UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2018 1:30 pm to 3:30 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Accretion on to compact objects

(a) Consider a disk around a black hole of mass M accreting at a constant rate $\dot{M}$. Recalling that the Schwarzchild radius is $2GM/c^2$ (where c is the speed of light), show that the energy released per unit time via accretion is $\sim \dot{M}c^2$.

(b) Next consider a magnetised neutron star of mass M exhibiting a dipole field. The star is encircled by a disk aligned with the dipole so that the magnetic field strength at the disk midplane is $B \sim \mu/r^3$, where μ is the dipole moment. The disk accretes at a rate $\dot{M} = 2\pi r \Sigma v$ (where Σ is the disk's surface density and v is the radial speed of the accretion flow) and its angular thickness $\epsilon = H/r$ is constant, where H is the disk scale height.

The star's magnetic field truncates the disk at a radius r_m. Suppose that the disk is disrupted when magnetic pressure $B^2/(8\pi)$ is greater than the 'ram pressure' $\frac{1}{2}\rho v^2$ of an accretion flow in radial free fall from infinity (where ρ is density). Hence show that

$$r_m \sim \left(\frac{\epsilon^2 \mu^4}{GM\dot{M}^2} \right)^{1/7}.$$

(c) The magnetic dipole rotates at a rate Ω_m. Near r_m the disk plasma will be accelerated to this rate and possibly flung from the system in what is termed 'propeller flow'. Give a condition on r_m for this to occur.

(d) Briefly comment on the long-term evolution of the neutron star due to its magnetic connection to the disk.

(e) The evolution of the disk is governed by the following equations

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r}\frac{\partial}{\partial r}\left[\left(\frac{dh}{dr} \right)^{-1} \frac{\partial \mathcal{G}}{\partial r} \right], \quad \mathcal{G} = -2\pi \overline{\nu} \Sigma r^3 \frac{d\Omega}{dr},$$

where $h(r)$ is the specific angular momentum, $\mathcal{G}$ the viscous torque, and $\overline{\nu}$ the turbulent kinematic viscosity (a constant). The disk receives mass at its outer boundary r_{out} at a rate $\dot{M}$, and receives a nonzero magnetic torque T_m at its inner radius $r = r_{\text{in}}$ (not necessarily r_m).

Assume the disk is in Keplerian rotation and in steady state. Find an expression for Σ in terms of $\overline{\nu}$ and the dimensionless parameter $\lambda = T_m/[h(r_{\text{in}})\dot{M}]$. Plot Σ as a function of r for $\lambda \gg 1$ and $\lambda \ll 1$. What regimes do these cases correspond to?

Suppose $r_{\text{in}} = r_m$ and the magnetic torque T_m is approximately $r^2B^2H/(4\pi)$ evaluated at r_m, i.e. the azimuthally and vertically averaged Maxwell stress. Using part (b), find an estimate of the value of λ.

2 Vertical structure of a slowly cooling disk

The governing equations of a thin astrophysical disk composed of ideal gas may be written as

$$D_t\rho = -\rho\nabla\cdot\mathbf{u}, \qquad D_t\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi,$$
$$\frac{1}{\gamma-1}\left(D_tP - \frac{\gamma P}{\rho}D_t\rho\right) = \mathcal{H}-\mathcal{C}, \qquad P = \frac{k}{\mu m_p}\rho T,$$

where $D_t = \partial_t + \mathbf{u}\cdot\nabla$ is the total derivative, and ρ, $\mathbf{u}$, P, and T are density, velocity, pressure, and temperature respectively, while γ, k, μ, and m_p are adiabatic index, Boltzmann's constant, molecular weight, and the mass of a proton. The heating and cooling rates are $\mathcal{H} = (9/4)\alpha\Omega P$ and $\mathcal{C} = AT^{1+\beta}$, where Ω is the disk's orbital frequency and α, A and β are positive constants. Finally, the gravitational potential of the central star is $\Phi = -MG/r$, where M is the star's mass, and r is spherical radius.

(a) Suppose the disk is thin. Give the approximate expression for Φ usually employed when describing thin disks.

(b) At a fixed cylindrical radius, write down the equations controlling the equilibrium vertical structure of the disk. Solve these equations, given that $T = T_0$ (a constant) at $z = 0$. Write your solutions for T, ρ, and P in terms of disk semi-thickness H defined so that there is vacuum for $|z| > H$.

(c) Suppose the equilibrium is slightly perturbed. Give an order of magnitude treatment showing that the timescale upon which vertical equilibrium is re-established is $\sim \Omega^{-1}$.

