\wp(z)$; that is, there exists a rational function R for which $\phi(z) = R(\wp(z))$.

9B Algebraic Curves

Write an essay on curves of genus one (over an algebraically closed field k of characteristic zero). Legendre's normal form should not be discussed.

10B Logic, Computation and Set Theory

What is a wellfounded relation, and how does wellfoundedness underpin wellfounded induction?

A formula $\phi(x,y)$ with two free variables *defines an* $\in$-*automorphism* if for all x there is a unique y, the function f, defined by $y = f(x)$ if and only if $\phi(x,y)$, is a permutation of the universe, and we have $(\forall xy)(x \in y \leftrightarrow f(x) \in f(y))$.

Use wellfounded induction over $\in$ to prove that all formulæ defining $\in$-automorphisms are equivalent to $x = y$.

11D Probability and Measure

State the first and second Borel-Cantelli Lemmas and the Kolmogorov 0-1 law.

Let $(X_n)_{n \geqslant 1}$ be a sequence of independent random variables with distribution given by

$$\mathbb{P}(X_n = n) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0),$$

and set $S_n = \sum_{i=1}^{n} X_i$.

(a) Show that there exist constants $0 \leqslant c_1 \leqslant c_2 \leqslant \infty$ such that $\liminf_n (S_n/n) = c_1$, almost surely and $\limsup_n (S_n/n) = c_2$ almost surely.

(b) Let $Y_k = \sum_{i=k+1}^{2k} X_i$ and $\tilde{Y}_k = \sum_{i=1}^{k} Z_i^{(k)}$, where $(Z_i^{(k)})_{i=1}^{k}$ are independent with

$$\mathbb{P}(Z_i^{(k)} = k) = \frac{1}{2k} = 1 - \mathbb{P}(Z_i^{(k)} = 0), \quad 1 \leqslant i \leqslant k,$$

and suppose that $\alpha \in \mathbb{Z}^+$.

Use the fact that $\mathbb{P}(Y_k \geqslant \alpha k) \geqslant \mathbb{P}(\tilde{Y}_k \geqslant \alpha k)$ to show that there exists $p_\alpha > 0$ such that $\mathbb{P}(Y_k \geqslant \alpha k) \geqslant p_\alpha$ for all sufficiently large k.

[*You may use the Poisson approximation to the binomial distribution without proof.*]

By considering a suitable subsequence of (Y_k), or otherwise, show that $c_2 = \infty$.

(c) Show that $c_1 \leqslant 1$. Consider an appropriately chosen sequence of random times T_i, with $2T_i \leqslant T_{i+1}$, for which $(S_{T_i}/T_i) \leqslant 3c_1/2$. Using the fact that the random variables (Y_{T_i}) are independent, and by considering the events $\{Y_{T_i} = 0\}$, or otherwise, show that $c_1 = 0$.

12D Applied Probability

Define a *renewal process* and a *renewal reward process*.

State and prove the strong law of large numbers for these processes.

[*You may assume the strong law of large numbers for independent, identically-distributed random variables.*]

State and prove Little's formula.

Customers arrive according to a Poisson process with rate ν at a single server, but a restricted waiting room causes those who arrive when n customers are already present to be lost. Accepted customers have service times which are independent and identically-distributed with mean α and independent of the arrival process. Let P_j be the equilibrium probability that an arriving customer finds j customers already present.

Using Little's formula, or otherwise, determine a relationship between P_0, P_n, ν and α.

13E Information Theory

State the Kraft inequality. Prove that it gives a necessary and sufficient condition for the existence of a prefix-free code with given codeword lengths.

14D Optimization and Control

Consider the scalar system with plant equation $x_{t+1} = x_t + u_t$, $t = 0, 1, \ldots$ and cost

$$C_s(x_0, u_0, u_1, \ldots) = \sum_{t=0}^{s} \left[u_t^2 + \frac{4}{3} x_t^2 \right] .$$

Show from first principles that $\min_{u_0, u_1, \ldots} C_s = V_s x_0^2$, where $V_0 = 4/3$ and for $s = 0, 1, \ldots$,

$$V_{s+1} = 4/3 + V_s/(1 + V_s) .$$

Show that $V_s \to 2$ as $s \to \infty$.

Prove that C_∞ is minimized by the stationary control, $u_t = -2x_t/3$ for all t.

Consider the stationary policy π_0 that has $u_t = -x_t$ for all t. What is the value of C_∞ under this policy?

Consider the following algorithm, in which steps 1 and 2 are repeated as many times as desired.

1. For a given stationary policy π_n, for which $u_t = k_n x_t$ for all t, determine the value of C_∞ under this policy as $V^{\pi_n} x_0^2$ by solving for V^{π_n} in

$$V^{\pi_n} = k_n^2 + 4/3 + (1 + k_n)^2 V^{\pi_n} .$$

2. Now find k_{n+1} as the minimizer of

$$k_{n+1}^2 + 4/3 + (1 + k_{n+1})^2 V^{\pi_n}$$

and define π_{n+1} as the policy for which $u_t = k_{n+1} x_t$ for all t.

Explain why π_{n+1} is guaranteed to be a better policy than π_n.

Let π_0 be the stationary policy with $u_t = -x_t$. Determine π_1 and verify that it minimizes C_∞ to within 0.2% of its optimum.

15E Principles of Statistics

Write an account, with appropriate examples, of **one** of the following:

(a) Inference in multi-parameter exponential families;

(b) Asymptotic properties of maximum-likelihood estimators and their use in hypothesis testing;

(c) Bootstrap inference.

16D Stochastic Financial Models

Write an essay on the mean-variance approach to portfolio selection in a one-period model. Your essay should contrast the solution in the case when all the assets are risky with that for the case when there is a riskless asset.

17K Dynamical Systems

Define the rotation number $\rho(f)$ of an orientation-preserving circle map f and the rotation number $\rho(F)$ of a lift F of f. Prove that $\rho(f)$ and $\rho(F)$ are well-defined. Prove also that $\rho(F)$ is a continuous function of F.

State without proof the main consequence of $\rho(f)$ being rational.

18A Partial Differential Equations

Write an essay on **one** of the following two topics:

(a) The notion of *well-posedness* for initial and boundary value problems for differential equations. Your answer should include a definition and give examples and state precise theorems for some specific problems.

(b) The concepts of *distribution* and *tempered distribution* and their use in the study of partial differential equations.

19L Methods of Mathematical Physics

Show that $\int_0^\pi \mathrm{e}^{ix\cos t}\,dt$ satisfies the differential equation

$$xy'' + y' + xy = 0,$$

and find a second solution, in the form of an integral, for $x > 0$.

Show, by finding the asymptotic behaviour as $x \to +\infty$, that your two solutions are linearly independent.

20K Numerical Analysis

Write an essay on the computation of eigenvalues and eigenvectors of matrices.

21F Electrodynamics

The Liénard-Wiechert potential for a particle of charge q, assumed to be moving non-relativistically along the trajectory $y^\mu(\tau)$, τ being the proper time along the trajectory, is

$$A^\mu(x,t) \;=\; \frac{\mu_0 q}{4\pi}\,\frac{dy^\mu/d\tau}{(x-y(\tau))_\nu dy^\nu/d\tau}\Big|_{\tau=\tau_0}.$$

Explain how to calculate τ_0 given $x^\mu = (x,t)$ and $y^\mu = (y,t')$.

Derive Larmor's formula for the rate at which electromagnetic energy is radiated from a particle of charge q undergoing an acceleration a.

Suppose that one considers the classical non-relativistic hydrogen atom with an electron of mass m and charge $-e$ orbiting a fixed proton of charge $+e$, in a circular orbit of radius r_0. What is the total energy of the electron? As the electron is accelerated towards the proton it will radiate, thereby losing energy and causing the orbit to decay. Derive a formula for the lifetime of the orbit.

22F Foundations of Quantum Mechanics

(i) The two states of a spin-$\frac{1}{2}$ particle corresponding to spin pointing along the z axis are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$. Explain why the states

$$|\uparrow,\theta\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle, \qquad |\downarrow,\theta\rangle = -\sin\frac{\theta}{2}\,|\uparrow\rangle + \cos\frac{\theta}{2}\,|\downarrow\rangle$$

correspond to the spins being aligned along a direction at an angle θ to the z direction.

The spin-0 state of two spin-$\frac{1}{2}$ particles is

$$|0\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\Big).$$

Show that this is independent of the direction chosen to define $|\uparrow\rangle_{1,2}$, $|\downarrow\rangle_{1,2}$. If the spin of particle 1 along some direction is measured to be $\frac{1}{2}\hbar$ show that the spin of particle 2 along the same direction is determined, giving its value.

[*The Pauli matrices are given by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$]

(ii) Starting from the commutation relation for angular momentum in the form

$$[J_3, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_3,$$

obtain the possible values of j, m, where $m\hbar$ are the eigenvalues of J_3 and $j(j+1)\hbar^2$ are the eigenvalues of $\mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$. Show that the corresponding normalized eigenvectors, $|j,m\rangle$, satisfy

$$J_\pm|j,m\rangle = \hbar\left((j \mp m)(j \pm m + 1)\right)^{1/2}|j,m\pm 1\rangle,$$

and that

$$\frac{1}{n!}J_-^n|j,j\rangle = \hbar^n\left(\frac{(2j)!}{n!(2j-n)!}\right)^{1/2}|j,j-n\rangle, \qquad n \le 2j.$$

The state $|w\rangle$ is defined by

$$|w\rangle = e^{wJ_-/\hbar}|j,j\rangle,$$

for any complex w. By expanding the exponential show that $\langle w|w\rangle = (1+|w|^2)^{2j}$. Verify that

$$e^{-wJ_-/\hbar}J_3\,e^{wJ_-/\hbar} = J_3 - wJ_-,$$

and hence show that

$$J_3|w\rangle = \hbar\left(j - w\frac{\partial}{\partial w}\right)|w\rangle.$$

If $H = \alpha J_3$ verify that $|e^{i\alpha t}\rangle\,e^{-ij\alpha t}$ is a solution of the time-dependent Schrödinger equation.

23F Statistical Physics

Given that the free energy F can be written in terms of the partition function Z as $F = -kT \log Z$ show that the entropy S and internal energy E are given by

$$S \;=\; k \,\frac{\partial (T \log Z)}{\partial T} \;, \qquad E \;=\; kT^2 \,\frac{\partial \log Z}{\partial T} \;.$$

A system of particles has Hamiltonian $H(\mathbf{p}, \mathbf{q})$ where $\mathbf{p}$ is the set of particle momenta and $\mathbf{q}$ the set of particle coordinates. Write down the expression for the classical partition function Z_C for this system in equilibrium at temperature T. In a particular case H is given by

$$H(\mathbf{p}, \mathbf{q}) \;=\; p_\alpha A_{\alpha\beta}(\mathbf{q}) p_\beta + \; q_\alpha B_{\alpha\beta}(\mathbf{q}) q_\beta \;.$$

Let H be a homogeneous function in all the p_α, $1 \le \alpha \le N$, and in a subset of the q_α, $1 \le \alpha \le M$ $(M \le N)$. Derive the principle of equipartition for this system.

A system consists of N weakly interacting harmonic oscillators each with Hamiltonian

$$h(p, q) \;=\; \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 \;.$$

Using equipartition calculate the classical specific heat of the system, $C_C(T)$. Also calculate the classical entropy $S_C(T)$.

Write down the expression for the quantum partition function of the system and derive expressions for the specific heat $C(T)$ and the entropy $S(T)$ in terms of N and the parameter $\theta = \hbar\omega / kT$. Show for $\theta \ll 1$ that

$$C(T) \;=\; C_C(T) + O\left(\theta\right) \;, \qquad S(T) \;=\; S_C(T) + S_0 + O\left(\theta\right) \;,$$

where S_0 should be calculated. Comment briefly on the physical significance of the constant S_0 and why it is non-zero.

24J Applications of Quantum Mechanics

Derive the Bloch form of the wave function $\psi(x)$ of an electron moving in a one-dimensional crystal lattice.

The potential in such an N-atom lattice is modelled by

$$V(x) = \sum_n \Big(-\frac{\hbar^2 U}{2m} \delta(x - nL) \Big).$$

Assuming that $\psi(x)$ is continuous across each lattice site, and applying periodic boundary conditions, derive an equation for the allowed electron energy levels. Show that for suitable values of UL they have a band structure, and calculate the number of levels in each band when $UL > 2$. Verify that when $UL \gg 1$ the levels are very close to those corresponding to a solitary atom.

Describe briefly how the band structure in a real 3-dimensional crystal differs from that of this simple model.

25J General Relativity

Discuss how Einstein's theory of gravitation reduces to Newton's in the limit of weak fields. Your answer should include discussion of:

(a) the field equations;

(b) the motion of a point particle;

(c) the motion of a pressureless fluid.

[*The metric in a weak gravitational field, with Newtonian potential ϕ, may be taken as*

$$ds^2 = dx^2 + dy^2 + dz^2 - (1 + 2\phi) dt^2.$$

The Riemann tensor is

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc}.\Big]$$

26H Fluid Dynamics II

Starting from the steady planar vorticity equation

$$\mathbf{u} \,.\nabla\omega = \nu\nabla^2\omega,$$

outline briefly the derivation of the boundary layer equation

$$uu_x + vu_y = UdU/dx + \nu u_{yy},$$

explaining the significance of the symbols used.

Viscous fluid occupies the region $y > 0$ with rigid stationary walls along $y = 0$ for $x > 0$ and $x < 0$. There is a line sink at the origin of strength πQ, $Q > 0$, with $Q/\nu \gg 1$. Assuming that vorticity is confined to boundary layers along the rigid walls:

(a) Find the flow outside the boundary layers.

(b) Explain why the boundary layer thickness δ along the wall $x > 0$ is proportional to x, and deduce that

$$\delta = \left(\frac{\nu}{Q}\right)^{\frac{1}{2}} x \; .$$

(c) Show that the boundary layer equation admits a solution having stream function

$$\psi = (\nu Q)^{1/2} f(\eta) \quad \text{with} \quad \eta = y/\delta \; .$$

Find the equation and boundary conditions satisfied by f .

(d) Verify that a solution is

$$f' = \frac{6}{1 + \cosh(\eta\sqrt{2} + c)} - 1,$$

provided that c has one of two values to be determined. Should the positive or negative value be chosen?

27L Waves in Fluid and Solid Media

Derive the ray-tracing equations governing the evolution of a wave packet $\phi(\mathbf{x}, t) = A(\mathbf{x}, t)\exp\{i\psi(\mathbf{x}, t)\}$ in a slowly varying medium, stating the conditions under which the equations are valid.

Consider now a stationary obstacle in a steadily moving homogeneous two-dimensional medium which has the dispersion relation

$$\omega(k_1, k_2) = \alpha\left(k_1^2 + k_2^2\right)^{1/4} - Vk_1,$$

where $(V, 0)$ is the velocity of the medium. The obstacle generates a steady wave system. Writing $(k_1, k_2) = \kappa(\cos\phi, \sin\phi)$, show that the wave satisfies

$$\kappa = \frac{\alpha^2}{V^2\cos^2\phi}.$$

Show that the group velocity of these waves can be expressed as

$$\mathbf{c}_g = V(\tfrac{1}{2}\cos^2\phi - 1, \tfrac{1}{2}\cos\phi\sin\phi).$$

Deduce that the waves occupy a wedge of semi-angle $\sin^{-1}\frac{1}{3}$ about the negative x_1-axis.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

List of Courses

A2/7 **Geometry of Surfaces**

(i) Give the definition of the curvature $\kappa(t)$ of a plane curve $\gamma : [a,b] \longrightarrow \mathbf{R}^2$. Show that, if $\gamma : [a,b] \longrightarrow \mathbf{R}^2$ is a simple closed curve, then

$$\int_a^b \kappa(t)\,\|\dot\gamma(t)\|\,dt = 2\pi.$$

(ii) Give the definition of a geodesic on a parametrized surface in $\mathbf{R}^3$. Derive the differential equations characterizing geodesics. Show that a great circle on the unit sphere is a geodesic.

A3/7 **Geometry of Surfaces**

(i) Give the definition of the surface area of a parametrized surface in $\mathbf{R}^3$ and show that it does not depend on the parametrization.

(ii) Let $\varphi(u) > 0$ be a differentiable function of u. Consider the surface of revolution:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto f(u,v) = \begin{pmatrix} \varphi(u)\cos(v) \\ \varphi(u)\sin(v) \\ u \end{pmatrix}.$$

Find a formula for each of the following:

(a) The first fundamental form.

(b) The unit normal.

(c) The second fundamental form.

(d) The Gaussian curvature.

A4/7 **Geometry of Surfaces**

Write an essay on the Gauss-Bonnet theorem. Make sure that your essay contains a precise statement of the theorem, in its local form, and a discussion of some of its applications, including the global Gauss-Bonnet theorem.

A1/8 **Graph Theory**

(i) Show that any graph G with minimal degree δ contains a cycle of length at least $\delta+1$. Give examples to show that, for each possible value of δ, there is a graph with minimal degree δ but no cycle of length greater than $\delta+1$.

(ii) Let K_N be the complete graph with N vertices labelled $v_1, v_2, \ldots, v_N$. Prove, from first principles, that there are N^{N-2} different spanning trees in K_N. In how many of these spanning trees does the vertex v_1 have degree 1?

A spanning tree in K_N is chosen at random, with each of the N^{N-2} trees being equally likely. Show that the average number of vertices of degree 1 in the random tree is approximately N/e when N is large.

Find the average degree of vertices in the random tree.

A2/8 **Graph Theory**

(i) Prove that any graph G drawn on a compact surface S with negative Euler characteristic $E(S)$ has a vertex colouring that uses at most

$$h = \lfloor \tfrac{1}{2}(7+\sqrt{49-24E(S)})\rfloor$$

colours.

Briefly discuss whether the result is still true when $E(S) \geqslant 0$.

(ii) Prove that a graph G is k edge-connected if and only if the removal of no set of less than k edges from G disconnects G.

[*If you use any form of Menger's theorem, you must prove it.*]

Let G be a minimal example of a graph that requires $k+1$ colours for a vertex colouring. Show that G must be k edge-connected.

A4/9 **Graph Theory**

Write an essay on extremal graph theory. Your essay should include proofs of at least two major results and a discussion of variations on these results or their proofs.

A1/9 **Number Theory**

(i) Describe Euclid's algorithm.

Find, in the RSA algorithm, the deciphering key corresponding to the enciphering key $7, 527$.

(ii) Explain what is meant by a primitive root modulo an odd prime p.

Show that, if g is a primitive root modulo p, then all primitive roots modulo p are given by g^m, where $1 \leqslant m < p$ and $(m, p-1) = 1$.

Verify, by Euler's criterion, that 3 is a primitive root modulo 17. Hence find all primitive roots modulo 17.

A3/9 **Number Theory**

(i) State the law of quadratic reciprocity.

Prove that 5 is a quadratic residue modulo primes $p \equiv \pm 1 \pmod{10}$ and a quadratic non-residue modulo primes $p \equiv \pm 3 \pmod{10}$.

Determine whether 5 is a quadratic residue or non-residue modulo 119 and modulo 539.

(ii) Explain what is meant by the continued fraction of a real number θ. Define the convergents to θ and write down the recurrence relations satisfied by their numerators and denominators.

Use the continued fraction method to find two solutions in positive integers x, y of the equation $x^2 - 15y^2 = 1$.

A4/10 **Number Theory**

Attempt **one** of the following:

(i) Discuss pseudoprimes and primality testing. Find all bases for which 57 is a Fermat pseudoprime. Determine whether 57 is also an Euler pseudoprime for these bases.

(ii) Write a brief account of various methods for factoring large numbers. Use Fermat factorization to find the factors of 10033. Would Pollard's $p-1$ method also be practical in this instance?

(iii) Show that $\sum 1/p_n$ is divergent, where p_n denotes the *n-th* prime.

Write a brief account of basic properties of the Riemann zeta-function.

State the prime number theorem. Show that it implies that for all sufficiently large positive integers n there is a prime p satisfying $n < p \leqslant 2n$.

A1/10 **Coding and Cryptography**

(i) Explain briefly how and why a signature scheme is used. Describe the el Gamal scheme.

(ii) Define a cyclic code. Define the generator of a cyclic code and show that it exists. Prove a necessary and sufficient condition for a polynomial to be the generator of a cyclic code of length n.

What is the BCH code? Show that the BCH code associated with $\{\beta, \beta^2\}$, where β is a root of $X^3 + X + 1$ in an appropriate field, is Hamming's original code.

A2/9 **Coding and Cryptography**

(i) Give brief answers to the following questions.

(a) What is a stream cypher?

(b) Explain briefly why a one-time pad is safe if used only once but becomes unsafe if used many times.

(c) What is a feedback register of length d? What is a linear feedback register of length d?

(d) A cypher stream is given by a linear feedback register of known length d. Show that, given plain text and cyphered text of length $2d$, we can find the complete cypher stream.

(e) State and prove a similar result for a general feedback register.

(ii) Describe the construction of a Reed-Muller code. Establish its information rate and its weight.

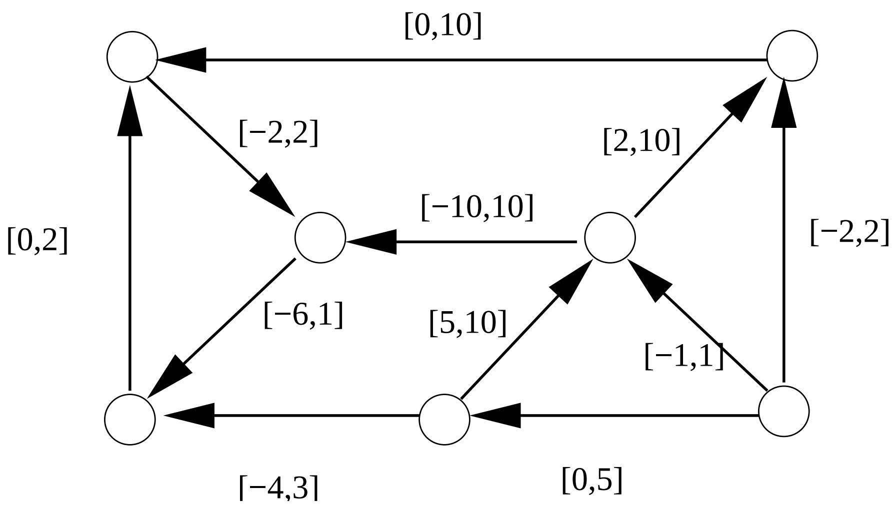
A2/10 **Algorithms and Networks**

(i) Let G be a directed network with nodes N and arcs A. Let $S \subset N$ be a subset of the nodes, x be a flow on G, and y be the divergence of x. Describe carefully what is meant by a *cut* $Q = [S, N \backslash S]$. Define the *arc-cut incidence* e_Q, and the *flux of x across Q*. Define also the *divergence $y(S)$ of S*. Show that $y(S) = x.e_Q$.

Now suppose that capacity constraints are specified on each of the arcs. Define the *upper cut capacity $c^+(Q)$ of Q*. State the feasible distribution problem for a specified divergence b, and show that the problem only has a solution if $b(N) = 0$ and $b(S) \leqslant c^+(Q)$ for all cuts $Q = [S, N \backslash S]$.

(ii) Describe an algorithm to find a feasible distribution given a specified divergence b and capacity constraints on each arc. Explain what happens when no feasible distribution exists.

Illustrate the algorithm by either finding a feasible circulation, or demonstrating that one does not exist, in the network given below. Arcs are labelled with capacity constraint intervals.



A3/10 **Algorithms and Networks**

(i) Let P be the problem

$$\text{minimize} \quad f(x) \qquad \text{subject to} \quad h(x) = b, \quad x \in X.$$

Explain carefully what it means for the problem P to be *Strong Lagrangian.*

Outline the main steps in a proof that a quadratic programming problem

$$\text{minimize} \quad \frac{1}{2}x^T Q x + c^T x \qquad \text{subject to} \quad Ax \geqslant b\,,$$

where Q is a symmetric positive semi-definite matrix, is Strong Lagrangian.

[*You should carefully state the results you need, but should not prove them.*]

(ii) Consider the quadratic programming problem:

$$\text{minimize} \quad x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 - 4x_2$$

$$\text{subject to} \quad 3x_1 + 2x_2 \leqslant 6, \quad x_1 + x_2 \geqslant 1\,.$$

State necessary and sufficient conditions for (x_1, x_2) to be optimal, and use the active-set algorithm (explaining your steps briefly) to solve the problem starting with initial condition $(2, 0)$. Demonstrate that the solution you have found is optimal by showing that it satisfies the necessary and sufficient conditions stated previously.

A4/11 **Algorithms and Networks**

State the optimal distribution problem. Carefully describe the simplex-on-a-graph algorithm for solving optimal distribution problems when the flow in each arc in the network is constrained to lie in the interval $[0, \infty)$. Explain how the algorithm can be initialised if there is no obvious feasible solution with which to begin. Describe the adjustments that are needed for the algorithm to cope with more general capacity constraints $x(j) \in [c^-(j), c^+(j)]$ for each arc j (where $c^{\pm}(j)$ may be finite or infinite).

A1/13 **Computational Statistics and Statistical Modelling**

(i) Assume that the n-dimensional observation vector Y may be written as

$$Y = X\beta + \epsilon \quad ,$$

where X is a given $n \times p$ matrix of rank p, β is an unknown vector, and

$$\epsilon \sim N_n(0, \sigma^2 I).$$

Let $Q(\beta) = (Y - X\beta)^T(Y - X\beta)$. Find $\widehat{\beta}$, the least-squares estimator of β, and show that

$$Q(\widehat{\beta}) = Y^T(I - H)Y \quad ,$$

where H is a matrix that you should define.

(ii) Show that $\sum_i H_{ii} = p$. Show further for the special case of

$$Y_i = \beta_1 + \beta_2 x_i + \beta_3 z_i + \epsilon_i, \quad 1 \leqslant i \leqslant n,$$

where $\Sigma x_i = 0, \quad \Sigma z_i = 0$, that

$$H \quad = \quad \frac{1}{n}\mathbf{1}\mathbf{1}^T + axx^T + b(xz^T + zx^T) + czz^T \quad ;$$

here, $\mathbf{1}$ is a vector of which every element is one, and a, b, c, are constants that you should derive.

Hence show that, if $\widehat{Y} = X\widehat{\beta}$ is the vector of fitted values, then

$$\frac{1}{\sigma^2}\mathrm{var}(\widehat{Y}_i) = \frac{1}{n} + ax_i^2 + 2bx_iz_i + cz_i^2, \quad 1 \leqslant i \leqslant n.$$

A2/12 **Computational Statistics and Statistical Modelling**

(i) Suppose that $Y_1, \ldots, Y_n$ are independent random variables, and that Y_i has probability density function

$$f(y_i|\theta_i, \phi) = \exp[(y_i\theta_i - b(\theta_i))/\phi + c(y_i, \phi)].$$

Assume that $E(Y_i) = \mu_i$, and that $g(\mu_i) = \beta^T x_i$, where $g(\cdot)$ is a known 'link' function, $x_1, \ldots, x_n$ are known covariates, and β is an unknown vector. Show that

$$\mathbb{E}(Y_i) = b'(\theta_i),\ \mathrm{var}(Y_i) = \phi b''(\theta_i) = V_i,\ \text{say},$$

and hence

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} \frac{(y_i - \mu_i)x_i}{g'(\mu_i)V_i},\ \text{where } l = l(\beta, \phi) \text{ is the log-likelihood.}$$

(ii) The table below shows the number of train miles (in millions) and the number of collisions involving British Rail passenger trains between 1970 and 1984. Give a detailed interpretation of the R output that is shown under this table:

	year	collisions	miles
1	1970	3	281
2	1971	6	276
3	1972	4	268
4	1973	7	269
5	1974	6	281
6	1975	2	271
7	1976	2	265
8	1977	4	264
9	1978	1	267
10	1979	7	265
11	1980	3	267
12	1981	5	260
13	1982	6	231
14	1983	1	249

```
Call:

glm(formula = collisions ~ year + log(miles), family = poisson)

Coefficients:
                  Estimate  Std. Error  z value  Pr(>|z|)
     (Intercept) 127.14453   121.37796    1.048     0.295
     year         -0.05398     0.05175   -1.043     0.297
     log(miles)   -3.41654     4.18616   -0.816     0.414

(Dispersion parameter for poisson family taken to be 1)

      Null deviance:  15.937 on 13 degrees of freedom

Residual deviance:  14.843 on 11 degrees of freedom

Number of Fisher Scoring iterations:  4
```

A4/14 **Computational Statistics and Statistical Modelling**

(i) Assume that independent observations $Y_1, \ldots, Y_n$ are such that

$$Y_i \sim \text{Binomial}(t_i, \pi_i), \; \log \frac{\pi_i}{1-\pi_i} = \beta^T x_i \quad \text{for } 1 \leqslant i \leqslant n \quad ,$$

where $x_1, \ldots, x_n$ are given covariates. Discuss carefully how to estimate β, and how to test that the model fits.

(ii) Carmichael *et al.* (1989) collected data on the numbers of 5-year old children with "dmft", i.e. with 5 or more decayed, missing or filled teeth, classified by social class, and by whether or not their tap water was fluoridated or non-fluoridated. The numbers of such children with dmft, and the total numbers, are given in the table below:

dmft

Social Class	Fluoridated	Non-fluoridated
I	12/117	12/56
II	26/170	48/146
III	11/52	29/64
Unclassified	24/118	49/104

A (slightly edited) version of the *R* output is given below. Explain carefully what model is being fitted, whether it does actually fit, and what the parameter estimates and Std. Errors are telling you. (You may assume that the factors SClass (social class) and Fl (with/without) have been correctly set up.)

```
Call:

    glm(formula = Yes/Total ~ SClass + Fl, family = binomial,
          weights = Total)

Coefficients:

                Estimate  Std.  Error  z value
    (Intercept) -2.2716         0.2396   -9.480
    SClassII     0.5099         0.2628    1.940
    SClassIII    0.9857         0.3021    3.262
    SClassUnc    1.0020         0.2684    3.734
    Flwithout    1.0813         0.1694    6.383

(Dispersion parameter for binomial family taken to be 1)

    Null deviance:  68.53785 on 7 degrees of freedom

Residual deviance:  0.64225 on 3 degrees of freedom

Number of Fisher Scoring iterations:  3
```

Here 'Yes' is the vector of numbers with dmft, taking values $12, 12, \ldots, 24, 49$, 'Total' is the vector of Total in each category, taking values $117, 56, \ldots, 118, 104$, and SClass, Fl are the factors corresponding to Social class and Fluoride status, defined in the obvious way.

A1/14 **Quantum Physics**

(i) A spinless quantum mechanical particle of mass m moving in two dimensions is confined to a square box with sides of length L. Write down the energy eigenfunctions for the particle and the associated energies.

Show that, for large L, the number of states in the energy range $E \to E + dE$ is $\rho(E)dE$, where

$$\rho(E) = \frac{mL^2}{2\pi\hbar^2}.$$

(ii) If, instead, the particle is an electron with magnetic moment μ moving in an external magnetic field, H, show that

$$\begin{aligned} \rho(E) =& \frac{mL^2}{2\pi\hbar^2}, \qquad & -\mu H < E < \mu H \\ =& \frac{mL^2}{\pi\hbar^2}, \qquad & \mu H < E. \end{aligned}$$

Let there be N electrons in the box. Explain briefly how to construct the ground state of the system. Let E_F be the Fermi energy. Show that when $E_F > \mu H$,

$$N = \frac{mL^2}{\pi\hbar^2}\, E_F.$$

Show also that the magnetic moment, M, of the system in the ground state is

$$M = \frac{\mu^2 mL^2}{\pi\hbar^2}\, H,$$

and that the ground state energy is

$$\frac{1}{2}\frac{\pi\hbar^2}{mL^2}\, N^2 - \frac{1}{2} MH.$$

A2/14 **Quantum Physics**

(i) Each particle in a system of N identical fermions has a set of energy levels, E_i, with degeneracy g_i, where $1 \leq i < \infty$. Explain why, in thermal equilibrium, the average number of particles with energy E_i is

$$N_i = \frac{g_i}{e^{\beta(E_i-\mu)}+1}.$$

The physical significance of the parameters β and μ should be made clear.

(ii) A simple model of a crystal consists of a linear array of sites with separation a. At the nth site an electron may occupy either of two states with probability amplitudes b_n and c_n, respectively. The time-dependent Schrödinger equation governing the amplitudes gives

$$i\hbar\dot{b}_n = E_0 b_n - A(b_{n+1}+b_{n-1}+c_{n+1}+c_{n-1}),$$
$$i\hbar\dot{c}_n = E_1 c_n - A(b_{n+1}+b_{n-1}+c_{n+1}+c_{n-1}),$$

where $A > 0$.

By examining solutions of the form

$$\begin{pmatrix} b_n \\ c_n \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} e^{i(kna-Et/\hbar)},$$

show that the energies of the electron fall into two bands given by

$$E = \frac{1}{2}(E_0+E_1-4A\cos ka) \pm \frac{1}{2}\sqrt{(E_0-E_1)^2+16A^2\cos^2 ka}.$$

Describe briefly how the energy band structure for electrons in real crystalline materials can be used to explain why they are insulators, conductors or semiconductors.

A4/16 **Quantum Physics**

A harmonic oscillator of frequency ω is in thermal equilibrium with a heat bath at temperature T. Show that the mean number of quanta n in the oscillator is

$$n = \frac{1}{e^{\hbar\omega/kT}-1}.$$

Use this result to show that the density of photons of frequency ω for cavity radiation at temperature T is

$$n(\omega) = \frac{\omega^2}{\pi^2 c^3}\,\frac{1}{e^{\hbar\omega/kT}-1}.$$

By considering this system in thermal equilibrium with a set of distinguishable atoms, derive formulae for the Einstein A and B coefficients.

Give a brief description of the operation of a laser.

A1/16 **Statistical Physics and Cosmology**

(i) Introducing the concept of a co-moving distance co-ordinate, explain briefly how the velocity of a galaxy in an isotropic and homogeneous universe is determined by the scale factor $a(t)$. How is the scale factor related to the Hubble constant H_0?

A homogeneous and isotropic universe has an energy density $\rho(t)c^2$ and a pressure $P(t)$. Use the relation $dE = -PdV$ to derive the "fluid equation"

$$\dot{\rho} = -3\left(\rho + \frac{P}{c^2}\right)\left(\frac{\dot{a}}{a}\right),$$

where the overdot indicates differentiation with respect to time, t. Given that $a(t)$ satisfies the "acceleration equation"

$$\ddot{a} = -\frac{4\pi G}{3}\, a\left(\rho + \frac{3P}{c^2}\right),$$

show that the quantity

$$k = c^{-2}\left(\frac{8\pi G}{3}\rho a^2 \; - \dot{a}^2\right)$$

is time-independent.

The pressure P is related to ρ by the "equation of state"

$$P = \sigma\rho c^2, \quad |\sigma| < 1 \; .$$

Given that $a(t_0) = 1$, find $a(t)$ for $k = 0$, and hence show that $a(0) = 0$.

(ii) What is meant by the expression "the Hubble time"?

Assuming that $a(0) = 0$ and $a(t_0) = 1$, where t_0 is the time now (of the current cosmological era), obtain a formula for the radius R_0 of the observable universe.

Given that

$$a(t) \quad = \quad \left(\frac{t}{t_0}\right)^{\alpha}$$

for constant α, find the values of α for which R_0 is finite. Given that R_0 is finite, show that the age of the universe is less than the Hubble time. Explain briefly, and qualitatively, why this result is to be expected as long as

$$\rho + 3\frac{P}{c^2} > 0.$$

A3/14 **Statistical Physics and Cosmology**

(i) A spherically symmetric star has pressure $P(r)$ and mass density $\rho(r)$, where r is distance from the star's centre. Stating without proof any theorems you may need, show that mechanical equilibrium implies the Newtonian pressure support equation

$$P' = -\frac{Gm\rho}{r^2} ,$$

where $m(r)$ is the mass within radius r and $P' = dP/dr$.

Write down an integral expression for the total gravitational potential energy, E_{gr}. Use this to derive the "virial theorem"

$$E_{gr} = -3\langle P\rangle V ,$$

when $\langle P\rangle$ is the average pressure.

(ii) Given that the total kinetic energy, E_{kin}, of a spherically symmetric star is related to its average pressure by the formula

$$E_{kin} = \alpha\langle P\rangle V \qquad (*)$$

for constant α, use the virial theorem (stated in part (i)) to determine the condition on α needed for gravitational binding. State the relation between pressure P and "internal energy" U for an ideal gas of non-relativistic particles. What is the corresponding relation for ultra-relativistic particles? Hence show that the formula $(*)$ applies in these cases, and determine the values of α.

Why does your result imply a maximum mass for any star, whatever the source of its pressure? What is the maximum mass, approximately, for stars supported by

(a) thermal pressure,

(b) electron degeneracy pressure (White Dwarf),

(c) neutron degeneracy pressure (Neutron Star).

A White Dwarf can accrete matter from a companion star until its mass exceeds the Chandrasekar limit. Explain briefly the process by which it then evolves into a neutron star.

A4/18 **Statistical Physics and Cosmology**

(i) Given that $g(p)dp$ is the number of eigenstates of a gas particle with momentum between p and $p + dp$, write down the Bose-Einstein distribution $\bar{n}(p)$ for the average number of particles with momentum between p and $p + dp$, as a function of temperature T and chemical potential μ.

Given that $\mu = 0$ and $g(p) = 8\pi \frac{Vp^2}{h^3}$ for a gas of photons, obtain a formula for the energy density ρ_T at temperature T in the form

$$\rho_T = \int_0^\infty \epsilon_T(\nu) d\nu,$$

where $\epsilon_T(\nu)$ is a function of the photon frequency ν that you should determine. Hence show that the value ν_{peak} of ν at the maximum of $\epsilon_T(\nu)$ is proportional to T.

A thermally isolated photon gas undergoes a slow change of its volume V. Why is $\bar{n}(p)$ unaffected by this change? Use this fact to show that VT^3 remains constant.

(ii) According to the "Hot Big Bang" theory, the Universe evolved by expansion from an earlier state in which it was filled with a gas of electrons, protons and photons (with $n_e = n_p$) at thermal equilibrium at a temperature T such that

$$2m_e c^2 \gg kT \gg B \ ,$$

where m_e is the electron mass and B is the binding energy of a hydrogen atom. Why must the composition have been different when $kT \gg 2m_e c^2$? Why must it change as the temperature falls to $kT \ll B$? Why does this lead to a thermal decoupling of radiation from matter?

The baryon number of the Universe can be taken to be the number of protons, either as free particles or as hydrogen atom nuclei. Let n_b be the baryon number density and n_γ the photon number density. Why is the ratio $\eta = n_b/n_\gamma$ unchanged by the expansion of the universe? Given that $\eta \ll 1$, obtain an estimate of the temperature T_D at which decoupling occurs, as a function of η and B. How does this decoupling lead to the concept of a "surface of last scattering" and a prediction of a Cosmic Microwave Background Radiation (CMBR)?

A1/17 **Symmetries and Groups in Physics**

(i) Let $h : G \to G'$ be a surjective homomorphism between two groups, G and G'. If $D' : G' \to GL(\mathbb{C}^n)$ is a representation of G', show that $D(g) = D'(h(g))$ for $g \in G$ is a representation of G and, if D' is irreducible, show that D is also irreducible. Show further that $\widetilde{D}(\widetilde{g}) = D'\left(\widetilde{h}(\widetilde{g})\right)$ is a representation of $G/\ker(h)$, where $\widetilde{h}(\widetilde{g}) = h(g)$ for $g \in G$ and $\widetilde{g} \in G/\ker(h)$ (with $g \in \widetilde{g}$). Deduce that the characters $\chi, \widetilde{\chi}, \chi'$ of $D, \widetilde{D}, D'$, respectively, satisfy

$$\chi(g) = \widetilde{\chi}(\widetilde{g}) = \chi'(h(g)) .$$

(ii) D_4 is the symmetry group of rotations and reflections of a square. If c is a rotation by $\pi/2$ about the centre of the square and b is a reflection in one of its symmetry axes, then $D_4 = \{e, c, c^2, c^3, b, bc, bc^2, bc^3\}$. Given that the conjugacy classes are $\{e\}$ $\{c^2\}$, $\{c, c^3\}$ $\{b, bc^2\}$ and $\{bc, bc^3\}$ derive the character table of D_4.

Let H_0 be the Hamiltonian of a particle moving in a central potential. The $SO(3)$ symmetry ensures that the energy eigenvalues of H_0 are the same for all the angular momentum eigenstates in a given irreducible $SO(3)$ representation. Therefore, the energy eigenvalues of H_0 are labelled E_{nl} with $n \in \mathbb{N}$ and $l \in \mathbb{N}_0$, $l < n$. Assume now that in a crystal the symmetry is reduced to a D_4 symmetry by an additional term H_1 of the total Hamiltonian, $H = H_0 + H_1$. Find how the H_0 eigenstates in the $SO(3)$ irreducible representation with $l = 2$ (the D-wave orbital) decompose into irreducible representations of H. You may assume that the character, $g(\theta)$, of a group element of $SO(3)$, in a representation labelled by l is given by

$$\chi(g_\theta) = 1 + 2\cos\theta + 2\cos(2\theta) + \ldots + 2\cos(l\theta),$$

where θ is a rotation angle and $l(l+1)$ is the eigenvalue of the total angular momentum, $\mathbf{L}^2$.

A3/15 **Symmetries and Groups in Physics**

(i) The pions form an isospin triplet with $\pi^+ = |1,1\rangle$, $\pi^0 = |1,0\rangle$ and $\pi^- = |1,-1\rangle$, whilst the nucleons form an isospin doublet with $p = |\frac{1}{2},\frac{1}{2}\rangle$ and $n = |\frac{1}{2},-\frac{1}{2}\rangle$. Consider the isospin representation of two-particle states spanned by the basis

$$T = \{|\pi^+ p\rangle,\ |\pi^+ n\rangle,\ |\pi^0 p\rangle,\ |\pi^0 n\rangle,\ |\pi^- p\rangle,\ |\pi^- n\rangle\}\,.$$

State which irreducible representations are contained in this representation and explain why $|\pi^+ p\rangle$ is an isospin eigenstate.

Using

$$I_- \,|j,m\rangle = \sqrt{(j-m+1)(j+m)}\,|j,m-1\rangle,$$

where I_- is the isospin ladder operator, write the isospin eigenstates in terms of the basis, T.

(ii) The Lie algebra $su(2)$ of generators of $SU(2)$ is spanned by the operators $\{J_+, J_-, J_3\}$ satisfying the commutator algebra $[J_+, J_-] = 2J_3$ and $[J_3, J_\pm] = \pm J_\pm$. Let Ψ_j be an eigenvector of J_3: $J_3\left(\Psi_j\right) = j\Psi_j$ such that $J_+\Psi_j = 0$. The vector space $V_j = \mathrm{span}\{J_-^n \Psi_j : n \in \mathbb{N}_0\}$ together with the action of an arbitrary $su(2)$ operator A on V_j defined by

$$J_-\left(J_-^n\Psi_j\right) = J_-^{n+1}\Psi_j\ , \qquad A\left(J_-^n\Psi_j\right) = [A, J_-]\left(J_-^{n-1}\Psi_j\right) + J_-\left(A\left(J_-^{n-1}\Psi_j\right)\right)\ ,$$

forms a representation (not necessarily reducible) of $su(2)$. Show that if $J_-^n\Psi_j$ is non-trivial then it is an eigenvector of J_3 and find its eigenvalue. Given that $[J_+, J_-^n] = \alpha_n J_-^{n-1}J_3 + \beta_n J_-^{n-1}$ show that α_n and β_n satisfy

$$\alpha_n = \alpha_{n-1} + 2, \qquad \beta_n = \beta_{n-1} - \alpha_{n-1}.$$

By solving these equations evaluate $[J_+, J_-^n]$. Show that $J_+J_-^{2j+1}\Psi_j = 0$. Hence show that $J_-^{2j+1}\Psi_j$ is contained in a proper sub-representation of V_j.

A1/18 **Transport Processes**

(i) The diffusion equation for a spherically-symmetric concentration field $C(r,t)$ is

$$C_t = \frac{D}{r^2}\left(r^2 C_r\right)_r\,, \tag{1}$$

where r is the radial coordinate. Find and sketch the similarity solution to (1) which satisfies $C \to 0$ as $r \to \infty$ and $\int_0^\infty 4\pi r^2 C(r,t)dr = M =$ constant, assuming it to be of the form

$$C = \frac{M}{(Dt)^a}F(\eta), \quad \eta = \frac{r}{(Dt)^b},$$

where a and b are numbers to be found.

(ii) A two-dimensional piece of heat-conducting material occupies the region $a \leqslant r \leqslant b,\ -\pi/2 \leqslant \theta \leqslant \pi/2$ (in plane polar coordinates). The surfaces $r = a,\ \theta = -\pi/2,\ \theta = \pi/2$ are maintained at a constant temperature T_1; at the surface $r = b$ the boundary condition on temperature $T(r,\theta)$ is

$$T_r + \beta T = 0,$$

where $\beta > 0$ is a constant. Show that the temperature, which satisfies the steady heat conduction equation

$$T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta} = 0,$$

is given by a Fourier series of the form

$$\frac{T}{T_1} = K + \sum_{n=0}^{\infty} \cos\left(\alpha_n\theta\right)\left[A_n\left(\frac{r}{a}\right)^{2n+1} + B_n\left(\frac{a}{r}\right)^{2n+1}\right],$$

where $K,\ \alpha_n,\ A_n,\ B_n$ are to be found.

In the limits $a/b \ll 1$ and $\beta b \ll 1$, show that

$$\int_{-\pi/2}^{\pi/2} T_r r d\theta \approx -\pi\beta b T_1,$$

given that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Explain how, in these limits, you could have obtained this result much more simply.

A3/16 **Transport Processes**

(i) Incompressible fluid of kinematic viscosity ν occupies a parallel-sided channel $0 \leqslant y \leqslant h_0$, $-\infty < x < \infty$. The wall $y = 0$ is moving parallel to itself, in the x-direction, with velocity Re $\left\{Ue^{i\omega t}\right\}$, where t is time and U, ω are real constants. The fluid velocity $u(y,t)$ satisfies the equation

$$u_t = \nu u_{yy};$$

write down the boundary conditions satisfied by u.

Assuming that

$$u = \operatorname{Re}\left\{a \sinh[b(1-\eta)]e^{i\omega t}\right\},$$

where $\eta = y/h_0$, find the complex constants a, b. Calculate the velocity (in real, not complex, form) in the limit $h_0(\omega/\nu)^{1/2} \to 0$.

(ii) Incompressible fluid of viscosity μ fills the narrow gap between the rigid plane $y = 0$, which moves parallel to itself in the x-direction with constant speed U, and the rigid wavy wall $y = h(x)$, which is at rest. The length-scale, L, over which h varies is much larger than a typical value, h_0, of h.

Assume that inertia is negligible, and therefore that the governing equations for the velocity field (u, v) and the pressure p are

$$u_x + v_y = 0,\ p_x = \mu\left(u_{xx} + u_{yy}\right), p_y = \mu\left(v_{xx} + v_{yy}\right).$$

Use scaling arguments to show that these equations reduce approximately to

$$p_x = \mu u_{yy}, \quad p_y = 0.$$

Hence calculate the velocity $u(x,y)$, the flow rate

$$Q = \int_0^h u dy,$$

and the viscous shear stress exerted by the fluid on the plane wall,

$$\tau = -\mu u_y|_{y=0}$$

in terms of p_x, h, U and μ.

Now assume that $h = h_0(1 + \epsilon \sin kx)$, where $\epsilon \ll 1$ and $kh_0 \ll 1$, and that p is periodic in x with wavelength $2\pi/k$. Show that

$$Q = \frac{h_0 U}{2}\left(1 - \frac{3}{2}\epsilon^2 + O\left(\epsilon^4\right)\right)$$

and calculate τ correct to $O\left(\epsilon^2\right)$. Does increasing the amplitude ϵ of the corrugation cause an increase or a decrease in the force required to move the plane $y = 0$ at the chosen speed U?

A4/19 **Transport Processes**

Fluid flows in the x-direction past the infinite plane $y = 0$ with uniform but time-dependent velocity $U(t) = U_0 G(t/t_0)$, where G is a positive function with timescale t_0. A long region of the plane, $0 < x < L$, is heated and has temperature $T_0\left(1 + \gamma (x/L)^n\right)$, where $T_0,\ \gamma,\ n$ are constants $[\gamma = O(1)]$; the remainder of the plane is insulating $(T_y = 0)$. The fluid temperature far from the heated region is T_0. A thermal boundary layer is formed over the heated region. The full advection–diffusion equation for temperature $T(x, y, t)$ is

$$T_t + U(t)T_x = D\left(T_{yy} + T_{xx}\right), \tag{1}$$

where D is the thermal diffusivity. By considering the steady case $(G \equiv 1)$, derive a scale for the thickness of the boundary layer, and explain why the term T_{xx} in (1) can be neglected if $U_0 L/D \gg 1$.

Neglecting T_{xx}, use the change of variables

$$\tau = \frac{t}{t_0}, \quad \xi = \frac{x}{L}, \quad \eta = y\left[\frac{U(t)}{Dx}\right]^{1/2}, \quad \frac{T - T_0}{T_0} = \gamma\left(\frac{x}{L}\right)^n f(\xi, \eta, \tau)$$

to transform the governing equation to

$$f_{\eta\eta} + \frac{1}{2}\eta f_\eta - nf = \xi f_\xi + \frac{L\xi}{t_0 U_0}\left(\frac{G_\tau}{2G^2}\eta f_\eta + \frac{1}{G} f_\tau\right). \tag{2}$$

Write down the boundary conditions to be satisfied by f in the region $0 < \xi < 1$.

In the case in which U is slowly-varying, so $\epsilon = \frac{L}{t_0 U_0} \ll 1$, consider a solution for f in the form

$$f = f_0(\eta) + \epsilon f_1(\xi, \eta, \tau) + O\left(\epsilon^2\right).$$

Explain why f_0 is independent of ξ and τ.

Henceforth take $n = \frac{1}{2}$. Calculate $f_0(\eta)$ and show that

$$f_1 = \frac{G_\tau \xi}{G^2} g(\eta) \quad ,$$

where g satisfies the ordinary differential equation

$$g'' + \frac{1}{2}\eta g' - \frac{3}{2} g = \frac{-\eta}{4}\int_\eta^\infty e^{-u^2/4} du.$$

State the boundary conditions on $g(\eta)$.

The heat transfer per unit length of the heated region is $-DT_y|_{y=0}$. Use the above results to show that the total rate of heat transfer is

$$T_0\left[DLU(t)\right]^{1/2} \frac{\gamma}{2}\left\{\sqrt{\pi} - \frac{\epsilon G_\tau}{G^2} g'(0) + O\left(\epsilon^2\right)\right\}.$$

A1/19 **Theoretical Geophysics**

(i) From the surface of a flat Earth, an explosive source emits P-waves downward into a horizontal homogeneous elastic layer of uniform thickness h and P-wave speed α_1 overlying a lower layer of infinite depth and P-wave speed α_2, where $\alpha_2 > \alpha_1$. A line of seismometers on the surface records the travel time t as a function of distance x from the source for the various arrivals along different ray paths.

Sketch the ray paths associated with the direct, reflected and head waves arriving at a given position. Calculate the travel times $t(x)$ of the direct and reflected waves, and sketch the corresponding travel-time curves. Hence explain how to estimate α_1 and h from the recorded arrival times. Explain briefly why head waves are only observed beyond a minimum distance x_c from the source and why they have a travel-time curve of the form $t = t_c + (x - x_c)/\alpha_2$ for $x > x_c$.
[*You need not calculate x_c or t_c.*]

(ii) A plane SH-wave in a homogeneous elastic solid has displacement proportional to $\exp[i(kx+mz-\omega t)]$. Express the slowness vector $\mathbf{s}$ in terms of the wavevector $\mathbf{k} = (k, 0, m)$ and ω. Deduce an equation for m in terms of k, ω and the S-wave speed β.

A homogeneous elastic layer of uniform thickness h, S-wave speed β_1 and shear modulus μ_1 has a stress-free surface $z = 0$ and overlies a lower layer of infinite depth, S-wave speed β_2 ($> \beta_1$) and shear modulus μ_2. Find the vertical structure of Love waves with displacement proportional to $\exp[i(kx - \omega t)]$, and show that the horizontal phase speed c obeys

$$\tan\left[\left(\frac{1}{\beta_1^2} - \frac{1}{c^2}\right)^{1/2} \omega h\right] = \frac{\mu_2}{\mu_1}\left(\frac{1/c^2 - 1/\beta_2^2}{1/\beta_1^2 - 1/c^2}\right)^{1/2} .$$

By sketching both sides of the equation as a function of c in $\beta_1 \leqslant c \leqslant \beta_2$ show that at least one mode exists for every value of ω.

A2/16 **Theoretical Geophysics**

(i) In a reference frame rotating with constant angular velocity $\boldsymbol{\Omega}$ the equations of motion for an inviscid, incompressible fluid of density ρ in a gravitational field $\mathbf{g} = -\nabla\Phi$ are

$$\rho\frac{D\mathbf{u}}{Dt} + 2\rho\boldsymbol{\Omega}\wedge\mathbf{u} = -\nabla p + \rho\mathbf{g}\ , \qquad \nabla\cdot\mathbf{u} = 0\ .$$

Define the Rossby number and explain what is meant by geostrophic flow.

Derive the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla\mathbf{u} + \frac{\nabla\rho\wedge\nabla p}{\rho^2}.$$

[*Recall that* $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}^2) - \mathbf{u}\wedge(\nabla\wedge\mathbf{u})$.]

Give a physical interpretation for the term $(\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla\mathbf{u}$.

(ii) Consider the rotating fluid of part (i), but now let ρ be constant and absorb the effects of gravity into a modified pressure $P = p - \rho\mathbf{g}\cdot\mathbf{x}$. State the *linearized* equations of motion and the *linearized* vorticity equation for small-amplitude motions (inertial waves).

Use the linearized equations of motion to show that

$$\nabla^2 P = 2\rho\boldsymbol{\Omega}\cdot\boldsymbol{\omega}\ .$$

Calculate the time derivative of the curl of the linearized vorticity equation. Hence show that

$$\frac{\partial^2}{\partial t^2}(\nabla^2\mathbf{u}) = -(2\boldsymbol{\Omega}\cdot\nabla)^2\mathbf{u}\ .$$

Deduce the dispersion relation for waves proportional to $\exp[i(\mathbf{k}\cdot\mathbf{x} - nt)]$. Show that $|n| \leq 2\Omega$. Show further that if $n = 2\Omega$ then $P = 0$.

A4/20 **Theoretical Geophysics**

Write down expressions for the phase speed c and group velocity c_g in one dimension for general waves of the form $A\exp[i(kx - \omega t)]$ with dispersion relation $\omega(k)$. Briefly indicate the physical significance of c and c_g for a wavetrain of finite size.

The dispersion relation for internal gravity waves with wavenumber $\mathbf{k} = (k, 0, m)$ in an incompressible stratified fluid with constant buoyancy frequency N is

$$\omega = \frac{\pm Nk}{(k^2 + m^2)^{1/2}}.$$

Calculate the group velocity $\mathbf{c}_g$ and show that it is perpendicular to $\mathbf{k}$. Show further that the horizontal components of $\mathbf{k}/\omega$ and $\mathbf{c}_g$ have the same sign and that the vertical components have the opposite sign.

The vertical velocity w of small-amplitude internal gravity waves is governed by

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 w\right) + N^2 \nabla_h^2 w = 0\ , \qquad (*)$$

where ∇_h^2 is the horizontal part of the Laplacian and N is constant.

Find separable solutions to $(*)$ of the form $w(x, z, t) = X(x - Ut)Z(z)$ corresponding to waves with constant horizontal phase speed $U > 0$. Comment on the nature of these solutions for $0 < k < N/U$ and for $k > N/U$.

A semi-infinite stratified fluid occupies the region $z > h(x, t)$ above a moving lower boundary $z = h(x, t)$. Construct the solution to $(*)$ for the case $h = \epsilon \sin[k(x - Ut)]$, where ϵ and k are constants and $\epsilon \ll 1$.

Sketch the orientation of the wavecrests, the propagation direction and the group velocity for the case $0 < k < N/U$.

A2/17 **Mathematical Methods**

(i) A certain physical quantity $q(x)$ can be represented by the series $\sum\limits_{n=0}^{\infty} c_n x^n$ in $0 \leqslant x < x_0$, but the series diverges for $x > x_0$. Describe the Euler transformation to a new series which may enable $q(x)$ to be computed for $x > x_0$. Give the first four terms of the new series.

Describe briefly the disadvantages of the method.

(ii) The series $\sum\limits_{1}^{\infty} c_r$ has partial sums $S_n = \sum\limits_{1}^{n} c_r$. Describe Shanks' method to approximate S_n by

$$S_n = A + BC^n \ , \qquad (*)$$

giving expressions for A, B and C.

Denote by B_N and C_N the values of B and C respectively derived from these expressions using S_{N-1}, S_N and S_{N+1} for some fixed N. Now let $A^{(n)}$ be the value of A obtained from $(*)$ with $B = B_N, C = C_N$. Show that, if $|C_N| < 1$,

$$\sum_{1}^{\infty} c_r = \lim_{n \to \infty} A^{(n)} \ .$$

If, in fact, the partial sums satisfy

$$S_n = a + \alpha c^n + \beta d^n \ ,$$

with $1 > |c| > |d|$, show that

$$A^{(n)} = A + \gamma d^n + o(d^n) \ ,$$

where γ is to be found.

A3/17 **Mathematical Methods**

(i) The function $y(x)$ satisfies the differential equation

$$y'' + by' + cy = 0 \ , \quad 0 < x < 1 \ ,$$

where b and c are constants, with boundary conditions $y(0) = 0, \ y'(0) = 1$. By integrating this equation or otherwise, show that y must also satisfy the integral equation

$$y(x) = g(x) + \int_0^1 K(x,t)y(t)dt \ ,$$

and find the functions $g(x)$ and $K(x,t)$.

(ii) Solve the integral equation

$$\varphi(x) = 1 + \lambda^2 \int_0^x (x-t)\varphi(t)dt \ , \quad x > 0 \ , \quad \lambda \text{ real} \ ,$$

by finding an ordinary differential equation satisfied by $\varphi(x)$ together with boundary conditions.

Now solve the integral equation by the method of successive approximations and show that the two solutions are the same.

A4/21 **Mathematical Methods**

The equation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \ ,$$

where $\mathbf{A}$ is a real square matrix and $\mathbf{x}$ a column vector, has a simple eigenvalue $\lambda = \mu$ with corresponding right-eigenvector $\mathbf{x} = \mathbf{X}$. Show how to find expressions for the perturbed eigenvalue and right-eigenvector solutions of

$$\mathbf{A}\mathbf{x} + \epsilon \mathbf{b}(\mathbf{x}) = \lambda \mathbf{x} \ , \quad |\epsilon| \ll 1 \ ,$$

to first order in ϵ, where $\mathbf{b}$ is a vector function of $\mathbf{x}$. State clearly any assumptions you make.

If $\mathbf{A}$ is $(n \times n)$ and has a complete set of right-eigenvectors $\mathbf{X}^{(j)}, \ j = 1, 2, ...n$, which span $\mathbb{R}^n$ and correspond to separate eigenvalues $\mu^{(j)}, \ j = 1, 2, ...n$, find an expression for the first-order perturbation to $\mathbf{X}^{(1)}$ in terms of the $\left\{\mathbf{X}^{(j)}\right\}$ and the corresponding left-eigenvectors of $\mathbf{A}$.

Find the normalised eigenfunctions and eigenvalues of the equation

$$\frac{d^2y}{dx^2} + \lambda y = 0, \ 0 < x < 1 \ ,$$

with $y(0) = y(1) = 0$. Let these be the zeroth order approximations to the eigenfunctions of

$$\frac{d^2y}{dx^2} + \lambda y + \epsilon b(y) = 0, \ 0 < x < 1 \ ,$$

with $y(0) = y(1) = 0$ and where b is a function of y. Show that the first-order perturbations of the eigenvalues are given by

$$\lambda_n^{(1)} = -\epsilon\sqrt{2} \int_0^1 \sin(n\pi x) \ \ b\left(\sqrt{2}\sin n\pi x\right) dx \ .$$

A2/18 **Nonlinear Waves**

(i) Establish two conservation laws for the MKdV equation

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

State sufficient boundary conditions that u should satisfy for the conservation laws to be valid.

(ii) The equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\Big(\rho V\Big) = 0$$

models traffic flow on a single-lane road, where $\rho(x,t)$ represents the density of cars, and V is a given function of ρ. By considering the rate of change of the integral

$$\int_a^b \rho \, dx,$$

show that V represents the velocity of the cars.

Suppose now that $V = 1 - \rho$ (in suitable units), and that $0 \leqslant \rho \leqslant 1$ everywhere. Assume that a queue is building up at a traffic light at $x = 1$, so that, when the light turns green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0 & \text{for} \quad x < 0 \quad \text{and} \quad x > 1 \\ x & \text{for} \quad 0 \leqslant x < 1. \end{cases}$$

For this problem, find and sketch the characteristics in the (x,t) plane, for $t > 0$, paying particular attention to those emerging from the point $(1,0)$. Show that a shock forms at $t = \frac{1}{2}$. Find the density of cars $\rho(x,t)$ for $0 < t < \frac{1}{2}$, and all x.

A3/18 **Nonlinear Waves**

(i) The so-called breather solution of the sine-Gordon equation is

$$\phi(x,t) = 4\tan^{-1}\left(\frac{(1-\lambda^2)^{\frac{1}{2}}}{\lambda}\frac{\sin \lambda t}{\cosh(1-\lambda^2)^{\frac{1}{2}}x}\right), \qquad 0 < \lambda < 1.$$

Describe qualitatively the behaviour of $\phi(x,t)$, for $\lambda \ll 1$, when $|x| \gg \ln(2/\lambda)$, when $|x| \ll 1$, and when $\cosh x \approx \frac{1}{\lambda}|\sin \lambda t|$. Explain how this solution can be interpreted in terms of motion of a kink and an antikink. Estimate the greatest separation of the kink and antikink.

(ii) The field $\psi(x,t)$ obeys the nonlinear wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + \frac{dU}{d\psi} = 0,$$

where the potential U has the form

$$U(\psi) = \frac{1}{2}(\psi - \psi^3)^2.$$

Show that $\psi = 0$ and $\psi = 1$ are stable constant solutions.

Find a steady wave solution $\psi = f(x - vt)$ satisfying the boundary conditions $\psi \to 0$ as $x \to -\infty$, $\psi \to 1$ as $x \to \infty$. What constraint is there on the velocity v?

A1/1 B1/1 **Markov Chains**

(i) Let $X = (X_n : 0 \leqslant n \leqslant N)$ be an irreducible Markov chain on the finite state space S with transition matrix $P = (p_{ij})$ and invariant distribution π. What does it mean to say that X is reversible in equilibrium?

Show that X is reversible in equilibrium if and only if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

(ii) A finite connected graph G has vertex set V and edge set E, and has neither loops nor multiple edges. A particle performs a random walk on V, moving at each step to a randomly chosen neighbour of the current position, each such neighbour being picked with equal probability, independently of all previous moves. Show that the unique invariant distribution is given by $\pi_v = d_v/(2|E|)$ where d_v is the degree of vertex v.

A rook performs a random walk on a chessboard; at each step, it is equally likely to make any of the moves which are legal for a rook. What is the mean recurrence time of a corner square. (You should give a clear statement of any general theorem used.)

[*A chessboard is an* 8×8 *square grid. A legal move is one of any length parallel to the axes.*]

A2/1 **Markov Chains**

(i) The fire alarm in Mill Lane is set off at random times. The probability of an alarm during the time-interval $(u, u+h)$ is $\lambda(u)h + o(h)$ where the 'intensity function' $\lambda(u)$ may vary with the time u. Let $N(t)$ be the number of alarms by time t, and set $N(0) = 0$. Show, subject to reasonable extra assumptions to be stated clearly, that $p_i(t) = P(N(t) = i)$ satisfies

$$p_0'(t) = -\lambda(t)p_0(t), \quad p_i'(t) = \lambda(t)\{p_{i-1}(t) - p_i(t)\}, \quad i \geqslant 1.$$

Deduce that $N(t)$ has the Poisson distribution with parameter $\Lambda(t) = \int_0^t \lambda(u)du$.

(ii) The fire alarm in Clarkson Road is different. The number $M(t)$ of alarms by time t is such that

$$P(M(t+h) = m+1 \mid M(t) = m) = \lambda_m h + o(h) ,$$

where $\lambda_m = \alpha m + \beta$, $m \geqslant 0$, and $\alpha, \beta > 0$. Show, subject to suitable extra conditions, that $p_m(t) = P(M(t) = m)$ satisfies a set of differential-difference equations to be specified. Deduce without solving these equations in their entirety that $M(t)$ has mean $\beta(e^{\alpha t} - 1)/\alpha$, and find the variance of $M(t)$.

A3/1 B3/1 **Markov Chains**

(i) Explain what is meant by the *transition semigroup* $\{P_t\}$ of a Markov chain X in continuous time. If the state space is finite, show under assumptions to be stated clearly, that $P_t' = GP_t$ for some matrix G. Show that a distribution π satisfies $\pi G = 0$ if and only if $\pi P_t = \pi$ for all $t \geqslant 0$, and explain the importance of such π.

(ii) Let X be a continuous-time Markov chain on the state space $S = \{1, 2\}$ with generator

$$G = \begin{pmatrix} -\beta & \beta \\ \gamma & -\gamma \end{pmatrix}, \qquad \text{where} \quad \beta, \gamma > 0.$$

Show that the transition semigroup $P_t = \exp(tG)$ is given by

$$(\beta+\gamma)P_t = \begin{pmatrix} \gamma + \beta h(t) & \beta(1-h(t)) \\ \gamma(1-h(t)) & \beta + \gamma h(t) \end{pmatrix},$$

where $h(t) = e^{-t(\beta+\gamma)}$.

For $0 < \alpha < 1$, let

$$H(\alpha) = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}.$$

For a continuous-time chain X, let M be a matrix with (i,j) entry $P(X(1) = j \mid X(0) = i)$, for $i, j \in S$. Show that there is a chain X with $M = H(\alpha)$ if and only if $\alpha > \frac{1}{2}$.

A4/1 **Markov Chains**

Write an essay on the convergence to equilibrium of a discrete-time Markov chain on a countable state-space. You should include a discussion of the existence of invariant distributions, and of the limit theorem in the non-null recurrent case.

A1/2 B1/2 **Principles of Dynamics**

(i) Show that Newton's equations in Cartesian coordinates, for a system of N particles at positions $\mathbf{x}_i(t), i = 1, 2 \dots N$, in a potential $V(\mathbf{x}, t)$, imply Lagrange's equations in a generalised coordinate system

$$q_j = q_j(\mathbf{x}_i, t) \quad , \quad j = 1, 2 \dots 3N;$$

that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad , \quad j = 1, 2 \dots 3N,$$

where $L = T - V$, $T(q, \dot{q}, t)$ being the total kinetic energy and $V(q, t)$ the total potential energy.

(ii) Consider a light rod of length L, free to rotate in a vertical plane (the xz plane), but with one end P forced to move in the x-direction. The other end of the rod is attached to a heavy mass M upon which gravity acts in the negative z direction.

(a) Write down the Lagrangian for the system.

(b) Show that, if P is stationary, the rod has two equilibrium positions, one stable and the other unstable.

(c) The end at P is now forced to move with constant acceleration, $\ddot{x} = A$. Show that, once more, there is one stable equilibrium value of the angle the rod makes with the vertical, and find it.

A2/2 B2/1 **Principles of Dynamics**

(i) An axially symmetric top rotates freely about a fixed point O on its axis. The principal moments of inertia are A, A, C and the centre of gravity G is a distance h from O.

Define the three Euler angles θ, ϕ and ψ, specifying the orientation of the top. Use Lagrange's equations to show that there are three conserved quantities in the motion. Interpret them physically.

(ii) Initially the top is spinning with angular speed n about OG, with OG vertical, before it is slightly disturbed.

Show that, in the subsequent motion, θ stays close to zero if $C^2n^2 > 4mghA$, but if this condition fails then θ attains a maximum value given approximately by

$$\cos\theta \approx \frac{C^2n^2}{2mghA} - 1.$$

Why is this only an approximation?

A3/2 **Principles of Dynamics**

(i) **(a)** Write down Hamilton's equations for a dynamical system. Under what condition is the Hamiltonian a constant of the motion? What is the condition for one of the momenta to be a constant of the motion?

(b) Explain the notion of an adiabatic invariant. Give an expression, in terms of Hamiltonian variables, for one such invariant.

(ii) A mass m is attached to one end of a straight spring with potential energy $\frac{1}{2}kr^2$, where k is a constant and r is the length. The other end is fixed at a point O. Neglecting gravity, consider a general motion of the mass in a plane containing O. Show that the Hamiltonian is given by

$$H = \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}kr^2, \tag{1}$$

where θ is the angle made by the spring relative to a fixed direction, and p_θ, p_r are the generalised momenta. Show that p_θ and the energy E are constants of the motion, using Hamilton's equations.

If the mass moves in a non-circular orbit, and the spring constant k is slowly varied, the orbit gradually changes. Write down the appropriate adiabatic invariant $J\,(E, p_\theta, k, m)$. Show that J is proportional to

$$\sqrt{mk}\,(r_+ - r_-)^2\,,$$

where

$$r_\pm^2 = \frac{E}{k} \pm \sqrt{\left(\frac{E}{k}\right)^2 - \frac{p_\theta^2}{mk}}.$$

Consider an orbit for which p_θ is zero. Show that, as k is slowly varied, the energy $E \propto k^\alpha$, for a constant α which should be found.

