CS624 - Analysis of Algorithms

Medians and Order Statistics

September 30, 2019

Midterm Exam I

- The midterm exam will take place on Thursday, October 17, in class.
- Covered material: Induction, runtime analysis, sorting (mergesort, insertion sort, quicksort, heapsort, lower bounds), heaps, medians and order statistics (maybe, depends on whether you get your HW in time).
- The class on October 15 will be partly a review class.
- Prepare your own questions to ask me!!!

Midterm Exam I

- Probably 4-5 questions. Assume every topic will be covered.
- Bring class notes and homework assignments.
- No books, no computers, no cellphones/smartphones/tablets, strictly no friends.
- It will count towards 20% of your final grade.

Medians and Order Statistics

- i^{th} order statistic: i^{th} smallest element of a set of n elements.
- Minimum: first order statistic.
- Maximum: n^{th} order statistic.
- Median: "half-way point" of the set.
- Unique, when n is odd – occurs at i = (n+1)/2.
- Two medians when n is even.
- Lower median, at i = n/2.
- Upper median, at i = n/2+1.
- For consistency, "median" will refer to the lower median.

Selection Problem

- Selection problem:
  - Input: A set A of n **distinct** numbers and a number k, with $1 \leq k \leq n$.
  - Output: the element $x \in A$ that is larger than exactly k-1 other elements of A (the k^{th} order statistics).
- Can be solved in $O(n \log n)$ time. How?
- We will study faster linear-time algorithms.
  - For the special cases when k = 1 and k = n.
  - For the general problem.

Minimum (Maximum)

- Minimum (Maximum) can be found in $\Theta(n)$ time.
- Simply scan all the elements and find the smallest (largest).
- Some applications need to determine both the maximum and minimum of a set of elements.
- Example: Graphics program trying to fit a set of points onto a rectangular display.
- Independent determination of maximum and minimum requires 2n-2 comparisons.
- Can we reduce this number?

Simultaneous Minimum and Maximum

- Maintain minimum and maximum elements seen so far
- Process elements in pairs
- Compare the smaller to the current minimum and the larger to the current maximum
- Update current minimum and maximum based on the outcomes
- No. of comparisons per pair $= 3$. How?
- No. of pairs $\leq \lfloor n/2 \rfloor$.

Simultaneous Minimum and Maximum

- For odd n: initialize min and max to A[1]. Pair the remaining elements. So, no. of pairs $= \lfloor n/2 \rfloor$
- For even n: initialize min to the smaller of the first pair and max to the larger. So, remaining no. of pairs $= (n-2)/2 < \lfloor n/2 \rfloor$.
- Total no. of comparisons, $C \leq 3\lfloor n/2 \rfloor$.
- For odd n: $C = 3\lfloor n/2 \rfloor$.
- For even n: $C = 3(n-2)/2 + 1$ (For the initial comparison). $= 3n/2 - 2 < 3\lfloor n/2 \rfloor$

Finding k^{th} SmallestElement

- Why can't we use a similar method for any order statistics in a linear time?
- The cost of finding the k^{th} order statistic using either of these methods is $\Theta(kn)$. If k is fixed, this is $\Theta(n)$.
- If k is not fixed, this is not so good. For instance, suppose we want to find the median.
- Then k is about $n/2$, and the cost would be quadratic.
- Worse than sorting the array...
- Even using heaps won't do better than sorting the array for finding the median.

General Selection Problem

- Find the i^{th} order statistics.
- Seems more difficult than Minimum or Maximum.
- Yet, has solutions with same asymptotic complexity as Minimum and Maximum.
- We will study an algorithm for the general problem with expected linear-time complexity (independent of k).
- A second algorithm, whose worst-case complexity is linear, can be found in the text.

