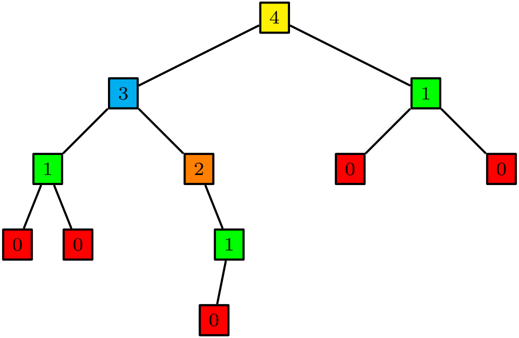
CS624 - Analysis of Algorithms

Heaps

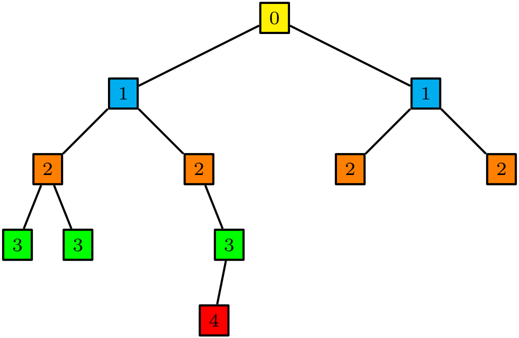
September 17, 2019

Define the height of a node in a tree as the number of edges on the longest path from that node down to a leaf



The height of the tree, H, is the height of its root.

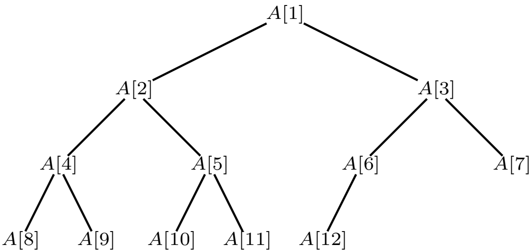
The *level* of the root is 0, the children of the root are at level 1. In general, the children of a node of level k are at level $k+1$.



Heaps and Pre-Heaps

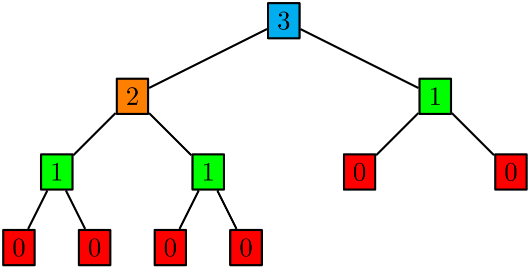
- In a binary tree, there are at most 2^k nodes at level k.
- If the highest level is completely filled in, that level contains 2^H nodes, and the tree contains $1 + 2 + 4 + \cdots + 2^H = 2^{H+1} - 1$ nodes.
- A *heap* is a special kind of a binary tree (do not confuse with the CS term related to memory allocation!).
- Let us define a $pre - heap$ as follows:
  - All leaves are on at most two adjacent levels.
  - Except maybe the lowest level, all the levels are completely filled. The leaves on the lowest level are filled in, without gaps, from the left.

Pre-Heap – Example



- Notice that it can be represented as a simple array.
- The children of the node holding $A[n]$ are the nodes holding $A[2n]$ and $A[2n+1]$, and the parent of the the node holding $A[n]$ is $A\left[\lfloor n/2 \rfloor\right]$.

Pre-Heap – Another Example



- The nodes are tagged by their height.
- All the levels less than 3 are completely filled in, and there are a total of $2^3 - 1$ nodes at those levels.

Pre-Heap Properties

- If we have a pre-heap with n nodes, denote its height by H.
- As seen above, we must have $2^H \leq n \leq 2^{H+1} - 1 < 2^{H+1}$. Equivalently, $H = \lfloor \log_2 n \rfloor$

Lemma

In a pre-heap with n elements, there are $\left\lceil \frac{n}{2} \right\rceil$ leaves.

Proof.

