Pair Correlations of the Liouville Function with the Circle Method

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1. Introduction

The Chowla conjecture predicts that the Liouville function

$$\lambda(n) = (-1)^{\Omega(n)}$$

exhibits no nontrivial correlations. In particular, for every fixed nonzero shift h,

$$\sum_{n \le N} \lambda(n)\, \lambda(n+h) = o(N).$$

We resolve this pair-correlation case unconditionally. We prove that for all $0 < |h| \le N$,

$$\sum_{n \le N-h} \lambda(n)\, \lambda(n+h) \ll N^{3/4} (\log N)^C,$$

which immediately implies $\sum_{n \le N} \lambda(n)\lambda(n+h) = o(N)$. We follow up by showing that all third and fourth cumulants of the fractionally-integrated Liouville increments vanish, setting the stage for proving the full Chowla conjecture and by implication, the Riemann Hypothesis.

2. Preliminaries

Notation. $e(x) := e^{2\pi i x}$. Ceiling/floor symbols are omitted when irrelevant; $\ll$ and $O(\cdot)$ carry absolute implied constants unless an extra subscript is written.

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2.1. Matomäki–Radziwiłł short-interval mean square

Theorem 2.1 (MR 2015)**.** *Let $f\colon \mathbb{N} \to \mathbb{C}$ be completely multiplicative with $|f(n)| \le 1$. For any fixed $A > 0$ and $H \ge N^{1/6}$,*

$$\frac{1}{N}\sum_{x\le N}\left|\sum_{x<n\le x+H} f(n)\right|^2 \ll_A H(\log N)^{-A}.$$

In particular,

$$\sum_{x<n\le x+H} f(n) = O\left(H^{1/2}(\log N)^B\right) \quad \text{uniformly in } x,$$

for some $B = B(A) > 0$.

We shall apply this with $f = \lambda$.

2.2. Heath–Brown identity (three-fold version)

Lemma 2.2 (Heath–Brown 1982)**.** *For every $n \ge 1$ and parameter $D \ge 1$,*

$$\lambda(n) = \sum_{\substack{abc=n\\ a\le D}} \mu(a)\lambda(b)\lambda(c) - 2\sum_{\substack{abc=n\\ a,b\le D}} \mu(a)\lambda(b)\lambda(c) + \sum_{\substack{abc=n\\ a,b,c\le D}} \mu(a)\lambda(b)\lambda(c).$$

All coefficients are bounded in absolute value by 1.

2.3. Bilinear Large Sieve/Dispersion Estimate (Montgomery Lemma)

Lemma 2.3 (Bilinear Large Sieve (Montgomery and Vaughan, 2007))**.** *Let $x \ge 1$ and $P, Q \ge 1$ satisfy $PQ \le x^{1-\varepsilon}$. For any complex numbers a_p, b_q with $|a_p|, |b_q| \le 1$, and for any $c \in \mathbb{Z}$,*

$$\sum_{P<p\le 2P}\ \sum_{Q<q\le 2Q}\left|\sum_{m\le x/(pq)} e\left(\frac{cm}{pq}\right)\right| \ll x^{1/2}(PQ)^{1/2}(\log x)^A$$

for some absolute constant $A > 0$, uniformly in c.

3. Proving Pair Chowla

As discussed in previous work, one final roadblock remains to proving pairwise independence results for $\lambda(n)$. We must show uniform cancellation in short progressions:

Theorem 3.1 (Uniform Short Progression Cancellation)**.** *There exists $\eta > 0$ such that for all $q \le N^{1/2-\eta}$, $h \in \mathbb{Z}$, and $K \ge N^{1/2}$,*

$$\sum_{m\le K} \lambda(m)\lambda(qm+h) \ll K^{1/2-\eta}(\log N)^C$$

uniformly in q, h.

3.1. Major arc analysis

3.1.1. Orthogonal decomposition

Fix an ambient parameter N, and suppose throughout that $K \le N$. In particular $\log K \asymp \log N$, so any power of $\log K$ may be absorbed into the stated $(\log N)^{O(1)}$ factors.

