

Introductions to BesselK

Introduction to the Bessel functions

General

The Bessel functions have been known since the 18th century when mathematicians and scientists started to describe physical processes through differential equations. Many different-looking processes satisfy the same partial differential equations. These equations were named Laplace, d'Alembert (wave), Poisson, Helmholtz, and heat (diffusion) equations. Different methods were used to investigate these equations. The most powerful was the separation of variables method, which in polar coordinates often leads to ordinary differential equations of special structure:

$$z^2 w'' + z w' + (z^2 - n^2)w = 0.$$

This equation with concrete values of the parameter n appeared in the articles by F. W. Bessel (1816, 1824) who built two partial solutions w_1 and w_2 of the previous equation in the form of series:

$$w_1 = z^n \sum_{j=0}^{\infty} a_j z^{2j} + z^{-n} \sum_{j=0}^{\infty} b_j z^{2j}, \quad \sum_{k=0}^{\infty} \left(a_{2k} z^{2k} + a_{2k+1} z^{2k+1} \right) + z^{-n} \left(\sum_{k=0}^{\infty} b_{2k} z^{2k} + \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1} \right).$$

Substituting the series into the differential equation produces the following solutions:

$$w_1 = z^n \sum_{k=0}^{\infty} A_k z^{2k}; A_0 = \frac{2^{-n}}{\Gamma(n+1)}, A_1 = -\frac{2^{-n-2}}{\Gamma(n+2)}, A_k = -\frac{1}{4k} \frac{\Gamma(n-k)}{\Gamma(n+k+1)} A_{k-1}.$$

$$w_2 = z^{-n} \sum_{k=0}^{\infty} B_k z^{2k}; B_0 = \frac{2^n}{\Gamma(-n)}, B_1 = -\frac{2^{n-2}}{\Gamma(-n+1)}, B_k = -\frac{1}{4k} \frac{\Gamma(-n-k)}{\Gamma(-n+k+1)} B_{k-1}.$$

O. Schlömilch (1857) used the name Bessel functions for these solutions, E. Lommel (1868) considered n as an arbitrary real parameter, and H. Hankel (1869) considered complex values for n . The two independent solutions of the differential equation were notated as J_n and J_{-n} .

For integer index n , the functions J_n and J_{-n} coincide or have different signs. In such cases, the second linear independent solution of the previous differential equation was introduced by C. G. Neumann (1867) as the limit case of the following special linear combination of the functions J_n and J_{-n} :

$$Y_n = \lim_{m \rightarrow n} \frac{\cos(\pi m) J_m - J_{-m}}{\sin(\pi m)}; n \notin \mathbb{Z}.$$

J. Watson (1867) introduced the notation Y_n for this function. Other authors (H. Hankel (1869), H. Weber (1873), and L. Schlöfli (1875)) investigated its properties. In particular, the general solution of the previous differential equation for all values of the parameter n can be presented by the formula:

$$z^2 w'' + z w' + (z^2 - n^2)w = 0; w = c_1 J_n + c_2 Y_n,$$

where c_1 and c_2 are arbitrary complex constants.

In a similar way, A. B. Basset (1888) and H. M. MacDonald (1899) introduced the modified Bessel functions $I_n(z)$ and $K_n(z)$, which satisfy the modified Bessel differential equation:

$$z^2 w'' + z w' - (z^2 + n^2)w = 0; \quad w = c_1 I_n(z) + c_2 K_n(z)$$

The first differential equation can be converted into the last one by changing the independent variable z to \tilde{z} .

Definitions of Bessel functions

The Bessel functions of the first kind $J_n(z)$ and $I_n(z)$ are defined as sums of the following infinite series:

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+n+1)\Gamma(k)} \left(\frac{z}{2}\right)^{2k+n}$$

$$I_n(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+n+1)\Gamma(k)} \left(\frac{z}{2}\right)^{2k+n}$$

These sums are convergent everywhere in the complex z -plane. The Bessel functions of the second kind $K_n(z)$ and $Y_n(z)$ for noninteger parameter n are defined as special linear combinations of the last two functions:

$$K_n(z) = \frac{\pi \csc \pi n}{2} [I_{-n}(z) - I_n(z)]; \quad n \notin \mathbb{Z}$$

$$Y_n(z) = \csc \pi n [\cos \pi n I_n(z) - J_{-n}(z)]; \quad n \notin \mathbb{Z}.$$

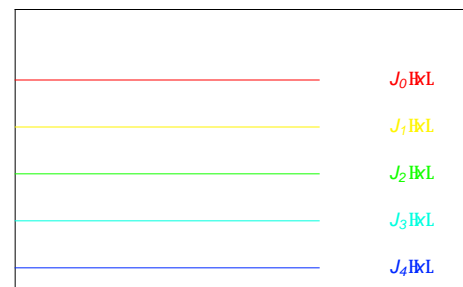
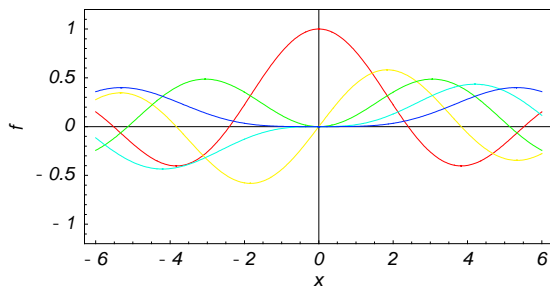
In the case of integer index n , the right-hand sides of the previous expressions give removable indeterminate values of the type $\frac{0}{0}$. In this case, the Bessel functions $K_n(z)$ and $Y_n(z)$ are defined through the following limits:

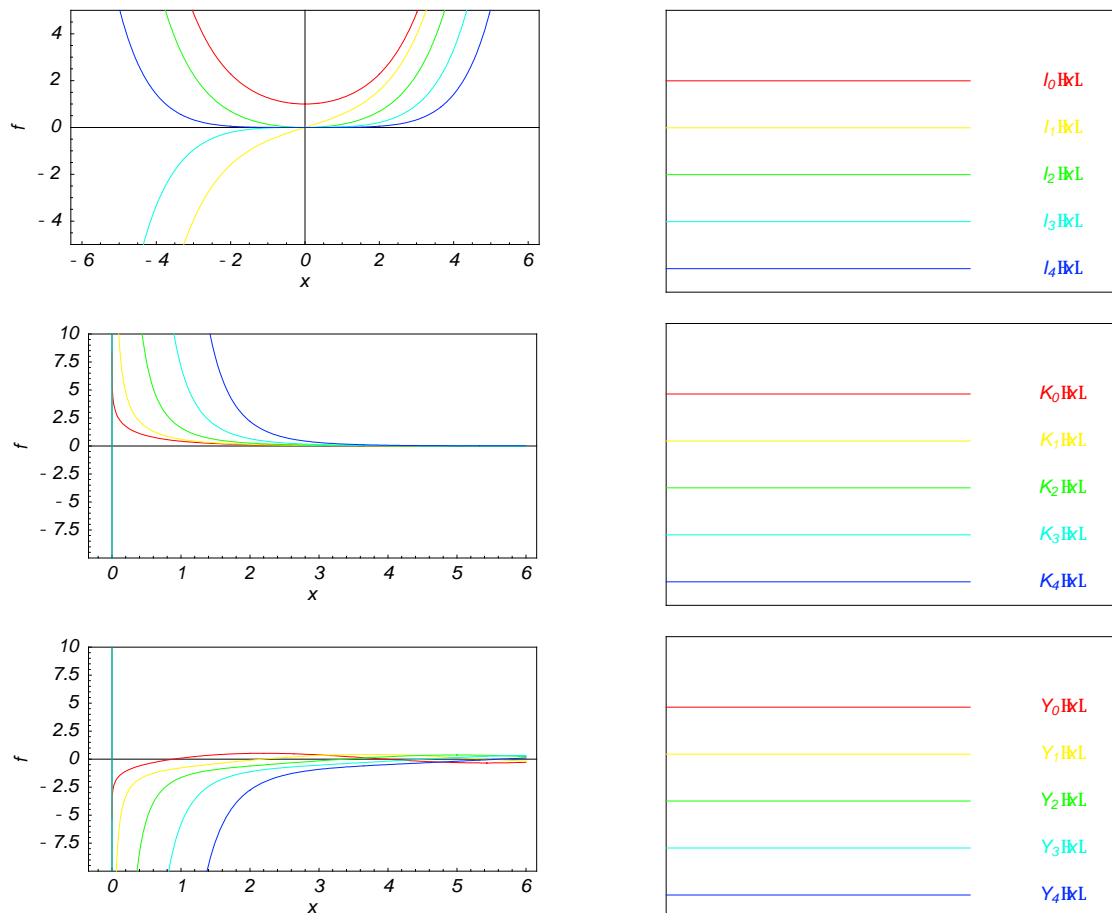
$$K_n(z) = \lim_{m \rightarrow n} K_m(z); \quad n \in \mathbb{Z}$$

$$Y_n(z) = \lim_{m \rightarrow n} Y_m(z); \quad n \in \mathbb{Z}.$$

A quick look at the Bessel functions

Here is a quick look at the graphics for the Bessel functions along the real axis.