[*You may assume without proof that*

$$\int_{r_-}^{r_+} dr\sqrt{\left(1-\frac{r^2}{r_+^2}\right)\left(1-\frac{r_-^2}{r^2}\right)} = \frac{\pi}{4r_+}\,(r_+ - r_-)^2\,.\Big]$$

A4/2 **Principles of Dynamics**

(i) Consider a particle of charge q and mass m, moving in a stationary magnetic field $\mathbf{B}$. Show that Lagrange's equations applied to the Lagrangian

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 + q \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}),$$

where $\mathbf{A}$ is the vector potential such that $\mathbf{B} = \operatorname{curl} \mathbf{A}$, lead to the correct Lorentz force law. Compute the canonical momentum $\mathbf{p}$, and show that the Hamiltonian is $H = \frac{1}{2} m \dot{\mathbf{r}}^2$.

(ii) Expressing the velocity components $\dot{r}_i$ in terms of the canonical momenta and co-ordinates for the above system, derive the following formulae for Poisson brackets:

(a) $\{FG, H\} = F\{G, H\} + \{F, H\} G$, for any functions F, G, H;

(b) $\{m\dot{r}_i, m\dot{r}_j\} = q\epsilon_{ijk} B_k$;

(c) $\{m\dot{r}_i, r_j\} = -\delta_{ij}$;

(d) $\{m\dot{r}_i, f(r_j)\} = -\frac{\partial}{\partial r_i} f(r_j)$.

Now consider a particle moving in the field of a magnetic monopole,

$$B_i = g \frac{r_i}{r^3}.$$

Show that $\{H, \mathbf{J}\} = 0$, where $\mathbf{J} = m\mathbf{r} \wedge \dot{\mathbf{r}} - gq\hat{\mathbf{r}}$. Explain why this means that $\mathbf{J}$ is conserved.

Show that, if $g = 0$, conservation of $\mathbf{J}$ implies that the particle moves in a plane perpendicular to $\mathbf{J}$. What type of surface does the particle move on if $g \neq 0$?

A1/3 **Functional Analysis**

(i) Define the adjoint of a bounded, linear map $u : H \to H$ on the Hilbert space H. Find the adjoint of the map

$$u : H \to H \; ; \quad x \mapsto \phi(x)a$$

where $a, b \in H$ and $\phi \in H^*$ is the linear map $x \mapsto \langle b, x \rangle$.

Now let J be an **incomplete** inner product space and $u : J \to J$ a bounded, linear map. Is it always true that there is an adjoint $u^* : J \to J$?

(ii) Let $\mathcal{H}$ be the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ on the unit disc $\mathbb{D}$ for which

$$\int\int_{\mathbb{D}} |f(z)|^2 \, dx \, dy \; < \; \infty \qquad (z = x + iy).$$

You may assume that this is a Hilbert space for the inner product:

$$\langle f, g \rangle = \int\int_{\mathbb{D}} \overline{f(z)} g(z) \; dx \, dy \; .$$

Show that the functions $u_k : z \mapsto \alpha_k z^k$ $(k = 0, 1, 2, \ldots)$ form an orthonormal sequence in $\mathcal{H}$ when the constants α_k are chosen appropriately.

Prove carefully that every function $f \in \mathcal{H}$ can be written as the sum of a convergent series $\sum_{k=0}^{\infty} f_k u_k$ in $\mathcal{H}$ with $f_k \in \mathbb{C}$.

For each smooth curve γ in the disc $\mathbb{D}$ starting from 0, prove that

$$\phi : \mathcal{H} \to \mathbb{C} \; ; \quad f \mapsto \int_{\gamma} f(z) \; dz$$

is a continuous, linear map. Show that the norm of ϕ satisfies

$$||\phi||^2 = \frac{1}{\pi} \log \left(\frac{1}{1 - |w|^2} \right) \; ,$$

where w is the endpoint of γ.

A2/3 B2/2 **Functional Analysis**

(i) State the Stone-Weierstrass theorem for complex-valued functions. Use it to show that the trigonometric polynomials are dense in the space $C(\mathbb{T})$ of continuous, complex-valued functions on the unit circle $\mathbb{T}$ with the uniform norm.

Show further that, for $f \in C(\mathbb{T})$, the nth Fourier coefficient

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta$$

tends to 0 as $|n|$ tends to infinity.

(ii) **(a)** Let X be a normed space with the property that the series $\sum_{n=1}^{\infty} x_n$ converges whenever (x_n) is a sequence in X with $\sum_{n=1}^{\infty} ||x_n||$ convergent. Show that X is a Banach space.

(b) Let K be a compact metric space and L a closed subset of K. Let $R : C(K) \to C(L)$ be the map sending $f \in C(K)$ to its restriction $R(f) = f|L$ to L. Show that R is a bounded, linear map and that its image is a subalgebra of $C(L)$ separating the points of L.

Show further that, for each function g in the image of R, there is a function $f \in C(K)$ with $R(f) = g$ and $||f||_\infty = ||g||_\infty$. Deduce that every continuous, complex-valued function on L can be extended to a continuous function on all of K.

A3/3 B3/2 **Functional Analysis**

(i) Define the notion of a measurable function between measurable spaces. Show that a continuous function $\mathbb{R}^2 \to \mathbb{R}$ is measurable with respect to the Borel σ-fields on $\mathbb{R}^2$ and $\mathbb{R}$.

By using this, or otherwise, show that, when $f, g : X \to \mathbb{R}$ are measurable with respect to some σ-field $\mathcal{F}$ on X and the Borel σ-field on $\mathbb{R}$, then $f + g$ is also measurable.

(ii) State the Monotone Convergence Theorem for $[0, \infty]$-valued functions. Prove the Dominated Convergence Theorem.

[*You may assume the Monotone Convergence Theorem but any other results about integration that you use will need to be stated carefully and proved.*]

Let X be the real Banach space of continuous real-valued functions on $[0, 1]$ with the uniform norm. Fix $u \in X$ and define

$$T : X \to \mathbb{R} \; ; \quad f \mapsto \int_0^1 f(t)u(t)\, dt \; .$$

Show that T is a bounded, linear map with norm

$$||T|| = \int_0^1 |u(t)|\, dt \; .$$

Is it true, for every choice of u, that there is function $f \in X$ with $||f|| = 1$ and $||T(f)|| = ||T||$?

A4/3 **Functional Analysis**

Write an account of the classical sequence spaces: ℓ_p $(1 \leqslant p \leqslant \infty)$ and c_0. You should define them, prove that they are Banach spaces, and discuss their properties, including their dual spaces. Show that ℓ_∞ is inseparable but that c_0 and ℓ_p for $1 \leqslant p < \infty$ are separable.

Prove that, if $T : X \to Y$ is an isomorphism between two Banach spaces, then

$$T^* : Y^* \to X^* \; ; \quad f \mapsto f \circ T$$

is an isomorphism between their duals.

Hence, or otherwise, show that no two of the spaces $c_0, \ell_1, \ell_2, \ell_\infty$ are isomorphic.

A1/4 B1/3 **Groups, Rings and Fields**

(i) Define the notion of a Sylow p-subgroup of a finite group G, and state a theorem concerning the number of them and the relation between them.

(ii) Show that any group of order 30 has a non-trivial normal subgroup. Is it true that every group of order 30 is commutative?

A2/4 B2/3 **Groups, Rings and Fields**

(i) Show that the ring $k = \mathbf{F}_2[X]/(X^2 + X + 1)$ is a field. How many elements does it have?

(ii) Let k be as in (i). By considering what happens to a chosen basis of the vector space k^2, or otherwise, find the order of the groups $GL_2(k)$ and $SL_2(k)$.

By considering the set of lines in k^2, or otherwise, show that $SL_2(k)$ is a subgroup of the symmetric group S_5, and identify this subgroup.

A3/4 **Groups, Rings and Fields**

(i) Let G be the cyclic subgroup of $GL_2(\mathbf{C})$ generated by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$,

acting on the polynomial ring $\mathbf{C}[X, Y]$. Determine the ring of invariants $\mathbf{C}[X, Y]^G$.

(ii) Determine $\mathbf{C}[X, Y]^G$ when G is the cyclic group generated by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

[*Hint: consider the eigenvectors.*]

A4/4 **Groups, Rings and Fields**

Show that the ring $\mathbf{Z}[\omega]$ is Euclidean, where $\omega = \exp(2\pi i/3)$.

Show that a prime number $p \neq 3$ is reducible in $\mathbf{Z}[\omega]$ if and only if $p \equiv 1 (\text{mod } 3)$.

Which prime numbers p can be written in the form $p = a^2 + ab + b^2$ with $a, b \in \mathbf{Z}$ (and why)?

A1/5 B1/4 **Electromagnetism**

(i) Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a current sheet, $\mathbf{J}$, with unit normal to the sheet $\mathbf{n}$, are

$$\mathbf{n} \wedge \mathbf{B}_2 - \mathbf{n} \wedge \mathbf{B}_1 = \mu_0 \mathbf{J}.$$

State without proof the force per unit area on $\mathbf{J}$.

(ii) Conducting gas occupies the infinite slab $0 \leqslant x \leqslant a$. It carries a steady current $\mathbf{j} = (0,0,j)$ and a magnetic field $\mathbf{B} = (0,B,0)$ where $\mathbf{j}, \mathbf{B}$ depend only on x. The pressure is $p(x)$. The equation of hydrostatic equilibrium is $\nabla p = \mathbf{j} \wedge \mathbf{B}$. Write down the equations to be solved in this case. Show that $p + (1/2\mu_0)B^2$ is independent of x. Using the suffixes 1,2 to denote values at $x = 0, a$, respectively, verify that your results are in agreement with those of Part (i) in the case of $a \to 0$.

Suppose that

$$j(x) = \frac{\pi j_0}{2a} \sin\left(\frac{\pi x}{a}\right), \quad B_1 = 0, \quad p_2 = 0.$$

Find $B(x)$ everywhere in the slab.

A2/5 **Electromagnetism**

(i) Write down the expression for the electrostatic potential $\phi(\mathbf{r})$ due to a distribution of charge $\rho(\mathbf{r})$ contained in a volume V. Perform the multipole expansion of $\phi(\mathbf{r})$ taken only as far as the dipole term.

(ii) If the volume V is the sphere $|\mathbf{r}| \leqslant a$ and the charge distribution is given by

$$\rho(\mathbf{r}) = \begin{cases} r^2 \cos\theta & r \leqslant a \\ 0 & r > a \end{cases},$$

where r, θ are spherical polar coordinates, calculate the charge and dipole moment. Hence deduce ϕ as far as the dipole term.

Obtain an exact solution for $r > a$ by solving the boundary value problem using trial solutions of the forms

$$\phi = \frac{A\cos\theta}{r^2} \quad \text{for} \quad r > a,$$

and

$$\phi = Br\cos\theta + Cr^4\cos\theta \quad \text{for} \quad r < a.$$

Show that the solution obtained from the multipole expansion is in fact exact for $r > a$.

[*You may use without proof the result*

$$\nabla^2(r^k \cos\theta) = (k+2)(k-1)r^{k-2}\cos\theta, \qquad k \in \mathbb{N}.]$$

A3/5 B3/3 **Electromagnetism**

(i) Develop the theory of electromagnetic waves starting from Maxwell equations in vacuum. You should relate the wave-speed c to ϵ_0 and μ_0 and establish the existence of plane, plane-polarized waves in which $\mathbf{E}$ takes the form

$$\mathbf{E} = (E_0 \cos(kz - \omega t), 0, 0) \ .$$

You should give the form of the magnetic field $\mathbf{B}$ in this case.

(ii) Starting from Maxwell's equation, establish Poynting's theorem.

$$-\mathbf{j} \cdot \mathbf{E} \quad = \quad \frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} \ ,$$

where $W = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$ and $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B}$. Give physical interpretations of W, S and the theorem.

Compute the averages over space and time of W and $\mathbf{S}$ for the plane wave described in (i) and relate them. Comment on the result.

A4/5 **Electromagnetism**

Write down the form of Ohm's Law that applies to a conductor if at a point $\mathbf{r}$ it is moving with velocity $\mathbf{v}(\mathbf{r})$.

Use two of Maxwell's equations to prove that

$$\int_C (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot d\mathbf{r} \quad = \quad -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad ,$$

where $C(t)$ is a moving closed loop, $\mathbf{v}$ is the velocity at the point $\mathbf{r}$ on C, and S is a surface spanning C. The time derivative on the right hand side accounts for changes in both C and $\mathbf{B}$. Explain briefly the physical importance of this result.

Find and sketch the magnetic field $\mathbf{B}$ described in the vector potential

$$\mathbf{A}(r, \theta, z) \quad = \quad (0, \tfrac{1}{2}\, brz, 0)$$

in cylindrical polar coordinates (r, θ, z) , where $b > 0$ is constant.

A conducting circular loop of radius a and resistance R lies in the plane $z = h(t)$ with its centre on the z-axis.

Find the magnitude and direction of the current induced in the loop as $h(t)$ changes with time, neglecting self-inductance.

At time $t = 0$ the loop is at rest at $z = 0$. For time $t > 0$ the loop moves with constant velocity $dh/dt = v > 0$. Ignoring the inertia of the loop, use energy considerations to find the force $F(t)$ necessary to maintain this motion.

[*In cylindrical polar coordinates*

$$\text{curl } \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right) . \Big]$$

A1/6 **Dynamics of Differential Equations**

(i) Given a differential equation $\dot{x} = f(x)$ for $x \in \mathbb{R}^n$, explain what it means to say that the solution is given by a flow $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Define the orbit, $o(x)$, through a point x and the ω-limit set, $\omega(x)$, of x. Define also a homoclinic orbit to a fixed point x_0. Sketch a flow in $\mathbb{R}^2$ with a homoclinic orbit, and identify (without detailed justification) the ω-limit sets $\omega(x)$ for each point x in your diagram.

(ii) Consider the differential equations

$$\dot{x} = zy, \qquad \dot{y} = -zx, \qquad \dot{z} = -z^2\,.$$

Transform the equations to polar coordinates (r, θ) in the (x, y) plane. Solve the equation for z to find $z(t)$, and hence find $\theta(t)$. Hence, or otherwise, determine (with justification) the ω-limit set for all points $(x_0, y_0, z_0) \in \mathbb{R}^3$.

A2/6 B2/4 **Dynamics of Differential Equations**

(i) Define a Liapounov function for a flow ϕ on $\mathbb{R}^n$. Explain what it means for a fixed point of the flow to be Liapounov stable. State and prove Liapounov's first stability theorem.

(ii) Consider the damped pendulum

$$\ddot{\theta} + k\dot{\theta} + \sin\theta = 0,$$

where $k > 0$. Show that there are just two fixed points (considering the phase space as an infinite cylinder), and that one of these is the origin and is Liapounov stable. Show further that the origin is asymptotically stable, and that the the ω-limit set of each point in the phase space is one or other of the two fixed points, justifying your answer carefully.

[*You should state carefully any theorems you use in your answer, but you need not prove them.*]

A3/6 B3/4 **Dynamics of Differential Equations**

(i) Define a hyperbolic fixed point x_0 of a flow determined by a differential equation $\dot{x} = f(x)$ where $x \in R^n$ and f is C^1 (i.e. differentiable). State the Hartman-Grobman Theorem for flow near a hyperbolic fixed point. For nonlinear flows in R^2 with a hyperbolic fixed point x_0, does the theorem necessarily allow us to distinguish, on the basis of the linearized flow near x_0 between (a) a stable focus and a stable node; and (b) a saddle and a stable node? Justify your answers briefly.

(ii) Show that the system:

$$\begin{aligned}\dot{x} &= -(\mu+1) + (\mu-3)x - y + 6x^2 + 12xy + 5y^2 - 2x^3 - 6x^2y - 5xy^2,\\ \dot{y} &= 2 - 2x + (\mu-5)y + 4xy + 6y^2 - 2x^2y - 6xy^2 - 5y^3\end{aligned}$$

has a fixed point $(x_0, 0)$ on the x-axis. Show that there is a bifurcation at $\mu = 0$ and determine the stability of the fixed point for $\mu > 0$ and for $\mu < 0$.

Make a linear change of variables of the form $u = x - x_0 + \alpha y$, $v = x - x_0 + \beta y$, where α and β are constants to be determined, to bring the system into the form:

$$\begin{aligned}\dot{u} &= v + u[\mu - (u^2 + v^2)]\\ \dot{v} &= -u + v[\mu - (u^2 + v^2)]\end{aligned}$$

and hence determine whether the periodic orbit produced in the bifurcation is stable or unstable, and whether it exists in $\mu < 0$ or $\mu > 0$.

A4/6 **Dynamics of Differential Equations**

Write a short essay about periodic orbits in flows in two dimensions. Your essay should include criteria for the existence and non-existence of periodic orbits, and should mention (with sketches) at least two bifurcations that create or destroy periodic orbits in flows as a parameter is altered (though a detailed analysis of any bifurcation is not required).

A1/7 B1/12 **Logic, Computation and Set Theory**

(i) What is the *Halting Problem*? What is an *unsolvable* problem?

(ii) Prove that the Halting Problem is unsolvable. Is it decidable whether or not a machine halts with input zero?

B2/11 **Logic, Computation and Set Theory**

Let U be an arbitrary set, and $\mathcal{P}(U)$ the power set of U. For X a subset of $\mathcal{P}(U)$, the *dual* $X^\vee$ of X is the set $\{y \subseteq U : (\forall x \in X)(y \cap x \neq \emptyset)\}$.

(i) Show that $X \subseteq Y \to Y^\vee \subseteq X^\vee$.

Show that for $\{X_i : i \in I\}$ a family of subsets of $\mathcal{P}(U)$

$$\left(\bigcup\{X_i : i \in I\}\right)^\vee = \bigcap\{X_i^\vee : i \in I\}.$$

(ii) Consider $S = \{X \subseteq \mathcal{P}(U) : X \subseteq X^\vee\}$. Show that $S, \subseteq$ is a chain-complete poset.

State Zorn's lemma and use it to deduce that there exists X with $X = X^\vee$.

Show that if $X = X^\vee$ then the following hold:

X is closed under superset; for all $U' \subseteq U$, X contains either U' or $U \setminus U'$.

A3/8 B3/11 **Logic, Computation and Set Theory**

(i) Write down a set of axioms for the theory of dense linear order with a bottom element but no top element.

(ii) Prove that this theory has, up to isomorphism, precisely one countable model.

A4/8 B4/10 **Logic, Computation and Set Theory**

What is a wellfounded relation, and how does wellfoundedness underpin wellfounded induction?

A formula $\phi(x, y)$ with two free variables *defines an $\in$-automorphism* if for all x there is a unique y, the function f, defined by $y = f(x)$ if and only if $\phi(x, y)$, is a permutation of the universe, and we have $(\forall xy)(x \in y \leftrightarrow f(x) \in f(y))$.

Use wellfounded induction over $\in$ to prove that all formulæ defining $\in$-automorphisms are equivalent to $x = y$.

A1/12 B1/15 **Principles of Statistics**

(i) What are the main approaches by which prior distributions are specified in Bayesian inference?

Define the risk function of a decision rule d. Given a prior distribution, define what is meant by a Bayes decision rule and explain how this is obtained from the posterior distribution.

(ii) Dashing late into King's Cross, I discover that Harry must have already boarded the Hogwarts Express. I must therefore make my own way onto platform nine and three-quarters. Unusually, there are two guards on duty, and I will ask one of them for directions. It is safe to assume that one guard is a Wizard, who will certainly be able to direct me, and the other a Muggle, who will certainly not. But which is which? Before choosing one of them to ask for directions to platform nine and three-quarters, I have just enough time to ask one of them "Are you a Wizard?", and on the basis of their answer I must make my choice of which guard to ask for directions. I know that a Wizard will answer this question truthfully, but that a Muggle will, with probability $\frac{1}{3}$, answer it untruthfully.

Failure to catch the Hogwarts Express results in a loss which I measure as 1000 galleons, there being no loss associated with catching up with Harry on the train.

Write down an exhaustive set of non-randomised decision rules for my problem and, by drawing the associated risk set, determine my minimax decision rule.

My prior probability is $\frac{2}{3}$ that the guard I ask "Are you a Wizard?" is indeed a Wizard. What is my Bayes decision rule?

A2/11 B2/16 **Principles of Statistics**

(i) Let $X_1, \ldots, X_n$ be independent, identically-distributed $N(\mu, \mu^2)$ random variables, $\mu > 0$.

Find a minimal sufficient statistic for μ.

Let $T_1 = n^{-1} \sum_{i=1}^n X_i$ and $T_2 = \sqrt{n^{-1} \sum_{i=1}^n X_i^2}$. Write down the distribution of X_i/μ, and hence show that $Z = T_1/T_2$ is ancillary. Explain briefly why the Conditionality Principle would lead to inference about μ being drawn from the conditional distribution of T_2 given Z.

What is the maximum likelihood estimator of μ?

(ii) Describe briefly the Bayesian approach to predictive inference.

Let $Z_1, \ldots, Z_n$ be independent, identically-distributed $N(\mu, \sigma^2)$ random variables, with μ, σ^2 both unknown. Derive the maximum likelihood estimators $\widehat{\mu}, \widehat{\sigma}^2$ of μ, σ^2 based on $Z_1, \ldots, Z_n$, and state, without proof, their joint distribution.

Suppose that it is required to construct a prediction interval $I_{1-\alpha} \equiv I_{1-\alpha}(Z_1, \ldots, Z_n)$ for a future, independent, random variable Z_0 with the same $N(\mu, \sigma^2)$ distribution, such that

$$P(Z_0 \in I_{1-\alpha}) = 1 - \alpha,$$

with the probability over the *joint* distribution of $Z_0, Z_1, \ldots, Z_n$. Let

$$I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2) = \left[\bar{Z}_n - z_{\alpha/2}\sigma\sqrt{1+1/n},\ \bar{Z}_n + z_{\alpha/2}\sigma\sqrt{1+1/n}\right],$$

where $\bar{Z}_n = n^{-1} \sum_{i=1}^n Z_i$, and $\Phi(z_\beta) = 1 - \beta$, with Φ the distribution function of $N(0,1)$.

Show that $P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \sigma^2)) = 1 - \alpha$.

By considering the distribution of $(Z_0 - \bar{Z}_n)/\left(\widehat{\sigma}\sqrt{\frac{n+1}{n-1}}\right)$, or otherwise, show that

$$P(Z_0 \in I_{1-\alpha}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) < 1 - \alpha,$$

and show how to construct an interval $I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)$ with

$$P(Z_0 \in I_{1-\gamma}(Z_1, \ldots, Z_n; \widehat{\sigma}^2)) = 1 - \alpha.$$

[Hint: if Y has the t-distribution with m degrees of freedom and $t_\beta^{(m)}$ is defined by $P(Y < t_\beta^{(m)}) = 1 - \beta$ then $t_\beta > z_\beta$ for $\beta < \frac{1}{2}$.]

A3/12 B3/15 **Principles of Statistics**

(i) Explain what is meant by a *uniformly most powerful unbiased test* of a null hypothesis against an alternative.

Let $Y_1, \ldots, Y_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, with σ^2 known. Explain how to construct a uniformly most powerful unbiased size α test of the null hypothesis that $\mu = 0$ against the alternative that $\mu \neq 0$.

(ii) Outline briefly the Bayesian approach to hypothesis testing based on *Bayes factors*.

Let the distribution of $Y_1, \ldots, Y_n$ be as in (i) above, and suppose we wish to test, as in (i), $\mu = 0$ against the alternative $\mu \neq 0$. Suppose we assume a $N(0, \tau^2)$ prior for μ under the alternative. Find the form of the Bayes factor B, and show that, for fixed n, $B \to \infty$ as $\tau \to \infty$.

A4/13 B4/15 **Principles of Statistics**

Write an account, with appropriate examples, of **one** of the following:

(a) Inference in multi-parameter exponential families;

(b) Asymptotic properties of maximum-likelihood estimators and their use in hypothesis testing;

(c) Bootstrap inference.

A1/11 B1/16 **Stochastic Financial Models**

(i) The price of the stock in the binomial model at time r, $1 \leqslant r \leqslant n$, is $S_r = S_0 \prod_{j=1}^{r} Y_j$, where $Y_1, Y_2, \ldots, Y_n$ are independent, identically-distributed random variables with $\mathbb{P}(Y_1 = u) = p = 1 - \mathbb{P}(Y_1 = d)$ and the initial price S_0 is a constant. Denote the fixed interest rate on the bank account by ρ, where $u > 1 + \rho > d > 0$, and let the discount factor $\alpha = 1/(1+\rho)$. Determine the unique value $p = \overline{p}$ for which the sequence $\{\alpha^r S_r,\, 0 \leqslant r \leqslant n\}$ is a martingale.

Explain briefly the significance of $\overline{p}$ for the pricing of contingent claims in the model.

(ii) Let T_a denote the first time that a standard Brownian motion reaches the level $a > 0$. Prove that for $t > 0$,

$$\mathbb{P}(T_a \leqslant t) = 2\left[1 - \Phi\left(a/\sqrt{t}\right)\right],$$

where Φ is the standard normal distribution function.

Suppose that A_t and B_t represent the prices at time t of two different stocks with initial prices 1 and 2, respectively; the prices evolve so that they may be represented as $A_t = e^{\sigma_1 X_t + \mu t}$ and $B_t = 2e^{\sigma_2 Y_t + \mu t}$, respectively, where $\{X_t\}_{t \geqslant 0}$ and $\{Y_t\}_{t \geqslant 0}$ are independent standard Brownian motions and σ_1, σ_2 and μ are constants. Let T denote the first time, if ever, that the prices of the two stocks are the same. Determine $\mathbb{P}(T \leqslant t)$, for $t > 0$.

A3/11 B3/16 **Stochastic Financial Models**

(i) Suppose that Z is a random variable having the normal distribution with $\mathbb{E}Z = \beta$ and Var $Z = \tau^2$.

For positive constants a, c show that

$$\mathbb{E}\left(ae^Z - c\right)_+ = ae^{\left(\beta+\tau^2/2\right)}\Phi\left(\frac{\log(a/c)+\beta}{\tau}+\tau\right) - c\Phi\left(\frac{\log(a/c)+\beta}{\tau}\right),$$

where Φ is the standard normal distribution function.

In the context of the Black-Scholes model, derive the formula for the price at time 0 of a European call option on the stock at strike price c and maturity time t_0 when the interest rate is ρ and the volatility of the stock is σ.

(ii) Let p denote the price of the call option in the Black-Scholes model specified in (i). Show that $\frac{\partial p}{\partial \rho} > 0$ and sketch carefully the dependence of p on the volatility σ (when the other parameters in the model are held fixed).

Now suppose that it is observed that the interest rate lies in the range $0 < \rho < \rho_0$ and when it changes it is linked to the volatility by the formula $\sigma = \ln(\rho_0/\rho)$. Consider a call option at strike price $c = S_0$, where S_0 is the stock price at time 0. There is a small increase $\Delta\rho$ in the interest rate; will the price of the option increase or decrease (assuming that the stock price is unaffected)? Justify your answer carefully.

[*You may assume that the function* $\phi(x)/\Phi(x)$ *is decreasing in* x, $-\infty < x < \infty$, *and increases to* $+\infty$ *as* $x \to -\infty$, *where* Φ *is the standard-normal distribution function and* $\phi = \Phi'$.]

A4/12 B4/16 **Stochastic Financial Models**

Write an essay on the mean-variance approach to portfolio selection in a one-period model. Your essay should contrast the solution in the case when all the assets are risky with that for the case when there is a riskless asset.

A2/13 B2/21 **Foundations of Quantum Mechanics**

(i) Hermitian operators $\hat{x}$, $\hat{p}$, satisfy $[\hat{x},\hat{p}] = i\hbar$. The eigenvectors $|p\rangle$, satisfy $\hat{p}|p\rangle = p|p\rangle$ and $\langle p'|p\rangle = \delta(p'-p)$. By differentiating with respect to b verify that

$$e^{-ib\hat{x}/\hbar}\hat{p}\, e^{ib\hat{x}/\hbar} = \hat{p} + b$$

and hence show that

$$e^{ib\hat{x}/\hbar}|p\rangle = |p+b\rangle.$$

Show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

and

$$\langle p|\hat{p}|\psi\rangle = p\,\langle p|\psi\rangle\,.$$

(ii) A quantum system has Hamiltonian $H = H_0 + H_1$, where H_1 is a small perturbation. The eigenvalues of H_0 are ϵ_n. Give (without derivation) the formulae for the first order and second order perturbations in the energy level of a non-degenerate state. Suppose that the rth energy level of H_0 has j degenerate states. Explain how to determine the eigenvalues of H corresponding to these states to first order in H_1.

In a particular quantum system an orthonormal basis of states is given by $|n_1,n_2\rangle$, where n_i are integers. The Hamiltonian is given by

$$H = \sum_{n_1,n_2}(n_1^2+n_2^2)\,|n_1,n_2\rangle\langle n_1,n_2| + \sum_{n_1,n_2,n'_1,n'_2}\lambda_{|n_1-n'_1|,|n_2-n'_2|}\,|n_1,n_2\rangle\langle n'_1,n'_2|\,,$$

where $\lambda_{r,s} = \lambda_{s,r}$, $\lambda_{0,0} = 0$ and $\lambda_{r,s} = 0$ unless r and s are both even.

Obtain an expression for the ground state energy to second order in the perturbation, $\lambda_{r,s}$. Find the energy eigenvalues of the first excited state to first order in the perturbation. Determine a matrix (which depends on two independent parameters) whose eigenvalues give the first order energy shift of the second excited state.

A3/13 B3/21 **Foundations of Quantum Mechanics**

(i) Write the Hamiltonian for the harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

in terms of creation and annihilation operators, defined by

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - i\frac{p}{m\omega}\right), \qquad a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x + i\frac{p}{m\omega}\right).$$

Obtain an expression for $[a^\dagger, a]$ by using the usual commutation relation between p and x. Deduce the quantized energy levels for this system.

(ii) Define the number operator, N, in terms of creation and annihilation operators, $a^\dagger$ and a. The normalized eigenvector of N with eigenvalue n is $|n\rangle$. Show that $n \geq 0$.

Determine $a|n\rangle$ and $a^\dagger|n\rangle$ in the basis defined by $\{|n\rangle\}$.

Show that

$$a^{\dagger m} a^m |n\rangle = \begin{cases} \frac{n!}{(n-m)!}|n\rangle, & m \leq n\,, \\ 0\,, & m > n\,. \end{cases}$$

Verify the relation

$$|0\rangle\langle 0| = \sum_{m=0} \frac{1}{m!}(-1)^m a^{\dagger m} a^m\,,$$

by considering the action of both sides of the equation on an arbitrary basis vector.

A4/15 B4/22 **Foundations of Quantum Mechanics**

(i) The two states of a spin-$\frac{1}{2}$ particle corresponding to spin pointing along the z axis are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$. Explain why the states

$$|\uparrow,\theta\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + \sin\frac{\theta}{2}\,|\downarrow\rangle, \qquad |\downarrow,\theta\rangle = -\sin\frac{\theta}{2}\,|\uparrow\rangle + \cos\frac{\theta}{2}\,|\downarrow\rangle$$

correspond to the spins being aligned along a direction at an angle θ to the z direction.

The spin-0 state of two spin-$\frac{1}{2}$ particles is

$$|0\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\Big).$$

Show that this is independent of the direction chosen to define $|\uparrow\rangle_{1,2}$, $|\downarrow\rangle_{1,2}$. If the spin of particle 1 along some direction is measured to be $\frac{1}{2}\hbar$ show that the spin of particle 2 along the same direction is determined, giving its value.

[*The Pauli matrices are given by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \Bigg]$$

(ii) Starting from the commutation relation for angular momentum in the form

$$[J_3, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_3,$$

obtain the possible values of j, m, where $m\hbar$ are the eigenvalues of J_3 and $j(j+1)\hbar^2$ are the eigenvalues of $\mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$. Show that the corresponding normalized eigenvectors, $|j,m\rangle$, satisfy

$$J_\pm|j,m\rangle = \hbar\left((j\mp m)(j\pm m+1)\right)^{1/2}|j,m\pm1\rangle,$$

and that

$$\frac{1}{n!}J_-^n|j,j\rangle = \hbar^n\left(\frac{(2j)!}{n!(2j-n)!}\right)^{1/2}|j,j-n\rangle, \qquad n \le 2j.$$

The state $|w\rangle$ is defined by

$$|w\rangle = e^{wJ_-/\hbar}|j,j\rangle,$$

for any complex w. By expanding the exponential show that $\langle w|w\rangle = (1+|w|^2)^{2j}$. Verify that

$$e^{-wJ_-/\hbar}J_3\,e^{wJ_-/\hbar} = J_3 - wJ_-,$$

and hence show that

$$J_3|w\rangle = \hbar\left(j - w\frac{\partial}{\partial w}\right)|w\rangle.$$

If $H = \alpha J_3$ verify that $|e^{i\alpha t}\rangle\, e^{-ij\alpha t}$ is a solution of the time-dependent Schrödinger equation.

A1/15 B1/24 **General Relativity**

(i) The metric of any two-dimensional curved space, rotationally symmetric about a point P, can by suitable choice of coordinates be written locally in the form

$$ds^2 = e^{2\phi(r)}(dr^2 + r^2 d\theta^2),$$

where $r = 0$ at P, $r > 0$ away from P, and $0 \leqslant \theta < 2\pi$. Labelling the coordinates as $(x^1, x^2) = (r, \theta)$, show that the Christoffel symbols $\Gamma^1_{12}, \Gamma^2_{11}$ and Γ^2_{22} are each zero, and compute the non-zero Christoffel symbols $\Gamma^1_{11}, \Gamma^1_{22}$ and $\Gamma^2_{12} = \Gamma^2_{21}$.

The Ricci tensor R_{ab} $(a, b = 1, 2)$ is defined by

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^d_{ac}\Gamma^c_{bd},$$

where a comma denotes a partial derivative. Show that $R_{12} = 0$ and that

$$R_{11} = -\phi'' - r^{-1}\phi', \qquad R_{22} = r^2 R_{11}.$$

(ii) Suppose further that, in a neighbourhood of P, the Ricci scalar R takes the constant value -2. Find a second order differential equation, which you should denote by $(*)$, for $\phi(r)$.

This space of constant Ricci scalar can, by a suitable coordinate transformation $r \to \chi(r)$, leaving θ invariant, be written locally as

$$ds^2 = d\chi^2 + \sinh^2 \chi d\theta^2$$

By studying this coordinate transformation, or otherwise, find $\cosh\chi$ and $\sinh\chi$ in terms of r (up to a constant of integration). Deduce that

$$e^{\phi(r)} = \frac{2A}{(1 - A^2 r^2)} \quad , \quad (0 \leqslant Ar < 1),$$

where A is a positive constant and verify that your equation $(*)$ for ϕ holds.

[*Note that*

$$\int \frac{d\chi}{\sinh\chi} = \text{const.} + \frac{1}{2}\log\bigl(\cosh\chi - 1\bigr) - \frac{1}{2}\log\bigl(\cosh\chi + 1\bigr).\Bigr]$$

A2/15 B2/23 **General Relativity**

(i) Show that the geodesic equation follows from a variational principle with Lagrangian

$$L = g_{ab}\dot{x}^a\dot{x}^b$$

where the path of the particle is $x^a(\lambda)$, and λ is an affine parameter along that path.

(ii) The Schwarzschild metric is given by

$$ds^2 = dr^2\left(1-\frac{2M}{r}\right)^{-1} + r^2(d\theta^2+\sin^2\theta d\phi^2) - \left(1-\frac{2M}{r}\right)dt^2.$$

Consider a photon which moves within the equatorial plane $\theta = \frac{\pi}{2}$. Using the above Lagrangian, or otherwise, show that

$$\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right) = E, \quad \text{and} \quad r^2\left(\frac{d\phi}{d\lambda}\right) = h,$$

for constants E and h. Deduce that

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1-\frac{2M}{r}\right). \qquad (*)$$

Assume further that the photon approaches from infinity. Show that the impact parameter b is given by

$$b = \frac{h}{E} \quad .$$

By considering the equation $(*)$, or otherwise

(a) show that, if $b^2 > 27M^2$, the photon is deflected but not captured by the black hole;

(b) show that, if $b^2 < 27M^2$, the photon is captured;

(c) describe, with justification, the qualitative form of the photon's orbit in the case $b^2 = 27M^2$.

A4/17 B4/25 **General Relativity**

Discuss how Einstein's theory of gravitation reduces to Newton's in the limit of weak fields. Your answer should include discussion of:

(a) the field equations;

(b) the motion of a point particle;

(c) the motion of a pressureless fluid.

[*The metric in a weak gravitational field, with Newtonian potential ϕ, may be taken as*

$$ds^2 = dx^2 + dy^2 + dz^2 - (1+2\phi)dt^2.$$

The Riemann tensor is

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc}.$$]

A1/20 B1/20 **Numerical Analysis**

(i) Let A be a symmetric $n \times n$ matrix such that

$$A_{k,k} > \sum_{\substack{l=1 \\ l \neq k}}^{n} |A_{k,l}| \qquad 1 \leqslant k \leqslant n.$$

Prove that it is positive definite.

(ii) Prove that both Jacobi and Gauss-Seidel methods for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, where the matrix A obeys the conditions of (i), converge.

[*You may quote the Householder-John theorem without proof.*]

A2/19 B2/19 **Numerical Analysis**

(i) Define m-step BDF (backward differential formula) methods for the numerical solution of ordinary differential equations and derive explicitly their coefficients.

(ii) Prove that the linear stability domain of the two-step BDF method includes the interval $(-\infty, 0)$.

A3/19 B3/20 **Numerical Analysis**

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is discretized by the finite-difference method

$$u_m^{n+1} - \frac{1}{2}(\mu - \alpha)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}(\mu + \alpha)(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

where $u_m^n \approx u(m\Delta x, n\Delta t), \mu = \Delta t/(\Delta x)^2$ and α is a constant. Derive the order of magnitude (as a power of Δx) of the local error for different choices of α.

(ii) Investigate the stability of the above finite-difference method for different values of α by the Fourier technique.

A4/22 B4/20 **Numerical Analysis**

Write an essay on the computation of eigenvalues and eigenvectors of matrices.

B1/5 **Combinatorics**

Let $\mathcal{A} \subset [n]^{(r)}$ where $r \leqslant n/2$. Prove that, if $\mathcal{A}$ is 1-intersecting, then $|\mathcal{A}| \leqslant \binom{n-1}{r-1}$. State an upper bound on $|\mathcal{A}|$ that is valid if $\mathcal{A}$ is t-intersecting and n is large compared to r and t.

Let $\mathcal{B} \subset \mathcal{P}([n])$ be maximal 1-intersecting; that is, $\mathcal{B}$ is 1-intersecting but if $\mathcal{B} \subset \mathcal{C} \subset \mathcal{P}([n])$ and $\mathcal{B} \neq \mathcal{C}$ then $\mathcal{C}$ is not 1-intersecting. Show that $|\mathcal{B}| = 2^{n-1}$.

Let $\mathcal{B} \subset \mathcal{P}([n])$ be 2-intersecting. Show that $|\mathcal{B}| \geqslant 2^{n-2}$ is possible. Can the inequality be strict?

B2/5 **Combinatorics**

As usual, $R_k^{(r)}(m)$ denotes the smallest integer n such that every k-colouring of $[n]^{(r)}$ yields a monochromatic m-subset $M \in [n]^{(m)}$. Prove that $R_2^{(2)}(m) > 2^{m/2}$ for $m \geqslant 3$.

Let $\mathcal{P}([n])$ have the colex order, and for $a, b \in \mathcal{P}([n])$ let $\delta(a,b) = \max a \triangle b$; thus $a < b$ means $\delta(a,b) \in b$. Show that if $a < b < c$ then $\delta(a,b) \neq \delta(b,c)$, and that $\delta(a,c) = \max\{\delta(a,b), \delta(b,c)\}$.

Given a red-blue colouring of $[n]^{(2)}$, the 4-colouring

$$\chi : \mathcal{P}([n])^{(3)} \to \{\text{red}, \text{blue}\} \times \{0,1\}$$

is defined as follows:

$$\chi(\{a,b,c\}) = \begin{cases} (\text{red}, 0) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is red} \;\; \text{and } \delta(a,b) < \delta(b,c) \\ (\text{red}, 1) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is red} \;\; \text{and } \delta(a,b) > \delta(b,c) \\ (\text{blue}, 0) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is blue and } \delta(a,b) < \delta(b,c) \\ (\text{blue}, 1) & \text{if } \{\delta(a,b), \delta(b,c)\} \text{ is blue and } \delta(a,b) > \delta(b,c) \end{cases}$$

where $a < b < c$. Show that if $M = \{a_0, a_1, \ldots, a_m\} \in \mathcal{P}([n])^{(m+1)}$ is monochromatic then $\{\delta_1, \ldots, \delta_m\} \in [n]^{(m)}$ is monochromatic, where $\delta_i = \delta(a_{i-1}, a_i)$ and $a_0 < a_1 < \cdots < a_m$.

Deduce that $R_4^{(3)}(m+1) > 2^{2^{m/2}}$ for $m \geqslant 3$.

B4/1 **Combinatorics**

Write an essay on extremal graph theory. You should give proofs of at least two major theorems and you should also include a description of alternative proofs or of further results.

B1/6 **Representation Theory**

Compute the character table of A_5 (begin by listing the conjugacy classes and their orders).

[*It is not enough to write down the result; you must justify your answer.*]

B2/6 **Representation Theory**

(i) Let G be a group, and X and Y finite G-sets. Define the permutation representation $\mathbf{C}[X]$ and compute its character. Show that

$$\dim \operatorname{Hom}_G(\mathbf{C}[X], \mathbf{C}[Y])$$

is equal to the number of G-orbits in $X \times Y$.

(ii) Let $G = S_n$ $(n \geqslant 4)$, $X = \{1, \ldots, n\}$, and

$$Z = \{\, \{i, j\} \subseteq X \mid i \neq j\}$$

be the set of 2-element subsets of X. Decompose $\mathbf{C}[Z]$ into irreducibles, and determine the dimension of each irreducible constituent.

B3/5 **Representation Theory**

Let $G = SU_2$, and V_n be the vector space of homogeneous polynomials of degree n in the variables x and y.

(i) Define the action of G on V_n, and prove that V_n is an irreducible representation of G.

(ii) Decompose $V_4 \otimes V_3$ into irreducible representations of SU_2. Briefly justify your answer.

(iii) SU_2 acts on the vector space $M_3(\mathbf{C})$ of complex 3×3 matrices via

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} X \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}, \qquad X \in M_3(\mathbf{C}).$$

Decompose this representation into irreducible representations.

B4/2 **Representation Theory**

Let G be the Heisenberg group of order p^3. This is the subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, x \in \mathbf{F}_p \right\}$$

of 3×3 matrices over the finite field $\mathbf{F}_p$ (p prime). Let H be the subgroup of G of such matrices with $a = 0$.

(i) Find all one dimensional representations of G.

[*You may assume without proof that* $[G, G]$ *is equal to the set of matrices in* G *with* $a = b = 0$.]

(ii) Let $\psi : \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{C}^*$ be a non-trivial one dimensional representation of $\mathbf{F}_p$, and define a one dimensional representation ρ of H by

$$\rho \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \psi(x).$$

Show that $V_\psi = \mathrm{Ind}_H^G(\rho)$ is irreducible.

(iii) List all the irreducible representations of G and explain why your list is complete.

B1/7 **Galois Theory**

Prove that the Galois group G of the polynomial $X^6 + 3$ over $\mathbf{Q}$ is of order 6. By explicitly describing the elements of G, show that they have orders $1, 2$ or 3. Hence deduce that G is isomorphic to S_3.

Why does it follow that $X^6 + 3$ is reducible over the finite field $\mathbf{F}_p$, for all primes p?

B3/6 **Galois Theory**

Let $\mathbf{F}_p$ be the finite field with p elements (p a prime), and let k be a finite extension of $\mathbf{F}_p$. Define the Frobenius automorphism $\sigma : k \longrightarrow k$, verifying that it is an $\mathbf{F}_p$-automorphism of k.

Suppose $f = X^{p+1} + X^p + 1 \in \mathbf{F}_p[X]$ and that K is its splitting field over $\mathbf{F}_p$. Why are the zeros of f distinct? If α is any zero of f in K, show that $\sigma(\alpha) = -\frac{1}{\alpha+1}$. Prove that f has at most two zeros in $\mathbf{F}_p$ and that $\sigma^3 = id$. Deduce that the Galois group of f over $\mathbf{F}_p$ is a cyclic group of order three.

B4/3 **Galois Theory**

Define the concept of separability and normality for algebraic field extensions. Suppose $K = k(\alpha)$ is a simple algebraic extension of k, and that $\mathrm{Aut}(K/k)$ denotes the group of k-automorphisms of K. Prove that
$|\mathrm{Aut}(K/k)| \leqslant [K : k]$, with equality if and only if K/k is normal and separable.

[*You may assume that the splitting field of a separable polynomial $f \in k[X]$ is normal and separable over k.*]

Suppose now that G is a finite group of automorphisms of a field F, and $E = F^G$ is the fixed subfield. Prove:

(i) F/E is separable.

(ii) $G = \mathrm{Aut}(F/E)$ and $[F : E] = |G|$.

(iii) F/E is normal.

[*The Primitive Element Theorem for finite separable extensions may be used without proof.*]

B1/8 **Differentiable Manifolds**

Define an immersion and an embedding of one manifold in another. State a necessary and sufficient condition for an immersion to be an embedding and prove its necessity.

Assuming the existence of "bump functions" on Euclidean spaces, state and prove a version of Whitney's embedding theorem.

Deduce that $\mathbb{RP}^n$ embeds in $\mathbb{R}^{(n+1)^2}$.

B2/7 **Differentiable Manifolds**

State Stokes' Theorem.

Prove that, if M^m is a compact connected manifold and $\Phi : U \to \mathbb{R}^m$ is a surjective chart on M, then for any $\omega \in \Omega^m(M)$ there is $\eta \in \Omega^{m-1}(M)$ such that $\operatorname{supp}(\omega + d\eta) \subseteq \Phi^{-1}(\mathbf{B}^m)$, where $\mathbf{B}^m$ is the unit ball in $\mathbb{R}^m$.

[*You may assume that, if* $\omega \in \Omega^m(\mathbb{R}^m)$ *with* $\operatorname{supp}(\omega) \subseteq \mathbf{B}^m$ *and* $\int\limits_{\mathbb{R}^m} \omega = 0$*, then* $\exists\, \eta \in \Omega^{m-1}(\mathbb{R}^m)$ *with* $\operatorname{supp}(\eta) \subseteq \mathbf{B}^m$ *such that* $d\eta = \omega$*.*]

By considering the m-form

$$\omega = x_1 dx_2 \wedge ... \wedge dx_{m+1} + \cdots + x_{m+1} dx_1 \wedge ... \wedge dx_m$$

on $\mathbb{R}^{m+1}$, or otherwise, deduce that $H^m(S^m) \cong \mathbb{R}$.

B4/4 **Differentiable Manifolds**

Describe the Mayer-Vietoris exact sequence for forms on a manifold M and show how to derive from it the Mayer-Vietoris exact sequence for the de Rham cohomology.

Calculate $H^*(\mathbb{RP}^n)$.

B2/8 **Algebraic Topology**

Show that the fundamental group of the 2-torus $S^1 \times S^1$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

Show that an injective continuous map from the circle S^1 to itself induces multiplication by ± 1 on the fundamental group.

Show that there is no retraction from the solid torus $S^1 \times D^2$ to its boundary.

B3/7 **Algebraic Topology**

Write down the Mayer-Vietoris sequence and describe all the maps involved.

Use the Mayer-Vietoris sequence to compute the homology of the n-sphere S^n for all n.

B4/5 **Algebraic Topology**

Write an essay on the definition of simplicial homology groups. The essay should include a discussion of orientations, of the action of a simplicial map and a proof of $\partial^2 = 0$.

B1/9 **Number Fields**

Let $K = \mathbf{Q}(\alpha)$ be a number field, where $\alpha \in \mathcal{O}_K$. Let f be the (normalized) minimal polynomial of α over $\mathbf{Q}$. Show that the discriminant $\operatorname{disc}(f)$ of f is equal to $(\mathcal{O}_K : \mathbf{Z}[\alpha])^2 D_K$.

Show that $f(x) = x^3 + 5x^2 - 19$ is irreducible over $\mathbf{Q}$. Determine $\operatorname{disc}(f)$ and the ring of algebraic integers $\mathcal{O}_K$ of $K = \mathbf{Q}(\alpha)$, where $\alpha \in \mathbf{C}$ is a root of f.

B2/9 **Number Fields**

Determine the ideal class group of $\mathbf{Q}(\sqrt{-11})$.

Find all solutions of the diophantine equation

$$y^2 + 11 = x^3 \qquad (x, y \in \mathbf{Z}) \; .$$

[*Minkowski's bound is* $n!n^{-n}(4/\pi)^{r_2}|D_k|^{1/2}$.]

B4/6 **Number Fields**

For a prime number $p > 2$, set $\zeta = e^{2\pi i/p}, K = \mathbf{Q}(\zeta)$ and $K^+ = \mathbf{Q}(\zeta + \zeta^{-1})$.

(a) Show that the (normalized) minimal polynomial of $\zeta - 1$ over $\mathbf{Q}$ is equal to

$$f(x) = \frac{(x+1)^p - 1}{x}.$$

(b) Determine the degrees $[K : \mathbf{Q}]$ and $[K^+ : \mathbf{Q}]$.

(c) Show that

$$\prod_{j=1}^{p-1} (1 - \zeta^j) = p.$$

(d) Show that $\operatorname{disc}(f) = (-1)^{\frac{p-1}{2}} p^{p-2}$.

(e) Show that K contains $\mathbf{Q}(\sqrt{p^*})$, where $p^* = (-1)^{\frac{p-1}{2}} p$.

(f) If $j, k \in \mathbf{Z}$ are not divisible by p, show that $\frac{1-\zeta^j}{1-\zeta^k}$ lies in $\mathcal{O}_K^*$.

(g) Show that the ideal $(p) = p\mathcal{O}_K$ is equal to $(1 - \zeta)^{p-1}$.

B1/10 **Hilbert Spaces**

State and prove the Riesz representation theorem for bounded linear functionals on a Hilbert space H.

[*You may assume, without proof, that* $H = E \oplus E^{\perp}$, *for every closed subspace* E *of* H.]

Prove that, for every $T \in \mathcal{B}(H)$, there is a unique $T^* \in \mathcal{B}(H)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for every $x, y \in H$. Prove that $\|T^*T\| = \|T\|^2$ for every $T \in \mathcal{B}(H)$.

Define a *normal* operator $T \in \mathcal{B}(H)$. Prove that T is normal if and only if $\|Tx\| = \|T^*x\|$ for every $x \in H$. Deduce that every point in the spectrum of a normal operator T is an approximate eigenvalue of T.

[*You may assume, without proof, any general criterion for the invertibility of a bounded linear operator on* H.]

B3/8 **Hilbert Spaces**

Let T be a bounded linear operator on a Hilbert space H. Define what it means to say that T is (i) *compact*, and (ii) *Fredholm*. What is the *index*, $\mathrm{ind}(T)$, of a Fredholm operator T?

Let S, T be bounded linear operators on H. Prove that S and T are Fredholm if and only if both ST and TS are Fredholm. Prove also that if S is invertible and T is Fredholm then $\mathrm{ind}(ST) = \mathrm{ind}(TS) = \mathrm{ind}(T)$.

Let K be a compact linear operator on H. Prove that $I + K$ is Fredholm with index zero.

B4/7 **Hilbert Spaces**

Write an essay on the use of Hermite functions in the elementary theory of the Fourier transform on $L^2(\mathbb{R})$.

[*You should assume, without proof, any results that you need concerning the approximation of functions by Hermite functions.*]

B1/11 **Riemann Surfaces**

Recall that an *automorphism* of a Riemann surface is a bijective analytic map onto itself, and that the inverse map is then guaranteed to be analytic.

Let Δ denote the disc $\{z \in \mathbb{C} \big| |z| < 1\}$, and let $\Delta^* = \Delta - \{0\}$.

(a) Prove that an automorphism $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ is a Euclidian rotation.

[*Hint: Apply the maximum modulus principle to the functions* $\phi(z)/z$ *and* $\phi^{-1}(z)/z$.]

(b) Prove that a holomorphic map $\phi : \Delta^* \to \Delta$ extends to the entire disc, and use this to conclude that any automorphism of Δ^* is a Euclidean rotation.

[*You may use the result stated in part (a).*]

(c) Define an analytic map between Riemann surfaces. Show that a continuous map between Riemann surfaces, known to be analytic everywhere except perhaps at a single point P, is, in fact, analytic everywhere.

B3/9 **Riemann Surfaces**

Let $f : X \to Y$ be a nonconstant holomorphic map between compact connected Riemann surfaces. Define the *valency* of f at a point, and the *degree* of f.

Define the *genus* of a compact connected Riemann surface X (assuming the existence of a triangulation).

State the Riemann-Hurwitz theorem. Show that a holomorphic non-constant self-map of a compact Riemann surface of genus $g > 1$ is bijective, with holomorphic inverse. Verify that the Riemann surface in $\mathbb{C}^2$ described in the equation $w^4 = z^4 - 1$ is non-singular, and describe its topological type.

[*Note: The description can be in the form of a picture or in words. If you apply Riemann-Hurwitz, explain first how you compactify the surface.*]

B4/8 **Riemann Surfaces**

Let λ and μ be fixed, non-zero complex numbers, with $\lambda/\mu \notin \mathbb{R}$, and let $\Lambda = \mathbb{Z}\mu + \mathbb{Z}\lambda$ be the lattice they generate in $\mathbb{C}$. The series

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}\Big[\frac{1}{(z - m\lambda - n\mu)^2} - \frac{1}{(m\lambda + n\mu)^2}\Big],$$

with the sum taken over all pairs $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ other than (0,0), is known to converge to an *elliptic function*, meaning a meromorphic function $\wp : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ satisfying $\wp(z) = \wp(z+\mu) = \wp(z+\lambda)$ for all $z \in \mathbb{C}$. ($\wp$ is called the *Weierstrass function.*)

(a) Find the three zeros of $\wp'$ modulo Λ, explaining why there are no others.

(b) Show that, for any number $a \in \mathbb{C}$, other than the three values $\wp(\lambda/2), \wp(\mu/2)$ and $\wp((\lambda+\mu)/2)$, the equation $\wp(z) = a$ has exactly two solutions, modulo Λ; whereas, for each of the specified values, it has a single solution.

[*In (a) and (b), you may use, without proof, any known results about valencies and degrees of holomorphic maps between compact Riemann surfaces, provided you state them correctly.*]

(c) Prove that every *even elliptic function* $\phi(z)$ is a rational function of $\wp(z)$; that is, there exists a rational function R for which $\phi(z) = R(\wp(z))$.

B2/10 **Algebraic Curves**

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map given by $f(X_0 : X_1 : X_2) = (X_1X_2 : X_0X_2 : X_0X_1)$. Determine whether f is defined at the following points: $(1 : 1 : 1), (0 : 1 : 1), (0 : 0 : 1)$.

Let $C \subset \mathbb{P}^2$ be the curve defined by $X_1^2X_2 - X_0^3 = 0$. Define a bijective morphism $\alpha : \mathbb{P}^1 \to C$. Prove that α is not an isomorphism.

B3/10 **Algebraic Curves**

Let C be the projective curve (over an algebraically closed field k of characteristic zero) defined by the affine equation

$$x^5 + y^5 = 1 \ .$$

Determine the points at infinity of C and show that C is smooth.

Determine the divisors of the rational functions $x, y \in k(C)$.

Show that $\omega = dx/y^4$ is a regular differential on C.

Compute the divisor of ω. What is the genus of C?

B4/9 **Algebraic Curves**

Write an essay on curves of genus one (over an algebraically closed field k of characteristic zero). Legendre's normal form should not be discussed.

B1/13 **Probability and Measure**

State and prove Hölder's Inequality.

[*Jensen's inequality, and other standard results, may be assumed.*]

Let (X_n) be a sequence of random variables bounded in L_p for some $p > 1$. Prove that (X_n) is uniformly integrable.

Suppose that $X \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $p \in (1, \infty)$. Show that $X \in L_r(\Omega, \mathcal{F}, \mathbb{P})$ for all $1 \leqslant r < p$ and that $||X||_r$ is an increasing function of r on $[1, p]$.

Show further that $\lim\limits_{r \to 1^+} ||X||_r = ||X||_1$.

B2/12 **Probability and Measure**

(a) Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}\left((0,1)\right)$ be the Borel σ-field and let $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. What is the distribution of the random variable Z, where $Z(\omega) = 2\omega - 1$?

Let $\omega = \sum_{n=1}^{\infty} 2^{-n} R_n(\omega)$ be the binary expansion of the point $\omega \in \Omega$ and set $U(\omega) = \sum\limits_{n\ odd} 2^{-n} Q_n(\omega)$, where $Q_n(\omega) = 2R_n(\omega) - 1$. Find a random variable V independent of U such that U and V are identically distributed and $U + \frac{1}{2}V$ is uniformly distributed on $(-1, 1)$.

(b) Now suppose that on some probability triple X and Y are independent, identically-distributed random variables such that $X + \frac{1}{2}Y$ is uniformly distributed on $(-1, 1)$.

Let ϕ be the characteristic function of X. Calculate $\phi(t)/\phi(t/4)$. Show that the distribution of X must be the same as the distribution of the random variable U in (a).

B3/12 **Probability and Measure**

State and prove Birkhoff's almost-everywhere ergodic theorem.

[*You need not prove convergence in L_p and the maximal ergodic lemma may be assumed provided that it is clearly stated.*]

Let $\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}([0,1))$ be the Borel σ-field and let $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. Give an example of an ergodic measure-preserving map $\theta : \Omega \to \Omega$ (you need not prove it is ergodic).

Let $f(x) = x$ for $x \in [0,1)$. Find (at least for all x outside a set of measure zero)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (f \circ \theta^{i-1})(x).$$

Briefly justify your answer.

B4/11 **Probability and Measure**

State the first and second Borel-Cantelli Lemmas and the Kolmogorov 0-1 law.

Let $(X_n)_{n\geqslant 1}$ be a sequence of independent random variables with distribution given by

$$\mathbb{P}(X_n = n) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0),$$

and set $S_n = \sum_{i=1}^{n} X_i$.

(a) Show that there exist constants $0 \leqslant c_1 \leqslant c_2 \leqslant \infty$ such that $\liminf_n (S_n/n) = c_1$, almost surely and $\limsup_n (S_n/n) = c_2$ almost surely.

(b) Let $Y_k = \sum_{i=k+1}^{2k} X_i$ and $\tilde{Y}_k = \sum_{i=1}^{k} Z_i^{(k)}$, where $(Z_i^{(k)})_{i=1}^{k}$ are independent with

$$\mathbb{P}(Z_i^{(k)} = k) = \frac{1}{2k} = 1 - \mathbb{P}(Z_i^{(k)} = 0), \quad 1 \leqslant i \leqslant k,$$

and suppose that $\alpha \in \mathbb{Z}^+$.

Use the fact that $\mathbb{P}(Y_k \geqslant \alpha k) \geqslant \mathbb{P}(\tilde{Y}_k \geqslant \alpha k)$ to show that there exists $p_\alpha > 0$ such that $\mathbb{P}(Y_k \geqslant \alpha k) \geqslant p_\alpha$ for all sufficiently large k.

[*You may use the Poisson approximation to the binomial distribution without proof.*]

By considering a suitable subsequence of (Y_k), or otherwise, show that $c_2 = \infty$.

(c) Show that $c_1 \leqslant 1$. Consider an appropriately chosen sequence of random times T_i, with $2T_i \leqslant T_{i+1}$, for which $(S_{T_i}/T_i) \leqslant 3c_1/2$. Using the fact that the random variables (Y_{T_i}) are independent, and by considering the events $\{Y_{T_i} = 0\}$, or otherwise, show that $c_1 = 0$.

B2/13 **Applied Probability**

Let M be a Poisson random measure on $E = \mathbb{R} \times [0, \pi)$ with constant intensity λ. For $(x, \theta) \in E$, denote by $l(x, \theta)$ the line in $\mathbb{R}^2$ obtained by rotating the line $\{(x, y) : y \in \mathbb{R}\}$ through an angle θ about the origin.

Consider the line process $L = M \circ\ l^{-1}$.

(i) What is the distribution of the number of lines intersecting the disc $\{z \in \mathbb{R}^2 : |z| \leqslant a\}$?

(ii) What is the distribution of the distance from the origin to the nearest line?

(iii) What is the distribution of the distance from the origin to the kth nearest line?

B3/13 **Applied Probability**

Consider an $M/G/1$ queue with arrival rate λ and traffic intensity less than 1. Prove that the moment-generating function of a typical busy period, $M_B(\theta)$, satisfies

$$M_B(\theta) = M_S(\theta - \lambda + \lambda\ M_B(\theta)),$$

where $M_S(\theta)$ is the moment-generating function of a typical service time.

If service times are exponentially distributed with parameter $\mu > \lambda$, show that

$$M_B(\theta) = \frac{\lambda + \mu - \theta - \{(\lambda + \mu - \theta)^2 - 4\lambda\mu\ \}^{1/2}}{2\lambda}$$

for all sufficiently small but positive values of θ.

B4/12 **Applied Probability**

Define a *renewal process* and a *renewal reward process.*

State and prove the strong law of large numbers for these processes.

[*You may assume the strong law of large numbers for independent, identically-distributed random variables.*]

State and prove Little's formula.

Customers arrive according to a Poisson process with rate ν at a single server, but a restricted waiting room causes those who arrive when n customers are already present to be lost. Accepted customers have service times which are independent and identically-distributed with mean α and independent of the arrival process. Let P_j be the equilibrium probability that an arriving customer finds j customers already present.

Using Little's formula, or otherwise, determine a relationship between P_0, P_n, ν and α.

B1/14 **Information Theory**

Let $p_1, \ldots, p_n$ be a probability distribution, with $p^* = \max_i [p_i]$. Prove that

$$(i) -\sum_i p_i \log p_i \geqslant -p^* \log p^* - (1-p^*) \log(1-p^*);$$

$$(ii) -\sum_i p_i \log p_i \geqslant \log(1/p^*); \text{ and}$$

$$(iii) -\sum_i p_i \log p_i \geqslant 2(1-p^*).$$

All logarithms are to base 2.

[*Hint: To prove (iii), it is convenient to use (i) for* $p^* \geqslant \frac{1}{2}$ *and (ii) for* $p^* \leqslant \frac{1}{2}$.]

Random variables X and Y with values x and y from finite 'alphabets' I and J represent the input and output of a transmission channel, with the conditional probability $p(x \mid y) = \mathbb{P}(X = x \mid Y = y)$. Let $h(p(\cdot \mid y))$ denote the entropy of the conditional distribution $p(\cdot \mid y)$, $y \in J$, and $h(X \mid Y)$ denote the conditional entropy of X given Y. Define the ideal observer decoding rule as a map $f : J \to I$ such that $p(f(y) \mid y) = \max_{x \in I} p(x \mid y)$ for all $y \in J$. Show that under this rule the error probability

$$\pi_{\text{er}}(y) = \sum_{\substack{x \in I \\ x \neq f(y)}} p(x \mid y)$$

satisfies $\pi_{\text{er}}(y) \leqslant \frac{1}{2} h(p(\cdot \mid y))$, and the expected value satisfies

$$\mathbb{E}\pi_{\text{er}}(Y) \leqslant \tfrac{1}{2} h(X \mid Y).$$

B2/14 **Information Theory**

A subset $\mathcal{C}$ of the Hamming space $\{0,1\}^n$ of cardinality $|\mathcal{C}| = r$ and with the minimal (Hamming) distance $\min\left[d(x,x') : x,x' \in \mathcal{C}, x \neq x'\right] = \delta$ is called an $[n,r,\delta]$-code (not necessarily linear). An $[n,r,\delta]$-code is called *maximal* if it is not contained in any $[n,r+1,\delta]$-code. Prove that an $[n,r,\delta]$-code is maximal if and only if for any $y \in \{0,1\}^n$ there exists $x \in \mathcal{C}$ such that $d(x,y) < \delta$. Conclude that if there are δ or more changes made in a codeword then the new word is closer to some other codeword than to the original one.

Suppose that a maximal $[n,r,\delta]$-code is used for transmitting information via a binary memoryless channel with the error probability p, and the receiver uses the maximum likelihood decoder. Prove that the probability of erroneous decoding, $\pi_{\rm err}^{\rm ml}$, obeys the bounds

$$1 - b(n,\delta-1) \leqslant \pi_{\rm err}^{\rm ml} \leqslant 1 - b(n,[(\delta-1)/2]),$$

where

$$b(n,m) = \sum_{0 \leqslant k \leqslant m} \binom{n}{k} p^k (1-p)^{n-k}$$

is a partial binomial sum and $[\,\cdot\,]$ is the integer part.

B4/13 **Information Theory**

State the Kraft inequality. Prove that it gives a necessary and sufficient condition for the existence of a prefix-free code with given codeword lengths.

B2/15 **Optimization and Control**

A street trader wishes to dispose of k counterfeit Swiss watches. If he offers one for sale at price u he will sell it with probability ae^{-u}. Here a is known and less than 1. Subsequent to each attempted sale (successful or not) there is a probability $1-\beta$ that he will be arrested and can make no more sales. His aim is to choose the prices at which he offers the watches so as to maximize the expected values of his sales up until the time he is arrested or has sold all k watches.

Let $V(k)$ be the maximum expected amount he can obtain when he has k watches remaining and has not yet been arrested. Explain why $V(k)$ is the solution to

$$V(k) = \max_{u>0} \left\{ ae^{-u}[u + \beta V(k-1)] + (1 - ae^{-u})\beta V(k) \right\} .$$

Denote the optimal price by u_k and show that

$$u_k = 1 + \beta V(k) - \beta V(k-1)$$

and that

$$V(k) = ae^{-u_k}/(1-\beta) .$$

Show inductively that $V(k)$ is a nondecreasing and concave function of k.

B3/14 **Optimization and Control**

A file of X Mb is to be transmitted over a communications link. At each time t the sender can choose a transmission rate, $u(t)$, within the range $[0,1]$ Mb per second. The charge for transmitting at rate $u(t)$ at time t is $u(t)p(t)$. The function p is fully known at time 0. If it takes a total time T to transmit the file then there is a delay cost of γT^2, $\gamma > 0$. Thus u and T are to be chosen to minimize

$$\int_0^T u(t)p(t)dt + \gamma T^2 ,$$

where $u(t) \in [0,1]$, $dx(t)/dt = -u(t)$, $x(0) = X$ and $x(T) = 0$. Quoting and applying appropriate results of Pontryagin's maximum principle show that a property of the optimal policy is that there exists p^* such that $u(t) = 1$ if $p(t) < p^*$ and $u(t) = 0$ if $p(t) > p^*$.

Show that the optimal p^* and T are related by $p^* = p(T) + 2\gamma T$.

Suppose $p(t) = t + 1/t$ and $X = 1$. For what value of γ is it optimal to transmit at a constant rate 1 between times 1/2 and 3/2?

B4/14 **Optimization and Control**

Consider the scalar system with plant equation $x_{t+1} = x_t + u_t$, $t = 0, 1, \ldots$ and cost

$$C_s(x_0, u_0, u_1, \ldots) = \sum_{t=0}^{s} \left[u_t^2 + \frac{4}{3} x_t^2 \right] .$$

Show from first principles that $\min_{u_0, u_1, \ldots} C_s = V_s x_0^2$, where $V_0 = 4/3$ and for $s = 0, 1, \ldots$,

$$V_{s+1} = 4/3 + V_s/(1 + V_s) .$$

Show that $V_s \to 2$ as $s \to \infty$.

Prove that C_∞ is minimized by the stationary control, $u_t = -2x_t/3$ for all t.

Consider the stationary policy π_0 that has $u_t = -x_t$ for all t. What is the value of C_∞ under this policy?

Consider the following algorithm, in which steps 1 and 2 are repeated as many times as desired.

1. For a given stationary policy π_n, for which $u_t = k_n x_t$ for all t, determine the value of C_∞ under this policy as $V^{\pi_n} x_0^2$ by solving for V^{π_n} in

$$V^{\pi_n} = k_n^2 + 4/3 + (1 + k_n)^2 V^{\pi_n} .$$

2. Now find k_{n+1} as the minimizer of

$$k_{n+1}^2 + 4/3 + (1 + k_{n+1})^2 V^{\pi_n}$$

   and define π_{n+1} as the policy for which $u_t = k_{n+1} x_t$ for all t.

Explain why π_{n+1} is guaranteed to be a better policy than π_n.

Let π_0 be the stationary policy with $u_t = -x_t$. Determine π_1 and verify that it minimizes C_∞ to within 0.2% of its optimum.

B1/17 **Dynamical Systems**

Define topological conjugacy and C^1-conjugacy.

Let a, b be real numbers with $a > b > 0$ and let F_a, F_b be the maps of $(0, \infty)$ to itself given by $F_a(x) = ax, F_b(x) = bx$. For which pairs a, b are F_a and F_b topologically conjugate? Would the answer be the same for C^1-conjugacy? Justify your statements.

B3/17 **Dynamical Systems**

If $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ show that $A^{n+2} = A^{n+1} + A^n$ for all $n \geqslant 0$. Show that A^5 has trace 11 and deduce that the subshift map defined by A has just two cycles of exact period 5. What are they?

B4/17 **Dynamical Systems**

Define the rotation number $\rho(f)$ of an orientation-preserving circle map f and the rotation number $\rho(F)$ of a lift F of f. Prove that $\rho(f)$ and $\rho(F)$ are well-defined. Prove also that $\rho(F)$ is a continuous function of F.

State without proof the main consequence of $\rho(f)$ being rational.

B1/18 **Partial Differential Equations**

(a) Solve the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2$$

together with the boundary condition on the x-axis:

$$u(x,0) = f(x) \; ,$$

where f is a smooth function. You should discuss the domain on which the solution is smooth. For which functions f can the solution be extended to give a smooth solution on the upper half plane $\{y > 0\}$?

