CS624 - Analysis of Algorithms

Sorting

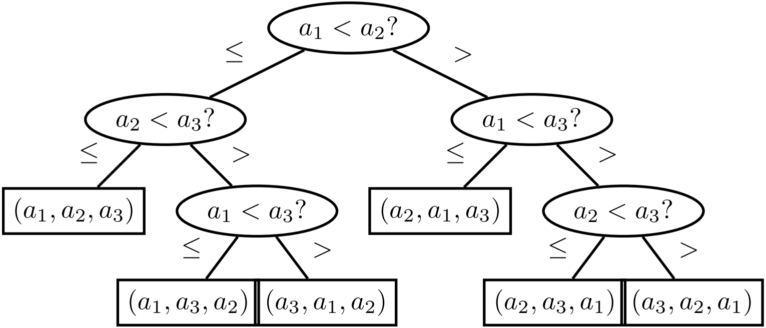
September 26, 2019

A Little More on Sorting

- How well can we really do?
- Is there a sorting method whose worst case runtime is $O(n)$?
- Obviously we can't do better than that (why?).
- For the class of algorithms we've seen so far the answer is no. The lower bound really is $O(n \log n)$.
- These sorting algorithms are based on comparisons and can be modeled as binary decision trees.

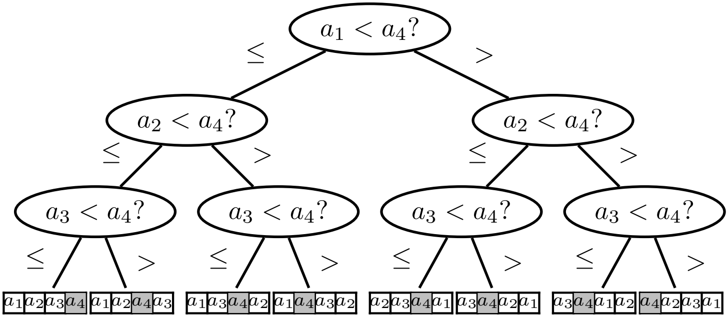
A Simple Example

The run of InsertionSort on an array of 3 elements: $\{a_1, a_2, a_3\}$:



Another Simple Example

The run of Quicksort on an array of elements: $\{a_1, a_2, a_3, a_4\}$, first partition:



Bound on Sorting Algorithms

Theorem

In a sorting algorithm modeled by a binary decision tree, the worst-case running time is $\Omega(n \log n)$.

Proof.

The worst-case running time is bounded below by the depth of the decision tree. The number of leaves in the decision tree must be the number of possible permutations, which is $n!$. The depth of a binary tree with L leaves is $\Omega(\log L)$. Therefore the depth of the decision tree is

$$\Omega(\log n!) = \Omega(n \log n)$$

□

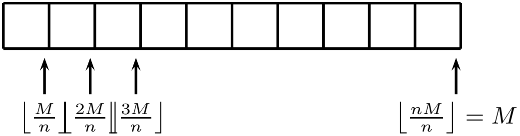
Still, Can We Do Better?

- When our model is not based on comparisons we can do better.
- Example: BucketSort.
- Simple case: We have a set of integers 1..n in some random order.
- How do we sort this?

A More Complicated Example

- Given a set of integers $\{a_1, a_2, \ldots, a_n\}$.
- Each integer is in the range of 1...M where $M \geq n$.
- Create an array A[1..n] where the elements are sets of integers.
- Each set is called a bucket.
- Each integer in the original sequence will be put in its appropriate bucket.