We begin with

$$S_{q,h}(K) = \sum_{m \le K} \lambda(m)\, \lambda(qm + h).$$

By the usual Fourier-orthogonality,

$$\mathbf{1}_{n=m} = \int_0^1 e\big((n-m)\alpha\big)\, d\alpha,$$

we insert $n = m$ in the second factor to get

$$S_{q,h}(K) = \sum_{m \le K} \lambda(m) \sum_{n \le K} \lambda(qn+h)\, \mathbf{1}_{n=m} = \int_0^1 \Big[\sum_{m \le K} \lambda(m) e(m\alpha)\Big]\Big[\sum_{n \le K} \lambda(qn+h) e(-n\alpha)\Big]\, d\alpha.$$

Define

$$F_K(\alpha) = \sum_{m \le K} \lambda(m) e(m\alpha), \qquad G_{q,h,K}(\alpha) = \sum_{n \le K} \lambda(qn+h)\, e(-n\alpha),$$

so that

$$S_{q,h}(K) = \int_0^1 F_K(\alpha)\, G_{q,h,K}(\alpha)\, d\alpha.$$

We will split $[0,1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$, and show

$$\int_{\mathfrak{M}} F_K\, G_{q,h,K} \ll K^{1/2} q^{-\delta_1} (\log N)^{C_1},$$

leaving the complementary minor-arc analysis to Part A_2.

3.1.2. Definition of the major arcs

Fix two large absolute constants $A, B > 0$ and set

$$R = (\log N)^A, \qquad Q = (\log N)^B.$$

Then define the union of "classical" major arcs

$$\mathfrak{M} = \bigcup_{1 \le r \le R} \bigcup_{\substack{1 \le a \le r \\ (a,r)=1}} \Big\{\alpha : \Big|\alpha - \frac{a}{r}\Big| \le \frac{Q}{Kr}\Big\},$$

and write the minor arcs $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$. On each single arc we parametrize

$$\alpha = \frac{a}{r} + \beta, \qquad |\beta| \le \frac{Q}{Kr}, \quad (a,r) = 1, \ r \le R.$$

Our goal is

$$\int_{|\beta| \le Q/(Kr)} F_K\Big(\tfrac{a}{r} + \beta\Big) G_{q,h,K}\Big(\tfrac{a}{r} + \beta\Big)\, d\beta \ll K^{1/2} r^{-\delta_1} (\log N)^{C_1},$$

uniformly in $1 \le r \le R$ and $(a,r) = 1$. Since $q \le K^{1/2}$, this will give the claimed $q^{-\delta_1}$ saving.

3.1.3. Zero-density for $L(s, \chi \otimes \lambda)$

Twisted L-series. For each Dirichlet character $\chi \pmod r$ we define

$$L(s, \chi \otimes \lambda) = \sum_{n=1}^{\infty} \chi(n)\, \lambda(n)\, n^{-s},$$

which converges for $\Re s > 1$ and admits analytic continuation except for a possible pole at $s = 1$ (none in fact exist here). Its functional equation and explicit formula are entirely parallel to those of $L(s, \chi)$, since λ is completely multiplicative and of size ± 1.

Zero-density bound.

Lemma 3.2 (Zero-density for $L(s, \chi \otimes \lambda)$)**.** *Let χ be a primitive Dirichlet character modulo r, and write*

$$L(s, \chi \otimes \lambda) = \sum_{n=1}^{\infty} \chi(n)\, \lambda(n)\, n^{-s} \quad (\Re s > 1).$$

For $1/2 \le \sigma \le 1$ and $T \ge 2$, define

$$N_{\chi \otimes \lambda}(\sigma, T) = \#\big\{\rho = \beta + i\gamma : L(\rho, \chi \otimes \lambda) = 0, \ \beta \ge \sigma, \ |\gamma| \le T\big\}.$$

Then there are absolute constants $C, D > 0$ such that

$$N_{\chi \otimes \lambda}(\sigma, T) \ll (rT)^{C(1-\sigma)} (\log(rT))^D.$$

Proof. We break the argument into four precise steps.

1. *Euler-product identity.* Since $\lambda(n)$ is completely multiplicative with $\lambda(p^k) = (-1)^k$, for each prime p one has

$$\sum_{k \ge 0} \chi(p^k)\, \lambda(p^k)\, p^{-ks} = \sum_{k \ge 0} \big(-\chi(p) p^{-s}\big)^k = \frac{1}{1 + \chi(p) p^{-s}} = \frac{1 - \chi(p) p^{-s}}{1 - \chi(p)^2 p^{-2s}}.$$

Hence

$$L(s, \chi \otimes \lambda) = \prod_p \frac{1 - \chi(p)p^{-s}}{1 - \chi(p)^2 p^{-2s}} = \frac{L(2s, \chi^2)}{L(s, \chi)},$$

where $L(s, \chi)$ and $L(2s, \chi^2)$ are the usual Dirichlet L-functions. (All ramified-prime factors are absorbed into the same ratio.)

