Homework - Week 5

Adam Collado

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Question 3:

4.1.3: Recognizing well-defined algebraic functions and their ranges.

Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

B) $f(x) = 1/(x^2 - 4)$

When $x = 2$ or $x = -2$, $1/(x^2 - 4)$ becomes $1/0$. This is not possible, therefore this is not a function of $\mathbb{R}$ to $\mathbb{R}$ and is not well-defined.

C) $f(x) = \sqrt{x^2}$

Since x^2 will always produce either a positive number or a 0 and therefore will also produce a real number. This makes the function well-defined, with a range of 0 to $\mathbb{R}^+$.

4.1.5: Range of a function.

Express the range of each function using roster notation.

B)

Let $A = \{2, 3, 4, 5\}$ $f : A \to \mathbb{Z}$ such that $f(x) = x^2$

Solution: $\{4, 9, 16, 25\}$

D)

$f : \{0, 1\}^5 \to \mathbb{Z}$.

For $x \in \{0, 1\}^5$, $f(x)$ is the number of 1's that occur in x. For example $f(01101) = 3$, because there are three 1's in the string '01101'.

Solution: $\{0, 1, 2, 3, 4, 5\}$

H)

Let $A = \{1, 2, 3\}$ $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$ where $f(x, y) = (y, x)$

Solution: $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

I)

Let $A = \{1, 2, 3\}$ $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$ where $f(x, y) = (y, x + 1)$

Solution: $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

L)

Let $A = \{1, 2, 3\}$ $f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - \{1\}$

Solution: $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$

Question 4:

4.2.2: Properties of algebraic functions.

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

C) $h : \mathbb{Z} \to \mathbb{Z}$**.** $h(x) = x^3$

This function is one-to-one.

This function is NOT onto. The integer 4 can never be a result of $h(x)$ since $h(1) = 1$ and $h(2) = 8$.

G) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$**,** $f(x, y) = (x + 1, 2y)$

This function is one-to-one.

This function is NOT onto. While x+1 is onto, the expression $2y$ can never be 1 since $2(0) = 0$ and $2(1) = 2$.

K) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$**,** $f(x, y) = 2^x + y$

This function is NOT one-to-one. $(1, 3)$ and $(2, 1)$ both map to the output 5.

This function is NOT onto. The elements 1 and 2 in the co-domain $\mathbb{Z}^+$ can't be mapped to by $f(x, y)$. This can be seen when you input the lowest values that can be represented for $f(x, y)$, that being $f(1, 1)$. These values map to the value of 3.

4.2.4 Properties of functions on strings and power sets

For each of the functions below, indicate whether the function is onto, one-to-one, neither, or both. If the function is not onto or not one-to-one, give an example showing why.

B) $f : \{0, 1\}^3 \to \{0, 1\}^3$

The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.

Solution: This function is neither one-to-one nor onto. Since the first bit of the input string is always replaced with 1, no elements in the $\{0, 1\}^3$ that start with a 0 can be mapped to, making this not onto. And since multiple values in the domain can be mapped to the same value in the target (for instance, $\{001\} = 101$ and $\{101\} = 101$), then this function can't be one-to-one.

C) $f : \{0,1\}^3 \to \{0,1\}^3$

The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.

Solution: This function is both one-to-one and onto.

D) $F : \{0,1\}^3 \to \{0,1\}^4$

The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, f(100) = 1001.

Solution: This function is one-to-one but NOT onto. Onto functions must have a domain that is greater than or equal to it's co-domain such that $|A| \geq |B|$. Since $|\{0,1\}^3| < |\{0,1\}^4|$, this function cannot be onto.

G)

Let A be defined to be the set $\{1,2,3,4,5,6,7,8\}$ and let $B = \{1\}$. $f : P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - B$. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

Solution: This function is NEITHER one-to-one NOR onto. The elements $\{1,2\}$ and $\{2\}$ will both match with $\{2\}$ in the co-domain, such that $f(\{1,2\}) = f(\{2\}) = \{2\}$. Therefore showing it can't be one-to-one.

There is no element in the $P(A)$ where $f(x) = \{1\}$. Therefore, it can't be onto.

Give an example of a function from the set of integers to the set of positive integers that is:

one-to-one, but not onto.

$$f(x) = \begin{cases} 2x+3 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

onto, but not one-to-one.

$$f(x) = |x| + 1$$

one-to-one, and onto.

$$f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

neither one-to-one nor onto.

$$f(x) = x^2$$

Question 5:

4.3.2 Finding inverses of functions.

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}

C) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = 2x + 3$

Solution: The inverse of the function is well-defined

$$\begin{aligned} f(x) &= 2x + 3 \\ y &= 2x + 3 \\ y - 3 &= 2x \\ \frac{y-3}{2} &= x \\ f^-1x &= \frac{y-3}{2} \end{aligned}$$

D)

Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$

$f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

For $X \subseteq A$, $f(X) = |X|$. Recall that for a finite set $A, P(A)$ denotes the power set of A which is the set of all subsets of A.