Connections within the group of Bessel functions and with other function groups

Representations through more general functions

The Bessel functions $J_n(x)$, $I_n(x)$, $K_n(x)$, and $Y_n(x)$ are particular cases of more general functions: hypergeometric and Meijer G functions.

In particular, the functions $J_n(x)$ and $I_n(x)$ can be represented through the regularized hypergeometric functions ${}_0F_1$ (without any restrictions on the parameter n):

$$J_n(x) \sim \frac{1}{2} \sqrt{\frac{2}{\pi x}} {}_0F_1\left(\frac{3}{2}; \frac{x^2}{4}\right) \quad I_n(x) \sim \frac{1}{2} \sqrt{\frac{2}{\pi x}} {}_0F_1\left(\frac{3}{2}; \frac{x^2}{4}\right)$$

Similar formulas, but with restrictions on the parameter n , represent $J_n(x)$ and $I_n(x)$ through the classical hypergeometric function ${}_0F_1$:

$$J_n(x) \sim \frac{1}{\Gamma(n+1)} {}_0F_1\left(\frac{3}{2}; \frac{x^2}{4}\right) \quad I_n(x) \sim \frac{1}{\Gamma(n+1)} {}_0F_1\left(\frac{3}{2}; \frac{x^2}{4}\right)$$

The functions $J_n(x)$ and $I_n(x)$ can also be represented through the hypergeometric functions ${}_1F_1$ by the following formulas:

$$J_n(x) \sim \frac{x^n}{2^n \Gamma(n+1)} {}_1F_1\left(n+1; 2n+1; -\frac{x^2}{4}\right) \quad I_n(x) \sim \frac{x^n}{2^n \Gamma(n+1)} {}_1F_1\left(n+1; 2n+1; \frac{x^2}{4}\right)$$

$$J_n \text{HLS} \frac{z^n}{2^n \text{GH}+1\text{L}} \lim_{a \otimes \mathbb{N}} {}_1F_1 J a; n+1; -\frac{z^2}{4a} \text{N} \quad I_n \text{HLS} \frac{z^n}{2^n \text{GH}+1\text{L}} \lim_{a \otimes \mathbb{N}} {}_1F_1 J a; n+1; \frac{z^2}{4a} \text{N}.$$

Similar formulas for other Bessel functions $K_n \text{HL}$ and $Y_n \text{HL}$ always include restrictions on the parameter, namely $n \in \mathbb{Z}$:

$$K_n \text{HLS} \text{p cscH pL} \left(2^{n-1} z^{-n} {}_0\tilde{F}_1 \left(; 1-n; \frac{z^2}{4} \right) - 2^{-n-1} z^n {}_0\tilde{F}_1 \left(; n+1; \frac{z^2}{4} \right) \right) \bullet; n \in \mathbb{Z}$$

$$K_n \text{HLS} 2^{n-1} \text{GHL} z^{-n} {}_0F_1 \left(; 1-n; \frac{z^2}{4} \right) + 2^{-n-1} \text{GHL} z^n {}_0F_1 \left(; n+1; \frac{z^2}{4} \right) \bullet; n \in \mathbb{Z}$$

$$K_n \text{HLS} 2^{n-1} \text{GHL} z^{-n} \tilde{a}^{-z} {}_1F_1 \left(\frac{1}{2} - n; 1-2n; 2z \right) + 2^{-n-1} \text{GHL} z^n \tilde{a}^{-z} {}_1F_1 \left(\frac{1}{2} + n; 1+2n; 2z \right) \bullet; n \in \mathbb{Z}$$

$$Y_n \text{HLS} 2^{-n} z^n \text{cotH pL} {}_0\tilde{F}_1 \left(; n+1; -\frac{z^2}{4} \right) - 2^n z^{-n} \text{cscH pL} {}_0\tilde{F}_1 \left(; 1-n; -\frac{z^2}{4} \right) \bullet; n \in \mathbb{Z}$$

$$Y_n \text{HLS} -\frac{2^n z^{-n} \text{GHL}}{\text{p}} {}_0F_1 \left(; 1-n; -\frac{z^2}{4} \right) - \frac{2^{-n} z^n \text{cosH pLGHL}}{\text{p}} {}_0F_1 \left(; n+1; -\frac{z^2}{4} \right) \bullet; n \in \mathbb{Z}$$

$$Y_n \text{HLS} -\frac{2^{-n} \text{cosH pLGHL} \tilde{a}^{-z} z^n}{\text{p}} {}_1F_1 \left(n+\frac{1}{2}; 2n+1; 2\tilde{a}z \right) - \frac{2^n \text{GHL} z^{-n} \tilde{a}^{-z}}{\text{p}} {}_1F_1 \left(\frac{1}{2}-n; 1-2n; 2\tilde{a}z \right) \bullet; n \in \mathbb{Z}.$$

In the case of integer n , the right-hand sides of the preceding six formulas evaluate to removable indeterminate expressions of the type \mathbb{N} , $-\mathbb{N}$. The limit of the right-hand sides exists and produces complicated series expansions including logarithmic and polygamma functions. These difficulties can be removed by using the generalized Meijer G function. The generalized Meijer G function allows representation of all four Bessel functions for all values of the parameter n by the following simple formulas:

$$J_n \text{HLS} G_{0,2}^{1,0} \left(\frac{z}{2}, \frac{1}{2} \left| \frac{n}{2}, -\frac{n}{2} \right. \right)$$

$$I_n \text{HLS} \text{p} G_{1,3}^{1,0} \left(\frac{z}{2}, \frac{1}{2} \left| \frac{n+1}{2}, \frac{n}{2}, -\frac{n}{2}, \frac{n+1}{2} \right. \right)$$

$$K_n \text{HLS} \frac{1}{2} G_{0,2}^{2,0} \left(\frac{z}{2}, \frac{1}{2} \left| \frac{n}{2}, -\frac{n}{2} \right. \right)$$

$$Y_n \text{HLS} G_{1,3}^{2,0} \left(\frac{z}{2}, \frac{1}{2} \left| -\frac{1}{2} \text{H}+1\text{L}, \frac{n}{2}, -\frac{n}{2}, -\frac{1}{2} \text{H}+1\text{L} \right. \right).$$

The classical Meijer G function is less convenient because it can lead to additional restrictions:

$$J_n \text{HLS} 2^{-n} z^n G_{0,2}^{1,0} \left(\frac{z^2}{4} \left| 0, -n \right. \right)$$

$$I_n \text{HL} \check{S} p 2^{-n} z^n G_{1,3}^{1,0} \left(\frac{z^2}{4} \left| \begin{matrix} \frac{1}{2} \\ 0, -n, \frac{1}{2} \end{matrix} \right. \right)$$

$$K_n \text{HL} \check{S} \frac{1}{2} G_{0,2}^{2,0} \left(\frac{z^2}{4} \left| \begin{matrix} \frac{n}{2}, -\frac{n}{2} \end{matrix} \right. \right); \text{Re } n > 0$$

$$Y_n \text{HL} \check{S} G_{1,3}^{2,0} \left(\frac{z^2}{4} \left| \begin{matrix} -\frac{1}{2} n + 1 \\ \frac{n}{2}, -\frac{n}{2}, -\frac{1}{2} n + 1 \end{matrix} \right. \right); \text{Re } n > 0.$$

Representations through other Bessel functions

Each of the Bessel functions can be represented through other Bessel functions:

$$J_n \text{HL} \check{S} \frac{z^n}{n!} I_n \text{HL} - J_n \text{HL} \check{S} \frac{n!}{z^n} I_n \text{HL}$$

$$I_n \text{HL} \check{S} \frac{z^n}{n!} J_n \text{HL} - I_n \text{HL} \check{S} \frac{n!}{z^n} J_n \text{HL}$$

$$J_n \text{HL} \check{S} \csc \frac{\pi n}{2} Y_{-n} \text{HL} - \cot \frac{\pi n}{2} Y_n \text{HL}$$

$$I_n \text{HL} \check{S} \frac{z^n}{n!} \text{H} \csc \frac{\pi n}{2} Y_{-n} \text{HL} - \cot \frac{\pi n}{2} Y_n \text{HL}$$

$$K_n \text{HL} \check{S} \frac{p}{2} \left(\frac{n! \cos \frac{\pi n}{2}}{z^{2n}} - 1 \right) \csc \frac{\pi n}{2} I_n \text{HL} - \frac{p n!}{2 z^n} Y_n \text{HL}; n \notin \mathbb{Z}$$

$$K_n \text{HL} \check{S} \frac{p}{2} \left(\frac{n! \cos \frac{\pi n}{2}}{z^n} - \frac{z^n}{n!} \right) \csc \frac{\pi n}{2} J_n \text{HL} - \frac{p n!}{2 z^n} Y_n \text{HL}; n \notin \mathbb{Z}$$

$$Y_n \text{HL} \check{S} \frac{z^{-2n}}{p} (1 - 2 z^n K_n \text{HL} - \frac{1}{2} z^{2n} - p J_n \text{HL} \csc \frac{\pi n}{2} - \frac{1}{2} z^{2n} + p z^{2n} J_n \text{HL} \cot \frac{\pi n}{2}); n \notin \mathbb{Z}$$

$$Y_n \text{HL} \check{S} \frac{n!}{p} z^{-n} (p \csc \frac{\pi n}{2} z^{2n} \cos \frac{\pi n}{2} - n! z^{2n} W_n \text{HL} - 2 n! z^{2n} K_n \text{HL}); n \notin \mathbb{Z}.$$

The best-known properties and formulas for Bessel functions

Real values for real arguments

For real values of parameter n and positive argument z , the values of all four Bessel functions $J_n \text{HL}$, $I_n \text{HL}$, $K_n \text{HL}$, and $Y_n \text{HL}$ are real.