General Selection Problem

- Modeled after randomized quicksort.
- Exploits the abilities of Randomized-Partition (RP).
- It uses Partition repeatedly, except that at each step, we only have to use recursion on one side of the partitioned set.
- We hope that in the "average case", we recurse on a subarray that is about half the size of the previous subarray.
- Then the total cost will be the cost of the partitions, which will be roughly some constant times

$$n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots = 2n$$

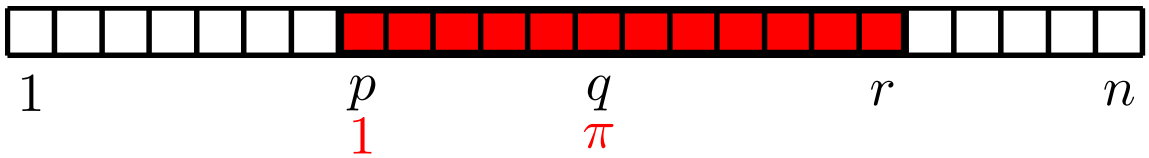
- So the total cost on average should be $O(n)$.

Randomized Select

Algorithm 1 RandomizedSelect(A,p,r,i)

```
1:  if p = r then
2:      return A[p]
3:  end if
4:  q ← RandomizedPartition(A, p, r)
5:  π ← q − p + 1
6:  if i = π then
7:      return A[q]
8:  else
9:      if i < π then
10:         return RandomizedSelect(A, p, q − 1, i)
11:     end if
12: else
13:     return RandomizedSelect(A, q + 1, r, i − π)
14: end if
```

Randomized Select



Notation used in the algorithm RandomizedSelect:

- p, q, and r are indices in the original array A.
- π is the 1-based index of the pivot $A[q]$ in the subarray $A[p \ldots r]$. (*Be careful – π here is used to denote just an ordinary variable.*

Analysis

We see that Randomized-Select is divided into 3 cases:

1. $\pi < i$, so we search for the $(i - \pi)^{th}$ element in A[q+1..r]
2. The pivot $\pi = i$. We finish and exit.
3. $\pi > i$, so we search for the i^{th} element in A[p..q-1].

Analysis

- Worst-case Complexity:
  - $\Theta(n^2)$ – As we could get unlucky and always recurse on a subarray that is only one element smaller than the previous subarray.
- Average-case Complexity:
  - $\Theta(n)$ – Intuition: Because the pivot is chosen at random, we expect that we get rid of half of the list each time we choose a random pivot q.
- Why $\Theta(n)$ and not $\Theta(nlogn)$?

Average Runtime Analysis

- Denote the average runtime of RandomizedSelect(A,1,n,i) by $C(n, i)$.
- We will find an upper bound for $C(n, i)$:
- $T(n) = \max\{C(n, i) : 1 \leq i \leq n\}$.
- That is, $T(n)$ is the worst average-case time of computing *any* i^{th} element of an array of size n using RandomizedSelect.
- We will prove that $T(n) = O(n)$.
- First of all – the cost of partition is $O(n)$, so we can bound it by a*n for some a.

Average Runtime Analysis

Therefore:

$$
\begin{aligned}
C(n,i) &\leq an + \frac{1}{n}\left(\sum_{\pi=1}^{i-1} C(n-\pi, i-\pi) + \sum_{\pi=i+1}^{n} C(\pi-1, i)\right) \\
&\leq an + \frac{1}{n}\left(\sum_{\pi=1}^{i-1} T(n-\pi) + \sum_{\pi=i+1}^{n} T(\pi-1)\right) \\
&\leq \max\{an + \frac{1}{n}\left(\sum_{\pi=1}^{i-1} T(n-\pi) + \sum_{\pi=i+1}^{n} T(\pi-1)\right) : 1 \leq i \leq n\} \\
&= an + \max\{\frac{1}{n}\left(\sum_{\pi=1}^{i-1} T(n-\pi) + \sum_{\pi=i+1}^{n} T(\pi-1)\right) : 1 \leq i \leq n\}
\end{aligned}
$$

Average Runtime Analysis – Explanation

- The call to RandomizedSelect has two parts: The partition which is the *an* part, and the recursive call whose cost varies depending on the location of the pivot which we denote π.
- We assume that the pivot is equally likely to wind up in any of the n positions in the array, and we average over all those n possibilities.
- That accounts for the factor $\frac{1}{n}$ just outside the big parenthesized term on the right-hand side.
- Inside the parentheses is the sum of all the possibilities that can happen.
- The first term is if the pivot falls before i, the second term is if the pivot falls after i.
- In the third case (where the pivot is exactly i) we just end the run...