- Some leaves are at level H, and some are at level $H - 1$.
- Since the number of nodes at level $H - 1$ or less is $2^H - 1$, the number of leaves at level H is $n - (2^H - 1)$.
- The parent of node n is node $\left\lfloor \frac{n}{2} \right\rfloor$, and that node is the last node of height 1.

□

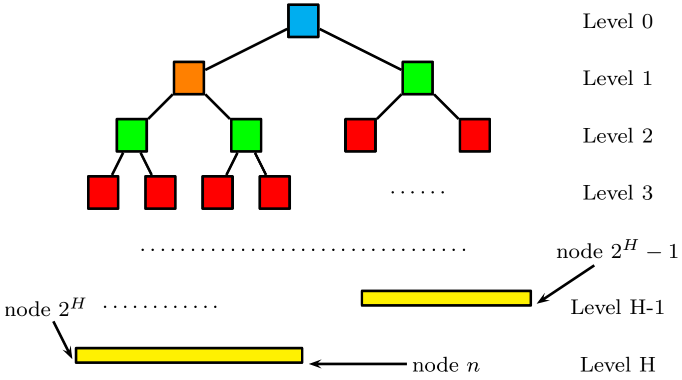
Proof (Cont.)

- So all the rest of the nodes at level $H-1$ are of height 0, i.e., are leaf nodes.
- Therefore the number of leaves at level $H-1$ is $(2^H-1)-\left\lfloor \frac{n}{2} \right\rfloor$.
- Hence the total number of leaves is

$$n-(2^H-1)+(2^H-1)-\left\lfloor \frac{n}{2} \right\rfloor = n-\left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$$

$\square$

Pre-Heap Properties



Pre-Heap Properties

Corollary

In a pre-heap with height H, there are at most 2^H leaves.

Proof.

If n is the number of elements in the pre-heap, we know that $2^H \leq n \leq 2^{H+1} - 1 < 2^{H+1}$. Then by the Lemma, the number of leaves is

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{2^{H+1}}{2} = 2^H$$

□

Pre-Heap Properties

Theorem

In a pre-heap with n elements, there are at most $\frac{n}{2^h}$ nodes at height h.

Proof.

- We have just seen that there are at most 2^H leaves in such a tree, and the leaves are just the nodes at height 0.
- If we take away the leaves, we have a smaller pre-heap with at most 2^{H-1} leaves, and these leaves are exactly the nodes at height 1 in the original tree.
- Continuing, we see that there are at most 2^{H-h} nodes at height h in the original tree, therefore $2^{H-h} = \frac{2^H}{2^h} \leq \frac{n}{2^h}$

□

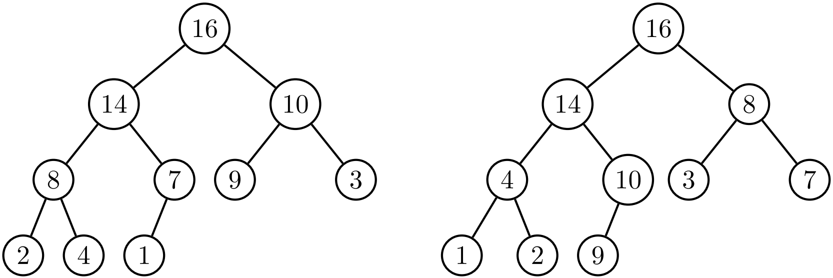
Definition

- A *heap* is a binary tree with a key in each node.
- The keys must be comparable.
- Additionally, the heap must have the following properties:
  - All leaves are on at most two adjacent levels.
  - With the possible exception of the lowest level, all the levels are completely filled. The leaves on the lowest level are filled in, without gaps, from the left.
  - The key at each node is greater than or equal to the key in any descendant of that node.

Heaps

- Note that another way of phrasing the third condition would be:
  - The key in the root is greater than or equal to that of its children, and its left and right subtrees are again heaps.
- Thus every heap is a pre-heap.
- Even though the shape of a heap containing n elements is uniquely determined (since it is a pre-heap), the arrangement of those n elements is not.