Simple values at zero

The Bessel functions $J_n \text{HL}$, $I_n \text{HL}$, $K_n \text{HL}$, and $Y_n \text{HL}$ have rather simple values for the argument $z \check{S} 0$:

$$J_0 \text{HL} \check{S} 1$$

$$I_0 \text{HL} \check{S} 1$$

$$K_0 \text{HL} \check{S} \infty$$

$$Y_0 \text{HLS} - \mathbb{Y}$$

$$J_n \text{HLS} \ 0 \bullet; \operatorname{Re} \text{HL} > 0$$

$$I_n \text{HLS} \ 0 \bullet; \operatorname{Re} \text{HL} > 0$$

$$K_n \text{HLS} \ \mathbb{Y}$$

$$Y_n \text{HLS} \ \mathbb{Y}.$$

Specific values for specialized parameters

In the case of half-integer n ($n = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots$) all Bessel functions $J_n \text{HL}$, $I_n \text{HL}$, $K_n \text{HL}$ and $Y_n \text{HL}$ can be expressed through sine, cosine, or exponential functions multiplied by rational and square root functions. Modulo simple factors, these are the so-called spherical Bessel functions, for example:

$$\begin{aligned} J_{-\frac{1}{2}} \text{HLS} &= \sqrt{\frac{2}{p}} \frac{\cos \text{HL}}{\sqrt{z}} \quad I_{-\frac{1}{2}} \text{HLS} = \sqrt{\frac{2}{p}} \frac{\cosh \text{HL}}{\sqrt{z}} \quad Y_{-\frac{1}{2}} \text{HLS} = \sqrt{\frac{2}{p}} \frac{\sin \text{HL}}{\sqrt{z}} \\ J_{\frac{1}{2}} \text{HLS} &= \sqrt{\frac{2}{p}} \frac{\sin \text{HL}}{\sqrt{z}} \quad I_{\frac{1}{2}} \text{HLS} = \sqrt{\frac{2}{p}} \frac{\sinh \text{HL}}{\sqrt{z}} \quad Y_{\frac{1}{2}} \text{HLS} = \sqrt{\frac{2}{p}} \frac{\cos \text{HL}}{\sqrt{z}} \\ K_{\frac{1}{2}} \text{HLS} &= K_{-\frac{1}{2}} \text{HLS} = \sqrt{\frac{p}{2}} \frac{\tilde{a}^{-z}}{\sqrt{z}}. \end{aligned}$$

The previous formulas are particular cases of the following, more general formulas:

$$\begin{aligned} J_n \text{HLS} &= \sqrt{\frac{2}{p}} \frac{1}{\sqrt{z}} \left(\cos \left(\frac{p}{2} \left(n - \frac{1}{2} \right) - z \right) \hat{a}^{\frac{f^2 n^2 - 3}{4} v} \frac{\mathbb{H} \ 1L^j J_2 j + n^{\alpha} + \frac{1}{2} N! \ \mathbb{H} \ z L^{-2j-1}}{\mathbb{H} \ j+1L! J_- 2 j + n^{\alpha} - \frac{3}{2} N!} - \right. \\ &\quad \left. \sin \left(\frac{p}{2} \left(n - \frac{1}{2} \right) - z \right) \hat{a}^{\frac{f^2 n^2 - 1}{4} v} \frac{\mathbb{H} \ 1L^j J_2 j + n^{\alpha} - \frac{1}{2} N!}{\mathbb{H} \ jL! J_- 2 j + n^{\alpha} - \frac{1}{2} N! \ \mathbb{H} \ z L^{2j}} \right) \bullet; n - \frac{1}{2} \hat{\mathbb{I}} \ Z \\ I_n \text{HLS} &= \frac{1}{\sqrt{z}} \hat{a}^{\frac{p}{2} \tilde{a} J_{\frac{1}{2}} - nN} \sqrt{\frac{2}{p}} \left(\sinh \left(\frac{p}{2} \left(\frac{1}{2} - n \right) - z \right) \hat{a}^{\frac{f^2 n^2 - 1}{4} v} \frac{J \ n^{\alpha} + 2k - \frac{1}{2} N!}{\mathbb{H} \ kL! J \ n^{\alpha} - 2k - \frac{1}{2} N! \ \mathbb{H} \ z L^{2k}} + \right. \\ &\quad \left. \cosh \left(\frac{p}{2} \left(\frac{1}{2} - n \right) - z \right) \hat{a}^{\frac{f^2 n^2 - 3}{4} v} \frac{J \ n^{\alpha} + 2k + \frac{1}{2} N! \ \mathbb{H} \ z L^{-2k-1}}{\mathbb{H} \ k+1L! J \ n^{\alpha} - 2k - \frac{3}{2} N!} \right) \bullet; n - \frac{1}{2} \hat{\mathbb{I}} \ Z \\ Y_n \text{HLS} &= \sqrt{\frac{2}{p}} \frac{\mathbb{H} \ 1L^{n+\frac{1}{2}}}{\sqrt{z}} \left(\sin \left(\frac{p}{2} \left(n + \frac{1}{2} \right) + z \right) \hat{a}^{\frac{f^2 n^2 - 1}{4} v} \frac{\mathbb{H} \ 1L^j J \ n^{\alpha} + 2j - \frac{1}{2} N! \ \mathbb{H} \ z L^{-2j}}{\mathbb{H} \ jL! J \ n^{\alpha} - 2j - \frac{1}{2} N!} + \right. \\ &\quad \left. \cos \left(\frac{p}{2} \left(n + \frac{1}{2} \right) + z \right) \hat{a}^{\frac{f^2 n^2 - 3}{4} v} \frac{\mathbb{H} \ 1L^j J \ n^{\alpha} + 2j + \frac{1}{2} N! \ \mathbb{H} \ z L^{-2j-1}}{\mathbb{H} \ j+1L! J \ n^{\alpha} - 2j - \frac{3}{2} N!} \right) \bullet; n - \frac{1}{2} \hat{\mathbb{I}} \ Z \end{aligned}$$

$$K_n \text{BesselS} \sqrt{\frac{p}{2}} \frac{\tilde{a}^{-z} \frac{f n^{\alpha} - \frac{1}{2} v}{\sqrt{z}}}{\frac{\tilde{a}}{\sqrt{z}} \frac{J_j + n^{\alpha} - \frac{1}{2} N!}{j! j - j + n^{\alpha} - \frac{1}{2} N!}} \text{BesselZ}^j \bullet; n - \frac{1}{2} \hat{I} \text{Z}.$$

It can be shown that for other values of the parameters n , the Bessel functions cannot be represented through elementary functions. But for values n equal to $\pm \frac{1}{3}$, $\pm \frac{4}{3}$, $\pm \frac{7}{3}$, $\frac{1}{4}$, and $\pm \frac{2}{3}$, $\pm \frac{5}{3}$, $\pm \frac{8}{3}$, $\frac{1}{4}$, all Bessel functions can be converted into other known special functions, the Airy functions and their derivatives, for example:

$$J_{-\frac{1}{3}} \text{BesselS} \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(3 \text{Ai} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) + \sqrt{3} \text{Bi} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) \right)$$

$$J_{\frac{1}{3}} \text{BesselS} \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(3 \text{Ai} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) - \sqrt{3} \text{Bi} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) \right)$$

$$I_{-\frac{1}{3}} \text{BesselS} \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(3 \text{Ai} \left(\left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) + \sqrt{3} \text{Bi} \left(\left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) \right)$$

$$I_{\frac{1}{3}} \text{BesselS} \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(\sqrt{3} \text{Bi} \left(\left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) - 3 \text{Ai} \left(\left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) \right)$$

$$Y_{-\frac{1}{3}} \text{BesselS} \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(\sqrt{3} \text{Ai} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) - 3 \text{Bi} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) \right)$$

$$Y_{\frac{1}{3}} \text{BesselS} - \frac{1}{2^{2/3} \sqrt[3]{3} \sqrt[3]{z}} \left(\sqrt{3} \text{Ai} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) + 3 \text{Bi} \left(- \left(\frac{3}{2} \right)^{2/3} z^{2/3} \right) \right)$$

$$K_{\pm \frac{1}{3}} \text{BesselS} \frac{\sqrt[3]{2} \sqrt[3]{3} p}{\sqrt[3]{z}} \text{Ai} \left(\left(\frac{3}{2} \right)^{2/3} z^{2/3} \right).$$

Analyticity

All four Bessel functions $J_n \text{Bessel}$, $I_n \text{Bessel}$, $K_n \text{Bessel}$, and $Y_n \text{Bessel}$ are defined for all complex values of the parameter n and variable z , and they are analytical functions of n and z over the whole complex n - and z -planes.

