A1f                    **Vectors and Matrices: Study Sheet**                    Michaelmas 2016

## Exercises on Suffix Notation and the Summation Convention

*These exercises are designed to help to familiarise students with this topic and are not intended as part of the Examples Sheets for supervision.*

*Free suffices and dummy suffices are assumed to range/sum through $1, 2, \ldots, n$ unless stated otherwise.*

1. Write each of the following using the summation convention:

   (a) $d\theta = \frac{\partial \theta}{\partial x_1}\, dx_1 + \frac{\partial \theta}{\partial x_2}\, dx_2 + \ldots + \frac{\partial \theta}{\partial x_n}\, dx_n$;

   (b) $\frac{d\overline{x}_k}{dt} = \frac{\partial \overline{x}_k}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \overline{x}_k}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial \overline{x}_k}{\partial x_n}\frac{dx_n}{dt}$;

   (c) $(x_1)^2 + (x_2)^2 + (x_3)^2 + \ldots + (x_n)^2$;

   (d) $\displaystyle\sum_{p=1}^{3}\sum_{q=1}^{3} g_{pq}dx_p dx_q$.

2. Write the following expressions in full:

   (a) $a_{jk}x_k$;

   (b) $A_{pq}A_{qr}$;

   (c) $\frac{\partial}{\partial x_k}(\sqrt{g}A_k)$ with $n = 3$;

   (d) $A_{jk}B_{pk}C_j$ with $n = 2$.

3. Write each of the following using the summation convention.

   (a) $a_1 x_1 x_3 + a_2 x_2 x_3 + \ldots + a_n x_n x_3$ ;

   (b) $A_{21}B_1 + A_{22}B_2 + A_{23}B_3 + \ldots + A_{2n}B_n$ ;

   (c) $g_{21}g_{11} + g_{22}g_{21} + g_{23}g_{31} + g_{24}g_{41}$ .

4. If $n = 2$, write out the system of equations represented by $a_{pq}x_q = b_p$.

*From now on assume $n = 3$ unless stated otherwise.*

5. If
$$
\begin{array}{rclcrclcrcl}
a_{11} & = & 1, & a_{12} & = & -1, & a_{13} & = & 0, \\
a_{21} & = & -2, & a_{22} & = & 3, & a_{23} & = & 1, \\
a_{31} & = & 2, & a_{32} & = & 0, & a_{33} & = & 4,
\end{array}
$$
   show that
$$
\begin{array}{rclcrclcrcl}
a_{ii} & = & 8, & a_{i1}a_{i2} & = & -7, & a_{i2}a_{i3} & = & 3, \\
a_{1i}a_{2i} & = & -5, & a_{2i}a_{3i} & = & 0, & a_{i1}a_{2i} & = & -6.
\end{array}
$$

6. If the numbers $a_{ij}$ are as given in question (5) above and if $b_1 = 1$, $b_2 = -1$ and $b_3 = 4$, show that
$$
a_{1i}b_i = 2, \quad a_{j1}b_j = 11, \quad a_{ji}a_{i1}b_j = 49.
$$

   *Hint:* for the last part, first evaluate $a_{j1}b_j, a_{j2}b_j$ and $a_{j3}b_j$.

7. Show that $\delta_{ij}b_j = \delta_{ji}b_j = b_i$ .

8. If the numbers $a_{ij}$ are as given in (5) above, evaluate

   (a) $a_{1j}\delta_{1j}$,   (b) $a_{12}\delta_{ii}$,   (c) $a_{1i}a_{2k}\delta_{ik}$ .

9. Assume the suffix $i$ takes all integral values from 0 to $\infty$, and that $a_i$ and $b_i$ are defined by
$$
a_i = x^i, \quad \text{and} \quad b_i = \frac{1}{i!}
$$
   respectively, where $x$ is a constant and, by definition, $0! = 1$. Show that
$$
a_i\, b_i = e^x .
$$

10. If the quantities $e_{ij}$, and $e'_{ij}$ satisfy the relation

$$l_{\ell i}\, l_{mj}\, e_{ij} = e'_{\ell m}\,,$$

and if

$$l_{ki}\, l_{kj} = \delta_{ij}\,,$$

show that

$$e_{ij} = l_{\ell i} l_{mj} e'_{\ell m}\,.$$

*Hint:* multiply the first equation by $l_{\ell p} l_{mq}$.

11. Without referring to your notes, prove that

$$\delta_{ij}\epsilon_{ijk} = 0\,, \quad \epsilon_{ijk}\epsilon_{rjk} = 2\delta_{ir} \quad \text{and} \quad \epsilon_{ijk}\epsilon_{ijk} = 6\,.$$

## Answers

1. (a) $d\theta = \frac{\partial\theta}{\partial x_j}\, dx_j$;

   (b) $\frac{d\overline{x}_k}{dt} = \frac{\partial\overline{x}_k}{\partial x_m}\, \frac{dx_m}{dt}$;

   (c) $x_k x_k$;

   (d) $g_{pq} dx_p dx_q$ with $n = 3$.

2. (a)
$$\sum_{k=1}^{n} a_{jk}x_k = a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n\,;$$

   (b)
$$\sum_{q=1}^{n} A_{pq}A_{qr} = A_{p1}A_{1r} + A_{p2}A_{2r} + \ldots + A_{pn}A_{nr}\,;$$

   (c)
$$\sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\sqrt{g}A_k) = \frac{\partial}{\partial x_1}(\sqrt{g}A_1) + \frac{\partial}{\partial x_2}(\sqrt{g}A_2) + \frac{\partial}{\partial x_3}(\sqrt{g}A_3)\,;$$

   (d)
$$\sum_{j,k=1}^{2} A_{jk}B_{pk}C_j = \sum_{k=1}^{2}(A_{1k}B_{pk}C_1 + A_{2k}B_{pk}C_2)$$
$$= A_{11}B_{p1}C_1 + A_{12}B_{p2}C_1 + A_{21}B_{p1}C_2 + A_{22}B_{p2}C_2\,.$$

3. (a) $a_k x_k x_3$;

   (b) $A_{2j}B_j$;

   (c) $g_{2p}g_{p1}$ with $n = 4$.

8. (a) 1;

   (b) -3;

   (c) -5.