# 1819-108-C1-W4-FirstExam

Renāts Jakubovskis

February 2019

and, on differentiating twice with respect to $\mu$, we find

$$\frac{\delta^2 lnP}{\delta\mu^2} = -\frac{N}{\delta^2}$$

This is independent of the $x_i$ and so its expectation value is also equal to $-N/\mu^2$. With b set equal to zero in (31.17), Fisher‘s inequality thus states that, for any unbiased estimator $\hat{\mu}$ of the population mean,

$$V[\hat{\mu}] \geq \tfrac{\delta^2}{N}.$$

Since $V[\overline{x}] = \delta^2/N$, the sample mean $\overline{x}$ is a minimum-variance estimator of $\mu$. ◄

### 31.3.2 Fisher‘s inequality

As mentioned above, Fisher‘s inequality provides a lower limit on the variance of any estimator $\hat{a}$ of the quantity a; it reads

$$V[\hat{a}] \geq \Big(1 + \frac{\delta b}{\delta a}\Big)^2 \Big/ E\Big[-\frac{\delta^2 lnP}{\delta a^2}\Big], \tag{31.18}$$

where P stands for the population $P(x|a)$ and b is the bias of the estimator. We now present a proof of this inequality. Since the derivation is somewhat complicated, and many of the details are unimportant, this section can be omitted on a first reading. Nevertheless, some aspects of the proof will be useful when the efficiency of maximum-likelihood estimators is discussed in section 31.5.

► *Prove Fisher‘s inequality (31.18).*

The normalisation of P(x—a) is given by

$$\int P(x|a) d^N x = 1, \tag{31.19}$$

where $d^N x = dx_1 dx_2 ... dx_N$ and the integral extends over all the allowed values of the sample items $x_i$. Differentiating (31.19) with respect to the parameter a, we obtain

$$\int \frac{\delta P)}{\delta a} d^N x = \int \frac{\delta lnP}{\delta a} P d^N x = 0. \tag{31.20}$$

We note that the second integral is simply the expectation value of $\delta lnP/\delta a$, where the average is taken over all possible samples $x_i, i = 1, 2, ..., N$. Further, by equating the two expressions for $\delta E[\hat{a}]/\delta a$ obtained by differentiating (31.15) and (31.14) with respect to a we obtain, dropping the functional dependencies, a second relationship,

$$1 + \frac{\delta b}{\delta a} \int \hat{a} \frac{\delta P)}{\delta a} d^N x = \int \hat{a} \frac{\delta lnP}{\delta a} P d^N x. \tag{31.21}$$

Now, multiplying (31.20) by $\alpha(a)$, where $\alpha(a)$ is any function of a, and subtracting the result from (31.21), we obtain

$$\textstyle\int [\hat{a} - \alpha(a)] \frac{\delta lnP}{\delta a} P d^N x = 1 + \frac{\delta b}{\delta a}.$$

At this point we must invoke the Schwarz inequality proved in subsection 8.1.3. The proof

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\documentclass{report}
\usepackage[utf8]{inputenc}
\usepackage{geometry}
\usepackage{manfnt}
\usepackage{wasysym}
\usepackage{amssymb}
\geometry{verbose,a4paper,tmargin=2cm,bmargin=2cm,lmargin=2.5cm,rmargin=1.5cm}

\title{1819-108-C1-W4-FirstExam}
\author{Renāts Jakubovskis}
\date{February 2019}

\begin{document}

\maketitle

\begin{center}
    \LARGE 31.3 ESTIMATORS AND SAMPLING DISTRIBUTIONS
\end{center}
\line (1,0) {470}



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\end{tabular}



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\setcounter{page}{1233}
\end{document}
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