Poles and essential singularities

For fixed n , the functions $J_n \text{Bessel}$, $I_n \text{Bessel}$, $K_n \text{Bessel}$, and $Y_n \text{Bessel}$ have an essential singularity at $z \tilde{S} \infty$. At the same time, the point $z \tilde{S} \infty$ is a branch point (except in the case of integer n for the two functions $J_n \text{Bessel}$ and $I_n \text{Bessel}$).

For fixed integer n , the functions $J_n \text{Bessel}$ and $I_n \text{Bessel}$ are entire functions of z .

For fixed z , the functions $J_n \text{Bessel}$, $I_n \text{Bessel}$, $K_n \text{Bessel}$, and $Y_n \text{Bessel}$ are entire functions of n and have only one essential singular point at $n = \infty$.

Branch points and branch cuts

For fixed noninteger n , the functions $J_n(z)$, $I_n(z)$, $K_n(z)$, and $Y_n(z)$ have two branch points: $z=0$, $z=\infty$, and one straight line branch cut between them.

For fixed integer n , only the functions $Y_n(z)$ and $K_n(z)$ have two branch points: $z=0$, $z=\infty$, and one straight line branch cut between them.

For cases where the functions $J_n(z)$, $I_n(z)$, $K_n(z)$, and $Y_n(z)$ have branch cuts, the branch cuts are single-valued functions on the z -plane cut along the interval $[-\infty, 0]$, where they are continuous from above:

$$\lim_{\epsilon \rightarrow 0^+} J_n(x + i\epsilon) \sim J_n(x); \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} I_n(x + i\epsilon) \sim I_n(x); \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} Y_n(x + i\epsilon) \sim Y_n(x); \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} K_n(x + i\epsilon) \sim K_n(x); \quad x < 0.$$

These functions have discontinuities that are described by the following formulas:

$$\lim_{\epsilon \rightarrow 0^+} J_n(x - i\epsilon) \sim J_n(x) - 2\pi i J_n(x); \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} I_n(x - i\epsilon) \sim I_n(x) - 2\pi i I_n(x); \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} Y_n(x - i\epsilon) \sim Y_n(x) - 2\pi i \coth(\pi n) J_n(x) - \csc(\pi n) J_{-n}(x); \quad n \notin \mathbb{Z}; \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} Y_n(x - i\epsilon) \sim Y_n(x) - 4\pi i J_n(x); \quad n \in \mathbb{Z}; \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} K_n(x - i\epsilon) \sim K_n(x) - \frac{\pi \csc(\pi n)}{2} I_{-n}(x) - 2\pi i I_n(x); \quad n \notin \mathbb{Z}; \quad x < 0$$

$$\lim_{\epsilon \rightarrow 0^+} K_n(x - i\epsilon) \sim K_n(x) - \pi i I_n(x) + K_n(x); \quad n \in \mathbb{Z}; \quad x < 0.$$

Periodicity

All Bessel functions $J_n(z)$, $I_n(z)$, $K_n(z)$, and $Y_n(z)$ do not have periodicity.

Parity and symmetry

All Bessel functions $J_n(z)$, $I_n(z)$, $K_n(z)$, and $Y_n(z)$ have mirror symmetry (ignoring the interval $(-\infty, 0)$):

$$J_n(z) \sim \overline{J_n(z)}; \quad z \in \mathbb{R}, \quad 0 \leq z < \infty \quad Y_n(z) \sim \overline{Y_n(z)}; \quad z \in \mathbb{R}, \quad 0 \leq z < \infty$$

$$I_n(z) \sim \overline{I_n(z)}; \quad z \in \mathbb{R}, \quad 0 \leq z < \infty \quad K_n(z) \sim \overline{K_n(z)}; \quad z \in \mathbb{R}, \quad 0 \leq z < \infty$$

The two Bessel functions of the first kind have special parity (either odd or even) in each variable:

$$J_n(z) \sim (-1)^n J_n(z); \quad J_{-n}(z) \sim (-1)^n J_n(z); \quad n \in \mathbb{Z}$$

$$I_n(z) \sim (-1)^n I_n(z); \quad I_{-n}(z) \sim I_n(z); \quad n \in \mathbb{Z}.$$

The two Bessel functions of the second kind have special parity (either odd or even) only in their parameter:

$$Y_{-n}(z) \stackrel{?}{=} (-1)^n Y_n(z); \quad n \in \mathbb{Z}$$

$$K_{-n}(z) \stackrel{?}{=} K_n(z)$$

Series representations

The Bessel functions $J_n(z)$, $I_n(z)$, $K_n(z)$, and $Y_n(z)$ have the following series expansions (which converge in the whole complex z -plane):

$$J_n(z) \sim \frac{1}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(1 - \frac{z^2}{4(n+1)} + \frac{z^4}{32(n+1)(n+2)} - \frac{1}{4} \right); \quad n \in \mathbb{C}$$

$$J_n(z) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(n+1+k)!} \frac{z^{2k+n}}{2^{2k+n}}$$

$$I_n(z) \sim \frac{1}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{1}{k!} \left(1 + \frac{z^2}{4(n+1)} + \frac{z^4}{32(n+1)(n+2)} + \frac{1}{4} \right); \quad n \in \mathbb{C}$$

$$I_n(z) \sim \sum_{k=0}^{\infty} \frac{1}{\Gamma(n+1+k)!} \frac{z^{2k+n}}{2^{2k+n}}$$

$$K_n(z) \sim \frac{1}{2} \left(\Gamma(n) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(1 + \frac{z^2}{4(n-1)} + \frac{z^4}{32(n-1)(n-2)} + \frac{1}{4} \right) + \Gamma(n) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(1 + \frac{z^2}{4(n+1)} + \frac{z^4}{32(n+1)(n+2)} + \frac{1}{4} \right) \right);$$

$$n \in \mathbb{C}, \quad n \neq 0, \pm 1$$

$$K_n(z) \sim \frac{p \cosh p}{2} \left(\sum_{k=0}^{\infty} \frac{1}{\Gamma(n-k+1)!} \frac{z^{2k-n}}{2^{2k-n}} - \sum_{k=0}^{\infty} \frac{1}{\Gamma(n+k+1)!} \frac{z^{2k+n}}{2^{2k+n}} \right); \quad n \in \mathbb{Z}$$

$$Y_n(z) \sim \frac{\cosh p \Gamma(n)}{p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(1 - \frac{z^2}{4(n+1)} + \frac{z^4}{32(n+1)(n+2)} - \frac{1}{4} \right) - \frac{\Gamma(n)}{p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(1 - \frac{z^2}{4(n-1)} + \frac{z^4}{32(n-1)(n-2)} - \frac{1}{4} \right);$$

$$n \in \mathbb{C}, \quad n \neq 0, \pm 1$$

$$Y_n(z) \sim \cosh p \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(n+k+1)!} \frac{z^{2k+n}}{2^{2k+n}} - \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(n-k+1)!} \frac{z^{2k-n}}{2^{2k-n}} \right); \quad n \in \mathbb{Z}$$

The last four formulas have restrictions that do not allow their right sides to become indeterminate expressions for integer n .

In such cases, evaluation of the limit from the right sides leads to much more complicated representations, for example:

$$K_0(z) \sim \left(-\gamma + \frac{1}{4} \ln z^2 + \frac{1}{128} \ln^2 z + \frac{1}{4} \right) - \log \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}} \left(1 + \frac{z^2}{4} + \frac{z^4}{64} + \frac{1}{4} \right); \quad z \in \mathbb{C}$$

$$K_1(z) \sim \frac{1}{z} + \frac{z}{4} \left(2\gamma - 1 + \frac{1}{8} \left(2\gamma - \frac{5}{2} \right) z^2 + \frac{1}{192} \left(2\gamma - \frac{10}{3} \right) z^4 + \frac{1}{4} \right) + \frac{z}{2} \log \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}} \left(1 + \frac{z^2}{8} + \frac{z^4}{192} + \frac{1}{4} \right); \quad z \in \mathbb{C}$$

