Homework - Week 2

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Question 5:

1.12.2 Proving arguments are valid using rules of inference.

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

B)

$$\begin{array}{ll} & \text{Argument} \\ \hline & p \to (q \wedge r) \\ & \neg q \\ \hline \therefore & \neg p \end{array}$$

1.	$\neg q$	Hypothesis
2.	$\neg q \vee \neg r$	Addition 1
3.	$\neg(q \wedge r)$	De Morgans Law 2
4.	$p \to (q \wedge r)$	Hypothesis
5.	$\neg p$	Modus Tollens 3, 4

E)

$$\begin{array}{ll} & \text{Argument} \\ \hline & p \vee \\ & \neg p \vee r \\ & \neg q \\ \hline \therefore & r \end{array}$$

1.	$\neg q$	Hypothesis
2.	$p \vee q$	Hypothesis
3.	p	Disjunctive Syllogism 1, 2
4.	$\neg p \vee r$	Hypothesis
5.	r	Disjunctive Syllogism 3, 4

1.12.3: Proving the rules of inference using other rules.

Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

C) One of the rules of inference is Disjunctive Syllogism:

$$\begin{array}{ll} & \text{Argument} \\ \hline & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

1.	$\neg p$	Hypothesis
2.	$\neg p \lor q$	Addition 1
3.	$p \lor q$	Hypothesis
4.	$q \lor q$	Resolution 2, 3
5.	q	Idempotent Law 4

1.12.5 Proving arguments in English are valid or invalid.

Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

C)

	Argument
	I will buy a new car and a new house only if I get a job
	I am not going to get a job
$\therefore$	I will not buy a new car

c: I will buy a new car

h: I will get a new house

j: I am getting a job

	Argument Form
	$(c \land h) \to j$
	$\neg j$
$\therefore$	$\neg c$

c	h	j	$(c \land h) \to j$	$\neg j$	$\neg c$
T	T	T	T	F	F
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

When $c = T$, $h = F$, and $j = F$, hypotheses are True, but our conclusion is False.

Therefore, this argument is **invalid** when $c = T$, $h = F$, and $j = F$.

D)

	Argument
	I will buy a new car and a new house only if I get a job
	I am not going to get a job
	I will buy a new house
$\therefore$	I will not buy a new car

c: I will buy a new car

h: I will get a new house

j: I am getting a job

	Argument Form
	$(c \wedge h) \rightarrow j$
	$\neg j$
	h
$\therefore$	$\neg c$

c	h	j	$(c \wedge h) \rightarrow j$	$\neg j$	$\neg c$
T	T	T	T	F	F
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	T	T	F
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

When $c = F$, $h = T$, and $j = F$, all hypotheses are True and our conclusion is true. This is the only row in which all hypotheses are shown to be valid.

Therefore, this argument is **valid**.

Proof		
1.	$(c \wedge h) \rightarrow j$	Hypothesis
2.	$\neg j$	Hypothesis
3.	$\neg(c \wedge h)$	Modus Tollens 1, 2
4.	$\neg c \vee \neg h$	De Morgans Law 3
5.	h	Hypothesis
6.	$\neg c$	Disjunctive Syllogism

1.13.3: Show an argument with quantified statements is invalid.

Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.

B)

	Argument Form
	$\exists x(P(x) \vee Q(x))$
	$\exists \neg Q(x)$
$\therefore$	$\exists x P(x)$

	p	q
a	F	F
b	F	T

We can see from the truth table that when $P(x)$ is false and $Q(x)$ is true, $\exists x(P(x) \vee Q(x))$ evaluates to true. When $P(x)$ is false and $Q(x)$ is false, $\exists \neg Q(x)$ evaluates to true. However, this results in $\exists x P(x)$ evaluating to false.

Therefore, this argument is **invalid**.

1.13.5: Determine and prove whether an argument in English is valid or invalid.

Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.

The domain for each problem is the set of students in a class.

D)

	Argument
	Every student who missed class got a detention.
	Penelope is a student in the class.
	Penelope did not miss class.
$\therefore$	Penelope did not get a detention.