(b) Solve the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

together with the boundary condition on the unit circle:

$$u(x,y) = x \qquad \text{when} \qquad x^2 + y^2 = 1.$$

B2/17 **Partial Differential Equations**

Define the Schwartz space $\mathcal{S}(\mathbb{R})$ and the corresponding space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

Use the Fourier transform to give an integral formula for the solution of the equation

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} + u = f \tag{$*$}$$

for $f \in \mathcal{S}(\mathbb{R})$. Prove that your solution lies in $\mathcal{S}(\mathbb{R})$. Is your formula the unique solution to $(*)$ in the Schwartz space?

Deduce from this formula an integral expression for the fundamental solution of the operator $P = -\frac{d^2}{dx^2} + \frac{d}{dx} + 1$.

Let K be the function:

$$K(x) = \begin{cases} \frac{1}{\sqrt{5}} e^{-(\sqrt{5}-1)x/2} & \text{for } x \geqslant 0, \\ \frac{1}{\sqrt{5}} e^{(\sqrt{5}+1)x/2} & \text{for } x \leqslant 0. \end{cases}$$

Using the definition of distributional derivatives verify that this function is a fundamental solution for P.

B3/18 **Partial Differential Equations**

Write down a formula for the solution $u = u(t, x)$, for $t > 0$ and $x \in \mathbb{R}^n$, of the initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad u(0, x) = f(x),$$

for f a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. State (without proof) a theorem which ensures that this formula is the unique solution in some class of functions (which should be explicitly described).

By writing $u = e^t v$, or otherwise, solve the initial value problem

$$\frac{\partial v}{\partial t} + v - \Delta v = 0, \qquad v(0, x) = g(x), \tag{$\dagger$}$$

for g a bounded continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and give a class of functions in which your solution is the unique one.

Hence, or otherwise, prove that for all $t > 0$:

$$\sup_{x \in \mathbb{R}^n} v(t, x) \leqslant \sup_{x \in \mathbb{R}^n} g(x)$$

and deduce that the solutions $v_1(t, x)$ and $v_2(t, x)$ of ($\dagger$) corresponding to initial values $g_1(x)$ and $g_2(x)$ satisfy, for $t > 0$,

$$\sup_{x \in \mathbb{R}^n} |v_1(t, x) - v_2(t, x)| \leqslant \sup_{x \in \mathbb{R}^n} |g_1(x) - g_2(x)|.$$

B4/18 **Partial Differential Equations**

Write an essay on **one** of the following two topics:

(a) The notion of *well-posedness* for initial and boundary value problems for differential equations. Your answer should include a definition and give examples and state precise theorems for some specific problems.

(b) The concepts of *distribution* and *tempered distribution* and their use in the study of partial differential equations.

B1/19 Methods of Mathematical Physics

State and prove the convolution theorem for Laplace transforms.

Use the convolution theorem to prove that the Beta function

$$B(p,q) = \int_0^1 (1-\tau)^{p-1}\tau^{q-1}d\tau$$

may be written in terms of the Gamma function as

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \ .$$

B2/18 Methods of Mathematical Physics

The Bessel function $J_\nu(z)$ is defined, for $|\arg z| < \pi/2$, by

$$J_\nu(z) = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} \mathrm{e}^{(t-t^{-1})z/2} t^{-\nu-1} dt \ ,$$

where the path of integration is the Hankel contour and $t^{-\nu-1}$ is the principal branch.

Use the method of steepest descent to show that, as $z \to +\infty$,

$$J_\nu(z) \sim (2/\pi z)^{\frac{1}{2}} \cos(z - \pi\nu/2 - \pi/4) \ .$$

You should give a rough sketch of the steepest descent paths.

B3/19 Methods of Mathematical Physics

Consider the integral

$$\int_0^\infty \frac{t^z \mathrm{e}^{-at}}{1+t}\, dt \ ,$$

where t^z is the principal branch and a is a positive constant. State the region of the complex z–plane in which the integral defines a holomorphic function.

Show how the analytic continuation of this function can be obtained by means of an alternative integral representation using the Hankel contour.

Hence show that the analytic continuation is holomorphic except for simple poles at $z = -1, -2, \ldots$, and that the residue at $z = -n$ is

$$(-1)^{n-1} \sum_{r=0}^{n-1} \frac{a^r}{r!} \ .$$

B4/19 **Methods of Mathematical Physics**

Show that $\int_0^\pi e^{ix\cos t}\,dt$ satisfies the differential equation

$$xy'' + y' + xy = 0,$$

and find a second solution, in the form of an integral, for $x > 0$.

Show, by finding the asymptotic behaviour as $x \to +\infty$, that your two solutions are linearly independent.

B1/21 **Electrodynamics**

Explain the multipole expansion in electrostatics, and devise formulae for the total charge, dipole moments and quadrupole moments given by a static charge distribution $\rho(\mathbf{r})$.

A nucleus is modelled as a uniform distribution of charge inside the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1.$$

The total charge of the nucleus is Q. What are the dipole moments and quadrupole moments of this distribution?

Describe qualitatively what happens if the nucleus starts to oscillate.

B2/20 **Electrodynamics**

In a superconductor, there are superconducting charge carriers with number density n, mass m and charge q. Starting from the quantum mechanical wavefunction $\Psi = Re^{i\Phi}$ (with real R and Φ), construct a formula for the electric current and explain carefully why your result is gauge invariant.

Now show that inside a superconductor a static magnetic field obeys the equation

$$\nabla^2\,\mathbf{B}=\frac{\mu_0\,n\,q^2}{m}\mathbf{B}.$$

A superconductor occupies the region $z>0$, while for $z<0$ there is a vacuum with a constant magnetic field in the x direction. Show that the magnetic field cannot penetrate deep into the superconductor.

B4/21 **Electrodynamics**

The Liénard-Wiechert potential for a particle of charge q, assumed to be moving non-relativistically along the trajectory $y^\mu(\tau)$, τ being the proper time along the trajectory, is

$$A^\mu(x,t) \;=\; \frac{\mu_0 q}{4\pi}\;\frac{dy^\mu/d\tau}{(x-y(\tau))_\nu dy^\nu/d\tau}\Big|_{\tau=\tau_0}.$$

Explain how to calculate τ_0 given $x^\mu=(x,t)$ and $y^\mu=(y,t')$.

Derive Larmor's formula for the rate at which electromagnetic energy is radiated from a particle of charge q undergoing an acceleration a.

Suppose that one considers the classical non-relativistic hydrogen atom with an electron of mass m and charge $-e$ orbiting a fixed proton of charge $+e$, in a circular orbit of radius r_0. What is the total energy of the electron? As the electron is accelerated towards the proton it will radiate, thereby losing energy and causing the orbit to decay. Derive a formula for the lifetime of the orbit.

B1/22 **Statistical Physics**

Write down the first law of thermodynamics in differential form for an infinitesimal reversible change in terms of the increments dE, dS and dV, where E, S and V are to be defined. Briefly give an interpretation of each term and deduce that

$$P = -\left(\frac{\partial E}{\partial V}\right)_S , \qquad T = \left(\frac{\partial E}{\partial S}\right)_V .$$

Define the specific heat at constant volume C_V and show that for an adiabatic change

$$C_V\, dT + \left(\left(\frac{\partial E}{\partial V}\right)_T + P\right) dV = 0 .$$

Derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V ,$$

where T is temperature and hence show that

$$\left(\frac{\partial E}{\partial V}\right)_T = -P + T\left(\frac{\partial P}{\partial T}\right)_V .$$

An imperfect gas of volume V obeys the van der Waals equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT ,$$

where a and b are non-negative constants. Show that

$$\left(\frac{\partial C_V}{\partial V}\right)_T = 0 ,$$

and deduce that C_V is a function of T only. It can further be shown that in this case C_V is independent of T. Hence show that

$$T(V - b)^{R/C_V}$$

is constant on adiabatic curves.

B3/22 **Statistical Physics**

A system consists of N weakly interacting non-relativistic fermions, each of mass m, in a three-dimensional volume, V. Derive the Fermi-Dirac distribution

$$n(\epsilon) \;=\; KVg\frac{\epsilon^{1/2}}{\exp((\epsilon-\mu)/kT)+1}\ ,$$

where $n(\epsilon)d\epsilon$ is the number of particles with energy in $(\epsilon, \epsilon + d\epsilon)$ and $K = 2\pi(2m)^{3/2}/h^3$. Explain the physical significance of g.

Explain how the constant μ is determined by the number of particles N and write down expressions for N and the internal energy E in terms of $n(\epsilon)$.

Show that, in the limit $\kappa \equiv e^{-\mu/kT} \gg 1$,

$$N \;=\; \frac{V}{A\kappa}\left(1 - \frac{1}{2\sqrt{2}\kappa} + O\left(\frac{1}{\kappa^2}\right)\right)\ ,$$

where $A = h^3/g(2\pi mkT)^{3/2}$.

Show also that in this limit

$$E \;=\; \frac{3}{2}NkT\left(1 + \frac{1}{4\sqrt{2}\kappa} + O\left(\frac{1}{\kappa^2}\right)\right)\ .$$

Deduce that the condition $\kappa \gg 1$ implies that $AN/V \ll 1$. Discuss briefly whether or not this latter condition is satisfied in a white dwarf star and in a dilute electron gas at room temperature.

$\left[\ \textit{Note that } \int_0^\infty du\, e^{-u^2 a} = \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]$.

B4/23 **Statistical Physics**

Given that the free energy F can be written in terms of the partition function Z as $F = -kT \log Z$ show that the entropy S and internal energy E are given by

$$S = k \frac{\partial (T \log Z)}{\partial T} , \qquad E = kT^2 \frac{\partial \log Z}{\partial T} .$$

A system of particles has Hamiltonian $H(\mathbf{p}, \mathbf{q})$ where $\mathbf{p}$ is the set of particle momenta and $\mathbf{q}$ the set of particle coordinates. Write down the expression for the classical partition function Z_C for this system in equilibrium at temperature T. In a particular case H is given by

$$H(\mathbf{p}, \mathbf{q}) = p_\alpha A_{\alpha\beta}(\mathbf{q}) p_\beta + q_\alpha B_{\alpha\beta}(\mathbf{q}) q_\beta .$$

Let H be a homogeneous function in all the p_α, $1 \le \alpha \le N$, and in a subset of the q_α, $1 \le \alpha \le M$ $(M \le N)$. Derive the principle of equipartition for this system.

A system consists of N weakly interacting harmonic oscillators each with Hamiltonian

$$h(p, q) = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 .$$

Using equipartition calculate the classical specific heat of the system, $C_C(T)$. Also calculate the classical entropy $S_C(T)$.

Write down the expression for the quantum partition function of the system and derive expressions for the specific heat $C(T)$ and the entropy $S(T)$ in terms of N and the parameter $\theta = \hbar\omega/kT$. Show for $\theta \ll 1$ that

$$C(T) = C_C(T) + O(\theta) , \qquad S(T) = S_C(T) + S_0 + O(\theta) ,$$

where S_0 should be calculated. Comment briefly on the physical significance of the constant S_0 and why it is non-zero.

B1/23 Applications of Quantum Mechanics

A steady beam of particles, having wavenumber k and moving in the z direction, scatters on a spherically-symmetric potential. Write down the asymptotic form of the wave function at large r.

The incoming wave is written as a partial-wave series

$$\sum_{\ell=0}^{\infty} \chi_\ell(kr) P_\ell(\cos\theta).$$

Show that for large r

$$\chi_\ell(kr) \sim \frac{\ell+\frac{1}{2}}{ikr}\left(e^{ikr} - (-1)^\ell e^{-ikr}\right)$$

and calculate $\chi_0(kr)$ and $\chi_1(kr)$ for all r.

Write down the second-order differential equation satisfied by the $\chi_\ell(kr)$. Construct a second linearly-independent solution for each ℓ that is singular at $r=0$ and, when it is suitably normalised, has large-r behaviour

$$\frac{\ell+\frac{1}{2}}{ikr}\left(e^{ikr} + (-1)^\ell e^{-ikr}\right).$$

B2/22 Applications of Quantum Mechanics

A particle of charge e moves freely within a cubical box of side a. Its initial wavefunction is

$$(2/a)^{-\frac{3}{2}} \sin(\pi x/a)\sin(\pi y/a)\sin(\pi z/a).$$

A uniform electric field $\mathcal{E}$ in the x direction is switched on for a time T. Derive from first principles the probability, correct to order $\mathcal{E}^2$, that after the field has been switched off the wave function will be found to be

$$(2/a)^{-\frac{3}{2}} \sin(2\pi x/a)\sin(\pi y/a)\sin(\pi z/a).$$

B3/23 Applications of Quantum Mechanics

Write down the commutation relations satisfied by the cartesian components of the total angular momentum operator $\mathbf{J}$.

In quantum mechanics an operator $\mathbf{V}$ is said to be a vector operator if, under rotations, its components transform in the same way as those of the coordinate operator $\mathbf{r}$. Show from first principles that this implies that its cartesian components satisfy the commutation relations

$$[J_j, V_k] = i\epsilon_{jkl} V_l \ .$$

Hence calculate the total angular momentum of the nonvanishing states $V_j|0\rangle$, where $|0\rangle$ is the vacuum state.

B4/24 Applications of Quantum Mechanics

Derive the Bloch form of the wave function $\psi(x)$ of an electron moving in a one-dimensional crystal lattice.

The potential in such an N-atom lattice is modelled by

$$V(x) = \sum_n \Big(-\frac{\hbar^2 U}{2m} \delta(x - nL) \Big).$$

Assuming that $\psi(x)$ is continuous across each lattice site, and applying periodic boundary conditions, derive an equation for the allowed electron energy levels. Show that for suitable values of UL they have a band structure, and calculate the number of levels in each band when $UL > 2$. Verify that when $UL \gg 1$ the levels are very close to those corresponding to a solitary atom.

Describe briefly how the band structure in a real 3-dimensional crystal differs from that of this simple model.

B1/25 **Fluid Dynamics II**

The energy equation for the motion of a viscous, incompressible fluid states that

$$\frac{d}{dt}\int_{V(t)}\frac{1}{2}\rho u^2 dV=\int_{S(t)}u_i\sigma_{ij}n_j dS-2\mu\int_{V(t)}e_{ij}e_{ij}dV.$$

Interpret each term in this equation and explain the meaning of the symbols used.

For steady rectilinear flow in a (not necessarily circular) pipe having rigid stationary walls, deduce a relation between the viscous dissipation per unit length of the pipe, the pressure gradient G, and the volume flux Q.

Starting from the Navier-Stokes equations, calculate the velocity field for steady rectilinear flow in a circular pipe of radius a. Using the relationship derived above, or otherwise, find in terms of G the viscous dissipation per unit length for this flow.

[*In cylindrical polar coordinates,*

$$\nabla^2 w(r)=\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right).]$$

B2/24 **Fluid Dynamics II**

Explain what is meant by a Stokes flow and show that, in such a flow, in the absence of body forces, $\partial\sigma_{ij}/\partial x_j=0$, where σ_{ij} is the stress tensor.

State and prove the *reciprocal theorem* for Stokes flow.

When a rigid sphere of radius a translates with velocity $\mathbf{U}$ through unbounded fluid at rest at infinity, it may be shown that the traction per unit area, $\boldsymbol{\sigma}\cdot\mathbf{n}$, exerted by the sphere on the fluid, has the uniform value $3\mu\mathbf{U}/2a$ over the sphere surface. Find the drag on the sphere.

Suppose that the same sphere is free of external forces and is placed with its centre at the origin in an unbounded Stokes flow given in the absence of the sphere as $\mathbf{u}_s(\mathbf{x})$. By applying the reciprocal theorem to the perturbation to the flow generated by the presence of the sphere, and assuming this to tend to zero sufficiently rapidly at infinity, show that the instantaneous velocity of the centre of the sphere is

$$\mathbf{V}=\frac{1}{4\pi a^2}\int_{r=a}\mathbf{u}_s(\mathbf{x})dS.$$

B3/24 **Fluid Dynamics II**

A planar flow of an inviscid, incompressible fluid is everywhere in the x-direction and has velocity profile

$$u = \begin{cases} U & y > 0, \\ 0 & y < 0. \end{cases}$$

By examining linear perturbations to the vortex sheet at $y = 0$ that have the form $e^{ikx - i\omega t}$, show that

$$\omega = \frac{1}{2} kU(1 \pm i)$$

and deduce the temporal stability of the sheet to disturbances of wave number k.

Use this result to determine also the spatial growth rate and propagation speed of disturbances of frequency ω introduced at a fixed spatial position.

B4/26 **Fluid Dynamics II**

Starting from the steady planar vorticity equation

$$\mathbf{u} \,.\nabla\omega = \nu\nabla^2\omega,$$

outline briefly the derivation of the boundary layer equation

$$uu_x + vu_y = UdU/dx + \nu u_{yy},$$

explaining the significance of the symbols used.

Viscous fluid occupies the region $y > 0$ with rigid stationary walls along $y = 0$ for $x > 0$ and $x < 0$. There is a line sink at the origin of strength πQ, $Q > 0$, with $Q/\nu \gg 1$. Assuming that vorticity is confined to boundary layers along the rigid walls:

(a) Find the flow outside the boundary layers.

(b) Explain why the boundary layer thickness δ along the wall $x > 0$ is proportional to x, and deduce that

$$\delta = \left(\frac{\nu}{Q}\right)^{\frac{1}{2}} x \ .$$

(c) Show that the boundary layer equation admits a solution having stream function

$$\psi = (\nu Q)^{1/2} f(\eta) \quad \text{with} \quad \eta = y/\delta \ .$$

Find the equation and boundary conditions satisfied by f .

(d) Verify that a solution is

$$f' = \frac{6}{1 + \cosh(\eta\sqrt{2} + c)} - 1,$$

provided that c has one of two values to be determined. Should the positive or negative value be chosen?

B1/26 **Waves in Fluid and Solid Media**

Derive Riemann's equations for finite amplitude, one-dimensional sound waves in a perfect gas with ratio of specific heats γ.

At time $t = 0$ the gas is at rest and has uniform density ρ_0, pressure p_0 and sound speed c_0. A piston initially at $x = 0$ starts moving backwards at time $t = 0$ with displacement $x = -a \sin \omega t$, where a and ω are positive constants. Explain briefly how to find the resulting disturbance using a graphical construction in the xt-plane, and show that prior to any shock forming $c = c_0 + \frac{1}{2}(\gamma - 1)u$.

For small amplitude a, show that the excess pressure $\Delta p = p - p_0$ and the excess sound speed $\Delta c = c - c_0$ are related by

$$\frac{\Delta p}{p_0} = \frac{2\gamma}{\gamma - 1} \frac{\Delta c}{c_0} + \frac{\gamma(\gamma + 1)}{(\gamma - 1)^2} \left(\frac{\Delta c}{c_0} \right)^2 + O\left(\left(\frac{\Delta c}{c_0} \right)^3 \right).$$

Deduce that the time-averaged pressure on the face of the piston exceeds p_0 by

$$\frac{1}{8} \rho_0 a^2 \omega^2 (\gamma + 1) + O(a^3).$$

B2/25 **Waves in Fluid and Solid Media**

A semi-infinite elastic medium with shear modulus μ_1 and shear-wave speed c_1 lies in $y < 0$. Above it there is a layer $0 \leq y \leqslant h$ of a second elastic medium with shear modulus μ_2 and shear-wave speed c_2 $(< c_1)$. The top boundary $y = h$ is stress-free. Consider a monochromatic shear wave propagating at speed c with wavenumber k in the x-direction and with displacements only in the z-direction.

Obtain the dispersion relation

$$\tan kh\theta = \frac{\mu_1 c_2}{\mu_2 c_1} \frac{1}{\theta} \left(\frac{c_1^2}{c_2^2} - 1 - \theta^2 \right)^{1/2}, \qquad \text{where} \quad \theta = \sqrt{\frac{c^2}{c_2^2} - 1}.$$

Deduce that the modes have a cut-off frequency $\pi n c_1 c_2 / h \sqrt{c_1^2 - c_2^2}$ where they propagate at speed $c = c_1$.

B3/25 **Waves in Fluid and Solid Media**

Consider the equation

$$\phi_{tt} + \alpha^2 \phi_{xxxx} + \beta^2 \phi = 0, \qquad (*)$$

where α and β are real constants. Find the dispersion relation for waves of frequency ω and wavenumber k. Find the phase velocity $c(k)$ and the group velocity $c_g(k)$ and sketch graphs of these functions.

Multiplying equation $(*)$ by ϕ_t, obtain an equation of the form

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} = 0$$

where A and B are expressions involving ϕ and its derivatives. Give a physical interpretation of this equation.

Evaluate the time-averaged energy $\langle E \rangle$ and energy flux $\langle I \rangle$ of a monochromatic wave $\phi = \cos(kx - wt)$, and show that

$$\langle I \rangle = c_g \langle E \rangle .$$

B4/27 **Waves in Fluid and Solid Media**

Derive the ray-tracing equations governing the evolution of a wave packet $\phi(\mathbf{x}, t) = A(\mathbf{x}, t) \exp\{i\psi(\mathbf{x}, t)\}$ in a slowly varying medium, stating the conditions under which the equations are valid.

Consider now a stationary obstacle in a steadily moving homogeneous two-dimensional medium which has the dispersion relation

$$\omega(k_1, k_2) = \alpha \left(k_1^2 + k_2^2\right)^{1/4} - V k_1,$$

where $(V, 0)$ is the velocity of the medium. The obstacle generates a steady wave system. Writing $(k_1, k_2) = \kappa(\cos\phi, \sin\phi)$, show that the wave satisfies

$$\kappa = \frac{\alpha^2}{V^2 \cos^2 \phi}.$$

Show that the group velocity of these waves can be expressed as

$$\mathbf{c}_g = V\left(\tfrac{1}{2}\cos^2\phi - 1, \tfrac{1}{2}\cos\phi \sin\phi\right).$$

Deduce that the waves occupy a wedge of semi-angle $\sin^{-1}\frac{1}{3}$ about the negative x_1-axis.

MATHEMATICAL TRIPOS Part II 2002

List of Courses

A2/7 **Geometry of Surfaces**

(i)

Consider the surface

$$z = \frac{1}{2}(\lambda x^2 + \mu y^2) + h(x, y),$$

where $h(x, y)$ is a term of order at least 3 in x, y. Calculate the first fundamental form at $x = y = 0$.

(ii) Calculate the second fundamental form, at $x = y = 0$, of the surface given in Part (i). Calculate the Gaussian curvature. Explain why your answer is consistent with Gauss' "Theorema Egregium".

A3/7 **Geometry of Surfaces**

(i) State what it means for surfaces $f : U \to \mathbb{R}^3$ and $g : V \to \mathbb{R}^3$ to be isometric.

Let $f : U \to \mathbb{R}^3$ be a surface, $\phi : V \to U$ a diffeomorphism, and let $g = f \circ \phi : V \to \mathbb{R}^3$.

State a formula comparing the first fundamental forms of f and g.

(ii) Give a proof of the formula referred to at the end of part (i). Deduce that "isometry" is an equivalence relation.

The *catenoid* and the *helicoid* are the surfaces defined by

$$(u, v) \to (u \cos v,\ u \sin v,\ v)$$

and

$$(\vartheta, z) \to (\cosh z \cos \vartheta,\ \cosh z \sin \vartheta,\ z).$$

Show that the catenoid and the helicoid are isometric.

A4/7 **Geometry of Surfaces**

Write an essay on the Euler number of topological surfaces. Your essay should include a definition of subdivision, some examples of surfaces and their Euler numbers, and a discussion of the statement and significance of the Gauss–Bonnet theorem.

A1/8 **Graph Theory**

(i) State and prove a necessary and sufficient condition for a graph to be Eulerian (that is, to have an Eulerian circuit).

Prove that, given any connected non-Eulerian graph G, there is an Eulerian graph H and a vertex $v \in H$ such that $G = H - v$.

(ii) Let G be a connected plane graph with n vertices, e edges and f faces. Prove that $n - e + f = 2$. Deduce that $e \leq g(n-2)/(g-2)$, where g is the smallest face size.

The *crossing number* $c(G)$ of a non-planar graph G is the minimum number of edge-crossings needed when drawing the graph in the plane. (The crossing of three edges at the same point is not allowed.) Show that if G has n vertices and e edges then $c(G) \geq e - 3n + 6$. Find $c(K_6)$.

A2/8 **Graph Theory**

(i) Define the chromatic polynomial $p(G\,;t)$ of the graph G, and establish the standard identity

$$p(G\,;t) \;=\; p(G-e\,;t) \;-\; p(G/e\,;t),$$

where e is an edge of G. Deduce that, if G has n vertices and m edges, then

$$p(G\,;t) = a_n t^n - a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + (-1)^n a_0\,,$$

where $a_n = 1$, $a_{n-1} = m$ and $a_j \geq 0$ for $0 \leq j \leq n$.

(ii) Let G and $p(G\,;t)$ be as in Part (i). Show that if G has k components $G_1, \ldots, G_k$ then $p(G\,;t) \;=\; \prod_{i=1}^{k} p(G_i\,;t)$. Deduce that $a_k > 0$ and $a_j = 0$ for $0 \leq j < k$.

Show that if G is a tree then $p(G\,;t) = t(t-1)^{n-1}$. Must the converse hold? Justify your answer.

Show that if $p(G\,;t) = p(T_r(n)\,;t)$, where $T_r(n)$ is a Turán graph, then $G = T_r(n)$.

A4/9 **Graph Theory**

Write an essay on connectivity in graphs.

Your essay should include proofs of at least two major theorems, along with a discussion of one or two significant corollaries.

A1/9 **Number Theory**

(i) Let p be a prime number. Prove that the multiplicative group of the field with p elements is cyclic.

(ii) Let p be an odd prime, and let $k \geqslant 1$ be an integer. Prove that we have $x^2 \equiv 1 \bmod p^k$ if and only if either $x \equiv 1 \bmod p^k$ or $x \equiv -1 \bmod p^k$. Is this statement true when $p = 2$?

Let m be an odd positive integer, and let r be the number of distinct prime factors of m. Prove that there are precisely 2^r different integers x satisfying $x^2 \equiv 1 \bmod m$ and $0 < x < m$.

A3/9 **Number Theory**

(i) Let $\pi(x)$ denote the number of primes $\leqslant x$, where x is a positive real number. State and prove Legendre's formula relating $\pi(x)$ to $\pi(\sqrt{x})$. Use this formula to compute $\pi(100) - \pi(10)$.

(ii) Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, where s is a real number greater than 1. Prove the following two assertions rigorously, assuming always that $s > 1$.

(a) $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$, where the product is taken over all primes p;

(b) $\zeta(s) = \dfrac{1}{1 - 2^{1-s}} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$.

Explain why (b) enables us to define $\zeta(s)$ for $0 < s < 1$. Deduce from (b) that $\lim_{s \to 1} (s-1)\zeta(s) = 1$.

A4/10 **Number Theory**

Write an essay on quadratic reciprocity. Your essay should include (i) a proof of the law of quadratic reciprocity for the Legendre symbol, (ii) a proof of the law of quadratic reciprocity for the Jacobi symbol, and (iii) a comment on why this latter law is useful in primality testing.

A1/10 **Coding and Cryptography**

(i) Describe the original Hamming code of length 7. Show how to encode a message word, and how to decode a received word involving at most one error. Explain why the procedure works.

(ii) What is a linear binary code? What is its dual code? What is a cyclic binary code? Explain how cyclic binary codes of length n correspond to polynomials in $\mathbb{F}_2[X]$ dividing $X^n + 1$. Show that the dual of a cyclic code of length n is cyclic of length n.

Using the factorization

$$X^7 + 1 = (X + 1)(X^3 + X + 1)(X^3 + X^2 + 1)$$

in $\mathbb{F}_2[X]$, find all cyclic binary codes of length 7. Identify those which are Hamming codes and their duals. Justify your answer.

A2/9 **Coding and Cryptography**

(i) Explain the idea of public key cryptography. Give an example of a public key system, explaining how it works.

(ii) What is a general feedback register of length d with initial fill $(X_0, \ldots, X_{d-1})$? What is the maximal period of such a register, and why? What does it mean for such a register to be linear?

Describe and justify the Berlekamp-Massey algorithm for breaking a cypher stream arising from a general linear feedback register of unknown length.

Use the Berlekamp-Massey algorithm to find a linear recurrence in $\mathbb{F}_2$ with first eight terms 1, 1, 0, 0, 1, 0, 1, 1.

A2/10 **Algorithms and Networks**

(i) Let G be a directed network with nodes N, arcs A and capacities specified on each of the arcs. Define the terms *feasible flow*, *divergence, cut, upper* and *lower cut capacities.* Given two disjoint sets of nodes N^+ and N^-, what does it mean to say that a cut Q separates N^+ from N^-? Prove that the flux of a feasible flow x from N^+ to N^- is bounded above by the upper capacity of Q, for any cut Q separating N^+ from N^-.

(ii) Define the maximum-flow and minimum-cut problems. State the max-flow min-cut theorem and outline the main steps of the maximum-flow algorithm. Use the algorithm to find the maximum flow between the nodes 1 and 5 in a network whose node set is $\{1, 2, \ldots, 5\}$, where the lower capacity of each arc is 0 and the upper capacity c_{ij} of the directed arc joining node i to node j is given by the (i, j)-entry in the matrix

$$\begin{pmatrix} 0 & 7 & 9 & 8 & 0 \\ 0 & 0 & 6 & 8 & 4 \\ 0 & 9 & 0 & 2 & 10 \\ 0 & 3 & 7 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

[*The painted-network theorem can be used without proof but should be stated clearly. You may assume in your description of the maximum-flow algorithm that you are given an initial feasible flow.*]

A3/10 **Algorithms and Networks**

(i) Consider the unconstrained geometric programme GP

$$\text{minimise} \quad g(t) = \sum_{i=1}^{n} c_i \, \Pi_{j=1}^{m} \, t_j^{a_{ij}}$$

$$\text{subject to} \quad t_j > 0 \quad j = 1, \ldots, m.$$

State the dual problem to GP. Give a careful statement of the AM-GM inequality, and use it to prove the primal-dual inequality for GP.

(ii) Define min-path and max-tension problems. State and outline the proof of the max-tension min-path theorem.

A company has branches in five cities A, B, C, D and E. The fares for direct flights between these cities are as follows:

	A	B	C	D	E
A	–	50	40	25	10
B	50	–	20	90	25
C	40	20	–	10	25
D	25	90	10	–	55
E	10	25	25	55	–

Formulate this as a min-path problem. Illustrate the max-tension min-path algorithm by finding the cost of travelling by the cheapest routes between D and each of the other cities.

A4/11 **Algorithms and Networks**

Write an essay on Strong Lagrangian problems. You should give an account of duality and how it relates to the Strong Lagrangian property. In particular, establish carefully the relationship between the Strong Lagrangian property and supporting hyperplanes.

Also, give an example of a class of problems that are Strong Lagrangian. [*You should explain carefully why your example has the Strong Lagrangian property.*]

A1/13 **Computational Statistics and Statistical Modelling**

(i) Suppose $Y_1, \ldots, Y_n$ are independent Poisson variables, and

$$\mathbb{E}(Y_i) = \mu_i \ , \ \log \mu_i = \alpha + \beta^T x_i \ , \ 1 \leqslant i \leqslant n$$

where α, β are unknown parameters, and $x_1, \ldots, x_n$ are given covariates, each of dimension p. Obtain the maximum-likelihood equations for α, β, and explain briefly how you would check the validity of this model.

(ii) The data below show $y_1, \ldots, y_{33}$, which are the monthly accident counts on a major US highway for each of the 12 months of 1970, then for each of the 12 months of 1971, and finally for the first 9 months of 1972. The data-set is followed by the (slightly edited) *R* output. You may assume that the factors 'Year' and 'month' have been set up in the appropriate fashion. Give a careful interpretation of this *R* output, and explain (a) how you would derive the corresponding standardised residuals, and (b) how you would predict the number of accidents in October 1972.

```
52 37 49 29 31 32 28 34 32 39 50 63
35 22 27 27 34 23 42 30 36 56 48 40
33 26 31 25 23 20 25 20 36
```

> first.glm _ glm(y~ Year + month, poisson) ; summary(first.glm)

Call:

glm(formula = y ~ Year + month, family = poisson)

```
Coefficients:
                Estimate   Std. Error   z value   Pr(>|z|)
(Intercept)      3.81969   0.09896      38.600    < 2e-16 ***
Year1971        -0.12516   0.06694      -1.870    0.061521 .
Year1972        -0.28794   0.08267      -3.483    0.000496 ***
month2          -0.34484   0.14176      -2.433    0.014994 *
month3          -0.11466   0.13296      -0.862    0.388459
month4          -0.39304   0.14380      -2.733    0.006271 **
month5          -0.31015   0.14034      -2.210    0.027108 *
month6          -0.47000   0.14719      -3.193    0.001408 **
month7          -0.23361   0.13732      -1.701    0.088889 .
month8          -0.35667   0.14226      -2.507    0.012168 *
month9          -0.14310   0.13397      -1.068    0.285444
month10          0.10167   0.13903       0.731    0.464628
month11          0.13276   0.13788       0.963    0.335639
month12          0.18252   0.13607       1.341    0.179812
```

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'

(Dispersion parameter for poisson family taken to be 1)

```
    Null deviance:     101.143   on 32 degrees of freedom
Residual deviance:      27.273   on 19 degrees of freedom
```

Number of Fisher Scoring iterations: 3

A2/12 **Computational Statistics and Statistical Modelling**

(i) Suppose that the random variable Y has density function of the form

$$f(y|\theta, \phi) = \exp\left[\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right]$$

where $\phi > 0$. Show that Y has expectation $b'(\theta)$ and variance $\phi b''(\theta)$.

(ii) Suppose now that $Y_1, \ldots, Y_n$ are independent negative exponential variables, with Y_i having density function $f(y_i|\mu_i) = \frac{1}{\mu_i} e^{-y_i/\mu_i}$ for $y_i > 0$. Suppose further that $g(\mu_i) = \beta^T x_i$ for $1 \leqslant i \leqslant n$, where $g(\cdot)$ is a known 'link' function, and $x_1, \ldots, x_n$ are given covariate vectors, each of dimension p. Discuss carefully the problem of finding $\hat{\beta}$, the maximum-likelihood estimator of β, firstly for the case $g(\mu_i) = 1/\mu_i$, and secondly for the case $g(\mu) = \log \mu_i$; in both cases you should state the large-sample distribution of $\hat{\beta}$.

[*Any standard theorems used need not be proved.*]

A4/14 **Computational Statistics and Statistical Modelling**

Assume that the n-dimensional observation vector Y may be written as $Y = X\beta + \epsilon$, where X is a given $n \times p$ matrix of rank p, β is an unknown vector, with $\beta^T = (\beta_1, \ldots, \beta_p)$, and

$$\epsilon \sim N_n(0, \sigma^2 I) \qquad (*)$$

where σ^2 is unknown. Find $\hat{\beta}$, the least-squares estimator of β, and describe (without proof) how you would test

$$H_0 : \beta_\nu = 0$$

for a given ν.

Indicate briefly two plots that you could use as a check of the assumption $(*)$.

Continued opposite

Sulphur dioxide is one of the major air pollutants. A data-set presented by Sokal and Rohlf (1981) was collected on 41 US cities in 1969-71, corresponding to the following variables:

Y = sulphur dioxide content of air in micrograms per cubic metre

$X1$ = average annual temperature in degrees Fahrenheit

$X2$ = number of manufacturing enterprises employing 20 or more workers

$X3$ = population size (1970 census) in thousands

$X4$ = average annual wind speed in miles per hour

$X5$ = average annual precipitation in inches

$X6$ = average annual of days with precipitation per year.

Interpret the R output that follows below, quoting any standard theorems that you need to use.

> next.lm _ lm(log(Y) ~ X1 + X2 + X3 + X4 + X5 + X6)

> summary(next.lm)