$$\begin{aligned}
& K_n \mathbb{H} \mathbb{L} \tilde{S} \mathbb{H} \mathbb{L}^{n-1} \log \frac{z}{2} \frac{z^n}{2} \frac{\mathbb{I}_2^{\frac{z}{2}} \tilde{M}^k}{k! \mathbb{H} + n\mathbb{L}} + \\
& \frac{1}{2} \frac{z^{-n-1}}{2} \mathbb{H} \mathbb{L}^k \mathbb{H} - k - 1\mathbb{L}! \frac{z^{2k}}{2} \frac{\mathbb{H} \mathbb{L}^n}{2} \frac{z^n}{2} \frac{\mathbb{I}_2^{\frac{z}{2}}}{k! \mathbb{H} + n\mathbb{L}} \frac{y\mathbb{H} + 1\mathbb{L} + y\mathbb{H} + n + 1\mathbb{L}}{2} \frac{z^{2k}}{2} \mathbb{K} - 0 \mathbb{H} \mathbb{L} \mathbb{N} \\
& Y_0 \mathbb{H} \mathbb{L} \tilde{S} \frac{2}{p} \log \frac{z}{2} \mathbb{K} - 0 \left(1 - \frac{z^2}{4} + \frac{z^4}{64} + \frac{1}{4} \right) - \frac{1}{p} \left(-2\dot{y} + \frac{1}{4} \mathbb{H} 2 + 2\dot{y} \mathbb{L} z^2 + \frac{1}{64} \mathbb{H} - 2\dot{y} \mathbb{L} z^4 + \frac{1}{4} \right) \mathbb{H} \mathbb{L} \mathbb{N} \\
& Y_1 \mathbb{H} \mathbb{L} \mu \frac{z}{p} \log \frac{z}{2} \mathbb{K} - 0 \left(1 - \frac{z^2}{8} + \frac{z^4}{192} + \frac{1}{4} \right) - \frac{2}{p z} - \frac{z}{2p} \left(-2\dot{y} + 1 + \frac{1}{8} \left(-\frac{5}{2} + 2\dot{y} \right) z^2 + \frac{1}{192} \left(\frac{10}{3} - 2\dot{y} \right) z^4 + \frac{1}{4} \right) \mathbb{H} \mathbb{L} \mathbb{N} \\
& Y_n \mathbb{H} \mathbb{L} \tilde{S} - \frac{1}{p} \frac{z^{-n-1}}{2} \mathbb{H} - k - 1\mathbb{L}! \frac{z^{2k}}{2} \mathbb{K} - 0 + \\
& \frac{2}{p} \log \frac{z}{2} \frac{z^n}{2} \frac{\mathbb{I}_2^{\frac{z}{2}}}{k! \mathbb{H} + n\mathbb{L}} \frac{\mathbb{H} \mathbb{L}^k}{2} \mathbb{K} - 0 - \frac{1}{p} \frac{z^n}{2} \frac{\mathbb{I}_2^{\frac{z}{2}}}{k! \mathbb{H} + n\mathbb{L}} \frac{\mathbb{H} \mathbb{L}^k \mathbb{H} y\mathbb{H} + 1\mathbb{L} + y\mathbb{H} + n + 1\mathbb{L}}{2} \frac{z^{2k}}{2} \mathbb{K} - 0 \mathbb{H} \mathbb{L} \mathbb{N}.
\end{aligned}$$

Interestingly, closed-form expressions for the truncated version of the Taylor series at the origin can be expressed through the generalized hypergeometric function ${}_1F_2$ and the Meijer G function, for example:

$$J_n \mathbb{H} \mathbb{L} \tilde{S} F_{\mathbb{Y}} \mathbb{H}, n\mathbb{L}; \left(\left(F_n \mathbb{H}, n\mathbb{L} \tilde{S} \hat{a} \frac{n \mathbb{H} \mathbb{L}^k \mathbb{I}_2^{\frac{z}{2}} \tilde{M}^{k+n}}{G\mathbb{H} + n + 1\mathbb{L}k!} \tilde{S} J_n \mathbb{H} \mathbb{L} + \frac{\mathbb{H} \mathbb{L}^n 2^{-2n-2} z^{2n+n+2}}{G\mathbb{H} + n + 2\mathbb{L}\mathbb{H} + 1\mathbb{L}!} {}_1F_2 \left(1; n+2, n+n+2; -\frac{z^2}{4} \right) \right) \mathbb{I} \mathbb{H} \mathbb{L} \mathbb{N} \right)$$

$$K_n \mathbb{H} \mathbb{L} \tilde{S} F_{\mathbb{Y}} \mathbb{H}, n\mathbb{L};$$

$$\left(\left(F_m \mathbb{H}, n\mathbb{L} \tilde{S} \frac{1}{2} \left(G\mathbb{H} \mathbb{L} \mathbb{K} - 0 \frac{z^{-n-m}}{2} \hat{a} \frac{\mathbb{I}_2^{\frac{z}{2}} \tilde{M}^k}{\mathbb{H} - n\mathbb{L}_k k!} + G\mathbb{H} n\mathbb{L} \mathbb{K} - 0 \frac{z^n}{2} \hat{a} \frac{\mathbb{I}_2^{\frac{z}{2}} \tilde{M}^k}{\mathbb{H} + 1\mathbb{L}_k k!} \right) \tilde{S} K_n \mathbb{H} \mathbb{L} + \frac{p}{\sin \mathbb{H} p \mathbb{L} \mathbb{H} + 1\mathbb{L}!} \left(\frac{2^{-2m-n-3} z^{2m+n+2}}{G\mathbb{H} + n + 2\mathbb{L}} \right. \right. \\
\left. \left. {}_1F_2 \left(1; m+2, m+n+2; \frac{z^2}{4} \right) - \frac{2^{-2m+n-3} z^{2m-n+2}}{G\mathbb{H} - n + 2\mathbb{L}} {}_1F_2 \left(1; m+2, m-n+2; \frac{z^2}{4} \right) \right) \right) \mathbb{I} \mathbb{H} \mathbb{L} \mathbb{N} \right); n \mathbb{I} \mathbb{Z}$$

$$K_n \mathbb{H} \mathbb{L} \tilde{S} F_{\mathbb{Y}} \mathbb{H}, n\mathbb{L};$$

$$\left(\left(F_m \mathbb{H}, n\mathbb{L} \tilde{S} \frac{\mathbb{H} \mathbb{L}^n}{2} \frac{z^n}{2} \frac{m}{k! \mathbb{H} + n\mathbb{L}!} \frac{y\mathbb{H} + 1\mathbb{L} + y\mathbb{H} + n + 1\mathbb{L}}{2} \frac{z^{2k}}{2} \mathbb{K} - 0 + \mathbb{H} \mathbb{L}^{n-1} \log \frac{z}{2} \mathbb{K} - 0 I_n \mathbb{H} \mathbb{L} + \frac{1}{2} \hat{a} \frac{\mathbb{H} \mathbb{L}^k \mathbb{H} k + n - 1\mathbb{L}!}{k!} \frac{z^{2k-n}}{2} \tilde{S} \right. \right. \\
\left. \left. z^n \mathbb{I} z^2 \tilde{M}^{\frac{n}{2}} K_n \left(\sqrt{z^2} \right) - \log \left(\frac{z^2}{4} \right) \frac{\mathbb{H} \mathbb{L}^n 2^{-2m-n-3} z^{2\mathbb{H}n+1\mathbb{L}+n}}{\mathbb{H}n + 1\mathbb{L}! \mathbb{H}n + n + 1\mathbb{L}!} {}_1F_2 \left(1; m+2, m+n+2; \frac{z^2}{4} \right) + \right. \right. \\
\left. \left. \mathbb{H} \mathbb{L}^{n-1} I_n \mathbb{H} \mathbb{L} \log \frac{z}{2} \mathbb{K} - 0 + \frac{1}{2} \mathbb{H} \mathbb{L}^n \log \left(\frac{z^2}{4} \right) I_n \mathbb{H} \mathbb{L} - \frac{\mathbb{H} \mathbb{L}^n}{2} \frac{z^n}{2} G_{2,4}^{2,2} \left(\frac{z^2}{4} \left| \begin{matrix} m+1, m+1 \\ m+1, m+1, 0, -n \end{matrix} \right. \right) \right) \mathbb{I} \mathbb{H} \mathbb{L} \mathbb{N} \right).$$

