# CS624 - Analysis of Algorithms

BFS

October 31, 2019

# Graphs – Basic Definitions

- Graph $G = (V, E)$
- V = set of vertices, E = set of edges $\subseteq (V \times V)$
- Undirected graph: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
- Directed graph: $(u, v)$ is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
- Weighted graph: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbb{R}$.
- Dense graph: $|E| \approx |V|^2$.
- Sparse: $|E| \ll |V|2$.
- $|E| = O(|V|^2)$

# Graphs – Basic Definitions

- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is symmetric if G is undirected, not necessarily so if G is directed.
- G is connected if there is a path between every pair of vertices.
- In this case $|E| \geq |V| - 1$.
- Furthermore, if $|E| = |V| - 1$, then G is a tree.
- Other definitions in Appendix B (B.4 and B.5) as needed.

# Graph Search Algorithms

- Searching a graph: Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms:
  - Breadth-first Search (BFS).
  - Depth-first Search (DFS).

# Breadth-First Search (BFS)

- Let $G$ be an undirected graph.
- One way to represent a graph is by a set *adjacency lists*, one for each vertex.
- For each vertex $v \in V$, we have a list $Adj[v]$ consisting of those vertices $u$ such that $(v, u) \in E$.
- It is actually a set, but usually implemented as a list.
- This representation works just as well for directed graphs.
- In this case, the edge $(v, u)$ means the edge starting from $v$ and ending at $u$.
- BFS scans the graph $G$, starting from some arbitrary node $s$.
- The key mechanism in this algorithm is the use of a queue, denoted by $Q$.

# The BFS Algorithm

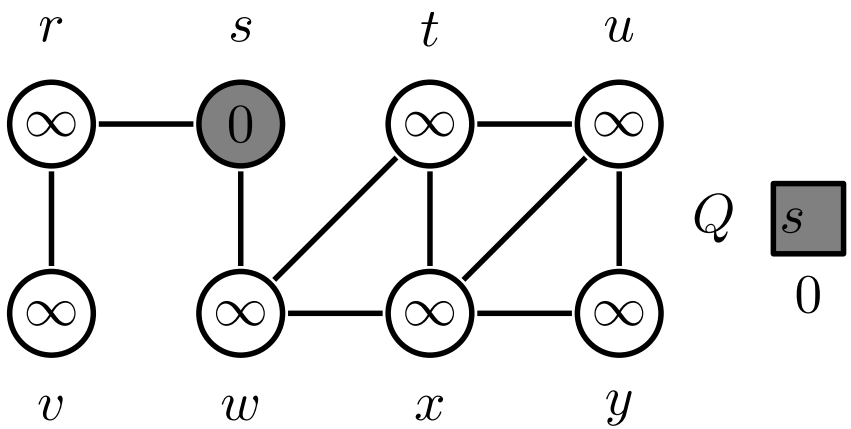
**Algorithm 1** $BFS(G, s)$

```
1:  for each vertex u ∈ V[G] \ s do
2:      Color[u] ← White
3:      d[u] ← ∞
4:      π[u] ← nil
5:  end for
6:  Color[s] ← Gray
7:  d[s] ← 0
8:  π[s] ← nil
9:  Q ← ∅
10: Enqueue(Q, s)
11: while Q ≠ ∅ do
12:     u ← Dequeue(Q)
13:     for each v ∈ Adj[u] do
14:         if Color[v] == White then
15:             Color[v] ← Gray
16:             d[v] ← d[u] + 1
17:             π[v] ← u
18:             Mark the edge from π[v] to u as a "tree edge".
19:             Enqueue(Q, v)
20:         end if
21:     end for
22:     Color[u] ← Black
23: end while
```

