MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 1.30 to 4.30

PAPER 9

COMPLEX DYNAMICS

*Attempt **QUESTIONS 1 and 2**, and **TWO** other questions.*

Questions 1 and 2 each carry 30% of the total marks;
Questions 3, 4 and 5 each carry 20% of the total marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on the partitioning of the complex sphere into the Fatou and Julia sets of a rational map. You should briefly consider the case of rational maps of degree one, discuss the main properties of the Fatou and Julia sets, and mention some (or all) of the following topics: equicontinuity, the Arzéla-Ascoli Theorem, and normal families of meromorphic maps.

2 Give an informal sketch of the Mandelbrot set, and briefly discuss its relevance to the iteration of quadratic polynomials.

Let $P(z) = z - z^2$. Show that P is conjugate to the map $z \mapsto z^2 + c$ for one, and only one, value of c and describe where this c lies in the Mandelbrot set.

For each fixed point ζ of P in $\mathbb{C}_\infty$ decide whether or not there is a neighbourhood $\mathcal{N}$ of ζ such that the iterates P^n of P converge uniformly to ζ on $\mathcal{N}$. Let $\Delta = \{z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{1}{2}\}$. By considering the iterates P^n on Δ, or otherwise, show that Δ lies in the Fatou set $F(P)$ of P. Show that the component F_Δ of $F(P)$ that contains Δ is a completely invariant component of $F(P)$, and that $P^n \to 0$ on F_Δ. Show that 0 lies in the Julia set of P.

3 Define the chordal metric on the complex sphere $\mathbb{C}_\infty$. Show that a rational map $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is Lipschitz with respect to the chordal metric.

Let R and S be rational maps, each of degree at least two. Show that if R and S commute (that is, if $R\big(S(z)\big) = S\big(R(z)\big)$ for all z), then R and S have the same Fatou and Julia sets.
[You may use Montel's Theorem on three omitted values without proof providing that it is carefully stated.]

Give an example of rational maps R and S, each of degree one, which commute but which have different Julia sets.

Give an example of two rational maps, each of degree one, which have the same Julia sets but which do not commute.

Give an example of distinct rational maps R and S which commute, such that each has degree at least two, and such that neither is a power of z.

4 Let P be a non-constant polynomial. Show that if m divides n then the polynomial $P^m(z) - z$ divides $P^n(z) - z$.

The sequence $w_1, w_2, \ldots, w_n$ is a *cycle of length* n for P if these points are distinct, and if $P(w_j) = w_{j+1}$ for $j = 1, 2, \ldots, n$, where $w_{n+1} = w_1$. Show that if one point in a cycle of length n for P is an attracting fixed point of P^n, then so is every point in the cycle.

Let $P(z) = z^2 + c$. Show that there is a unique value of c such that P has no cycles of length two, and find this value of c. Comment on the four solutions of $P^2(z) = z$ for this value of c.

5 Define what is meant by a critical point, and the multiplicity of a critical point, of a rational map R. Show that if R has degree d, then R has exactly $2d - 2$ critical points in $\mathbb{C}_\infty$.

Show that if z_0 is a critical point of a polynomial P, then z_0 is a critical point of every iterate of P. Show also that if z_0 is a critical point of some iterate P^n, then z_0 is mapped to a critical point of P by some iterate of P.

Let $P(z) = z^2 + i$. Show that the finite critical point of P is in the Julia set of P.

END OF PAPER