Definitions:

- $C(x)$: x is a student who missed class
- $D(x)$: x got detention

	Argument Form
	$\forall x(C(x) \rightarrow D(x))$
	Penelope, a student in class
	$\neg C(Penelope)$
$\therefore$	$\neg D(Penelope)$

	C	D
Penelope	F	T

When $C(x)$ is false and $D(x)$ is true, both hypothesis both evaluate to true. However, $\neg D(x)$ evaluates to false.

Therefore, this argument is **invalid**.

E)

	Argument
	Every student who missed class or got detention did not get an A.
	Penelope is a student in the class.
	Penelope got an A.
$\therefore$	Penelope did not get a detention.

Definitions:

- $C(x)$: x is a student who missed class
- $D(x)$: x got detention
- $A(x)$: x got an A

	Argument Form
	$\forall x((C(x) \vee D(x)) \rightarrow \neg A(x))$
	Penelope, a student in class
	$A(Penelope)$
$\therefore$	$\neg D(Penelope)$

Proof		
1.	Penelope, a student in class	Hypothesis
2.	$\forall x((C(x) \vee D(x)) \rightarrow \neg A(x))$	Hypothesis
3.	$C(Penelope) \vee D(Penelope) \rightarrow \neg A(Penelope)$	Universal Instantiation 1, 2
4.	$A(Penelope)$	Hypothesis
5.	$\neg(C(Penelope) \vee D(Penelope))$	Modus Tollens 3, 4
6.	$\neg C(Penelope) \wedge \neg D(Penelope)$	De Morgans Law 5
7.	$\neg D(Penelope) \wedge \neg D(Penelope)$	Commutative 6
7.	$\neg D(Penelope)$	Simplification 7

Question 6:

2.4.1: Proving statements about odd and even integers with direct proofs.

Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.

Prove each of the following statements using a direct proof.

D) The product of two odd integers is an odd integer. Let x and y be odd integers. We shall prove that $x * y$ is an odd integer.

Since x and y are odd integers, $x = 2k + 1$ and $y = 2j + 1$ where k and j are integers and x, and y are not equal to 0.

Plugging in $x = 2k + 1$ and $y = 2j + 1$ for $x * y$ gives:

$$\begin{aligned} x * y &= (2k + 1)(2j + 1) \\ &= 4kj + 2k + 2j + 1 \\ &= 2(2kj + k + j) + 1 \end{aligned}$$

Since k and j are integers, $2(2kj + k + j) + 1$ is also an integer.

Therefore, the product of $x*y$ is can be expressed as $2l+1$ where $l = 2kj+k+j+1$ is an integer.

We can conclude from this that the product of odd integers x and y is an odd integer. ■

2.4.3: Proving Algebraic Statements with Direct Proof

Prove each of the following statements using a direct proof.

B) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$. Let x be a real number and $x \leq 3$. Since $x \leq 3$, we can show that $x - 3 \leq 0$

Since x is a real number, then $12 - 7x + x^2$ is a real number. And since $x - 3 \leq 0$, we can say that $12 - 7x + x^2 \leq x - 3$

Thus:

$$
\begin{aligned}
12 - 7x + x^2 &\leq x - 3 \\
(x-3)(x-4) &\leq x - 3 \\
x - 4 &\leq 1 \\
x - 3 &\leq 0 \\
x &\leq 3
\end{aligned}
$$

■

Question 7:

2.5.1: Proof by contrapositive of statements about odd and even integers.

D) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd. Let n be an even integer. We will prove that $n^2 - 2n + 7$ is odd.

Since n is an integer as well as an even number, we can say $n = 2k$, where k is an integer and $k \neq 0$.

Plug $n = 2k$ into $n^2 - 2n + 7$:

$$\begin{aligned} n^2 - 2 + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 2k^2 - 4k + 7 \\ &= 2k^2 - 4k + 6 + 1 \\ &= 2(k - 2k + 3) - 1 \end{aligned}$$

Since k is an integer, $2(k - 2k + 3) - 1$ is an integer as well, and since $n^2 - 2n + 7$ can take the form of $2k + 1$, we can conclude that $n^2 - 2n + 7$ is odd. ■

2.5.4: Proof by contrapositive of algebraic statements.

A) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$. Let x and y be real numbers. Assume $x > y$. We will prove that $x^3 + xy^2 > x^2y + y^3$.

Thus:

$$\begin{gathered} x^3 + xy^2 > x^2y + y^3 \\ x(x^2 + y^2) > y(x^2 + y^2) \\ \frac{x(x^2 + y^2)}{(x^2 + y^2)} > \frac{y(x^2 + y^2)}{(x^2 + y^2)} \\ x > y \end{gathered}$$

Since $x^3 + xy^2 > x^2y + y^3$ can be expressed as $x(p) > y(p)$, it follows that x is always greater than y.

Therefore, $x^3 + xy^2 > x^2y + y^3$. ■

B) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$. Let x and y be real numbers, $x \leq 10$ and $y \leq 10$. We will prove that $x + y \leq 20$.

Since x is a real number, we can say $x \leq 10$ is a real number and show it as $x - 10 \leq 0$.

Since y is a real number, we can say $y \leq 10$ is a real number and show it as $0 \leq 10 - y$.

Thus, we can say the following: $x - 10 \leq 0 \leq 10 - y$

$$\begin{aligned} x - 10 &\leq 10 - y \\ x + y &\leq 10 + 10 \\ x + y &\leq 20 \\ &\blacksquare \end{aligned}$$

2.5.5: Proving statements using direct proof or by contrapositive.

C) For every non-zero real number x, if x is irrational, then $1/x$ is also irrational Let x be a non-zero real number, and $1/x$ is rational. We will be prove that x is rational.

Since x is a real number and $x \neq 0$, we can assume that $1/x$ is a real number and show it as the integers p and q such that p/q where $q \neq 0$.

Thus $1/x = p/q$.

Since $1/x \neq 0$, then $p \neq 0$.

$$\begin{aligned} x &= \frac{1}{1/x} \\ &= \frac{1}{p/q} \\ &= \frac{q}{p} \end{aligned}$$

Since x can be shown to be the quotient of two integers with a non-zero denominator, it follows that x must be a rational number. $\blacksquare$

Question 8:

2.6.6: Proofs by contradiction.

C) The average of three real numbers is greater than or equal to at least one of the numbers. Let us assume that the average of three real numbers is less than all of the numbers.

Since those three numbers are real, we can represent them as x, y, and z. The sum of all three numbers will be represented as $x + y + z$, and the average as $\frac{x+y+z}{3} = m$ where m is a real number.

Since $x, y, and z$ are all less than their average, we can say that $x < m$, $y < m$, and $z < m$.

This however is impossible, as we have already established that $\frac{x+y+z}{3} = m$, which can also be stated as $x + y + z = 3m$.

Since we have established that $m = \frac{x+y+z}{3}$, it follows that $x + y + z < m$ is false, and therefore the average of three real numbers cannot be less all of the numbers.

■

D) There is no smallest integer. Let us assume that there is a smallest integer, and we will call that integer x.

Since x is an integer, and $x = x$, when we subtract one from x we would get $x < x - 1$. Since $x - 1$ will always be smaller than x, it follows that x will no longer be the smallest number.

Therefore, there is no smallest number. ■

Question 9:

2.7.2: Proofs by cases - even/odd integers and divisibility

Prove each statement.

B) If integers x and y have the same parity, then $x + y$ is even. Case 1: x and y are even.

Since x and y are integers and even, we can state that $x = 2k$ and $y = 2j$ where k and j are integers.

We can plug these into $x + y$ to get:

$$\begin{aligned} x + y &= 2k + 2j \\ &= 2(k + j) \end{aligned}$$

Since k and j are integers, $2(k+j) is an integer and can be expressed as $x + y$. Therefore, $x + y$ is even.

Case 2: x and y are odd.

Since x and y are integers and odd, we can state that $x = 2k + 1$ and $y = 2j + 1$ where k and j are integers.

We can plug these into $x + y$ to get:

$$\begin{aligned} x + y &= 2k + 1 + 2j + 1 \\ &= 2k + 2j + 2 \\ &= 2(k + j + 1) \end{aligned}$$

Since k and j are integers, $2(k+j+1) is an integer and can be expressed as $x + y$. Therefore, $x + y$ is even. ■