Mathematical Tripos Part IA **Michaelmas Term 2018**
Vectors and Matrices **Dr. J.M. Evans**

Example Sheet 1

1. Let D be the interior of the circle $|z - 1 - i| = 1$. Show, by using suitable inequalities for $|z_1 \pm z_2|$, that if $z \in D$ then
$$\sqrt{5} - 1 < |z - 3| < \sqrt{5} + 1\,.$$
Obtain the same result geometrically [start by considering the line through the centre of the circle and the point 3].

2. Given $|z| = 1$ and $\arg z = \theta$, find both algebraically and geometrically the modulus-argument forms of
$$\text{(i)}\quad 1 + z, \qquad \text{(ii)}\quad 1 - z\,.$$
Show that the locus of w as z varies with $|z| = 1$, where w is given by
$$w^2 = \frac{1 - z}{1 + z}\,,$$
is a pair of straight lines.

3. Consider a triangle in the complex plane with vertices at 0, z_1 and z_2. Write down an expression for the general point on the median through z_1, and a similar expression for the general point on the median through z_2. Show that the three medians of the triangle are concurrent.

4. Express
$$I = \frac{z^5 - 1}{z - 1}$$
as a polynomial in z. By considering the complex fifth root of unity ω, obtain the four factors of I linear in z. Hence write I as the product of two real quadratic factors. By considering the term in z^2 in the identity so obtained for I, show that
$$4 \cos\frac{\pi}{5}\,\sin\frac{\pi}{10} = 1\ .$$

5. Find all complex numbers z that satisfy $\sin z = 2$.

6. (a) Let $z, a, b \in \mathbb{C}$ $(a \neq b)$ correspond to points P, A, B in the Argand diagram. Let C_λ be the locus of P defined by
$$PA/PB = \lambda\,,$$
where λ is a fixed real positive constant. Show that C_λ is a circle if $\lambda \neq 1$, and find its centre and radius. What happens if $\lambda = 1$?

   (b) For the case $a = -b = p,\ p \in \mathbb{R}$, and for each fixed $\mu \in \mathbb{R}$, show that the curve
$$S_\mu = \left\{ z \in \mathbb{C} : |z - i\mu| = \sqrt{p^2 + \mu^2} \right\}$$
is a circle passing through A and B with its centre on the perpendicular bisector of AB.
Show that the circles C_λ and S_μ intersect orthogonally for all λ, μ.

7. Show by vector methods that the altitudes of a triangle are concurrent.
Hint: let the altitudes AD, BE of $\triangle ABC$ meet at H, and show that CH is perpendicular to AB.

8. Given that vectors $\mathbf{x}$ and $\mathbf{y}$ satisfy
$$\mathbf{x} + \mathbf{y}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{a}\ ,$$
for a fixed vector $\mathbf{a}$, show that
$$(\mathbf{x} \cdot \mathbf{y})^2 = \frac{|\mathbf{a}|^2 - |\mathbf{x}|^2}{2 + |\mathbf{y}|^2}\,.$$

Use an inequality involving $\mathbf{x}\cdot\mathbf{y}$ and the lengths of $\mathbf{x}$ and $\mathbf{y}$ to deduce that

$$|\mathbf{x}|(1+|\mathbf{y}|^2) \geqslant |\mathbf{a}| \geqslant |\mathbf{x}| \ .$$

Explain the circumstances in which either of the inequalities above become equalities, and describe the relation between $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{a}$ in these circumstances.

9. (a) In $\triangle ABC$, let $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{CA} = \mathbf{w}$. Show that

$$\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{w} = \mathbf{w}\times\mathbf{u}\,,$$

and hence obtain the sine rule for $\triangle ABC$.

(b) Given any three vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ such that

$$\mathbf{p}\times\mathbf{q} = \mathbf{q}\times\mathbf{r} = \mathbf{r}\times\mathbf{p}\,,$$

and $|\mathbf{p}\times\mathbf{q}| \neq 0$, show that

$$\mathbf{p}+\mathbf{q}+\mathbf{r} = \mathbf{0}\,.$$

10. Show that the line through the points $\mathbf{a}$ and $\mathbf{b}$ has equation

$$\mathbf{r} \ = \ (1-\lambda)\mathbf{a} \ + \ \lambda\mathbf{b},$$

and that the plane through the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ has the equation

$$\mathbf{r} \ = \ (1-\mu-\nu)\mathbf{a} \ + \ \mu\mathbf{b} \ + \ \nu\mathbf{c},$$

where λ, μ and ν are scalars. Obtain forms of these equations that do not involve λ, μ, ν.

11. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be fixed vectors in three dimensions. For each of the following equations, find all solutions for $\mathbf{r}$:

$$\text{(i)} \ \ \mathbf{r} + \mathbf{r}\times\mathbf{d} = \mathbf{c}\,; \qquad \text{(ii)} \ \ \mathbf{r} + (\mathbf{r}\cdot\mathbf{a})\,\mathbf{b} = \mathbf{c}\,.$$

[In (ii), consider separately the cases $\mathbf{a}\cdot\mathbf{b} \neq -1$ and $\mathbf{a}\cdot\mathbf{b} = -1$.]

12. (a) Using the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, show that

$$\begin{aligned} &\text{(i)} \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}),\\ &\text{(ii)} \quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}. \end{aligned}$$

Relate the case $\mathbf{c} = \mathbf{a}$, $\mathbf{d} = \mathbf{b}$ of (i) to a well-known trigonometric identity.
Evaluate $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ in two distinct ways and compare the results to find an explicit linear combination of the four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ that is zero.

(b) Given $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, show that

$$[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.$$

13. The vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are defined in terms of the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ by

$$\begin{aligned} \mathbf{e}_r &= \cos\phi\sin\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k}\,,\\ \mathbf{e}_\theta &= \cos\phi\cos\theta\,\mathbf{i} + \sin\phi\cos\theta\,\mathbf{j} - \sin\theta\,\mathbf{k}\,,\\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} \end{aligned}$$

where θ and ϕ are real. Show, as efficiently as possible, that $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are an orthornormal right-handed set.

Comments to: J.M.Evans@damtp.cam.ac.uk