Call: lm(formula = log(Y) ~ X1 + X2 + X3 + X4 + X5 + X6)

```
Residuals:
     Min        1Q    Median        3Q       Max
-0.79548  -0.25538  -0.01968   0.28328   0.98029
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> $\|t\|$)	
(Intercept)	7.2532456	1.4483686	5.008	1.68e-05	***
X1	-0.0599017	0.0190138	-3.150	0.00339	**
X2	0.0012639	0.0004820	2.622	0.01298	*
X3	-0.0007077	0.0004632	-1.528	0.13580	
X4	-0.1697171	0.0555563	-3.055	0.00436	**
X5	0.0173723	0.0111036	1.565	0.12695	
X6	0.0004347	0.0049591	0.088	0.93066	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'

Residual standard error: 0.448 on 34 degrees of freedom

Multiple R-Squared: 0.6541

F-statistic: 10.72 on 6 and 34 degrees of freedom, p-value: 1.126e-06

A1/14 **Quantum Physics**

(i) A system of N identical non-interacting bosons has energy levels E_i with degeneracy g_i, $1 \leq i < \infty$, for each particle. Show that in thermal equilibrium the number of particles N_i with energy E_i is given by

$$N_i = \frac{g_i}{e^{\beta(E_i-\mu)} - 1} \quad ,$$

where β and μ are parameters whose physical significance should be briefly explained.

(ii) A photon moves in a cubical box of side L. Assuming periodic boundary conditions, show that, for large L, the number of photon states lying in the frequency range $\omega \to \omega + d\omega$ is $\rho(\omega)d\omega$ where

$$\rho(\omega) = L^3 \left(\frac{\omega^2}{\pi^2 c^3}\right) \quad .$$

If the box is filled with thermal radiation at temperature T, show that the number of photons per unit volume in the frequency range $\omega \to \omega + d\omega$ is $n(\omega)d\omega$ where

$$n(\omega) = \left(\frac{\omega^2}{\pi^2 c^3}\right) \frac{1}{e^{\hbar\omega/kT} - 1} \quad .$$

Calculate the energy density W of the thermal radiation. Show that the pressure P exerted on the surface of the box satisfies

$$P = \frac{1}{3} W \quad .$$

[*You may use the result* $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$.]

A2/14 **Quantum Physics**

(i) A simple model of a one-dimensional crystal consists of an infinite array of sites equally spaced with separation a . An electron occupies the nth site with a probability amplitude c_n . The time-dependent Schrödinger equation governing these amplitudes is

$$i\hbar\frac{dc_n}{dt} = E_0 c_n - A(c_{n-1} + c_{n+1}) \quad ,$$

where E_0 is the energy of an electron at an isolated site and the amplitude for transition between neighbouring sites is $A > 0$. By examining a solution of the form

$$c_n = e^{ikan - iEt/\hbar} \quad ,$$

show that E, the energy of the electron in the crystal, lies in a band

$$E_0 - 2A \leq E \leq E_0 + 2A \ .$$

Identify the Brillouin zone for this model and explain its significance.

(ii) In the above model the electron is now subject to an electric field $\mathcal{E}$ in the direction of increasing n . Given that the charge on the electron is $-e$ write down the new form of the time-dependent Schrödinger equation for the probability amplitudes. Show that it has a solution of the form

$$c_n = \exp\left\{-\frac{i}{\hbar}\int_0^t \epsilon(t')dt' + i(k - \frac{e\mathcal{E}t}{\hbar})na\right\} \quad ,$$

where

$$\epsilon(t) = E_0 - 2A\cos\left((k - \frac{e\mathcal{E}t}{\hbar})a\right) \quad .$$

Explain briefly how to interpret this result and use it to show that the dynamical behaviour of an electron near the bottom of the energy band is the same as that for a free particle in the presence of an electric field with an effective mass $m^* = \hbar^2/(2Aa^2)$.

A4/16 **Quantum Physics**

Explain how the energy band structure for electrons determines the conductivity properties of crystalline materials.

A semiconductor has a conduction band with a lower edge E_c and a valence band with an upper edge E_v . Assuming that the density of states for electrons in the conduction band is

$$\rho_c(E) = B_c(E - E_c)^{\frac{1}{2}} \ , \qquad E > E_c \ \ ,$$

and in the valence band is

$$\rho_v(E) = B_v(E_v - E)^{\frac{1}{2}} \ , \qquad E < E_v \ \ ,$$

where B_c and B_v are constants characteristic of the semiconductor, explain why at low temperatures the chemical potential for electrons lies close to the mid-point of the gap between the two bands.

Describe what is meant by the doping of a semiconductor and explain the distinction between n-type and p-type semiconductors, and discuss the low temperature limit of the chemical potential in both cases. Show that, whatever the degree and type of doping,

$$n_e n_p = B_c B_v [\Gamma(3/2)]^2 (kT)^3 e^{-(E_c - E_v)/kT} \ \ ,$$

where n_e is the density of electrons in the conduction band and n_p is the density of holes in the valence band.

A1/16 **Statistical Physics and Cosmology**

(i) Consider a one-dimensional model universe with "stars" distributed at random on the x-axis, and choose the origin to coincide with one of the stars; call this star the "home-star." Home-star astronomers have discovered that all other stars are receding from them with a velocity $v(x)$, that depends on the position x of the star. Assuming non-relativistic addition of velocities, show how the assumption of homogeneity implies that $v(x) = H_0 x$ for some constant H_0.

In attempting to understand the history of their one-dimensional universe, home-star astronomers seek to determine the velocity $v(t)$ at time t of a star at position $x(t)$. Assuming homogeneity, show how $x(t)$ is determined in terms of a scale factor $a(t)$ and hence deduce that $v(t) = H(t)x(t)$ for some function $H(t)$. What is the relation between $H(t)$ and H_0?

(ii) Consider a three-dimensional homogeneous and isotropic universe with mass density $\rho(t)$, pressure $p(t)$ and scale factor $a(t)$. Given that $E(t)$ is the energy in volume $V(t)$, show how the relation $dE = -p\,dV$ yields the "fluid" equation

$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)H,$$

where $H = \dot{a}/a$.

Show how conservation of energy applied to a test particle at the boundary of a spherical fluid element yields the Friedmann equation

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = -kc^2$$

for constant k. Hence obtain an equation for the acceleration $\ddot{a}$ in terms of ρ, p and a.

A model universe has mass density and pressure

$$\rho = \frac{\rho_0}{a^3} + \rho_1, \qquad p = -\rho_1 c^2,$$

where ρ_0 is constant. What does the fluid equation imply about ρ_1? Show that the acceleration $\ddot{a}$ vanishes if

$$a = \left(\frac{\rho_0}{2\rho_1}\right)^{\frac{1}{3}}.$$

Hence show that this universe is static and determine the sign of the constant k.

A3/14 **Statistical Physics and Cosmology**

(i) Write down the first law of thermodynamics for the change dU in the internal energy $U(N,V,S)$ of a gas of N particles in a volume V with entropy S.

Given that

$$PV = (\gamma - 1)U,$$

where P is the pressure, use the first law to show that PV^{γ} is constant at constant N and S.

Write down the Boyle-Charles law for a non-relativistic ideal gas and hence deduce that the temperature T is proportional to $V^{1-\gamma}$ at constant N and S.

State the principle of equipartition of energy and use it to deduce that

$$U = \frac{3}{2}NkT.$$

Hence deduce the value of γ. Show that this value of γ is such that the ratio E_i/kT is unchanged by a change of volume at constant N and S, where E_i is the energy of the i-th one particle eigenstate of a non-relativistic ideal gas.

(ii) A classical gas of non-relativistic particles of mass m at absolute temperature T and number density n has a chemical potential

$$\mu = mc^2 - kT \ln\left(\frac{g_s}{n}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right),$$

where g_s is the particle's spin degeneracy factor. What condition on n is needed for the validity of this formula and why?

Thermal and chemical equilibrium between two species of non-relativistic particles a and b is maintained by the reaction

$$a + \alpha \leftrightarrow b + \beta,$$

where α and β are massless particles with zero chemical potential. Given that particles a and b have masses m_a and m_b respectively, but equal spin degeneracy factors, find the number density ratio n_a/n_b as a function of m_a, m_b and T. Given that $m_a > m_b$ but $m_a - m_b \ll m_b$ show that

$$\frac{n_a}{n_b} \approx f\left(\frac{(m_a - m_b)c^2}{kT}\right)$$

for some function f which you should determine.

Explain how a reaction of the above type is relevant to a determination of the neutron to proton ratio in the early universe and why this ratio does not fall rapidly to zero as the universe cools. Explain briefly the process of primordial nucleosynthesis by which neutrons are converted into stable helium nuclei. Let

$$Y_{He} = \frac{\rho_{He}}{\rho}$$

be the fraction of the universe that ends up in helium. Compute Y_{He} as a function of the ratio $r = n_a/n_b$ at the time of nucleosynthesis.

A4/18 **Statistical Physics and Cosmology**

What is an ideal gas? Explain how the microstates of an ideal gas of indistinguishable particles can be labelled by a set of integers. What range of values do these integers take for (a) a boson gas and (b) a Fermi gas?

Let E_i be the energy of the i-th one-particle energy eigenstate of an ideal gas in thermal equilibrium at temperature T and let $p_i(n_i)$ be the probability that there are n_i particles of the gas in this state. Given that

$$p_i(n_i) = e^{-\beta E_i n_i}/Z_i \quad (\beta = \frac{1}{kT}),$$

determine the normalization factor Z_i for (a) a boson gas and (b) a Fermi gas. Hence obtain an expression for $\bar{n}_i$, the average number of particles in the i-th one-particle energy eigenstate for both cases (a) and (b).

In the case of a Fermi gas, write down (without proof) the generalization of your formula for $\bar{n}_i$ to a gas at non-zero chemical potential μ. Show how it leads to the concept of a Fermi energy ϵ_F for a gas at zero temperature. How is ϵ_F related to the Fermi momentum p_F for (a) a non-relativistic gas and (b) an ultra-relativistic gas?

In an approximation in which the discrete set of energies E_i is replaced with a continuous set with momentum p, the density of one-particle states with momentum in the range p to $p + dp$ is $g(p)dp$. Explain briefly why

$$g(p) \propto p^2 V, \tag{$*$}$$

where V is the volume of the gas. Using this formula, obtain an expression for the total energy E of an ultra-relativistic gas at zero chemical potential as an integral over p. Hence show that

$$\frac{E}{V} \propto T^{\alpha},$$

where α is a number that you should compute. Why does this result apply to a photon gas?

Using the formula ($*$) for a non-relativistic Fermi gas at zero temperature, obtain an expression for the particle number density n in terms of the Fermi momentum and provide a physical interpretation of this formula in terms of the typical de Broglie wavelength. Obtain an analogous formula for the (internal) energy density and hence show that the pressure P behaves as

$$P \propto n^{\gamma}$$

where γ is a number that you should compute. [*You need not prove any relation between the pressure and the energy density you use.*] What is the origin of this pressure given that $T = 0$ by assumption? Explain briefly and qualitatively how it is relevant to the stability of white dwarf stars.

A1/17 **Symmetries and Groups in Physics**

(i) Let H be a normal subgroup of the group G. Let G/H denote the group of cosets $\tilde{g} = gH$ for $g \in G$. If $D : G \to GL(\mathbb{C}^n)$ is a representation of G with $D(h_1) = D(h_2)$ for all $h_1, h_2 \in H$ show that $\tilde{D}(\tilde{g}) = D(g)$ is well-defined and that it is a representation of G/H. Show further that $\tilde{D}(\tilde{g})$ is irreducible if and only if $D(g)$ is irreducible.

(ii) For a matrix $U \in SU(2)$ define the linear map $\Phi_U : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_U(\mathbf{x}).\boldsymbol{\sigma} = U\mathbf{x}.\boldsymbol{\sigma}U^\dagger$ with $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ as the vector of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\|\Phi_U(\mathbf{x})\| = \|\mathbf{x}\|$. Because of the linearity of Φ_U there exists a matrix $R(U)$ such that $\Phi_U(\mathbf{x}) = R(U)\mathbf{x}$. Given that any $SU(2)$ matrix can be written as

$$U = \cos\alpha \, I - i \sin\alpha \, \mathbf{n}.\boldsymbol{\sigma} \quad ,$$

where $\alpha \in [0, \pi]$ and $\mathbf{n}$ is a unit vector, deduce that $R(U) \in SO(3)$ for all $U \in SU(2)$. Compute $R(U)\mathbf{n}$ and $R(U)\mathbf{x}$ in the case that $\mathbf{x}.\mathbf{n} = 0$ and deduce that $R(U)$ is the matrix of a rotation about $\mathbf{n}$ with angle 2α.

[*Hint:* $\mathbf{m}.\boldsymbol{\sigma} \; \mathbf{n}.\boldsymbol{\sigma} = \mathbf{m}.\mathbf{n} \, I + i(\mathbf{m} \times \mathbf{n}).\boldsymbol{\sigma}$.]

Show that $R(U)$ defines a surjective homomorphism $\Theta : SU(2) \to SO(3)$ and find the kernel of Θ.

A3/15 **Symmetries and Groups in Physics**

(i) Let D_6 denote the symmetry group of rotations and reflections of a regular hexagon. The elements of D_6 are given by $\{e, c, c^2, c^3, c^4, c^5, b, bc, bc^2, bc^3, bc^4, bc^5\}$ with $c^6 = b^2 = e$ and $cb = bc^5$. The conjugacy classes of D_6 are $\{e\}$, $\{c, c^5\}$, $\{c^2, c^4\}$, $\{c^3\}$, $\{b, bc^2, bc^4\}$ and $\{bc, bc^3, bc^5\}$.

Show that the character table of D_6 is

D_6	e	$\{c, c^5\}$	$\{c^2, c^4\}$	$\{c^3\}$	$\{b, bc^2, bc^4\}$	$\{bc, bc^3, bc^5\}$
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2	1	-1	-2	0	0
χ_6	2	-1	-1	2	0	0

(ii) Show that the character of an $SO(3)$ rotation with angle θ in the $2l+1$ dimensional irreducible representation of $SO(3)$ is given by

$$\chi_l(\theta) = 1 + 2\cos\theta + 2\cos(2\theta) + \ldots + 2\cos((l-1)\theta) + 2\cos(l\theta) \; .$$

For a hexagonal crystal of atoms find how the degeneracy of the D-wave orbital states ($l = 2$) in the atomic central potential is split by the crystal potential with D_6 symmetry and give the new degeneracies.

By using the fact that D_3 is isomorphic to $D_6/\{e, c^3\}$, or otherwise, find the degeneracies of eigenstates if the hexagonal symmetry is broken to the subgroup D_3 by a deformation. The introduction of a magnetic field further reduces the symmetry to C_3. What will the degeneracies of the energy eigenstates be now?

A1/18 **Transport Processes**

(i) Material of thermal diffusivity D occupies the semi-infinite region $x > 0$ and is initially at uniform temperature T_0. For time $t > 0$ the temperature at $x = 0$ is held at a constant value $T_1 > T_0$. Given that the temperature $T(x,t)$ in $x > 0$ satisfies the diffusion equation $T_t = DT_{xx}$, write down the equation and the boundary and initial conditions satisfied by the dimensionless temperature $\theta = (T - T_0) / (T_1 - T_0)$.

Use dimensional analysis to show that the lengthscale of the region in which T is significantly different from T_0 is proportional to $(Dt)^{1/2}$. Hence show that this problem has a similarity solution

$$\theta = \operatorname{erfc}(\xi/2) \equiv \frac{2}{\sqrt{\pi}} \int_{\xi/2}^{\infty} e^{-u^2} du \ ,$$

where $\xi = x/(Dt)^{1/2}$.

What is the rate of heat input, $-DT_x$, across the plane $x = 0$?

(ii) Consider the same problem as in Part (i) except that the boundary condition at $x = 0$ is replaced by one of constant rate of heat input Q. Show that $\theta(\xi, t)$ satisfies the partial differential equation

$$\theta_{\xi\xi} + \frac{\xi}{2}\theta_\xi = t\theta_t$$

and write down the boundary conditions on $\theta(\xi, t)$. Deduce that the problem has a similarity solution of the form

$$\theta = \frac{Q(t/D)^{1/2}}{T_1 - T_0} f(\xi).$$

Derive the ordinary differential equation and boundary conditions satisfied by $f(\xi)$. Differentiate this equation once to obtain

$$f''' + \frac{\xi}{2} f'' = 0$$

and solve for $f'(\xi)$. Hence show that

$$f(\xi) = \frac{2}{\sqrt{\pi}} e^{-\xi^2/4} - \xi \operatorname{erfc}(\xi/2) \ .$$

Sketch the temperature distribution $T(x,t)$ for various times t, and calculate $T(0,t)$ explicitly.

A3/16 **Transport Processes**

(i) A layer of fluid of depth $h(x,t)$, density ρ and viscosity μ sits on top of a rigid horizontal plane at $y=0$. Gravity g acts vertically and surface tension is negligible.

Assuming that the horizontal velocity component u and pressure p satisfy the lubrication equations

$$0 = -p_x + \mu u_{yy}$$
$$0 = -p_y - \rho g,$$

together with appropriate boundary conditions at $y=0$ and $y=h$ (which should be stated), show that h satisfies the partial differential equation

$$h_t = \frac{g}{3\nu}\left(h^3 h_x\right)_x\,, \qquad (*)$$

where $\nu = \mu/\rho$.

(ii) A two-dimensional blob of the above fluid has fixed area A and time-varying width $2X(t)$, such that

$$A = \int_{-X(t)}^{X(t)} h(x,t)\,dx.$$

The blob spreads under gravity.

Use scaling arguments to show that, after an initial transient, $X(t)$ is proportional to $t^{1/5}$ and $h(0,t)$ is proportional to $t^{-1/5}$. Hence show that equation $(*)$ of Part (i) has a similarity solution of the form

$$h(x,t) = \left(\frac{A^2\nu}{gt}\right)^{1/5} H(\xi), \quad \text{where} \quad \xi = \frac{x}{\left(A^3 gt/\nu\right)^{1/5}},$$

and find the differential equation satisfied by $H(\xi)$.

Deduce that

$$H = \begin{cases} \left[\frac{9}{10}\left(\xi_0^2 - \xi^2\right)\right]^{1/3} & \text{in} \quad -\xi_0 < \xi < \xi_0 \\ 0 & \text{in} \quad |\xi| > \xi_0\ , \end{cases}$$

where

$$X(t) = \xi_0 \left(\frac{A^3 gt}{\nu}\right)^{1/5}.$$

Express ξ_0 in terms of the integral

$$I = \int_{-1}^{1} \left(1-u^2\right)^{1/3} du.$$

A4/19 **Transport Processes**

(a) A biological vessel is modelled two-dimensionally as a fluid-filled channel bounded by parallel plane walls $y = \pm a$, embedded in an infinite region of fluid-saturated tissue. In the tissue a solute has concentration $C^{out}(y,t)$, diffuses with diffusivity D and is consumed by biological activity at a rate kC^{out} per unit volume, where D and k are constants. By considering the solute balance in a slice of tissue of infinitesimal thickness, show that

$$C_t^{out} = DC_{yy}^{out} - kC^{out}.$$

A *steady* concentration profile $C^{out}(y)$ results from a flux $\beta\left(C^{in} - C_a^{out}\right)$, per unit area of wall, of solute from the channel into the tissue, where C^{in} is a constant concentration of solute that is maintained in the channel and $C_a^{out} = C^{out}(a)$. Write down the boundary conditions satisfied by $C^{out}(y)$. Solve for $C^{out}(y)$ and show that

$$C_a^{out} = \frac{\gamma}{\gamma+1}C^{in}, \qquad (*)$$

where $\gamma = \beta/\sqrt{kD}$.

(b) Now let the solute be supplied by steady flow down the channel from one end, $x = 0$, with the channel taken to be semi-infinite in the x-direction. The cross-sectionally averaged velocity in the channel $u(x)$ varies due to a flux of fluid from the tissue to the channel (by osmosis) equal to $\lambda\left(C^{in} - C_a^{out}\right)$ per unit area. Neglect both the variation of $C^{in}(x)$ across the channel and diffusion in the x-direction.

By considering conservation of fluid, show that

$$au_x = \lambda\left(C^{in} - C_a^{out}\right)$$

and write down the corresponding equation derived from conservation of solute. Deduce that

$$u(\lambda C^{in} + \beta) = u_0(\lambda C_0^{in} + \beta)\ ,$$

where $u_0 = u(0)$ and $C_0^{in} = C^{in}(0)$.

Assuming that equation $(*)$ still holds, even though C^{out} is now a function of x as well as y, show that $u(x)$ satisfies the ordinary differential equation

$$(\gamma+1)auu_x + \beta u = u_0\left(\lambda C_0^{in} + \beta\right).$$

Find scales $\hat{x}$ and $\hat{u}$ such that the dimensionless variables $U = u/\hat{u}$ and $X = x/\hat{x}$ satisfy

$$UU_X + U = 1.$$

Derive the solution $(1-U)e^U = Ae^{-X}$ and find the constant A.

To what values do u and C_{in} tend as $x \to \infty$?

A1/19 **Theoretical Geophysics**

(i) In a reference frame rotating about a vertical axis with constant angular velocity $f/2$ the horizontal components of the momentum equation for a shallow layer of inviscid, incompressible fluid of constant density ρ are

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad ,$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} \quad ,$$

where u, v and P are independent of the vertical coordinate z.

Define the Rossby number Ro for a flow with typical velocity U and lengthscale L. What is the approximate form of the above equations when $Ro \ll 1$?

Show that the solution to the approximate equations is given by a streamfunction ψ proportional to P.

Conservation of potential vorticity for such a flow is represented by

$$\frac{D}{Dt}\frac{\zeta + f}{h} = 0,$$

where ζ is the vertical component of relative vorticity and $h(x, y)$ is the thickness of the layer. Explain briefly why the potential vorticity of a column of fluid should be conserved.

(ii) Suppose that the thickness of the rotating, shallow-layer flow in Part (i) is $h(y) = H_0 \exp(-\alpha y)$ where H_0 and α are constants. By linearising the equation of conservation of potential vorticity about $u = v = \zeta = 0$, show that the stream function for small disturbances to the state of rest obeys

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + \beta\frac{\partial \psi}{\partial x} = 0 \quad ,$$

where β is a constant that should be found.

Obtain the dispersion relationship for plane-wave solutions of the form $\psi \propto \exp[i(kx + ly - \omega t)]$. Hence calculate the group velocity.

Show that if $\beta > 0$ then the phase of these waves always propagates to the left (negative x direction) but that the energy may propagate to either left or right.

A2/16 **Theoretical Geophysics**

(i) State the equations that relate strain to displacement and stress to strain in a linear, isotropic elastic solid.

In the absence of body forces, the Euler equation for infinitesimal deformations of a solid of density ρ is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} \quad .$$

Derive an equation for $\mathbf{u}(\mathbf{x}, t)$ in a linear, isotropic, homogeneous elastic solid. Hence show that both the dilatation $\theta = \boldsymbol{\nabla} \cdot \mathbf{u}$ and the rotation $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ satisfy wave equations and find the corresponding wave speeds α and β.

(ii) The ray parameter $p = r \sin i / v$ is constant along seismic rays in a spherically symmetric Earth, where $v(r)$ is the relevant wave speed (α or β) and $i(r)$ is the angle between the ray and the local radial direction.

Express $\tan i$ and $\sec i$ in terms of p and the variable $\eta(r) = r/v$. Hence show that the angular distance and travel time between a surface source and receiver, both at radius R, are given by

$$\Delta(p) = 2 \int_{r_m}^{R} \frac{p}{r} \frac{dr}{(\eta^2 - p^2)^{1/2}} \quad , \qquad T(p) = 2 \int_{r_m}^{R} \frac{\eta^2}{r} \frac{dr}{(\eta^2 - p^2)^{1/2}} \quad ,$$

where r_m is the minimum radius attained by the ray. What is $\eta(r_m)$?

A simple Earth model has a solid mantle in $R/2 < r < R$ and a liquid core in $r < R/2$. If $\alpha(r) = A/r$ in the mantle, where A is a constant, find $\Delta(p)$ and $T(p)$ for P-arrivals (direct paths lying entirely in the mantle), and show that

$$T = \frac{R^2 \sin \Delta}{A} \quad .$$

[*You may assume that* $\displaystyle\int \frac{du}{u\sqrt{u-1}} = 2\cos^{-1}\left(\frac{1}{\sqrt{u}}\right)$.]

Sketch the $T - \Delta$ curves for P and PcP arrivals on the same diagram and explain briefly why they terminate at $\Delta = \cos^{-1} \frac{1}{4}$.

A4/20 **Theoretical Geophysics**

The equation of motion for small displacements $\mathbf{u}$ in a homogeneous, isotropic, elastic material is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \mu \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u}) \quad ,$$

where λ and μ are the Lamé constants. Derive the conditions satisfied by the polarisation $\mathbf{P}$ and (real) vector slowness $\mathbf{s}$ of plane-wave solutions $\mathbf{u} = \mathbf{P}f(\mathbf{s} \cdot \mathbf{x} - t)$, where f is an arbitrary scalar function. Describe the division of these waves into P-waves, SH-waves and SV-waves.

A plane harmonic SV-wave of the form

$$\mathbf{u} = (s_3, 0, -s_1) \exp[i\omega(s_1 x_1 + s_3 x_3 - t)]$$

travelling through homogeneous elastic material of P-wave speed α and S-wave speed β is incident from $x_3 < 0$ on the boundary $x_3 = 0$ of rigid material in $x_3 > 0$ in which the displacement is identically zero.

Write down the form of the reflected wavefield in $x_3 < 0$. Calculate the amplitudes of the reflected waves in terms of the components of the slowness vectors.

Derive expressions for the components of the incident and reflected slowness vectors, in terms of the wavespeeds and the angle of incidence θ_0. Hence show that there is no reflected SV-wave if

$$\sin^2 \theta_0 = \frac{\beta^2}{\alpha^2 + \beta^2} \quad .$$

Sketch the rays produced if the region $x_3 > 0$ is fluid instead of rigid.

A2/17 **Mathematical Methods**

(i) Show that the equation

$$\epsilon x^4 - x^2 + 5x - 6 = 0\,, \quad |\epsilon| \ll 1\ ,$$

has roots in the neighbourhood of $x = 2$ and $x = 3$. Find the first two terms of an expansion in ϵ for each of these roots.

Find a suitable series expansion for the other two roots and calculate the first two terms in each case.

(ii) Describe, giving reasons for the steps taken, how the leading-order approximation for $\lambda \gg 1$ to an integral of the form

$$I(\lambda) \equiv \int_A^B f(t) e^{i\lambda g(t)} dt\ ,$$

where λ and g are real, may be found by the method of stationary phase. Consider the cases where (a) $g'(t)$ has one simple zero at $t = t_0$ with $A < t_0 < B$; (b) $g'(t)$ has more than one simple zero in $A < t < B$; and (c) $g'(t)$ has only a simple zero at $t = B$. What is the order of magnitude of $I(\lambda)$ if $g'(t)$ is non-zero for $A \le t \le B$?

Use the method of stationary phase to find the leading-order approximation to

$$J(\lambda) \equiv \int_0^1 \sin[\lambda(2t^4 - t)]\, dt$$

for $\lambda \gg 1$.

[*You may use the fact that* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2} du = \sqrt{\pi} e^{i\pi/4}$.]

A3/17 **Mathematical Methods**

(i) State the Fredholm alternative for Fredholm integral equations of the second kind.

Show that the integral equation

$$\phi(x) - \lambda \int_0^1 (x+t)\phi(t)dt = f(x)\,, \quad 0 \leqslant x \leqslant 1\ ,$$

where f is a continuous function, has a unique solution for ϕ if $\lambda \neq -6 \pm 4\sqrt{3}$. Derive this solution.

(ii) Describe the WKB method for finding approximate solutions $f(x)$ of the equation

$$\frac{d^2 f(x)}{dx^2} + q(\epsilon x) f(x) = 0\ ,$$

where q is an arbitrary non-zero, differentiable function and ϵ is a small parameter. Obtain these solutions in terms of an exponential with slowly varying exponent and slowly varying amplitude.

Hence, by means of a suitable change of independent variable, find approximate solutions $w(t)$ of the equation

$$\frac{d^2 w}{dt^2} + \lambda^2 t w = 0\ ,$$

in $t > 0$, where λ is a large parameter.

A4/21 **Mathematical Methods**

State Watson's lemma giving an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_1 = \int_0^A f(t) e^{-\lambda t} dt\,, \quad A > 0\ .$$

Show how this result may be used to find an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_2 = \int_{-A}^B f(t) e^{-\lambda t^2} dt\,, \quad A > 0,\, B > 0\ .$$

Hence derive Laplace's method for obtaining an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_3 = \int_a^b f(t) e^{\lambda \phi(t)} dt\ ,$$

where $\phi(t)$ is differentiable, for the cases: (i) $\phi'(t) < 0$ in $a \leq t \leq b$; and (ii) $\phi'(t)$ has a simple zero at $t = c$ with $a < c < b$ and $\phi''(c) < 0$.

Find the first two terms in the asymptotic expansion as $x \to \infty$ of

$$I_4 = \int_{-\infty}^{\infty} \log(1+t^2) e^{-xt^2} dt\ .$$

[*You may leave your answer expressed in terms of* Γ-*functions.*]

A2/18 **Nonlinear Waves**

(i) Find a travelling wave solution of unchanging shape for the modified Burgers equation (with $\alpha > 0$)

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$$

with $u = 0$ far ahead of the wave and $u = 1$ far behind. What is the velocity of the wave? Sketch the shape of the wave.

(ii) Explain why the method of characteristics, when applied to an equation of the type

$$\frac{\partial u}{\partial t} + c(u) \frac{\partial u}{\partial x} = 0 \, ,$$

with initial data $u(x, 0) = f(x)$, sometimes gives a multi-valued solution. State the shock-fitting algorithm that gives a single-valued solution, and explain how it is justified.

Consider the equation above, with $c(u) = u^2$. Suppose that

$$u(x, 0) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases} \quad .$$

Sketch the characteristics in the (x, t) plane. Show that a shock forms immediately, and calculate the velocity at which it moves.

A3/18 **Nonlinear Waves**

(i) Show that the equation

$$\frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi}{\partial x^2} + 1 - \phi^2 = 0$$

has two solutions which are independent of both x and t. Show that one of these is linearly stable. Show that the other solution is linearly unstable, and find the range of wavenumbers that exhibit the instability.

Sketch the nonlinear evolution of the unstable solution after it receives a small, smooth, localized perturbation in the direction towards the stable solution.

(ii) Show that the equations

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} &= e^{-u+v} \quad , \\ -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} &= e^{-u-v} \end{aligned}$$

are a Bäcklund pair for the equations

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-2u}\,, \quad \frac{\partial^2 v}{\partial x \partial y} = 0\,.$$

By choosing v to be a suitable constant, and using the Bäcklund pair, find a solution of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-2u}$$

which is non-singular in the region $y < 4x$ of the (x, y) plane and has the value $u = 0$ at $x = \frac{1}{2}$, $y = 0$.

A1/1 B1/1 **Markov Chains**

(i) We are given a finite set of airports. Assume that between any two airports, i and j, there are $a_{ij} = a_{ji}$ flights in each direction on every day. A confused traveller takes one flight per day, choosing at random from all available flights. Starting from i, how many days on average will pass until the traveller returns again to i? Be careful to allow for the case where there may be no flights at all between two given airports.

(ii) Consider the infinite tree T with root R, where, for all $m \geqslant 0$, all vertices at distance 2^m from R have degree 3, and where all other vertices (except R) have degree 2. Show that the random walk on T is recurrent.

A2/1 **Markov Chains**

(i) In each of the following cases, the state-space I and non-zero transition rates q_{ij} $(i \neq j)$ of a continuous-time Markov chain are given. Determine in which cases the chain is explosive.

(a)	$I = \{1, 2, 3, \ldots\}$,	$q_{i,i+1} = i^2$,	$i \in I$,
(b)	$I = \mathbb{Z}$,	$q_{i,i-1} = q_{i,i+1} = 2^i$,	$i \in I$.

(ii) Children arrive at a see-saw according to a Poisson process of rate 1. Initially there are no children. The first child to arrive waits at the see-saw. When the second child arrives, they play on the see-saw. When the third child arrives, they all decide to go and play on the merry-go-round. The cycle then repeats. Show that the number of children at the see-saw evolves as a Markov Chain and determine its generator matrix. Find the probability that there are no children at the see-saw at time t.

Hence obtain the identity

$$\sum_{n=0}^{\infty} e^{-t} \frac{t^{3n}}{(3n)!} = \frac{1}{3} + \frac{2}{3}\, e^{-\frac{3}{2}t} \cos \frac{\sqrt{3}}{2} t \ .$$

A3/1 B3/1 **Markov Chains**

(i) Consider the continuous-time Markov chain $(X_t)_{t\geqslant 0}$ on $\{1,2,3,4,5,6,7\}$ with generator matrix

$$Q = \begin{pmatrix} -6 & 2 & 0 & 0 & 0 & 4 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -6 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

Compute the probability, starting from state 3, that X_t hits state 2 eventually.

Deduce that

$$\lim_{t\to\infty} \mathbb{P}(X_t = 2 | X_0 = 3) = \frac{4}{15}.$$

[*Justification of standard arguments is not expected.*]

(ii) A colony of cells contains immature and mature cells. Each immature cell, after an exponential time of parameter 2, becomes a mature cell. Each mature cell, after an exponential time of parameter 3, divides into two immature cells. Suppose we begin with one immature cell and let $n(t)$ denote the expected number of immature cells at time t. Show that

$$n(t) = (4e^t + 3e^{-6t})/7.$$

A4/1 **Markov Chains**

Write an essay on the long-time behaviour of discrete-time Markov chains on a finite state space. Your essay should include discussion of the convergence of probabilities as well as almost-sure behaviour. You should also explain what happens when the chain is not irreducible.

A1/2 B1/2 **Principles of Dynamics**

(i) Derive Hamilton's equations from Lagrange's equations. Show that the Hamiltonian H is constant if the Lagrangian L does not depend explicitly on time.

(ii) A particle of mass m is constrained to move under gravity, which acts in the negative z-direction, on the spheroidal surface $\epsilon^{-2}(x^2+y^2)+z^2=l^2$, with $0<\epsilon\leqslant 1$. If θ,ϕ parametrize the surface so that

$$x=\epsilon l\sin\theta\cos\phi,\ \ y=\epsilon l\sin\theta\sin\phi,\ \ z=l\cos\theta,$$

find the Hamiltonian $H(\theta,\phi,p_\theta,p_\phi)$.

Show that the energy

$$E=\frac{p_\theta^2}{2ml^2(\epsilon^2\cos^2\theta+\sin^2\theta)}+\frac{\alpha}{\sin^2\theta}+mgl\cos\theta$$

is a constant of the motion, where α is a non-negative constant.

Rewrite this equation as

$$\frac{1}{2}\dot{\theta}^2+V_{\text{eff}}(\theta)=0$$

and sketch $V_{\text{eff}}(\theta)$ for $\epsilon=1$ and $\alpha>0$, identifying the maximal and minimal values of $\theta(t)$ for fixed α and E. If ϵ is now taken not to be unity, how do these values depend on ϵ?

A2/2 B2/1 **Principles of Dynamics**

(i) A number N of non-interacting particles move in one dimension in a potential $V(x,t)$. Write down the Hamiltonian and Hamilton's equations for one particle.

At time t, the number density of particles in phase space is $f(x,p,t)$. Write down the time derivative of f along a particle's trajectory. By equating the rate of change of the number of particles in a fixed domain V in phase space to the flux into V across its boundary, deduce that f is a constant along any particle's trajectory.

(ii) Suppose that $V(x)=\frac{1}{2}m\omega^2x^2$, and particles are injected in such a manner that the phase space density is a constant f_1 at any point of phase space corresponding to a particle energy being smaller than E_1 and zero elsewhere. How many particles are present?

Suppose now that the potential is very slowly altered to the square well form

$$V(x)=\begin{cases}0, & -L<x<L\\ \infty & \text{elsewhere}\ .\end{cases}$$

Show that the greatest particle energy is now

$$E_2=\frac{\pi^2}{8}\frac{E_1^2}{mL^2\omega^2}.$$

A3/2 **Principles of Dynamics**

(i) Show that Hamilton's equations follow from the variational principle

$$\delta \int_{t_1}^{t_2} [p\dot{q} - H(q,p,t)]\, dt = 0$$

under the restrictions $\delta q(t_1) = \delta q(t_2) = \delta p(t_1) = \delta p(t_2) = 0$. Comment on the difference from the variational principle for Lagrange's equations.

(ii) Suppose we transform from p and q to $p' = p'(q,p,t)$ and $q' = q'(q,p,t)$, with

$$p'\dot{q}' - H' = p\dot{q} - H + \frac{\mathrm{d}}{\mathrm{d}t}F(q,p,q',p',t),$$

where H' is the new Hamiltonian. Show that p' and q' obey Hamilton's equations with Hamiltonian H'.

Show that the time independent generating function $F = F_1(q,q') = q'/q$ takes the Hamiltonian

$$H = \frac{1}{2q^2} + \frac{1}{2}p^2 q^4$$

to harmonic oscillator form. Show that q' and p' obey the Poisson bracket relation

$$\{q', p'\} = 1.$$

A4/2 **Principles of Dynamics**

Explain how the orientation of a rigid body can be specified by means of the three Eulerian angles, θ, ϕ and ψ.

An axisymmetric top of mass M has principal moments of inertia A, A and C, and is spinning with angular speed n about its axis of symmetry. Its centre of mass lies a distance h from the fixed point of support. Initially the axis of symmetry points vertically upwards. It then suffers a small disturbance. For what values of the spin is the initial configuration stable?

If the spin is such that the initial configuration is unstable, what is the lowest angle reached by the symmetry axis in the nutation of the top? Find the maximum and minimum values of the precessional angular velocity $\dot{\phi}$.

A1/3 **Functional Analysis**

(i) Let $P_r(e^{i\theta})$ be the real part of $\dfrac{1+re^{i\theta}}{1-re^{i\theta}}$. Establish the following properties of P_r for $0 \leqslant r < 1$:

(a) $0 < P_r(e^{i\theta}) = P_r(e^{-i\theta}) \leqslant \dfrac{1+r}{1-r}$;

(b) $P_r(e^{i\theta}) \leqslant P_r(e^{i\delta})$ for $0 < \delta \leqslant |\theta| \leqslant \pi$;

(c) $P_r(e^{i\theta}) \to 0$, uniformly on $0 < \delta \leqslant |\theta| \leqslant \pi$, as r increases to 1.

(ii) Suppose that $f \in L^1(\mathbf{T})$, where $\mathbf{T}$ is the unit circle $\{e^{i\theta} : -\pi \leqslant \theta \leqslant \pi\}$. By definition, $\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta$. Let

$$P_r(f)(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i(\theta-t)}) f(e^{it})\, dt.$$

Show that $P_r(f)$ is a continuous function on $\mathbf{T}$, and that $\|P_r(f)\|_1 \leqslant \|f\|_1$.

[*You may assume without proof that* $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i\theta})\, d\theta = 1$.]

Show that $P_r(f) \to f$, uniformly on $\mathbf{T}$ as r increases to 1, if and only if f is a continuous function on $\mathbf{T}$.

Show that $\|P_r(f) - f\|_1 \to 0$ as r increases to 1.

A2/3 B2/2 **Functional Analysis**

(i) State and prove the parallelogram law for Hilbert spaces.

Suppose that K is a closed linear subspace of a Hilbert space H and that $x \in H$. Show that x is orthogonal to K if and only if 0 is the nearest point to x in K.

(ii) Suppose that H is a Hilbert space and that ϕ is a continuous linear functional on H with $\|\phi\| = 1$. Show that there is a sequence (h_n) of unit vectors in H with $\phi(h_n)$ real and $\phi(h_n) > 1 - 1/n$.

Show that h_n converges to a unit vector h, and that $\phi(h) = 1$.

Show that h is orthogonal to N, the null space of ϕ, and also that $H = N \oplus \mathrm{span}(h)$.

Show that $\phi(k) = \langle k, h\rangle$, for all $k \in H$.

A3/3 B3/2 **Functional Analysis**

(i) Suppose that (f_n) is a decreasing sequence of continuous real-valued functions on a compact metric space (X,d) which converges pointwise to 0. By considering sets of the form $B_n = \{x : f_n(x) < \epsilon\}$, for $\epsilon > 0$, or otherwise, show that f_n converges uniformly to 0.

Can the condition that (f_n) is decreasing be dropped? Can the condition that (X,d) is compact be dropped? Justify your answers.

(ii) Suppose that k is a positive integer. Define polynomials p_n recursively by

$$p_0 = 0, \qquad p_{n+1}(t) = p_n(t) + (t - p_n^k(t))/k.$$

Show that $0 \leqslant p_n(t) \leqslant p_{n+1}(t) \leqslant t^{1/k}$, for $t \in [0,1]$, and show that $p_n(t)$ converges to $t^{1/k}$ uniformly on $[0,1]$.

[*You may wish to use the identity* $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \ldots + b^{k-1})$.]

Suppose that A is a closed subalgebra of the algebra $C(X)$ of continuous real-valued functions on a compact metric space (X,d), equipped with the uniform norm, and suppose that A has the property that for each $x \in X$ there exists $a \in A$ with $a(x) \neq 0$. Show that there exists $h \in A$ such that $0 < h(x) \leqslant 1$ for all $x \in X$.

Show that $h^{1/k} \in A$ for each positive integer k, and show that A contains the constant functions.

A4/3 **Functional Analysis**

Define the *distribution function* Φ_f of a non-negative measurable function f on the interval $I = [0,1]$. Show that Φ_f is a decreasing non-negative function on $[0,\infty]$ which is continuous on the right.

Define the *Lebesgue integral* $\int_I f\,dm$. Show that $\int_I f\,dm = 0$ if and only if $f = 0$ almost everywhere.

Suppose that f is a non-negative Riemann integrable function on $[0,1]$. Show that there are an increasing sequence (g_n) and a decreasing sequence (h_n) of non-negative step functions with $g_n \leqslant f \leqslant h_n$ such that $\int_0^1 (h_n(x) - g_n(x))\,dx \to 0$.

Show that the functions $g = \lim_n g_n$ and $h = \lim_n h_n$ are equal almost everywhere, that f is measurable and that the Lebesgue integral $\int_I f\,dm$ is equal to the Riemann integral $\int_0^1 f(x)\,dx$.

Suppose that j is a Riemann integrable function on $[0,1]$ and that $j(x) > 0$ for all x. Show that $\int_0^1 j(x)\,dx > 0$.

A1/4 **Groups, Rings and Fields**

(i) What is a Sylow subgroup? State Sylow's Theorems.

Show that any group of order 33 is cyclic.

(ii) Prove the existence part of Sylow's Theorems.

[*You may use without proof any arithmetic results about binomial coefficients which you need.*]

Show that a group of order p^2q, where p and q are distinct primes, is not simple. Is it always abelian? Give a proof or a counterexample.

B1/3 **Groups, Rings and Fields**

State Sylow's Theorems. Prove the existence part of Sylow's Theorems.

Show that any group of order 33 is cyclic.

Show that a group of order p^2q, where p and q are distinct primes, is not simple. Is it always abelian? Give a proof or a counterexample.

A2/4 B2/3 **Groups, Rings and Fields**

(i) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(ii) What are the units in $\mathbb{Z}[i]$? What are the primes in $\mathbb{Z}[i]$? Justify your answers.

Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

A3/4 **Groups, Rings and Fields**

(i) What does it mean for a ring to be Noetherian? State Hilbert's Basis Theorem. Give an example of a Noetherian ring which is not a principal ideal domain.

(ii) Prove Hilbert's Basis Theorem.

Is it true that if the ring $R[X]$ is Noetherian, then so is R?

A4/4 **Groups, Rings and Fields**

Let F be a finite field. Show that there is a unique prime p for which F contains the field $\mathbb{F}_p$ of p elements. Prove that F contains p^n elements, for some $n \in \mathbb{N}$. Show that $x^{p^n} = x$ for all $x \in F$, and hence find a polynomial $f \in \mathbb{F}_p[X]$ such that F is the splitting field of f. Show that, up to isomorphism, F is the unique field $\mathbb{F}_{p^n}$ of size p^n.

[*Standard results about splitting fields may be assumed.*]

Prove that the mapping sending x to x^p is an automorphism of $\mathbb{F}_{p^n}$. Deduce that the Galois group Gal $(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic of order n. For which m is $\mathbb{F}_{p^m}$ a subfield of $\mathbb{F}_{p^n}$?

A1/5 B1/4 **Electromagnetism**

(i) Show that, in a region where there is no magnetic field and the charge density vanishes, the electric field can be expressed either as minus the gradient of a scalar potential ϕ or as the curl of a vector potential $\mathbf{A}$. Verify that the electric field derived from

$$\mathbf{A} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \wedge \mathbf{r}}{r^3}$$

is that of an electrostatic dipole with dipole moment $\mathbf{p}$.

[*You may assume the following identities:*

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \wedge (\nabla \wedge \mathbf{b}) + \mathbf{b} \wedge (\nabla \wedge \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a},$$

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a}.]$$

(ii) An infinite conducting cylinder of radius a is held at zero potential in the presence of a line charge parallel to the axis of the cylinder at distance $s_0 > a$, with charge density q per unit length. Show that the electric field outside the cylinder is equivalent to that produced by replacing the cylinder with suitably chosen image charges.

A2/5 **Electromagnetism**

(i) Show that the Lorentz force corresponds to a curvature force and the gradient of a magnetic pressure, and that it can be written as the divergence of a second rank tensor, the Maxwell stress tensor.

Consider the potential field $\mathbf{B}$ given by $\mathbf{B} = -\nabla\Phi$, where

$$\Phi(x, y) = \left(\frac{B_0}{k}\right) \cos kx \; e^{-ky},$$

referred to cartesian coordinates (x, y, z). Obtain the Maxwell stress tensor and verify that its divergence vanishes.

(ii) The magnetic field in a stellar atmosphere is maintained by steady currents and the Lorentz force vanishes. Show that there is a scalar field α such that $\nabla \wedge \mathbf{B} = \alpha\mathbf{B}$ and $\mathbf{B} \cdot \nabla\alpha = 0$. Show further that if α is constant, then $\nabla^2\mathbf{B} + \alpha^2\mathbf{B} = 0$. Obtain a solution in the form $\mathbf{B} = (B_1(z), B_2(z), 0)$; describe the structure of this field and sketch its variation in the z-direction.

A3/5 B3/3 **Electromagnetism**

(i) A plane electromagnetic wave in a vacuum has an electric field

$$\mathbf{E} = (E_1, E_2, 0)\cos(kz - \omega t),$$

referred to cartesian axes (x, y, z). Show that this wave is plane polarized and find the orientation of the plane of polarization. Obtain the corresponding plane polarized magnetic field and calculate the rate at which energy is transported by the wave.

(ii) Suppose instead that

$$\mathbf{E} = (E_1\cos(kz - \omega t),\ E_2\cos(kz - \omega t + \phi), 0),$$

with ϕ a constant, $0 < \phi < \pi$. Show that, if the axes are now rotated through an angle ψ so as to obtain an elliptically polarized wave with an electric field

$$\mathbf{E}' = (F_1\cos(kz - \omega t + \chi),\ F_2\sin(kz - \omega t + \chi), 0),$$

then

$$\tan 2\psi = \frac{2E_1E_2\cos\phi}{E_1^2 - E_2^2}.$$

Show also that if $E_1 = E_2 = E$ there is an elliptically polarized wave with

$$\mathbf{E}' = \sqrt{2}E\left(\cos(kz - \omega t + \tfrac{1}{2}\phi)\cos\tfrac{1}{2}\phi,\ \sin(kz - \omega t + \tfrac{1}{2}\phi)\sin\tfrac{1}{2}\phi, 0\right).$$

A4/5 **Electromagnetism**

State the four integral relationships between the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ and explain their physical significance. Derive Maxwell's equations from these relationships and show that $\mathbf{E}$ and $\mathbf{B}$ can be described by a scalar potential ϕ and a vector potential $\mathbf{A}$ which satisfy the inhomogeneous wave equations

$$\nabla^2\phi - \epsilon_0\mu_0\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0},$$

$$\nabla^2\mathbf{A} - \epsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}.$$

If the current $\mathbf{j}$ satisfies Ohm's law and the charge density $\rho = 0$, show that plane waves of the form

$$\mathbf{A} = A(z, t)e^{i\omega t}\hat{\mathbf{x}},$$

where $\hat{\mathbf{x}}$ is a unit vector in the x-direction of cartesian axes (x, y, z), are damped. Find an approximate expression for $A(z, t)$ when $\omega \ll \sigma/\epsilon_0$, where σ is the electrical conductivity.

A1/6 **Dynamics of Differential Equations**

(i) A system in $\mathbb{R}^2$ obeys the equations:

$$\dot{x} = x - x^5 - 2xy^4 - 2y^3(a - x^2) \ ,$$
$$\dot{y} = y - x^4 y - 2y^5 + x^3(a - x^2) \ ,$$

where a is a positive constant.

By considering the quantity $V = \alpha x^4 + \beta y^4$, where α and β are appropriately chosen, show that if $a > 1$ then there is a unique fixed point and a unique limit cycle. How many fixed points are there when $a < 1$?

(ii) Consider the second order system

$$\ddot{x} - (a - bx^2)\dot{x} + x - x^3 = 0 \ ,$$

where a, b are constants.

(a) Find the fixed points and determine their stability.

(b) Show that if the fixed point at the origin is unstable and $3a > b$ then there are no limit cycles.

[*You may find it helpful to use the Liénard coordinate* $z = \dot{x} - ax + \frac{1}{3}bx^3$.]

A2/6 B2/4 **Dynamics of Differential Equations**

(i) Define the terms *stable manifold* and *unstable manifold* of a hyperbolic fixed point $\mathbf{x}_0$ of a dynamical system. State carefully the stable manifold theorem.

Give an approximation, correct to fourth order in $|\mathbf{x}|$, for the stable and unstable manifolds of the origin for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x + x^2 - y^2 \\ -y + x^2 \end{pmatrix} .$$

(ii) State, without proof, the centre manifold theorem. Show that the fixed point at the origin of the system

$$\begin{aligned} \dot{x} &= y - x + ax^3 \ , \\ \dot{y} &= rx - y - zy \ , \\ \dot{z} &= -z + xy \ , \end{aligned}$$

where a is a constant, is non-hyperbolic at $r = 1$.

Using new coordinates $v = x + y\,,\ w = x - y$, find the centre manifold in the form

$$w = \alpha v^3 + \dots\,, \quad z = \beta v^2 + \gamma v^4 + \dots$$

for constants α, β, γ to be determined. Hence find the evolution equation on the centre manifold in the form

$$\dot{v} = \frac{1}{8}(a-1)v^3 + \left(\frac{(3a+1)(a+1)}{128} + \frac{(a-1)}{32} \right) v^5 + \dots\ .$$

Ignoring higher order terms, give conditions on a that guarantee that the origin is asymptotically stable.

A3/6 B3/4 **Dynamics of Differential Equations**

(i) Define the Floquet multiplier and Liapunov exponent for a periodic orbit $\hat{\mathbf{x}}(t)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Show that one multiplier is always unity, and that the other is given by

$$\exp\left(\int_0^T \nabla\cdot \mathbf{f}(\hat{\mathbf{x}}(t))dt\right) \,, \qquad (*)$$

where T is the period of the orbit.

The Van der Pol oscillator $\ddot{x} + \epsilon\dot{x}(x^2-1) + x = 0\,,\ 0 < \epsilon \ll 1$ has a limit cycle $\hat{x}(t) \approx 2\sin t$. Show using $(*)$ that this orbit is stable.

(ii) Show, by considering the normal form for a Hopf bifurcation from a fixed point $\mathbf{x}_0(\mu)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mu)$, that in some neighbourhood of the bifurcation the periodic orbit is stable when it exists in the range of μ for which $\mathbf{x}_0$ is unstable, and unstable in the opposite case.

Now consider the system

$$\left.\begin{aligned} \dot{x} &= x(1-y) + \mu x \\ \dot{y} &= y(x-1) - \mu x \end{aligned}\right\} \qquad x > 0\ .$$

Show that the fixed point $(1+\mu\,,\ 1+\mu)$ has a Hopf bifurcation when $\mu = 0$, and is unstable (stable) when $\mu > 0$ ($\mu < 0$).

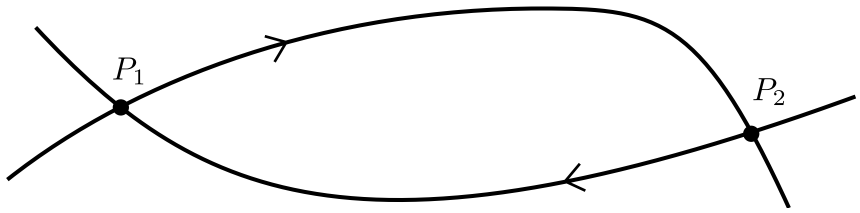
Suppose that a periodic orbit exists in $\mu > 0$. Show without solving for the orbit that the result of part (i) shows that such an orbit is unstable. Define a similar result for $\mu < 0$.

What do you conclude about the existence of periodic orbits when $\mu \neq 0$? Check your answer by applying Dulac's criterion to the system, using the weighting $\rho = e^{-(x+y)}$.

A4/6 **Dynamics of Differential Equations**

Define the terms *homoclinic orbit*, *heteroclinic orbit* and *heteroclinic loop*. In the case of a dynamical system that possesses a homoclinic orbit, explain, without detailed calculation, how to calculate its stability.

A second order dynamical system depends on two parameters μ_1 and μ_2. When $\mu_1 = \mu_2 = 0$ there is a heteroclinic loop between the points P_1, P_2 as in the diagram.



When μ_1, μ_2 are small there are trajectories that pass close to the fixed points P_1, P_2:

By adapting the method used above for trajectories near homoclinic orbits, show that the distances y_n, y_{n+1} to the stable manifold at P_1 on successive returns are related to z_n, z_{n+1}, the corresponding distances near P_2, by coupled equations of the form

$$\left.\begin{aligned} z_n &= (y_n)^{\gamma_1} + \mu_1, \\ y_{n+1} &= (z_n)^{\gamma_2} + \mu_2, \end{aligned}\right\}$$

where any arbitrary constants have been removed by rescaling, and γ_1, γ_2 depend on conditions near P_1, P_2. Show from these equations that there is a stable heteroclinic orbit ($\mu_1 = \mu_2 = 0$) if $\gamma_1\gamma_2 > 1$. Show also that in the marginal situation $\gamma_1 = 2$, $\gamma_2 = \frac{1}{2}$ there can be a stable fixed point for small positive y, z if $\mu_2 < 0$, $\mu_2^2 < \mu_1$. Explain carefully the form of the orbit of the original dynamical system represented by the solution of the above map when $\mu_2^2 = \mu_1$.

A1/7 B1/12 **Logic, Computation and Set Theory**

(i) State the Knaster-Tarski fixed point theorem. Use it to prove the Cantor-Bernstein Theorem; that is, if there exist injections $A \to B$ and $B \to A$ for two sets A and B then there exists a bijection $A \to B$.

(ii) Let A be an arbitrary set and suppose given a subset R of $PA \times A$. We define a subset $B \subseteq A$ to be R-*closed* just if whenever $(S, a) \in R$ and $S \subseteq B$ then $a \in B$. Show that the set of all R-closed subsets of A is a complete poset in the inclusion ordering.

Now assume that A is itself equipped with a partial ordering $\leqslant$.

(a) Suppose R satisfies the condition that if $b \geqslant a \in A$ then $(\{b\}, a) \in R$. Show that if B is R-closed then $c \leqslant b \in B$ implies $c \in B$.

(b) Suppose that R satisfies the following condition. Whenever $(S, a) \in R$ and $b \leqslant a$ then there exists $T \subseteq A$ such that $(T, b) \in R$, and for every $t \in T$ we have (i) $(\{b\}, t) \in R$, and (ii) $t \leqslant s$ for some $s \in S$. Let B and C be R-closed subsets of A. Show that the set

$$[B \to C] = \{a \in A \,|\, \forall b \leqslant a\, (b \in B \Rightarrow b \in C)\}$$

is R-closed.

B2/11 **Logic, Computation and Set Theory**

Explain what is meant by a *structure* for a first-order language and by a *model* for a first-order theory. If T is a first-order theory whose axioms are all universal sentences (that is, sentences of the form $(\forall x_1 \ldots x_n)p$ where p is quantifier-free), show that every substructure of a T-model is a T-model.

Now let T be an arbitrary first-order theory in a language L, and let M be an L-structure satisfying all the universal sentences which are derivable from the axioms of T. If p is a quantifier-free formula (with free variables $x_1, \ldots, x_n$ say) whose interpretation $[p]_M$ is a nonempty subset of M^n, show that $T \cup \{(\exists x_1 \cdots x_n)p\}$ is consistent.

Let L' be the language obtained from L by adjoining a new constant $\widehat{a}$ for each element a of M, and let

$$T' = T \cup \{p[\widehat{a}_1, \ldots, \widehat{a}_n / x_1, \ldots, x_n] \,|\, p \text{ is quantifier-free and } (a_1, \ldots, a_n) \in [p]_M\}.$$

Show that T' has a model. [*You may use the Completeness and Compactness Theorems.*] Explain briefly why any such model contains a substructure isomorphic to M.

A3/8 B3/11 **Logic, Computation and Set Theory**

(i) Explain briefly what is meant by the terms *register machine* and *computable function*.

Let u be the universal computable function $u(m,n) = f_m(n)$ and s a total computable function with $f_{s(m,n)}(k) = f_m(n,k)$. Here $f_m(n)$ and $f_m(n,k)$ are the unary and binary functions computed by the m-th register machine program P_m. Suppose $h : \mathbb{N} \to \mathbb{N}$ is a total computable function. By considering the function

$$g(m,n) = u(h(s(m,m)),n)$$

show that there is a number a such that $f_a = f_{h(a)}$.

(ii) Let P be the set of all partial functions $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Consider the mapping $\Phi : P \to P$ defined by

$$\Phi(g)(m,n) = \begin{cases} n+1 & \text{if } m = 0, \\ g(m-1,1) & \text{if } m > 0,\ n = 0 \text{ and } g(m-1,1) \text{ defined}, \\ g(m-1,g(m,n-1)) & \text{if } mn > 0 \text{ and } g(m-1,g(m,n-1)) \text{ defined}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(a) Show that any fixed point of Φ is a total function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Deduce that Φ has a unique fixed point.
[*The Bourbaki-Witt Theorem may be assumed if stated precisely.*]

(b) It follows from standard closure properties of the computable functions that there is a computable function ψ such that

$$\psi(p,m,n) = \Phi(f_p)(m,n).$$

Assuming this, show that there is a total computable function h such that

$$\Phi(f_p) = f_{h(p)} \text{ for all } p.$$

Deduce that the fixed point of Φ is computable.

A4/8 **Logic, Computation and Set Theory**

Let P be a set of primitive propositions. Let $L(P)$ denote the set of all compound propositions over P, and let S be a subset of $L(P)$. Consider the relation $\preceq_S$ on $L(P)$ defined by

$$s \preceq_S t \text{ if and only if } S \cup \{s\} \vdash t.$$

Prove that $\preceq_S$ is reflexive and transitive. Deduce that if we define $\approx_S$ by ($s \approx_S t$ if and only if $s \preceq_S t$ and $t \preceq_S s$), then $\approx_S$ is an equivalence relation and the quotient $B_S = L(P)/\approx_S$ is partially ordered by the relation $\leqslant_S$ induced by $\preccurlyeq_S$ (that is, $[s] \leqslant_S [t]$ if and only if $s \preccurlyeq_S t$, where square brackets denote equivalence classes).

Assuming the result that B_S is a Boolean algebra with lattice operations induced by the logical operations on $L(P)$ (that is, $[s] \wedge [t] = [s \wedge t]$, etc.), show that there is a bijection between the following two sets:
(a) The set of lattice homomorphisms $B_S \to \{0,1\}$.
(b) The set of models of the propositional theory S.

Deduce that the completeness theorem for propositional logic is equivalent to the assertion that, for any Boolean algebra B with more than one element, there exists a homomorphism $B \to \{0,1\}$.

[*You may assume the result that the completeness theorem implies the compactness theorem.*]

B4/10 **Logic, Computation and Set Theory**

Explain what is meant by a *well-ordering* of a set.

Without assuming Zorn's Lemma, show that the power-set of any well-ordered set can be given a total (linear) ordering.

By a *selection function* for a set A, we mean a function $f : PA \to PA$ such that $f(B) \subset B$ for all $B \subset A$, $f(B) \neq \emptyset$ for all $B \neq \emptyset$, and $f(B) \neq B$ if B has more than one element. Suppose given a selection function f. Given a mapping $g : \alpha \to [0,1]$ for some ordinal α, we define a subset $B(f,g) \subset A$ recursively as follows:

$$B(f,g) = A \quad \text{if } \alpha = 0,$$
$$B(f,g) = f(B(f,g|_\beta)) \quad \text{if } \alpha = \beta^+ \text{ and } g(\beta) = 1,$$
$$B(f,g) = B(f,g|_\beta) \backslash f(B(f,g|_\beta)) \quad \text{if } \alpha = \beta^+ \text{and } g(\beta) = 0,$$
$$B(f,g) = \bigcap \{B(f,g|_\beta) \mid \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Show that, for any $x \in A$ and any ordinal α, there exists a function g with domain α such that $x \in B(f,g)$.
[*It may help to observe that g is uniquely determined by x and α, though you need not show this explicitly.*]

Show also that there exists α such that, for every g with domain α, $B(f,g)$ is either empty or a singleton.

Deduce that the assertion 'Every set has a selection function' implies that every set can be totally ordered.
[*Hartogs' Lemma may be assumed, provided you state it precisely.*]

A1/12 B1/15 **Principles of Statistics**

(i) Explain in detail the *minimax* and *Bayes* principles of decision theory.

Show that if $d(X)$ is a Bayes decision rule for a prior density $\pi(\theta)$ and has constant risk function, then $d(X)$ is minimax.

(ii) Let $X_1, \ldots, X_p$ be independent random variables, with $X_i \sim N(\mu_i, 1)$, $i = 1, \ldots, p$.

Consider estimating $\mu = (\mu_1, \ldots, \mu_p)^T$ by $d = (d_1, \ldots, d_p)^T$, with loss function

$$L(\mu, d) = \sum_{i=1}^{p} (\mu_i - d_i)^2 \ .$$

What is the risk function of $X = (X_1, \ldots, X_p)^T$?

Consider the class of estimators of μ of the form

$$d^a(X) = \left(1 - \frac{a}{X^T X}\right) X \ ,$$

indexed by $a \geqslant 0$. Find the risk function of $d^a(X)$ in terms of $E\left(1/X^T X\right)$, which you should not attempt to evaluate, and deduce that X is inadmissible. What is the optimal value of a?

[*You may assume Stein's Lemma, that for suitably behaved real-valued functions h,*

$$E\left\{(X_i - \mu_i) h(X)\right\} = E\left\{\frac{\partial h(X)}{\partial X_i}\right\} . \quad]$$

A2/11 B2/16 **Principles of Statistics**

(i) Let X be a random variable with density function $f(x;\theta)$. Consider testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1$.

What is the form of the optimal size α classical hypothesis test?

Compare the form of the test with the Bayesian test based on the Bayes factor, *and* with the Bayes decision rule under the 0-1 loss function, under which a loss of 1 is incurred for an incorrect decision and a loss of 0 is incurred for a correct decision.

(ii) What does it mean to say that a family of densities $\{f(x;\theta), \theta \in \Theta\}$ with real scalar parameter θ is of *monotone likelihood ratio*?

Suppose X has a distribution from a family which is of monotone likelihood ratio with respect to a statistic $t(X)$ and that it is required to test $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$.

State, without proof, a theorem which establishes the existence of a uniformly most powerful test and describe in detail the form of the test.

Let $X_1, \ldots, X_n$ be independent, identically distributed $U(0, \theta)$, $\theta > 0$. Find a uniformly most powerful size α test of $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$, and find its power function. Show that we may construct a different, randomised, size α test with the same power function for $\theta \geqslant \theta_0$.

A3/12 B3/15 **Principles of Statistics**

(i) Describe in detail how to perform the Wald, score and likelihood ratio tests of a *simple* null hypothesis $H_0 : \theta = \theta_0$ given a random sample $X_1, \ldots, X_n$ from a regular one-parameter density $f(x; \theta)$. In each case you should specify the asymptotic null distribution of the test statistic.

(ii) Let $X_1, \ldots, X_n$ be an independent, identically distributed sample from a distribution F, and let $\hat{\theta}(X_1, \ldots, X_n)$ be an estimator of a parameter θ of F.

Explain what is meant by: (a) the *empirical distribution function* of the sample; (b) the *bootstrap estimator* of the *bias* of $\hat{\theta}$, based on the empirical distribution function. Explain how a bootstrap estimator of the *distribution function* of $\hat{\theta} - \theta$ may be used to construct an approximate $1 - \alpha$ confidence interval for θ.

Suppose the parameter of interest is $\theta = \mu^2$, where μ is the mean of F, and the estimator is $\hat{\theta} = \bar{X}^2$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ is the sample mean.

Derive an *explicit* expression for the bootstrap estimator of the bias of $\hat{\theta}$ and show that it is biased as an estimator of the true bias of $\hat{\theta}$.

Let $\hat{\theta}_i$ be the value of the estimator $\hat{\theta}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ computed from the sample of size $n-1$ obtained by deleting X_i and let $\hat{\theta}_{.} = n^{-1} \sum_{i=1}^{n} \hat{\theta}_i$. The *jackknife* estimator of the bias of $\hat{\theta}$ is

$$b_J = (n-1)\ (\hat{\theta}_{.} - \hat{\theta})\ .$$

Derive the jackknife estimator b_J for the case $\hat{\theta} = \bar{X}^2$, and show that, as an estimator of the true bias of $\hat{\theta}$, it is unbiased.

A4/13 B4/15 **Principles of Statistics**

(a) Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from a one-parameter distribution with density function

$$f(x; \theta) = h(x) g(\theta) \exp\{\theta t(x)\}\ ,\ x \in \mathbb{R}.$$

Explain in detail how you would test

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta \neq \theta_0\ .$$

What is the general form of a conjugate prior density for θ in a Bayesian analysis of this distribution?

(b) Let Y_1, Y_2 be independent Poisson random variables, with means $(1-\psi)\lambda$ and $\psi\lambda$ respectively, with λ *known.*

Explain why the Conditionality Principle leads to inference about ψ being drawn from the conditional distribution of Y_2, given $Y_1 + Y_2$. What is this conditional distribution?

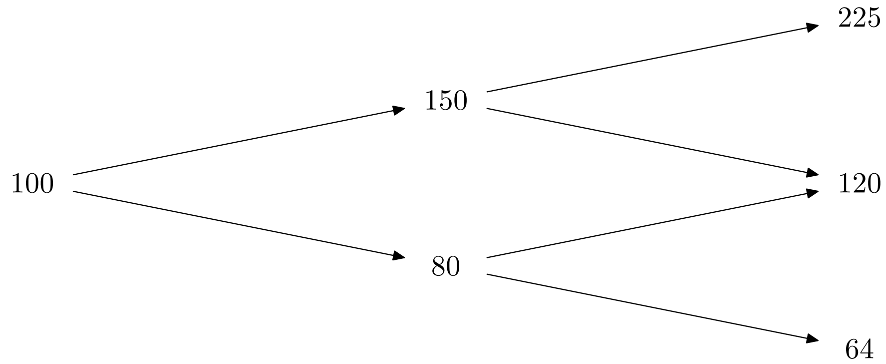
(c) Suppose Y_1, Y_2 have distributions as in (b), but that λ is now *unknown.*

Explain in detail how you would test $H_0 : \psi = \psi_0$ against $H_1 : \psi \neq \psi_0$, and describe the optimality properties of your test.

[*Any general results you use should be stated clearly, but need not be proved.*]

A1/11 B1/16 **Stochastic Financial Models**

(i) The prices, S_i, of a stock in a binomial model at times $i = 0, 1, 2$ are represented by the following binomial tree.



The fixed interest rate per period is $1/5$ and the probability that the stock price increases in a period is $1/3$. Find the price at time 0 of a European call option with strike price 78 and expiry time 2.

Explain briefly the ideas underlying your calculations.

(ii) Consider an investor in a one-period model who may invest in s assets, all of which are risky, with a random return vector $\boldsymbol{R}$ having mean $\mathbb{E}\boldsymbol{R} = \boldsymbol{r}$ and positive-definite covariance matrix $\boldsymbol{V}$; assume that not all the assets have the same expected return. Show that any minimum-variance portfolio is equivalent to the investor dividing his wealth between two portfolios, the global minimum-variance portfolio and the diversified portfolio, both of which should be specified clearly in terms of $\boldsymbol{r}$ and $\boldsymbol{V}$.

Now suppose that $\boldsymbol{R} = (R_1, R_2, \ldots, R_s)^{\top}$ where $R_1, R_2, \ldots, R_s$ are independent random variables with R_i having the exponential distribution with probability density function $\lambda_i e^{-\lambda_i x}$, $x \geqslant 0$, where $\lambda_i > 0$, $1 \leqslant i \leqslant s$. Determine the global minimum-variance portfolio and the diversified portfolio explicitly.

Consider further the situation when the investor has the utility function $u(x) = 1 - e^{-x}$, where x denotes his wealth. Suppose that he acts to maximize the expected utility of his final wealth, and that his initial wealth is $w > 0$. Show that he now divides his wealth between the diversified portfolio and the *uniform* portfolio, in which wealth is apportioned equally between the assets, and determine the amounts that he invests in each.

A3/11 B3/16 **Stochastic Financial Models**

(i) Explain briefly what it means to say that a stochastic process $\{W_t,\ t \geqslant 0\}$ is a standard Brownian motion.

Let $\{W_t,\ t \geqslant 0\}$ be a standard Brownian motion and let a, b be real numbers. What condition must a and b satisfy to ensure that the process e^{aW_t+bt} is a martingale? Justify your answer carefully.

(ii) At the beginning of each of the years $r = 0, 1, \ldots, n-1$ an investor has income X_r, of which he invests a proportion ρ_r, $0 \leqslant \rho_r \leqslant 1$, and consumes the rest during the year. His income at the beginning of the next year is

$$X_{r+1} = X_r + \rho_r X_r W_r,$$

where $W_0, \ldots, W_{n-1}$ are independent positive random variables with finite means and $X_0 \geqslant 0$ is a constant. He decides on ρ_r after he has observed both X_r and W_r at the beginning of year r, but at that time he does not have any knowledge of the value of W_s, for any $s > r$. The investor retires in year n and consumes his entire income during that year. He wishes to determine the investment policy that maximizes his expected total consumption

$$\mathbb{E}\left[\sum_{r=0}^{n-1}(1-\rho_r)X_r + X_n\right].$$

Prove that the optimal policy may be expressed in terms of the numbers $b_0, b_1, \ldots,$ b_n where $b_n = 1$, $b_r = b_{r+1} + \mathbb{E}\max\left(b_{r+1}W_r, 1\right)$, for $r < n$, and determine the optimal expected total consumption.

A4/12 B4/16 **Stochastic Financial Models**

Write an essay on the Black–Scholes formula for the price of a European call option on a stock. Your account should include a derivation of the formula and a careful analysis of its dependence on the parameters of the model.

A2/13 B2/21 **Foundations of Quantum Mechanics**

(i) A Hamiltonian H_0 has energy eigenvalues E_r and corresponding non-degenerate eigenstates $|r\rangle$. Show that under a small change in the Hamiltonian δH,

$$\delta|r\rangle = \sum_{s \neq r} \frac{\langle s|\delta H|r\rangle}{E_r - E_s}|s\rangle,$$

and derive the related formula for the change in the energy eigenvalue E_r to first and second order in δH.

(ii) The Hamiltonian for a particle moving in one dimension is $H = H_0 + \lambda H'$, where $H_0 = p^2/2m + V(x)$, $H' = p/m$ and λ is small. Show that

$$\frac{i}{\hbar}[H_0, x] = H'$$

and hence that

$$\delta E_r = -\lambda^2 \frac{i}{\hbar}\langle r|H'x|r\rangle = \lambda^2 \frac{i}{\hbar}\langle r|xH'|r\rangle$$

to second order in λ.

Deduce that δE_r is independent of the particular state $|r\rangle$ and explain why this change in energy is exact to all orders in λ.

A3/13 B3/21 **Foundations of Quantum Mechanics**

(i) Two particles with angular momenta j_1, j_2 and basis states $|j_1\ m_1\rangle, |j_2\ m_2\rangle$ are combined to give total angular momentum j and basis states $|j\ m\rangle$. State the possible values of j, m and show how a state with $j = m = j_1 + j_2$ can be constructed. Briefly describe, for a general allowed value of j, what the Clebsch-Gordan coefficients are.

(ii) If the angular momenta j_1 and j_2 are both 1 show that the combined state $|2\ 0\rangle$ is

$$|2\ 0\rangle = \sqrt{\frac{1}{6}}\Big(|1\ 1\rangle|1\ {-1}\rangle + |1\ {-1}\rangle|1\ 1\rangle\Big) + \sqrt{\frac{2}{3}}|1\ 0\rangle|1\ 0\rangle.$$

Determine the corresponding expressions for the combined states $|1\ 0\rangle$ and $|0\ 0\rangle$, assuming that they are respectively antisymmetric and symmetric under interchange of the two particles.

If the combined system is in state $|0\ 0\rangle$ what is the probability that measurements of the z-component of angular momentum for either constituent particle will give the value of 1?

[*Hint:* $J_\pm|j\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m \pm 1\rangle$.]

A4/15 B4/22 **Foundations of Quantum Mechanics**

Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have (a) spin zero, (b) spin 1/2.

The stationary Schrödinger equation for one particle in the potential

$$-\frac{2e^2}{4\pi\epsilon_0 r}$$

has normalized, spherically symmetric, real wave functions $\psi_n(\mathbf{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. What are the consequences of the Pauli exclusion principle for the ground state of the helium atom? Assuming that wavefunctions which are not spherically symmetric can be ignored, what are the states of the first excited energy level of the helium atom?
[*You may assume here that the electrons are non-interacting.*]

Show that, taking into account the interaction between the two electrons, the estimate for the energy of the ground state of the helium atom is

$$2E_0 + \frac{e^2}{4\pi\epsilon_0}\int \frac{d^3\mathbf{r}_1\, d^3\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\psi_0^2(\mathbf{r}_1)\psi_0^2(\mathbf{r}_2).$$

A1/15 B1/24 **General Relativity**

(i) Given a covariant vector field V_a, define the curvature tensor $R^a{}_{bcd}$ by

$$V_{a;bc} - V_{a;cb} = V_e R^e{}_{abc}. \qquad (*)$$

Express $R^e{}_{abc}$ in terms of the Christoffel symbols and their derivatives. Show that

$$R^e{}_{abc} = -R^e{}_{acb}.$$

Further, by setting $V_a = \partial\phi/\partial x^a$, deduce that

$$R^e{}_{abc} + R^e{}_{cab} + R^e{}_{bca} = 0.$$

(ii) Write down an expression similar to $(*)$ given in Part (i) for the quantity

$$g_{ab;cd} - g_{ab;dc}$$

and hence show that

$$R_{eabc} = -R_{aebc}.$$

Define the Ricci tensor, show that it is symmetric and write down the contracted Bianchi identities.

In certain spacetimes of dimension $n \geq 2$, R_{abcd} takes the form

$$R_{abcd} = K(x^e)[g_{ac}g_{bd} - g_{ad}g_{bc}].$$

Obtain the Ricci tensor and Ricci scalar. Deduce that K is a constant in such spacetimes if the dimension n is greater than 2.

A2/15 B2/23 **General Relativity**

(i) Consider the line element describing the interior of a star,

$$ds^2 = e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{2\gamma(r)}dt^2 \ ,$$

defined for $0 \le r \le r_0$ by

$$e^{-2\alpha(r)} = 1 - Ar^2$$

and

$$e^{\gamma(r)} = \frac{3}{2}e^{-\alpha_0} - \frac{1}{2}e^{-\alpha(r)}.$$

Here $A = 2M/r_0^3$, M is the mass of the star, and α_0 is defined to be $\alpha(r_0)$.

The star is made of a perfect fluid with energy-momentum tensor

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab}.$$

Here u^a is the 4-velocity of the fluid which is at rest, the density ρ is constant throughout the star ($0 \le r \le r_0$) and the pressure $p = p(r)$ depends only on the radial coordinate. Write down the Einstein field equations and show that (in geometrical units with $G = c = 1$) they may equivalently be written as

$$R_{ab} = 8\pi(p+\rho)u_a u_b + 4\pi(p-\rho)g_{ab}.$$

(ii) Using the formulae below, or otherwise, show that for $0 \le r \le r_0$ one has

$$\rho = \frac{3A}{8\pi}, \qquad p(r) = \frac{3A}{8\pi}\left(\frac{e^{-\alpha(r)} - e^{-\alpha_0}}{3e^{-\alpha_0} - e^{-\alpha(r)}}\right).$$

[*The non-zero components of the Ricci tensor are:*

$$R_{11} = -\gamma'' + \alpha'\gamma' - \gamma'^2 + \frac{2\alpha'}{r}, \quad R_{22} = e^{-2\alpha}[(\alpha'-\gamma')r - 1] + 1,$$

$$R_{33} = \sin^2\theta R_{22}, \quad R_{44} = e^{2\gamma-2\alpha}[\gamma'' - \alpha'\gamma' + \gamma'^2 + \frac{2\gamma'}{r}].$$

Note that

$$\alpha' = Are^{2\alpha}, \ \ \gamma' = \frac{1}{2}Are^{\alpha-\gamma}, \ \ \gamma'' = \frac{1}{2}Ae^{\alpha-\gamma} + \frac{1}{2}A^2r^2e^{3\alpha-\gamma} - \frac{1}{4}A^2r^2e^{2\alpha-2\gamma}. \quad]$$

A4/17 B4/25 **General Relativity**

With respect to the Schwarzschild coordinates (r, θ, ϕ, t), the Schwarzschild geometry is given by

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \left(1 - \frac{r_s}{r}\right)dt^2,$$

where $r_s = 2M$ is the Schwarzschild radius and M is the Schwarzschild mass. Show that, by a suitable choice of (θ, ϕ), the general geodesic can regarded as moving in the equatorial plane $\theta = \pi/2$. Obtain the equations governing timelike and null geodesics in terms of $u(\phi)$, where $u = 1/r$.

Discuss light bending and perihelion precession in the solar system.

A1/20 B1/20 **Numerical Analysis**

(i) Let A be an $n \times n$ symmetric real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$, where $\|\mathbf{v}_l\| = 1$. Given $\mathbf{x}^{(0)} \in \mathbb{R}^n$, $\|\mathbf{x}^{(0)}\| = 1$, the sequence $\mathbf{x}^{(k)}$ is generated in the following manner. We set

$$\mu = \mathbf{x}^{(k)\,T} A \mathbf{x}^{(k)},$$

$$\mathbf{y} = (A - \mu I)^{-1} \mathbf{x}^{(k)},$$

$$\mathbf{x}^{(k+1)} = \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

Show that if

$$\mathbf{x}^{(k)} = c^{-1} \left(\mathbf{v}_1 + \alpha \sum_{l=2}^{n} d_l \mathbf{v}_l \right),$$

where α is a real scalar and c is chosen so that $\|\mathbf{x}^{(k)}\| = 1$, then

$$\mu = c^{-2} \left(\lambda_1 + \alpha^2 \sum_{j=2}^{n} \lambda_j d_j^2 \right).$$

Give an explicit expression for c.

(ii) Use the above result to prove that, if $|\alpha|$ is small,

$$\mathbf{x}^{(k+1)} = \tilde{c}^{-1} \left(\mathbf{v}_1 + \alpha^3 \sum_{l=2}^{n} \tilde{d}_l \mathbf{v}_l \right) + O(\alpha^4)$$

and obtain the numbers $\tilde{c}$ and $\tilde{d}_2, \ldots, \tilde{d}_n$.

A2/19 B2/19 **Numerical Analysis**

(i)

Given the finite-difference method

$$\sum_{k=-r}^{s} \alpha_k u_{m+k}^{n+1} = \sum_{k=-r}^{s} \beta_k u_{m+k}^{n}, \quad m, n \in \mathbb{Z},\ n \geqslant 0,$$

define

$$H(z) = \frac{\sum\limits_{k=-r}^{s} \beta_k z^k}{\sum\limits_{k=-r}^{s} \alpha_k z^k}.$$

Prove that this method is stable if and only if

$$|H(e^{i\theta})| \leqslant 1, \quad -\pi \leqslant \theta \leqslant \pi.$$

[*You may quote without proof known properties of the Fourier transform.*]

(ii) Find the range of the parameter μ such that the method

$$(1-2\mu)u_{m-1}^{n+1} + 4\mu u_m^{n+1} + (1-2\mu)u_{m+1}^{n+1} = u_{m-1}^{n} + u_{m+1}^{n}$$

is stable. Supposing that this method is used to solve the diffusion equation for $u(x,t)$, determine the order of magnitude of the local error as a power of Δx.

A3/19 B3/20 **Numerical Analysis**

(i) Determine the order of the multistep method

$$\mathbf{y}_{n+2} - (1+\alpha)\mathbf{y}_{n+1} + \alpha\mathbf{y}_n = h[\tfrac{1}{12}(5+\alpha)\mathbf{f}_{n+2} + \tfrac{2}{3}(1-\alpha)\mathbf{f}_{n+1} - \tfrac{1}{12}(1+5\alpha)\mathbf{f}_n]$$

for the solution of ordinary differential equations for different choices of α in the range $-1 \leqslant \alpha \leqslant 1$.

(ii) Prove that no such choice of α results in a method whose linear stability domain includes the interval $(-\infty,\ 0)$.

A4/22 B4/20 **Numerical Analysis**

Write an essay on the method of conjugate gradients. You should describe the algorithm, present an analysis of its properties and discuss its advantages.

[*Any theorems quoted should be stated precisely but need not be proved.*]

B1/5 **Combinatorics**

Prove that every graph G on $n \geqslant 3$ vertices with minimal degree $\delta(G) \geqslant \frac{n}{2}$ is Hamiltonian. For each $n \geqslant 3$, give an example to show that this result does not remain true if we weaken the condition to $\delta(G) \geqslant \frac{n}{2} - 1$ (n even) or $\delta(G) \geqslant \frac{n-1}{2}$ (n odd).

Now let G be a connected graph (with at least 2 vertices) without a cutvertex. Does G Hamiltonian imply G Eulerian? Does G Eulerian imply G Hamiltonian? Justify your answers.

B2/5 **Combinatorics**

State and prove the local LYM inequality. Explain carefully when equality holds.

Define the colex order and state the Kruskal-Katona theorem. Deduce that, if n and r are fixed positive integers with $1 \leqslant r \leqslant n-1$, then for every $1 \leqslant m \leqslant \binom{n}{r}$ we have

$$\min\{|\partial\mathcal{A}| : \mathcal{A} \subset [n]^{(r)}, |\mathcal{A}| = m\} = \min\{|\partial\mathcal{A}| : \mathcal{A} \subset [n+1]^{(r)}, |\mathcal{A}| = m\}.$$

By a suitable choice of n, r and m, show that this result does not remain true if we replace the lower shadow ∂A with the upper shadow $\partial^+\mathcal{A}$.

B4/1 **Combinatorics**

Write an essay on Ramsey theory. You should include the finite and infinite versions of Ramsey's theorem, together with a discussion of upper and lower bounds in the finite case.

[*You may restrict your attention to colourings by just 2 colours.*]

B1/6 **Representation Theory**

Construct the character table of the symmetric group S_5, explaining the steps in your construction.

Use the character table to show that the alternating group A_5 is the only non-trivial normal subgroup of S_5.

B2/6 **Representation Theory**

State and prove Schur's Lemma. Deduce that the centre of a finite group G with a faithful irreducible complex representation ρ is cyclic and that $Z(\rho(G))$ consists of scalar transformations.

Let G be the subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle.$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of dimension 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible complex representations.

Show finally that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

B3/5 **Representation Theory**

Let G be a finite group acting on a finite set X. Define the permutation representation $(\rho, \mathbb{C}[X])$ of G and compute its character π_X. Prove that $\langle \pi_X, 1_G \rangle_G$ equals the number of orbits of G on X. If G acts also on the finite set Y, with character π_Y, show that $\langle \pi_X, \pi_Y \rangle_G$ equals the number of orbits of G on $X \times Y$.

Now let G be the symmetric group S_n acting naturally on the set $X = \{1, \ldots, n\}$, and let X_r be the set of all r-element subsets of X. Let π_r be the permutation character of G on X_r. Prove that

$$\langle \pi_k, \pi_\ell \rangle_G = \ell + 1 \text{ for } 0 \leqslant \ell \leqslant k \leqslant n/2.$$

Deduce that the class functions

$$\chi_r = \pi_r - \pi_{r-1}$$

are irreducible characters of S_n, for $1 \leqslant r \leqslant n/2$.

B4/2 **Representation Theory**

Write an essay on the representation theory of SU_2.

B1/7 **Galois Theory**

Let $F \subset K$ be a finite extension of fields and let G be the group of F-automorphisms of K. State a result relating the order of G to the degree $[K : F]$.

Now let $K = k(X_1, \ldots, X_4)$ be the field of rational functions in four variables over a field k and let $F = k(s_1, \ldots, s_4)$ where $s_1, \ldots, s_4$ are the elementary symmetric polynomials in $k[X_1, \ldots, X_4]$. Show that the degree $[K : F] \leqslant 4!$ and deduce that F is the fixed field of the natural action of the symmetric group S_4 on K.

Show that $X_1X_3 + X_2X_4$ has a cubic minimum polynomial over F. Let $G = \langle \sigma, \tau \rangle \subset S_4$ be the dihedral group generated by the permutations $\sigma = (1234)$ and $\tau = (13)$. Show that the fixed field of G is $F(X_1X_3 + X_2X_4)$. Find the fixed field of the subgroup $H = \langle \sigma^2, \tau \rangle$.

B3/6 **Galois Theory**

Show that the polynomial $f(X) = X^5 + 27X + 16$ has no rational roots. Show that the splitting field of f over the finite field $\mathbb{F}_3$ is an extension of degree 4. Hence deduce that f is irreducible over the rationals. Prove that f has precisely two (non-multiple) roots over the finite field $\mathbb{F}_7$. Find the Galois group of f over the rationals.

[*You may assume any general results you need including the fact that* A_5 *is the only index 2 subgroup of* S_5.]

B4/3 **Galois Theory**

Suppose K, L are fields and $\sigma_1, \ldots, \sigma_m$ are distinct embeddings of K into L. Prove that there do not exist elements $\lambda_1, \ldots, \lambda_m$ of L (not all zero) such that $\lambda_1\sigma_1(x) + \ldots + \lambda_m\sigma_m(x) = 0$ for all $x \in K$. Deduce that if K/k is a finite extension of fields, and $\sigma_1, \ldots, \sigma_m$ are distinct k-automorphisms of K, then $m \leqslant [K : k]$.

Suppose now that K is a Galois extension of k with Galois group cyclic of order n, where n is not divisible by the characteristic. If k contains a primitive nth root of unity, prove that K is a radical extension of k. Explain briefly the relevance of this result to the problem of solubility of cubics by radicals.

B1/8 Differentiable Manifolds

What is meant by a "bump function" on $\mathbb{R}^n$? If U is an open subset of a manifold M, prove that there is a bump function on M with support contained in U.

Prove the following.

(i) Given an open covering $\mathcal{U}$ of a compact manifold M, there is a partition of unity on M subordinate to $\mathcal{U}$.

(ii) Every compact manifold may be embedded in some Euclidean space.

B2/7 Differentiable Manifolds

State, giving your reasons, whether the following are true or false.

(a) Diffeomorphic connected manifolds must have the same dimension.

(b) Every non-zero vector bundle has a nowhere-zero section.

(c) Every projective space admits a volume form.

(d) If a manifold M has Euler characteristic zero, then M is orientable.

B4/4 Differentiable Manifolds

State and prove Stokes' Theorem for compact oriented manifolds-with-boundary.

[*You may assume results relating local forms on the manifold with those on its boundary provided you state them clearly.*]

Deduce that every differentiable map of the unit ball in $\mathbb{R}^n$ to itself has a fixed point.

B2/8 **Algebraic Topology**

Show that the fundamental group G of the Klein bottle is infinite. Show that G contains an abelian subgroup of finite index. Show that G is not abelian.

B3/7 **Algebraic Topology**

For a finite simplicial complex X, let $b_i(X)$ denote the rank of the finitely generated abelian group H_iX. Define the Euler characteristic $\chi(X)$ by the formula

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Let a_i denote the number of i-simplices in X, for each $i \geqslant 0$. Show that

$$\chi(X) = \sum_i (-1)^i a_i.$$

B4/5 **Algebraic Topology**

State the Mayer-Vietoris theorem for a finite simplicial complex X which is the union of closed subcomplexes A and B. Define all the maps in the long exact sequence. Prove that the sequence is exact at the term H_iX, for every $i \geqslant 0$.

B1/9 **Number Fields**

Explain what is meant by an integral basis $\omega_1, \ldots, \omega_n$ of a number field K. Give an expression for the discriminant of K in terms of the traces of the $\omega_i\omega_j$.

Let $K = \mathbb{Q}(i, \sqrt{2})$. By computing the traces $T_{K/k}(\theta)$, where k runs through the three quadratic subfields of K, show that the algebraic integers θ in K have the form $\frac{1}{2}(\alpha + \beta\sqrt{2})$, where $\alpha = a + ib$ and $\beta = c + id$ are Gaussian integers. By further computing the norm $N_{K/k}(\theta)$, where $k = \mathbb{Q}(\sqrt{2})$, show that a and b are even and that $c \equiv d \pmod 2$. Hence prove that an integral basis for K is $1, i, \sqrt{2}, \frac{1}{2}(1+i)\sqrt{2}$.

Calculate the discriminant of K.

B2/9 **Number Fields**

Let $K = \mathbb{Q}(\sqrt{35})$. By Dedekind's theorem, or otherwise, show that the ideal equations

$$2 = [2, \omega]^2, \quad 5 = [5, \omega]^2, \quad [\omega] = [2, \omega][5, \omega]$$

hold in K, where $\omega = 5 + \sqrt{35}$. Deduce that K has class number 2.

Verify that $1 + \omega$ is the fundamental unit in K. Hence show that the complete solution in integers x, y of the equation $x^2 - 35y^2 = -10$ is given by

$$x + \sqrt{35}y = \pm\omega(1 + \omega)^n \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Calculate the particular solution x, y for $n = 1$.
[*It can be assumed that the Minkowski constant for K is $\frac{1}{2}$.*]

B4/6 **Number Fields**

Write an essay on one of the following topics.

(i) Dirichlet's unit theorem and the Pell equation.

(ii) Ideals and the fundamental theorem of arithmetic.

(iii) Dedekind's theorem and the factorisation of primes. (You should treat explicitly either the case of quadratic fields or that of the cyclotomic field.)

B1/10 **Hilbert Spaces**

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. Define what it means for T to be *bounded below*. Prove that, if $LT = I$ for some $L \in \mathcal{B}(H)$, then T is bounded below.

Prove that an operator $T \in \mathcal{B}(H)$ is invertible if and only if both T and T^* are bounded below.

Let H be the sequence space ℓ^2. Define the operators S, R on H by setting

$$S(\xi) = (0, \xi_1, \xi_2, \xi_3, \dots), \quad R(\xi) = (\xi_2, \xi_3, \xi_4, \dots),$$

for all $\xi = (\xi_1, \xi_2, \xi_3, \dots) \in \ell^2$. Check that $RS = I$ but $SR \neq I$. Let $D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. For each $\lambda \in D$, explain why $I - \lambda R$ is invertible, and define

$$R(\lambda) = (I - \lambda R)^{-1} R.$$

Show that, for all $\lambda \in D$, we have $R(\lambda)(S - \lambda I) = I$, but $(S - \lambda I)R(\lambda) \neq I$. Deduce that, for all $\lambda \in D$, the operator $S - \lambda I$ is bounded below, but is not invertible. Deduce also that $\operatorname{Sp} S = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$.

Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and for $n = 1, 2, \dots$, define the element x_n of ℓ^2 by

$$x_n = n^{-1/2}(\lambda^{-1}, \lambda^{-2}, \dots, \lambda^{-n}, 0, 0, \dots).$$

Prove that $\|x_n\| = 1$ but that $(S - \lambda I)x_n \to 0$ as $n \to \infty$. Deduce that, for $|\lambda| = 1$, $S - \lambda I$ is not bounded below.

B3/8 **Hilbert Spaces**

Let H be an infinite-dimensional, separable Hilbert space. Let T be a compact linear operator on H, and let λ be a non-zero, approximate eigenvalue of T. Prove that λ is an eigenvalue, and that the corresponding eigenspace $E_\lambda(T) = \{x \in H : Tx = \lambda x\}$ is finite-dimensional.

Let S be a compact, self-adjoint operator on H. Prove that there is an orthonormal basis $(e_n)_{n \geqslant 0}$ of H, and a sequence $(\lambda_n)_{n \geqslant 0}$ in $\mathbb{C}$, such that (i) $Se_n = \lambda_n e_n$ $(n \geqslant 0)$ and (ii) $\lambda_n \to 0$ as $n \to \infty$.

Now let S be compact, self-adjoint and *injective*. Let R be a bounded self-adjoint operator on H such that $RS = SR$. Prove that H has an orthonormal basis $(e_n)_{n \geqslant 1}$, where, for every n, e_n is an eigenvector, both of S and of R.

[*You may assume, without proof, results about self-adjoint operators on finite-dimensional spaces.*]

B4/7 **Hilbert Spaces**

Throughout this question, H is an infinite-dimensional, separable Hilbert space. You may use, without proof, any theorems about compact operators that you require.

Define a *Fredholm operator* T, on a Hilbert space H, and define the *index* of T.

(i) Prove that if T is Fredholm then $\operatorname{im} T$ is closed.

(ii) Let $F \in \mathcal{B}(H)$ and let F have finite rank. Prove that F^* also has finite rank.

(iii) Let $T = I + F$, where I is the identity operator on H and F has finite rank; let $E = \operatorname{im} F + \operatorname{im} F^*$. By considering $T|E$ and $T|E^\perp$ (or otherwise) prove that T is Fredholm with $\operatorname{ind} T = 0$.

(iv) Let $T \in \mathcal{B}(H)$ be Fredholm with $\operatorname{ind} T = 0$. Prove that $T = A + F$, where A is invertible and F has finite rank.

[*You may wish to note that T effects an isomorphism from $(\ker T)^\perp$ onto $\operatorname{im} T$; also $\ker T$ and $(\operatorname{im} T)^\perp$ have the same finite dimension.*]

(v) Deduce from (iii) and (iv) that $T \in \mathcal{B}(H)$ is Fredholm with $\operatorname{ind} T = 0$ if and only if $T = A + K$ with A invertible and K compact.

(vi) Explain briefly, by considering suitable shift operators on ℓ^2 (i.e. *not* using any theorems about Fredholm operators) that, for each $k \in \mathbb{Z}$, there is a Fredholm operator S on H with $\operatorname{ind} S = k$.

B1/11 **Riemann Surfaces**

(a) Define the notions of (abstract) Riemann surface, holomorphic map, and biholomorphic map between Riemann surfaces.

(b) Prove the following theorem on the local form of a holomorphic map.

For a holomorphic map $f : R \to S$ between Riemann surfaces, which is not constant near a point $r \in R$, there exist neighbourhoods U of r in R and V of $f(r)$ in S, together with biholomorphic identifications $\phi : U \to \Delta$, $\psi : V \to \Delta$, such that $(\psi \circ f)(x) = \phi(x)^n$, for all $x \in U$.

(c) Prove further that a non-constant holomorphic map between compact, connected Riemann surfaces is surjective.

(d) Deduce from (c) the fundamental theorem of algebra.

B3/9 **Riemann Surfaces**

Let α_1, α_2 be two non-zero complex numbers with $\alpha_1/\alpha_2 \notin \mathbb{R}$. Let L be the lattice $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \subset \mathbb{C}$. A meromorphic function f on $\mathbb{C}$ is *elliptic* if $f(z+\lambda) = f(z)$, for all $z \in \mathbb{C}$ and $\lambda \in L$. The *Weierstrass functions* $\wp(z), \zeta(z), \sigma(z)$ are defined by the following properties:

- $\wp(z)$ is elliptic, has double poles at the points of L and no other poles, and $\wp(z) = 1/z^2 + O(z^2)$ near 0;
- $\zeta'(z) = -\wp(z)$, and $\zeta(z) = 1/z + O(z^3)$ near 0;
- $\sigma(z)$ is odd, and $\sigma'(z)/\sigma(z) = \zeta(z)$, and $\sigma(z)/z \to 1$ as $z \to 0$.

Prove the following.
(a) $\wp$, and hence ζ and σ, are uniquely determined by these properties. You are *not* expected to prove the existence of $\wp, \zeta, \sigma$, and you may use Liouville's theorem without proof.
(b) $\zeta(z+\alpha_i) = \zeta(z) + 2\eta_i$, and $\sigma(z+\alpha_i) = k_i e^{2\eta_i z}\sigma(z)$, for some constants η_i, k_i $(i = 1, 2)$.
(c) σ is holomorphic, has simple zeroes at the points of L, and has no other zeroes.
(d) Given $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{C}$ with $a_1 + \ldots + a_n = b_1 + \ldots + b_n$, the function

$$\frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}$$

is elliptic.
(e) $\wp(u) - \wp(v) = -\dfrac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}$.
(f) Deduce from (e), or otherwise, that $\dfrac{1}{2}\dfrac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} = \zeta(u+v) - \zeta(u) - \zeta(v)$.

B4/8 **Riemann Surfaces**

A holomorphic map $p : S \to T$ between Riemann surfaces is called a covering map if every $t \in T$ has a neighbourhood V for which $p^{-1}(V)$ breaks up as a disjoint union of open subsets U_α on which $p : U_\alpha \to V$ is biholomorphic.

(a) Suppose that $f : R \to T$ is any holomorphic map of connected Riemann surfaces, R is simply connected and $p : S \to T$ is a covering map. By considering the lifts of paths from T to S, or otherwise, prove that f lifts to a holomorphic map $\widetilde{f} : R \to S$, i.e. that there exists an $\widetilde{f}$ with $f = p \circ \widetilde{f}$.

(b) Write down a biholomorphic map from the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ onto a half-plane. Show that the unit disk Δ uniformizes the punctured unit disk $\Delta^\times = \Delta - \{0\}$ by constructing an explicit covering map $p : \Delta \to \Delta^\times$.

(c) Using the uniformization theorem, or otherwise, prove that any holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus greater than one is constant.

B2/10 **Algebraic Curves**

For $N \geq 1$, let V_N be the (irreducible) projective plane curve $V_N : X^N + Y^N + Z^N = 0$ over an algebraically closed field of characteristic zero.

Show that V_N is smooth (non-singular). For $m, n \geq 1$, let $\alpha_{m,n} : V_{mn} \to V_m$ be the morphism $\alpha_{m,n}(X : Y : Z) = (X^n : Y^n : Z^n)$. Determine the degree of $\alpha_{m,n}$, its points of ramification and the corresponding ramification indices.

Applying the Riemann–Hurwitz formula to $\alpha_{1,n}$, determine the genus of V_n.

B3/10 **Algebraic Curves**

Let $f = f(x, y)$ be an irreducible polynomial of degree $n \geq 2$ (over an algebraically closed field of characteristic zero) and $V_0 = \{f = 0\} \subset \mathbb{A}^2$ the corresponding affine plane curve. Assume that V_0 is smooth (non-singular) and that the projectivization $V \subset \mathbb{P}^2$ of V_0 intersects the line at infinity $\mathbb{P}^2 - \mathbb{A}^2$ in n distinct points. Show that V is smooth and determine the divisor of the rational differential $\omega = \dfrac{dx}{f'_y}$ on V. Deduce a formula for the genus of V.

B4/9 **Algebraic Curves**

Write an essay on the Riemann–Roch theorem and some of its applications.

B1/13 **Probability and Measure**

State and prove Dynkin's π-system lemma.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (A_n) be a sequence of independent events such that $\lim_{n\to\infty} \mathbb{P}(A_n) = p$. Let $\mathcal{G} = \sigma(A_1, A_2, \ldots)$. Prove that

$$\lim_{n\to\infty} \mathbb{P}(G \cap A_n) = p\mathbb{P}(G)$$

for all $G \in \mathcal{G}$.

B2/12 **Probability and Measure**

Let (X_n) be a sequence of non-negative random variables on a common probability space with $\mathbb{E}X_n \leqslant 1$, such that $X_n \to 0$ almost surely. Determine which of the following statements are necessarily true, justifying your answers carefully:

(a) $\mathbb{P}(X_n \geqslant 1) \to 0$ as $n \to \infty$;

(b) $\mathbb{E}X_n \to 0$ as $n \to \infty$;

(c) $\mathbb{E}(\sin(X_n)) \to 0$ as $n \to \infty$;

(d) $\mathbb{E}(\sqrt{X_n}) \to 0$ as $n \to \infty$.

[*Standard limit theorems for integrals, and results about uniform integrability, may be used without proof provided that they are clearly stated.*]

B3/12 **Probability and Measure**

Derive the characteristic function of a real-valued random variable which is normally distributed with mean μ and variance σ^2. What does it mean to say that an $\mathbb{R}^n$-valued random variable has a *multivariate Gaussian distribution*? Prove that the distribution of such a random variable is determined by its ($\mathbb{R}^n$-valued) mean and its covariance matrix.

Let X and Y be random variables defined on the same probability space such that (X, Y) has a Gaussian distribution. Show that X and Y are independent if and only if $\mathrm{cov}(X, Y) = 0$. Show that, even if they are not independent, one may always write $X = aY + Z$ for some constant a and some random variable Z independent of Y.

[*The inversion theorem for characteristic functions and standard results about independence may be assumed.*]

B4/11 **Probability and Measure**

State Birkhoff's Almost Everywhere Ergodic Theorem for measure-preserving transformations. Define what it means for a sequence of random variables to be *stationary*. Explain *briefly* how the stationarity of a sequence of random variables implies that a particular transformation is measure-preserving.

A bag contains one white ball and one black ball. At each stage of a process one ball is picked from the bag (uniformly at random) and then returned to the bag together with another ball of the same colour. Let X_n be a random variable which takes the value 0 if the nth ball added to the bag is white and 1 if it is black.

(a) Show that the sequence $X_1, X_2, X_3, \ldots$ is stationary and hence that the proportion of black balls in the bag converges almost surely to some random variable R.

(b) Find the distribution of R.

[*The fact that almost-sure convergence implies convergence in distribution may be used without proof.*]

B2/13 **Applied Probability**

Two enthusiastic probability students, Ros and Guil, sit an examination which starts at time 0 and ends at time T; they both decide to use the time to attempt a proof of a difficult theorem which carries a lot of extra marks.

Ros' strategy is to write the proof continuously at a constant speed λ lines per unit time. In a time interval of length δt he has a probability $\mu\delta t + o(\delta t)$ of realising he has made a mistake. If that happens he instantly panics, erases everything he has written and starts all over again.

Guil, on the other hand, keeps cool and thinks carefully about what he is doing. In a time interval of length δt, he has a probability $\lambda\delta t + o(\delta t)$ of writing the next line of proof and for each line he has written a probability $\mu\delta t + o(\delta t)$ of finding a mistake in that line, independently of all other lines he has written. When a mistake is found, he erases that line and carries on as usual, hoping for the best.

Both Ros and Guil realise that, even if they manage to finish the proof, they will not recognise that they have done so and will carry on writing as much as they can.

(a) Calculate $p_l(t)$, the probability that, for Ros, the length of his completed proof at time $t \geqslant l/\lambda$ is at least l.

(b) Let $q_n(t)$ be the probability that Guil has n lines of proof at time $t > 0$. Show that

$$\frac{\partial Q}{\partial t} = (s-1)(\lambda Q - \mu \frac{\partial Q}{\partial s}),$$

where $Q(s,t) = \sum_{n=0}^{\infty} s^n q_n(t)$.

(c) Suppose now that every time Ros starts all over again, the time until the next mistake has distribution F, independently of the past history. Write down a renewal-type integral equation satisfied by $l(t)$, the expected length of Ros' proof at time t. What is the expected length of proof produced by him at the end of the examination if F is the exponential distribution with mean $1/\mu$?

(d) What is the expected length of proof produced by Guil at the end of the examination if each line that he writes survives for a length of time with distribution F, independently of all other lines?

B3/13 **Applied Probability**

(a) Define a renewal process and a discrete renewal process.

(b) State and prove the Discrete Renewal Theorem.

(c) The sequence $\mathbf{u} = \{u_n : n \geqslant 0\}$ satisfies

$$u_0 = 1, \quad u_n = \sum_{i=1}^{n} f_i u_{n-i}, \qquad \text{for } n \geqslant 1$$

for some collection of non-negative numbers $(f_i : i \in \mathbb{N})$ summing to 1. Let $U(s) = \sum_{n=1}^{\infty} u_n s^n$, $F(s) = \sum_{n=1}^{\infty} f_n s^n$. Show that

$$F(s) = \frac{U(s)}{1 + U(s)}.$$

Give a probabilistic interpretation of the numbers u_n, f_n and $m_n = \sum_{i=1}^{n} u_i$.

(d) Let the sequence u_n be given by

$$u_{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}, \quad u_{2n+1} = 0, \quad n \geqslant 1.$$

How is this related to the simple symmetric random walk on the integers $\mathbb{Z}$ starting from the origin, and its subsequent returns to the origin? Determine $F(s)$ in this case, either by calculating $U(s)$ or by showing that F satisfies the quadratic equation

$$F^2 - 2F + s^2 = 0, \qquad \text{for} \quad 0 \leqslant s < 1.$$

B4/12 **Applied Probability**

Define a Poisson random measure. State and prove the Product Theorem for the jump times J_n of a Poisson process with constant rate λ and independent random variables Y_n with law μ. Write down the corresponding result for a Poisson process Π in a space $E = \mathbb{R}^d$ with rate $\lambda(x)$ $(x \in E)$ when we associate with each $X \in \Pi$ an independent random variable m_X with density $\rho(X, dm)$.

Prove Campbell's Theorem, i.e. show that if M is a Poisson random measure on the space E with intensity measure ν and $a : E \to \mathbb{R}$ is a bounded measurable function then

$$\mathbf{E}[e^{\theta\Sigma}] = \exp\left(\int_E (e^{\theta a(y)} - 1)\nu(dy)\right),$$

where

$$\Sigma = \int_E a(y)M(dy) = \sum_{X\in\Pi} a(X).$$

Stars are scattered over three-dimensional space $\mathbb{R}^3$ in a Poisson process Π with density $\nu(X)$ $(X \in \mathbb{R}^3)$. Masses of the stars are independent random variables; the mass m_X of a star at X has the density $\rho(X, dm)$. The gravitational potential at the origin is given by

$$F = \sum_{X\in\Pi} \frac{Gm_X}{|X|},$$

where G is a constant. Find the moment generating function $\mathbf{E}[e^{\theta F}]$.

A galaxy occupies a sphere of radius R centred at the origin. The density of stars is $\nu(\mathbf{x}) = 1/|\mathbf{x}|$ for points $\mathbf{x}$ inside the sphere; the mass of each star has the exponential distribution with mean M. Calculate the expected potential due to the galaxy at the origin. Let C be a positive constant. Find the distribution of the distance from the origin to the nearest star whose contribution to the potential F is at least C.

B1/14 **Information Theory**

(a) Define the entropy $h(X)$ and the mutual entropy $i(X,Y)$ of random variables X and Y. Prove the inequality

$$0 \leqslant i(X,Y) \leqslant \min\{h(X), h(Y)\}.$$

[*You may assume the Gibbs inequality.*]

(b) Let X be a random variable and let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a random vector.

(i) Prove or disprove by producing a counterexample the inequality

$$i(X, \mathbf{Y}) \leqslant \sum_{j=1}^{n} i(X, Y_j),$$

first under the assumption that $Y_1, \ldots, Y_n$ are independent random variables, and then under the assumption that $Y_1, \ldots, Y_n$ are conditionally independent given X.

(ii) Prove or disprove by producing a counterexample the inequality

$$i(X, \mathbf{Y}) \geqslant \sum_{j=1}^{n} i(X, Y_j),$$

first under the assumption that $Y_1, \ldots, Y_n$ are independent random variables, and then under the assumption that $Y_1, \ldots, Y_n$ are conditionally independent given X.

B2/14 **Information Theory**

Define the binary Hamming code of length $n = 2^{\ell} - 1$ and its dual. Prove that the Hamming code is perfect. Prove that in the dual code:

(i) The weight of any non-zero codeword equals $2^{\ell-1}$;

(ii) The distance between any pair of words equals $2^{\ell-1}$.

[*You may quote results from the course provided that they are carefully stated.*]

B4/13 **Information Theory**

Define the Huffman binary encoding procedure and prove its optimality among decipherable codes.

B2/15 **Optimization and Control**

State Pontryagin's maximum principle (PMP) for the problem of minimizing

$$\int_0^T c(x(t), u(t))\, dt + K(x(T)) ,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $dx/dt = a(x(t), u(t))$; here, $x(0)$ and T are given, and $x(T)$ is unconstrained.

Consider the two-dimensional problem in which $dx_1/dt = x_2$, $dx_2/dt = u$, $c(x,u) = \frac{1}{2}u^2$ and $K(x(T)) = \frac{1}{2}qx_1(T)^2$, $q > 0$. Show that, by use of a variable $z(t) = x_1(t) + x_2(t)(T-t)$, one can rewrite this problem as an equivalent one-dimensional problem.

Use PMP to solve this one-dimensional problem, showing that the optimal control can be expressed as $u(t) = -qz(T)(T-t)$, where $z(T) = z(0)/(1 + \frac{1}{3}qT^3)$.

Express $u(t)$ in a feedback form of $u(t) = k(t)z(t)$ for some $k(t)$.

Suppose that the initial state $x(0)$ is perturbed by a small amount to $x(0) + (\epsilon_1, \epsilon_2)$. Give an expression (in terms of ϵ_1, ϵ_2, $x(0)$, q and T) for the increase in minimal cost.

B3/14 **Optimization and Control**

Consider a scalar system with $x_{t+1} = (x_t + u_t)\xi_t$, where $\xi_0, \xi_1, \ldots$ is a sequence of independent random variables, uniform on the interval $[-a, a]$, with $a \leqslant 1$. We wish to choose $u_0, \ldots, u_{h-1}$ to minimize the expected value of

$$\sum_{t=0}^{h-1}(c + x_t^2 + u_t^2) + 3x_h^2 ,$$

where u_t is chosen knowing x_t but not ξ_t. Prove that the minimal expected cost can be written $V_h(x_0) = hc + x_0^2\Pi_h$ and derive a recurrence for calculating $\Pi_1, \ldots, \Pi_h$.

How does your answer change if u_t is constrained to lie in the set $\mathcal{U}(x_t) = \{u : |u + x_t| < |x_t|\}$?

Consider a stopping problem for which there are two options in state x_t, $t \geqslant 0$:

(1) stop: paying a terminal cost $3x_t^2$; no further costs are incurred;

(2) continue: choosing $u_t \in \mathcal{U}(x_t)$, paying $c + u_t^2 + x_t^2$, and moving to state $x_{t+1} = (x_t + u_t)\xi_t$.

Consider the problem of minimizing total expected cost subject to the constraint that no more than h continuation steps are allowed. Suppose $a = 1$. Show that an optimal policy stops if and only if either h continuation steps have already been taken or $x^2 \leqslant 2c/3$.

[*Hint: Use induction on h to show that a one-step-look-ahead rule is optimal. You should not need to find the optimal u_t for the continuation steps.*]

B4/14 **Optimization and Control**

A discrete-time decision process is defined on a finite set of states I as follows. Upon entry to state i_t at time t the decision-maker observes a variable ξ_t. He then chooses the next state freely within I, at a cost of $c(i_t, \xi_t, i_{t+1})$. Here $\{\xi_0, \xi_1, \ldots\}$ is a sequence of integer-valued, identically distributed random variables. Suppose there exist $\{\phi_i : i \in I\}$ and λ such that for all $i \in I$

$$\phi_i + \lambda = \sum_{k \in \mathbb{Z}} P(\xi_t = k) \min_{i' \in I} \left[c(i, k, i') + \phi_{i'} \right] .$$

Let π denote a policy. Show that

$$\lambda = \inf_{\pi} \limsup_{t \to \infty} E_{\pi} \left[\frac{1}{t} \sum_{s=0}^{t-1} c(i_s, \xi_s, i_{s+1}) \right] .$$

At the start of each month a boat manufacturer receives orders for 1, 2 or 3 boats. These numbers are equally likely and independent from month to month. He can produce j boats in a month at a cost of $6+3j$ units. All orders are filled at the end of the month in which they are ordered. It is possible to make extra boats, ending the month with a stock of i unsold boats, but i cannot be more than 2, and a holding cost of ci is incurred during any month that starts with i unsold boats in stock. Write down an optimality equation that can be used to find the long-run expected average-cost.

Let π be the policy of only ever producing sufficient boats to fill the present month's orders. Show that it is optimal if and only if $c \geqslant 2$.

Suppose $c < 2$. Starting from π, what policy is obtained after applying one step of the policy-improvement algorithm?

B1/17 **Dynamical Systems**

Let f_c be the map of the closed interval [0,1] to itself given by

$$f_c(x) = cx(1-x), \text{ where } 0 \leqslant c \leqslant 4.$$

Sketch the graphs of f_c and (without proof) of f_c^2, find their fixed points, and determine which of the fixed points of f_c are attractors. Does your argument work for $c = 3$?

B3/17 **Dynamical Systems**

Let $\mathcal{A}$ be a finite alphabet of letters and Σ either the semi-infinite space or the doubly infinite space of sequences whose elements are drawn from $\mathcal{A}$. Define the natural topology on Σ. If W is a set of words, denote by Σ_W the subspace of Σ consisting of those sequences none of whose subsequences is in W. Prove that Σ_W is a closed subspace of Σ; and state and prove a necessary and sufficient condition for a closed subspace of Σ to have the form Σ_W for some W.

If $\mathcal{A} = \{0, 1\}$ and

$$W = \{000,\ 111,\ 010,\ 101\}$$

what is the space Σ_W?

B4/17 **Dynamical Systems**

Let $\mathcal{S}$ be a metric space, F a map of $\mathcal{S}$ to itself and P a point of $\mathcal{S}$. Define an *attractor* for F and an *omega point* of the orbit of P under F.

Let f be the map of $\mathbb{R}$ to itself given by

$$f(x) = x + \frac{1}{2} + c\sin^2 2\pi x,$$

where $c > 0$ is so small that $f'(x) > 0$ for all x, and let F be the map of $\mathbb{R}/\mathbb{Z}$ to itself induced by f. What points if any are
(a) attractors for F^2,
(b) omega points of the orbit of some point P under F ?
Is the cycle $\{0,\ \frac{1}{2}\}$ an attractor?

In the notation of the first two sentences, let $\mathcal{C}$ be a cycle of order M and assume that F is continuous. Prove that $\mathcal{C}$ is an attractor for F if and only if each point of $\mathcal{C}$ is an attractor for F^M.

B1/18 **Partial Differential Equations**

(a) Solve the equation, for a function $u(x, y)$,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad (*)$$

together with the boundary condition on the x-axis:

$$u(x, 0) = x.$$

Find for which real numbers a it is possible to solve $(*)$ with the following boundary condition specified on the line $y = ax$:

$$u(x, ax) = x.$$

Explain your answer in terms of the notion of *characteristic hypersurface*, which should be defined.

(b) Solve the equation

$$\frac{\partial u}{\partial x} + (1 + u)\frac{\partial u}{\partial y} = 0$$

with the boundary condition on the x-axis

$$u(x, 0) = x,$$

in the domain $\mathcal{D} = \{(x, y) : 0 < y < (x+1)^2/4, \ \ -1 < x < \infty\}$. Sketch the characteristics.

B2/17 **Partial Differential Equations**

(a) Define the convolution $f * g$ of two functions. Write down a formula for a solution $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ to the initial value problem

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

together with the boundary condition

$$u(0, x) = f(x)$$

for f a bounded continuous function on $\mathbb{R}^n$. Comment briefly on the uniqueness of the solution.

(b) State and prove the Duhamel principle giving the solution (for $t > 0$) to the equation

$$\frac{\partial u}{\partial t} - \Delta u = g$$

together with the boundary condition

$$u(0, x) = f(x)$$

in terms of your answer to (a).

(c) Show that if $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is the solution to

$$\frac{\partial v}{\partial t} - \Delta v = G$$

together with the boundary condition

$$v(0, x) = f(x)$$

with $G(t, x) \leq g(t, x)$ for all (t, x) then $v(t, x) \leq u(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$.

Finally show that if in addition there exists a point (t_0, x_0) at which there is strict inequality in the assumption i.e.

$$G(t_0, x_0) < g(t_0, x_0),$$

then in fact

$$v(t, x) < u(t, x)$$

whenever $t > t_0$.

B3/18 **Partial Differential Equations**

Define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. State the Fourier inversion theorem for the Fourier transform of a Schwartz function.

Consider the initial value problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u = 0 \ , \ x \in \mathbb{R}^n \ , \ 0 < t < \infty \ ,$$

$$u(0,x) = f(x) \ , \qquad \frac{\partial u}{\partial t}(0,x) = 0$$

for f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Show that the solution can be written as

$$u(t,x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(t,\xi) d\xi \ ,$$

where

$$\hat{u}(t,\xi) = \cos\left(t\sqrt{1+|\xi|^2}\right) \hat{f}(\xi)$$

and

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

State the Plancherel-Parseval theorem and hence deduce that

$$\int_{\mathbb{R}^n} |u(t,x)|^2 dx \leq \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

B4/18 **Partial Differential Equations**

Discuss the notion of *fundamental solution* for a linear partial differential equation with constant coefficients.

B1/19 **Methods of Mathematical Physics**

State the Riemann-Lebesgue lemma as applied to the integral

$$\int_a^b g(u)e^{ixu}du \ ,$$

where $g'(u)$ is continuous and $a, b \in \mathbb{R}$.

Use this lemma to show that, as $x \to +\infty$,

$$\int_a^b (u-a)^{\lambda-1} f(u)e^{ixu}du \sim f(a)\, e^{ixa}\, e^{i\pi\lambda/2}\, \Gamma(\lambda)\, x^{-\lambda} \ ,$$

where $f(u)$ is holomorphic, $f(a) \neq 0$ and $0 < \lambda < 1$. You should explain each step of your argument, but detailed analysis is not required.

Hence find the leading order asymptotic behaviour as $x \to +\infty$ of

$$\int_0^1 \frac{e^{ixt^2}}{(1-t^2)^{\frac{1}{2}}}\, dt \ .$$

B2/18 **Methods of Mathematical Physics**

Show that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{t^{z-1}}{t-a}\, dt = \pi i a^{z-1} \ ,$$

where a is real and positive, $0 < \mathrm{Re}\,(z) < 1$ and $\mathcal{P}$ denotes the Cauchy principal value; the principal branches of t^z etc. are implied. Deduce that

$$\int_0^{\infty} \frac{t^{z-1}}{t+a}\, dt = \pi a^{z-1} \mathrm{cosec}\, \pi z \qquad (*)$$

and that

$$\mathcal{P}\int_0^{\infty} \frac{t^{z-1}}{t-a}\, dt = -\pi a^{z-1} \cot \pi z \ .$$

Use $(*)$ to show that, if $\mathrm{Im}\,(b) > 0$, then

$$\int_0^{\infty} \frac{t^{z-1}}{t-b}\, dt = -\pi b^{z-1}(\cot \pi z - i) \ .$$

What is the value of this integral if $\mathrm{Im}\,(b) < 0$?

B3/19 **Methods of Mathematical Physics**

Show that the equation

$$zw'' + w' + (\lambda - z)w = 0$$

has solutions of the form

$$w(z) = \int_\gamma (t-1)^{(\lambda-1)/2}(t+1)^{-(\lambda+1)/2} e^{zt} dt.$$

Give examples of possible choices of γ to provide two independent solutions, assuming $\mathrm{Re}(z) > 0$. Distinguish between the cases $\mathrm{Re}\,\lambda > -1$ and $\mathrm{Re}\,\lambda < 1$. Comment on the case $-1 < \mathrm{Re}\ \lambda < 1$ and on the case that λ is an odd integer.

Show that, if $\mathrm{Re}\,\lambda < 1$, there is a solution $w_1(z)$ that is bounded as $z \to +\infty$, and that, in this limit,

$$w_1(z) \sim A\, e^{-z} z^{(\lambda-1)/2} \left(1 - \frac{(1-\lambda)^2}{8z}\right) ,$$

where A is a constant.

B4/19 **Methods of Mathematical Physics**

Let

$$I(\lambda, a) = \int_{-i\infty}^{i\infty} \frac{e^{\lambda(t^3-3t)}}{t^2 - a^2}\, dt ,$$

where λ is real, a is real and non-zero, and the path of integration runs up the imaginary axis. Show that, if $a^2 > 1$,

$$I(\lambda, a) \sim \frac{ie^{-2\lambda}}{1-a^2}\sqrt{\frac{\pi}{3\lambda}}$$

as $\lambda \to +\infty$ and sketch the relevant steepest descent path.

What is the corresponding result if $a^2 < 1$?

B1/21 **Electrodynamics**

Explain how one can write Maxwell's equations in relativistic form by introducing an antisymmetric field strength tensor F_{ab}.

In an inertial frame S, the electric and magnetic fields are $\mathbf{E}$ and $\mathbf{B}$. Suppose that there is a second inertial frame S' moving with velocity v along the x-axis relative to S. Derive the rules for finding the electric and magnetic fields $\mathbf{E}'$ and $\mathbf{B}'$ in the frame S'. Show that $|\mathbf{E}|^2 - |\mathbf{B}|^2$ and $\mathbf{E} \cdot \mathbf{B}$ are invariant under Lorentz transformations.

Suppose that $\mathbf{E} = E_0(0\,,1\,,0)$ and $\mathbf{B} = E_0(0\,,\cos\theta\,,\sin\theta)$, where $0 \leq \theta < \pi/2$. At what velocity must an observer be moving in the frame S for the electric and magnetic fields to appear to be parallel?

Comment on the case $\theta = \pi/2$.

B2/20 **Electrodynamics**

A particle of rest mass m and charge q moves in an electromagnetic field given by a potential A_a along a trajectory $x^a(\tau)$, where τ is the proper time along the particle's worldline. The action for such a particle is

$$I = \int \left(m\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b} - qA_a\dot{x}^a \right) d\tau\,.$$

Show that the Euler-Lagrange equations resulting from this action reproduce the relativistic equation of motion for the particle.

Suppose that the particle is moving in the electrostatic field of a fixed point charge Q with radial electric field E_r given by

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}\,.$$

Show that one can choose a gauge such that $A_i = 0$ and only $A_0 \neq 0$. Find A_0.

Assume that the particle executes planar motion, which in spherical polar coordinates $(r\,,\theta\,,\phi)$ can be taken to be in the plane $\theta = \pi/2$. Derive the equations of motion for t and ϕ.

By using the fact that $\eta_{ab}\dot{x}^a\dot{x}^b = -1$, find the equation of motion for r, and hence show that the shape of the orbit is described by

$$\frac{dr}{d\phi} = \pm\frac{r^2}{\ell}\sqrt{\left(E + \frac{\gamma}{r}\right)^2 - 1 - \frac{\ell^2}{r^2}}\,,$$

where E (> 1) and ℓ are constants of integration and γ is to be determined.

By putting $u = 1/r$ or otherwise, show that if $\gamma^2 < \ell^2$ then the orbits are bounded and generally not closed, and show that the angle between successive minimal values of r is $2\pi(1 - \gamma^2/\ell^2)^{-1/2}$.

B4/21 **Electrodynamics**

Derive Larmor's formula for the rate at which radiation is produced by a particle of charge q moving along a trajectory $\mathbf{x}(t)$.

A non-relativistic particle of mass m, charge q and energy E is incident along a radial line in a central potential $V(r)$. The potential is vanishingly small for r very large, but increases without bound as $r \to 0$. Show that the total amount of energy $\mathcal{E}$ radiated by the particle is

$$\mathcal{E} = \frac{\mu_0 q^2}{3\pi m^2}\sqrt{\frac{m}{2}}\int_{r_0}^{\infty} \frac{1}{\sqrt{E - V(r)}}\left(\frac{dV}{dr}\right)^2 dr\,,$$

where $V(r_0) = E$.

Suppose that V is the Coulomb potential $V(r) = A/r$. Evaluate $\mathcal{E}$.

B1/22 **Statistical Physics**

A simple model for a rubber molecule consists of a one-dimensional chain of n links each of fixed length b and each of which is oriented in either the positive or negative direction. A unique state i of the molecule is designated by giving the orientation ± 1 of each link. If there are n_+ links oriented in the positive direction and n_- links oriented in the negative direction then $n = n_+ + n_-$ and the length of the molecule is $l = (n_+ - n_-)b$. The length of the molecule associated with state i is l_i.

What is the range of l?

What is the number of states with n, n_+, n_- fixed?

Consider an ensemble of A copies of the molecule in which a_i members are in state i and write down the expression for the mean length L.

By introducing a Lagrange multiplier τ for L show that the most probable configuration for the $\{a_i\}$ with given length L is found by maximizing

$$\log\left(\frac{A!}{\prod_i a_i!}\right) \;+\; \tau \sum_i \; a_i l_i \; - \alpha \sum_i a_i .$$

Hence show that the most probable configuration is given by

$$p_i \;=\; \frac{e^{\tau l_i}}{Z} \;,$$

where p_i is the probability for finding an ensemble member in the state i and Z is the partition function which should be defined.

Show that Z can be expressed as

$$Z \;=\; \sum_l g(l)\, e^{\tau l} \;,$$

where the meaning of $g(l)$ should be explained.

Hence show that Z is given by

$$Z \;=\; \sum_{n_+=0}^{n} \frac{n!}{n_+!\, n_-!} \,(e^{\tau b})^{n_+} (e^{-\tau b})^{n_-} \;, \qquad n_+ + n_- = n,$$

and therefore that the free energy G for the system is

$$G \;=\; -nkT \log(2\cosh \tau b) \;.$$

Show that τ is determined by

$$L \;=\; -\frac{1}{kT}\left(\frac{\partial G}{\partial \tau}\right)_n \;,$$

and hence that the equation of state is

$$\tanh \tau b \;=\; \frac{L}{nb} \;.$$

What are the independent variables on which G depends?

Explain why the tension in the rubber molecule is $kT\tau$.

B3/22 **Statistical Physics**

A system consisting of non-interacting bosons has single-particle levels uniquely labelled by r with energies ϵ_r, $\epsilon_r \geq 0$. Show that the free energy in the grand canonical ensemble is

$$F = kT \sum_r \log(1 - e^{-\beta(\epsilon_r - \mu)}) \ .$$

What is the maximum value for μ?

A system of N bosons in a large volume V has one energy level of energy zero and a large number $M \gg 1$ of energy levels of the same energy ϵ, where M takes the form $M = AV$ with A a positive constant. What are the dimensions of A?

Show that the free energy is

$$F = kT \left(\log(1 - e^{\beta\mu}) + AV \log(1 - e^{-\beta(\epsilon - \mu)}) \right) \ .$$

The numbers of particles with energies $0, \epsilon$ are respectively N_0, N_ϵ. Write down expressions for N_0, N_ϵ in terms of μ.

At temperature T what is the maximum number of bosons N_ϵ^{max} in the normal phase (the state with energy ϵ)? Explain what happens when $N > N_\epsilon^{max}$.

Given N and T calculate the transition temperature T_B at which Bose condensation occurs.

For $T > T_B$ show that $\mu = \epsilon(T_B - T)/T_B$. What is the value of μ for $T < T_B$?

Calculate the mean energy E for (a) $T > T_B$ (b) $T < T_B$, and show that the heat capacity of the system at constant volume is

$$C_V = \begin{cases} \dfrac{1}{kT^2} \dfrac{AV\epsilon^2}{(e^{\beta\epsilon} - 1)^2} & T < T_B \\ 0 & T > T_B . \end{cases}$$

B4/23 **Statistical Physics**

A perfect gas in equilibrium in a volume V has quantum stationary states $|i\rangle$ with energies E_i. In a Boltzmann distribution, the probability that the system is in state $|i\rangle$ is $\rho_i = Z^{-1}e^{-E_i/kT}$. The entropy is defined to be $S = -k\sum_i \rho_i \log \rho_i$.

For two nearby states establish the equation

$$dE = TdS - PdV ,$$

where E and P should be defined.

For reversible changes show that

$$dS = \frac{\delta Q}{T} ,$$

where δQ is the amount of heat transferred in the exchange.

Define C_V, the heat capacity at constant volume.

A system with constant heat capacity C_V initially at temperature T is heated at constant volume to a temperature Θ. Show that the change in entropy is $\Delta S = C_V \log(\Theta/T)$.

Explain what is meant by isothermal and adiabatic transitions.

Briefly, describe the Carnot cycle and define its efficiency. Explain briefly why no heat engine can be more efficient than one whose operation is based on a Carnot cycle.

Three identical bodies with constant heat capacity at fixed volume C_V, are initially at temperatures T_1, T_2, T_3, respectively. Heat engines operate between the bodies with no input of work or heat from the outside and the respective temperatures are changed to $\Theta_1, \Theta_2, \Theta_3$, the volume of the bodies remaining constant. Show that, if the heat engines operate on a Carnot cycle, then

$$\Theta_1\Theta_2\Theta_3 = A , \qquad \Theta_1 + \Theta_2 + \Theta_3 = B ,$$

where $A = T_1T_2T_3$ and $B = T_1 + T_2 + T_3$.

Hence show that the maximum temperature to which any one of the bodies can be raised is Θ where

$$\Theta + 2\left(\frac{A}{\Theta}\right)^{1/2} = B .$$

Show that a solution is $\Theta = T$ if initially $T_1 = T_2 = T_3 = T$. Do you expect there to be any other solutions?

Find Θ if initially $T_1 = 300\,\mathrm{K},\ T_2 = 300\,\mathrm{K},\ T_3 = 100\,\mathrm{K}$.

[*Hint: Choose to maximize one temperature and impose the constraints above using Lagrange multipliers.*]

B1/23 **Applications of Quantum Mechanics**

A quantum system, with Hamiltonian H_0, has continuous energy eigenstates $|E\rangle$ for all $E \geq 0$, and also a discrete eigenstate $|0\rangle$, with $H_0|0\rangle = E_0|0\rangle$, $\langle 0|0\rangle = 1$, $E_0 > 0$. A time-independent perturbation H_1, such that $\langle E|H_1|0\rangle \neq 0$, is added to H_0. If the system is initially in the state $|0\rangle$ obtain the formula for the decay rate

$$w = \frac{2\pi}{\hbar}\, \rho(E_0)\, \big|\langle E_0|H_1|0\rangle\big|^2 \ ,$$

where ρ is the density of states.

[*You may assume that* $\frac{1}{t}\big(\frac{\sin\frac{1}{2}\omega t}{\frac{1}{2}\omega}\big)^2$ *behaves like* $2\pi\,\delta(\omega)$ *for large* t.]

Assume that, for a particle moving in one dimension,

$$H_0 = E_0|0\rangle\langle 0| + \int_{-\infty}^{\infty} p^2 |p\rangle\langle p|\, dp\,, \qquad H_1 = f\int_{-\infty}^{\infty} \big(|p\rangle\langle 0| + |0\rangle\langle p|\big)\, dp\,,$$

where $\langle p'|p\rangle = \delta(p'-p)$, and f is constant. Obtain w in this case.

B2/22 **Applications of Quantum Mechanics**

Define the reciprocal lattice for a lattice L with lattice vectors $\boldsymbol{\ell}$.

A beam of electrons, with wave vector $\mathbf{k}$, is incident on a Bravais lattice L with a large number of atoms, N. If the scattering amplitude for scattering on an individual atom in the direction $\hat{\mathbf{k}}'$ is $f(\hat{\mathbf{k}}')$, show that the scattering amplitude for the whole lattice is

$$\sum_{\boldsymbol{\ell}\in L} e^{i\mathbf{q}\cdot\boldsymbol{\ell}} f(\hat{\mathbf{k}}')\,, \qquad \mathbf{q} = \mathbf{k} - |\mathbf{k}|\,\hat{\mathbf{k}}'\,.$$

Derive the formula for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = N|f(\hat{\mathbf{k}}')|^2\Delta(\mathbf{q})\,,$$

obtaining an explicit form for $\Delta(\mathbf{q})$. Show that $\Delta(\mathbf{q})$ is strongly peaked when $\mathbf{q} = \mathbf{g}$, a reciprocal lattice vector. Show that this leads to the Bragg formula $2d\sin\frac{\theta}{2} = \lambda$, where θ is the scattering angle, λ the electron wavelength and d the separation between planes of atoms in the lattice.

B3/23 **Applications of Quantum Mechanics**

A periodic potential is expressed as $V(\mathbf{x}) = \sum_{\mathbf{g}} a_{\mathbf{g}}\, e^{i\mathbf{g}\cdot\mathbf{x}}$, where $\{\mathbf{g}\}$ are reciprocal lattice vectors and $a_{\mathbf{g}}{}^* = a_{-\mathbf{g}}$, $a_{\mathbf{0}} = 0$. In the nearly free electron model explain why it is appropriate, near the boundaries of energy bands, to consider a Bloch wave state

$$|\psi_{\mathbf{k}}\rangle = \sum_r \alpha_r\, |\mathbf{k}_r\rangle\,, \qquad \mathbf{k}_r = \mathbf{k} + \mathbf{g}_r\,,$$

where $|\mathbf{k}\rangle$ is a free electron state for wave vector $\mathbf{k}$, $\langle\mathbf{k}'|\mathbf{k}\rangle = \delta_{\mathbf{k}'\mathbf{k}}$, and the sum is restricted to reciprocal lattice vectors $\mathbf{g}_r$ such that $|\mathbf{k}_r| \approx |\mathbf{k}|$. Obtain a determinantal formula for the possible energies $E(\mathbf{k})$ corresponding to Bloch wave states of this form.

[*You may take* $\mathbf{g}_1 = \mathbf{0}$ *and assume* $e^{i\mathbf{b}\cdot\mathbf{x}}|\mathbf{k}\rangle = |\mathbf{k}+\mathbf{b}\rangle$ *for any* $\mathbf{b}$.]

Suppose the sum is restricted to just $\mathbf{k}$ and $\mathbf{k}+\mathbf{g}$. Show that there is a gap $2|a_{\mathbf{g}}|$ between energy bands. Setting $\mathbf{k} = -\frac{1}{2}\mathbf{g} + \mathbf{q}$, show that there are two Bloch wave states with energies near the boundaries of the energy bands

$$E_{\pm}(\mathbf{k}) \approx \frac{\hbar^2|\mathbf{g}|^2}{8m} \pm |a_{\mathbf{g}}| + \frac{\hbar^2|\mathbf{q}|^2}{2m} \pm \frac{\hbar^4}{8m^2|a_{\mathbf{g}}|}\,(\mathbf{q}\cdot\mathbf{g})^2\,.$$

What is meant by effective mass? Determine the value of the effective mass at the top and the bottom of the adjacent energy bands if $\mathbf{q}$ is parallel to $\mathbf{g}$.

B4/24 **Applications of Quantum Mechanics**

Explain the variational method for computing the ground state energy for a quantum Hamiltonian.

For the one-dimensional Hamiltonian

$$H = \tfrac{1}{2}p^2 + \lambda x^4\,,$$

obtain an approximate form for the ground state energy by considering as a trial state the state $|w\rangle$ defined by $a|w\rangle = 0$, where $\langle w|w\rangle = 1$ and $a = (w/2\hbar)^{\frac{1}{2}}(x + ip/w)$.

[*It is useful to note that* $\langle w|(a+a^\dagger)^4|w\rangle = \langle w|(a^2a^{\dagger 2} + aa^\dagger aa^\dagger)|w\rangle$.]

Explain why the states $a^\dagger|w\rangle$ may be used as trial states for calculating the first excited energy level.

B1/25 **Fluid Dynamics II**

State the minimum dissipation theorem for Stokes flow in a bounded domain.

Fluid of density ρ and viscosity μ fills an infinite cylindrical annulus $a \le r \le b$ between a fixed cylinder $r = a$ and a cylinder $r = b$ which rotates about its axis with constant angular velocity Ω. In cylindrical polar coordinates (r, θ, z), the fluid velocity is $\mathbf{u} = (0, v(r), 0)$. The Reynolds number $\rho\Omega b^2/\mu$ is not necessarily small. Show that $v(r) = Ar + B/r$, where A and B are constants to be determined.

[*You may assume that* $\nabla^2\mathbf{u} = (0, \nabla^2 v - v/r^2, 0)$ *and* $(\mathbf{u}\cdot\nabla)\mathbf{u} = (-v^2/r,\ 0,\ 0)$.]

Show that the outer cylinder exerts a couple G_0 per unit length on the fluid, where

$$G_0 = \frac{4\pi\mu\Omega a^2 b^2}{b^2 - a^2}.$$

[*You may assume that, in standard notation,* $e_{r\theta} = \dfrac{r}{2}\dfrac{d}{dr}\left(\dfrac{v}{r}\right)$.]

Suppose now that $b \ge \sqrt{2}a$ and that the cylinder $r = a$ is replaced by a fixed cylinder whose cross-section is a square of side $2a$ centred on $r = 0$, all other conditions being unchanged. The flow may still be assumed steady. Explaining your argument carefully, show that the couple G now required to maintain the motion of the outer cylinder is greater than G_0.

B2/24 **Fluid Dynamics II**

A thin layer of liquid of kinematic viscosity ν flows under the influence of gravity down a plane inclined at an angle α to the horizontal $(0 \le \alpha \le \pi/2)$. With origin O on the plane, and axes Ox down the line of steepest slope and Oy normal to the plane, the free surface is given by $y = h(x,t)$, where $|\partial h/\partial x| \ll 1$. The pressure distribution in the liquid may be assumed to be hydrostatic. Using the approximations of lubrication theory, show that

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left\{h^3(\cos\alpha\frac{\partial h}{\partial x} - \sin\alpha)\right\}.$$

Now suppose that

$$h = h_0 + \eta(x,t)\ ,$$

where

$$\eta(x,0) = \eta_0 e^{-x^2/a^2}$$

and h_0, η_0 and a are constants with $\eta_0 \ll a, h_0$. Show that, to leading order,

$$\eta(x,t) = \frac{a\eta_0}{(a^2+4Dt)^{1/2}}\exp\left\{-\frac{(x-Ut)^2}{a^2+4Dt}\right\}\ ,$$

where U and D are constants to be determined.

Explain in physical terms the meaning of this solution.

B3/24 **Fluid Dynamics II**

(i) Suppose that, with spherical polar coordinates, the Stokes streamfunction

$$\Psi_\lambda(r,\theta) = r^\lambda \sin^2\theta\cos\theta$$

represents a Stokes flow and thus satisfies the equation $D^2(D^2\Psi_\lambda)=0$, where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \;.$$

Show that the possible values of λ are 5, 3, 0 and -2. For which of these values is the corresponding flow irrotational? Sketch the streamlines of the flow for the case $\lambda = 3$.

(ii) A spherical drop of liquid of viscosity μ_1, radius a and centre at $r=0$, is suspended in another liquid of viscosity μ_2 which flows with streamfunction

$$\Psi \sim \Psi_\infty(r,\theta) = \alpha r^3\sin^2\theta\cos\theta$$

far from the drop. The two liquids are of equal densities, surface tension is sufficiently strong to keep the drop spherical, and inertia is negligible. Show that

$$\Psi = \begin{cases} (Ar^5 + Br^3)\sin^2\theta\cos\theta & (r<a), \\ (\alpha r^3 + C + D/r^2)\sin^2\theta\cos\theta & (r>a) \end{cases}$$

and obtain four equations determining the constants A, B, C and D. (You need not solve these equations.)

[*You may assume, with standard notation, that*

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} \quad , \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \quad , \quad e_{r\theta} = \frac{1}{2}\left\{ r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right\} .]$$

B4/26 **Fluid Dynamics II**

Write an essay on boundary-layer theory and its application to the generation of lift in aerodynamics.

You should include discussion of the derivation of the boundary-layer equation, the similarity transformation leading to the Falkner–Skan equation, the influence of an adverse pressure gradient, and the mechanism(s) by which circulation is generated in flow past bodies with a sharp trailing edge.

B1/26 **Waves in Fluid and Solid Media**

Starting from the equations governing sound waves linearized about a state with density ρ_0 and sound speed c_0, derive the acoustic energy equation, giving expressions for the local energy density E and energy flux $\mathbf{I}$.

A sphere executes small-amplitude vibrations, with its radius varying according to

$$r(t) = a + \mathrm{Re}\left(\epsilon e^{i\omega t}\right),$$

with $0 < \epsilon \ll a$. Find an expression for the velocity potential of the sound, $\tilde{\phi}(r,t)$. Show that the time-averaged rate of working by the surface of the sphere is

$$2\pi a^2 \rho_0 \omega^2 \epsilon^2 c_0 \frac{\omega^2 a^2}{c_0^2 + \omega^2 a^2}.$$

Calculate the value at $r = a$ of the dimensionless ratio $c_0\overline{E}/|\overline{\mathbf{I}}|$, where the overbars denote time-averaged values, and comment briefly on the limits $c_0 \ll \omega a$ and $c_0 \gg \omega a$.

B2/25 **Waves in Fluid and Solid Media**

Starting from the equations for one-dimensional unsteady flow of a perfect gas of uniform entropy, show that the Riemann invariants,

$$R_{\pm} = u \pm \frac{2}{\gamma - 1}(c - c_0),$$

are constant on characteristics $C_{\pm}$ given by $\dfrac{dx}{dt} = u \pm c$, where $u(x,t)$ is the velocity of the gas, $c(x,t)$ is the local speed of sound and γ is the specific heat ratio.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas and the piston are initially at rest. At time $t = 0$ the piston starts moving to the left at a constant speed V. Find $u(x,t)$ and $c(x,t)$ in the three regions

$$\begin{aligned} &\text{(i)} \quad c_0 t \le x, \\ &\text{(ii)} \quad at \le x < c_0 t, \\ &\text{(iii)} \ -Vt \le x < at, \end{aligned}$$

where $a = c_0 - \frac{1}{2}(\gamma+1)V$. What is the largest value of V for which c is positive throughout region (iii)? What happens if V exceeds this value?

B3/25 **Waves in Fluid and Solid Media**

Consider the equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} - \frac{\partial^3 \phi}{\partial x^3} = 0.$$

Find the dispersion relation for waves of frequency ω and wavenumber k. Do the wave crests move faster or slower than a packet of waves?

Write down the solution with initial value

$$\phi(x, 0) = \int_{-\infty}^{\infty} A(k) e^{ikx} \, dk \ ,$$

where $A(k)$ is real and $A(-k) = A(k)$.

Use the method of stationary phase to obtain an approximation to $\phi(x, t)$ for large t, with x/t having the constant value V. Explain, using the notion of group velocity, the constraint that must be placed on V.

B4/27 **Waves in Fluid and Solid Media**

Write down the equation governing linearized displacements $\mathbf{u}(\mathbf{x}, t)$ in a uniform elastic medium of density ρ and Lamé constants λ and μ. Derive solutions for monochromatic plane P and S waves, and find the corresponding wave speeds c_P and c_S.

Such an elastic solid occupies the half-space $z > 0$, and the boundary $z = 0$ is clamped rigidly so that $\mathbf{u}(x, y, 0, t) = \mathbf{0}$. A plane SV-wave with frequency ω and wavenumber $(k, 0, -m)$ is incident on the boundary. At some angles of incidence, there results both a reflected SV-wave with frequency ω' and wavenumber $(k', 0, m')$ and a reflected P-wave with frequency ω'' and wavenumber $(k'', 0, m'')$. Relate the frequencies and wavenumbers of the reflected waves to those of the incident wave. At what angles of incidence will there be a reflected P-wave?

Find the amplitudes of the reflected waves as multiples of the amplitude of the incident wave. Confirm that these amplitudes give the sum of the time-averaged vertical fluxes of energy of the reflected waves equal to the time-averaged vertical flux of energy of the incident wave.

[*Results concerning the energy flux, energy density and kinetic energy density in a plane elastic wave may be quoted without proof.*]

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Monday 3 June 2002 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question is divided into Part ***(i)*** *and Part* ***(ii)****, which may or may not be related. Candidates may attempt either or both Parts of any question, but must not attempt Parts from more than* ***SIX*** *questions. If you submit answers to Parts of more than six questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same, with Part ***(ii)*** *of each question carrying twice as many marks as Part* ***(i)****. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked ***C,D,E, . . . , M*** *according to the letter affixed to each question. (For example,* ***2G, 19G*** *should be in one bundle and* ***7J, 9J*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *Parts of* ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) We are given a finite set of airports. Assume that between any two airports, i and j, there are $a_{ij} = a_{ji}$ flights in each direction on every day. A confused traveller takes one flight per day, choosing at random from all available flights. Starting from i, how many days on average will pass until the traveller returns again to i? Be careful to allow for the case where there may be no flights at all between two given airports.

(ii) Consider the infinite tree T with root R, where, for all $m \geqslant 0$, all vertices at distance 2^m from R have degree 3, and where all other vertices (except R) have degree 2. Show that the random walk on T is recurrent.

2G Principles of Dynamics

(i) Derive Hamilton's equations from Lagrange's equations. Show that the Hamiltonian H is constant if the Lagrangian L does not depend explicitly on time.

(ii) A particle of mass m is constrained to move under gravity, which acts in the negative z-direction, on the spheroidal surface $\epsilon^{-2}(x^2 + y^2) + z^2 = l^2$, with $0 < \epsilon \leqslant 1$. If θ, ϕ parametrize the surface so that

$$x = \epsilon l \sin\theta \cos\phi, \;\; y = \epsilon l \sin\theta \sin\phi, \;\; z = l\cos\theta,$$

find the Hamiltonian $H(\theta, \phi, p_\theta, p_\phi)$.

Show that the energy

$$E = \frac{p_\theta^2}{2ml^2(\epsilon^2\cos^2\theta + \sin^2\theta)} + \frac{\alpha}{\sin^2\theta} + mgl\cos\theta$$

is a constant of the motion, where α is a non-negative constant.

Rewrite this equation as

$$\frac{1}{2}\dot{\theta}^2 + V_{\text{eff}}(\theta) = 0$$

and sketch $V_{\text{eff}}(\theta)$ for $\epsilon = 1$ and $\alpha > 0$, identifying the maximal and minimal values of $\theta(t)$ for fixed α and E. If ϵ is now taken not to be unity, how do these values depend on ϵ?

3K Functional Analysis

(i) Let $P_r(e^{i\theta})$ be the real part of $\dfrac{1+re^{i\theta}}{1-re^{i\theta}}$. Establish the following properties of P_r for $0 \leqslant r < 1$:

(a) $0 < P_r(e^{i\theta}) = P_r(e^{-i\theta}) \leqslant \dfrac{1+r}{1-r}$;

(b) $P_r(e^{i\theta}) \leqslant P_r(e^{i\delta})$ for $0 < \delta \leqslant |\theta| \leqslant \pi$;

(c) $P_r(e^{i\theta}) \to 0$, uniformly on $0 < \delta \leqslant |\theta| \leqslant \pi$, as r increases to 1.

(ii) Suppose that $f \in L^1(\mathbf{T})$, where $\mathbf{T}$ is the unit circle $\{e^{i\theta} : -\pi \leqslant \theta \leqslant \pi\}$. By definition, $\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\, d\theta$. Let

$$P_r(f)(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i(\theta-t)})f(e^{it})\, dt.$$

Show that $P_r(f)$ is a continuous function on $\mathbf{T}$, and that $\|P_r(f)\|_1 \leqslant \|f\|_1$.

[*You may assume without proof that* $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i\theta})\, d\theta = 1$.]

Show that $P_r(f) \to f$, uniformly on $\mathbf{T}$ as r increases to 1, if and only if f is a continuous function on $\mathbf{T}$.

Show that $\|P_r(f) - f\|_1 \to 0$ as r increases to 1.

4H Groups, Rings and Fields

(i) What is a Sylow subgroup? State Sylow's Theorems.

Show that any group of order 33 is cyclic.

(ii) Prove the existence part of Sylow's Theorems.

[*You may use without proof any arithmetic results about binomial coefficients which you need.*]

Show that a group of order p^2q, where p and q are distinct primes, is not simple. Is it always abelian? Give a proof or a counterexample.

5D Electromagnetism

(i) Show that, in a region where there is no magnetic field and the charge density vanishes, the electric field can be expressed either as minus the gradient of a scalar potential ϕ or as the curl of a vector potential $\mathbf{A}$. Verify that the electric field derived from

$$\mathbf{A} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p} \wedge \mathbf{r}}{r^3}$$

is that of an electrostatic dipole with dipole moment $\mathbf{p}$.

[*You may assume the following identities:*

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \wedge (\nabla \wedge \mathbf{b}) + \mathbf{b} \wedge (\nabla \wedge \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a},$$

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a}.]$$

(ii) An infinite conducting cylinder of radius a is held at zero potential in the presence of a line charge parallel to the axis of the cylinder at distance $s_0 > a$, with charge density q per unit length. Show that the electric field outside the cylinder is equivalent to that produced by replacing the cylinder with suitably chosen image charges.

6F Dynamics of Differential Equations

(i) A system in $\mathbb{R}^2$ obeys the equations:

$$\begin{aligned} \dot{x} &= x - x^5 - 2xy^4 - 2y^3(a - x^2) \ , \\ \dot{y} &= y - x^4 y - 2y^5 + x^3(a - x^2) \ , \end{aligned}$$

where a is a positive constant.

By considering the quantity $V = \alpha x^4 + \beta y^4$, where α and β are appropriately chosen, show that if $a > 1$ then there is a unique fixed point and a unique limit cycle. How many fixed points are there when $a < 1$?

(ii) Consider the second order system

$$\ddot{x} - (a - bx^2)\dot{x} + x - x^3 = 0 \ ,$$

where a, b are constants.

(a) Find the fixed points and determine their stability.

(b) Show that if the fixed point at the origin is unstable and $3a > b$ then there are no limit cycles.

[*You may find it helpful to use the Liénard coordinate* $z = \dot{x} - ax + \frac{1}{3}bx^3$.]

7J Logic, Computation and Set Theory

(i) State the Knaster-Tarski fixed point theorem. Use it to prove the Cantor-Bernstein Theorem; that is, if there exist injections $A \to B$ and $B \to A$ for two sets A and B then there exists a bijection $A \to B$.

(ii) Let A be an arbitrary set and suppose given a subset R of $PA \times A$. We define a subset $B \subseteq A$ to be R-*closed* just if whenever $(S, a) \in R$ and $S \subseteq B$ then $a \in B$. Show that the set of all R-closed subsets of A is a complete poset in the inclusion ordering.

Now assume that A is itself equipped with a partial ordering $\leqslant$.

(a) Suppose R satisfies the condition that if $b \geqslant a \in A$ then $(\{b\}, a) \in R$. Show that if B is R-closed then $c \leqslant b \in B$ implies $c \in B$.

(b) Suppose that R satisfies the following condition. Whenever $(S, a) \in R$ and $b \leqslant a$ then there exists $T \subseteq A$ such that $(T, b) \in R$, and for every $t \in T$ we have (i) $(\{b\}, t) \in R$, and (ii) $t \leqslant s$ for some $s \in S$. Let B and C be R-closed subsets of A. Show that the set

$$[B \to C] = \{a \in A \,|\, \forall b \leqslant a \,(b \in B \Rightarrow b \in C)\}$$

is R-closed.

8H Graph Theory

(i) State and prove a necessary and sufficient condition for a graph to be Eulerian (that is, to have an Eulerian circuit).

Prove that, given any connected non-Eulerian graph G, there is an Eulerian graph H and a vertex $v \in H$ such that $G = H - v$.

(ii) Let G be a connected plane graph with n vertices, e edges and f faces. Prove that $n - e + f = 2$. Deduce that $e \le g(n-2)/(g-2)$, where g is the smallest face size.

The *crossing number* $c(G)$ of a non-planar graph G is the minimum number of edge-crossings needed when drawing the graph in the plane. (The crossing of three edges at the same point is not allowed.) Show that if G has n vertices and e edges then $c(G) \ge e - 3n + 6$. Find $c(K_6)$.

9J Number Theory

(i) Let p be a prime number. Prove that the multiplicative group of the field with p elements is cyclic.

(ii) Let p be an odd prime, and let $k \geqslant 1$ be an integer. Prove that we have $x^2 \equiv 1 \bmod p^k$ if and only if either $x \equiv 1 \bmod p^k$ or $x \equiv -1 \bmod p^k$. Is this statement true when $p = 2$?

Let m be an odd positive integer, and let r be the number of distinct prime factors of m. Prove that there are precisely 2^r different integers x satisfying $x^2 \equiv 1 \bmod m$ and $0 < x < m$.

10H Coding and Cryptography

(i) Describe the original Hamming code of length 7. Show how to encode a message word, and how to decode a received word involving at most one error. Explain why the procedure works.

(ii) What is a linear binary code? What is its dual code? What is a cyclic binary code? Explain how cyclic binary codes of length n correspond to polynomials in $\mathbb{F}_2[X]$ dividing $X^n + 1$. Show that the dual of a cyclic code of length n is cyclic of length n.

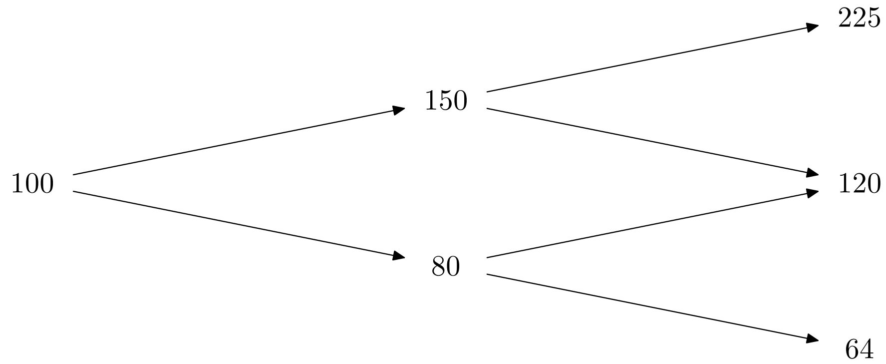
Using the factorization

$$X^7 + 1 = (X + 1)(X^3 + X + 1)(X^3 + X^2 + 1)$$

in $\mathbb{F}_2[X]$, find all cyclic binary codes of length 7. Identify those which are Hamming codes and their duals. Justify your answer.

11L Stochastic Financial Models

(i) The prices, S_i, of a stock in a binomial model at times $i = 0, 1, 2$ are represented by the following binomial tree.



The fixed interest rate per period is $1/5$ and the probability that the stock price increases in a period is $1/3$. Find the price at time 0 of a European call option with strike price 78 and expiry time 2.

Explain briefly the ideas underlying your calculations.

(ii) Consider an investor in a one-period model who may invest in s assets, all of which are risky, with a random return vector $\boldsymbol{R}$ having mean $\mathbb{E}\boldsymbol{R} = \boldsymbol{r}$ and positive-definite covariance matrix $\boldsymbol{V}$; assume that not all the assets have the same expected return. Show that any minimum-variance portfolio is equivalent to the investor dividing his wealth between two portfolios, the global minimum-variance portfolio and the diversified portfolio, both of which should be specified clearly in terms of $\boldsymbol{r}$ and $\boldsymbol{V}$.

Now suppose that $\boldsymbol{R} = (R_1, R_2, \ldots, R_s)^\top$ where $R_1, R_2, \ldots, R_s$ are independent random variables with R_i having the exponential distribution with probability density function $\lambda_i e^{-\lambda_i x}$, $x \geqslant 0$, where $\lambda_i > 0$, $1 \leqslant i \leqslant s$. Determine the global minimum-variance portfolio and the diversified portfolio explicitly.

Consider further the situation when the investor has the utility function $u(x) = 1 - e^{-x}$, where x denotes his wealth. Suppose that he acts to maximize the expected utility of his final wealth, and that his initial wealth is $w > 0$. Show that he now divides his wealth between the diversified portfolio and the *uniform* portfolio, in which wealth is apportioned equally between the assets, and determine the amounts that he invests in each.

12M Principles of Statistics

(i) Explain in detail the *minimax* and *Bayes* principles of decision theory.

Show that if $d(X)$ is a Bayes decision rule for a prior density $\pi(\theta)$ and has constant risk function, then $d(X)$ is minimax.

(ii) Let $X_1, \ldots, X_p$ be independent random variables, with $X_i \sim N(\mu_i, 1)$, $i = 1, \ldots, p$.

Consider estimating $\mu = (\mu_1, \ldots, \mu_p)^T$ by $d = (d_1, \ldots, d_p)^T$, with loss function

$$L(\mu, d) = \sum_{i=1}^{p} (\mu_i - d_i)^2 \ .$$

What is the risk function of $X = (X_1, \ldots, X_p)^T$?

Consider the class of estimators of μ of the form

$$d^a(X) = \left(1 - \frac{a}{X^T X}\right) X \ ,$$

indexed by $a \geqslant 0$. Find the risk function of $d^a(X)$ in terms of $E\left(1/X^T X\right)$, which you should not attempt to evaluate, and deduce that X is inadmissible. What is the optimal value of a?

[*You may assume Stein's Lemma, that for suitably behaved real-valued functions h,*

$$E\left\{(X_i - \mu_i) h(X)\right\} = E\left\{\frac{\partial h(X)}{\partial X_i}\right\}. \quad]$$

13L Computational Statistics and Statistical Modelling

(i) Suppose $Y_1, \ldots, Y_n$ are independent Poisson variables, and

$$\mathbb{E}(Y_i) = \mu_i \ , \ \log \mu_i = \alpha + \beta^T x_i \ , \ 1 \leqslant i \leqslant n$$

where α, β are unknown parameters, and $x_1, \ldots, x_n$ are given covariates, each of dimension p. Obtain the maximum-likelihood equations for α, β, and explain briefly how you would check the validity of this model.

(ii) The data below show $y_1, \ldots, y_{33}$, which are the monthly accident counts on a major US highway for each of the 12 months of 1970, then for each of the 12 months of 1971, and finally for the first 9 months of 1972. The data-set is followed by the (slightly edited) R output. You may assume that the factors 'Year' and 'month' have been set up in the appropriate fashion. Give a careful interpretation of this R output, and explain (a) how you would derive the corresponding standardised residuals, and (b) how you would predict the number of accidents in October 1972.