Illustration of Bucketsort



The largest number bucket $A[j]$ can hold is $\left\lfloor \frac{jM}{n} \right\rfloor$. Therefore the index j of the bucket that we want to place the number a_k in must satisfy

$$\left\lfloor \frac{(j-1)M}{n} \right\rfloor + 1 \leq a_k \leq \left\lfloor \frac{jM}{n} \right\rfloor$$

Since we always have $x - 1 < \lfloor x \rfloor$, this yields
$\frac{(j-1)M}{n} < a_k \leq \frac{jM}{n} \Rightarrow j - 1 < \frac{a_k n}{M} \leq j \Rightarrow j = \left\lceil \frac{a_k n}{M} \right\rceil$

Bucketsort analysis

- Problem: Elements are not necessarily uniformly distributed in buckets.
- Some buckets are empty, some may contain several elements.
- What is the average cost of sorting the buckets?
- Suppose we use InsertionSort to sort each bucket (good for small buckets).
- Do not assume that since the average number of elements per bucket is $O(1)$ it means that the average runtime is O(n).

Bucketsort analysis

- If bucket i has n_i elements, sorting it takes $O(n_i^2)$
- We can average over all the buckets.
- Since the distribution of elements in buckets is random we can average on n_1 (since it doesn't matter which bin we pick).
- The expected value of n_1 is
$$\sum_{j=0}^{n} (\text{the probability that } j \text{ numbers land in bucket } 1) \cdot j^2$$

Bucketsort analysis

- The probability of j items landing in bucket 1 is the probability of selecting j items out of n, $\binom{n}{j}$.
- The probability of a particular combination is: $\left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j}$.
- The probability of any j elements landing in bucket 1 is: $\left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j}$.
- The expected runtime of sorting n_1 is then $\sum_{j=0}^{n} \left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j} j^2$

Bucketsort analysis

This looks like a binomial generating function that has been differentiated. So let us set:

$$f(x) = \sum_{j=0}^{n} \left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j} x^j$$

Then we have

$$f'(x) = \sum_{j=0}^{n} \left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j} j x^{j-1}$$

We can't just differentiate again, because we would get $j(j-1)$. So we multiply by x first:

$$xf'(x) = \sum_{j=0}^{n} \left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j} j x^j$$

and then we can differentiate:

$$\left(xf'(x)\right)' = \sum_{j=0}^{n} \left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j} j^2 x^{j-1}$$

Bucketsort analysis

- Let us set $g(x) = \big(xf'(x)\big)'$.
- Then we see that the expected value of n_1^2 is just $g(1)$.
- The closed form of f follows from the binomial theorem:

$$
\begin{aligned}
f(x) &= \sum_{j=0}^{n} \left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j} x^j \\
&= \sum_{j=0}^{n} \left(\frac{x}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \binom{n}{j} \qquad = \left(1 + \frac{x-1}{n}\right)^n
\end{aligned}
$$

Bucketsort analysis

Going back to g:

$$f'(x) = n\left(1 + \frac{x-1}{n}\right)^{n-1} \cdot \frac{1}{n} = \left(1 + \frac{x-1}{n}\right)^{n-1}$$

and then

$$\begin{aligned} g(x) = (xf'(x))' &= \left(x\left(1 + \frac{x-1}{n}\right)^{n-1}\right)' \\ &= \left(1 + \frac{x-1}{n}\right)^{n-1} + (n-1)x\left(1 + \frac{x-1}{n}\right)^{n-2} \cdot \frac{1}{n} \\ &= \left(1 + \frac{x-1}{n}\right)^{n-1} + \left(1 - \frac{1}{n}\right)x\left(1 + \frac{x-1}{n}\right)^{n-2} \end{aligned}$$

By substituting 1 for x, we get $g(1) = 1 + \left(1 - \frac{1}{n}\right) = 2 - \frac{1}{n}$. That is the expected value of n_1^2, and in fact is the expected value of n_i^2 for any i. In short – the average time for sorting each bucket in bucketsort is O(1) and the overall expected runtime is O(n).

So... Why Not Always Bucketsort?

- And what happened to our lower bound?
- We are not using a binary decision tree!
- This is only because we know something about the input.
- Also – we did only average case analysis. What is the worst case?