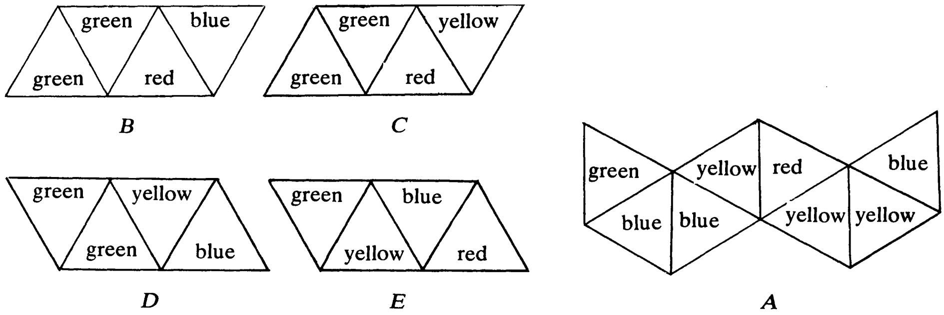

A coloring scheme for the pieces of Cleopatra's Pyramid that produces a puzzle slightly more complex than Instant Insanity. (Pieces are formed by folding with the colors out.)

FIGURE 2.

the pieces. Although this puzzle can be solved readily by trial and error, perhaps the reader might like to apply the graph techniques [2] used to solve Instant Insanity.

Serendipity occurs in a rather unique way in this topic. Several months after determining that the coloring scheme above is best, a colleague pointed out that the puzzle could be solved with the altered rule: only one color can appear on each face. In fact, the solution is unique with this rule. Furthermore, two solutions of the original puzzle can be obtained from the latter puzzle by merely rotating each small tetrahedra about the line normal to its hidden face. Either make all rotations clockwise or make all rotations counterclockwise.

Recall that there are $4(12)^5$ and $2(12)^4$ distinct arrangements for Cleopatra's Pyramid and Instant Insanity, respectively, and that there are 19 and 1 solutions respectively. Hence, although the ultimate goal of a unique solution to Cleopatra's Pyramid has eluded us, we have still constructed a puzzle 24/19 times as difficult as Instant Insanity.

References

[1] T. A. Brown, A note on "Instant Insanity", this MAGAZINE, 41 (1968) 68.
[2] A. P. Grecoŝ and R. W. Gibberd, A diagrammatic solution to "Instant Insanity" problem, this MAGAZINE, 44 (1971) 71.
[3] J. V. Devertes, Many Facets of Graph Theory, Lecture Notes in Mathematics, vol. 110, Springer-Verlag, New York, p. 283.
[4] B. L. Swartz, An improved solution to Instant Insanity, this MAGAZINE, 43 (1970) 70.

Another Strategy for SIM

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The game of SIM [1, 2, 3] is based on the fact that if the $\binom{6}{2} = 15$ lines determined by 6 points are colored with two colors, then a monochromatic triangle must occur. In SIM, two players choose a color and alternately color one of the 15 lines. The first to complete a monochromatic triangle loses the game. In this paper we present a strategy for the second player and introduce the projective geometry dual game which we call TRI-NOT.

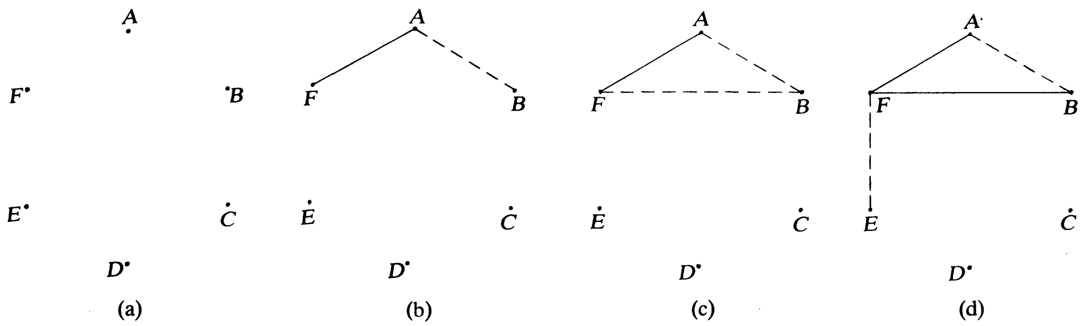


FIGURE 1.

There are several strategies for SIM in print **[4, 5]**. However, the strategy presented here is the first easily used strategy, for it requires calculations on only three plays. For convenience lable the 6 points $A, B, \ldots, F$ as in FIGURE 1(a). Let R (red — denoted by a solid line) be the first player, B (blue — denoted by a broken line) the second player. There are four rules for B to follow. Rules 1 and 2 actually are special cases of Rule 3, but are easily stated and allow the first 2 moves by B to be made without counting.

