UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 9 June, 2014 1:30 pm to 3:30 pm

PAPER 57

DYNAMICS OF ASTROPHYSICAL DISCS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Photoevaporation of a protoplanetary disk

The ideal gas in the surface layer of a thin disk becomes irradiated by its central star so that it attains a temperature of 10^4 K, which corresponds to a sound speed $c \approx 10$ km s^{-1}. You may assume that such gas is in circular Keplerian orbits while this occurs.

(a) At the gravitational radius r_g the sound speed of the gas in the surface layer equals its orbital speed. Show that $r_g = GM/c^2$, and explain why the gas beyond this radius is gravitationally unbound and may leave the disk in a wind. (Here M is the mass of the star.)

If $M_\odot \sim 2 \times 10^{33}$ g, $G \sim 7 \times 10^{-8}$ cm^3 g^{-1} s^{-2}, and 1 AU $\sim 10^{13}$ cm, give an estimate for r_g in the protosolar nebula (in AU).

(b) Mass and angular momentum conservation in the photoevaporated disk are governed by the equations

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r}\frac{\partial \mathcal{F}_{\rm acc}}{\partial r} - \mathcal{W}, \qquad \mathcal{F}_{\rm acc} = 6\pi r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\nu\Sigma\right)$$

where $\mathcal{F}_{\rm acc}$ is the radial mass accretion rate and $\mathcal{W}$ is the mass loss rate due to the photoevaporative wind. A functional form for $\mathcal{W}$ is

$$\mathcal{W} = \begin{cases} 0, & r < r_g, \\ \mathcal{W}_0\,(r/r_g)^{-5/2,}, & r \geqslant r_g, \end{cases}$$

for $\mathcal{W}_0$ a constant $\sim 10^{-12}$ g cm^{-2} s^{-1}.

Show that the total mass loss rate from the disk due to the wind is $\dot{M}_w = 4\pi\mathcal{W}_0 r_g^2$. Hence give an estimate for the dispersal time, in years, of a protosolar nebula with initial mass $10^{-3}M_\odot$.

(c) Consider a steady state scenario in which mass is fed into the disk at a radius $\gg r_g$ at a rate $\dot{M}_{\rm d} \geqslant \dot{M}_w$. Obtain an expression for $\mathcal{F}_{\rm acc}$ in terms of $\dot{M}_{\rm d}$, $\dot{M}_w$, and r_g, then sketch its form as a function of r. Give a physical reason for its shape.

Compute the corresponding surface density profile Σ for fixed ν, assuming there is no viscous stress at the surface of the star (which we may assume is at $r = 0$). Sketch the form of Σ for the special case $\dot{M}_d = \dot{M}_w$.

Discuss what might happen to the inner disk at late times when the wind mass-loss rate approaches the accretion rate.

2 Gravitational instability in a layer of dust

The dynamics of a 2D layer of dust in a gaseous accretion disk can be modelled in the shearing sheet approximation via the following equations

$$\partial_t \sigma + \mathbf{v} \cdot \nabla \sigma = -\sigma \nabla \cdot \mathbf{v},$$
$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\Omega \mathbf{e}_z \times \mathbf{v} = -\nabla \Phi_t - \nabla \Phi_{sg} - \frac{1}{\sigma} \nabla P - \epsilon \Omega (\mathbf{v} - \mathbf{U}_g).$$

Here σ, $\mathbf{v}$, P, and Φ_{sg} denote the surface density, 2D velocity, vertically integrated pressure, and gravitational potential of the dust, while $\Phi_t = -(3/2)\Omega^2 x^2$ is the tidal potential, $\mathbf{U}_g = (-3/2)\Omega\, x\, \mathbf{e}_y$ is the velocity of the disk's gas, and ϵ measures the strength of the gas drag. We assume that the dust does not alter the motion of the gas. Finally, Φ_{sg} can be obtained from Poisson's equation

$$\nabla^2 \Phi_{sg} = 4\pi G \sigma\, \delta(z),$$

and the dust pressure from $P = c^2 \sigma$, where c is the rms speed of the dust particles.