Example – Two Heaps With the Same Set of Keys



The Heapify Procedure

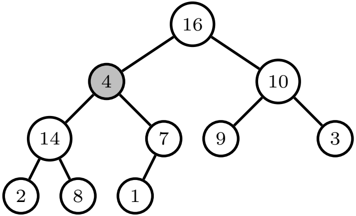
- The fundamental procedure to build a heap.
- We have a binary tree in the shape of a heap (but perhaps not actually a heap).
- We represent the tree as an array $A[1..n]$, where n is the size of the heap.
- We look at node i (holding the value $A[i]$). We assume that:
  - The tree rooted at $l = Left(i)$ is a heap.
  - The tree rooted at $r = Right(i)$ is a heap.
- However, we do not assume that the tree rooted at i is a heap.
- Heapify works by letting the value $A[i]$ "float down" to its proper position in the heap.

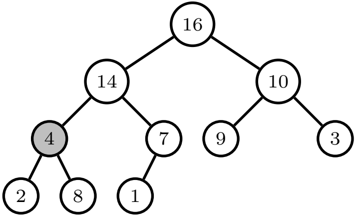
The Heapify Procedure

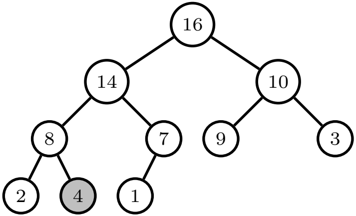
Algorithm 1 Heapify(A,i)

```
l ← Left(i)
r ← Right(i)
if l ≤ Heapsize[A] and A[l] > A[i] then
    largest ← l
else
    largest ← i
end if
if r ≤ heapsize[A] and A[r] > A[largest] then
    largest ← r
end if
if largest ≠ i then
    exchangeA[i] ↔ A[largest]
    Heapify(A, largest)
end if
```

Heapify – Example







Running Time of Heapify

The time needed to run Heapify on a subtree of size n rooted at a given node i is

- time $\Theta(1)$ to fix up the relationships among the elements $A[i]$, $A\big[Left(i)\big]$, and $A\big[Right(i)\big]$
- time to run Heapify on a subtree rooted at one of the children of node i.
- That subtree has size at most $2n/3$ – the worst case occurs when the last row of the tree is exactly half full.

So the running time $T(n)$ can be characterized by the recurrence

$$T(n) \leq T(2n/3) + \Theta(1)$$

This falls into case 2 of the "master theorem", and so we must have $T(n) = O(\log n)$

Building a Heap

The heap is built from the bottom up, starting at the first non-leaf node.

Algorithm 2 BuildHeap(A)

1: $heapsize[A] \leftarrow length[A]$
2: **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ to 1 **do**
3: $\quad Heapify(A, i)$
4: **end for**

To prove that this is correct We use the following loop invariant:

Lemma

At the start of each iteration of the for loop, each node $i+1, i+2, \ldots, n$ is the root of a heap.

Proof of Correctness

Proof.

- On the first iteration of the loop, $i = \lfloor n/2 \rfloor$. Each node $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$ is a leaf and is thus the root of a trivial heap.
- Inductive step – going from $i + 1$ to i, we assume that each element $i + 1, i + 2, \ldots, n$ is the root of a heap.
- Therefore each of the two children of node i (i.e., nodes $2i$ and $2i + 1$) is the root of a heap.
- Therefore the call to $Heapify(A, i)$ makes i the root of a heap.
- Further, all nodes which are not descendants of i are untouched by the call to $Heapify(A, i)$, (Do you see why?) and so we can conclude that each node $i, i + 1, \ldots, n$ is now the root of a heap.