(d) Consider now a non-turbulent disk that is cooling on a very long timescale $\gg \Omega^{-1}$. Using part (c), argue that the following equations adequately describe the slow evolution of the disk's vertical structure:

$$\partial_z P = -\Omega^2 z\rho, \qquad \frac{1}{\gamma-1}\left(\partial_t P - \frac{\gamma P}{\rho}\partial_t\rho\right) = -AT^{1+\beta},$$

with $P = k\rho T/(\mu m_p)$.

(e) Change independent variables from (z, t) to (ξ, t), where the new similarity variable ξ is defined through $z = \xi\eta(t)$, with η yet to be determined but satisfying $\eta = 1$ at $t = 0$. Next assume the solution has the following form:

$$\rho = \widetilde{\rho}(\xi)/\eta(t), \qquad T = \widetilde{T}(\xi)\eta(t)^2, \qquad P = \widetilde{P}(\xi)\eta(t).$$

Show that

$$\eta = (1 + Ct)^{-1/(1+2\beta)},$$

where C is a positive constant. Write down a set of equations in ξ for $\widetilde{\rho}$, $\widetilde{T}$, and $\widetilde{P}$.

Using part (b), or otherwise, solve these equations given that at $t = 0$ the midplane temperature is T_0. Find an expression for $T(z, t)$ in terms of a time-dependent semi-thickness $H(t)$, the form of which you should give. Briefly describe in words how the disk evolves.

3 Oscillatory convection in protoplanetary disks

Consider an accretion disk with a radially varying thermal structure. A local patch of the disk may be represented in the shearing sheet by the governing equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla P - 2\Omega\,\mathbf{e}_z \times \mathbf{u} + 3\Omega^2 x\,\mathbf{e}_x - N^2\,\theta\,\mathbf{e}_x,$$
$$\partial_t \theta + \mathbf{u}\cdot\nabla\theta = u_x + \xi\nabla^2\theta, \qquad \nabla\cdot\mathbf{u} = 0,$$

where $\mathbf{u}$, P, and ρ_0 are the velocity, pressure, and density, respectively. In addition, θ is the 'potential temperature' perturbation, and ξ is thermal diffusivity. Finally, Ω is the orbital frequency of the shearing sheet and N is the radial buoyancy frequency of the gas. All of ρ_0, Ω, N, and ξ are constants.

(a) Derive the vorticity equation:

$$\partial_t \boldsymbol{\omega} + \nabla\cdot\mathbf{T} = -N^2\nabla\theta\times\mathbf{e}_x,$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u} + 2\,\Omega\,\mathbf{e}_z$, and the components of the tensor $\mathbf{T}$ are given by $T_{ij} = u_i\omega_j - \omega_i u_j$. What can be said about the x-component of vorticity?

[*You may need the identities:*

$$\tfrac{1}{2}\nabla\mathbf{A}^2 = \mathbf{A}\cdot\nabla\mathbf{A} + \mathbf{A}\times\nabla\times\mathbf{A},$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + \mathbf{B}\cdot\nabla\mathbf{A} - \mathbf{A}\cdot\nabla\mathbf{B}.]$$

(b) Demonstrate that a steady solution to the governing equations is $\mathbf{u} = -(3/2)\Omega\,x\,\mathbf{e}_y$, $\theta = 0$, and $P =$ a constant.

(c) Perturb this steady state with disturbances $\mathbf{u}'$, P', and θ' proportional to $\exp(st + \mathrm{i}kz)$, where s is a (possibly complex) growth rate and k a (real) vertical wavenumber.

Write down the linearised equations governing the perturbations. Show that $u_z' = P' = 0$. Hence derive the dispersion relation:

$$s^3 + \beta s^2 + (N^2 + \Omega^2)s + \beta\Omega^2 = 0,$$

where $\beta = \xi k^2 > 0$.

(d) Suppose that $n^2 = N^2/\Omega^2$ and $|n^2| \ll 1$. Consider the expansion, $s = s_0 + s_1 n^2 + \dots$.

Show that the dispersion relation supports two epicycles with $s_0 = \pm\mathrm{i}\Omega$ plus a third energy mode that decays at a rate you must find.

Discard the third mode and find the next order correction s_1 for the two epicyclic modes. Show that their instability criterion is $N^2 < 0$ and that their maximum growth rate to leading order is $-(1/4)N^2/\Omega$.

END OF PAPER