(a) Consider small axisymmetric perturbations σ', $\mathbf{v}'$, Φ'_{sg} around the equilibrium $\mathbf{v} = \mathbf{U}_g$ with $\sigma = \sigma_0$ a constant. Assuming the perturbations are $\propto \mathrm{e}^{ikx+st}$, derive the following dispersion relation:

$$s^3 + 2s^2 \epsilon \Omega + (\varpi^2 + \epsilon^2 \Omega^2) s + \epsilon \Omega (\varpi^2 - \Omega^2) = 0,$$

where

$$\varpi^2 = \Omega^2 - 2\pi G \sigma_0 |k| + c^2 k^2.$$

You may assume that for such disturbances, $\Phi'_{sg} = -(2\pi G/|k|)\, \sigma'$.

(b) Show that when the dust particles decouple from the gas ($\epsilon = 0$), the dust is unstable when $\varpi^2 < 0$. Confirm that instability is assured when $Q < 1$, where $Q = c\Omega/(\pi G \sigma_0)$.

(c) Suppose $\varpi^2 > 0$ and that the particles are only weakly coupled to the gas, i.e. $0 < \epsilon \ll 1$. By making an expansion $s = s_0 + s_1 \epsilon + \ldots$, show that there exist two density waves that are weakly damped.

(d) For small ϵ, the third 'secular' mode has an expansion $s = s_1 \epsilon + \mathcal{O}(\epsilon^2)$. Show that this mode grows when $\varpi^2 < \Omega^2$. Hence demonstrate that the dust is always unstable for sufficiently small k.

(e) If the dust layer has radial extent L, derive the rough instability criterion $Q \lesssim L/H$, where $H = c/\Omega$ is the scale height.

(f) Suppose the background fluid flow is turbulent. How do you think this will alter the onset of instability?

3 The vertically stratified MRI

A fully ionised gas in the vertically stratified shearing sheet obeys the momentum and induction equations,

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + 2\Omega\mathbf{e}_z \times \mathbf{v} = -\nabla\Phi_t - \frac{1}{\rho}\nabla\Psi + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi\rho},$$
$$\partial_t \mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{v} - (\nabla\cdot\mathbf{v})\mathbf{B},$$

where $\Phi_t = -(3/2)\Omega^2 x^2 + (1/2)\Omega^2 z^2$ is the tidal potential and Ψ is the combined gas and magnetic pressure. Suppose the box is penetrated by a uniform magnetic field $B_0\mathbf{e}_z$, and exhibits a vertical density profile of $\rho = \rho_0 h$, where ρ_0 is the midplane density and $h = h(z/H)$ is a dimensionless function. The equilibrium velocity is $\mathbf{v}_0 = -(3/2)\Omega\, x\mathbf{e}_y$.

The equilibrium state is perturbed by a disturbance of the following form

$$\mathbf{v}' = \mathrm{e}^{st}\, F\, \left(v'_x,\, v'_y,\, 0\right),$$
$$\mathbf{B}' = B_0\, \mathrm{e}^{st} \left(\frac{1}{\mathrm{i}k}\frac{dF}{dz}\right) \left(b'_x,\, b'_y,\, 0\right),$$

where v'_x, v'_y, b'_x, and b'_y are (complex) constants, $F = F(z/H)$ is a (real) dimensionless function, bounded on the surface of the disk, and k is a (real) wavenumber. We do not perturb the density or pressure.

(a) By determining the solvability condition for the horizontal components of the linearised momentum and induction equation, show that if F satisfies

$$\frac{d^2F}{dz^2} + k^2\, h\, F = 0 \qquad (**)$$

then we recover the incompressible MRI dispersion relation

$$s^4 + (\Omega^2 + 2v_A^2 k^2)s^2 + v_A^2 k^2 (v_A^2 k^2 - 3\Omega^2) = 0,$$

where $v_A = B_0/\sqrt{4\pi\rho}$.

(b) Suppose the vertical structure of the disk is given by $h = \mathrm{sech}^2(z/H)$. Verify that Eq. $(**)$ admits discrete solutions

$$F_n = P_n[\tanh(z/H)], \qquad k_n = \sqrt{n^2+n}/H,$$

where P_n is a Legendre polynomial of integer order n. Hence show that the MRI is stabilised when $\beta < 4/3$, where $\beta = 2\Omega^2 H^2/v_A^2$. Give a physical explanation for why the disk is stabilised for sufficiently strong magnetic fields.

(c) If $\beta = 24$ what is n for the fastest growing mode?

[*Hint: The Legendre equation is*

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dF}{d\xi}\right] + \lambda F = 0$$

and admits solutions that are bounded at $\xi = \pm 1$ *if* $\lambda = n(n+1)$. *These solutions are the Legendre polynomials,* P_n.]

END OF PAPER