Asymptotic series expansions

The asymptotic behavior of the Bessel functions $J_n \mathbb{H} \mathbb{L}$, $I_n \mathbb{H} \mathbb{L}$, $K_n \mathbb{H} \mathbb{L}$, and $Y_n \mathbb{H} \mathbb{L}$ can be described by the following formulas (which show only the main terms):

$$J_n \mathbb{H} \mathbb{L} \mu \frac{\sqrt{2}}{\sqrt{p}} z^n I_z^2 \tilde{M}^{\frac{2n+1}{4}} \left(\cos \left(\sqrt{z^2 - \frac{p \mathbb{H} n + 1 \mathbb{L}}{4}} \right) \left(1 + O \left(\frac{1}{z^2} \right) \right) + \frac{1 - 4n^2}{8 \sqrt{z^2}} \sin \left(\sqrt{z^2 - \frac{p \mathbb{H} n + 1 \mathbb{L}}{4}} \right) \left(1 + O \left(\frac{1}{z^2} \right) \right) \right) \bullet; \mathbb{H} z^{\mathbb{H}} \mathbb{L} \mathbb{L}$$

$$I_n \mathbb{H} \mathbb{L} \mu \frac{1}{\sqrt{2p}} z^n I - z^2 \tilde{M}^{\frac{2n+1}{4}} \left(\exp \left(- \tilde{a} \left(\frac{\mathbb{H} n + 1 \mathbb{L} p}{4} - \sqrt{-z^2} \right) \right) \left(1 + O \left(\frac{1}{z} \right) \right) + \exp \left(\tilde{a} \left(\frac{\mathbb{H} n + 1 \mathbb{L} p}{4} - \sqrt{-z^2} \right) \right) \left(1 + O \left(\frac{1}{z} \right) \right) \right) \bullet; \mathbb{H} z^{\mathbb{H}} \mathbb{L} \mathbb{L}$$

$$K_n \mathbb{H} \mathbb{L} \mu \sqrt{\frac{p}{2}} \frac{\tilde{a}^{-z}}{\sqrt{z}} \left(1 + O \left(\frac{1}{z} \right) \right) \bullet; \mathbb{H} z^{\mathbb{H}} \mathbb{L} \mathbb{L}$$

$$Y_n \mathbb{H} \mathbb{L} \mu \sqrt{\frac{2}{p}} z^{-n} I_z^2 \tilde{M}^{\frac{2n+1}{4}} \csc \mathbb{H} p n \mathbb{L} \left(z^{2n} \cos \mathbb{H} p n \mathbb{L} \cos \left(\sqrt{z^2 - \frac{\mathbb{H} + 2 n \mathbb{L} p}{4}} \right) - I_z^2 \tilde{M} \cos \left(\sqrt{z^2 - \frac{\mathbb{H} - 2 n \mathbb{L} p}{4}} \right) \right) \left(1 + O \left(\frac{1}{z^2} \right) \right) - \frac{4n^2 - 1}{8 \sqrt{z^2}} \left(z^{2n} \cos \mathbb{H} p n \mathbb{L} \sin \left(\sqrt{z^2 - \frac{\mathbb{H} + 2 n \mathbb{L} p}{4}} \right) - I_z^2 \tilde{M} \sin \left(\sqrt{z^2 - \frac{\mathbb{H} - 2 n \mathbb{L} p}{4}} \right) \right) \left(1 + O \left(\frac{1}{z^2} \right) \right) \bullet; \mathbb{H} z^{\mathbb{H}} \mathbb{L} \mathbb{L}$$

The previous formulas are valid for any direction approaching the point z to infinity ($z^{\mathbb{H}} \mathbb{L} \mathbb{L}$ in particular cases, when $\text{Arg} \mathbb{H} \mathbb{L} < \frac{p}{2}$ or $\text{Arg} \mathbb{H} \mathbb{L} < p$, the second and fourth formulas can be simplified to the following forms:

$$I_n \mathbb{H} \mathbb{L} \mu \frac{\tilde{a}^z}{\sqrt{2p} z} \left(1 + O \left(\frac{1}{z} \right) \right) \bullet; \text{Arg} \mathbb{H} \mathbb{L} < \frac{p}{2} \quad \mathbb{H} z^{\mathbb{H}} \mathbb{L} \mathbb{L}$$

$$Y_n \mathbb{H} \mathbb{L} \mu \sqrt{\frac{2}{p}} \frac{1}{\sqrt{z}} \left(\frac{4n^2 - 1}{8z} \cos \left(z - \frac{\mathbb{H} + 2 n \mathbb{L} p}{4} \right) \left(1 + O \left(\frac{1}{z^2} \right) \right) + \sin \left(z - \frac{\mathbb{H} + 2 n \mathbb{L} p}{4} \right) \left(1 + O \left(\frac{1}{z^2} \right) \right) \right) \bullet; \text{Arg} \mathbb{H} \mathbb{L} < p \quad \mathbb{H} z^{\mathbb{H}} \mathbb{L} \mathbb{L}$$

Integral representations

The Bessel functions $J_n \mathbb{H} \mathbb{L}$, $I_n \mathbb{H} \mathbb{L}$, $K_n \mathbb{H} \mathbb{L}$, and $Y_n \mathbb{H} \mathbb{L}$ have simple integral representations through the cosine (or the hyperbolic cosine or exponential function) and power functions in the integrand:

$$J_n \mathbb{H} \mathbb{L} \tilde{S} \frac{2^{1-n} z^n}{\sqrt{p} \Gamma(n + \frac{1}{2})} \tilde{a}^{\frac{1}{2}} I_1 - t^2 \tilde{M}^{\frac{1}{2}} \cos \mathbb{H} t \mathbb{L} \hat{a} t \bullet; \text{Re} \mathbb{H} \mathbb{L} > -\frac{1}{2}$$

$$I_n \mathbb{H} \mathbb{L} \tilde{S} \frac{2^{1-n} z^n}{\sqrt{p} \Gamma(n + \frac{1}{2})} \tilde{a}^{\frac{1}{2}} I_1 - t^2 \tilde{M}^{\frac{1}{2}} \cosh \mathbb{H} t \mathbb{L} \hat{a} t \bullet; \text{Re} \mathbb{H} \mathbb{L} > -\frac{1}{2}$$

$$K_n \mathbb{H} \mathbb{L} \tilde{S} \frac{\sqrt{p} z^n}{2^n \Gamma(n + \frac{1}{2})} \tilde{a}^{\frac{1}{2}} \tilde{a}^{-z/t} I_1 t^2 - 1 \tilde{M}^{\frac{1}{2}} \hat{a} t \bullet; \text{Re} \mathbb{H} \mathbb{L} > -\frac{1}{2} \quad \text{Re} \mathbb{H} \mathbb{L} > 0$$

$$Y_n \mathbb{H} \mathbb{L} \tilde{S} - \frac{2^{n+1} z^{-n}}{\sqrt{p} \Gamma(\frac{1}{2} - n)} \tilde{a}^{\frac{1}{2}} I_1 t^2 - 1 \tilde{M}^{\frac{1}{2}} \cos \mathbb{H} t \mathbb{L} \hat{a} t \bullet; \text{Re} \mathbb{H} \mathbb{L} < \frac{1}{2} \quad z > 0.$$

Transformations

The argument of the Bessel functions $J_n \mathbb{H} \mathbb{L}$, $I_n \mathbb{H} \mathbb{L}$, $K_n \mathbb{H} \mathbb{L}$, and $Y_n \mathbb{H} \mathbb{L}$ sometimes can be simplified through formulas that remove square roots from the arguments. For the Bessel functions of the second kind $K_n \mathbb{H} \mathbb{L}$ and $Y_n \mathbb{H} \mathbb{L}$ with integer index n , this operation is realized by special formulas that include logarithms:

$$J_n \left(\sqrt{z^2} \right) \check{\mathbb{S}} z^{-n} I_{z^2 \check{\mathbb{M}}^2} J_n \mathbb{H} \mathbb{L}$$

$$I_n \left(\sqrt{z^2} \right) \check{\mathbb{S}} z^{-n} I_{z^2 \check{\mathbb{M}}^2} I_n \mathbb{H} \mathbb{L}$$

$$K_n \left(\sqrt{z^2} \right) \check{\mathbb{S}} z^n I_{z^2 \check{\mathbb{M}}^2} K_n \mathbb{H} \mathbb{L} - \frac{p \operatorname{csc} \mathbb{H} n \mathbb{L}}{2} \left(z^{-n} I_{z^2 \check{\mathbb{M}}^2} - z^n I_{z^2 \check{\mathbb{M}}^2} \right) I_n \mathbb{H} \mathbb{L}; n \hat{\mathbb{I}} \mathbb{Z}$$

$$K_n \left(\sqrt{z^2} \right) \check{\mathbb{S}} \left(\frac{\sqrt{z^2}}{z} \right)^n \left(K_n \mathbb{H} \mathbb{L} - \mathbb{H} 1 \mathbb{L}^n \left(\log \left(\sqrt{z^2} \right) - \log \mathbb{H} \mathbb{L} \right) I_n \mathbb{H} \mathbb{L} \right); n \hat{\mathbb{I}} \mathbb{Z}$$

$$Y_n \left(\sqrt{z^2} \right) \check{\mathbb{S}} z^n I_{z^2 \check{\mathbb{M}}^2} Y_n \mathbb{H} \mathbb{L} + \cot \mathbb{H} n \mathbb{L} \left(z^{-n} I_{z^2 \check{\mathbb{M}}^2} - z^n I_{z^2 \check{\mathbb{M}}^2} \right) J_n \mathbb{H} \mathbb{L}; n \hat{\mathbb{I}} \mathbb{Z}$$

$$Y_n \left(\sqrt{z^2} \right) \check{\mathbb{S}} \left(\frac{\sqrt{z^2}}{z} \right)^n \left(Y_n \mathbb{H} \mathbb{L} + \frac{2 \left(\log \left(\sqrt{z^2} \right) - \log \mathbb{H} \mathbb{L} \right)}{p} J_n \mathbb{H} \mathbb{L} \right); n \hat{\mathbb{I}} \mathbb{Z}.$$