# The BFS Algorithm

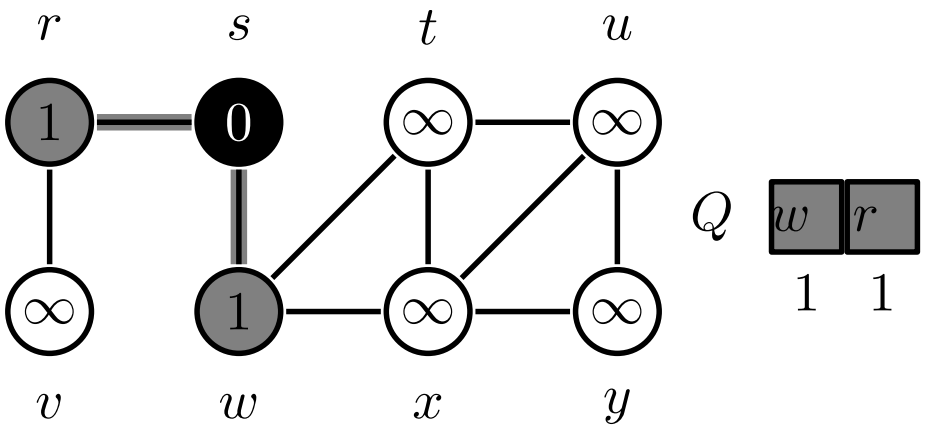
- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
- A vertex is "discovered" the first time it is encountered during the search.
- A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
- White – Undiscovered.
- Gray – Discovered but not finished.
- Black – Finished.
- Colors are required only to reason about the algorithm. Can be implemented without colors.

# The BFS Algorithm – Example



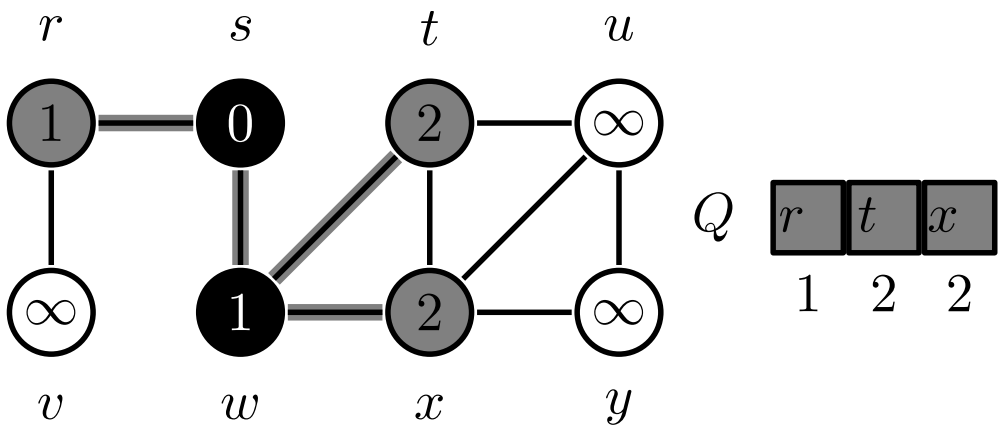

- Note that all nodes are initially colored white.
- A node is colored gray when it is placed on the queue.

# The BFS Algorithm – Example



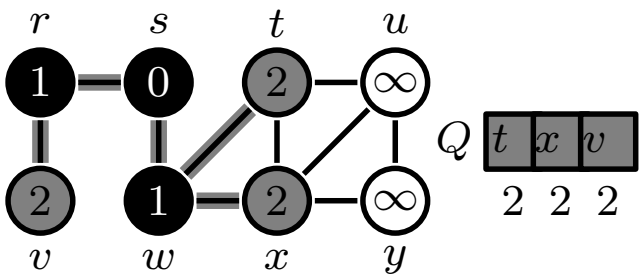

- A node is colored black when taken off the queue.
- Nodes colored white have not yet been visited. The nodes colored black are "finished" and the nodes colored gray are still being processed.