Rule 1. On move #2, B plays any line adjacent to the line chosen by R (FIGURE 1(b)).

Rule 2. On move #4 there are two cases:

(a) if R has not played to complete a triangle, i.e., BF in FIGURE 1(b), then B plays BF (FIGURE 1(c)).

(b) if R has played BF, then B plays any line containing the vertex F, i.e., B uses the vertex with red degree 2 (FIGURE 1(d)).

On moves #6, #8, and #10 a counting scheme is employed based on the playable lines, those that do not lead to an immediate loss. Let $\mathscr{R}$ be the lines playable by R only, $\mathscr{B}$, those playable by B only, $\mathscr{R} \vee \mathscr{B}$, those playable by either R or B, and $\mathscr{N}$, those not playable by either R or B. Suppose P, Q, S are three of the six vertices and that B has already played QS. Suppose that B is ready to choose his next move. If both PQ and PS are in $(\mathscr{R} \vee \mathscr{B}) \cup \mathscr{B}$, i.e., both playable by B, we say line PS is "lost by B" upon play of PQ. A brief experience with SIM suggest that a player should "lose" as few lines as possible. With this experience in mind we define several variables which will be used in the strategy.

Let x_{ij} be the number of lines "lost by B" upon play of line ij (where $i, j \in \{A, B, C, D, E, F\}$); x_{ij} is the number of lines which would be transferred from $(\mathscr{R} \vee \mathscr{B}) \cup \mathscr{B}$ to $\mathscr{R} \cup \mathscr{N}$ if B plays line ij. Our strategy will select a line from $\mathscr{R} \vee \mathscr{B}$ with minimum x value for moves #6, #8 and #10. If there is not a unique line in $\mathscr{R} \vee \mathscr{B}$ with minimum x value, we will need a finer distinction between essentially different lines with minimum x values.

If P, Q, S are three vertices and B can play all three of PQ, QS, PS, then if B plays PQ, x_{QS} and x_{PS} are each increased by 1. If R has played QS, and both R and B can play each of PQ and PS, then if B plays PQ the number $\bar{x}_{PS}$ (the number of lines lost by R) is reduced by 1. If neither of these apply, there is no change in the x or $\bar{x}$ numbers.

Hence we define

$$y_{ijk} = \begin{cases} 2, & \text{if lines } ik \text{ and } jk \in (\mathscr{R} \vee \mathscr{B}) \cup \mathscr{B}, \\ 1, & \text{if one of } ik, jk \text{ is red, the other in } \mathscr{R} \cup (\mathscr{R} \vee \mathscr{B}), \\ 0, & \text{otherwise,} \end{cases}$$

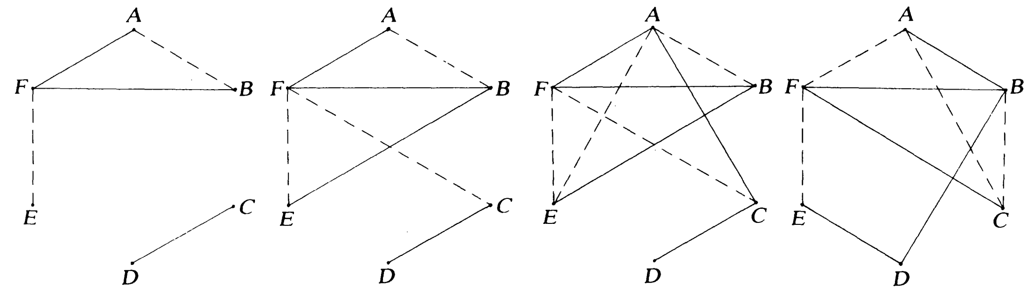
and let $z_{ij} = \Sigma_{k \neq i,j}\, y_{ijk}$. This permits us to return to the rules for the strategy.

Rule 3. On moves #6 and #8, B selects the subset of $\mathscr{R} \vee \mathscr{B}$ with minimum x_{ij} value and from this subset any line with minimum z_{ij} value.

Rule 3 is not nearly as difficult to use as it is to state precisely. B merely calculates the x value for all lines in $\mathcal{R} \vee \mathcal{B}$ and changes the entries in $\mathcal{R} \vee \mathcal{B}$ to ordered pairs such as CD, 2 where $2 = x_{CD}$. If there are several with minimum x value then the z_{ij} are calculated. This is easily done by omitting i, j and proceeding in sequence around the remaining vertices adding 0, 1 or 2 for each vertex. An example will be given below to exhibit the counting scheme.