Solution: The inverse for this function is NOT well-defined.

$f^{-1}(1)$ would produce multiple values from the domain $P(A)$ such as $\{0\}$, $\{1\}$ and other such single element sets.

G) $f : \{0, 1\}^3 \to \{0, 1\}^3$.

The output of f is obtained by taking the input string and reversing the bits. For example. $f(011) = 110$.

Solution: The inverse of this function is well-defined.

The function of f is to reverse it's input. It follows that it's inverse function, f^-1, performs the same operation such that $f^-1 = f$.

I) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$

Solution: The inverse of this function is well-defined.

$$\begin{aligned} f(x, y) &= (x + 5, y - 2) \\ f^{-1}(x, y) &= (x - 5, y + 2) \end{aligned}$$

4.4.8 Explicit formulas for compositions of functions

The domain and target set of functions f, g, and h are $\mathbb{Z}$. The functions are defined as:

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$
- $h(x) = x^2 + 1$

Give an explicit formula for each function given below.

C) $f \circ h$

$$\begin{aligned} f \circ h(x) &= 2(x) + 3 \\ &= 2(x^2 + 1) + 3 \\ &= 2x^2 + 2 + 3 \\ &= 2x^2 + 5 \end{aligned}$$

D) $h \circ f$

$$\begin{aligned} h \circ f &= x^2 + 1 \\ &= (2x + 3)^2 + 1 \\ &= 4x^2 + 6x + 6x + 9 + 1 \\ &= 4x^2 + 12x + 10 \end{aligned}$$

4.4.2: Composition of functions on integers.

Consider three functions f, g, and h, whose domain and target are $\mathbb{Z}$. Let:

$f(x = x^2)$

$g(x) = 2^x$

$h(x) = \lceil \frac{x}{5} \rceil$

B) Evaluate $(f \circ h)(52)$

$$\begin{aligned} (f \circ h)(52) &= x^2 \\ &= \lceil \frac{x}{5} \rceil^2 \\ &= \lceil \frac{52}{5} \rceil^2 \\ &= 11^2 \\ &= 121 \end{aligned}$$

C) Evaluate $(g \circ h \circ f)(4)$

$$
\begin{aligned}
(g \circ h \circ f)(4) &= 2^{x} \\
&= 2^{\lceil \frac{x^2}{5} \rceil} \\
&= 2^{\lceil \frac{4^2}{5} \rceil} \\
&= 2^{\lceil \frac{16}{5} \rceil} \\
&= 2^{4} \\
&= 16
\end{aligned}
$$

D) Give a mathematical expression for $h \circ f$**.**

$$
\begin{aligned}
g \circ h &= \lceil \frac{x}{5} \rceil \\
&= \lceil \frac{x^2}{5} \rceil
\end{aligned}
$$

4.4.6: Composition of functions on sets of strings.

Define the following functions f, g, and h:

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example $f(001) = 101$ and $f(110) = 110$.
- $g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

C) What is $(h \circ f)(010)$**?**

Solution: $(h \circ f)(010) = h(f(010)) = h(110) = 111$

D) What is the range of $h \circ f$**?**

Solution: $\{101, 111\}$

E) What is the range of $g \circ f$**?**

Solution: $\{001, 011, 101, 111\}$

Extra Credit

4.4.4: Composition of onto and one-to-one functions.

Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

C) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

No, this is not possible. Imagine we have two functions, f and g who's domains are all non-negative numbers. Lets assume f is not one-to-one, when $f(1) = 1$ and $f(2) = 1$. If that is the case, then $(g \circ f)(1) = g(f(1)) = g(1)$. This would also apply to $(g \circ f)(2) = g(f(2)) = g(1)$. So when x = 1 or x = 2, $g(f(x))$ will always evaluate to $g(1)$, thereby making it not one to one.

D) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Yes, this is possible. Imagine we have two function, a function f $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by $f(x) = x + 0$ and a function $g : \mathbb{R} \to \mathbb{R}^+$ given by \$f(x) = x^2 \$. In this case, the function g is not one-to-one. Now lets assume that the composition $g \circ f$ is one-to-one. While g is not one-to-one, it is because negative numbers will map to the same elements as positive numbers in the co-domain of g. However, since f has a domain of all positive real numbers, it's image will only be the subset of positive values in the domain of g. Thus, $g \circ f$ can still be one-to-one.