```
52 37  49  29  31  32  28  34  32  39  50  63
35 22  27  27  34  23  42  30  36  56  48  40
33 26  31  25  23  20  25  20  36
```

> first.glm _ glm(y~ Year + month, poisson) ; summary(first.glm)

Call:

glm(formula = y ~ Year + month, family = poisson)

```
Coefficients:
                  Estimate   Std. Error   z value   Pr(>|z|)
(Intercept)        3.81969   0.09896      38.600    < 2e-16 ***
Year1971          -0.12516   0.06694      -1.870    0.061521 .
Year1972          -0.28794   0.08267      -3.483    0.000496 ***
month2            -0.34484   0.14176      -2.433    0.014994 *
month3            -0.11466   0.13296      -0.862    0.388459
month4            -0.39304   0.14380      -2.733    0.006271 **
month5            -0.31015   0.14034      -2.210    0.027108 *
month6            -0.47000   0.14719      -3.193    0.001408 **
month7            -0.23361   0.13732      -1.701    0.088889 .
month8            -0.35667   0.14226      -2.507    0.012168 *
month9            -0.14310   0.13397      -1.068    0.285444
month10            0.10167   0.13903       0.731    0.464628
month11            0.13276   0.13788       0.963    0.335639
month12            0.18252   0.13607       1.341    0.179812
```

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'

(Dispersion parameter for poisson family taken to be 1)