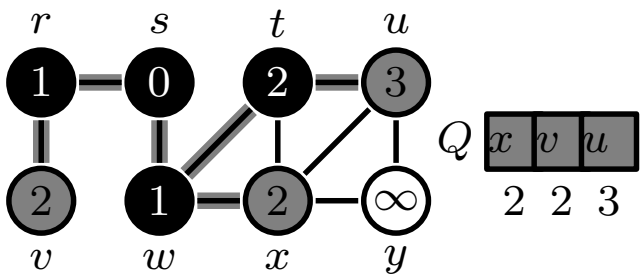
# The BFS Algorithm – Example

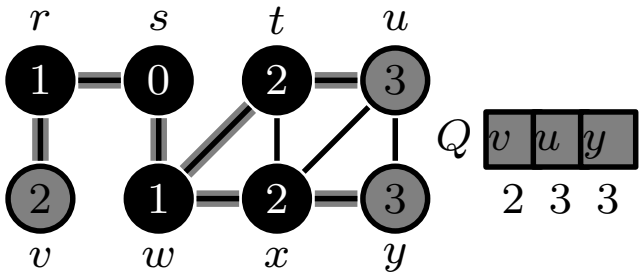


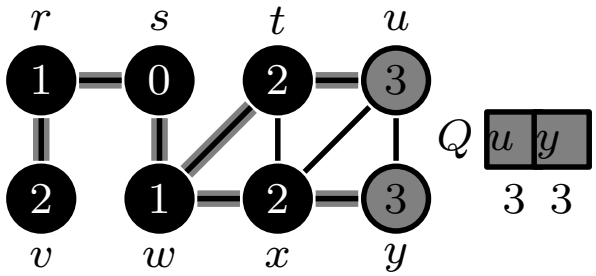

- When a node is placed on the queue, the edge from the first node in the queue (which is being taken off the queue) to that node is marked as a *tree edge* in the breadth-first tree.
- These edges actually do form a tree (called the breadth-first tree) whose root is the start node $s$.

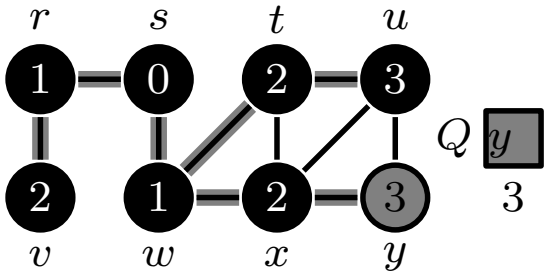
# The BFS Algorithm – Example

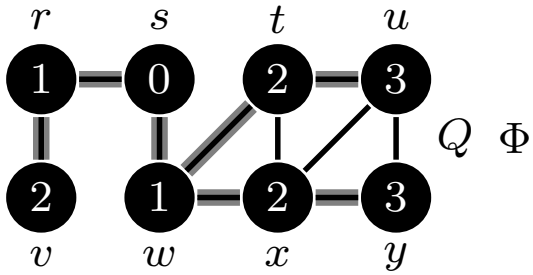


# The BFS Algorithm – Runtime and Properties

- Each node is visited once and each edge is examined at most twice.
- Therefore the cost is $O(|V| + |E|)$.
- Proof of correctness:

**Lemma**

*If $G$ is connected, then the breadth-first tree constructed by this algorithm*

- *Really is a tree*
- *It contains all the nodes in the graph.*

# The BFS Algorithm – Proof of Correctness

**Proof.**

- A node becomes the target of a tree edge when it is placed on the queue.
- Since that only happens once, no node is the target of two tree edges.
- Next, let us show that every node that is processed by the algorithm is reachable by a chain of tree edges from the root. It is enough to prove the following statement:
- When a node is placed on the queue, it is reachable by a chain of tree edges from the root.
- It is clearly true at the beginning: There is only one node in the queue and it is the root. The rest can be shown by induction.

# The BFS Algorithm – Proof of Correctness

## Cont.

- Suppose it is true up to some point.
- When the next node $v$ is placed on the queue, $v$ is an endpoint of an edge whose other endpoint is the node at the head of the queue, and that edge is made a tree edge.
- By the inductive assumption, the node at the head of the queue is reachable by a path of tree edges from the root.
- Appending the new edge to the path gives a path of tree edges from the root to $v$.

# The BFS Algorithm – Proof of Correctness

## Cont.

- Every node that is processed by the algorithm is reachable by a chain of edges from the root – so the edges form a tree.
- Suppose there was one node $v$ that was not reached by this process.
- Since $G$ is connected, there would have to be a path from the root to $v$.
- On that path there is a *first* node ($w$) which was not in the tree.
- That node might be $v$, or it might come earlier in the path.
- That means that the edge in the path leading to that node starts from a node in the tree.
- At some point, that node in the tree was at the head of the queue.
- Therefore, $w$ would have been placed in the queue by the algorithm, and the edge to $w$ would have been a tree edge – a contradiction.

