Homework - Week 11

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Contents

Question 5

a) Prove for any positive integer n, 3 divide $n^3 + 2n$

Theorem: For any positive integer n, 3 divide $n^3 + 2n$

Base Case: n = 1

P(1) is true due to the following:

$$P(1) = n^3 + 2n = 1^3 + 2(1) = 1 + 2 = 3$$

3 divides 3, thus P(1) is shown to be true.

Inductive Step: We will show that when P(k) is true for any positive integer k, 3 divide $k^3 + 2k$, it follows that P(k+1), the statement 3 divide $(k+1)^3 + 2(k+1)$ is also true.

This can be shown with the following:

$$\begin{aligned} P(k+1) &= (k+1)^3 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \quad \text{by algebra} \\ &= k^3 + 3k + 3k^2 + 3k + 3 \end{aligned}$$

Using the inductive hypothesis, we can conclude that 3 divides the term $k^3 + 3k$. We can also see that the reminder of the term, $3k^2 + 3k + 3$, can also be divided by 3.

Therefore, 3 divide $(k+1)^3 + 2(k+1)$ ■

b) Use strong induction to prove that any positive integer $n(n \geq 2)$ can be written as a product of primes.

Theorem: For any positive integer n, where n ≥ 2, that number can be written as a product of primes.

Base Case: n = 2, n = 4

It's clear that 2 is a prime number as it's a product of itself (2 * 1) and only itself.

We use n = 4 since 4 is the first composite number greater than 2.

4 = 2 * 2. This shows that 4 can be written as a product of primes.

Inductive Step: If we assume that for any number 2 to k, P(k) can be written as a product of primes, then it follows that k+1 can also be written as a product of primes.

First we have to consider that for any k+1, it can be either a prime number or a composite. If it's a prime number, then k+1 can only be a product of it's self, k+1. This already satisfies the theorem, much in the same way the base case has.

Next we have to consider k+1 being a composite number. In this case, we know that k+1 must be a product of two integers, a and b where a and b are greater than 2 and less than k. And since we've shown that a and b can be written as a product of primes in our inductive hypothesis, it follows that k+1 can also be written as a product of primes. ■

Question 6

7.4.1: Components of an inductive proof.

Define P(n) to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

a) Verify that P(3) is true.

Solution: The equation is proof showing how to ge the sum of squares for numbers 1 to n. If this is so, then P(3) should calculate to:

$$1^2 + 2^2 + 3^2 = 14$$

And we can see after calculating for P(3):

$$P(3) = \frac{n(n+1)(2n+1)}{6} = \frac{3(3+1)(2(3)+1)}{6} = \frac{3(4)(7)}{6} = \frac{84}{6} = 14$$

Thus, P(3) = 14, and therefore proves true.

b) Express P(k).

Solution:

$$P(k) = \frac{k(k+1)(2k+1)}{6}$$

c) Express P(K + 1).

Solution::

$$P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

d) In the inductive proof that for every positive integer n, what must be proven in the base case?

The base case must prove that the theorem is true for the first value of the sequence. In the case of this theorem our base case would be P(1), since the equation states that j=1 to n.

e) In an inductive proof that for every positive integer n, what must be proven in the inductive step?

The inductive step must prove that if the theorem holds for P(k) (which is assume to be true), then it holds true that P(k + 1).

f) What would be the inductive hypothesis in the inductive step from your previous answer?

The inductive hypothesis would be the statement P(k) being true.

g) Prove by induction that for any positive n.

First we will prove the base case that was stated in part d of this question.

Base Case: $n = 1$

$$P(1) = 1^2 = \frac{n(n+1)(2n+1)}{6} = \frac{(1+1)(2+1)}{6} = \frac{(2)(3)}{6} = \frac{6}{6} = 1$$

Therefore, $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2(1)+1)}{6}$

Inductive Step: Suppose that for a positive integer k:

$$\sum_{j=1}^{k} j^2 \frac{k(k+1)(2k+1)}{6}$$

We will show that for when $P(k+1)$:

$$\sum_{j=1}^{k+1} j^2 \frac{(k+1)(k+2)(2k+3)}{6}$$

We will start with the left side of the equation and continue:

$$
\begin{aligned}
\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} + (k+1)^2 && \text{separating the last} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{by inductive step} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} && \text{by algebra} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}
$$

Therefore, $\sum_{j=1}^{k+1} j^2 \frac{(k+1)(k+2)(2k+3)}{6}$ ■

7.4.3

Prove each of the following statements using mathematical induction.

c)

Theroem: For any positive integer n, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Base Case: n = 1

We can show the following:

$$\frac{1}{1^2} = 1 \leq 2 - \frac{1}{1} = 1$$

Thus proving P(1)

Inductive Step: We will show that when P(k) is true for any positive integer k, $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$, it follow that P(k+1), $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$ will also hold true.

We will show this with the following:

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \text{by inductive hypothesis} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} && \text{by algebra} \\
&= 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)} \\
&= 2 - \frac{k}{k(k+1)} \\
&= 2 - \frac{1}{k+1}
\end{aligned}$$

Therefore, $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$ ■

7.5.1

a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Theroem: For any positive integer n, 4 evenly divides $3^{2n} - 1$

Base Case: n = 1

For P(1):

$$P(1) = 3^{2(1)} - 1 = 9 - 1 = 8$$

4 evenly divides 8, therefore P(1) is true.

Inductive Step: We will show that if for any positive integer k, for P(k), 4 evenly divides $3^{2k} - 1$ is true, then it will follow that for P(k+1), 4 evenly divides $3^{2k+1} - 1$ is also true.

By the inductive hypothesis, $3^{2k} - 1$ is evenly divided by 4, it follows that $3^{2k} - 1 = 4m$, where m is an integer. We can then add 1 on both sides to get $3^{2k} = 4m + 1$, showing this statement is equivalent to the inductive hypothesis. Now we will show that $3^{2(k+1)} - 1$ can be expressed as 4 times as integer.

$$\begin{aligned}
3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\
&= 9 * 3^{2k} - 1 && \text{by algebra} \\
&= 9(4m + 1) - 1 && \text{by inductive hypothesis} \\
&= 36m + 9 - 1 \\
&= 36m + 8 \\
&= 4(9m + 2)
\end{aligned}$$

Since m is an integer, (9m + 2) is also an integer. Therefore $3^{2(k+1)} - 1$ is equal to 4 times an integer, which also means that it is divisible by 4 ■