```
    Null deviance:     101.143   on 32 degrees of freedom
Residual deviance:     27.273    on 19 degrees of freedom
```

Number of Fisher Scoring iterations: 3

14E Quantum Physics

(i) A system of N identical non-interacting bosons has energy levels E_i with degeneracy g_i, $1 \leq i < \infty$, for each particle. Show that in thermal equilibrium the number of particles N_i with energy E_i is given by

$$N_i = \frac{g_i}{e^{\beta(E_i - \mu)} - 1} \quad ,$$

where β and μ are parameters whose physical significance should be briefly explained.

(ii) A photon moves in a cubical box of side L. Assuming periodic boundary conditions, show that, for large L, the number of photon states lying in the frequency range $\omega \to \omega + d\omega$ is $\rho(\omega)d\omega$ where

$$\rho(\omega) = L^3 \left(\frac{\omega^2}{\pi^2 c^3} \right) \quad .$$

If the box is filled with thermal radiation at temperature T, show that the number of photons per unit volume in the frequency range $\omega \to \omega + d\omega$ is $n(\omega)d\omega$ where

$$n(\omega) = \left(\frac{\omega^2}{\pi^2 c^3} \right) \frac{1}{e^{\hbar\omega/kT} - 1} \quad .$$

Calculate the energy density W of the thermal radiation. Show that the pressure P exerted on the surface of the box satisfies

$$P = \frac{1}{3} W \quad .$$

[*You may use the result* $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$.]

15D General Relativity

(i) Given a covariant vector field V_a, define the curvature tensor $R^a{}_{bcd}$ by

$$V_{a;bc} - V_{a;cb} = V_e R^e{}_{abc}. \qquad (*)$$

Express $R^e{}_{abc}$ in terms of the Christoffel symbols and their derivatives. Show that

$$R^e{}_{abc} = -R^e{}_{acb}.$$

Further, by setting $V_a = \partial\phi/\partial x^a$, deduce that

$$R^e{}_{abc} + R^e{}_{cab} + R^e{}_{bca} = 0.$$

(ii) Write down an expression similar to $(*)$ given in Part (i) for the quantity

$$g_{ab;cd} - g_{ab;dc}$$

and hence show that

$$R_{eabc} = -R_{aebc}.$$

Define the Ricci tensor, show that it is symmetric and write down the contracted Bianchi identities.

In certain spacetimes of dimension $n \geq 2$, R_{abcd} takes the form

$$R_{abcd} = K(x^e)[g_{ac}g_{bd} - g_{ad}g_{bc}].$$

Obtain the Ricci tensor and Ricci scalar. Deduce that K is a constant in such spacetimes if the dimension n is greater than 2.

16D Statistical Physics and Cosmology

(i) Consider a one-dimensional model universe with "stars" distributed at random on the x-axis, and choose the origin to coincide with one of the stars; call this star the "home-star." Home-star astronomers have discovered that all other stars are receding from them with a velocity $v(x)$, that depends on the position x of the star. Assuming non-relativistic addition of velocities, show how the assumption of homogeneity implies that $v(x) = H_0 x$ for some constant H_0.

In attempting to understand the history of their one-dimensional universe, home-star astronomers seek to determine the velocity $v(t)$ at time t of a star at position $x(t)$. Assuming homogeneity, show how $x(t)$ is determined in terms of a scale factor $a(t)$ and hence deduce that $v(t) = H(t)x(t)$ for some function $H(t)$. What is the relation between $H(t)$ and H_0?

(ii) Consider a three-dimensional homogeneous and isotropic universe with mass density $\rho(t)$, pressure $p(t)$ and scale factor $a(t)$. Given that $E(t)$ is the energy in volume $V(t)$, show how the relation $dE = -p\,dV$ yields the "fluid" equation

$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)H,$$

where $H = \dot{a}/a$.

Show how conservation of energy applied to a test particle at the boundary of a spherical fluid element yields the Friedmann equation

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = -kc^2$$

for constant k. Hence obtain an equation for the acceleration $\ddot{a}$ in terms of ρ, p and a.

A model universe has mass density and pressure

$$\rho = \frac{\rho_0}{a^3} + \rho_1, \qquad p = -\rho_1 c^2,$$

where ρ_0 is constant. What does the fluid equation imply about ρ_1? Show that the acceleration $\ddot{a}$ vanishes if

$$a = \left(\frac{\rho_0}{2\rho_1}\right)^{\frac{1}{3}}.$$

Hence show that this universe is static and determine the sign of the constant k.

17E Symmetries and Groups in Physics

(i) Let H be a normal subgroup of the group G. Let G/H denote the group of cosets $\tilde{g} = gH$ for $g \in G$. If $D : G \to GL(\mathbb{C}^n)$ is a representation of G with $D(h_1) = D(h_2)$ for all $h_1, h_2 \in H$ show that $\tilde{D}(\tilde{g}) = D(g)$ is well-defined and that it is a representation of G/H. Show further that $\tilde{D}(\tilde{g})$ is irreducible if and only if $D(g)$ is irreducible.

(ii) For a matrix $U \in SU(2)$ define the linear map $\Phi_U : \mathbb{R}^3 \to \mathbb{R}^3$ by $\Phi_U(\mathbf{x}).\boldsymbol{\sigma} = U\mathbf{x}.\boldsymbol{\sigma}U^\dagger$ with $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ as the vector of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\|\Phi_U(\mathbf{x})\| = \|\mathbf{x}\|$. Because of the linearity of Φ_U there exists a matrix $R(U)$ such that $\Phi_U(\mathbf{x}) = R(U)\mathbf{x}$. Given that any $SU(2)$ matrix can be written as

$$U = \cos\alpha\, I - i \sin\alpha\, \mathbf{n}.\boldsymbol{\sigma} \quad ,$$

where $\alpha \in [0, \pi]$ and $\mathbf{n}$ is a unit vector, deduce that $R(U) \in SO(3)$ for all $U \in SU(2)$. Compute $R(U)\mathbf{n}$ and $R(U)\mathbf{x}$ in the case that $\mathbf{x}.\mathbf{n} = 0$ and deduce that $R(U)$ is the matrix of a rotation about $\mathbf{n}$ with angle 2α.

[*Hint:* $\mathbf{m}.\boldsymbol{\sigma}\ \mathbf{n}.\boldsymbol{\sigma} = \mathbf{m}.\mathbf{n}\, I + i(\mathbf{m} \times \mathbf{n}).\boldsymbol{\sigma}$.]

Show that $R(U)$ defines a surjective homomorphism $\Theta : SU(2) \to SO(3)$ and find the kernel of Θ.

18C Transport Processes

(i) Material of thermal diffusivity D occupies the semi-infinite region $x > 0$ and is initially at uniform temperature T_0. For time $t > 0$ the temperature at $x = 0$ is held at a constant value $T_1 > T_0$. Given that the temperature $T(x,t)$ in $x > 0$ satisfies the diffusion equation $T_t = DT_{xx}$, write down the equation and the boundary and initial conditions satisfied by the dimensionless temperature $\theta = (T - T_0)\,/\,(T_1 - T_0)$.

Use dimensional analysis to show that the lengthscale of the region in which T is significantly different from T_0 is proportional to $(Dt)^{1/2}$. Hence show that this problem has a similarity solution

$$\theta = \operatorname{erfc}(\xi/2) \equiv \frac{2}{\sqrt{\pi}} \int_{\xi/2}^{\infty} e^{-u^2}\, du \ ,$$

where $\xi = x/(Dt)^{1/2}$.

What is the rate of heat input, $-DT_x$, across the plane $x = 0$?

(ii) Consider the same problem as in Part (i) except that the boundary condition at $x = 0$ is replaced by one of constant rate of heat input Q. Show that $\theta(\xi, t)$ satisfies the partial differential equation

$$\theta_{\xi\xi} + \frac{\xi}{2}\theta_\xi = t\theta_t$$

and write down the boundary conditions on $\theta(\xi, t)$. Deduce that the problem has a similarity solution of the form

$$\theta = \frac{Q(t/D)^{1/2}}{T_1 - T_0} f(\xi).$$

Derive the ordinary differential equation and boundary conditions satisfied by $f(\xi)$. Differentiate this equation once to obtain

$$f''' + \frac{\xi}{2} f'' = 0$$

and solve for $f'(\xi)$. Hence show that

$$f(\xi) = \frac{2}{\sqrt{\pi}} e^{-\xi^2/4} - \xi \operatorname{erfc}(\xi/2) \ .$$

Sketch the temperature distribution $T(x,t)$ for various times t, and calculate $T(0,t)$ explicitly.

19G Theoretical Geophysics

(i) In a reference frame rotating about a vertical axis with constant angular velocity $f/2$ the horizontal components of the momentum equation for a shallow layer of inviscid, incompressible fluid of constant density ρ are

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad ,$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} \quad ,$$

where u, v and P are independent of the vertical coordinate z.

Define the Rossby number Ro for a flow with typical velocity U and lengthscale L. What is the approximate form of the above equations when $Ro \ll 1$?

Show that the solution to the approximate equations is given by a streamfunction ψ proportional to P.

Conservation of potential vorticity for such a flow is represented by

$$\frac{D}{Dt}\frac{\zeta + f}{h} = 0,$$

where ζ is the vertical component of relative vorticity and $h(x, y)$ is the thickness of the layer. Explain briefly why the potential vorticity of a column of fluid should be conserved.

(ii) Suppose that the thickness of the rotating, shallow-layer flow in Part (i) is $h(y) = H_0 \exp(-\alpha y)$ where H_0 and α are constants. By linearising the equation of conservation of potential vorticity about $u = v = \zeta = 0$, show that the stream function for small disturbances to the state of rest obeys

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + \beta\frac{\partial \psi}{\partial x} = 0 \quad ,$$

where β is a constant that should be found.

Obtain the dispersion relationship for plane-wave solutions of the form $\psi \propto \exp[i(kx + ly - \omega t)]$. Hence calculate the group velocity.

Show that if $\beta > 0$ then the phase of these waves always propagates to the left (negative x direction) but that the energy may propagate to either left or right.

20F Numerical Analysis

(i) Let A be an $n \times n$ symmetric real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots., \mathbf{v}_n$, where $\|\mathbf{v}_l\| = 1$. Given $\mathbf{x}^{(0)} \in \mathbb{R}^n$, $\|\mathbf{x}^{(0)}\| = 1$, the sequence $\mathbf{x}^{(k)}$ is generated in the following manner. We set

$$\mu = \mathbf{x}^{(k)\,T} A \mathbf{x}^{(k)},$$

$$\mathbf{y} = (A - \mu I)^{-1} \mathbf{x}^{(k)},$$

$$\mathbf{x}^{(k+1)} = \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

Show that if

$$\mathbf{x}^{(k)} = c^{-1} \left(\mathbf{v}_1 + \alpha \sum_{l=2}^{n} d_l \mathbf{v}_l \right),$$

where α is a real scalar and c is chosen so that $\|\mathbf{x}^{(k)}\| = 1$, then

$$\mu = c^{-2} \left(\lambda_1 + \alpha^2 \sum_{j=2}^{n} \lambda_j d_j^2 \right).$$

Give an explicit expression for c.

(ii) Use the above result to prove that, if $|\alpha|$ is small,

$$\mathbf{x}^{(k+1)} = \tilde{c}^{-1} \left(\mathbf{v}_1 + \alpha^3 \sum_{l=2}^{n} \tilde{d}_l \mathbf{v}_l \right) + O(\alpha^4)$$

and obtain the numbers $\tilde{c}$ and $\tilde{d}_2, \ldots, \tilde{d}_n$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Tuesday 4 June 2002 9 to 12

PAPER 2

Before you begin read these instructions carefully.

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Begin each answer on a separate sheet.

*Write legibly and on only **one** side of the paper.*

At the end of the examination:

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Attach a completed cover sheet to each bundle.

*Complete a master cover sheet listing **all** Parts of **all** questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) In each of the following cases, the state-space I and non-zero transition rates q_{ij} $(i \neq j)$ of a continuous-time Markov chain are given. Determine in which cases the chain is explosive.

(a) $I = \{1, 2, 3, \ldots\}$, $\quad q_{i,i+1} = i^2$, $\quad i \in I$,
(b) $I = \mathbb{Z}$, $\quad q_{i,i-1} = q_{i,i+1} = 2^i$, $\quad i \in I$.

(ii) Children arrive at a see-saw according to a Poisson process of rate 1. Initially there are no children. The first child to arrive waits at the see-saw. When the second child arrives, they play on the see-saw. When the third child arrives, they all decide to go and play on the merry-go-round. The cycle then repeats. Show that the number of children at the see-saw evolves as a Markov Chain and determine its generator matrix. Find the probability that there are no children at the see-saw at time t.

Hence obtain the identity

$$\sum_{n=0}^{\infty} e^{-t} \frac{t^{3n}}{(3n)!} = \frac{1}{3} + \frac{2}{3}\, e^{-\frac{3}{2}t} \cos \frac{\sqrt{3}}{2} t \ .$$

2G Principles of Dynamics

(i) A number N of non-interacting particles move in one dimension in a potential $V(x,t)$. Write down the Hamiltonian and Hamilton's equations for one particle.

At time t, the number density of particles in phase space is $f(x,p,t)$. Write down the time derivative of f along a particle's trajectory. By equating the rate of change of the number of particles in a fixed domain V in phase space to the flux into V across its boundary, deduce that f is a constant along any particle's trajectory.

(ii) Suppose that $V(x) = \frac{1}{2} m\omega^2 x^2$, and particles are injected in such a manner that the phase space density is a constant f_1 at any point of phase space corresponding to a particle energy being smaller than E_1 and zero elsewhere. How many particles are present?

Suppose now that the potential is very slowly altered to the square well form

$$V(x) = \begin{cases} 0, & -L < x < L \\ \infty & \text{elsewhere .} \end{cases}$$

Show that the greatest particle energy is now

$$E_2 = \frac{\pi^2}{8} \frac{E_1^2}{mL^2\omega^2}.$$

3K Functional Analysis

(i) State and prove the parallelogram law for Hilbert spaces.

Suppose that K is a closed linear subspace of a Hilbert space H and that $x \in H$. Show that x is orthogonal to K if and only if 0 is the nearest point to x in K.

(ii) Suppose that H is a Hilbert space and that ϕ is a continuous linear functional on H with $\|\phi\| = 1$. Show that there is a sequence (h_n) of unit vectors in H with $\phi(h_n)$ real and $\phi(h_n) > 1 - 1/n$.

Show that h_n converges to a unit vector h, and that $\phi(h) = 1$.

Show that h is orthogonal to N, the null space of ϕ, and also that $H = N \oplus \text{span}(h)$.

Show that $\phi(k) = \langle k, h \rangle$, for all $k \in H$.

4H Groups, Rings and Fields

(i) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(ii) What are the units in $\mathbb{Z}[i]$? What are the primes in $\mathbb{Z}[i]$? Justify your answers.

Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

5D Electromagnetism

(i) Show that the Lorentz force corresponds to a curvature force and the gradient of a magnetic pressure, and that it can be written as the divergence of a second rank tensor, the Maxwell stress tensor.

Consider the potential field $\mathbf{B}$ given by $\mathbf{B} = -\nabla\Phi$, where

$$\Phi(x, y) = \left(\frac{B_0}{k}\right) \cos kx \; e^{-ky},$$

referred to cartesian coordinates (x, y, z). Obtain the Maxwell stress tensor and verify that its divergence vanishes.

(ii) The magnetic field in a stellar atmosphere is maintained by steady currents and the Lorentz force vanishes. Show that there is a scalar field α such that $\nabla \wedge \mathbf{B} = \alpha\mathbf{B}$ and $\mathbf{B} \cdot \nabla\alpha = 0$. Show further that if α is constant, then $\nabla^2\mathbf{B} + \alpha^2\mathbf{B} = 0$. Obtain a solution in the form $\mathbf{B} = (B_1(z), B_2(z), 0)$; describe the structure of this field and sketch its variation in the z-direction.

6F Dynamics of Differential Equations

(i) Define the terms *stable manifold* and *unstable manifold* of a hyperbolic fixed point $\mathbf{x}_0$ of a dynamical system. State carefully the stable manifold theorem.

Give an approximation, correct to fourth order in $|\mathbf{x}|$, for the stable and unstable manifolds of the origin for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x + x^2 - y^2 \\ -y + x^2 \end{pmatrix} .$$

(ii) State, without proof, the centre manifold theorem. Show that the fixed point at the origin of the system

$$\begin{aligned} \dot{x} &= y - x + ax^3 \ , \\ \dot{y} &= rx - y - zy \ , \\ \dot{z} &= -z + xy \ , \end{aligned}$$

where a is a constant, is non-hyperbolic at $r = 1$.

Using new coordinates $v = x + y\ ,\ w = x - y$, find the centre manifold in the form

$$w = \alpha v^3 + \dots \ , \quad z = \beta v^2 + \gamma v^4 + \dots$$

for constants α, β, γ to be determined. Hence find the evolution equation on the centre manifold in the form

$$\dot{v} = \frac{1}{8}(a-1)v^3 + \left(\frac{(3a+1)(a+1)}{128} + \frac{(a-1)}{32} \right) v^5 + \dots \ .$$

Ignoring higher order terms, give conditions on a that guarantee that the origin is asymptotically stable.

7K Geometry of Surfaces

(i)

Consider the surface

$$z = \frac{1}{2}(\lambda x^2 + \mu y^2) + h(x, y),$$

where $h(x, y)$ is a term of order at least 3 in x, y. Calculate the first fundamental form at $x = y = 0$.

(ii) Calculate the second fundamental form, at $x = y = 0$, of the surface given in Part (i). Calculate the Gaussian curvature. Explain why your answer is consistent with Gauss' "Theorema Egregium".

8H Graph Theory

(i) Define the chromatic polynomial $p(G\,;t)$ of the graph G, and establish the standard identity

$$p(G\,;t) \;=\; p(G-e\,;t) \;-\; p(G/e\,;t),$$

where e is an edge of G. Deduce that, if G has n vertices and m edges, then

$$p(G\,;t) = a_n t^n - a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + (-1)^n a_0\,,$$

where $a_n = 1$, $a_{n-1} = m$ and $a_j \geq 0$ for $0 \leq j \leq n$.

(ii) Let G and $p(G\,;t)$ be as in Part (i). Show that if G has k components $G_1, \ldots, G_k$ then $p(G\,;t) \;=\; \prod_{i=1}^{k} p(G_i\,;t)$. Deduce that $a_k > 0$ and $a_j = 0$ for $0 \leq j < k$.

Show that if G is a tree then $p(G\,;t) = t(t-1)^{n-1}$. Must the converse hold? Justify your answer.

Show that if $p(G\,;t) = p(T_r(n)\,;t)$, where $T_r(n)$ is a Turán graph, then $G = T_r(n)$.

9H Coding and Cryptography

(i) Explain the idea of public key cryptography. Give an example of a public key system, explaining how it works.

(ii) What is a general feedback register of length d with initial fill $(X_0, \ldots, X_{d-1})$? What is the maximal period of such a register, and why? What does it mean for such a register to be linear?

Describe and justify the Berlekamp-Massey algorithm for breaking a cypher stream arising from a general linear feedback register of unknown length.

Use the Berlekamp-Massey algorithm to find a linear recurrence in $\mathbb{F}_2$ with first eight terms 1, 1, 0, 0, 1, 0, 1, 1.

10M Algorithms and Networks

(i) Let G be a directed network with nodes N, arcs A and capacities specified on each of the arcs. Define the terms *feasible flow*, *divergence, cut, upper* and *lower cut capacities.* Given two disjoint sets of nodes N^+ and N^-, what does it mean to say that a cut Q separates N^+ from N^-? Prove that the flux of a feasible flow x from N^+ to N^- is bounded above by the upper capacity of Q, for any cut Q separating N^+ from N^-.

(ii) Define the maximum-flow and minimum-cut problems. State the max-flow min-cut theorem and outline the main steps of the maximum-flow algorithm. Use the algorithm to find the maximum flow between the nodes 1 and 5 in a network whose node set is $\{1, 2, \ldots, 5\}$, where the lower capacity of each arc is 0 and the upper capacity c_{ij} of the directed arc joining node i to node j is given by the (i, j)-entry in the matrix

$$\begin{pmatrix} 0 & 7 & 9 & 8 & 0 \\ 0 & 0 & 6 & 8 & 4 \\ 0 & 9 & 0 & 2 & 10 \\ 0 & 3 & 7 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

[*The painted-network theorem can be used without proof but should be stated clearly. You may assume in your description of the maximum-flow algorithm that you are given an initial feasible flow.*]

11M Principles of Statistics

(i) Let X be a random variable with density function $f(x; \theta)$. Consider testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1$.

What is the form of the optimal size α classical hypothesis test?

Compare the form of the test with the Bayesian test based on the Bayes factor, *and* with the Bayes decision rule under the 0-1 loss function, under which a loss of 1 is incurred for an incorrect decision and a loss of 0 is incurred for a correct decision.

(ii) What does it mean to say that a family of densities $\{f(x; \theta), \theta \in \Theta\}$ with real scalar parameter θ is of *monotone likelihood ratio*?

Suppose X has a distribution from a family which is of monotone likelihood ratio with respect to a statistic $t(X)$ and that it is required to test $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$.

State, without proof, a theorem which establishes the existence of a uniformly most powerful test and describe in detail the form of the test.

Let $X_1, \ldots, X_n$ be independent, identically distributed $U(0, \theta)$, $\theta > 0$. Find a uniformly most powerful size α test of $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$, and find its power function. Show that we may construct a different, randomised, size α test with the same power function for $\theta \geqslant \theta_0$.

12L Computational Statistics and Statistical Modelling

(i) Suppose that the random variable Y has density function of the form

$$f(y|\theta, \phi) = \exp\left[\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right]$$

where $\phi > 0$. Show that Y has expectation $b'(\theta)$ and variance $\phi b''(\theta)$.

(ii) Suppose now that $Y_1, \ldots, Y_n$ are independent negative exponential variables, with Y_i having density function $f(y_i|\mu_i) = \frac{1}{\mu_i}e^{-y_i/\mu_i}$ for $y_i > 0$. Suppose further that $g(\mu_i) = \beta^T x_i$ for $1 \leqslant i \leqslant n$, where $g(\cdot)$ is a known 'link' function, and $x_1, \ldots, x_n$ are given covariate vectors, each of dimension p. Discuss carefully the problem of finding $\hat{\beta}$, the maximum-likelihood estimator of β, firstly for the case $g(\mu_i) = 1/\mu_i$, and secondly for the case $g(\mu) = \log \mu_i$; in both cases you should state the large-sample distribution of $\hat{\beta}$.

[*Any standard theorems used need not be proved.*]

13E Foundations of Quantum Mechanics

(i) A Hamiltonian H_0 has energy eigenvalues E_r and corresponding non-degenerate eigenstates $|r\rangle$. Show that under a small change in the Hamiltonian δH,

$$\delta|r\rangle = \sum_{s \neq r} \frac{\langle s|\delta H|r\rangle}{E_r - E_s}|s\rangle,$$

and derive the related formula for the change in the energy eigenvalue E_r to first and second order in δH.

(ii) The Hamiltonian for a particle moving in one dimension is $H = H_0 + \lambda H'$, where $H_0 = p^2/2m + V(x)$, $H' = p/m$ and λ is small. Show that

$$\frac{i}{\hbar}[H_0, x] = H'$$

and hence that

$$\delta E_r = -\lambda^2 \frac{i}{\hbar}\langle r|H'x|r\rangle = \lambda^2 \frac{i}{\hbar}\langle r|xH'|r\rangle$$

to second order in λ.

Deduce that δE_r is independent of the particular state $|r\rangle$ and explain why this change in energy is exact to all orders in λ.

14E Quantum Physics

(i) A simple model of a one-dimensional crystal consists of an infinite array of sites equally spaced with separation a . An electron occupies the nth site with a probability amplitude c_n . The time-dependent Schrödinger equation governing these amplitudes is

$$i\hbar \frac{dc_n}{dt} = E_0 c_n - A(c_{n-1} + c_{n+1}) \quad ,$$

where E_0 is the energy of an electron at an isolated site and the amplitude for transition between neighbouring sites is $A > 0$. By examining a solution of the form

$$c_n = e^{ikan - iEt/\hbar} \quad ,$$

show that E, the energy of the electron in the crystal, lies in a band

$$E_0 - 2A \leq E \leq E_0 + 2A \; .$$

Identify the Brillouin zone for this model and explain its significance.

(ii) In the above model the electron is now subject to an electric field $\mathcal{E}$ in the direction of increasing n . Given that the charge on the electron is $-e$ write down the new form of the time-dependent Schrödinger equation for the probability amplitudes. Show that it has a solution of the form

$$c_n = \exp\left\{ -\frac{i}{\hbar} \int_0^t \epsilon(t')dt' + i(k - \frac{e\mathcal{E}t}{\hbar})na \right\} \quad ,$$

where

$$\epsilon(t) = E_0 - 2A\cos\left((k - \frac{e\mathcal{E}t}{\hbar})a \right) \quad .$$

Explain briefly how to interpret this result and use it to show that the dynamical behaviour of an electron near the bottom of the energy band is the same as that for a free particle in the presence of an electric field with an effective mass $m^* = \hbar^2/(2Aa^2)$.

15D General Relativity

(i) Consider the line element describing the interior of a star,

$$ds^2 = e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{2\gamma(r)}dt^2 \ ,$$

defined for $0 \leq r \leq r_0$ by

$$e^{-2\alpha(r)} = 1 - Ar^2$$

and

$$e^{\gamma(r)} = \frac{3}{2}e^{-\alpha_0} - \frac{1}{2}e^{-\alpha(r)}.$$

Here $A = 2M/r_0^3$, M is the mass of the star, and α_0 is defined to be $\alpha(r_0)$.

The star is made of a perfect fluid with energy-momentum tensor

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab}.$$

Here u^a is the 4-velocity of the fluid which is at rest, the density ρ is constant throughout the star ($0 \leq r \leq r_0$) and the pressure $p = p(r)$ depends only on the radial coordinate. Write down the Einstein field equations and show that (in geometrical units with $G = c = 1$) they may equivalently be written as

$$R_{ab} = 8\pi(p+\rho)u_a u_b + 4\pi(p-\rho)g_{ab}.$$

(ii) Using the formulae below, or otherwise, show that for $0 \leq r \leq r_0$ one has

$$\rho = \frac{3A}{8\pi}, \quad p(r) = \frac{3A}{8\pi}\left(\frac{e^{-\alpha(r)} - e^{-\alpha_0}}{3e^{-\alpha_0} - e^{-\alpha(r)}}\right).$$

[*The non-zero components of the Ricci tensor are:*

$$R_{11} = -\gamma'' + \alpha'\gamma' - \gamma'^2 + \frac{2\alpha'}{r}, \quad R_{22} = e^{-2\alpha}[(\alpha' - \gamma')r - 1] + 1,$$

$$R_{33} = \sin^2\theta R_{22}, \quad R_{44} = e^{2\gamma - 2\alpha}[\gamma'' - \alpha'\gamma' + \gamma'^2 + \frac{2\gamma'}{r}].$$

Note that

$$\alpha' = Are^{2\alpha}, \quad \gamma' = \frac{1}{2}Are^{\alpha-\gamma}, \quad \gamma'' = \frac{1}{2}Ae^{\alpha-\gamma} + \frac{1}{2}A^2r^2e^{3\alpha-\gamma} - \frac{1}{4}A^2r^2e^{2\alpha-2\gamma}. \quad]$$

16G Theoretical Geophysics

(i) State the equations that relate strain to displacement and stress to strain in a linear, isotropic elastic solid.

In the absence of body forces, the Euler equation for infinitesimal deformations of a solid of density ρ is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} \quad .$$

Derive an equation for $\mathbf{u}(\mathbf{x}, t)$ in a linear, isotropic, homogeneous elastic solid. Hence show that both the dilatation $\theta = \boldsymbol{\nabla} \cdot \mathbf{u}$ and the rotation $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ satisfy wave equations and find the corresponding wave speeds α and β.

(ii) The ray parameter $p = r \sin i / v$ is constant along seismic rays in a spherically symmetric Earth, where $v(r)$ is the relevant wave speed (α or β) and $i(r)$ is the angle between the ray and the local radial direction.

Express $\tan i$ and $\sec i$ in terms of p and the variable $\eta(r) = r/v$. Hence show that the angular distance and travel time between a surface source and receiver, both at radius R, are given by

$$\Delta(p) = 2 \int_{r_m}^{R} \frac{p}{r} \frac{dr}{(\eta^2 - p^2)^{1/2}} \quad , \qquad T(p) = 2 \int_{r_m}^{R} \frac{\eta^2}{r} \frac{dr}{(\eta^2 - p^2)^{1/2}} \quad ,$$

where r_m is the minimum radius attained by the ray. What is $\eta(r_m)$?

A simple Earth model has a solid mantle in $R/2 < r < R$ and a liquid core in $r < R/2$. If $\alpha(r) = A/r$ in the mantle, where A is a constant, find $\Delta(p)$ and $T(p)$ for P-arrivals (direct paths lying entirely in the mantle), and show that

$$T = \frac{R^2 \sin \Delta}{A} \quad .$$

[*You may assume that* $\displaystyle\int \frac{du}{u\sqrt{u-1}} = 2\cos^{-1}\left(\frac{1}{\sqrt{u}}\right)$.]

Sketch the $T - \Delta$ curves for P and PcP arrivals on the same diagram and explain briefly why they terminate at $\Delta = \cos^{-1} \frac{1}{4}$.

17C Mathematical Methods

(i) Show that the equation

$$\epsilon x^4 - x^2 + 5x - 6 = 0\,, \quad |\epsilon| \ll 1\ ,$$

has roots in the neighbourhood of $x = 2$ and $x = 3$. Find the first two terms of an expansion in ϵ for each of these roots.

Find a suitable series expansion for the other two roots and calculate the first two terms in each case.

(ii) Describe, giving reasons for the steps taken, how the leading-order approximation for $\lambda \gg 1$ to an integral of the form

$$I(\lambda) \equiv \int_A^B f(t) e^{i\lambda g(t)} dt\ ,$$

where λ and g are real, may be found by the method of stationary phase. Consider the cases where (a) $g'(t)$ has one simple zero at $t = t_0$ with $A < t_0 < B$; (b) $g'(t)$ has more than one simple zero in $A < t < B$; and (c) $g'(t)$ has only a simple zero at $t = B$. What is the order of magnitude of $I(\lambda)$ if $g'(t)$ is non-zero for $A \le t \le B$?

Use the method of stationary phase to find the leading-order approximation to

$$J(\lambda) \equiv \int_0^1 \sin[\lambda(2t^4 - t)]\, dt$$

for $\lambda \gg 1$.

[*You may use the fact that* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2} du = \sqrt{\pi} e^{i\pi/4}$.]

18F Nonlinear Waves

(i) Find a travelling wave solution of unchanging shape for the modified Burgers equation (with $\alpha > 0$)

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$$

with $u = 0$ far ahead of the wave and $u = 1$ far behind. What is the velocity of the wave? Sketch the shape of the wave.

(ii) Explain why the method of characteristics, when applied to an equation of the type

$$\frac{\partial u}{\partial t} + c(u) \frac{\partial u}{\partial x} = 0 \,,$$

with initial data $u(x, 0) = f(x)$, sometimes gives a multi-valued solution. State the shock-fitting algorithm that gives a single-valued solution, and explain how it is justified.

Consider the equation above, with $c(u) = u^2$. Suppose that

$$u(x, 0) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases} \quad .$$

Sketch the characteristics in the (x, t) plane. Show that a shock forms immediately, and calculate the velocity at which it moves.

19F Numerical Analysis

(i)

Given the finite-difference method

$$\sum_{k=-r}^{s} \alpha_k u_{m+k}^{n+1} = \sum_{k=-r}^{s} \beta_k u_{m+k}^{n}, \quad m, n \in \mathbb{Z},\ n \geqslant 0,$$

define

$$H(z) = \frac{\sum_{k=-r}^{s} \beta_k z^k}{\sum_{k=-r}^{s} \alpha_k z^k}.$$

Prove that this method is stable if and only if

$$|H(e^{i\theta})| \leqslant 1, \quad -\pi \leqslant \theta \leqslant \pi.$$

[*You may quote without proof known properties of the Fourier transform.*]

(ii) Find the range of the parameter μ such that the method

$$(1 - 2\mu) u_{m-1}^{n+1} + 4\mu u_m^{n+1} + (1 - 2\mu) u_{m+1}^{n+1} = u_{m-1}^{n} + u_{m+1}^{n}$$

is stable. Supposing that this method is used to solve the diffusion equation for $u(x, t)$, determine the order of magnitude of the local error as a power of Δx.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Wednesday 5 June 2002 9 to 12

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It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) Consider the continuous-time Markov chain $(X_t)_{t\geqslant 0}$ on $\{1,2,3,4,5,6,7\}$ with generator matrix

$$Q = \begin{pmatrix} -6 & 2 & 0 & 0 & 0 & 4 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -6 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

Compute the probability, starting from state 3, that X_t hits state 2 eventually.

Deduce that

$$\lim_{t\to\infty} \mathbb{P}(X_t = 2 | X_0 = 3) = \frac{4}{15}.$$

[*Justification of standard arguments is not expected.*]

(ii) A colony of cells contains immature and mature cells. Each immature cell, after an exponential time of parameter 2, becomes a mature cell. Each mature cell, after an exponential time of parameter 3, divides into two immature cells. Suppose we begin with one immature cell and let $n(t)$ denote the expected number of immature cells at time t. Show that

$$n(t) = (4e^t + 3e^{-6t})/7.$$

2G Principles of Dynamics

(i) Show that Hamilton's equations follow from the variational principle

$$\delta \int_{t_1}^{t_2} [p\dot{q} - H(q,p,t)]\, dt = 0$$

under the restrictions $\delta q(t_1) = \delta q(t_2) = \delta p(t_1) = \delta p(t_2) = 0$. Comment on the difference from the variational principle for Lagrange's equations.

(ii) Suppose we transform from p and q to $p' = p'(q,p,t)$ and $q' = q'(q,p,t)$, with

$$p'\dot{q}' - H' = p\dot{q} - H + \frac{\mathrm{d}}{\mathrm{d}t} F(q,p,q',p',t),$$

where H' is the new Hamiltonian. Show that p' and q' obey Hamilton's equations with Hamiltonian H'.

Show that the time independent generating function $F = F_1(q,q') = q'/q$ takes the Hamiltonian

$$H = \frac{1}{2q^2} + \frac{1}{2}p^2 q^4$$

to harmonic oscillator form. Show that q' and p' obey the Poisson bracket relation

$$\{q', p'\} = 1.$$

3K Functional Analysis

(i) Suppose that (f_n) is a decreasing sequence of continuous real-valued functions on a compact metric space (X,d) which converges pointwise to 0. By considering sets of the form $B_n = \{x : f_n(x) < \epsilon\}$, for $\epsilon > 0$, or otherwise, show that f_n converges uniformly to 0.

Can the condition that (f_n) is decreasing be dropped? Can the condition that (X,d) is compact be dropped? Justify your answers.

(ii) Suppose that k is a positive integer. Define polynomials p_n recursively by

$$p_0 = 0, \qquad p_{n+1}(t) = p_n(t) + (t - p_n^k(t))/k.$$

Show that $0 \leqslant p_n(t) \leqslant p_{n+1}(t) \leqslant t^{1/k}$, for $t \in [0,1]$, and show that $p_n(t)$ converges to $t^{1/k}$ uniformly on $[0,1]$.

[*You may wish to use the identity* $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \ldots + b^{k-1})$.]

Suppose that A is a closed subalgebra of the algebra $C(X)$ of continuous real-valued functions on a compact metric space (X,d), equipped with the uniform norm, and suppose that A has the property that for each $x \in X$ there exists $a \in A$ with $a(x) \neq 0$. Show that there exists $h \in A$ such that $0 < h(x) \leqslant 1$ for all $x \in X$.

Show that $h^{1/k} \in A$ for each positive integer k, and show that A contains the constant functions.

4H Groups, Rings and Fields

(i) What does it mean for a ring to be Noetherian? State Hilbert's Basis Theorem. Give an example of a Noetherian ring which is not a principal ideal domain.

(ii) Prove Hilbert's Basis Theorem.

Is it true that if the ring $R[X]$ is Noetherian, then so is R?

5D Electromagnetism

(i) A plane electromagnetic wave in a vacuum has an electric field

$$\mathbf{E} = (E_1, E_2, 0)\cos(kz - \omega t),$$

referred to cartesian axes (x, y, z). Show that this wave is plane polarized and find the orientation of the plane of polarization. Obtain the corresponding plane polarized magnetic field and calculate the rate at which energy is transported by the wave.

(ii) Suppose instead that

$$\mathbf{E} = (E_1 \cos(kz - \omega t),\ E_2 \cos(kz - \omega t + \phi), 0),$$

with ϕ a constant, $0 < \phi < \pi$. Show that, if the axes are now rotated through an angle ψ so as to obtain an elliptically polarized wave with an electric field

$$\mathbf{E}' = (F_1 \cos(kz - \omega t + \chi),\ F_2 \sin(kz - \omega t + \chi), 0),$$

then

$$\tan 2\psi = \frac{2E_1 E_2 \cos\phi}{E_1^2 - E_2^2}.$$

Show also that if $E_1 = E_2 = E$ there is an elliptically polarized wave with

$$\mathbf{E}' = \sqrt{2}E\left(\cos(kz - \omega t + \tfrac{1}{2}\phi)\cos\tfrac{1}{2}\phi,\ \sin(kz - \omega t + \tfrac{1}{2}\phi)\sin\tfrac{1}{2}\phi, 0\right).$$

6F Dynamics of Differential Equations

(i) Define the Floquet multiplier and Liapunov exponent for a periodic orbit $\hat{\mathbf{x}}(t)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Show that one multiplier is always unity, and that the other is given by

$$\exp\left(\int_0^T \nabla \cdot \mathbf{f}(\hat{\mathbf{x}}(t))dt\right) , \qquad (*)$$

where T is the period of the orbit.

The Van der Pol oscillator $\ddot{x} + \epsilon \dot{x}(x^2 - 1) + x = 0\,,\ 0 < \epsilon \ll 1$ has a limit cycle $\hat{x}(t) \approx 2 \sin t$. Show using $(*)$ that this orbit is stable.

(ii) Show, by considering the normal form for a Hopf bifurcation from a fixed point $\mathbf{x}_0(\mu)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$, that in some neighbourhood of the bifurcation the periodic orbit is stable when it exists in the range of μ for which $\mathbf{x}_0$ is unstable, and unstable in the opposite case.

Now consider the system

$$\left.\begin{aligned} \dot{x} &= x(1-y) + \mu x \\ \dot{y} &= y(x-1) - \mu x \end{aligned}\right\} \qquad x > 0\ .$$

Show that the fixed point $(1+\mu\,,\ 1+\mu)$ has a Hopf bifurcation when $\mu = 0$, and is unstable (stable) when $\mu > 0$ ($\mu < 0$).

Suppose that a periodic orbit exists in $\mu > 0$. Show without solving for the orbit that the result of part (i) shows that such an orbit is unstable. Define a similar result for $\mu < 0$.

What do you conclude about the existence of periodic orbits when $\mu \neq 0$? Check your answer by applying Dulac's criterion to the system, using the weighting $\rho = e^{-(x+y)}$.

7K Geometry of Surfaces

(i) State what it means for surfaces $f : U \to \mathbb{R}^3$ and $g : V \to \mathbb{R}^3$ to be isometric.

Let $f : U \to \mathbb{R}^3$ be a surface, $\phi : V \to U$ a diffeomorphism, and let $g = f \circ \phi : V \to \mathbb{R}^3$.

State a formula comparing the first fundamental forms of f and g.

(ii) Give a proof of the formula referred to at the end of part (i). Deduce that "isometry" is an equivalence relation.

The *catenoid* and the *helicoid* are the surfaces defined by

$$(u, v) \to (u \cos v,\ u \sin v,\ v)$$

and

$$(\vartheta, z) \to (\cosh z \cos \vartheta,\ \cosh z \sin \vartheta,\ z).$$

Show that the catenoid and the helicoid are isometric.

8J Logic, Computation and Set Theory

(i) Explain briefly what is meant by the terms *register machine* and *computable function*.

Let u be the universal computable function $u(m,n) = f_m(n)$ and s a total computable function with $f_{s(m,n)}(k) = f_m(n,k)$. Here $f_m(n)$ and $f_m(n,k)$ are the unary and binary functions computed by the m-th register machine program P_m. Suppose $h : \mathbb{N} \to \mathbb{N}$ is a total computable function. By considering the function

$$g(m,n) = u(h(s(m,m)),n)$$

show that there is a number a such that $f_a = f_{h(a)}$.

(ii) Let P be the set of all partial functions $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Consider the mapping $\Phi : P \to P$ defined by

$$\Phi(g)(m,n) = \begin{cases} n+1 & \text{if } m = 0, \\ g(m-1,1) & \text{if } m > 0,\ n = 0 \text{ and } g(m-1,1) \text{ defined,} \\ g(m-1, g(m,n-1)) & \text{if } mn > 0 \text{ and } g(m-1,g(m,n-1)) \text{ defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(a) Show that any fixed point of Φ is a total function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Deduce that Φ has a unique fixed point.
[*The Bourbaki-Witt Theorem may be assumed if stated precisely.*]

(b) It follows from standard closure properties of the computable functions that there is a computable function ψ such that

$$\psi(p,m,n) = \Phi(f_p)(m,n).$$

Assuming this, show that there is a total computable function h such that

$$\Phi(f_p) = f_{h(p)} \text{ for all } p.$$

Deduce that the fixed point of Φ is computable.

9J Number Theory

(i) Let $\pi(x)$ denote the number of primes $\leqslant x$, where x is a positive real number. State and prove Legendre's formula relating $\pi(x)$ to $\pi(\sqrt{x})$. Use this formula to compute $\pi(100) - \pi(10)$.

(ii) Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, where s is a real number greater than 1. Prove the following two assertions rigorously, assuming always that $s > 1$.

(a) $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$, where the product is taken over all primes p;

(b) $\zeta(s) = \dfrac{1}{1 - 2^{1-s}} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$.

Explain why (b) enables us to define $\zeta(s)$ for $0 < s < 1$. Deduce from (b) that $\lim_{s \to 1} (s-1)\zeta(s) = 1$.

10M Algorithms and Networks

(i) Consider the unconstrained geometric programme GP

$$\text{minimise} \quad g(t) = \sum_{i=1}^{n} c_i \, \Pi_{j=1}^{m} \, t_j^{a_{ij}}$$

$$\text{subject to} \quad t_j > 0 \quad j = 1, \ldots, m.$$

State the dual problem to GP. Give a careful statement of the AM-GM inequality, and use it to prove the primal-dual inequality for GP.

(ii) Define min-path and max-tension problems. State and outline the proof of the max-tension min-path theorem.

A company has branches in five cities A, B, C, D and E. The fares for direct flights between these cities are as follows:

	A	B	C	D	E
A	–	50	40	25	10
B	50	–	20	90	25
C	40	20	–	10	25
D	25	90	10	–	55
E	10	25	25	55	–

Formulate this as a min-path problem. Illustrate the max-tension min-path algorithm by finding the cost of travelling by the cheapest routes between D and each of the other cities.

11L Stochastic Financial Models

(i) Explain briefly what it means to say that a stochastic process $\{W_t,\ t \geqslant 0\}$ is a standard Brownian motion.

Let $\{W_t,\ t \geqslant 0\}$ be a standard Brownian motion and let a, b be real numbers. What condition must a and b satisfy to ensure that the process e^{aW_t+bt} is a martingale? Justify your answer carefully.

(ii) At the beginning of each of the years $r = 0, 1, \ldots, n-1$ an investor has income X_r, of which he invests a proportion ρ_r, $0 \leqslant \rho_r \leqslant 1$, and consumes the rest during the year. His income at the beginning of the next year is

$$X_{r+1} = X_r + \rho_r X_r W_r,$$

where $W_0, \ldots, W_{n-1}$ are independent positive random variables with finite means and $X_0 \geqslant 0$ is a constant. He decides on ρ_r after he has observed both X_r and W_r at the beginning of year r, but at that time he does not have any knowledge of the value of W_s, for any $s > r$. The investor retires in year n and consumes his entire income during that year. He wishes to determine the investment policy that maximizes his expected total consumption

$$\mathbb{E}\left[\sum_{r=0}^{n-1}(1-\rho_r)X_r + X_n\right].$$

Prove that the optimal policy may be expressed in terms of the numbers $b_0, b_1, \ldots,$ b_n where $b_n = 1$, $b_r = b_{r+1} + \mathbb{E}\max(b_{r+1}W_r, 1)$, for $r < n$, and determine the optimal expected total consumption.

12M Principles of Statistics

(i) Describe in detail how to perform the Wald, score and likelihood ratio tests of a *simple* null hypothesis $H_0 : \theta = \theta_0$ given a random sample $X_1, \ldots, X_n$ from a regular one-parameter density $f(x;\theta)$. In each case you should specify the asymptotic null distribution of the test statistic.

(ii) Let $X_1, \ldots, X_n$ be an independent, identically distributed sample from a distribution F, and let $\hat{\theta}(X_1, \ldots, X_n)$ be an estimator of a parameter θ of F.

Explain what is meant by: (a) the *empirical distribution function* of the sample; (b) the *bootstrap estimator* of the *bias* of $\hat{\theta}$, based on the empirical distribution function. Explain how a bootstrap estimator of the *distribution function* of $\hat{\theta} - \theta$ may be used to construct an approximate $1 - \alpha$ confidence interval for θ.

Suppose the parameter of interest is $\theta = \mu^2$, where μ is the mean of F, and the estimator is $\hat{\theta} = \bar{X}^2$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ is the sample mean.

Derive an *explicit* expression for the bootstrap estimator of the bias of $\hat{\theta}$ and show that it is biased as an estimator of the true bias of $\hat{\theta}$.

Let $\hat{\theta}_i$ be the value of the estimator $\hat{\theta}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ computed from the sample of size $n-1$ obtained by deleting X_i and let $\hat{\theta}_. = n^{-1} \sum_{i=1}^{n} \hat{\theta}_i$. The *jackknife* estimator of the bias of $\hat{\theta}$ is

$$b_J = (n-1)\ (\hat{\theta}_. - \hat{\theta})\ .$$

Derive the jackknife estimator b_J for the case $\hat{\theta} = \bar{X}^2$, and show that, as an estimator of the true bias of $\hat{\theta}$, it is unbiased.

13E Foundations of Quantum Mechanics

(i) Two particles with angular momenta j_1, j_2 and basis states $|j_1\ m_1\rangle, |j_2\ m_2\rangle$ are combined to give total angular momentum j and basis states $|j\ m\rangle$. State the possible values of j, m and show how a state with $j = m = j_1 + j_2$ can be constructed. Briefly describe, for a general allowed value of j, what the Clebsch-Gordan coefficients are.

(ii) If the angular momenta j_1 and j_2 are both 1 show that the combined state $|2\ 0\rangle$ is

$$|2\ 0\rangle = \sqrt{\frac{1}{6}}\Big(|1\ 1\rangle|1\ {-1}\rangle + |1\ {-1}\rangle|1\ 1\rangle\Big) + \sqrt{\frac{2}{3}}|1\ 0\rangle|1\ 0\rangle.$$

Determine the corresponding expressions for the combined states $|1\ 0\rangle$ and $|0\ 0\rangle$, assuming that they are respectively antisymmetric and symmetric under interchange of the two particles.

If the combined system is in state $|0\ 0\rangle$ what is the probability that measurements of the z-component of angular momentum for either constituent particle will give the value of 1?

[*Hint:* $J_\pm |j\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m \pm 1\rangle$.]

14D Statistical Physics and Cosmology

(i) Write down the first law of thermodynamics for the change dU in the internal energy $U(N,V,S)$ of a gas of N particles in a volume V with entropy S.

Given that

$$PV = (\gamma - 1)U,$$

where P is the pressure, use the first law to show that PV^{γ} is constant at constant N and S.

Write down the Boyle-Charles law for a non-relativistic ideal gas and hence deduce that the temperature T is proportional to $V^{1-\gamma}$ at constant N and S.

State the principle of equipartition of energy and use it to deduce that

$$U = \frac{3}{2}NkT.$$

Hence deduce the value of γ. Show that this value of γ is such that the ratio E_i/kT is unchanged by a change of volume at constant N and S, where E_i is the energy of the i-th one particle eigenstate of a non-relativistic ideal gas.

(ii) A classical gas of non-relativistic particles of mass m at absolute temperature T and number density n has a chemical potential

$$\mu = mc^2 - kT \ln\left(\frac{g_s}{n}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right),$$

where g_s is the particle's spin degeneracy factor. What condition on n is needed for the validity of this formula and why?

Thermal and chemical equilibrium between two species of non-relativistic particles a and b is maintained by the reaction

$$a + \alpha \leftrightarrow b + \beta,$$

where α and β are massless particles with zero chemical potential. Given that particles a and b have masses m_a and m_b respectively, but equal spin degeneracy factors, find the number density ratio n_a/n_b as a function of m_a, m_b and T. Given that $m_a > m_b$ but $m_a - m_b \ll m_b$ show that

$$\frac{n_a}{n_b} \approx f\left(\frac{(m_a - m_b)c^2}{kT}\right)$$

for some function f which you should determine.

Explain how a reaction of the above type is relevant to a determination of the neutron to proton ratio in the early universe and why this ratio does not fall rapidly to zero as the universe cools. Explain briefly the process of primordial nucleosynthesis by which neutrons are converted into stable helium nuclei. Let

$$Y_{He} = \frac{\rho_{He}}{\rho}$$

be the fraction of the universe that ends up in helium. Compute Y_{He} as a function of the ratio $r = n_a/n_b$ at the time of nucleosynthesis.

15E Symmetries and Groups in Physics

(i) Let D_6 denote the symmetry group of rotations and reflections of a regular hexagon. The elements of D_6 are given by $\{e, c, c^2, c^3, c^4, c^5, b, bc, bc^2, bc^3, bc^4, bc^5\}$ with $c^6 = b^2 = e$ and $cb = bc^5$. The conjugacy classes of D_6 are $\{e\}$, $\{c, c^5\}$, $\{c^2, c^4\}$, $\{c^3\}$, $\{b, bc^2, bc^4\}$ and $\{bc, bc^3, bc^5\}$.

Show that the character table of D_6 is

D_6	e	$\{c, c^5\}$	$\{c^2, c^4\}$	$\{c^3\}$	$\{b, bc^2, bc^4\}$	$\{bc, bc^3, bc^5\}$
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2	1	-1	-2	0	0
χ_6	2	-1	-1	2	0	0

(ii) Show that the character of an $SO(3)$ rotation with angle θ in the $2l+1$ dimensional irreducible representation of $SO(3)$ is given by

$$\chi_l(\theta) = 1 + 2\cos\theta + 2\cos(2\theta) + \ldots + 2\cos((l-1)\theta) + 2\cos(l\theta)\ .$$

For a hexagonal crystal of atoms find how the degeneracy of the D-wave orbital states ($l = 2$) in the atomic central potential is split by the crystal potential with D_6 symmetry and give the new degeneracies.

By using the fact that D_3 is isomorphic to $D_6/\{e, c^3\}$, or otherwise, find the degeneracies of eigenstates if the hexagonal symmetry is broken to the subgroup D_3 by a deformation. The introduction of a magnetic field further reduces the symmetry to C_3. What will the degeneracies of the energy eigenstates be now?

16C Transport Processes

(i) A layer of fluid of depth $h(x,t)$, density ρ and viscosity μ sits on top of a rigid horizontal plane at $y = 0$. Gravity g acts vertically and surface tension is negligible.

Assuming that the horizontal velocity component u and pressure p satisfy the lubrication equations

$$0 = -p_x + \mu u_{yy}$$
$$0 = -p_y - \rho g,$$

together with appropriate boundary conditions at $y = 0$ and $y = h$ (which should be stated), show that h satisfies the partial differential equation

$$h_t = \frac{g}{3\nu}\left(h^3 h_x\right)_x , \qquad (*)$$

where $\nu = \mu/\rho$.

(ii) A two-dimensional blob of the above fluid has fixed area A and time-varying width $2X(t)$, such that

$$A = \int_{-X(t)}^{X(t)} h(x,t)\, dx.$$

The blob spreads under gravity.

Use scaling arguments to show that, after an initial transient, $X(t)$ is proportional to $t^{1/5}$ and $h(0,t)$ is proportional to $t^{-1/5}$. Hence show that equation $(*)$ of Part (i) has a similarity solution of the form

$$h(x,t) = \left(\frac{A^2\nu}{gt}\right)^{1/5} H(\xi), \quad \text{where} \quad \xi = \frac{x}{\left(A^3 gt/\nu\right)^{1/5}},$$

and find the differential equation satisfied by $H(\xi)$.

Deduce that

$$H = \begin{cases} \left[\frac{9}{10}\left(\xi_0^2 - \xi^2\right)\right]^{1/3} & \text{in } \; -\xi_0 < \xi < \xi_0 \\ 0 & \text{in } \; |\xi| > \xi_0 \; , \end{cases}$$

where

$$X(t) = \xi_0 \left(\frac{A^3 gt}{\nu}\right)^{1/5} .$$

Express ξ_0 in terms of the integral

$$I = \int_{-1}^{1} \left(1 - u^2\right)^{1/3} du.$$

17C Mathematical Methods

(i) State the Fredholm alternative for Fredholm integral equations of the second kind.

Show that the integral equation

$$\phi(x) - \lambda \int_0^1 (x+t)\phi(t)dt = f(x) , \quad 0 \leqslant x \leqslant 1 ,$$

where f is a continuous function, has a unique solution for ϕ if $\lambda \neq -6 \pm 4\sqrt{3}$. Derive this solution.

(ii) Describe the WKB method for finding approximate solutions $f(x)$ of the equation

$$\frac{d^2 f(x)}{dx^2} + q(\epsilon x) f(x) = 0 ,$$

where q is an arbitrary non-zero, differentiable function and ϵ is a small parameter. Obtain these solutions in terms of an exponential with slowly varying exponent and slowly varying amplitude.

Hence, by means of a suitable change of independent variable, find approximate solutions $w(t)$ of the equation

$$\frac{d^2 w}{dt^2} + \lambda^2 t w = 0 ,$$

in $t > 0$, where λ is a large parameter.

18F Nonlinear Waves

(i) Show that the equation

$$\frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi}{\partial x^2} + 1 - \phi^2 = 0$$

has two solutions which are independent of both x and t. Show that one of these is linearly stable. Show that the other solution is linearly unstable, and find the range of wavenumbers that exhibit the instability.

Sketch the nonlinear evolution of the unstable solution after it receives a small, smooth, localized perturbation in the direction towards the stable solution.

(ii) Show that the equations

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} &= e^{-u+v} \quad , \\ -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} &= e^{-u-v} \end{aligned}$$

are a Bäcklund pair for the equations

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-2u} \,, \quad \frac{\partial^2 v}{\partial x \partial y} = 0 \,.$$

By choosing v to be a suitable constant, and using the Bäcklund pair, find a solution of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-2u}$$

which is non-singular in the region $y < 4x$ of the (x, y) plane and has the value $u = 0$ at $x = \frac{1}{2}$, $y = 0$.

19F Numerical Analysis

(i) Determine the order of the multistep method

$$\mathbf{y}_{n+2} - (1+\alpha)\mathbf{y}_{n+1} + \alpha \mathbf{y}_n = h[\tfrac{1}{12}(5+\alpha)\mathbf{f}_{n+2} + \tfrac{2}{3}(1-\alpha)\mathbf{f}_{n+1} - \tfrac{1}{12}(1+5\alpha)\mathbf{f}_n]$$

for the solution of ordinary differential equations for different choices of α in the range $-1 \leqslant \alpha \leqslant 1$.

(ii) Prove that no such choice of α results in a method whose linear stability domain includes the interval $(-\infty,\ 0)$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Thursday 6 June 2002 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

*Candidates must not attempt more than **FOUR** questions. If you submit answers to more than four questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

*Write legibly and on only **one** side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

*Tie your answers in separate bundles, marked **C, D, E, ..., M** according to the letter affixed to each question. (For example, **19C, 21C** should be in one bundle and **12L, 14L** in another bundle.)*

Attach a completed cover sheet to each bundle.

*Complete a master cover sheet listing **all** questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

Write an essay on the long-time behaviour of discrete-time Markov chains on a finite state space. Your essay should include discussion of the convergence of probabilities as well as almost-sure behaviour. You should also explain what happens when the chain is not irreducible.

2F Principles of Dynamics

Explain how the orientation of a rigid body can be specified by means of the three Eulerian angles, θ, ϕ and ψ.

An axisymmetric top of mass M has principal moments of inertia A, A and C, and is spinning with angular speed n about its axis of symmetry. Its centre of mass lies a distance h from the fixed point of support. Initially the axis of symmetry points vertically upwards. It then suffers a small disturbance. For what values of the spin is the initial configuration stable?

If the spin is such that the initial configuration is unstable, what is the lowest angle reached by the symmetry axis in the nutation of the top? Find the maximum and minimum values of the precessional angular velocity $\dot{\phi}$.

3K Functional Analysis

Define the *distribution function* Φ_f of a non-negative measurable function f on the interval $I = [0,1]$. Show that Φ_f is a decreasing non-negative function on $[0,\infty]$ which is continuous on the right.

Define the *Lebesgue integral* $\int_I f\,dm$. Show that $\int_I f\,dm = 0$ if and only if $f = 0$ almost everywhere.

Suppose that f is a non-negative Riemann integrable function on $[0,1]$. Show that there are an increasing sequence (g_n) and a decreasing sequence (h_n) of non-negative step functions with $g_n \leqslant f \leqslant h_n$ such that $\int_0^1 (h_n(x) - g_n(x))\,dx \to 0$.

Show that the functions $g = \lim_n g_n$ and $h = \lim_n h_n$ are equal almost everywhere, that f is measurable and that the Lebesgue integral $\int_I f\,dm$ is equal to the Riemann integral $\int_0^1 f(x)\,dx$.

Suppose that j is a Riemann integrable function on $[0,1]$ and that $j(x) > 0$ for all x. Show that $\int_0^1 j(x)\,dx > 0$.

4H Groups, Rings and Fields

Let F be a finite field. Show that there is a unique prime p for which F contains the field $\mathbb{F}_p$ of p elements. Prove that F contains p^n elements, for some $n \in \mathbb{N}$. Show that $x^{p^n} = x$ for all $x \in F$, and hence find a polynomial $f \in \mathbb{F}_p[X]$ such that F is the splitting field of f. Show that, up to isomorphism, F is the unique field $\mathbb{F}_{p^n}$ of size p^n.

[*Standard results about splitting fields may be assumed.*]

Prove that the mapping sending x to x^p is an automorphism of $\mathbb{F}_{p^n}$. Deduce that the Galois group Gal $(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is cyclic of order n. For which m is $\mathbb{F}_{p^m}$ a subfield of $\mathbb{F}_{p^n}$?

5D Electromagnetism

State the four integral relationships between the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ and explain their physical significance. Derive Maxwell's equations from these relationships and show that $\mathbf{E}$ and $\mathbf{B}$ can be described by a scalar potential ϕ and a vector potential $\mathbf{A}$ which satisfy the inhomogeneous wave equations

$$\nabla^2\phi - \epsilon_0\mu_0\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0},$$

$$\nabla^2\mathbf{A} - \epsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}.$$

If the current $\mathbf{j}$ satisfies Ohm's law and the charge density $\rho = 0$, show that plane waves of the form

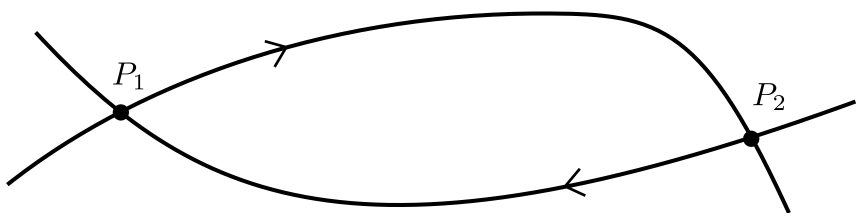
$$\mathbf{A} = A(z,t)e^{i\omega t}\hat{\mathbf{x}},$$

where $\hat{\mathbf{x}}$ is a unit vector in the x-direction of cartesian axes (x, y, z), are damped. Find an approximate expression for $A(z,t)$ when $\omega \ll \sigma/\epsilon_0$, where σ is the electrical conductivity.

6F Dynamics of Differential Equations

Define the terms *homoclinic orbit*, *heteroclinic orbit* and *heteroclinic loop*. In the case of a dynamical system that possesses a homoclinic orbit, explain, without detailed calculation, how to calculate its stability.

A second order dynamical system depends on two parameters μ_1 and μ_2. When $\mu_1 = \mu_2 = 0$ there is a heteroclinic loop between the points P_1, P_2 as in the diagram.



When μ_1, μ_2 are small there are trajectories that pass close to the fixed points P_1, P_2:

By adapting the method used above for trajectories near homoclinic orbits, show that the distances y_n, y_{n+1} to the stable manifold at P_1 on successive returns are related to z_n, z_{n+1}, the corresponding distances near P_2, by coupled equations of the form

$$\left.\begin{aligned} z_n &= (y_n)^{\gamma_1} + \mu_1, \\ y_{n+1} &= (z_n)^{\gamma_2} + \mu_2, \end{aligned}\right\}$$

where any arbitrary constants have been removed by rescaling, and γ_1, γ_2 depend on conditions near P_1, P_2. Show from these equations that there is a stable heteroclinic orbit ($\mu_1 = \mu_2 = 0$) if $\gamma_1\gamma_2 > 1$. Show also that in the marginal situation $\gamma_1 = 2$, $\gamma_2 = \frac{1}{2}$ there can be a stable fixed point for small positive y, z if $\mu_2 < 0$, $\mu_2^2 < \mu_1$. Explain carefully the form of the orbit of the original dynamical system represented by the solution of the above map when $\mu_2^2 = \mu_1$.

7K Geometry of Surfaces

Write an essay on the Euler number of topological surfaces. Your essay should include a definition of subdivision, some examples of surfaces and their Euler numbers, and a discussion of the statement and significance of the Gauss–Bonnet theorem.

8J Logic, Computation and Set Theory

Let P be a set of primitive propositions. Let $L(P)$ denote the set of all compound propositions over P, and let S be a subset of $L(P)$. Consider the relation $\preceq_S$ on $L(P)$ defined by

$$s \preceq_S t \text{ if and only if } S \cup \{s\} \vdash t.$$

Prove that $\preceq_S$ is reflexive and transitive. Deduce that if we define $\approx_S$ by ($s \approx_S t$ if and only if $s \preceq_S t$ and $t \preceq_S s$), then $\approx_S$ is an equivalence relation and the quotient $B_S = L(P)/\approx_S$ is partially ordered by the relation $\leqslant_S$ induced by $\preccurlyeq_S$ (that is, $[s] \leqslant_S [t]$ if and only if $s \preccurlyeq_S t$, where square brackets denote equivalence classes).

Assuming the result that B_S is a Boolean algebra with lattice operations induced by the logical operations on $L(P)$ (that is, $[s] \wedge [t] = [s \wedge t]$, etc.), show that there is a bijection between the following two sets:
(a) The set of lattice homomorphisms $B_S \to \{0,1\}$.
(b) The set of models of the propositional theory S.

Deduce that the completeness theorem for propositional logic is equivalent to the assertion that, for any Boolean algebra B with more than one element, there exists a homomorphism $B \to \{0,1\}$.

[*You may assume the result that the completeness theorem implies the compactness theorem.*]

9H Graph Theory

Write an essay on connectivity in graphs.

Your essay should include proofs of at least two major theorems, along with a discussion of one or two significant corollaries.

10J Number Theory

Write an essay on quadratic reciprocity. Your essay should include (i) a proof of the law of quadratic reciprocity for the Legendre symbol, (ii) a proof of the law of quadratic reciprocity for the Jacobi symbol, and (iii) a comment on why this latter law is useful in primality testing.

11M Algorithms and Networks

Write an essay on Strong Lagrangian problems. You should give an account of duality and how it relates to the Strong Lagrangian property. In particular, establish carefully the relationship between the Strong Lagrangian property and supporting hyperplanes.

Also, give an example of a class of problems that are Strong Lagrangian. [*You should explain carefully why your example has the Strong Lagrangian property.*]

12L Stochastic Financial Models

Write an essay on the Black–Scholes formula for the price of a European call option on a stock. Your account should include a derivation of the formula and a careful analysis of its dependence on the parameters of the model.

13M Principles of Statistics

(a) Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from a one-parameter distribution with density function

$$f(x;\theta) = h(x)g(\theta)\exp\{\theta t(x)\}\ ,\ x \in \mathbb{R}.$$

Explain in detail how you would test

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta \neq \theta_0\ .$$

What is the general form of a conjugate prior density for θ in a Bayesian analysis of this distribution?

(b) Let Y_1, Y_2 be independent Poisson random variables, with means $(1-\psi)\lambda$ and $\psi\lambda$ respectively, with λ *known.*

Explain why the Conditionality Principle leads to inference about ψ being drawn from the conditional distribution of Y_2, given Y_1+Y_2. What is this conditional distribution?

(c) Suppose Y_1, Y_2 have distributions as in (b), but that λ is now *unknown.*

Explain in detail how you would test $H_0 : \psi = \psi_0$ against $H_1 : \psi \neq \psi_0$, and describe the optimality properties of your test.

[*Any general results you use should be stated clearly, but need not be proved.*]

14L Computational Statistics and Statistical Modelling

Assume that the n-dimensional observation vector Y may be written as $Y = X\beta+\epsilon$, where X is a given $n\times p$ matrix of rank p, β is an unknown vector, with $\beta^T = (\beta_1, \ldots, \beta_p)$, and

$$\epsilon \sim N_n(0, \sigma^2 I) \qquad (*)$$

where σ^2 is unknown. Find $\hat{\beta}$, the least-squares estimator of β, and describe (without proof) how you would test

$$H_0 : \beta_\nu = 0$$

for a given ν.

Indicate briefly two plots that you could use as a check of the assumption $(*)$.

Continued opposite

Sulphur dioxide is one of the major air pollutants. A data-set presented by Sokal and Rohlf (1981) was collected on 41 US cities in 1969-71, corresponding to the following variables:

Y = sulphur dioxide content of air in micrograms per cubic metre

$X1$ = average annual temperature in degrees Fahrenheit

$X2$ = number of manufacturing enterprises employing 20 or more workers

$X3$ = population size (1970 census) in thousands

$X4$ = average annual wind speed in miles per hour

$X5$ = average annual precipitation in inches

$X6$ = average annual of days with precipitation per year.

Interpret the R output that follows below, quoting any standard theorems that you need to use.

> next.lm _ lm(log(Y) ~ X1 + X2 + X3 + X4 + X5 + X6)

> summary(next.lm)

Call: lm(formula = log(Y) ~ X1 + X2 + X3 + X4 + X5 + X6)

```
Residuals:
      Min        1Q    Median       3Q      Max
 -0.79548  -0.25538  -0.01968  0.28328  0.98029
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> |t|)	
(Intercept)	7.2532456	1.4483686	5.008	1.68e-05	***
X1	-0.0599017	0.0190138	-3.150	0.00339	**
X2	0.0012639	0.0004820	2.622	0.01298	*
X3	-0.0007077	0.0004632	-1.528	0.13580	
X4	-0.1697171	0.0555563	-3.055	0.00436	**
X5	0.0173723	0.0111036	1.565	0.12695	
X6	0.0004347	0.0049591	0.088	0.93066	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'

Residual standard error: 0.448 on 34 degrees of freedom

Multiple R-Squared: 0.6541

F-statistic: 10.72 on 6 and 34 degrees of freedom, p-value: 1.126e-06

15E Foundations of Quantum Mechanics

Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have (a) spin zero, (b) spin 1/2.

The stationary Schrödinger equation for one particle in the potential

$$-\frac{2e^2}{4\pi\epsilon_0 r}$$

has normalized, spherically symmetric, real wave functions $\psi_n(\mathbf{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. What are the consequences of the Pauli exclusion principle for the ground state of the helium atom? Assuming that wavefunctions which are not spherically symmetric can be ignored, what are the states of the first excited energy level of the helium atom?
[*You may assume here that the electrons are non-interacting.*]

Show that, taking into account the interaction between the two electrons, the estimate for the energy of the ground state of the helium atom is

$$2E_0 + \frac{e^2}{4\pi\epsilon_0}\int \frac{d^3\mathbf{r}_1\, d^3\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\psi_0^2(\mathbf{r}_1)\psi_0^2(\mathbf{r}_2).$$

16E Quantum Physics

Explain how the energy band structure for electrons determines the conductivity properties of crystalline materials.

A semiconductor has a conduction band with a lower edge E_c and a valence band with an upper edge E_v . Assuming that the density of states for electrons in the conduction band is

$$\rho_c(E) = B_c(E - E_c)^{\frac{1}{2}} \ , \qquad E > E_c \ \ ,$$

and in the valence band is

$$\rho_v(E) = B_v(E_v - E)^{\frac{1}{2}} \ , \qquad E < E_v \ \ ,$$

where B_c and B_v are constants characteristic of the semiconductor, explain why at low temperatures the chemical potential for electrons lies close to the mid-point of the gap between the two bands.

Describe what is meant by the doping of a semiconductor and explain the distinction between n-type and p-type semiconductors, and discuss the low temperature limit of the chemical potential in both cases. Show that, whatever the degree and type of doping,

$$n_e n_p = B_c B_v [\Gamma(3/2)]^2 (kT)^3 e^{-(E_c - E_v)/kT} \ \ ,$$

where n_e is the density of electrons in the conduction band and n_p is the density of holes in the valence band.

17D General Relativity

With respect to the Schwarzschild coordinates (r, θ, ϕ, t), the Schwarzschild geometry is given by

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - \left(1 - \frac{r_s}{r}\right) dt^2,$$

where $r_s = 2M$ is the Schwarzschild radius and M is the Schwarzschild mass. Show that, by a suitable choice of (θ, ϕ), the general geodesic can regarded as moving in the equatorial plane $\theta = \pi/2$. Obtain the equations governing timelike and null geodesics in terms of $u(\phi)$, where $u = 1/r$.

Discuss light bending and perihelion precession in the solar system.

18D Statistical Physics and Cosmology

What is an ideal gas? Explain how the microstates of an ideal gas of indistinguishable particles can be labelled by a set of integers. What range of values do these integers take for (a) a boson gas and (b) a Fermi gas?

Let E_i be the energy of the i-th one-particle energy eigenstate of an ideal gas in thermal equilibrium at temperature T and let $p_i(n_i)$ be the probability that there are n_i particles of the gas in this state. Given that

$$p_i(n_i) = e^{-\beta E_i n_i}/Z_i \quad (\beta = \frac{1}{kT}),$$

determine the normalization factor Z_i for (a) a boson gas and (b) a Fermi gas. Hence obtain an expression for $\bar{n}_i$, the average number of particles in the i-th one-particle energy eigenstate for both cases (a) and (b).

In the case of a Fermi gas, write down (without proof) the generalization of your formula for $\bar{n}_i$ to a gas at non-zero chemical potential μ. Show how it leads to the concept of a Fermi energy ϵ_F for a gas at zero temperature. How is ϵ_F related to the Fermi momentum p_F for (a) a non-relativistic gas and (b) an ultra-relativistic gas?

In an approximation in which the discrete set of energies E_i is replaced with a continuous set with momentum p, the density of one-particle states with momentum in the range p to $p + dp$ is $g(p)dp$. Explain briefly why

$$g(p) \propto p^2 V, \tag{$*$}$$

where V is the volume of the gas. Using this formula, obtain an expression for the total energy E of an ultra-relativistic gas at zero chemical potential as an integral over p. Hence show that

$$\frac{E}{V} \propto T^{\alpha},$$

where α is a number that you should compute. Why does this result apply to a photon gas?

Using the formula $(*)$ for a non-relativistic Fermi gas at zero temperature, obtain an expression for the particle number density n in terms of the Fermi momentum and provide a physical interpretation of this formula in terms of the typical de Broglie wavelength. Obtain an analogous formula for the (internal) energy density and hence show that the pressure P behaves as

$$P \propto n^{\gamma}$$

where γ is a number that you should compute. [*You need not prove any relation between the pressure and the energy density you use.*] What is the origin of this pressure given that $T = 0$ by assumption? Explain briefly and qualitatively how it is relevant to the stability of white dwarf stars.

19C Transport Processes

(a) A biological vessel is modelled two-dimensionally as a fluid-filled channel bounded by parallel plane walls $y = \pm a$, embedded in an infinite region of fluid-saturated tissue. In the tissue a solute has concentration $C^{out}(y,t)$, diffuses with diffusivity D and is consumed by biological activity at a rate kC^{out} per unit volume, where D and k are constants. By considering the solute balance in a slice of tissue of infinitesimal thickness, show that

$$C_t^{out} = DC_{yy}^{out} - kC^{out}.$$

A *steady* concentration profile $C^{out}(y)$ results from a flux $\beta\left(C^{in} - C_a^{out}\right)$, per unit area of wall, of solute from the channel into the tissue, where C^{in} is a constant concentration of solute that is maintained in the channel and $C_a^{out} = C^{out}(a)$. Write down the boundary conditions satisfied by $C^{out}(y)$. Solve for $C^{out}(y)$ and show that

$$C_a^{out} = \frac{\gamma}{\gamma+1} C^{in}, \qquad (*)$$

where $\gamma = \beta/\sqrt{kD}$.

(b) Now let the solute be supplied by steady flow down the channel from one end, $x = 0$, with the channel taken to be semi-infinite in the x-direction. The cross-sectionally averaged velocity in the channel $u(x)$ varies due to a flux of fluid from the tissue to the channel (by osmosis) equal to $\lambda\left(C^{in} - C_a^{out}\right)$ per unit area. Neglect both the variation of $C^{in}(x)$ across the channel and diffusion in the x-direction.

By considering conservation of fluid, show that

$$au_x = \lambda\left(C^{in} - C_a^{out}\right)$$

and write down the corresponding equation derived from conservation of solute. Deduce that

$$u(\lambda C^{in} + \beta) = u_0(\lambda C_0^{in} + \beta)\ ,$$

where $u_0 = u(0)$ and $C_0^{in} = C^{in}(0)$.

Assuming that equation $(*)$ still holds, even though C^{out} is now a function of x as well as y, show that $u(x)$ satisfies the ordinary differential equation

$$(\gamma+1)auu_x + \beta u = u_0\left(\lambda C_0^{in} + \beta\right).$$

Find scales $\hat{x}$ and $\hat{u}$ such that the dimensionless variables $U = u/\hat{u}$ and $X = x/\hat{x}$ satisfy

$$UU_X + U = 1.$$

Derive the solution $(1-U)e^U = Ae^{-X}$ and find the constant A.

To what values do u and C_{in} tend as $x \to \infty$?

20G Theoretical Geophysics

The equation of motion for small displacements $\mathbf{u}$ in a homogeneous, isotropic, elastic material is

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \mu \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u}) \quad ,$$

where λ and μ are the Lamé constants. Derive the conditions satisfied by the polarisation $\mathbf{P}$ and (real) vector slowness $\mathbf{s}$ of plane-wave solutions $\mathbf{u} = \mathbf{P} f(\mathbf{s} \cdot \mathbf{x} - t)$, where f is an arbitrary scalar function. Describe the division of these waves into P-waves, SH-waves and SV-waves.

A plane harmonic SV-wave of the form

$$\mathbf{u} = (s_3, 0, -s_1) \exp[i\omega(s_1 x_1 + s_3 x_3 - t)]$$

travelling through homogeneous elastic material of P-wave speed α and S-wave speed β is incident from $x_3 < 0$ on the boundary $x_3 = 0$ of rigid material in $x_3 > 0$ in which the displacement is identically zero.

Write down the form of the reflected wavefield in $x_3 < 0$. Calculate the amplitudes of the reflected waves in terms of the components of the slowness vectors.

Derive expressions for the components of the incident and reflected slowness vectors, in terms of the wavespeeds and the angle of incidence θ_0. Hence show that there is no reflected SV-wave if

$$\sin^2 \theta_0 = \frac{\beta^2}{\alpha^2 + \beta^2} \quad .$$

Sketch the rays produced if the region $x_3 > 0$ is fluid instead of rigid.

21C Mathematical Methods

State Watson's lemma giving an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_1 = \int_0^A f(t) e^{-\lambda t} dt \,, \quad A > 0 \;.$$

Show how this result may be used to find an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_2 = \int_{-A}^{B} f(t) e^{-\lambda t^2} dt \,, \quad A > 0,\, B > 0 \;.$$

Hence derive Laplace's method for obtaining an asymptotic expansion as $\lambda \to \infty$ for an integral of the form

$$I_3 = \int_a^b f(t) e^{\lambda \phi(t)} dt \;,$$

where $\phi(t)$ is differentiable, for the cases: (i) $\phi'(t) < 0$ in $a \le t \le b$; and (ii) $\phi'(t)$ has a simple zero at $t = c$ with $a < c < b$ and $\phi''(c) < 0$.

Find the first two terms in the asymptotic expansion as $x \to \infty$ of

$$I_4 = \int_{-\infty}^{\infty} \log(1 + t^2) e^{-x t^2} dt \;.$$

[*You may leave your answer expressed in terms of* Γ*-functions.*]

22F Numerical Analysis

Write an essay on the method of conjugate gradients. You should describe the algorithm, present an analysis of its properties and discuss its advantages.

[*Any theorems quoted should be stated precisely but need not be proved.*]

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Monday 3 June 2002 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***4D, 22D*** *should be in one bundle and* ***13L, 16L*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) We are given a finite set of airports. Assume that between any two airports, i and j, there are $a_{ij} = a_{ji}$ flights in each direction on every day. A confused traveller takes one flight per day, choosing at random from all available flights. Starting from i, how many days on average will pass until the traveller returns again to i? Be careful to allow for the case where there may be no flights at all between two given airports.

(ii) Consider the infinite tree T with root R, where, for all $m \geqslant 0$, all vertices at distance 2^m from R have degree 3, and where all other vertices (except R) have degree 2. Show that the random walk on T is recurrent.

2G Principles of Dynamics

(i) Derive Hamilton's equations from Lagrange's equations. Show that the Hamiltonian H is constant if the Lagrangian L does not depend explicitly on time.

(ii) A particle of mass m is constrained to move under gravity, which acts in the negative z-direction, on the spheroidal surface $\epsilon^{-2}(x^2+y^2)+z^2=l^2$, with $0<\epsilon\leqslant 1$. If θ,ϕ parametrize the surface so that

$$x=\epsilon l\sin\theta\cos\phi,\ y=\epsilon l\sin\theta\sin\phi,\ z=l\cos\theta,$$

find the Hamiltonian $H(\theta,\phi,p_\theta,p_\phi)$.

Show that the energy

$$E=\frac{p_\theta^2}{2ml^2(\epsilon^2\cos^2\theta+\sin^2\theta)}+\frac{\alpha}{\sin^2\theta}+mgl\cos\theta$$

is a constant of the motion, where α is a non-negative constant.

Rewrite this equation as

$$\frac{1}{2}\dot\theta^2+V_{\text{eff}}(\theta)=0$$

and sketch $V_{\text{eff}}(\theta)$ for $\epsilon=1$ and $\alpha>0$, identifying the maximal and minimal values of $\theta(t)$ for fixed α and E. If ϵ is now taken not to be unity, how do these values depend on ϵ?

3H Groups, Rings and Fields

State Sylow's Theorems. Prove the existence part of Sylow's Theorems.

Show that any group of order 33 is cyclic.

Show that a group of order p^2q, where p and q are distinct primes, is not simple. Is it always abelian? Give a proof or a counterexample.

4D Electromagnetism

(i) Show that, in a region where there is no magnetic field and the charge density vanishes, the electric field can be expressed either as minus the gradient of a scalar potential ϕ or as the curl of a vector potential $\mathbf{A}$. Verify that the electric field derived from

$$\mathbf{A} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\wedge\mathbf{r}}{r^3}$$

is that of an electrostatic dipole with dipole moment $\mathbf{p}$.

[*You may assume the following identities:*

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\wedge(\nabla\wedge\mathbf{b}) + \mathbf{b}\wedge(\nabla\wedge\mathbf{a}) + (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a},$$

$$\nabla\wedge(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot\mathbf{a}.]$$

(ii) An infinite conducting cylinder of radius a is held at zero potential in the presence of a line charge parallel to the axis of the cylinder at distance $s_0 > a$, with charge density q per unit length. Show that the electric field outside the cylinder is equivalent to that produced by replacing the cylinder with suitably chosen image charges.

5H Combinatorics

Prove that every graph G on $n \geqslant 3$ vertices with minimal degree $\delta(G) \geqslant \frac{n}{2}$ is Hamiltonian. For each $n \geqslant 3$, give an example to show that this result does not remain true if we weaken the condition to $\delta(G) \geqslant \frac{n}{2} - 1$ (n even) or $\delta(G) \geqslant \frac{n-1}{2}$ (n odd).

Now let G be a connected graph (with at least 2 vertices) without a cutvertex. Does G Hamiltonian imply G Eulerian? Does G Eulerian imply G Hamiltonian? Justify your answers.

6J Representation Theory

Construct the character table of the symmetric group S_5, explaining the steps in your construction.

Use the character table to show that the alternating group A_5 is the only non-trivial normal subgroup of S_5.

7J Galois Theory

Let $F \subset K$ be a finite extension of fields and let G be the group of F-automorphisms of K. State a result relating the order of G to the degree $[K : F]$.

Now let $K = k(X_1, \ldots, X_4)$ be the field of rational functions in four variables over a field k and let $F = k(s_1, \ldots, s_4)$ where $s_1, \ldots, s_4$ are the elementary symmetric polynomials in $k[X_1, \ldots, X_4]$. Show that the degree $[K : F] \leqslant 4!$ and deduce that F is the fixed field of the natural action of the symmetric group S_4 on K.

Show that $X_1X_3 + X_2X_4$ has a cubic minimum polynomial over F. Let $G = \langle \sigma, \tau \rangle \subset S_4$ be the dihedral group generated by the permutations $\sigma = (1234)$ and $\tau = (13)$. Show that the fixed field of G is $F(X_1X_3 + X_2X_4)$. Find the fixed field of the subgroup $H = \langle \sigma^2, \tau \rangle$.

8K Differentiable Manifolds

What is meant by a "bump function" on $\mathbb{R}^n$? If U is an open subset of a manifold M, prove that there is a bump function on M with support contained in U.

Prove the following.

(i) Given an open covering $\mathcal{U}$ of a compact manifold M, there is a partition of unity on M subordinate to $\mathcal{U}$.

(ii) Every compact manifold may be embedded in some Euclidean space.

9J Number Fields

Explain what is meant by an integral basis $\omega_1, \ldots, \omega_n$ of a number field K. Give an expression for the discriminant of K in terms of the traces of the $\omega_i\omega_j$.

Let $K = \mathbb{Q}(i, \sqrt{2})$. By computing the traces $T_{K/k}(\theta)$, where k runs through the three quadratic subfields of K, show that the algebraic integers θ in K have the form $\frac{1}{2}(\alpha + \beta\sqrt{2})$, where $\alpha = a + ib$ and $\beta = c + id$ are Gaussian integers. By further computing the norm $N_{K/k}(\theta)$, where $k = \mathbb{Q}(\sqrt{2})$, show that a and b are even and that $c \equiv d \pmod 2$. Hence prove that an integral basis for K is $1, i, \sqrt{2}, \frac{1}{2}(1+i)\sqrt{2}$.