Average Runtime Analysis – Explanation (Cont.)

- Once we have taken the maximum over i in the last line, we note that the final right-hand side is actually independent of i.
- Therefore since for each i, $C(n, i)$ on the left-hand side is $\leq$ this expression, the maximum of them all is as well.
- That is, we can take the maximum over i of the left-hand side, (using the previous equation) and get a term which we will use for proof by induction.

$$T(n) \leq an + \max\{\frac{1}{n}\left(\sum_{\pi=1}^{i-1} T(n-\pi) + \sum_{\pi=i+1}^{n} T(\pi-1)\right) : 1 \leq i \leq n\}$$

Average Runtime Analysis – Proof by Induction

- Base case: We can arrange that this is true for $n = 2$ by making sure (when we finally figure out an appropriate value for C) that $C \geq a$.
- Prove that the inductive hypothesis remains true for $k = n$. We have two things we can use:
  - the inductive hypothesis which we can assume is true for $1 \leq k < n$
  - the recursive inequality above.

Average Runtime Analysis – Proof by Induction

We start with the recursive inequality:

$$T(n) \leq an + \max\{\frac{1}{n}\left(\sum_{\pi=1}^{i-1} T(n-\pi) + \sum_{\pi=i+1}^{n} T(\pi-1)\right) : 1 \leq i \leq n\}$$

$$\leq an + \max\{\frac{C}{n}\left(\sum_{\pi=1}^{i-1}(n-\pi) + \sum_{\pi=i+1}^{n}(\pi-1)\right) : 1 \leq i \leq n\}$$

$$= an + \max\{\frac{C}{n}\left((i-1)n - \frac{(i-1)i}{2} + \frac{(n-1)n}{2} - \frac{(i-1)i}{2}\right) : 1 \leq i \leq n\}$$

$$= an + \max\{\frac{C}{n}\left((i-1)n - (i-1)i + \frac{(n-1)n}{2}\right) : 1 \leq i \leq n\}$$

Average Runtime Analysis – Proof by Induction

- We have to find the maximum of $(i-1)n-(i-1)i = -i^2+(n+1)i-n$ between $i=1$ and $i=n$.
- This is the kind of thing we've seen before: this is a concave function of i.
- in fact, it's an "upside-down parabola" – and so its maximum occurs where the derivative is 0.
- The derivative is simply -2i + (n+1) and this is 0 when $i = \frac{n+1}{2}$.
- So the maximum value of the expression $(i-1)n-(i-1)i$, which is also $(i-1)(n-i)$, is

$$\left(\frac{n+1}{2}-1\right)\left(n-\frac{n+1}{2}\right) = \frac{n-1}{2}\frac{n-1}{2} = \frac{(n-1)^2}{4}$$

Average Runtime Analysis – Proof by Induction

So we have

$$
\begin{aligned}
T(n) &\leq an + \frac{C}{n}\left(\frac{(n-1)^2}{4} + \frac{(n-1)n}{2}\right) \\
&= an + \frac{C}{n}\left(\frac{n^2-2n+1}{4} + \frac{n^2-n}{2}\right) \\
&= an + \frac{C}{n}\left(\frac{3n^2}{4} - n + \frac{1}{4}\right) \\
&= an + C\left(\frac{3n}{4} - 1 + \frac{1}{4n}\right) \\
&\leq an + C\frac{3n}{4} \qquad \text{for } n \geq 1 \\
&= \left(a + \frac{3}{4}C\right)n
\end{aligned}
$$

Average Runtime Analysis – Proof by Induction

So we can fix C once and for all so that

- $C \geq a$, and
- $a + (3/4)C \leq C$

(for instance, $C = 4a$ would work), then we get $T(n) \leq Cn$ and we are done.