If the argument of a Bessel function includes an explicit minus sign, the following formulas produce Bessel functions without the minus sign argument:

$$J_n \mathbb{H} z \mathbb{L} \check{\mathbb{S}} \mathbb{H} z \mathbb{L}^n z^{-n} J_n \mathbb{H} \mathbb{L}$$

$$I_n \mathbb{H} z \mathbb{L} \check{\mathbb{S}} \mathbb{H} z \mathbb{L}^n z^{-n} I_n \mathbb{H} \mathbb{L}$$

$$K_n \mathbb{H} z \mathbb{L} \check{\mathbb{S}} z^n K_n \mathbb{H} \mathbb{L} \mathbb{H} z \mathbb{L}^{-n} + \frac{p}{2} \mathbb{H} z \mathbb{L}^{-n} z^n - \mathbb{H} z \mathbb{L}^n z^{-n} L_n \mathbb{H} \mathbb{L} \operatorname{csc} \mathbb{H} n \mathbb{L}; n \hat{\mathbb{I}} \mathbb{Z}$$

$$K_n \mathbb{H} z \mathbb{L} \check{\mathbb{S}} \mathbb{H} 1 \mathbb{L}^n K_n \mathbb{H} \mathbb{L} + \mathbb{H} \log \mathbb{H} \mathbb{L} - \log \mathbb{H} z \mathbb{L} L_n \mathbb{H} \mathbb{L}; n \hat{\mathbb{I}} \mathbb{Z}$$

$$Y_n \mathbb{H} z \mathbb{L} \check{\mathbb{S}} z^n Y_n \mathbb{H} \mathbb{L} \mathbb{H} z \mathbb{L}^{-n} + \mathbb{H} z \mathbb{L}^n z^{-n} - \mathbb{H} z \mathbb{L}^{-n} z^n L_n \mathbb{H} \mathbb{L} \cot \mathbb{H} n \mathbb{L}; n \hat{\mathbb{I}} \mathbb{Z}$$

$$Y_n \mathbb{H} z \mathbb{L} \check{\mathbb{S}} \mathbb{H} 1 \mathbb{L}^n \left(Y_n \mathbb{H} \mathbb{L} - \frac{2}{p} \mathbb{H} \log \mathbb{H} \mathbb{L} - \log \mathbb{H} z \mathbb{L} L_n \mathbb{H} \mathbb{L} \right); n \hat{\mathbb{I}} \mathbb{Z}.$$

If the arguments of the Bessel functions include sums, the following formulas hold:

$$J_n \mathbb{H} z_1 + z_2 \mathbb{L} \check{\mathbb{S}} \hat{\mathbb{A}} \sum_{k=-\infty}^{\infty} J_{n-k} \mathbb{H} z_1 L J_k \mathbb{H} z_2 \mathbb{L}; \left| \frac{z_2}{z_1} \right| < 1 \hat{\mathbb{I}} \mathbb{Z}$$

$$I_n \mathbb{H} z_1 + z_2 \mathbb{L} \check{\mathbb{S}} \hat{\mathbb{A}} \sum_{k=-\infty}^{\infty} I_{n-k} \mathbb{H} z_1 L I_k \mathbb{H} z_2 \mathbb{L}; \left| \frac{z_2}{z_1} \right| < 1 \hat{\mathbb{I}} \mathbb{Z}$$

$$K_n \mathbb{H} z_1 + z_2 \mathbb{L} \check{\mathbb{S}} \hat{\mathbb{A}} \mathbb{H} 1 \mathbb{L}^k K_{n-k} \mathbb{H} z_1 L J_k \mathbb{H} z_2 \mathbb{L}; \left| \frac{z_2}{z_1} \right| < 1$$

$$Y_n H_1 + z_2 L \check{S} \hat{a} \sum_{k=-\infty}^{\infty} Y_{n-k} H_1 L J_k H_2 L; \left| \frac{z_2}{z_1} \right| < 1.$$

If arguments of the Bessel functions include products, the following formulas hold:

$$J_n H_1 z_2 L \check{S} z_1^n \hat{a} \sum_{k=0}^{\infty} \frac{H_1 L^k}{k!} I_{z_1^2 - 1}^k J_{k+n} H_2 L K \frac{z_2}{2}^k 0$$

$$I_n H_1 z_2 L \check{S} z_1^n \hat{a} \sum_{k=0}^{\infty} \frac{I_{z_1^2 - 1}^k}{k!} I_{k+n} H_2 L K \frac{z_2}{2}^k 0$$

$$K_n H_1 z_2 L \check{S} z_1^n \hat{a} \sum_{k=0}^{\infty} \frac{H_1 L^k}{k!} I_{z_1^2 - 1}^k K_{k+n} H_2 L K \frac{z_2}{2}^k 0; |z_1^2 - 1| < 1$$

$$Y_n H_1 z_2 L \check{S} z_1^n \hat{a} \sum_{k=0}^{\infty} \frac{H_1 L^k}{k!} I_{z_1^2 - 1}^k Y_{k+n} H_2 L K \frac{z_2}{2}^k 0; |z_1^2 - 1| < 1.$$

Identities

The Bessel functions $J_n H_1$, $I_n H_1$, $K_n H_1$, and $Y_n H_1$ satisfy the following recurrence identities:

$$J_n H_1 L \check{S} \frac{2 H_1 + 1 L}{z} J_{n+1} H_1 L - J_{n+2} H_1 L$$

$$J_n H_1 L \check{S} \frac{2 H_1 - 1 L}{z} J_{n-1} H_1 L - J_{n-2} H_1 L$$

$$I_n H_1 L \check{S} \frac{2 H_1 + 1 L}{z} I_{n+1} H_1 L + I_{n+2} H_1 L$$

$$I_n H_1 L \check{S} \frac{2 H_1 - 1 L}{z} I_{n-1} H_1 L - I_{n-2} H_1 L$$

$$K_n H_1 L \check{S} \frac{2 H_1 + 1 L}{z} K_{n+2} H_1 L - K_{n+1} H_1 L$$

$$K_n H_1 L \check{S} \frac{2 H_1 - 1 L}{z} K_{n-2} H_1 L + K_{n-1} H_1 L$$

$$Y_n H_1 L \check{S} \frac{2 H_1 + 1 L}{z} Y_{n+1} H_1 L - Y_{n+2} H_1 L$$

$$Y_n H_1 L \check{S} \frac{2 H_1 - 1 L}{z} Y_{n-1} H_1 L - Y_{n-2} H_1 L$$

The last eight identities can be generalized to the following recurrence identities with jump length n :

$$J_n H_1 L \check{S} C_n H_1, z L J_{n+n} H_1 L - C_{n-1} H_1, z L J_{n+n+1} H_1 L;$$

$$C_0 H_1, z L \check{S} 1 \hat{a} C_1 H_1, z L \check{S} \frac{2 H_1 + 1 L}{z} \hat{a} C_n H_1, z L \check{S} \frac{2 H_1 + n L}{z} C_{n-1} H_1, z L - C_{n-2} H_1, z L \hat{a} n \hat{a} N^+$$