Calculate the discriminant of K.

10K Hilbert Spaces

Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. Define what it means for T to be *bounded below*. Prove that, if $LT = I$ for some $L \in \mathcal{B}(H)$, then T is bounded below.

Prove that an operator $T \in \mathcal{B}(H)$ is invertible if and only if both T and T^* are bounded below.

Let H be the sequence space ℓ^2. Define the operators S, R on H by setting

$$S(\xi) = (0, \xi_1, \xi_2, \xi_3, \dots), \quad R(\xi) = (\xi_2, \xi_3, \xi_4, \dots),$$

for all $\xi = (\xi_1, \xi_2, \xi_3, \dots) \in \ell^2$. Check that $RS = I$ but $SR \neq I$. Let $D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. For each $\lambda \in D$, explain why $I - \lambda R$ is invertible, and define

$$R(\lambda) = (I - \lambda R)^{-1} R.$$

Show that, for all $\lambda \in D$, we have $R(\lambda)(S - \lambda I) = I$, but $(S - \lambda I)R(\lambda) \neq I$. Deduce that, for all $\lambda \in D$, the operator $S - \lambda I$ is bounded below, but is not invertible. Deduce also that $\operatorname{Sp} S = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$.

Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, and for $n = 1, 2, \dots$, define the element x_n of ℓ^2 by

$$x_n = n^{-1/2}(\lambda^{-1}, \lambda^{-2}, \dots, \lambda^{-n}, 0, 0, \dots).$$

Prove that $\|x_n\| = 1$ but that $(S - \lambda I)x_n \to 0$ as $n \to \infty$. Deduce that, for $|\lambda| = 1$, $S - \lambda I$ is not bounded below.

11K Riemann Surfaces

(a) Define the notions of (abstract) Riemann surface, holomorphic map, and biholomorphic map between Riemann surfaces.

(b) Prove the following theorem on the local form of a holomorphic map.

For a holomorphic map $f : R \to S$ between Riemann surfaces, which is not constant near a point $r \in R$, there exist neighbourhoods U of r in R and V of $f(r)$ in S, together with biholomorphic identifications $\phi : U \to \Delta$, $\psi : V \to \Delta$, such that $(\psi \circ f)(x) = \phi(x)^n$, for all $x \in U$.

(c) Prove further that a non-constant holomorphic map between compact, connected Riemann surfaces is surjective.

(d) Deduce from (c) the fundamental theorem of algebra.

12J Logic, Computation and Set Theory

(i) State the Knaster-Tarski fixed point theorem. Use it to prove the Cantor-Bernstein Theorem; that is, if there exist injections $A \to B$ and $B \to A$ for two sets A and B then there exists a bijection $A \to B$.

(ii) Let A be an arbitrary set and suppose given a subset R of $PA \times A$. We define a subset $B \subseteq A$ to be R-*closed* just if whenever $(S, a) \in R$ and $S \subseteq B$ then $a \in B$. Show that the set of all R-closed subsets of A is a complete poset in the inclusion ordering.

Now assume that A is itself equipped with a partial ordering $\leqslant$.

(a) Suppose R satisfies the condition that if $b \geqslant a \in A$ then $(\{b\}, a) \in R$. Show that if B is R-closed then $c \leqslant b \in B$ implies $c \in B$.

(b) Suppose that R satisfies the following condition. Whenever $(S, a) \in R$ and $b \leqslant a$ then there exists $T \subseteq A$ such that $(T, b) \in R$, and for every $t \in T$ we have (i) $(\{b\}, t) \in R$, and (ii) $t \leqslant s$ for some $s \in S$. Let B and C be R-closed subsets of A. Show that the set

$$[B \to C] = \{a \in A \,|\, \forall b \leqslant a \, (b \in B \Rightarrow b \in C)\}$$

is R-closed.

13L Probability and Measure

State and prove Dynkin's π-system lemma.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (A_n) be a sequence of independent events such that $\lim_{n\to\infty} \mathbb{P}(A_n) = p$. Let $\mathcal{G} = \sigma(A_1, A_2, \ldots)$. Prove that

$$\lim_{n\to\infty} \mathbb{P}(G \cap A_n) = p\mathbb{P}(G)$$

for all $G \in \mathcal{G}$.

14M Information Theory

(a) Define the entropy $h(X)$ and the mutual entropy $i(X,Y)$ of random variables X and Y. Prove the inequality

$$0 \leqslant i(X,Y) \leqslant \min\{h(X),h(Y)\}.$$

[*You may assume the Gibbs inequality.*]

(b) Let X be a random variable and let $\mathbf{Y}=(Y_1,\ldots,Y_n)$ be a random vector.

(i) Prove or disprove by producing a counterexample the inequality

$$i(X,\mathbf{Y}) \leqslant \sum_{j=1}^{n} i(X,Y_j),$$

first under the assumption that $Y_1,\ldots,Y_n$ are independent random variables, and then under the assumption that $Y_1,\ldots,Y_n$ are conditionally independent given X.

(ii) Prove or disprove by producing a counterexample the inequality

$$i(X,\mathbf{Y}) \geqslant \sum_{j=1}^{n} i(X,Y_j),$$

first under the assumption that $Y_1,\ldots,Y_n$ are independent random variables, and then under the assumption that $Y_1,\ldots,Y_n$ are conditionally independent given X.

15M Principles of Statistics

(i) Explain in detail the *minimax* and *Bayes* principles of decision theory.

Show that if $d(X)$ is a Bayes decision rule for a prior density $\pi(\theta)$ and has constant risk function, then $d(X)$ is minimax.

(ii) Let $X_1, \ldots, X_p$ be independent random variables, with $X_i \sim N(\mu_i, 1)$, $i = 1, \ldots, p$.

Consider estimating $\mu = (\mu_1, \ldots, \mu_p)^T$ by $d = (d_1, \ldots, d_p)^T$, with loss function

$$L(\mu, d) = \sum_{i=1}^{p} (\mu_i - d_i)^2 \ .$$

What is the risk function of $X = (X_1, \ldots, X_p)^T$?

Consider the class of estimators of μ of the form

$$d^a(X) = \left(1 - \frac{a}{X^T X}\right) X \ ,$$

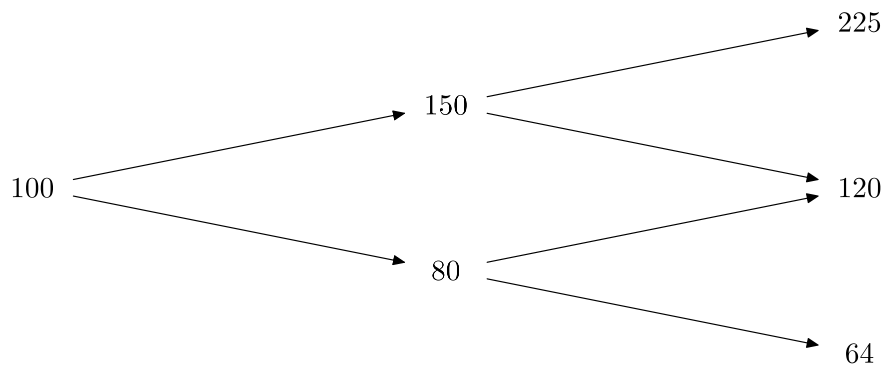
indexed by $a \geqslant 0$. Find the risk function of $d^a(X)$ in terms of $E\left(1/X^T X\right)$, which you should not attempt to evaluate, and deduce that X is inadmissible. What is the optimal value of a?

[*You may assume Stein's Lemma, that for suitably behaved real-valued functions h,*

$$E\left\{(X_i - \mu_i) h(X)\right\} = E\left\{\frac{\partial h(X)}{\partial X_i}\right\}. \quad]$$

16L Stochastic Financial Models

(i) The prices, S_i, of a stock in a binomial model at times $i = 0, 1, 2$ are represented by the following binomial tree.



The fixed interest rate per period is 1/5 and the probability that the stock price increases in a period is 1/3. Find the price at time 0 of a European call option with strike price 78 and expiry time 2.

Explain briefly the ideas underlying your calculations.

(ii) Consider an investor in a one-period model who may invest in s assets, all of which are risky, with a random return vector $\boldsymbol{R}$ having mean $\mathbb{E}\boldsymbol{R} = \boldsymbol{r}$ and positive-definite covariance matrix $\boldsymbol{V}$; assume that not all the assets have the same expected return. Show that any minimum-variance portfolio is equivalent to the investor dividing his wealth between two portfolios, the global minimum-variance portfolio and the diversified portfolio, both of which should be specified clearly in terms of $\boldsymbol{r}$ and $\boldsymbol{V}$.

Now suppose that $\boldsymbol{R} = (R_1, R_2, \ldots, R_s)^{\top}$ where $R_1, R_2, \ldots, R_s$ are independent random variables with R_i having the exponential distribution with probability density function $\lambda_i e^{-\lambda_i x}$, $x \geqslant 0$, where $\lambda_i > 0$, $1 \leqslant i \leqslant s$. Determine the global minimum-variance portfolio and the diversified portfolio explicitly.

Consider further the situation when the investor has the utility function $u(x) = 1 - e^{-x}$, where x denotes his wealth. Suppose that he acts to maximize the expected utility of his final wealth, and that his initial wealth is $w > 0$. Show that he now divides his wealth between the diversified portfolio and the *uniform* portfolio, in which wealth is apportioned equally between the assets, and determine the amounts that he invests in each.

17F Dynamical Systems

Let f_c be the map of the closed interval [0,1] to itself given by

$$f_c(x) = cx(1-x), \text{ where } 0 \leqslant c \leqslant 4.$$

Sketch the graphs of f_c and (without proof) of f_c^2, find their fixed points, and determine which of the fixed points of f_c are attractors. Does your argument work for $c = 3$?

18G Partial Differential Equations

(a) Solve the equation, for a function $u(x, y)$,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad (*)$$

together with the boundary condition on the x-axis:

$$u(x, 0) = x.$$

Find for which real numbers a it is possible to solve $(*)$ with the following boundary condition specified on the line $y = ax$:

$$u(x, ax) = x.$$

Explain your answer in terms of the notion of *characteristic hypersurface*, which should be defined.

(b) Solve the equation

$$\frac{\partial u}{\partial x} + (1 + u)\frac{\partial u}{\partial y} = 0$$

with the boundary condition on the x-axis

$$u(x, 0) = x,$$

in the domain $\mathcal{D} = \{(x, y) : 0 < y < (x+1)^2/4, \ \ -1 < x < \infty\}$. Sketch the characteristics.

19G Methods of Mathematical Physics

State the Riemann-Lebesgue lemma as applied to the integral

$$\int_a^b g(u)e^{ixu}du\ ,$$

where $g'(u)$ is continuous and $a, b \in \mathbb{R}$.

Use this lemma to show that, as $x \to +\infty$,

$$\int_a^b (u-a)^{\lambda-1} f(u)e^{ixu}du \sim f(a)\, e^{ixa}\, e^{i\pi\lambda/2}\, \Gamma(\lambda)\, x^{-\lambda}\ ,$$

where $f(u)$ is holomorphic, $f(a) \neq 0$ and $0 < \lambda < 1$. You should explain each step of your argument, but detailed analysis is not required.

Hence find the leading order asymptotic behaviour as $x \to +\infty$ of

$$\int_0^1 \frac{e^{ixt^2}}{(1-t^2)^{\frac{1}{2}}}\, dt\ .$$

20F Numerical Analysis

(i) Let A be an $n \times n$ symmetric real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots., \mathbf{v}_n$, where $\|\mathbf{v}_l\| = 1$. Given $\mathbf{x}^{(0)} \in \mathbb{R}^n$, $\|\mathbf{x}^{(0)}\| = 1$, the sequence $\mathbf{x}^{(k)}$ is generated in the following manner. We set

$$\mu = \mathbf{x}^{(k)\,T} A \mathbf{x}^{(k)},$$

$$\mathbf{y} = (A - \mu I)^{-1} \mathbf{x}^{(k)},$$

$$\mathbf{x}^{(k+1)} = \frac{\mathbf{y}}{\|\mathbf{y}\|}.$$

Show that if

$$\mathbf{x}^{(k)} = c^{-1} \left(\mathbf{v}_1 + \alpha \sum_{l=2}^{n} d_l \mathbf{v}_l \right),$$

where α is a real scalar and c is chosen so that $\|\mathbf{x}^{(k)}\| = 1$, then

$$\mu = c^{-2} \left(\lambda_1 + \alpha^2 \sum_{j=2}^{n} \lambda_j d_j^2 \right).$$

Give an explicit expression for c.

(ii) Use the above result to prove that, if $|\alpha|$ is small,

$$\mathbf{x}^{(k+1)} = \tilde{c}^{-1} \left(\mathbf{v}_1 + \alpha^3 \sum_{l=2}^{n} \tilde{d}_l \mathbf{v}_l \right) + O(\alpha^4)$$

and obtain the numbers $\tilde{c}$ and $\tilde{d}_2, \ldots, \tilde{d}_n$.

21E Electrodynamics

Explain how one can write Maxwell's equations in relativistic form by introducing an antisymmetric field strength tensor F_{ab}.

In an inertial frame S, the electric and magnetic fields are $\mathbf{E}$ and $\mathbf{B}$. Suppose that there is a second inertial frame S' moving with velocity v along the x-axis relative to S. Derive the rules for finding the electric and magnetic fields $\mathbf{E}'$ and $\mathbf{B}'$ in the frame S'. Show that $|\mathbf{E}|^2 - |\mathbf{B}|^2$ and $\mathbf{E} \cdot \mathbf{B}$ are invariant under Lorentz transformations.

Suppose that $\mathbf{E} = E_0(0\,,1\,,0)$ and $\mathbf{B} = E_0(0\,,\cos\theta\,,\sin\theta)$, where $0 \leq \theta < \pi/2$. At what velocity must an observer be moving in the frame S for the electric and magnetic fields to appear to be parallel?

Comment on the case $\theta = \pi/2$.

22D Statistical Physics

A simple model for a rubber molecule consists of a one-dimensional chain of n links each of fixed length b and each of which is oriented in either the positive or negative direction. A unique state i of the molecule is designated by giving the orientation ± 1 of each link. If there are n_+ links oriented in the positive direction and n_- links oriented in the negative direction then $n = n_+ + n_-$ and the length of the molecule is $l = (n_+ - n_-)b$. The length of the molecule associated with state i is l_i.

What is the range of l?

What is the number of states with n, n_+, n_- fixed?

Consider an ensemble of A copies of the molecule in which a_i members are in state i and write down the expression for the mean length L.

By introducing a Lagrange multiplier τ for L show that the most probable configuration for the $\{a_i\}$ with given length L is found by maximizing

$$\log\left(\frac{A!}{\prod_i a_i!}\right) \ + \ \tau \sum_i \ a_i l_i \ - \alpha \sum_i a_i .$$

Hence show that the most probable configuration is given by

$$p_i \ = \ \frac{e^{\tau l_i}}{Z} \ ,$$

where p_i is the probability for finding an ensemble member in the state i and Z is the partition function which should be defined.

Show that Z can be expressed as

$$Z \ = \ \sum_l g(l) \ e^{\tau l} \ ,$$

where the meaning of $g(l)$ should be explained.

Hence show that Z is given by

$$Z \ = \ \sum_{n_+=0}^{n} \frac{n!}{n_+! \, n_-!} \ (e^{\tau b})^{n_+} (e^{-\tau b})^{n_-} \ , \qquad n_+ + n_- = n,$$

and therefore that the free energy G for the system is

$$G \ = \ -nkT \log(2\cosh \tau b) \ .$$

Show that τ is determined by

$$L \ = \ -\frac{1}{kT}\left(\frac{\partial G}{\partial \tau}\right)_n \ ,$$

and hence that the equation of state is

$$\tanh \tau b \ = \ \frac{L}{nb} \ .$$

What are the independent variables on which G depends?

Explain why the tension in the rubber molecule is $kT\tau$.

23E Applications of Quantum Mechanics

A quantum system, with Hamiltonian H_0, has continuous energy eigenstates $|E\rangle$ for all $E \geq 0$, and also a discrete eigenstate $|0\rangle$, with $H_0|0\rangle = E_0|0\rangle$, $\langle 0|0\rangle = 1$, $E_0 > 0$. A time-independent perturbation H_1, such that $\langle E|H_1|0\rangle \neq 0$, is added to H_0. If the system is initially in the state $|0\rangle$ obtain the formula for the decay rate

$$w = \frac{2\pi}{\hbar}\,\rho(E_0)\,\big|\langle E_0|H_1|0\rangle\big|^2\,,$$

where ρ is the density of states.

[*You may assume that* $\frac{1}{t}\big(\frac{\sin\frac{1}{2}\omega t}{\frac{1}{2}\omega}\big)^2$ *behaves like* $2\pi\,\delta(\omega)$ *for large* t.]

Assume that, for a particle moving in one dimension,

$$H_0 = E_0|0\rangle\langle 0| + \int_{-\infty}^{\infty} p^2|p\rangle\langle p|\,dp\,, \qquad H_1 = f\int_{-\infty}^{\infty}\big(|p\rangle\langle 0| + |0\rangle\langle p|\big)\,dp\,,$$

where $\langle p'|p\rangle = \delta(p'-p)$, and f is constant. Obtain w in this case.

24D General Relativity

(i) Given a covariant vector field V_a, define the curvature tensor $R^a{}_{bcd}$ by

$$V_{a;bc} - V_{a;cb} = V_e R^e{}_{abc}. \qquad (*)$$

Express $R^e{}_{abc}$ in terms of the Christoffel symbols and their derivatives. Show that

$$R^e{}_{abc} = -R^e{}_{acb}.$$

Further, by setting $V_a = \partial\phi/\partial x^a$, deduce that

$$R^e{}_{abc} + R^e{}_{cab} + R^e{}_{bca} = 0.$$

(ii) Write down an expression similar to $(*)$ given in Part (i) for the quantity

$$g_{ab;cd} - g_{ab;dc}$$

and hence show that

$$R_{eabc} = -R_{aebc}.$$

Define the Ricci tensor, show that it is symmetric and write down the contracted Bianchi identities.

In certain spacetimes of dimension $n \geq 2$, R_{abcd} takes the form

$$R_{abcd} = K(x^e)[g_{ac}g_{bd} - g_{ad}g_{bc}].$$

Obtain the Ricci tensor and Ricci scalar. Deduce that K is a constant in such spacetimes if the dimension n is greater than 2.

25C Fluid Dynamics II

State the minimum dissipation theorem for Stokes flow in a bounded domain.

Fluid of density ρ and viscosity μ fills an infinite cylindrical annulus $a \leq r \leq b$ between a fixed cylinder $r = a$ and a cylinder $r = b$ which rotates about its axis with constant angular velocity Ω. In cylindrical polar coordinates (r, θ, z), the fluid velocity is $\mathbf{u} = (0, v(r), 0)$. The Reynolds number $\rho\Omega b^2/\mu$ is not necessarily small. Show that $v(r) = Ar + B/r$, where A and B are constants to be determined.

[*You may assume that* $\nabla^2\mathbf{u} = (0, \nabla^2 v - v/r^2, 0)$ *and* $(\mathbf{u}\cdot\nabla)\mathbf{u} = (-v^2/r, 0, 0)$.]

Show that the outer cylinder exerts a couple G_0 per unit length on the fluid, where

$$G_0 = \frac{4\pi\mu\Omega a^2 b^2}{b^2 - a^2}.$$

[*You may assume that, in standard notation,* $e_{r\theta} = \dfrac{r}{2}\dfrac{d}{dr}\left(\dfrac{v}{r}\right)$.]

Suppose now that $b \geq \sqrt{2}a$ and that the cylinder $r = a$ is replaced by a fixed cylinder whose cross-section is a square of side $2a$ centred on $r = 0$, all other conditions being unchanged. The flow may still be assumed steady. Explaining your argument carefully, show that the couple G now required to maintain the motion of the outer cylinder is greater than G_0.

26C Waves in Fluid and Solid Media

Starting from the equations governing sound waves linearized about a state with density ρ_0 and sound speed c_0, derive the acoustic energy equation, giving expressions for the local energy density E and energy flux $\mathbf{I}$.

A sphere executes small-amplitude vibrations, with its radius varying according to

$$r(t) = a + \mathrm{Re}\left(\epsilon e^{i\omega t}\right),$$

with $0 < \epsilon \ll a$. Find an expression for the velocity potential of the sound, $\tilde{\phi}(r, t)$. Show that the time-averaged rate of working by the surface of the sphere is

$$2\pi a^2 \rho_0 \omega^2 \epsilon^2 c_0 \frac{\omega^2 a^2}{c_0^2 + \omega^2 a^2}.$$

Calculate the value at $r = a$ of the dimensionless ratio $c_0\overline{E}/|\overline{\mathbf{I}}|$, where the overbars denote time-averaged values, and comment briefly on the limits $c_0 \ll \omega a$ and $c_0 \gg \omega a$.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Tuesday 4 June 2002 9 to 12

PAPER 2

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***1G, 17G*** *should be in one bundle and* ***13L, 15L*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1G Principles of Dynamics

(i) A number N of non-interacting particles move in one dimension in a potential $V(x,t)$. Write down the Hamiltonian and Hamilton's equations for one particle.

At time t, the number density of particles in phase space is $f(x,p,t)$. Write down the time derivative of f along a particle's trajectory. By equating the rate of change of the number of particles in a fixed domain V in phase space to the flux into V across its boundary, deduce that f is a constant along any particle's trajectory.

(ii) Suppose that $V(x)=\frac{1}{2}m\omega^2x^2$, and particles are injected in such a manner that the phase space density is a constant f_1 at any point of phase space corresponding to a particle energy being smaller than E_1 and zero elsewhere. How many particles are present?

Suppose now that the potential is very slowly altered to the square well form

$$V(x)=\begin{cases}0, & -L<x<L\\ \infty & \text{elsewhere}\ .\end{cases}$$

Show that the greatest particle energy is now

$$E_2=\frac{\pi^2}{8}\frac{E_1^2}{mL^2\omega^2}.$$

2K Functional Analysis

(i) State and prove the parallelogram law for Hilbert spaces.

Suppose that K is a closed linear subspace of a Hilbert space H and that $x\in H$. Show that x is orthogonal to K if and only if 0 is the nearest point to x in K.

(ii) Suppose that H is a Hilbert space and that ϕ is a continuous linear functional on H with $\|\phi\|=1$. Show that there is a sequence (h_n) of unit vectors in H with $\phi(h_n)$ real and $\phi(h_n)>1-1/n$.

Show that h_n converges to a unit vector h, and that $\phi(h)=1$.

Show that h is orthogonal to N, the null space of ϕ, and also that $H=N\oplus\mathrm{span}(h)$.

Show that $\phi(k)=\langle k,h\rangle$, for all $k\in H$.

3H Groups, Rings and Fields

(i) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(ii) What are the units in $\mathbb{Z}[i]$? What are the primes in $\mathbb{Z}[i]$? Justify your answers.

Factorize $11+7i$ into primes in $\mathbb{Z}[i]$.

4F Dynamics of Differential Equations

(i) Define the terms *stable manifold* and *unstable manifold* of a hyperbolic fixed point $\mathbf{x}_0$ of a dynamical system. State carefully the stable manifold theorem.

Give an approximation, correct to fourth order in $|\mathbf{x}|$, for the stable and unstable manifolds of the origin for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x + x^2 - y^2 \\ -y + x^2 \end{pmatrix} .$$

(ii) State, without proof, the centre manifold theorem. Show that the fixed point at the origin of the system

$$\begin{aligned} \dot{x} &= y - x + ax^3 \ , \\ \dot{y} &= rx - y - zy \ , \\ \dot{z} &= -z + xy \ , \end{aligned}$$

where a is a constant, is non-hyperbolic at $r = 1$.

Using new coordinates $v = x + y\,,\ w = x - y$, find the centre manifold in the form

$$w = \alpha v^3 + \dots \,, \quad z = \beta v^2 + \gamma v^4 + \dots$$

for constants α, β, γ to be determined. Hence find the evolution equation on the centre manifold in the form

$$\dot{v} = \frac{1}{8}(a-1)v^3 + \left(\frac{(3a+1)(a+1)}{128} + \frac{(a-1)}{32} \right) v^5 + \dots \ .$$

Ignoring higher order terms, give conditions on a that guarantee that the origin is asymptotically stable.

5H Combinatorics

State and prove the local LYM inequality. Explain carefully when equality holds.

Define the colex order and state the Kruskal-Katona theorem. Deduce that, if n and r are fixed positive integers with $1 \leqslant r \leqslant n-1$, then for every $1 \leqslant m \leqslant \binom{n}{r}$ we have

$$\min\{|\partial \mathcal{A}| : \mathcal{A} \subset [n]^{(r)}, |\mathcal{A}| = m\} = \min\{|\partial \mathcal{A}| : \mathcal{A} \subset [n+1]^{(r)}, |\mathcal{A}| = m\}.$$

By a suitable choice of n, r and m, show that this result does not remain true if we replace the lower shadow ∂A with the upper shadow $\partial^+ \mathcal{A}$.

6J Representation Theory

State and prove Schur's Lemma. Deduce that the centre of a finite group G with a faithful irreducible complex representation ρ is cyclic and that $Z(\rho(G))$ consists of scalar transformations.

Let G be the subgroup of order 18 of the symmetric group S_6 given by

$$G = \langle (123), (456), (23)(56) \rangle .$$

Show that G has a normal subgroup of order 9 and four normal subgroups of order 3. By considering quotients, show that G has two representations of dimension 1 and four inequivalent irreducible representations of degree 2. Deduce that G has no faithful irreducible complex representations.

Show finally that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible on restriction to H.

7K Differentiable Manifolds

State, giving your reasons, whether the following are true or false.

(a) Diffeomorphic connected manifolds must have the same dimension.

(b) Every non-zero vector bundle has a nowhere-zero section.

(c) Every projective space admits a volume form.

(d) If a manifold M has Euler characteristic zero, then M is orientable.

8J Algebraic Topology

Show that the fundamental group G of the Klein bottle is infinite. Show that G contains an abelian subgroup of finite index. Show that G is not abelian.

9J Number Fields

Let $K = \mathbb{Q}(\sqrt{35})$. By Dedekind's theorem, or otherwise, show that the ideal equations

$$2 = [2,\omega]^2, \quad 5 = [5,\omega]^2, \quad [\omega] = [2,\omega][5,\omega]$$

hold in K, where $\omega = 5 + \sqrt{35}$. Deduce that K has class number 2.

Verify that $1+\omega$ is the fundamental unit in K. Hence show that the complete solution in integers x, y of the equation $x^2 - 35y^2 = -10$ is given by

$$x + \sqrt{35}y = \pm\omega(1+\omega)^n \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Calculate the particular solution x, y for $n = 1$.
[*It can be assumed that the Minkowski constant for K is $\frac{1}{2}$.*]

10K Algebraic Curves

For $N \geq 1$, let V_N be the (irreducible) projective plane curve $V_N : X^N + Y^N + Z^N = 0$ over an algebraically closed field of characteristic zero.

Show that V_N is smooth (non-singular). For $m, n \geq 1$, let $\alpha_{m,n} : V_{mn} \to V_m$ be the morphism $\alpha_{m,n}(X : Y : Z) = (X^n : Y^n : Z^n)$. Determine the degree of $\alpha_{m,n}$, its points of ramification and the corresponding ramification indices.

Applying the Riemann–Hurwitz formula to $\alpha_{1,n}$, determine the genus of V_n.

11J Logic, Computation and Set Theory

Explain what is meant by a *structure* for a first-order language and by a *model* for a first-order theory. If T is a first-order theory whose axioms are all universal sentences (that is, sentences of the form $(\forall x_1 \ldots x_n)p$ where p is quantifier-free), show that every substructure of a T-model is a T-model.

Now let T be an arbitrary first-order theory in a language L, and let M be an L-structure satisfying all the universal sentences which are derivable from the axioms of T. If p is a quantifier-free formula (with free variables $x_1, \ldots, x_n$ say) whose interpretation $[p]_M$ is a nonempty subset of M^n, show that $T \cup \{(\exists x_1 \cdots x_n)p\}$ is consistent.

Let L' be the language obtained from L by adjoining a new constant $\widehat{a}$ for each element a of M, and let

$$T' = T \cup \{p[\widehat{a}_1, \ldots, \widehat{a}_n / x_1, \ldots, x_n] \,|\, p \text{ is quantifier-free and } (a_1, \ldots, a_n) \in [p]_M\}.$$

Show that T' has a model. [*You may use the Completeness and Compactness Theorems.*] Explain briefly why any such model contains a substructure isomorphic to M.

12L Probability and Measure

Let (X_n) be a sequence of non-negative random variables on a common probability space with $\mathbb{E}X_n \leqslant 1$, such that $X_n \to 0$ almost surely. Determine which of the following statements are necessarily true, justifying your answers carefully:

(a) $\mathbb{P}(X_n \geqslant 1) \to 0$ as $n \to \infty$;

(b) $\mathbb{E}X_n \to 0$ as $n \to \infty$;

(c) $\mathbb{E}(\sin(X_n)) \to 0$ as $n \to \infty$;

(d) $\mathbb{E}(\sqrt{X_n}) \to 0$ as $n \to \infty$.

[*Standard limit theorems for integrals, and results about uniform integrability, may be used without proof provided that they are clearly stated.*]

13L Applied Probability

Two enthusiastic probability students, Ros and Guil, sit an examination which starts at time 0 and ends at time T; they both decide to use the time to attempt a proof of a difficult theorem which carries a lot of extra marks.

Ros' strategy is to write the proof continuously at a constant speed λ lines per unit time. In a time interval of length δt he has a probability $\mu\delta t + o(\delta t)$ of realising he has made a mistake. If that happens he instantly panics, erases everything he has written and starts all over again.

Guil, on the other hand, keeps cool and thinks carefully about what he is doing. In a time interval of length δt, he has a probability $\lambda\delta t + o(\delta t)$ of writing the next line of proof and for each line he has written a probability $\mu\delta t + o(\delta t)$ of finding a mistake in that line, independently of all other lines he has written. When a mistake is found, he erases that line and carries on as usual, hoping for the best.

Both Ros and Guil realise that, even if they manage to finish the proof, they will not recognise that they have done so and will carry on writing as much as they can.

(a) Calculate $p_l(t)$, the probability that, for Ros, the length of his completed proof at time $t \geqslant l/\lambda$ is at least l.

(b) Let $q_n(t)$ be the probability that Guil has n lines of proof at time $t > 0$. Show that

$$\frac{\partial Q}{\partial t} = (s-1)(\lambda Q - \mu \frac{\partial Q}{\partial s}),$$

where $Q(s,t) = \sum_{n=0}^{\infty} s^n q_n(t)$.

(c) Suppose now that every time Ros starts all over again, the time until the next mistake has distribution F, independently of the past history. Write down a renewal-type integral equation satisfied by $l(t)$, the expected length of Ros' proof at time t. What is the expected length of proof produced by him at the end of the examination if F is the exponential distribution with mean $1/\mu$?

(d) What is the expected length of proof produced by Guil at the end of the examination if each line that he writes survives for a length of time with distribution F, independently of all other lines?

14M Information Theory

Define the binary Hamming code of length $n = 2^\ell - 1$ and its dual. Prove that the Hamming code is perfect. Prove that in the dual code:

(i) The weight of any non-zero codeword equals $2^{\ell-1}$;

(ii) The distance between any pair of words equals $2^{\ell-1}$.

[*You may quote results from the course provided that they are carefully stated.*]

15L Optimization and Control

State Pontryagin's maximum principle (PMP) for the problem of minimizing

$$\int_0^T c(x(t), u(t))\, dt + K(x(T))\,,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $dx/dt = a(x(t), u(t))$; here, $x(0)$ and T are given, and $x(T)$ is unconstrained.

Consider the two-dimensional problem in which $dx_1/dt = x_2$, $dx_2/dt = u$, $c(x,u) = \frac{1}{2}u^2$ and $K(x(T)) = \frac{1}{2}qx_1(T)^2$, $q > 0$. Show that, by use of a variable $z(t) = x_1(t) + x_2(t)(T-t)$, one can rewrite this problem as an equivalent one-dimensional problem.

Use PMP to solve this one-dimensional problem, showing that the optimal control can be expressed as $u(t) = -qz(T)(T-t)$, where $z(T) = z(0)/(1 + \frac{1}{3}qT^3)$.

Express $u(t)$ in a feedback form of $u(t) = k(t)z(t)$ for some $k(t)$.

Suppose that the initial state $x(0)$ is perturbed by a small amount to $x(0) + (\epsilon_1, \epsilon_2)$. Give an expression (in terms of ϵ_1, ϵ_2, $x(0)$, q and T) for the increase in minimal cost.

16M Principles of Statistics

(i) Let X be a random variable with density function $f(x;\theta)$. Consider testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative hypothesis $H_1 : \theta = \theta_1$.

What is the form of the optimal size α classical hypothesis test?

Compare the form of the test with the Bayesian test based on the Bayes factor, *and* with the Bayes decision rule under the 0-1 loss function, under which a loss of 1 is incurred for an incorrect decision and a loss of 0 is incurred for a correct decision.

(ii) What does it mean to say that a family of densities $\{f(x;\theta), \theta \in \Theta\}$ with real scalar parameter θ is of *monotone likelihood ratio*?

Suppose X has a distribution from a family which is of monotone likelihood ratio with respect to a statistic $t(X)$ and that it is required to test $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$.

State, without proof, a theorem which establishes the existence of a uniformly most powerful test and describe in detail the form of the test.

Let $X_1, \ldots, X_n$ be independent, identically distributed $U(0,\theta)$, $\theta > 0$. Find a uniformly most powerful size α test of $H_0 : \theta \leqslant \theta_0$ against $H_1 : \theta > \theta_0$, and find its power function. Show that we may construct a different, randomised, size α test with the same power function for $\theta \geqslant \theta_0$.

17G Partial Differential Equations

(a) Define the convolution $f * g$ of two functions. Write down a formula for a solution $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ to the initial value problem

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

together with the boundary condition

$$u(0, x) = f(x)$$

for f a bounded continuous function on $\mathbb{R}^n$. Comment briefly on the uniqueness of the solution.

(b) State and prove the Duhamel principle giving the solution (for $t > 0$) to the equation

$$\frac{\partial u}{\partial t} - \Delta u = g$$

together with the boundary condition

$$u(0, x) = f(x)$$

in terms of your answer to (a).

(c) Show that if $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is the solution to

$$\frac{\partial v}{\partial t} - \Delta v = G$$

together with the boundary condition

$$v(0, x) = f(x)$$

with $G(t, x) \leq g(t, x)$ for all (t, x) then $v(t, x) \leq u(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$.

Finally show that if in addition there exists a point (t_0, x_0) at which there is strict inequality in the assumption i.e.

$$G(t_0, x_0) < g(t_0, x_0),$$

then in fact

$$v(t, x) < u(t, x)$$

whenever $t > t_0$.

18G Methods of Mathematical Physics

Show that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{t^{z-1}}{t-a}\,dt = \pi i a^{z-1}\ ,$$

where a is real and positive, $0 < \mathrm{Re}\,(z) < 1$ and $\mathcal{P}$ denotes the Cauchy principal value; the principal branches of t^z etc. are implied. Deduce that

$$\int_0^{\infty} \frac{t^{z-1}}{t+a}\,dt = \pi a^{z-1}\mathrm{cosec}\,\pi z \qquad (*)$$

and that

$$\mathcal{P}\int_0^{\infty} \frac{t^{z-1}}{t-a}\,dt = -\pi a^{z-1}\cot \pi z\ .$$

Use $(*)$ to show that, if $\mathrm{Im}\,(b) > 0$, then

$$\int_0^{\infty} \frac{t^{z-1}}{t-b}\,dt = -\pi b^{z-1}(\cot \pi z - i)\ .$$

What is the value of this integral if $\mathrm{Im}\,(b) < 0$?

19F Numerical Analysis

(i)

Given the finite-difference method

$$\sum_{k=-r}^{s} \alpha_k u_{m+k}^{n+1} = \sum_{k=-r}^{s} \beta_k u_{m+k}^{n}, \quad m,n \in \mathbb{Z},\ n \geqslant 0,$$

define

$$H(z) = \frac{\sum\limits_{k=-r}^{s} \beta_k z^k}{\sum\limits_{k=-r}^{s} \alpha_k z^k}.$$

Prove that this method is stable if and only if

$$|H(e^{i\theta})| \leqslant 1, \quad -\pi \leqslant \theta \leqslant \pi.$$

[*You may quote without proof known properties of the Fourier transform.*]

(ii) Find the range of the parameter μ such that the method

$$(1-2\mu)u_{m-1}^{n+1} + 4\mu u_m^{n+1} + (1-2\mu)u_{m+1}^{n+1} = u_{m-1}^n + u_{m+1}^n$$

is stable. Supposing that this method is used to solve the diffusion equation for $u(x,t)$, determine the order of magnitude of the local error as a power of Δx.

20E Electrodynamics

A particle of rest mass m and charge q moves in an electromagnetic field given by a potential A_a along a trajectory $x^a(\tau)$, where τ is the proper time along the particle's worldline. The action for such a particle is

$$I = \int \left(m\sqrt{-\eta_{ab}\dot{x}^a\dot{x}^b} - qA_a\dot{x}^a \right) d\tau \,.$$

Show that the Euler-Lagrange equations resulting from this action reproduce the relativistic equation of motion for the particle.

Suppose that the particle is moving in the electrostatic field of a fixed point charge Q with radial electric field E_r given by

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} \,.$$

Show that one can choose a gauge such that $A_i = 0$ and only $A_0 \neq 0$. Find A_0.

Assume that the particle executes planar motion, which in spherical polar coordinates $(r\,,\theta\,,\phi)$ can be taken to be in the plane $\theta = \pi/2$. Derive the equations of motion for t and ϕ.

By using the fact that $\eta_{ab}\dot{x}^a\dot{x}^b = -1$, find the equation of motion for r, and hence show that the shape of the orbit is described by

$$\frac{dr}{d\phi} = \pm\frac{r^2}{\ell}\sqrt{\left(E + \frac{\gamma}{r}\right)^2 - 1 - \frac{\ell^2}{r^2}} \,,$$

where E (> 1) and ℓ are constants of integration and γ is to be determined.

By putting $u = 1/r$ or otherwise, show that if $\gamma^2 < \ell^2$ then the orbits are bounded and generally not closed, and show that the angle between successive minimal values of r is $2\pi(1 - \gamma^2/\ell^2)^{-1/2}$.

21E Foundations of Quantum Mechanics

(i) A Hamiltonian H_0 has energy eigenvalues E_r and corresponding non-degenerate eigenstates $|r\rangle$. Show that under a small change in the Hamiltonian δH,

$$\delta|r\rangle = \sum_{s \neq r} \frac{\langle s|\delta H|r\rangle}{E_r - E_s}|s\rangle,$$

and derive the related formula for the change in the energy eigenvalue E_r to first and second order in δH.

(ii) The Hamiltonian for a particle moving in one dimension is $H = H_0 + \lambda H'$, where $H_0 = p^2/2m + V(x)$, $H' = p/m$ and λ is small. Show that

$$\frac{i}{\hbar}[H_0, x] = H'$$

and hence that

$$\delta E_r = -\lambda^2 \frac{i}{\hbar}\langle r|H'x|r\rangle = \lambda^2 \frac{i}{\hbar}\langle r|xH'|r\rangle$$

to second order in λ.

Deduce that δE_r is independent of the particular state $|r\rangle$ and explain why this change in energy is exact to all orders in λ.

22E Applications of Quantum Mechanics

Define the reciprocal lattice for a lattice L with lattice vectors $\boldsymbol{\ell}$.

A beam of electrons, with wave vector $\mathbf{k}$, is incident on a Bravais lattice L with a large number of atoms, N. If the scattering amplitude for scattering on an individual atom in the direction $\hat{\mathbf{k}}'$ is $f(\hat{\mathbf{k}}')$, show that the scattering amplitude for the whole lattice is

$$\sum_{\boldsymbol{\ell} \in L} e^{i\mathbf{q}\cdot\boldsymbol{\ell}} f(\hat{\mathbf{k}}')\,, \qquad \mathbf{q} = \mathbf{k} - |\mathbf{k}|\,\hat{\mathbf{k}}'\,.$$

Derive the formula for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = N|f(\hat{\mathbf{k}}')|^2 \Delta(\mathbf{q})\,,$$

obtaining an explicit form for $\Delta(\mathbf{q})$. Show that $\Delta(\mathbf{q})$ is strongly peaked when $\mathbf{q} = \mathbf{g}$, a reciprocal lattice vector. Show that this leads to the Bragg formula $2d \sin\frac{\theta}{2} = \lambda$, where θ is the scattering angle, λ the electron wavelength and d the separation between planes of atoms in the lattice.

23D General Relativity

(i) Consider the line element describing the interior of a star,

$$ds^2 = e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{2\gamma(r)}dt^2 \ ,$$

defined for $0 \leq r \leq r_0$ by

$$e^{-2\alpha(r)} = 1 - Ar^2$$

and

$$e^{\gamma(r)} = \frac{3}{2}e^{-\alpha_0} - \frac{1}{2}e^{-\alpha(r)}.$$

Here $A = 2M/r_0^3$, M is the mass of the star, and α_0 is defined to be $\alpha(r_0)$.

The star is made of a perfect fluid with energy-momentum tensor

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab}.$$

Here u^a is the 4-velocity of the fluid which is at rest, the density ρ is constant throughout the star ($0 \leq r \leq r_0$) and the pressure $p = p(r)$ depends only on the radial coordinate. Write down the Einstein field equations and show that (in geometrical units with $G = c = 1$) they may equivalently be written as

$$R_{ab} = 8\pi(p+\rho)u_a u_b + 4\pi(p-\rho)g_{ab}.$$

(ii) Using the formulae below, or otherwise, show that for $0 \leq r \leq r_0$ one has

$$\rho = \frac{3A}{8\pi}, \quad p(r) = \frac{3A}{8\pi}\left(\frac{e^{-\alpha(r)} - e^{-\alpha_0}}{3e^{-\alpha_0} - e^{-\alpha(r)}}\right).$$

[*The non-zero components of the Ricci tensor are:*

$$R_{11} = -\gamma'' + \alpha'\gamma' - \gamma'^2 + \frac{2\alpha'}{r}, \quad R_{22} = e^{-2\alpha}[(\alpha'-\gamma')r - 1] + 1,$$

$$R_{33} = \sin^2\theta R_{22}, \quad R_{44} = e^{2\gamma-2\alpha}[\gamma'' - \alpha'\gamma' + \gamma'^2 + \frac{2\gamma'}{r}].$$

Note that

$$\alpha' = Are^{2\alpha}, \quad \gamma' = \frac{1}{2}Are^{\alpha-\gamma}, \quad \gamma'' = \frac{1}{2}Ae^{\alpha-\gamma} + \frac{1}{2}A^2r^2e^{3\alpha-\gamma} - \frac{1}{4}A^2r^2e^{2\alpha-2\gamma}. \quad]$$

24C Fluid Dynamics II

A thin layer of liquid of kinematic viscosity ν flows under the influence of gravity down a plane inclined at an angle α to the horizontal ($0 \le \alpha \le \pi/2$). With origin O on the plane, and axes Ox down the line of steepest slope and Oy normal to the plane, the free surface is given by $y = h(x,t)$, where $|\partial h/\partial x| \ll 1$. The pressure distribution in the liquid may be assumed to be hydrostatic. Using the approximations of lubrication theory, show that

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left\{h^3(\cos\alpha\frac{\partial h}{\partial x} - \sin\alpha)\right\}.$$

Now suppose that

$$h = h_0 + \eta(x,t) \ ,$$

where

$$\eta(x,0) = \eta_0 e^{-x^2/a^2}$$

and h_0, η_0 and a are constants with $\eta_0 \ll a, h_0$. Show that, to leading order,

$$\eta(x,t) = \frac{a\eta_0}{(a^2+4Dt)^{1/2}} \exp\left\{-\frac{(x-Ut)^2}{a^2+4Dt}\right\} \ ,$$

where U and D are constants to be determined.

Explain in physical terms the meaning of this solution.

25C Waves in Fluid and Solid Media

Starting from the equations for one-dimensional unsteady flow of a perfect gas of uniform entropy, show that the Riemann invariants,

$$R_\pm = u \pm \frac{2}{\gamma - 1}(c - c_0) \ ,$$

are constant on characteristics $C_\pm$ given by $\dfrac{dx}{dt} = u \pm c$, where $u(x,t)$ is the velocity of the gas, $c(x,t)$ is the local speed of sound and γ is the specific heat ratio.

Such a gas initially occupies the region $x > 0$ to the right of a piston in an infinitely long tube. The gas and the piston are initially at rest. At time $t = 0$ the piston starts moving to the left at a constant speed V. Find $u(x,t)$ and $c(x,t)$ in the three regions

(i) $c_0 t \le x$,

(ii) $at \le x < c_0 t$,

(iii) $-Vt \le x < at$,

where $a = c_0 - \frac{1}{2}(\gamma+1)V$. What is the largest value of V for which c is positive throughout region (iii)? What happens if V exceeds this value?

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Wednesday 5 June 2002 9 to 12

PAPER 3

Before you begin read these instructions carefully.

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***9K, 10K*** *should be in one bundle and* ***1M, 15M*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1M Markov Chains

(i) Consider the continuous-time Markov chain $(X_t)_{t\geqslant 0}$ on $\{1,2,3,4,5,6,7\}$ with generator matrix

$$Q = \begin{pmatrix} -6 & 2 & 0 & 0 & 0 & 4 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -6 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

Compute the probability, starting from state 3, that X_t hits state 2 eventually.

Deduce that

$$\lim_{t\to\infty} \mathbb{P}(X_t = 2 | X_0 = 3) = \frac{4}{15}.$$

[*Justification of standard arguments is not expected.*]

(ii) A colony of cells contains immature and mature cells. Each immature cell, after an exponential time of parameter 2, becomes a mature cell. Each mature cell, after an exponential time of parameter 3, divides into two immature cells. Suppose we begin with one immature cell and let $n(t)$ denote the expected number of immature cells at time t. Show that

$$n(t) = (4e^t + 3e^{-6t})/7.$$

2K Functional Analysis

(i) Suppose that (f_n) is a decreasing sequence of continuous real-valued functions on a compact metric space (X,d) which converges pointwise to 0. By considering sets of the form $B_n = \{x : f_n(x) < \epsilon\}$, for $\epsilon > 0$, or otherwise, show that f_n converges uniformly to 0.

Can the condition that (f_n) is decreasing be dropped? Can the condition that (X,d) is compact be dropped? Justify your answers.

(ii) Suppose that k is a positive integer. Define polynomials p_n recursively by

$$p_0 = 0, \qquad p_{n+1}(t) = p_n(t) + (t - p_n^k(t))/k.$$

Show that $0 \leqslant p_n(t) \leqslant p_{n+1}(t) \leqslant t^{1/k}$, for $t \in [0,1]$, and show that $p_n(t)$ converges to $t^{1/k}$ uniformly on $[0,1]$.

[*You may wish to use the identity* $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \ldots + b^{k-1})$.]

Suppose that A is a closed subalgebra of the algebra $C(X)$ of continuous real-valued functions on a compact metric space (X,d), equipped with the uniform norm, and suppose that A has the property that for each $x \in X$ there exists $a \in A$ with $a(x) \neq 0$. Show that there exists $h \in A$ such that $0 < h(x) \leqslant 1$ for all $x \in X$.

Show that $h^{1/k} \in A$ for each positive integer k, and show that A contains the constant functions.

3D Electromagnetism

(i) A plane electromagnetic wave in a vacuum has an electric field

$$\mathbf{E} = (E_1, E_2, 0)\cos(kz - \omega t),$$

referred to cartesian axes (x, y, z). Show that this wave is plane polarized and find the orientation of the plane of polarization. Obtain the corresponding plane polarized magnetic field and calculate the rate at which energy is transported by the wave.

(ii) Suppose instead that

$$\mathbf{E} = (E_1\cos(kz - \omega t),\ E_2\cos(kz - \omega t + \phi), 0),$$

with ϕ a constant, $0 < \phi < \pi$. Show that, if the axes are now rotated through an angle ψ so as to obtain an elliptically polarized wave with an electric field

$$\mathbf{E}' = (F_1\cos(kz - \omega t + \chi),\ F_2\sin(kz - \omega t + \chi), 0),$$

then

$$\tan 2\psi = \frac{2E_1E_2\cos\phi}{E_1^2 - E_2^2}.$$

Show also that if $E_1 = E_2 = E$ there is an elliptically polarized wave with

$$\mathbf{E}' = \sqrt{2}E\left(\cos(kz - \omega t + \tfrac{1}{2}\phi)\cos\tfrac{1}{2}\phi,\ \sin(kz - \omega t + \tfrac{1}{2}\phi)\sin\tfrac{1}{2}\phi, 0\right).$$

4F Dynamics of Differential Equations

(i) Define the Floquet multiplier and Liapunov exponent for a periodic orbit $\hat{\mathbf{x}}(t)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$. Show that one multiplier is always unity, and that the other is given by

$$\exp\left(\int_0^T \nabla\cdot \mathbf{f}(\hat{\mathbf{x}}(t))dt\right) \ , \qquad (*)$$

where T is the period of the orbit.

The Van der Pol oscillator $\ddot{x} + \epsilon\dot{x}(x^2 - 1) + x = 0\,,\ 0 < \epsilon \ll 1$ has a limit cycle $\hat{x}(t) \approx 2\sin t$. Show using $(*)$ that this orbit is stable.

(ii) Show, by considering the normal form for a Hopf bifurcation from a fixed point $\mathbf{x}_0(\mu)$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$, that in some neighbourhood of the bifurcation the periodic orbit is stable when it exists in the range of μ for which $\mathbf{x}_0$ is unstable, and unstable in the opposite case.

Now consider the system

$$\left.\begin{aligned} \dot{x} &= x(1-y) + \mu x \\ \dot{y} &= y(x-1) - \mu x \end{aligned}\right\} \qquad x > 0\ .$$

Show that the fixed point $(1+\mu\,,\ 1+\mu)$ has a Hopf bifurcation when $\mu = 0$, and is unstable (stable) when $\mu > 0$ $(\mu < 0)$.

Suppose that a periodic orbit exists in $\mu > 0$. Show without solving for the orbit that the result of part (i) shows that such an orbit is unstable. Define a similar result for $\mu < 0$.

What do you conclude about the existence of periodic orbits when $\mu \neq 0$? Check your answer by applying Dulac's criterion to the system, using the weighting $\rho = e^{-(x+y)}$.

5J Representation Theory

Let G be a finite group acting on a finite set X. Define the permutation representation $(\rho, \mathbb{C}[X])$ of G and compute its character π_X. Prove that $\langle \pi_X, 1_G\rangle_G$ equals the number of orbits of G on X. If G acts also on the finite set Y, with character π_Y, show that $\langle \pi_X, \pi_Y\rangle_G$ equals the number of orbits of G on $X \times Y$.

Now let G be the symmetric group S_n acting naturally on the set $X = \{1, \ldots, n\}$, and let X_r be the set of all r-element subsets of X. Let π_r be the permutation character of G on X_r. Prove that

$$\langle \pi_k, \pi_\ell\rangle_G = \ell + 1 \text{ for } 0 \leqslant \ell \leqslant k \leqslant n/2.$$

Deduce that the class functions

$$\chi_r = \pi_r - \pi_{r-1}$$

are irreducible characters of S_n, for $1 \leqslant r \leqslant n/2$.

6J Galois Theory

Show that the polynomial $f(X) = X^5 + 27X + 16$ has no rational roots. Show that the splitting field of f over the finite field $\mathbb{F}_3$ is an extension of degree 4. Hence deduce that f is irreducible over the rationals. Prove that f has precisely two (non-multiple) roots over the finite field $\mathbb{F}_7$. Find the Galois group of f over the rationals.

[*You may assume any general results you need including the fact that A_5 is the only index 2 subgroup of S_5.*]

7J Algebraic Topology

For a finite simplicial complex X, let $b_i(X)$ denote the rank of the finitely generated abelian group H_iX. Define the Euler characteristic $\chi(X)$ by the formula

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Let a_i denote the number of i-simplices in X, for each $i \geqslant 0$. Show that

$$\chi(X) = \sum_i (-1)^i a_i.$$

8K Hilbert Spaces

Let H be an infinite-dimensional, separable Hilbert space. Let T be a compact linear operator on H, and let λ be a non-zero, approximate eigenvalue of T. Prove that λ is an eigenvalue, and that the corresponding eigenspace $E_\lambda(T) = \{x \in H : Tx = \lambda x\}$ is finite-dimensional.

Let S be a compact, self-adjoint operator on H. Prove that there is an orthonormal basis $(e_n)_{n\geqslant 0}$ of H, and a sequence $(\lambda_n)_{n\geqslant 0}$ in $\mathbb{C}$, such that (i) $Se_n = \lambda_n e_n$ $(n \geqslant 0)$ and (ii) $\lambda_n \to 0$ as $n \to \infty$.

Now let S be compact, self-adjoint and *injective*. Let R be a bounded self-adjoint operator on H such that $RS = SR$. Prove that H has an orthonormal basis $(e_n)_{n\geqslant 1}$, where, for every n, e_n is an eigenvector, both of S and of R.

[*You may assume, without proof, results about self-adjoint operators on finite-dimensional spaces.*]

9K Riemann Surfaces

Let α_1, α_2 be two non-zero complex numbers with $\alpha_1/\alpha_2 \notin \mathbb{R}$. Let L be the lattice $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \subset \mathbb{C}$. A meromorphic function f on $\mathbb{C}$ is *elliptic* if $f(z+\lambda) = f(z)$, for all $z \in \mathbb{C}$ and $\lambda \in L$. The *Weierstrass functions* $\wp(z), \zeta(z), \sigma(z)$ are defined by the following properties:

- $\wp(z)$ is elliptic, has double poles at the points of L and no other poles, and $\wp(z) = 1/z^2 + O(z^2)$ near 0;
- $\zeta'(z) = -\wp(z)$, and $\zeta(z) = 1/z + O(z^3)$ near 0;
- $\sigma(z)$ is odd, and $\sigma'(z)/\sigma(z) = \zeta(z)$, and $\sigma(z)/z \to 1$ as $z \to 0$.

Prove the following.
(a) $\wp$, and hence ζ and σ, are uniquely determined by these properties. You are *not* expected to prove the existence of $\wp, \zeta, \sigma$, and you may use Liouville's theorem without proof.
(b) $\zeta(z+\alpha_i) = \zeta(z) + 2\eta_i$, and $\sigma(z+\alpha_i) = k_i e^{2\eta_i z}\sigma(z)$, for some constants η_i, k_i $(i = 1, 2)$.
(c) σ is holomorphic, has simple zeroes at the points of L, and has no other zeroes.
(d) Given $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{C}$ with $a_1 + \ldots + a_n = b_1 + \ldots + b_n$, the function

$$\frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}$$

is elliptic.
(e) $\wp(u) - \wp(v) = -\dfrac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}$.
(f) Deduce from (e), or otherwise, that $\dfrac{1}{2}\dfrac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} = \zeta(u+v) - \zeta(u) - \zeta(v)$.

10K Algebraic Curves

Let $f = f(x, y)$ be an irreducible polynomial of degree $n \geq 2$ (over an algebraically closed field of characteristic zero) and $V_0 = \{f = 0\} \subset \mathbb{A}^2$ the corresponding affine plane curve. Assume that V_0 is smooth (non-singular) and that the projectivization $V \subset \mathbb{P}^2$ of V_0 intersects the line at infinity $\mathbb{P}^2 - \mathbb{A}^2$ in n distinct points. Show that V is smooth and determine the divisor of the rational differential $\omega = \dfrac{dx}{f'_y}$ on V. Deduce a formula for the genus of V.

11J Logic, Computation and Set Theory

(i) Explain briefly what is meant by the terms *register machine* and *computable function*.

Let u be the universal computable function $u(m,n) = f_m(n)$ and s a total computable function with $f_{s(m,n)}(k) = f_m(n,k)$. Here $f_m(n)$ and $f_m(n,k)$ are the unary and binary functions computed by the m-th register machine program P_m. Suppose $h : \mathbb{N} \to \mathbb{N}$ is a total computable function. By considering the function

$$g(m,n) = u(h(s(m,m)),n)$$

show that there is a number a such that $f_a = f_{h(a)}$.

(ii) Let P be the set of all partial functions $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Consider the mapping $\Phi : P \to P$ defined by

$$\Phi(g)(m,n) = \begin{cases} n+1 & \text{if } m = 0, \\ g(m-1,1) & \text{if } m > 0,\ n = 0 \text{ and } g(m-1,1) \text{ defined}, \\ g(m-1, g(m,n-1)) & \text{if } mn > 0 \text{ and } g(m-1,g(m,n-1)) \text{ defined}, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(a) Show that any fixed point of Φ is a total function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Deduce that Φ has a unique fixed point.
[*The Bourbaki-Witt Theorem may be assumed if stated precisely.*]

(b) It follows from standard closure properties of the computable functions that there is a computable function ψ such that

$$\psi(p,m,n) = \Phi(f_p)(m,n).$$

Assuming this, show that there is a total computable function h such that

$$\Phi(f_p) = f_{h(p)} \text{ for all } p.$$

Deduce that the fixed point of Φ is computable.

12L Probability and Measure

Derive the characteristic function of a real-valued random variable which is normally distributed with mean μ and variance σ^2. What does it mean to say that an $\mathbb{R}^n$-valued random variable has a *multivariate Gaussian distribution*? Prove that the distribution of such a random variable is determined by its ($\mathbb{R}^n$-valued) mean and its covariance matrix.

Let X and Y be random variables defined on the same probability space such that (X,Y) has a Gaussian distribution. Show that X and Y are independent if and only if $\operatorname{cov}(X,Y) = 0$. Show that, even if they are not independent, one may always write $X = aY + Z$ for some constant a and some random variable Z independent of Y.

[*The inversion theorem for characteristic functions and standard results about independence may be assumed.*]

13L Applied Probability

(a) Define a renewal process and a discrete renewal process.

(b) State and prove the Discrete Renewal Theorem.

(c) The sequence $\mathbf{u} = \{u_n : n \geqslant 0\}$ satisfies

$$u_0 = 1, \quad u_n = \sum_{i=1}^{n} f_i u_{n-i}, \qquad \text{for } n \geqslant 1$$

for some collection of non-negative numbers $(f_i : i \in \mathbb{N})$ summing to 1. Let $U(s) = \sum_{n=1}^{\infty} u_n s^n$, $F(s) = \sum_{n=1}^{\infty} f_n s^n$. Show that

$$F(s) = \frac{U(s)}{1 + U(s)}.$$

Give a probabilistic interpretation of the numbers u_n, f_n and $m_n = \sum_{i=1}^{n} u_i$.

(d) Let the sequence u_n be given by

$$u_{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}, \quad u_{2n+1} = 0, \quad n \geqslant 1.$$

How is this related to the simple symmetric random walk on the integers $\mathbb{Z}$ starting from the origin, and its subsequent returns to the origin? Determine $F(s)$ in this case, either by calculating $U(s)$ or by showing that F satisfies the quadratic equation

$$F^2 - 2F + s^2 = 0, \qquad \text{for} \quad 0 \leqslant s < 1.$$

14L Optimization and Control

Consider a scalar system with $x_{t+1} = (x_t + u_t)\xi_t$, where $\xi_0, \xi_1, \ldots$ is a sequence of independent random variables, uniform on the interval $[-a, a]$, with $a \leqslant 1$. We wish to choose $u_0, \ldots, u_{h-1}$ to minimize the expected value of

$$\sum_{t=0}^{h-1}(c + x_t^2 + u_t^2) + 3x_h^2\,,$$

where u_t is chosen knowing x_t but not ξ_t. Prove that the minimal expected cost can be written $V_h(x_0) = hc + x_0^2\Pi_h$ and derive a recurrence for calculating $\Pi_1, \ldots, \Pi_h$.

How does your answer change if u_t is constrained to lie in the set $\mathcal{U}(x_t) = \{u : |u + x_t| < |x_t|\}$?

Consider a stopping problem for which there are two options in state x_t, $t \geqslant 0$:

(1) stop: paying a terminal cost $3x_t^2$; no further costs are incurred;

(2) continue: choosing $u_t \in \mathcal{U}(x_t)$, paying $c + u_t^2 + x_t^2$, and moving to state $x_{t+1} = (x_t + u_t)\xi_t$.

Consider the problem of minimizing total expected cost subject to the constraint that no more than h continuation steps are allowed. Suppose $a = 1$. Show that an optimal policy stops if and only if either h continuation steps have already been taken or $x^2 \leqslant 2c/3$.

[*Hint: Use induction on h to show that a one-step-look-ahead rule is optimal. You should not need to find the optimal u_t for the continuation steps.*]

15M Principles of Statistics

(i) Describe in detail how to perform the Wald, score and likelihood ratio tests of a *simple* null hypothesis $H_0 : \theta = \theta_0$ given a random sample $X_1, \ldots, X_n$ from a regular one-parameter density $f(x; \theta)$. In each case you should specify the asymptotic null distribution of the test statistic.

(ii) Let $X_1, \ldots, X_n$ be an independent, identically distributed sample from a distribution F, and let $\hat{\theta}(X_1, \ldots, X_n)$ be an estimator of a parameter θ of F.

Explain what is meant by: (a) the *empirical distribution function* of the sample; (b) the *bootstrap estimator* of the *bias* of $\hat{\theta}$, based on the empirical distribution function. Explain how a bootstrap estimator of the *distribution function* of $\hat{\theta} - \theta$ may be used to construct an approximate $1 - \alpha$ confidence interval for θ.

Suppose the parameter of interest is $\theta = \mu^2$, where μ is the mean of F, and the estimator is $\hat{\theta} = \bar{X}^2$, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ is the sample mean.

Derive an *explicit* expression for the bootstrap estimator of the bias of $\hat{\theta}$ and show that it is biased as an estimator of the true bias of $\hat{\theta}$.

Let $\hat{\theta}_i$ be the value of the estimator $\hat{\theta}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ computed from the sample of size $n-1$ obtained by deleting X_i and let $\hat{\theta}_{\cdot} = n^{-1} \sum_{i=1}^{n} \hat{\theta}_i$. The *jackknife* estimator of the bias of $\hat{\theta}$ is

$$b_J = (n-1)\left(\hat{\theta}_{\cdot} - \hat{\theta}\right) .$$

Derive the jackknife estimator b_J for the case $\hat{\theta} = \bar{X}^2$, and show that, as an estimator of the true bias of $\hat{\theta}$, it is unbiased.

16L Stochastic Financial Models

(i) Explain briefly what it means to say that a stochastic process $\{W_t,\ t \geqslant 0\}$ is a standard Brownian motion.

Let $\{W_t,\ t \geqslant 0\}$ be a standard Brownian motion and let a, b be real numbers. What condition must a and b satisfy to ensure that the process e^{aW_t+bt} is a martingale? Justify your answer carefully.

(ii) At the beginning of each of the years $r = 0, 1, \ldots, n-1$ an investor has income X_r, of which he invests a proportion ρ_r, $0 \leqslant \rho_r \leqslant 1$, and consumes the rest during the year. His income at the beginning of the next year is

$$X_{r+1} = X_r + \rho_r X_r W_r,$$

where $W_0, \ldots, W_{n-1}$ are independent positive random variables with finite means and $X_0 \geqslant 0$ is a constant. He decides on ρ_r after he has observed both X_r and W_r at the beginning of year r, but at that time he does not have any knowledge of the value of W_s, for any $s > r$. The investor retires in year n and consumes his entire income during that year. He wishes to determine the investment policy that maximizes his expected total consumption

$$\mathbb{E}\left[\sum_{r=0}^{n-1}(1-\rho_r)X_r + X_n\right].$$

Prove that the optimal policy may be expressed in terms of the numbers $b_0, b_1, \ldots,$ b_n where $b_n = 1$, $b_r = b_{r+1} + \mathbb{E}\max(b_{r+1}W_r, 1)$, for $r < n$, and determine the optimal expected total consumption.

17F Dynamical Systems

Let $\mathcal{A}$ be a finite alphabet of letters and Σ either the semi-infinite space or the doubly infinite space of sequences whose elements are drawn from $\mathcal{A}$. Define the natural topology on Σ. If W is a set of words, denote by Σ_W the subspace of Σ consisting of those sequences none of whose subsequences is in W. Prove that Σ_W is a closed subspace of Σ; and state and prove a necessary and sufficient condition for a closed subspace of Σ to have the form Σ_W for some W.

If $\mathcal{A} = \{0, 1\}$ and

$$W = \{000,\ 111,\ 010,\ 101\}$$

what is the space Σ_W?

18G Partial Differential Equations

Define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. State the Fourier inversion theorem for the Fourier transform of a Schwartz function.

Consider the initial value problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u = 0 \ , \ x \in \mathbb{R}^n \ , \ 0 < t < \infty \ ,$$

$$u(0,x) = f(x) \ , \qquad \frac{\partial u}{\partial t}(0,x) = 0$$

for f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Show that the solution can be written as

$$u(t,x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(t,\xi) d\xi \ ,$$

where

$$\hat{u}(t,\xi) = \cos\left(t\sqrt{1+|\xi|^2}\right) \hat{f}(\xi)$$

and

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

State the Plancherel-Parseval theorem and hence deduce that

$$\int_{\mathbb{R}^n} |u(t,x)|^2 dx \leq \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

19G Methods of Mathematical Physics

Show that the equation

$$zw'' + w' + (\lambda - z)w = 0$$

has solutions of the form

$$w(z) = \int_\gamma (t-1)^{(\lambda-1)/2}(t+1)^{-(\lambda+1)/2} e^{zt} dt.$$

Give examples of possible choices of γ to provide two independent solutions, assuming $\mathrm{Re}(z) > 0$. Distinguish between the cases $\mathrm{Re}\,\lambda > -1$ and $\mathrm{Re}\,\lambda < 1$. Comment on the case $-1 < \mathrm{Re}\ \lambda < 1$ and on the case that λ is an odd integer.

Show that, if $\mathrm{Re}\,\lambda < 1$, there is a solution $w_1(z)$ that is bounded as $z \to +\infty$, and that, in this limit,

$$w_1(z) \sim A\, e^{-z} z^{(\lambda-1)/2} \left(1 - \frac{(1-\lambda)^2}{8z}\right) ,$$

where A is a constant.

20F Numerical Analysis

(i) Determine the order of the multistep method

$$\mathbf{y}_{n+2} - (1+\alpha)\mathbf{y}_{n+1} + \alpha\mathbf{y}_n = h[\tfrac{1}{12}(5+\alpha)\mathbf{f}_{n+2} + \tfrac{2}{3}(1-\alpha)\mathbf{f}_{n+1} - \tfrac{1}{12}(1+5\alpha)\mathbf{f}_n]$$

for the solution of ordinary differential equations for different choices of α in the range $-1 \leqslant \alpha \leqslant 1$.

(ii) Prove that no such choice of α results in a method whose linear stability domain includes the interval $(-\infty,\ 0)$.

21E Foundations of Quantum Mechanics

(i) Two particles with angular momenta j_1, j_2 and basis states $|j_1\ m_1\rangle, |j_2\ m_2\rangle$ are combined to give total angular momentum j and basis states $|j\ m\rangle$. State the possible values of j, m and show how a state with $j = m = j_1 + j_2$ can be constructed. Briefly describe, for a general allowed value of j, what the Clebsch-Gordan coefficients are.

(ii) If the angular momenta j_1 and j_2 are both 1 show that the combined state $|2\ 0\rangle$ is

$$|2\ 0\rangle = \sqrt{\frac{1}{6}}\Big(|1\ 1\rangle|1\ {-1}\rangle + |1\ {-1}\rangle|1\ 1\rangle\Big) + \sqrt{\frac{2}{3}}|1\ 0\rangle|1\ 0\rangle.$$

Determine the corresponding expressions for the combined states $|1\ 0\rangle$ and $|0\ 0\rangle$, assuming that they are respectively antisymmetric and symmetric under interchange of the two particles.

If the combined system is in state $|0\ 0\rangle$ what is the probability that measurements of the z-component of angular momentum for either constituent particle will give the value of 1?

[*Hint:* $J_\pm |j\ m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j\ m \pm 1\rangle$.]

22D Statistical Physics

A system consisting of non-interacting bosons has single-particle levels uniquely labelled by r with energies $\epsilon_r, \;\; \epsilon_r \geq 0$. Show that the free energy in the grand canonical ensemble is

$$F \;=\; kT \sum_r \; \log(1 - e^{-\beta(\epsilon_r - \mu)}) \; .$$

What is the maximum value for μ?

A system of N bosons in a large volume V has one energy level of energy zero and a large number $M \gg 1$ of energy levels of the same energy ϵ, where M takes the form $M = AV$ with A a positive constant. What are the dimensions of A?

Show that the free energy is

$$F \;=\; kT \left(\log(1 - e^{\beta\mu}) + AV \log(1 - e^{-\beta(\epsilon - \mu)}) \right) \; .$$

The numbers of particles with energies $0, \epsilon$ are respectively N_0, N_ϵ. Write down expressions for N_0, N_ϵ in terms of μ.

At temperature T what is the maximum number of bosons N_ϵ^{max} in the normal phase (the state with energy ϵ)? Explain what happens when $N > N_\epsilon^{max}$.

Given N and T calculate the transition temperature T_B at which Bose condensation occurs.

For $T > T_B$ show that $\mu = \epsilon(T_B - T)/T_B$. What is the value of μ for $T < T_B$?