$\square$

Running Time of BuildHeap

- The number of elements of the heap at height h is $\leq \frac{n}{2^h}$, and the cost of running Heapify on a node of height h is $O(h)$.
- The root of a heap of n elements has height $\lfloor \log_2 n \rfloor$.
- Therefore the worst-case cost of running BuildHeap on a heap of n elements is bounded by

$$\sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h} O(h) = O\left(n \sum_{h=0}^{\lfloor \log_2 n \rfloor} \frac{h}{2^h} \right) = O(n)$$

since the sum converges, so we don't care what the upper bound of the summation is.

Heap Properties

- Heaps give us partial information about the order of elements in a set.
- We can tell immediately what the largest element is.
- They are really cheap to build and can be stored in a simple array.
- This makes them very useful for various applications.

Heapsort

Algorithm 3 Heapsort(A)

$BuildHeap(A)$
for $i \leftarrow length[A]$ to 2 **do**
    $exchange\ A[1] \leftrightarrow A[i]$
    $heapsize[A] \leftarrow heapsize[A] - 1$
    $Heapify(A, 1)$
end for

The call to BuildHeap takes time $O(n)$. Each of the $n - 1$ calls to Heapify takes time $O(\log n)$. Hence the total running time is (in the worst case) $O(n \log n)$.

Priority Queues

Definition

A *priority queue* is a data structure that maintains a set S of elements, each with an associated value called a *key*. (As usual, the keys must be comparable.) The priority queue supports the following operations:

- $\mathit{Insert}(S, x)$ inserts the element x into the set S.
- $\mathit{Maximum}(S)$ returns the element of S with the largest key.
- $\mathit{ExtractMax}(S)$ removes and returns the element of S with the largest key.
- $\mathit{IncreaseKey}(S, x, k)$ increases the value of element x's key to the new value k, which is assumed to be at least as large as x's current key value.

A priority queue can be implemented using a heap.

Priority Queue Operations

Algorithm 4 HeapMaximum(A)

```
return A[1]
```

Obviously, the run time is O(1).

Algorithm 5 HeapExtractMax(A)

```
if heapsize[A] < 1 then
    ERROR – heap underflow
end if
maxx ← A[1]
A[1] ← A[heapsize[A]]
heapsize[A] ← heapsize[A] − 1
Heapify(A, 1)
return maxx
```

Here the running time is dominated by the call to Heapify, so it is $O(\log n)$.

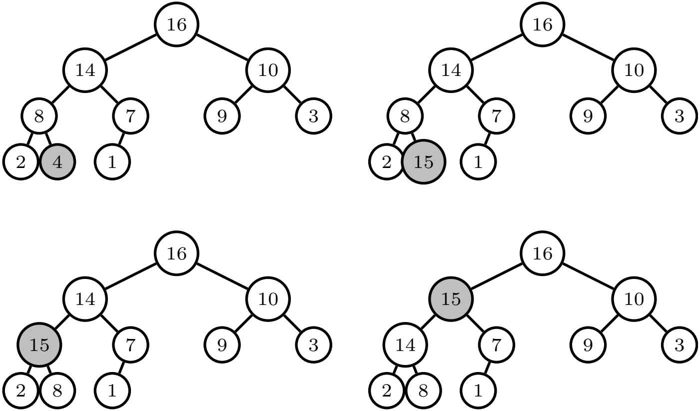
Priority Queue Operations

Algorithm 6 HeapIncreaseKey(A, i, key)

```
if key < A[i] then
    ERROR – new key is smaller than current key
end if
A[i] ← key
while i > 1 and A[Parent(i)] < A[i] do
    exchangeA[i] ↔ A[Parent(i)]
    i ← Parent(i)
end while
```

We just increase the key of $A[i]$, and then let that node "float up" to its proper position.

HeapIncreaseKey – Example



HeapInsert

Algorithm 7 HeapInsert(A, key)

1: $heapsize[A] \leftarrow heapsize[A] + 1$
2: $A[heapsize[A]] \leftarrow -\infty$
3: $HeapIncreaseKey(A, heapsize[A], key)$

The running time here is again $O(\log_2 n)$.
Thus, a heap supports any priority queue operation on a set of size n in $O(\log n)$ time.