$$\begin{aligned}
& J_n \mathbb{H} \tilde{S} C_n \mathbb{H}, z L_{n-n} \mathbb{H} L - C_{n-1} \mathbb{H}, z L_{n-n-1} \mathbb{H} L \bullet; \\
& C_0 \mathbb{H}, z L \tilde{S} 1 \dot{=} C_1 \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} - 1 \mathbb{L}}{z} \dot{=} C_n \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} n + n \mathbb{L}}{z} C_{n-1} \mathbb{H}, z L - C_{n-2} \mathbb{H}, z L \dot{=} n \hat{\mathbb{I}} N^+ \\
& I_n \mathbb{H} \tilde{S} C_n \mathbb{H}, z L_{n+n} \mathbb{H} L + C_{n-1} \mathbb{H}, z L_{n+n+1} \mathbb{H} L \bullet; \\
& C_0 \mathbb{H}, z L \tilde{S} 1 \dot{=} C_1 \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} + 1 \mathbb{L}}{z} \dot{=} C_n \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} + n \mathbb{L}}{z} C_{n-1} \mathbb{H}, z L + C_{n-2} \mathbb{H}, z L \dot{=} n \hat{\mathbb{I}} N^+ \\
& I_n \mathbb{H} \tilde{S} C_n \mathbb{H}, z L_{n-n} \mathbb{H} L + C_{n-1} \mathbb{H}, z L_{n-n-1} \mathbb{H} L \bullet; \\
& C_0 \mathbb{H}, z L \tilde{S} 1 \dot{=} C_1 \mathbb{H}, z L \tilde{S} - \frac{2 \mathbb{H} - 1 \mathbb{L}}{z} \dot{=} C_n \mathbb{H}, z L \tilde{S} - \frac{2 \mathbb{H} n + n \mathbb{L}}{z} C_{n-1} \mathbb{H}, z L + C_{n-2} \mathbb{H}, z L \dot{=} n \hat{\mathbb{I}} N^+ \\
& K_n \mathbb{H} \tilde{S} C_n \mathbb{H}, z L K_{n+n} \mathbb{H} L + C_{n-1} \mathbb{H}, z L K_{n+n+1} \mathbb{H} L \bullet; \\
& C_0 \mathbb{H}, z L \tilde{S} 1 \dot{=} C_1 \mathbb{H}, z L \tilde{S} - \frac{2 \mathbb{H} + 1 \mathbb{L}}{z} \dot{=} C_n \mathbb{H}, z L \tilde{S} - \frac{2 \mathbb{H} + n \mathbb{L}}{z} C_{n-1} \mathbb{H}, z L + C_{n-2} \mathbb{H}, z L \dot{=} n \hat{\mathbb{I}} N^+ \\
& K_n \mathbb{H} \tilde{S} C_n \mathbb{H}, z L K_{n-n} \mathbb{H} L + C_{n-1} \mathbb{H}, z L K_{n-n-1} \mathbb{H} L \bullet; \\
& C_0 \mathbb{H}, z L \tilde{S} 1 \dot{=} C_1 \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} - 1 \mathbb{L}}{z} \dot{=} C_n \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} n + n \mathbb{L}}{z} C_{n-1} \mathbb{H}, z L + C_{n-2} \mathbb{H}, z L \dot{=} n \hat{\mathbb{I}} N^+ \\
& Y_n \mathbb{H} \tilde{S} C_n \mathbb{H}, z L Y_{n+n} \mathbb{H} L - C_{n-1} \mathbb{H}, z L Y_{n+n+1} \mathbb{H} L \bullet; \\
& C_0 \mathbb{H}, z L \tilde{S} 1 \dot{=} C_1 \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} + 1 \mathbb{L}}{z} \dot{=} C_n \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} + n \mathbb{L}}{z} C_{n-1} \mathbb{H}, z L - C_{n-2} \mathbb{H}, z L \dot{=} n \hat{\mathbb{I}} N^+ \\
& Y_n \mathbb{H} \tilde{S} C_n \mathbb{H}, z L Y_{n-n} \mathbb{H} L - C_{n-1} \mathbb{H}, z L Y_{n-n-1} \mathbb{H} L \bullet; \\
& C_0 \mathbb{H}, z L \tilde{S} 1 \dot{=} C_1 \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} - 1 \mathbb{L}}{z} \dot{=} C_n \mathbb{H}, z L \tilde{S} \frac{2 \mathbb{H} n + n \mathbb{L}}{z} C_{n-1} \mathbb{H}, z L - C_{n-2} \mathbb{H}, z L \dot{=} n \hat{\mathbb{I}} N^+.
\end{aligned}$$

Simple representations of derivatives

The derivatives of all the four Bessel functions $J_n \mathbb{H}$, $I_n \mathbb{H}$, $K_n \mathbb{H}$, and $Y_n \mathbb{H}$ have rather simple and symmetrical representations that can be expressed through other Bessel functions with different indices:

$$\begin{aligned}
\frac{\mathbb{I} J_n \mathbb{H}}{\mathbb{I} z} & \tilde{S} \frac{1}{2} \mathbb{H}_{n-1} \mathbb{H} L - J_{n+1} \mathbb{H} L \\
\frac{\mathbb{I} I_n \mathbb{H}}{\mathbb{I} z} & \tilde{S} \frac{1}{2} \mathbb{H}_{n-1} \mathbb{H} L + I_{n+1} \mathbb{H} L \\
\frac{\mathbb{I} Y_n \mathbb{H}}{\mathbb{I} z} & \tilde{S} \frac{1}{2} \mathbb{H}_{n-1} \mathbb{H} L - Y_{n+1} \mathbb{H} L \\
\frac{\mathbb{I} K_n \mathbb{H}}{\mathbb{I} z} & \tilde{S} - \frac{1}{2} \mathbb{H}_{n-1} \mathbb{H} L + K_{n+1} \mathbb{H} L
\end{aligned}$$

But these derivatives can be represented in other forms, for example:

$$\begin{aligned}
\frac{\mathbb{I} J_n \mathbb{H}}{\mathbb{I} z} & \tilde{S} J_{n-1} \mathbb{H} L - \frac{n}{z} J_n \mathbb{H} \tilde{S} \frac{n}{z} J_n \mathbb{H} L - J_{n+1} \mathbb{H} L \\
\frac{\mathbb{I} I_n \mathbb{H}}{\mathbb{I} z} & \tilde{S} I_{n-1} \mathbb{H} L - \frac{n}{z} I_n \mathbb{H} \tilde{S} \frac{n}{z} I_n \mathbb{H} L + I_{n+1} \mathbb{H} L
\end{aligned}$$

$$\frac{{}_0F_1 \left(\begin{matrix} - \\ n+1 \end{matrix} ; -\frac{z}{n+1} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)} \sim \frac{{}_0F_1 \left(\begin{matrix} - \\ n-1 \end{matrix} ; -\frac{z}{n-1} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)} \sim \frac{{}_0F_1 \left(\begin{matrix} - \\ n+1 \end{matrix} ; -\frac{z}{n+1} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}$$

The symbolic n^{th} -order derivatives have more complicated representations through the regularized hypergeometric function ${}_2F_3$ or generalized Meijer G function:

$$\frac{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)} \sim 2^{n-2n} \sqrt{p} z^{n-n} \text{GH} + {}_1L_2 \tilde{F}_3 \left(\frac{n+1}{2}, \frac{n+2}{2}; \frac{n-n+1}{2}, \frac{n-n+2}{2}, n+1; -\frac{z^2}{4} \right); n \in \mathbb{N}$$

$$\frac{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)} \sim 2^{n-2n} \sqrt{p} z^{n-n} \text{GH} + {}_1L_2 \tilde{F}_3 \left(\frac{n+1}{2}, \frac{n+2}{2}; \frac{1-n+n}{2}, \frac{2-n+n}{2}, n+1; \frac{z^2}{4} \right); n \in \mathbb{N}$$

$$\frac{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)} \sim \frac{1}{2} G_{2,4}^{2,2} \left(\frac{z}{2}, \frac{1}{2} \left| \begin{matrix} \frac{1-n}{2}, -\frac{n}{2} \\ \frac{n-n}{2}, -\frac{n+n}{2}, \frac{1}{2}, 0 \end{matrix} \right. \right); n \in \mathbb{N}$$

$$\frac{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)} \sim G_{3,5}^{2,2} \left(\frac{z}{2}, \frac{1}{2} \left| \begin{matrix} \frac{1-n}{2}, -\frac{n}{2}, -\frac{n+n+1}{2} \\ \frac{n-n}{2}, -\frac{n+n}{2}, \frac{1}{2}, 0, -\frac{n+n+1}{2} \end{matrix} \right. \right); n \in \mathbb{N}$$

Differential equations

The Bessel functions J_n , Y_n , I_n , and K_n appeared as special solutions of two linear second-order differential equations (the so-called Bessel equation):

$$z^2 w'' + z w' + (z^2 - n^2) w = 0; w = c_1 J_n + c_2 Y_n$$

$$z^2 w'' + z w' - (z^2 + n^2) w = 0; w = c_1 I_n + c_2 K_n$$

where c_1 and c_2 are arbitrary constants.

Zeros

When n is real, the functions J_n and $J \frac{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}$ each have an infinite number of real zeros, all of which are simple with the possible exception of the zero $z = 0$:

$$J_n \neq 0; z \neq z_k \in \mathbb{R}; n \in \mathbb{R}; \text{Re}(z_k) \leq z_k$$

$$\frac{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)} \neq 0; z \neq z_k \in \mathbb{R}; n \in \mathbb{R}.$$

When $n \geq -1$, the zeros of J_n are all real. If $n < -1$ and n is not an integer, the number of complex zeros of J_n is $2 \lfloor -n \rfloor$; if $\lfloor -n \rfloor$ is odd, two of these zeros lie on the imaginary axis.

If $n > 0$, all zeros of $J \frac{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}{{}_0F_1 \left(\begin{matrix} - \\ n \end{matrix} ; -\frac{z}{n} \right)}$ are real.

The function K_n has no zeros in the region $|\arg z| \leq \frac{\pi}{2}$ for any real n .

When n is real, the functions Y_n^{HL} and $J \frac{Y_n^{\text{HL}}}{z}$ each have an infinite number of real zeros, all of which are simple with the possible exception of the zero $z = 0$:

$$Y_n^{\text{HL}} \not\equiv 0; z \not\equiv z_k \text{ if } k \in \mathbb{N} \text{ if } n \in \mathbb{R} \text{ if } \operatorname{Re} z_k > 0$$

$$\frac{Y_n^{\text{HL}}}{z} \not\equiv 0; z \not\equiv z_k \text{ if } k \in \mathbb{N} \text{ if } n \in \mathbb{R}.$$

Applications of Bessel functions

Applications of Bessel functions include mechanics, electrodynamics, electroengineering, solid state physics, and celestial mechanics.

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