Calculate the mean energy E for (a) $T > T_B$ (b) $T < T_B$, and show that the heat capacity of the system at constant volume is

$$C_V = \begin{cases} \dfrac{1}{kT^2} \dfrac{AV\epsilon^2}{(e^{\beta\epsilon} - 1)^2} & T < T_B \\ \qquad 0 & T > T_B . \end{cases}$$

23E Applications of Quantum Mechanics

A periodic potential is expressed as $V(\mathbf{x}) = \sum_{\mathbf{g}} a_{\mathbf{g}}\, e^{i\mathbf{g}\cdot\mathbf{x}}$, where $\{\mathbf{g}\}$ are reciprocal lattice vectors and $a_{\mathbf{g}}{}^* = a_{-\mathbf{g}}$, $a_{\mathbf{0}} = 0$. In the nearly free electron model explain why it is appropriate, near the boundaries of energy bands, to consider a Bloch wave state

$$|\psi_{\mathbf{k}}\rangle = \sum_r \alpha_r\, |\mathbf{k}_r\rangle\,, \qquad \mathbf{k}_r = \mathbf{k} + \mathbf{g}_r\,,$$

where $|\mathbf{k}\rangle$ is a free electron state for wave vector $\mathbf{k}$, $\langle \mathbf{k}'|\mathbf{k}\rangle = \delta_{\mathbf{k}'\mathbf{k}}$, and the sum is restricted to reciprocal lattice vectors $\mathbf{g}_r$ such that $|\mathbf{k}_r| \approx |\mathbf{k}|$. Obtain a determinantal formula for the possible energies $E(\mathbf{k})$ corresponding to Bloch wave states of this form.

[*You may take* $\mathbf{g}_1 = \mathbf{0}$ *and assume* $e^{i\mathbf{b}\cdot\mathbf{x}}|\mathbf{k}\rangle = |\mathbf{k}+\mathbf{b}\rangle$ *for any* $\mathbf{b}$.]

Suppose the sum is restricted to just $\mathbf{k}$ and $\mathbf{k}+\mathbf{g}$. Show that there is a gap $2|a_{\mathbf{g}}|$ between energy bands. Setting $\mathbf{k} = -\frac{1}{2}\mathbf{g} + \mathbf{q}$, show that there are two Bloch wave states with energies near the boundaries of the energy bands

$$E_{\pm}(\mathbf{k}) \approx \frac{\hbar^2|\mathbf{g}|^2}{8m} \pm |a_{\mathbf{g}}| + \frac{\hbar^2|\mathbf{q}|^2}{2m} \pm \frac{\hbar^4}{8m^2|a_{\mathbf{g}}|}\,(\mathbf{q}\cdot\mathbf{g})^2\,.$$

What is meant by effective mass? Determine the value of the effective mass at the top and the bottom of the adjacent energy bands if $\mathbf{q}$ is parallel to $\mathbf{g}$.

24C Fluid Dynamics II

(i) Suppose that, with spherical polar coordinates, the Stokes streamfunction

$$\Psi_\lambda(r,\theta) = r^\lambda \sin^2\theta\cos\theta$$

represents a Stokes flow and thus satisfies the equation $D^2(D^2\Psi_\lambda) = 0$, where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \ .$$

Show that the possible values of λ are 5, 3, 0 and -2. For which of these values is the corresponding flow irrotational? Sketch the streamlines of the flow for the case $\lambda = 3$.

(ii) A spherical drop of liquid of viscosity μ_1, radius a and centre at $r = 0$, is suspended in another liquid of viscosity μ_2 which flows with streamfunction

$$\Psi \sim \Psi_\infty(r,\theta) = \alpha r^3 \sin^2\theta\cos\theta$$

far from the drop. The two liquids are of equal densities, surface tension is sufficiently strong to keep the drop spherical, and inertia is negligible. Show that

$$\Psi = \begin{cases} (Ar^5 + Br^3)\sin^2\theta\cos\theta & (r < a), \\ (\alpha r^3 + C + D/r^2)\sin^2\theta\cos\theta & (r > a) \end{cases}$$

and obtain four equations determining the constants A, B, C and D. (You need not solve these equations.)

[*You may assume, with standard notation, that*

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} \quad , \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \quad , \quad e_{r\theta} = \frac{1}{2}\left\{ r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right\} .]$$

25C Waves in Fluid and Solid Media

Consider the equation

$$\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} - \frac{\partial^3\phi}{\partial x^3} = 0.$$

Find the dispersion relation for waves of frequency ω and wavenumber k. Do the wave crests move faster or slower than a packet of waves?

Write down the solution with initial value

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk \ ,$$

where $A(k)$ is real and $A(-k) = A(k)$.

Use the method of stationary phase to obtain an approximation to $\phi(x,t)$ for large t, with x/t having the constant value V. Explain, using the notion of group velocity, the constraint that must be placed on V.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative B

Thursday 6 June 2002 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Candidates must not attempt more than ***FOUR*** *questions. If you submit answers to more than four questions, your lowest scoring attempt(s) will be rejected.*

The number of marks for each question is the same. Additional credit will be given for a substantially complete answer.

Write legibly and on only **one** *side of the paper.*

Begin each answer on a separate sheet.

At the end of the examination:

Tie your answers in separate bundles, marked ***C, D, E, . . . , M*** *according to the letter affixed to each question. (For example,* ***23D, 25D*** *should be in one bundle and* ***2J, 6J*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1H Combinatorics

Write an essay on Ramsey theory. You should include the finite and infinite versions of Ramsey's theorem, together with a discussion of upper and lower bounds in the finite case.

[*You may restrict your attention to colourings by just 2 colours.*]

2J Representation Theory

Write an essay on the representation theory of SU_2.

3J Galois Theory

Suppose K, L are fields and $\sigma_1, \ldots, \sigma_m$ are distinct embeddings of K into L. Prove that there do not exist elements $\lambda_1, \ldots, \lambda_m$ of L (not all zero) such that $\lambda_1\sigma_1(x) + \ldots + \lambda_m\sigma_m(x) = 0$ for all $x \in K$. Deduce that if K/k is a finite extension of fields, and $\sigma_1, \ldots, \sigma_m$ are distinct k-automorphisms of K, then $m \leqslant [K : k]$.

Suppose now that K is a Galois extension of k with Galois group cyclic of order n, where n is not divisible by the characteristic. If k contains a primitive nth root of unity, prove that K is a radical extension of k. Explain briefly the relevance of this result to the problem of solubility of cubics by radicals.

4K Differentiable Manifolds

State and prove Stokes' Theorem for compact oriented manifolds-with-boundary.

[*You may assume results relating local forms on the manifold with those on its boundary provided you state them clearly.*]

Deduce that every differentiable map of the unit ball in $\mathbb{R}^n$ to itself has a fixed point.

5J Algebraic Topology

State the Mayer-Vietoris theorem for a finite simplicial complex X which is the union of closed subcomplexes A and B. Define all the maps in the long exact sequence. Prove that the sequence is exact at the term H_iX, for every $i \geqslant 0$.

6J Number Fields

Write an essay on one of the following topics.

(i) Dirichlet's unit theorem and the Pell equation.

(ii) Ideals and the fundamental theorem of arithmetic.

(iii) Dedekind's theorem and the factorisation of primes. (You should treat explicitly either the case of quadratic fields or that of the cyclotomic field.)

7K Hilbert Spaces

Throughout this question, H is an infinite-dimensional, separable Hilbert space. You may use, without proof, any theorems about compact operators that you require.

Define a *Fredholm operator* T, on a Hilbert space H, and define the *index* of T.

(i) Prove that if T is Fredholm then $\operatorname{im} T$ is closed.

(ii) Let $F \in \mathcal{B}(H)$ and let F have finite rank. Prove that F^* also has finite rank.

(iii) Let $T = I + F$, where I is the identity operator on H and F has finite rank; let $E = \operatorname{im} F + \operatorname{im} F^*$. By considering $T|E$ and $T|E^{\perp}$ (or otherwise) prove that T is Fredholm with $\operatorname{ind} T = 0$.

(iv) Let $T \in \mathcal{B}(H)$ be Fredholm with $\operatorname{ind} T = 0$. Prove that $T = A + F$, where A is invertible and F has finite rank.

[*You may wish to note that T effects an isomorphism from* $(\ker T)^{\perp}$ *onto* $\operatorname{im} T$*; also* $\ker T$ *and* $(\operatorname{im} T)^{\perp}$ *have the same finite dimension.*]

(v) Deduce from (iii) and (iv) that $T \in \mathcal{B}(H)$ is Fredholm with $\operatorname{ind} T = 0$ if and only if $T = A + K$ with A invertible and K compact.

(vi) Explain briefly, by considering suitable shift operators on ℓ^2 (i.e. *not* using any theorems about Fredholm operators) that, for each $k \in \mathbb{Z}$, there is a Fredholm operator S on H with $\operatorname{ind} S = k$.

8K Riemann Surfaces

A holomorphic map $p : S \to T$ between Riemann surfaces is called a covering map if every $t \in T$ has a neighbourhood V for which $p^{-1}(V)$ breaks up as a disjoint union of open subsets U_α on which $p : U_\alpha \to V$ is biholomorphic.

(a) Suppose that $f : R \to T$ is any holomorphic map of connected Riemann surfaces, R is simply connected and $p : S \to T$ is a covering map. By considering the lifts of paths from T to S, or otherwise, prove that f lifts to a holomorphic map $\widetilde{f} : R \to S$, i.e. that there exists an $\widetilde{f}$ with $f = p \circ \widetilde{f}$.

(b) Write down a biholomorphic map from the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ onto a half-plane. Show that the unit disk Δ uniformizes the punctured unit disk $\Delta^\times = \Delta - \{0\}$ by constructing an explicit covering map $p : \Delta \to \Delta^\times$.

(c) Using the uniformization theorem, or otherwise, prove that any holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus greater than one is constant.

9K Algebraic Curves

Write an essay on the Riemann–Roch theorem and some of its applications.

10J Logic, Computation and Set Theory

Explain what is meant by a *well-ordering* of a set.

Without assuming Zorn's Lemma, show that the power-set of any well-ordered set can be given a total (linear) ordering.

By a *selection function* for a set A, we mean a function $f : PA \to PA$ such that $f(B) \subset B$ for all $B \subset A$, $f(B) \neq \emptyset$ for all $B \neq \emptyset$, and $f(B) \neq B$ if B has more than one element. Suppose given a selection function f. Given a mapping $g : \alpha \to [0,1]$ for some ordinal α, we define a subset $B(f,g) \subset A$ recursively as follows:

$$
\begin{aligned}
&B(f,g) = A \quad \text{if } \alpha = 0, \\
&B(f,g) = f(B(f,g|_\beta)) \quad \text{if } \alpha = \beta^+ \text{ and } g(\beta) = 1, \\
&B(f,g) = B(f,g|_\beta) \backslash f(B(f,g|_\beta)) \quad \text{if } \alpha = \beta^+ \text{and } g(\beta) = 0, \\
&B(f,g) = \bigcap \{B(f,g|_\beta) \mid \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}
\end{aligned}
$$

Show that, for any $x \in A$ and any ordinal α, there exists a function g with domain α such that $x \in B(f,g)$.
[*It may help to observe that g is uniquely determined by x and α, though you need not show this explicitly.*]

Show also that there exists α such that, for every g with domain α, $B(f,g)$ is either empty or a singleton.

Deduce that the assertion 'Every set has a selection function' implies that every set can be totally ordered.
[*Hartogs' Lemma may be assumed, provided you state it precisely.*]

11L Probability and Measure

State Birkhoff's Almost Everywhere Ergodic Theorem for measure-preserving transformations. Define what it means for a sequence of random variables to be *stationary*. Explain *briefly* how the stationarity of a sequence of random variables implies that a particular transformation is measure-preserving.

A bag contains one white ball and one black ball. At each stage of a process one ball is picked from the bag (uniformly at random) and then returned to the bag together with another ball of the same colour. Let X_n be a random variable which takes the value 0 if the nth ball added to the bag is white and 1 if it is black.

(a) Show that the sequence $X_1, X_2, X_3, \ldots$ is stationary and hence that the proportion of black balls in the bag converges almost surely to some random variable R.

(b) Find the distribution of R.

[*The fact that almost-sure convergence implies convergence in distribution may be used without proof.*]

12L Applied Probability

Define a Poisson random measure. State and prove the Product Theorem for the jump times J_n of a Poisson process with constant rate λ and independent random variables Y_n with law μ. Write down the corresponding result for a Poisson process Π in a space $E = \mathbb{R}^d$ with rate $\lambda(x)$ $(x \in E)$ when we associate with each $X \in \Pi$ an independent random variable m_X with density $\rho(X, dm)$.

Prove Campbell's Theorem, i.e. show that if M is a Poisson random measure on the space E with intensity measure ν and $a : E \to \mathbb{R}$ is a bounded measurable function then

$$\mathbf{E}[e^{\theta\Sigma}] = \exp\left(\int_E (e^{\theta a(y)} - 1)\nu(dy)\right),$$

where

$$\Sigma = \int_E a(y)M(dy) = \sum_{X \in \Pi} a(X).$$

Stars are scattered over three-dimensional space $\mathbb{R}^3$ in a Poisson process Π with density $\nu(X)$ $(X \in \mathbb{R}^3)$. Masses of the stars are independent random variables; the mass m_X of a star at X has the density $\rho(X, dm)$. The gravitational potential at the origin is given by

$$F = \sum_{X \in \Pi} \frac{Gm_X}{|X|},$$

where G is a constant. Find the moment generating function $\mathbf{E}[e^{\theta F}]$.

A galaxy occupies a sphere of radius R centred at the origin. The density of stars is $\nu(\mathbf{x}) = 1/|\mathbf{x}|$ for points $\mathbf{x}$ inside the sphere; the mass of each star has the exponential distribution with mean M. Calculate the expected potential due to the galaxy at the origin. Let C be a positive constant. Find the distribution of the distance from the origin to the nearest star whose contribution to the potential F is at least C.

13M Information Theory

Define the Huffman binary encoding procedure and prove its optimality among decipherable codes.

14L Optimization and Control

A discrete-time decision process is defined on a finite set of states I as follows. Upon entry to state i_t at time t the decision-maker observes a variable ξ_t. He then chooses the next state freely within I, at a cost of $c(i_t, \xi_t, i_{t+1})$. Here $\{\xi_0, \xi_1, \ldots\}$ is a sequence of integer-valued, identically distributed random variables. Suppose there exist $\{\phi_i : i \in I\}$ and λ such that for all $i \in I$

$$\phi_i + \lambda = \sum_{k \in \mathbb{Z}} P(\xi_t = k) \min_{i' \in I} \left[c(i, k, i') + \phi_{i'} \right] .$$

Let π denote a policy. Show that

$$\lambda = \inf_{\pi} \limsup_{t \to \infty} E_{\pi} \left[\frac{1}{t} \sum_{s=0}^{t-1} c(i_s, \xi_s, i_{s+1}) \right] .$$

At the start of each month a boat manufacturer receives orders for 1, 2 or 3 boats. These numbers are equally likely and independent from month to month. He can produce j boats in a month at a cost of $6+3j$ units. All orders are filled at the end of the month in which they are ordered. It is possible to make extra boats, ending the month with a stock of i unsold boats, but i cannot be more than 2, and a holding cost of ci is incurred during any month that starts with i unsold boats in stock. Write down an optimality equation that can be used to find the long-run expected average-cost.

Let π be the policy of only ever producing sufficient boats to fill the present month's orders. Show that it is optimal if and only if $c \geqslant 2$.

Suppose $c < 2$. Starting from π, what policy is obtained after applying one step of the policy-improvement algorithm?

15M Principles of Statistics

(a) Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from a one-parameter distribution with density function

$$f(x;\theta) = h(x)g(\theta)\exp\{\theta t(x)\}\ ,\ x \in \mathbb{R}.$$

Explain in detail how you would test

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta \neq \theta_0\ .$$

What is the general form of a conjugate prior density for θ in a Bayesian analysis of this distribution?

(b) Let Y_1, Y_2 be independent Poisson random variables, with means $(1-\psi)\lambda$ and $\psi\lambda$ respectively, with λ *known.*

Explain why the Conditionality Principle leads to inference about ψ being drawn from the conditional distribution of Y_2, given Y_1+Y_2. What is this conditional distribution?

(c) Suppose Y_1, Y_2 have distributions as in (b), but that λ is now *unknown.*

Explain in detail how you would test $H_0 : \psi = \psi_0$ against $H_1 : \psi \neq \psi_0$, and describe the optimality properties of your test.

[*Any general results you use should be stated clearly, but need not be proved.*]

16L Stochastic Financial Models

Write an essay on the Black–Scholes formula for the price of a European call option on a stock. Your account should include a derivation of the formula and a careful analysis of its dependence on the parameters of the model.

17F Dynamical Systems

Let $\mathcal{S}$ be a metric space, F a map of $\mathcal{S}$ to itself and P a point of $\mathcal{S}$. Define an *attractor* for F and an *omega point* of the orbit of P under F.

Let f be the map of $\mathbb{R}$ to itself given by

$$f(x) = x + \frac{1}{2} + c\sin^2 2\pi x,$$

where $c > 0$ is so small that $f'(x) > 0$ for all x, and let F be the map of $\mathbb{R}/\mathbb{Z}$ to itself induced by f. What points if any are
(a) attractors for F^2,
(b) omega points of the orbit of some point P under F ?
Is the cycle $\{0, \frac{1}{2}\}$ an attractor?

In the notation of the first two sentences, let $\mathcal{C}$ be a cycle of order M and assume that F is continuous. Prove that $\mathcal{C}$ is an attractor for F if and only if each point of $\mathcal{C}$ is an attractor for F^M.

18G Partial Differential Equations

Discuss the notion of *fundamental solution* for a linear partial differential equation with constant coefficients.

19G Methods of Mathematical Physics

Let

$$I(\lambda, a) = \int_{-i\infty}^{i\infty} \frac{e^{\lambda(t^3-3t)}}{t^2 - a^2}\, dt\ ,$$

where λ is real, a is real and non-zero, and the path of integration runs up the imaginary axis. Show that, if $a^2 > 1$,

$$I(\lambda, a) \sim \frac{ie^{-2\lambda}}{1-a^2}\sqrt{\frac{\pi}{3\lambda}}$$

as $\lambda \to +\infty$ and sketch the relevant steepest descent path.

What is the corresponding result if $a^2 < 1$?

20F Numerical Analysis

Write an essay on the method of conjugate gradients. You should describe the algorithm, present an analysis of its properties and discuss its advantages.

[*Any theorems quoted should be stated precisely but need not be proved.*]

21E Electrodynamics

Derive Larmor's formula for the rate at which radiation is produced by a particle of charge q moving along a trajectory $\mathbf{x}(t)$.

A non-relativistic particle of mass m, charge q and energy E is incident along a radial line in a central potential $V(r)$. The potential is vanishingly small for r very large, but increases without bound as $r \to 0$. Show that the total amount of energy $\mathcal{E}$ radiated by the particle is

$$\mathcal{E} = \frac{\mu_0 q^2}{3\pi m^2}\sqrt{\frac{m}{2}}\int_{r_0}^{\infty} \frac{1}{\sqrt{E - V(r)}}\left(\frac{dV}{dr}\right)^2 dr\,,$$

where $V(r_0) = E$.

Suppose that V is the Coulomb potential $V(r) = A/r$. Evaluate $\mathcal{E}$.

22E Foundations of Quantum Mechanics

Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have (a) spin zero, (b) spin 1/2.

The stationary Schrödinger equation for one particle in the potential

$$-\frac{2e^2}{4\pi\epsilon_0 r}$$

has normalized, spherically symmetric, real wave functions $\psi_n(\mathbf{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. What are the consequences of the Pauli exclusion principle for the ground state of the helium atom? Assuming that wavefunctions which are not spherically symmetric can be ignored, what are the states of the first excited energy level of the helium atom?
[*You may assume here that the electrons are non-interacting.*]

Show that, taking into account the interaction between the two electrons, the estimate for the energy of the ground state of the helium atom is

$$2E_0 + \frac{e^2}{4\pi\epsilon_0}\int \frac{d^3\mathbf{r}_1\, d^3\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\psi_0^2(\mathbf{r}_1)\psi_0^2(\mathbf{r}_2).$$

23D Statistical Physics

A perfect gas in equilibrium in a volume V has quantum stationary states $|i\rangle$ with energies E_i. In a Boltzmann distribution, the probability that the system is in state $|i\rangle$ is $\rho_i = Z^{-1} e^{-E_i/kT}$. The entropy is defined to be $S = -k \sum_i \rho_i \log \rho_i$.

For two nearby states establish the equation

$$dE = TdS - PdV ,$$

where E and P should be defined.

For reversible changes show that

$$dS = \frac{\delta Q}{T} ,$$

where δQ is the amount of heat transferred in the exchange.

Define C_V, the heat capacity at constant volume.

A system with constant heat capacity C_V initially at temperature T is heated at constant volume to a temperature Θ. Show that the change in entropy is $\Delta S = C_V \log(\Theta/T)$.

Explain what is meant by isothermal and adiabatic transitions.

Briefly, describe the Carnot cycle and define its efficiency. Explain briefly why no heat engine can be more efficient than one whose operation is based on a Carnot cycle.

Three identical bodies with constant heat capacity at fixed volume C_V, are initially at temperatures T_1, T_2, T_3, respectively. Heat engines operate between the bodies with no input of work or heat from the outside and the respective temperatures are changed to $\Theta_1, \Theta_2, \Theta_3$, the volume of the bodies remaining constant. Show that, if the heat engines operate on a Carnot cycle, then

$$\Theta_1 \Theta_2 \Theta_3 = A , \qquad \Theta_1 + \Theta_2 + \Theta_3 = B ,$$

where $A = T_1 T_2 T_3$ and $B = T_1 + T_2 + T_3$.

Hence show that the maximum temperature to which any one of the bodies can be raised is Θ where

$$\Theta + 2 \left(\frac{A}{\Theta} \right)^{1/2} = B .$$

Show that a solution is $\Theta = T$ if initially $T_1 = T_2 = T_3 = T$. Do you expect there to be any other solutions?

Find Θ if initially $T_1 = 300\,\mathrm{K},\ T_2 = 300\,\mathrm{K},\ T_3 = 100\,\mathrm{K}$.

[*Hint: Choose to maximize one temperature and impose the constraints above using Lagrange multipliers.*]

24D Applications of Quantum Mechanics

Explain the variational method for computing the ground state energy for a quantum Hamiltonian.

For the one-dimensional Hamiltonian

$$H = \tfrac{1}{2}p^2 + \lambda x^4\,,$$

obtain an approximate form for the ground state energy by considering as a trial state the state $|w\rangle$ defined by $a|w\rangle = 0$, where $\langle w|w\rangle = 1$ and $a = (w/2\hbar)^{\frac{1}{2}}(x + ip/w)$.

[*It is useful to note that* $\langle w|(a+a^\dagger)^4|w\rangle = \langle w|(a^2a^{\dagger 2} + aa^\dagger aa^\dagger)|w\rangle$.]

Explain why the states $a^\dagger|w\rangle$ may be used as trial states for calculating the first excited energy level.

25D General Relativity

With respect to the Schwarzschild coordinates (r,θ,ϕ,t), the Schwarzschild geometry is given by

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \left(1 - \frac{r_s}{r}\right)dt^2,$$

where $r_s = 2M$ is the Schwarzschild radius and M is the Schwarzschild mass. Show that, by a suitable choice of (θ,ϕ), the general geodesic can regarded as moving in the equatorial plane $\theta = \pi/2$. Obtain the equations governing timelike and null geodesics in terms of $u(\phi)$, where $u = 1/r$.

Discuss light bending and perihelion precession in the solar system.

26C Fluid Dynamics II

Write an essay on boundary-layer theory and its application to the generation of lift in aerodynamics.

You should include discussion of the derivation of the boundary-layer equation, the similarity transformation leading to the Falkner–Skan equation, the influence of an adverse pressure gradient, and the mechanism(s) by which circulation is generated in flow past bodies with a sharp trailing edge.

27C Waves in Fluid and Solid Media

Write down the equation governing linearized displacements $\mathbf{u}(\mathbf{x}, t)$ in a uniform elastic medium of density ρ and Lamé constants λ and μ. Derive solutions for monochromatic plane P and S waves, and find the corresponding wave speeds c_P and c_S.

Such an elastic solid occupies the half-space $z > 0$, and the boundary $z = 0$ is clamped rigidly so that $\mathbf{u}(x, y, 0, t) = \mathbf{0}$. A plane SV-wave with frequency ω and wavenumber $(k, 0, -m)$ is incident on the boundary. At some angles of incidence, there results both a reflected SV-wave with frequency ω' and wavenumber $(k', 0, m')$ and a reflected P-wave with frequency ω'' and wavenumber $(k'', 0, m'')$. Relate the frequencies and wavenumbers of the reflected waves to those of the incident wave. At what angles of incidence will there be a reflected P-wave?

Find the amplitudes of the reflected waves as multiples of the amplitude of the incident wave. Confirm that these amplitudes give the sum of the time-averaged vertical fluxes of energy of the reflected waves equal to the time-averaged vertical flux of energy of the incident wave.

[*Results concerning the energy flux, energy density and kinetic energy density in a plane elastic wave may be quoted without proof.*]

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Monday 2 June 2003 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C, . . . , J** *according to the letter affixed to each question. (For example,* **4F, 8F** *should be in one bundle and* **1J, 11J** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

(i) Let $(X_n, Y_n)_{n \geqslant 0}$ be a simple symmetric random walk in $\mathbb{Z}^2$, starting from $(0,0)$, and set $T = \inf\{n \geqslant 0 : \max\{|X_n|, |Y_n|\} = 2\}$. Determine the quantities $\mathbb{E}(T)$ and $\mathbb{P}(X_T = 2 \text{ and } Y_T = 0)$.

(ii) Let $(X_n)_{n \geqslant 0}$ be a discrete-time Markov chain with state-space I and transition matrix P. What does it mean to say that a state $i \in I$ is recurrent? Prove that i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, where $p_{ii}^{(n)}$ denotes the (i,i) entry in P^n.

Show that the simple symmetric random walk in $\mathbb{Z}^2$ is recurrent.

2D Principles of Dynamics

(i) Consider N particles moving in 3 dimensions. The Cartesian coordinates of these particles are $x^A(t),\ A = 1, \ldots, 3N$. Now consider an invertible change of coordinates to coordinates $q^a(x^A, t), \quad a = 1, \ldots, 3N$, so that one may express x^A as $x^A(q^a, t)$. Show that the velocity of the system in Cartesian coordinates $\dot{x}^A(t)$ is given by the following expression:

$$\dot{x}^A(\dot{q}^a, q^a, t) = \sum_{b=1}^{3N} \dot{q}^b \frac{\partial x^A}{\partial q^b}(q^a, t) + \frac{\partial x^A}{\partial t}(q^a, t)\,.$$

Furthermore, show that Lagrange's equations in the two coordinate systems are related via

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = \sum_{A=1}^{3N} \frac{\partial x^A}{\partial q^a}\left(\frac{\partial L}{\partial x^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A}\right).$$

(ii) Now consider the case where there are $p < 3N$ constraints applied, $f^\ell(x^A, t) = 0,\ \ell = 1, \ldots, p$. By considering the $f^\ell,\ \ell = 1, \ldots, p$, and a set of independent coordinates $q^a, a = 1, \ldots, 3N - p$, as a set of $3N$ new coordinates, show that the Lagrange equations of the constrained system, i.e.

$$\frac{\partial L}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) + \sum_{\ell=1}^{p} \lambda^\ell \frac{\partial f^\ell}{\partial x^A} = 0, \quad A = 1, \ldots, 3N,$$
$$f^\ell = 0, \qquad \ell = 1, \ldots, p,$$

(where the λ^ℓ are Lagrange multipliers) imply Lagrange's equations for the unconstrained coordinates, i.e.

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0, \quad a = 1, \ldots, 3N - p\,.$$

3G Functional Analysis

(i) Let $T : H_1 \to H_2$ be a continuous linear map between two Hilbert spaces H_1, H_2. Define the adjoint T^* of T. Explain what it means to say that T is Hermitian or unitary.

Let $\phi : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function. Show that the map

$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

with $Tf(x) = \phi(x)f(x+1)$ is a continuous linear map and find its adjoint. When is T Hermitian? When is it unitary?

(ii) Let C be a closed, non-empty, convex subset of a real Hilbert space H. Show that there exists a unique point $x_o \in C$ with minimal norm. Show that x_o is characterised by the property

$$\langle x_o - x, x_o \rangle \leqslant 0 \qquad \text{for all } \; x \in C\,.$$

Does this result still hold when C is not closed or when C is not convex? Justify your answers.

4F Groups, Rings and Fields

(i) Let p be a prime number. Show that a group G of order p^n $(n \geqslant 2)$ has a nontrivial normal subgroup, that is, G is not a simple group.

(ii) Let p and q be primes, $p > q$. Show that a group G of order pq has a normal Sylow p-subgroup. If G has also a normal Sylow q-subgroup, show that G is cyclic. Give a necessary and sufficient condition on p and q for the existence of a non-abelian group of order pq. Justify your answer.

5A Electromagnetism

(i) Using Maxwell's equations as they apply to magnetostatics, show that the magnetic field $\mathbf{B}$ can be described in terms of a vector potential $\mathbf{A}$ on which the condition $\nabla \cdot \mathbf{A} = 0$ may be imposed. Hence derive an expression, valid at any point in space, for the vector potential due to a steady current distribution of density $\mathbf{j}$ that is non-zero only within a finite domain.

(ii) Verify that the vector potential $\mathbf{A}$ that you found in Part (i) satisfies $\nabla \cdot \mathbf{A} = 0$, and use it to obtain the Biot–Savart law expression for $\mathbf{B}$. What is the corresponding result for a steady surface current distribution of density $\mathbf{s}$?

In cylindrical polar coordinates (ρ, ϕ, z) (oriented so that $\mathbf{e}_\rho \times \mathbf{e}_\phi = \mathbf{e}_z$) a surface current

$$\mathbf{s} = s(\rho)\mathbf{e}_\phi$$

flows in the plane $z = 0$. Given that

$$s(\rho) = \begin{cases} 4I\left(1 + \frac{a^2}{\rho^2}\right)^{\frac{1}{2}} & a \leqslant \rho \leqslant 3a \\ 0 & \text{otherwise} \end{cases}$$

show that the magnetic field at the point $\mathbf{r} = a\mathbf{e}_z$ has z-component

$$B_z = \mu_0 I \log 5\,.$$

State, with justification, the full result for $\mathbf{B}$ at the point $\mathbf{r} = a\mathbf{e}_z$.

6D Dynamics of Differential Equations

(i) State and prove *Dulac's Criterion* for the non-existence of periodic orbits in $\mathbb{R}^2$. Hence show (choosing a weighting factor of the form $x^\alpha y^\beta$) that there are no periodic orbits of the equations

$$\dot{x} = x(2 - 6x^2 - 5y^2)\,, \quad \dot{y} = y(-3 + 10x^2 + 3y^2)\,.$$

(ii) State the *Poincaré–Bendixson Theorem*. A model of a chemical reaction (the Brusselator) is defined by the second order system

$$\dot{x} = a - x(1+b) + x^2 y\,, \quad \dot{y} = bx - x^2 y,$$

where a, b are positive parameters. Show that there is a unique fixed point. Show that, for a suitable choice of $p > 0$, trajectories enter the closed region bounded by $x = p$, $y = b/p$, $x + y = a + b/p$ and $y = 0$. Deduce that when $b > 1 + a^2$, the system has a periodic orbit.

7H Logic, Computation and Set Theory

(i) State Zorn's Lemma. Use Zorn's Lemma to prove that every real vector space has a basis.

(ii) State the Bourbaki–Witt Theorem, and use it to prove Zorn's Lemma, making clear where in the argument you appeal to the Axiom of Choice.

Conversely, deduce the Bourbaki–Witt Theorem from Zorn's Lemma.

If X is a non-empty poset in which every chain has an upper bound, must X be chain-complete?

8F Graph Theory

(i) State Brooks' Theorem, and prove it in the case of a 3-connected graph.

(ii) Let G be a bipartite graph, with vertex classes X and Y, each of order n. If G contains no cycle of length 4 show that

$$e(G) \leqslant \frac{n}{2}(1+\sqrt{4n-3}).$$

For which integers $n \leq 12$ are there examples where equality holds?

9G Number Theory

(i) Let p be an odd prime and k a strictly positive integer. Prove that the multiplicative group of relatively prime residue classes modulo p^k is cyclic.

[*You may assume that the result is true for* $k=1$.]

(ii) Let $n = p_1 p_2 \ldots p_r$, where $r \geqslant 2$ and $p_1, p_2, \ldots, p_r$ are distinct odd primes. Let B denote the set of all integers which are relatively prime to n. We recall that n is said to be an *Euler pseudo-prime to the base* $b \in B$ if

$$b^{(n-1)/2} \equiv \left(\frac{b}{n}\right) \mod n\,.$$

If n is an Euler pseudo-prime to the base $b_1 \in B$, but is not an Euler pseudo-prime to the base $b_2 \in B$, prove that n is not an Euler pseudo-prime to the base $b_1 b_2$. Let p denote any of the primes $p_1, p_2, \ldots, p_r$. Prove that there exists a $b \in B$ such that

$$\left(\frac{b}{p}\right) = -1 \qquad \text{and} \qquad b \equiv 1 \mod n/p\,,$$

and deduce that n is not an Euler pseudo-prime to this base b. Hence prove that n is not an Euler pseudo-prime to the base b for at least half of all the relatively prime residue classes $b \bmod n$.

10F Coding and Cryptography

(i) We work over the field of two elements. Define what is meant by a linear code of length n. What is meant by a generator matrix for a linear code?

Define what is meant by a parity check code of length n. Show that a code is linear if and only if it is a parity check code.

Give the original Hamming code in terms of parity checks and then find a generator matrix for it.

[*You may use results from the theory of vector spaces provided that you quote them correctly.*]

(ii) Suppose that $1/4 > \delta > 0$ and let $\alpha(n, n\delta)$ be the largest information rate of any binary error correcting code of length n which can correct $[n\delta]$ errors.

Show that

$$1 - H(2\delta) \leqslant \liminf_{n\to\infty} \alpha(n, n\delta) \leqslant 1 - H(\delta)$$

where

$$H(\eta) = -\eta \log_2 \eta - (1-\eta)\log_2(1-\eta).$$

[*You may assume any form of Stirling's theorem provided that you quote it correctly.*]

11J Stochastic Financial Models

(i) In the context of a single-period financial market with d traded assets, what is an *arbitrage?* What is an *equivalent martingale measure?*

A simple single-period financial market contains two assets, S^0 (a bond), and S^1 (a share). The period can be good, bad, or indifferent, with probabilities 1/3 each. At the beginning of the period, time 0, both assets are worth 1, i.e.

$$S_0^0 = 1 = S_0^1,$$

and at the end of the period, time 1, the share is worth

$$S_1^1 = \begin{cases} a & \text{if the period was bad,} \\ b & \text{if the period was indifferent,} \\ c & \text{if the period was good,} \end{cases}$$

where $a < b < c$. The bond is always worth 1 at the end of the period. Show that there is no arbitrage in this market if and only if $a < 1 < c$.

(ii) An agent with C^2 strictly increasing strictly concave utility U has wealth w_0 at time 0, and wishes to invest his wealth in shares and bonds so as to maximise his expected utility of wealth at time 1. Explain how the solution to his optimisation problem generates an equivalent martingale measure.

Assume now that $a = 3/4$, $b = 1$, and $c = 3/2$. Characterise all equivalent martingale measures for this problem. Characterise all equivalent martingale measures which arise as solutions of an agent's optimisation problem.

Calculate the largest and smallest possible prices for a European call option with strike 1 and expiry 1, as the pricing measure ranges over all equivalent martingale measures. Calculate the corresponding bounds when the pricing measure is restricted to the set arising from expected-utility-maximising agents' optimisation problems.

12I Principles of Statistics

(i) A public health official is seeking a rational policy of vaccination against a relatively mild ailment which causes absence from work. Surveys suggest that 60% of the population are already immune, but accurate tests to detect vulnerability in any individual are too costly for mass screening. A simple skin test has been developed, but is not completely reliable. A person who is immune to the ailment will have a negligible reaction to the skin test with probability 0.4, a moderate reaction with probability 0.5 and a strong reaction with probability 0.1. For a person who is vulnerable to the ailment the corresponding probabilities are 0.1, 0.4 and 0.5. It is estimated that the money-equivalent of work-hours lost from failing to vaccinate a vulnerable person is 20, that the unnecessary cost of vaccinating an immune person is 8, and that there is no cost associated with vaccinating a vulnerable person or failing to vaccinate an immune person. On the basis of the skin test, it must be decided whether to vaccinate or not. What is the Bayes decision rule that the health official should adopt?

(ii) A collection of I students each sit J exams. The ability of the ith student is represented by θ_i and the performance of the ith student on the jth exam is measured by X_{ij}. Assume that, given $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_I)$, an appropriate model is that the variables $\{X_{ij}\,,\, 1 \leqslant i \leqslant I,\, 1 \leqslant j \leqslant J\}$ are independent, and

$$X_{ij} \sim N(\theta_i, \tau^{-1}),$$

for a known positive constant τ. It is reasonable to assume, *a priori*, that the θ_i are independent with

$$\theta_i \sim N(\mu, \zeta^{-1}),$$

where μ and ζ are population parameters, known from experience with previous cohorts of students.

Compute the posterior distribution of $\boldsymbol{\theta}$ given the observed exam marks vector $\mathbf{X} = \{X_{ij}\,,\, 1 \leqslant i \leqslant I,\, 1 \leqslant j \leqslant J\}$.

Suppose now that τ is also unknown, but assumed to have a $\text{Gamma}(\alpha_0, \beta_0)$ distribution, for known α_0, β_0. Compute the posterior distribution of τ given $\boldsymbol{\theta}$ and $\mathbf{X}$. Find, up to a normalisation constant, the form of the marginal density of $\boldsymbol{\theta}$ given $\mathbf{X}$.

13I Computational Statistics and Statistical Modelling

(i) Suppose Y_i, $1 \leqslant i \leqslant n$, are independent binomial observations, with $Y_i \sim Bi(t_i, \pi_i)$, $1 \leqslant i \leqslant n$, where $t_1, \ldots, t_n$ are known, and we wish to fit the model

$$\omega : \log \frac{\pi_i}{1-\pi_i} = \mu + \beta^T x_i \quad \text{for each } i,$$

where $x_1, \ldots, x_n$ are given covariates, each of dimension p. Let $\hat{\mu}$, $\hat{\beta}$ be the maximum likelihood estimators of μ, β. Derive equations for $\hat{\mu}$, $\hat{\beta}$ and state without proof the form of the approximate distribution of $\hat{\beta}$.

(ii) In 1975, data were collected on the 3-year survival status of patients suffering from a type of cancer, yielding the following table

		survive?	
age in years	malignant	yes	no
under 50	no	77	10
under 50	yes	51	13
50-69	no	51	11
50-69	yes	38	20
70+	no	7	3
70+	yes	6	3

Here the second column represents whether the initial tumour was not malignant or was malignant.

Let Y_{ij} be the number surviving, for age group i and malignancy status j, for $i = 1, 2, 3$ and $j = 1, 2$, and let t_{ij} be the corresponding total number. Thus $Y_{11} = 77$, $t_{11} = 87$. Assume $Y_{ij} \sim Bi(t_{ij}, \pi_{ij})$, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 2$. The results from fitting the model

$$\log(\pi_{ij}/(1-\pi_{ij})) = \mu + \alpha_i + \beta_j$$

with $\alpha_1 = 0$, $\beta_1 = 0$ give $\hat{\beta}_2 = -0.7328$ (se $= 0.2985$), and deviance $= 0.4941$. What do you conclude?

Why do we take $\alpha_1 = 0$, $\beta_1 = 0$ in the model?

What "residuals" should you compute, and to which distribution would you refer them?

14C Quantum Physics

(i) An electron of mass m and spin $\frac{1}{2}$ moves freely inside a cubical box of side L . Verify that the energy eigenstates of the system are $\phi_{lmn}(\mathbf{r})\chi_{\pm}$ where the spatial wavefunction is given by

$$\phi_{lmn}(\mathbf{r}) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{l\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \quad ,$$

and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Give the corresponding energy eigenvalues.

A second electron is inserted into the box. Explain how the Pauli principle determines the structure of the wavefunctions associated with the lowest energy level and the first excited energy level. What are the values of the energy in these two levels and what are the corresponding degeneracies?

(ii) When the side of the box, L, is large, the number of eigenstates available to the electron with energy in the range $E \to E + dE$ is $\rho(E)dE$. Show that

$$\rho(E) = \frac{L^3}{\pi^2 \hbar^3} \sqrt{2m^3 E} \quad .$$

A large number, N, of electrons are inserted into the box. Explain how the ground state is constructed and define the Fermi energy, E_F. Show that in the ground state

$$N = \frac{2}{3} \frac{L^3}{\pi^2 \hbar^3} \sqrt{2m^3} \left(E_F\right)^{3/2} \quad .$$

When a magnetic field H in the z-direction is applied to the system, an electron with spin up acquires an additional energy $+\mu H$ and an electron with spin down an energy $-\mu H$, where $-\mu$ is the magnetic moment of the electron and $\mu > 0$. Describe, for the case $E_F > \mu H$, the structure of the ground state of the system of N electrons in the box and show that

$$N = \frac{1}{3} \frac{L^3}{\pi^2 \hbar^3} \sqrt{2m^3} \left((E_F + \mu H)^{3/2} + (E_F - \mu H)^{3/2} \right) \quad .$$

Calculate the induced magnetic moment, M, of the ground state of the system and show that for a *weak* magnetic field the magnetic moment is given by

$$M \approx \frac{3}{2} N \frac{\mu^2 H}{E_F} \quad .$$

15A General Relativity

(i) The worldline $x^a(\lambda)$ of a massive particle moving in a spacetime with metric g_{ab} obeys the geodesic equation

$$\frac{d^2x^a}{d\tau^2} + \left\{ {}_{b}{}^{a}{}_{c} \right\} \frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

where τ is the particle's proper time and $\left\{ {}_{b}{}^{a}{}_{c} \right\}$ are the Christoffel symbols; these are the equations of motion for the Lagrangian

$$L_1 = -m\sqrt{-g_{ab}\dot{x}^a\dot{x}^b}$$

where m is the particle's mass, and $\dot{x}^a = dx^a/d\lambda$. Why is the choice of worldline parameter λ irrelevant? Among all possible worldlines passing through points A and B, why is $x^a(\lambda)$ the one that extremizes the proper time elapsed between A and B?

Explain how the equations of motion for a massive particle may be obtained from the alternative Lagrangian

$$L_2 = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b \, .$$

What can you conclude from the fact that L_2 has no explicit dependence on λ? How are the equations of motion for a massless particle obtained from L_2?

(ii) A photon moves in the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - \left(1 - \frac{2M}{r}\right) dt^2.$$

Given that the motion is confined to the plane $\theta = \pi/2$, obtain the radial equation

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{h^2}{r^2}\left(1 - \frac{2M}{r}\right),$$

where E and h are constants, the physical meaning of which should be stated.

Setting $u = 1/r$, obtain the equation

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2.$$

Using the approximate solution

$$u = \frac{1}{b}\sin\phi + \frac{M}{2b^2}\left(3 + \cos 2\phi\right) + \dots,$$

obtain the standard formula for the deflection of light passing far from a body of mass M with impact parameter b. Reinstate factors of G and c to give your result in physical units.

16A Statistical Physics and Cosmology

(i) Explain briefly how the relative motion of galaxies in a homogeneous and isotropic universe is described in terms of the scale factor $a(t)$ (where t is time). In particular, show that the relative velocity $\mathbf{v}(t)$ of two galaxies is given in terms of their relative displacement $\mathbf{r}(t)$ by the formula $\mathbf{v}(t) = H(t)\mathbf{r}(t)$, where $H(t)$ is a function that you should determine in terms of $a(t)$. Given that $a(0) = 0$, obtain a formula for the distance $R(t)$ to the cosmological horizon at time t. Given further that $a(t) = (t/t_0)^\alpha$, for $0 < \alpha < 1$ and constant t_0, compute $R(t)$. Hence show that $R(t)/a(t) \to 0$ as $t \to 0$.

(ii) A homogeneous and isotropic model universe has energy density $\rho(t)c^2$ and pressure $P(t)$, where c is the speed of light. The evolution of its scale factor $a(t)$ is governed by the Friedmann equation

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2$$

where the overdot indicates differentiation with respect to t. Use the "Fluid" equation

$$\dot{\rho} = -3\left(\rho + \frac{P}{c^2}\right)\left(\frac{\dot{a}}{a}\right)$$

to obtain an equation for the acceleration $\ddot{a}(t)$. Assuming $\rho > 0$ and $P \geqslant 0$, show that ρa^3 cannot increase with time as long as $\dot{a} > 0$, nor decrease if $\dot{a} < 0$. Hence determine the late time behaviour of $a(t)$ for $k < 0$. For $k > 0$ show that an initially expanding universe must collapse to a "big crunch" at which $a \to 0$. How does $\dot{a}$ behave as $a \to 0$? Given that $P = 0$, determine the form of $a(t)$ near the big crunch. Discuss the qualitative late time behaviour for $k = 0$.

Cosmological models are often assumed to have an equation of state of the form $P = \sigma\rho c^2$ for constant σ. What physical principle requires $\sigma \leqslant 1$? Matter with $P = \rho c^2$ ($\sigma = 1$) is called "stiff matter" by cosmologists. Given that $k = 0$, determine $a(t)$ for a universe that contains only stiff matter. In our Universe, why would you expect stiff matter to be negligible now even if it were significant in the early Universe?

17C Symmetries and Groups in Physics

(i) Define the character χ of a representation D of a finite group G. Show that $<\chi \mid \chi>= 1$ if and only if D is irreducible, where

$$<\chi \mid \chi>= \frac{1}{\mid G \mid}\sum_{g\in G}\chi(g)\chi(g^{-1})\,.$$

If $\mid G \mid= 8$ and $<\chi \mid \chi>= 2$, what are the possible dimensions of the representation D?

(ii) State and prove Schur's first and second lemmas.

18E Transport Processes

(i) A solute occupying a domain V_0 has concentration $C(\boldsymbol{x}, t)$ and is created at a rate $S(\boldsymbol{x}, t)$ per unit volume; $\boldsymbol{J}(\boldsymbol{x}, t)$ is the flux of solute per unit area; $\boldsymbol{x}, t$ are position and time. Derive the transport equation

$$C_t + \nabla \cdot \boldsymbol{J} = S.$$

State Fick's Law of diffusion and hence write down the diffusion equation for $C(\boldsymbol{x}, t)$ for a case in which the solute flux occurs solely by diffusion, with diffusivity $D(\boldsymbol{x})$.

In a finite domain $0 \leqslant x \leqslant L$, D, S and the steady-state distribution of C depend only on x; C is equal to C_0 at $x = 0$ and $C_1 \neq C_0$ at $x = L$. Find $C(x)$ in the following two cases:

(a) $D = D_0$, $S = 0$,

(b) $D = D_1 x^{1/2}$, $S = 0$,

where D_0 and D_1 are positive constants.

Show that there is no steady solution satisfying the boundary conditions if $D = D_1 x$, $S = 0$.

(ii) For the problem of Part (i), consider the case $D = D_0$, $S = kC$, where D_0 and k are positive constants. Calculate the steady-state solution, $C = C_s(x)$, assuming that $\sqrt{k/D_0} \neq n\pi/L$ for any integer n.

Now let

$$C(x, 0) = C_0 \frac{\sin \alpha(L - x)}{\sin \alpha L},$$

where $\alpha = \sqrt{k/D_0}$. Find the equations, boundary and initial conditions satisfied by $C'(x, t) = C(x, t) - C_s(x)$. Solve the problem using separation of variables and show that

$$C'(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \exp\left[\left(\alpha^2 - \frac{n^2\pi^2}{L^2}\right) D_0 t\right],$$

for some constants A_n. Write down an integral expression for A_n, show that

$$A_1 = -\frac{2\pi C_1}{\alpha^2 L^2 - \pi^2},$$

and comment on the behaviour of the solution for large times in the two cases $\alpha L < \pi$ and $\alpha L > \pi$.

19E Theoretical Geophysics

(i) Explain the concepts of: traction on an element of surface; the stress tensor; the strain tensor in an elastic medium. Derive a relationship between the two tensors for a linear isotropic elastic medium, stating clearly any assumption you need to make.

(ii) State what is meant by an SH wave in a homogeneous isotropic elastic medium. An SH wave in a medium with shear modulus μ and density ρ is incident at angle θ on an interface with a medium with shear modulus μ' and density ρ'. Evaluate the form and amplitude of the reflected wave and transmitted wave. Comment on the case $c' \sin\theta / c > 1$, where $c^2 = \mu/\rho$ and $(c')^2 = \mu'/\rho'$.

20E Numerical Analysis

(i) The linear algebraic equations $A\mathbf{u} = \mathbf{b}$, where A is symmetric and positive-definite, are solved with the Gauss–Seidel method. Prove that the iteration always converges.

(ii) The Poisson equation $\nabla^2 u = f$ is given in the bounded, simply connected domain $\Omega \subseteq \mathbb{R}^2$, with zero Dirichlet boundary conditions on $\partial\Omega$. It is approximated by the five-point formula

$$U_{m-1,n} + U_{m,n-1} + U_{m+1,n} + U_{m,n+1} - 4U_{m,n} = (\Delta x)^2 f_{m,n},$$

where $U_{m,n} \approx u(m\Delta x, n\Delta x)$, $\quad f_{m,n} = f(m\Delta x, n\Delta x)$, and $(m\Delta x, n\Delta x)$ is in the interior of Ω.

Assume for the sake of simplicity that the intersection of $\partial\Omega$ with the grid consists only of grid points, so that no special arrangements are required near the boundary. Prove that the method can be written in a vector notation, $A\mathbf{u} = \mathbf{b}$ with a negative-definite matrix A.

UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Wednesday 4 June 2003 9 to 12

PAPER 2

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)***, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* **SIX** *questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)***. Additional credit will be given for a substantially complete answer to either Part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked **A, B, C,** *. . . ,* **J** *according to the letter affixed to each question. (For example,* **16E, 19E** *should be in one bundle and* **4F, 8F** *in another bundle.)*

Attach a completed cover sheet to each bundle listing the Parts of questions attempted.

Complete a master cover sheet listing separately **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1J Markov Chains

(i) What is meant by a Poisson process of rate λ? Show that if $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ are independent Poisson processes of rates λ and μ respectively, then $(X_t + Y_t)_{t\geqslant 0}$ is also a Poisson process, and determine its rate.

(ii) A Poisson process of rate λ is observed by someone who believes that the first holding time is longer than all subsequent holding times. How long on average will it take before the observer is proved wrong?

2D Principles of Dynamics

(i) The trajectory $\mathbf{x}(t)$ of a non-relativistic particle of mass m and charge q moving in an electromagnetic field obeys the Lorentz equation

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \frac{\dot{\mathbf{x}}}{c} \wedge \mathbf{B}) .$$

Show that this equation follows from the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\left(\phi - \frac{\dot{\mathbf{x}} \cdot \mathbf{A}}{c}\right)$$

where $\phi(\mathbf{x}, t)$ is the electromagnetic scalar potential and $\mathbf{A}(\mathbf{x}, t)$ the vector potential, so that

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \boldsymbol{\nabla}\phi \text{ and } \mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} .$$

(ii) Let $\mathbf{E} = 0$. Consider a particle moving in a constant magnetic field which points in the z direction. Show that the particle moves in a helix about an axis pointing in the z direction. Evaluate the radius of the helix.

3G Functional Analysis

(i) Define the dual of a normed vector space $(E, ||\cdot||)$. Show that the dual is always a complete normed space.

Prove that the vector space ℓ_1, consisting of those real sequences $(x_n)_{n=1}^{\infty}$ for which the norm

$$||(x_n)||_1 = \sum_{n=1}^{\infty} |x_n|$$

is finite, has the vector space ℓ_∞ of all bounded sequences as its dual.

(ii) State the Stone–Weierstrass approximation theorem.

Let K be a compact subset of $\mathbb{R}^n$. Show that every $f \in C_{\mathbb{R}}(K)$ can be uniformly approximated by a sequence of polynomials in n variables.

Let f be a continuous function on $[0,1] \times [0,1]$. Deduce that

$$\int_0^1 \left(\int_0^1 f(x,y)\, dx \right) dy = \int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx\,.$$

4F Groups, Rings and Fields

(i) In each of the following two cases, determine a highest common factor in $\mathbb{Z}[i]$:

(a) $3+4i,\ 4-3i$;

(b) $3+4i,\ 1+2i$.

(ii) State and prove the Eisenstein criterion for irreducibility of polynomials with integer coefficients. Show that, if p is prime, the polynomial

$$1 + x + \cdots + x^{p-1}$$

is irreducible over $\mathbb{Z}$.

5A Electromagnetism

(i) A plane electromagnetic wave has electric and magnetic fields

$$\mathbf{E} = \mathbf{E}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,, \qquad \mathbf{B} = \mathbf{B}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad (*)$$

for constant vectors $\mathbf{E}_0, \mathbf{B}_0$, constant positive angular frequency ω and constant wave-vector $\mathbf{k}$. Write down the vacuum Maxwell equations and show that they imply

$$\mathbf{k}\cdot\mathbf{E}_0 = 0\,, \qquad \mathbf{k}\cdot\mathbf{B}_0 = 0\,, \qquad \omega\mathbf{B}_0 = \mathbf{k}\times\mathbf{E}_0\,.$$

Show also that $|\mathbf{k}| = \omega/c$, where c is the speed of light.

(ii) State the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at the surface S of a perfect conductor. Let σ be the surface charge density and $\mathbf{s}$ the surface current density on S. How are σ and $\mathbf{s}$ related to $\mathbf{E}$ and $\mathbf{B}$?

A plane electromagnetic wave is incident from the half-space $x < 0$ upon the surface $x = 0$ of a perfectly conducting medium in $x > 0$. Given that the electric and magnetic fields of the incident wave take the form $(*)$ with

$$\mathbf{k} = k(\cos\theta, \sin\theta, 0) \qquad\qquad (0 < \theta < \pi/2)$$

and

$$\mathbf{E}_0 = \lambda\,(-\sin\theta, \cos\theta, 0)\,,$$

find $\mathbf{B}_0$.

Reflection of the incident wave at $x = 0$ produces a reflected wave with electric field

$$\mathbf{E}_0' \, e^{i\left(\mathbf{k}'\cdot\mathbf{r} - \omega t\right)}$$

with

$$\mathbf{k}' = k(-\cos\theta, \sin\theta, 0).$$

By considering the boundary conditions at $x = 0$ on the total electric field, show that

$$\mathbf{E}_0' = -\lambda\,(\sin\theta, \cos\theta, 0)\,.$$

Show further that the electric charge density on the surface $x = 0$ takes the form

$$\sigma = \sigma_0 e^{ik(y\sin\theta - ct)}$$

for a constant σ_0 that you should determine. Find the magnetic field of the reflected wave and hence the surface current density $\mathbf{s}$ on the surface $x = 0$.

6D Dynamics of Differential Equations

(i) What is a *Liapunov function*?

Consider the second order ODE

$$\dot{x} = y \ , \quad \dot{y} = -y - \sin^3 x \, .$$

By finding a suitable Liapunov function of the form $V(x, y) = f(x) + g(y)$, where f and g are to be determined, show that the origin is asymptotically stable. Using your form of V, find the greatest value of y_0 such that a trajectory through $(0, y_0)$ is guaranteed to tend to the origin as $t \to \infty$.

[*Any theorems you use need not be proved but should be clearly stated.*]

(ii) Explain the use of the stroboscopic method for investigating the dynamics of equations of the form $\ddot{x} + x = \epsilon f(x, \dot{x}, t)$, when $|\epsilon| \ll 1$. In particular, for $x = R\cos(t+\theta)$, $\dot{x} = -R\sin(t+\theta)$ derive the equations, correct to order ϵ,

$$\dot{R} = -\epsilon\langle f \sin(t+\theta)\rangle \ , \quad R\dot{\theta} = -\epsilon\langle f \cos(t+\theta)\rangle, \tag{$*$}$$

where the brackets denote an average over the period of the unperturbed oscillator.

Find the form of the right hand sides of these equations explicitly when $f = \Gamma x^2 \cos t - 3qx$, where $\Gamma > 0$, $q \neq 0$. Show that apart from the origin there is another fixed point of $(*)$, and determine its stability. Sketch the trajectories in (R, θ) space in the case $q > 0$. What do you deduce about the dynamics of the full equation?

[*You may assume that* $\langle \cos^2 t \rangle = \frac{1}{2}$, $\langle \cos^4 t \rangle = \frac{3}{8}$, $\langle \cos^2 t \sin^2 t \rangle = \frac{1}{8}$.]

7H Geometry of Surfaces

(i) What are geodesic polar coordinates at a point P on a surface M with a Riemannian metric ds^2 ?

Assume that

$$ds^2 = dr^2 + H(r, \theta)^2 d\theta^2,$$

for geodesic polar coordinates r, θ and some function H. What can you say about H and dH/dr at $r = 0$?

(ii) Given that the Gaussian curvature K may be computed by the formula $K = -H^{-1}\partial^2 H/\partial r^2$, show that for small R the area of the geodesic disc of radius R centred at P is

$$\pi R^2 - (\pi/12)KR^4 + a(R),$$

where $a(R)$ is a function satisfying $\lim\limits_{R \to 0} a(R)/R^4 = 0$.

8F Graph Theory

(i) State and prove a result of Euler relating the number of vertices, edges and faces of a plane graph. Use this result to exhibit a non-planar graph.

(ii) State the vertex form of Menger's Theorem and explain how it follows from an appropriate version of the Max-flow-min-cut Theorem. Let $k \geqslant 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

9F Coding and Cryptography

(i) Answer the following questions briefly but clearly.

(a) How does coding theory apply when the error rate $p > 1/2$?

(b) Give an example of a code which is not a linear code.

(c) Give an example of a linear code which is not a cyclic code.

(d) Give an example of a general feedback register with output k_j, and initial fill $(k_0, k_1, \ldots, k_N)$, such that

$$(k_n, k_{n+1}, \ldots, k_{n+N}) \neq (k_0, k_1, \ldots, k_N)$$

for all $n \geqslant 1$.

(e) Explain why the original Hamming code can not always correct two errors.

(ii) Describe the Rabin–Williams scheme for coding a message x as x^2 modulo a certain N. Show that, if N is chosen appropriately, breaking this code is equivalent to factorising the product of two primes.

10I Algorithms and Networks

(i) Consider a network with node set N and set of directed arcs A equipped with functions $d^+ : A \to \mathbb{Z}$ and $d^- : A \to \mathbb{Z}$ with $d^- \leqslant d^+$. Given $u : N \to \mathbb{R}$ we define the differential $\Delta u : A \to \mathbb{R}$ by $\Delta u(j) = u(i') - u(i)$ for $j = (i, i') \in A$. We say that Δu is a feasible differential if

$$d^-(j) \leqslant \Delta u(j) \leqslant d^+(j) \text{ for all } j \in A.$$

Write down a necessary and sufficient condition on d^+, d^- for the existence of a feasible differential and prove its necessity.

Assuming Minty's Lemma, describe an algorithm to construct a feasible differential and outline how this algorithm establishes the sufficiency of the condition you have given.

(ii) Let $E \subseteq S \times T$, where S, T are finite sets. A *matching* in E is a subset $M \subseteq E$ such that, for all $s, s' \in S$ and $t, t' \in T$,

$$\begin{aligned} (s,t), (s',t) \in M \quad &\text{implies} \quad s = s' \\ (s,t), (s,t') \in M \quad &\text{implies} \quad t = t' \,. \end{aligned}$$

A matching M is maximal if for any other matching M' with $M \subseteq M'$ we must have $M = M'$. Formulate the problem of finding a maximal matching in E in terms of an optimal distribution problem on a suitably defined network, and hence in terms of a standard linear optimization problem.

[*You may assume that the optimal distribution subject to integer constraints is integer-valued.*]

11I Principles of Statistics

(i) Outline briefly the Bayesian approach to hypothesis testing based on Bayes factors.

(ii) Let Y_1, Y_2 be independent random variables, both uniformly distributed on $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$. Find a minimal sufficient statistic for θ. Let $Y_{(1)} = \min\{Y_1, Y_2\}$, $Y_{(2)} = \max\{Y_1, Y_2\}$. Show that $R = Y_{(2)} - Y_{(1)}$ is ancilliary and explain why the Conditionality Principle would lead to inference about θ being drawn from the conditional distribution of $\frac{1}{2}\{Y_{(1)} + Y_{(2)}\}$ given R. Find the form of this conditional distribution.

12I Computational Statistics and Statistical Modelling

(i) Suppose $Y_1, \ldots, Y_n$ are independent Poisson variables, and

$$\mathbb{E}(Y_i) = \mu_i\,, \qquad \log \mu_i = \alpha + \beta t_i\,, \qquad \text{for} \quad i = 1, \ldots, n\,,$$

where α, β are two unknown parameters, and $t_1, \ldots, t_n$ are given covariates, each of dimension 1. Find equations for $\hat{\alpha}, \hat{\beta}$, the maximum likelihood estimators of α, β, and show how an estimate of $\text{var}(\hat{\beta})$ may be derived, quoting any standard theorems you may need.

(ii) By 31 December 2001, the number of new vCJD patients, classified by reported calendar year of onset, were

$$8, 10, 11, 14, 17, 29, 23$$

for the years

$$1994, \ldots, 2000 \quad \text{respectively.}$$

Discuss carefully the (slightly edited) R output for these data given below, quoting any standard theorems you may need.

```
> year
year
[1] 1994 1995 1996 1997 1998 1999 2000
> tot
[1]  8 10 11 14 17 29 23
>first.glm _ glm(tot ~ year, family = poisson)
> summary(first.glm)
Call:
glm(formula = tot ~ year, family = poisson)
Coefficients
                    Estimate Std. Error z value Pr(>|z|)
      (Intercept) -407.81285   99.35366  -4.105 4.05e-05
      year           0.20556    0.04973   4.133 3.57e-05


      (Dispersion parameter for poisson family taken to be 1)


      Null deviance: 20.7753  on 6  degrees of freedom
  Residual deviance:  2.7931  on 5  degrees of freedom


  Number of Fisher Scoring iterations: 3
```

13C Foundations of Quantum Mechanics

(i) Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture. Explain how the two pictures provide equivalent descriptions of observable results.

Derive the equation of motion for an operator in the Heisenberg picture.

(ii) For a particle moving in one dimension, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, and the state vector is $|\Psi\rangle$. Eigenstates of $\hat{x}$ and $\hat{p}$ satisfy

$$\langle x|p\rangle = \left(\frac{1}{2\pi\hbar}\right)^{1/2} e^{ipx/\hbar}, \quad \langle x|x'\rangle = \delta(x-x'), \quad \langle p|p'\rangle = \delta(p-p')\,.$$

Use standard methods in the Dirac formalism to show that

$$\begin{aligned}\langle x|\hat{p}|x'\rangle &= -i\hbar\frac{\partial}{\partial x}\delta(x-x')\\ \langle p|\hat{x}|p'\rangle &= i\hbar\frac{\partial}{\partial p}\delta(p-p')\,.\end{aligned}$$

Calculate $\langle x|\hat{H}|x'\rangle$ and express $\langle x|\hat{p}|\Psi\rangle$, $\langle x|\hat{H}|\Psi\rangle$ in terms of the position space wave function $\Psi(x)$.

Compute the momentum space Hamiltonian for the harmonic oscillator with potential $V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$.

14C Quantum Physics

(i) A system of N *distinguishable* non-interacting particles has energy levels E_i with degeneracy g_i, $1 \leqslant i < \infty$, for each particle. Show that in thermal equilibrium the number of particles N_i with energy E_i is given by

$$N_i = g_i e^{-\beta(E_i - \mu)},$$

where β and μ are parameters whose physical significance should be briefly explained.

A gas comprises a set of atoms with non-degenerate energy levels E_i, $1 \leqslant i < \infty$. Assume that the gas is dilute and the motion of the atoms can be neglected. For such a gas the atoms can be treated as distinguishable. Show that when the system is at temperature T, the number of atoms N_i at level i and the number N_j at level j satisfy

$$\frac{N_i}{N_j} = e^{-(E_i - E_j)/kT} \quad ,$$

where k is Boltzmann's constant.

(ii) A system of bosons has a set of energy levels W_a with degeneracy f_a, $1 \leqslant a < \infty$, for each particle. In thermal equilibrium at temperature T the number n_a of particles in level a is

$$n_a = \frac{f_a}{e^{(W_a - \mu)/kT} - 1} \quad .$$

What is the value of μ when the particles are photons?

Given that the density of states $\rho(\omega)$ for photons of frequency ω in a cubical box of side L is

$$\rho(\omega) = L^3 \frac{\omega^2}{\pi^2 c^3} \quad ,$$

where c is the speed of light, show that at temperature T the density of photons in the frequency range $\omega \to \omega + d\omega$ is $n(\omega)d\omega$ where

$$n(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1} \quad .$$

Deduce the energy density, $\epsilon(\omega)$, for photons of frequency ω .

The cubical box is occupied by the gas of atoms described in Part (i) in the presence of photons at temperature T . Consider the two atomic levels i and j where $E_i > E_j$ and $E_i - E_j = \hbar\omega$. The rate of spontaneous photon emission for the transition $i \to j$ is A_{ij}. The rate of absorption is $B_{ji}\,\epsilon(\omega)$ and the rate of stimulated emission is $B_{ij}\,\epsilon(\omega)$. Show that the requirement that these processes maintain the atoms and photons in thermal equilibrium yields the relations

$$B_{ij} = B_{ji}$$

and

$$A_{ij} = \left(\frac{\hbar\omega^3}{\pi^2 c^3}\right) B_{ij} \quad .$$

15A General Relativity

(i) What is a "stationary" metric? What distinguishes a stationary metric from a "static" metric?

A Killing vector field K^a of a metric g_{ab} satisfies

$$K_{a;b} + K_{b;a} = 0.$$

Show that this is equivalent to

$$g_{ab,c}K^c + g_{ac}K^c{}_{,b} + g_{cb}K^c{}_{,a} = 0.$$

Hence show that a constant vector field K^a with one non-zero component, K^4 say, is a Killing vector field if g_{ab} is independent of x^4.

(ii) Given that K^a is a Killing vector field, show that $K_a u^a$ is constant along the geodesic worldline of a massive particle with 4-velocity u^a. Hence find the energy ε of a particle of unit mass moving in a static spacetime with metric

$$ds^2 = h_{ij}dx^i dx^j - e^{2U}dt^2,$$

where h_{ij} and U are functions only of the space coordinates x^i. By considering a particle with speed small compared with that of light, and given that $U \ll 1$, show that $h_{ij} = \delta_{ij}$ to lowest order in the Newtonian approximation, and that U is the Newtonian potential.

A metric admits an antisymmetric tensor Y_{ab} satisfying

$$Y_{ab;c} + Y_{ac;b} = 0.$$

Given a geodesic $x^a(\lambda)$, let $s_a = Y_{ab}\,\dot{x}^b$. Show that s_a is parallelly propagated along the geodesic, and that it is orthogonal to the tangent vector of the geodesic. Hence show that the scalar

$$\phi = s^a s_a$$

is constant along the geodesic.

16E Theoretical Geophysics

(i) Explain briefly what is meant by the concepts of hydrostatic equilibrium and the buoyancy frequency. Evaluate an expression for the buoyancy frequency in an incompressible inviscid fluid with stable density profile $\rho(z)$.

(ii) Explain briefly what is meant by the Boussinesq approximation.

Write down the equations describing motions of small amplitude in an incompressible, stratified, Boussinesq fluid of constant buoyancy frequency.

Derive the resulting dispersion relationship for plane wave motion. Show that there is a maximum frequency for the waves and explain briefly why this is the case.

What would be the response to a solid body oscillating at a frequency in excess of the maximum?

17B Mathematical Methods

(i) Explain how to solve the Fredholm integral equation of the second kind,

$$f(x) = \mu \int_a^b K(x,t) f(t) dt \; + g(x) \; ,$$

in the case where $K(x,t)$ is of the separable (degenerate) form

$$K(x,t) = a_1(x) b_1(t) + a_2(x) b_2(t) \; .$$

(ii) For what values of the real constants λ and A does the equation

$$u(x) = \lambda \sin x \; + \; A \int_0^\pi (\cos x \cos t + \cos 2x \cos 2t) \, u(t) dt$$

have (a) a unique solution, (b) no solution?

18B Nonlinear Waves and Integrable Systems

(i) Write down the shock condition associated with the equation

$$\rho_t + q_x = 0,$$

where $q = q(\rho)$. Discuss briefly two possible heuristic approaches to justifying this shock condition.

(ii) According to shallow water theory, waves on a uniformly sloping beach are described by the equations

$$\begin{aligned} &\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0, \quad h = \alpha x + \eta, \end{aligned}$$

where α is the constant slope of the beach, g is the gravitational acceleration, $u(x,t)$ is the fluid velocity, and $\eta(x,t)$ is the elevation of the fluid surface above the undisturbed level.

Find the characteristic velocities and the characteristic form of the equations.

What are the Riemann variables and how do they vary with t on the characteristics?

19E Numerical Analysis

(i) Explain briefly what is meant by the *convergence* of a numerical method for ordinary differential equations.

(ii) Suppose the sufficiently-smooth function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ obeys the Lipschitz condition: there exists $\lambda > 0$ such that

$$||\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{y})|| \leqslant \lambda ||\mathbf{x} - \mathbf{y}||, \qquad \mathbf{x},\mathbf{y} \in \mathbb{R}^d, t \geqslant 0\,.$$

Prove from first principles, without using the Dahlquist equivalence theorem, that the trapezoidal rule

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n,\mathbf{y}_n) + \mathbf{f}(t_{n+1},\mathbf{y}_{n+1})]$$

for the solution of the ordinary differential equation

$$\mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \quad t \geqslant 0, \quad \mathbf{y}(0) = \mathbf{y}_0,$$

converges.