Rule 4. If at move #10 there is one line in $\mathcal{R}$, 2 in $\mathcal{R} \vee \mathcal{B}$ and 3 in $\mathcal{B}$, then select one line from $\mathcal{R} \vee \mathcal{B}$ and two lines from $\mathcal{B}$ such that all three lines can be played without a loss. Then play the line chosen from $\mathcal{R} \vee \mathcal{B}$ on move #10, the other two lines in any order on move #12 and #14. Otherwise use Rule 3 for move #10.

At the twelfth move, player B can select a move by inspection which will gurantee a win. (It may be necessary for the inexperienced player to construct the tree of possibilities starting with move #12. This is a relatively simple task. With some practice a player can pose several "If I play —, then R plays —, and I play — and guarantee a win" statements. Remember it is only necessary on move #12 to guarantee one more move than R can make!)



$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
	AC 1,4	
	AD	
	AE 1,4	
	BC	
	BD	
	BE	
	CE 1,5	
	CF 1,3	
	DE	
	DF	

(a)

$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
CE	*AC*, 1	
	AD, 1	
	AE, 0	
	BC, 1	
	BD, 1	
	DE, 1	
	DF, 1	

(b)

$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
CE	*BC*	*AD*
	BD	
	DE	
	DF	

(c)

$\mathcal{R}$	$\mathcal{R} \vee \mathcal{B}$	$\mathcal{B}$
AE	*CD*, 1,1	*AD*
	CE, 1,1	*BE*
		DF

(d)

FIGURE 2.

An example should help clarify these rules. Suppose the sequence of moves has been Red *AF*, Blue *AB*, Red *BF*, Blue *FE*, Red *CD*, so that the state of the game is as given in FIGURE 2(a). Note *A* and *B* are equivalent points as are *C* and *D*. So we need only consider *AC*, *AE*, *CE*, *CF*, each of which has x value 1. In calculating the z values, for *AC* we add 0 for *B*, 1 for *D*, 2 for *E* and 1 for *F*. Hence, $z_{AC} = 4$. For *CF*, we count 1 for *A*, 1 for *B*, 1 for *D* and 0 for *E*. Hence, $z_{CF} = 3$ and *CF* is the best move. Suppose B plays *CF* and R plays *BE*. The state of the game and the revised table will then be as in FIGURE 2(b).

There is no need to calculate the z values since $x_{AE} = 0$ is the unique minimum x value. So B plays *AE*. Suppose *R* plays *AC*. The state of the game and the revised table are then as shown in FIGURE 2(c). Actually, R has played quite well, forcing B to be very careful. Since the special case for move #10 governed by Rule 4 does not apply, B uses Rule 3 again. This requires B to play *BC*. Consider *DF* and AD played in that order by B. B can play both lines without losing. The only way that R can block this winning sequence is by playing DF on move #11. But in this case R loses BD and B plays DE on move #12 and BD on move #14. Hence B wins the game.

Although in this example Rule 4 degenerated to Rule 3, the improper use of Rule 3 on the 10th move can be disastrous. For example, consider the game situation given in FIGURE 2(d). Rule 3 would indicate that either choice of CD or CE would be acceptable for B. However, if B plays CE, Red can play CD and AE while B can play only one of AD and DF. Hence R wins!

Testing our strategy seems to require looking at the entire game tree. The only simplification possible is to reduce the number of cases to a minimum. There are several ways to "prune" the game tree to reasonable size. The first method is to prune branches that are topologically (graph theoretically) isomorphic. In SIM this method reduces the number of terminal vertices from DeLoach's 1.7×10^{10} [6] to Beeler's 2250 [7]. The second method of pruning a tree is to eliminate branches that are not permitted by the strategy. The third method of pruning trees is to eliminate all but one of several isomorphic states since we are interested only in the present state of the game and not in its history. Finally, those branches which result in a premature loss by R may also be pruned.

The first and third methods are relatively easy to check by hand. The second pruning method requires some calculations in the case of SIM. These calculations have been made by hand and by computer as a check, producing the following table of nonisomorphic moves after each move through move 9:

move number	1	2	3	4	5	6	7	8	9
nonisomorphic games	1	1	5	4	15	13	56	41	97

(The second pruning method is largely responsible for keeping the number of games associated with the "even move numbers" in this table no larger than for the preceding odd number.)

Most of the 97 cases after move #9 can be dealt with by direct reasoning. In fact there are only 19 cases in which R can block a winning move by B, forcing B to choose an alternate winning move. The process of checking isomorphisms and applying the strategy is too long to present here. However, the process has been done by hand several times, requiring about four hours each time.