# The BFS Algorithm – Proof of Correctness

**Lemma**

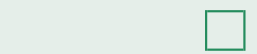
*If at any point in the execution of the BFS algorithm the queue consists of the vertices* $\{v_1, v_2, \ldots, v_n\}$*, where* $v_1$ *is at the head of the queue, then* $d[v_i] \leq d[v_{i+1}]$ *for* $1 \leq i \leq n-1$*, and* $d[v_n] \leq d[v_1| + 1$.

- In other words, the assigned depth numbers increase as one walks down the queue, and there are at most two different depths in the queue at any one time.
- If there are two, they are consecutive.

# The BFS Algorithm – Proof of Correctness

**Proof.**

- The result is true trivially at the start of the program, since there is only one element in the queue. The rest by induction.
- At any step, a vertex is added to the tail of the queue only when it is reachable from the vertex at the head (which is being taken off).
- The depth assigned to the new vertex at the tail is 1 more than that of the vertex at the head.
- By the inductive hypothesis it is greater than or equal to the depths of any other vertex on the queue, and no more than 1 greater than any of them.



# The BFS Algorithm – Proof of Correctness

## Lemma

*If two nodes in $G$ are joined by an edge in the graph (which might or might not be a tree edge), their $d$ values differ by at most 1.*

## Proof.

- Let the nodes be $v$ and $u$. One of them is reached first in the breadth-first walk.
- w.l.o.g, say $v$ is reached first. So $v$ is put on the queue first, and reaches the head of the queue before $u$ does. When $v$ reaches the head of the queue, there are two possibilities:
  - $u$ has not yet been reached. In that case, when we take $v$ off the queue, since there is an edge from $v$ to $u$, $u$ will be put on the queue and we will have $d[u] = d[v] + 1$.
  - $u$ has been reached and therefore is on the queue. In this case, we know from the previous lemma that $d[v] \leq d[u] \leq d[v] + 1$.

# The BFS Algorithm – Proof of Correctness

**Theorem**

*If $G$ is connected, then the breadth-first search tree gives the shortest path from the root to any node.*

**Proof.**

- We know there is a path in the tree from the root to any node.
- The depth of any node in the tree is the length of the path in the tree from the root to that node.
- So for each node $v$ in the tree, we have

  $d[v] =$ the length of the path in the tree from the root to $v$

  and let us set

  $s[v] =$ the length of the shortest path in $G$ from the root to $v$

# The BFS Algorithm – Proof of Correctness

## Cont.

- We are trying to prove that $d[v] = s[v]$ for all $v \in G$.
- We know just by the definition of $s[v]$ that $s[v] \leq d[v]$ for all $v$.
- Suppose there is at least one node for which the theorem is not true.
- All the nodes $w$ for which the statement of the theorem is not true satisfy $s[w] < d[w]$.
- Among all those nodes, pick one – call it $v$ – for which $s[v]$ is smallest.

# The BFS Algorithm – Proof of Correctness

## Cont.

- Let $u$ be the node preceding $v$ on a shortest path from the root to $v$.
- We have

$$d[v] > s[v]$$
$$s[v] = s[u] + 1$$
$$s[u] = d[u]$$

- Hence $d[v] > s[v] = s[u] + 1 = d[u] + 1$.
- But by former lemma, this is impossible.

# Print Shortest Path

We assume that $BFS(G, s)$ has already been run, so that each node $x$ has been assigned its depth $d[x]$.

**Algorithm 2** *PrintPath*$(G, s, v)$

```
1: if v = s then
2:     PRINT s
3: else
4:     if π[v] == nil then
5:         PRINT "no path from" s "to" v "exists"
6:     else
7:         PrintPath(G, s, π[v])
8:         PRINT v
9:     end if
10: end if
```

The cost of this algorithm is proportional to the number of vertices in the path, so it is $O\big(d[v]\big)$.