The strategy suggested in this note was originally obtained and tested using the projective geometry dual of SIM, which has been named TRI-NOT by the author's children. Instead of 6 points, no 3 collinear, TRI-NOT is defined by 6 lines, no 3 concurrent, and the $\binom{6}{2} = 15$ points of intersection. The dual of a triangle is a triangle, but instead of edges, it is the vertices that are colored. The first to color the vertices of a triangle loses the game. The advantage of TRI-NOT is that it can be made into a board game with reusable pegs or markers (FIGURE 3). Our strategy translates directly to a strategy for TRI-NOT via the duality principle.

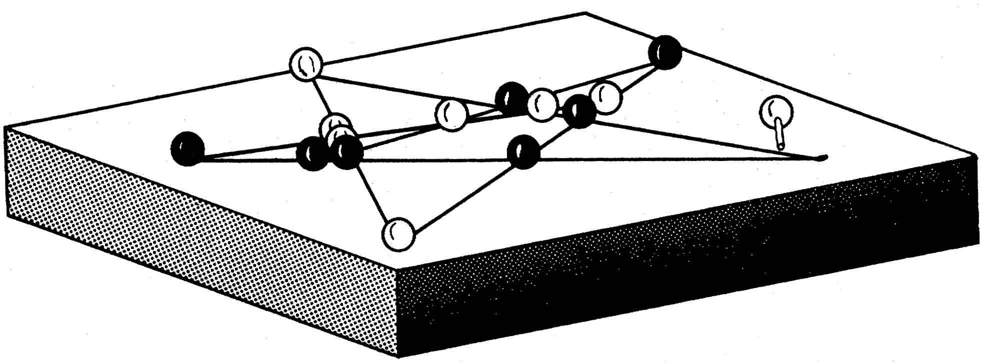


FIGURE 3.

References

[1] G. J. Simmons, The game of SIM, J. Rec. Math. 2, 2 (1969) 193–202.
[2] A. Napier, A new game in town, Empire Magazine, Denver Post (May 2, 1970).
[3] M. Gardner, Mathematical Games, Scientific American, February (1973), pp. 108–112.
[4] E. Mead, A. Rosa, C. Huang, The game of SIM: a winning strategy for the second player, Mathematical Report No. 58, March 1973, McMaster University.
[5] E. M. Rounds, S. S. Yau, A winning strategy for SIM, J. Rec. Math. 7, 3 (1974) 193–202.
[6] A. P. DeLoach, Some investigations into the game of SIM, J. Rec. Math., (1971) 36–41.
[7] M. Beeler, Artificial Intelligence Laboratory, MIT, Private Communication, March 1973.

Square Permutations

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This paper presents a new idea called square permutations, originally suggested by David Silverman. The development is elementary. It can be fully understood by a college sophomore, most of it even by a bright high school student. This illustrates my claim [**2**, **3**] that original mathematical research can take place at a much lower level than is commonly realized.

Let P denote a permutation of the integers $0, 1, 2, \ldots, n$. We shall write permutations in cycle notation. For example, the permutation

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$P(k)$	4	8	14	6	0	11	3	9	1	7	15	5	13	12	2	10

will be written $(0\ 4)(1\ 8)(2\ 14)(3\ 6)(5\ 11)(7\ 9)(10\ 15)(12\ 13)$. We shall say that P is a **square permutation** if and only if $k + P(k)$ is a perfect square for each value of k from 0 up to n. We shall say that n has a square permutation if and only if there is such a P. The permutation above is an example of a square permutation; it shows that 15 has a square permutation. For our main result we will prove below that every non-negative integer has a square permutation.

Lemma 1. *For $n = 1, 2, \ldots$, both $n^2 - 1$ and n^2 have square permutations.*

Proof. The permutations are, respectively, $(0)\ (1\quad n^2-1)(2\quad n^2-2)\ldots$ and $(0\quad n^2)(1\quad n^2-1)$ $(2\quad n^2-2)\ldots$, where the last cycle is $(\frac{1}{2}n^2)$ if n is even, and $(\frac{1}{2}(n^2-1)\quad \frac{1}{2}(n^2+1))$ if n is odd.

From small square permutations we can sometimes build up larger ones by appending cycles of length two whose elements sum to a square. For example, from $(0\ 4)\ (1\ 3)\ (2)$ (the permutation for 4 obtained from Lemma 1) we can get a square permutation for 11 by appending $(5\ 11)\ (6\ 10)\ (7\ 9)\ (8)$. This method generalizes:

Lemma 2. *Suppose that m has a square permutation, that $n > m \geqq 0$, and that for some integer r, $n + m = r^2 - 1$. Then n has a square permutation.*

Proof. Append to the square permutation for m the cycles $(m+1\quad n)(m+2\quad n-1)\ldots$; the last cycle is $(\frac{1}{2}r^2)$ or $(\frac{1}{2}(r^2-1)\quad \frac{1}{2}(r^2+1))$ according as r is even or odd.

Theorem. *Every non-negative integer has a square permutation.*