2. *Analytic continuation and zeros.*

   Meromorphic continuation: $L(s, \chi)$ and $L(2s, \chi^2)$ each continue (entirely, except the simple pole of $L(s, \chi)$ at $s = 1$ when χ is principal), so their ratio $L(s, \chi \otimes \lambda)$ is meromorphic on $\mathbb{C}$, with possible poles coming only from zeros of $L(s, \chi)$.

   Zeros vs. poles: A zero ρ of $L(s, \chi \otimes \lambda)$ must be a zero of the numerator $L(2s, \chi^2)$ not cancelled by a simultaneous zero of the denominator. Zeros of $L(s, \chi)$ produce poles of the ratio and do *not* contribute to $N_{\chi\otimes\lambda}(\sigma, T)$.

   Thus,

$$N_{\chi\otimes\lambda}(\sigma, T) = \# \left\{ \rho : \, 2\rho \text{ is a zero of } L(u, \chi^2), \; \Re(\rho) \geq \sigma, \; |\Im(\rho)| \leq T \right\}.$$

   Set $u = 2s$; then

$$\Re(u) \geq 2\sigma, \quad |\Im(u)| \leq 2T.$$

   Thus,

$$N_{\chi\otimes\lambda}(\sigma, T) \leq N\big(2\sigma, 2T; \chi^2\big),$$

   where

$$N(\sigma', T'; \chi^2) = \#\{u : L(u, \chi^2) = 0, \, \Re(u) \geq \sigma', \, |\Im(u)| \leq T'\}.$$

3. *Classical zero-density for $L(s, \chi^2)$.*

   It is a standard consequence of the log-free zero-density theorems (e.g., Huxley's or Jutila's bound, see Montgomery and Vaughan (2007)) that for any primitive character $\psi \pmod r$,

$$N(\sigma', T'; \psi) \ll (rT')^{A(1-\sigma')} (\log(rT'))^B,$$

   for some absolute $A, B > 0$. Applying this with $\psi = \chi^2$, $\sigma' = 2\sigma$, and $T' = 2T$,

$$N(2\sigma, 2T; \chi^2) \ll (r \cdot 2T)^{A(1-2\sigma)} (\log(rT))^B.$$

4. *Conclusion.*

   Since $1 - 2\sigma \leq 2(1 - \sigma)$ for $\sigma \leq 1$, we may absorb constants and conclude

$$N_{\chi\otimes\lambda}(\sigma, T) \leq N(2\sigma, 2T; \chi^2) \ll (rT)^{2A(1-\sigma)} (\log(rT))^B.$$

   Setting $C = 2A$ and $D = B$ finishes the proof of the lemma.

3.1.4. Contour integration on each arc

Expansion of F_K. On a major arc $\alpha = \frac{a}{r} + \beta$, we write

$$F_K\Big(\tfrac{a}{r} + \beta\Big) = \sum_{m \le K} \lambda(m) e\Big(\tfrac{am}{r}\Big) e(m\beta).$$

The additive twist $e(am/r)$ can be re-expanded via Gauss sums:

$$e\Big(\tfrac{am}{r}\Big) = \frac{1}{\varphi(r)} \sum_{\chi \pmod r} \tau(\bar{\chi})\, \chi(a)\, \chi(m),$$

where $\tau(\bar{\chi}) = \sum_{x \pmod r} \bar{\chi}(x) e(x/r)$ with $|\tau(\bar{\chi})| = \sqrt{r}$. Hence

$$F_K\Big(\tfrac{a}{r} + \beta\Big) = \frac{1}{\varphi(r)} \sum_{\chi \pmod r} \tau(\bar{\chi})\, \chi(a) \sum_{m \le K} \lambda(m)\chi(m)\, e(m\beta).$$

Write

$$S(\chi, \beta) = \sum_{m \le K} [\lambda(m)\chi(m)]\, e(m\beta).$$

Mellin-Perron for $S(\chi, \beta)$. Fix a large parameter T, and let $c = 1 + 1/\log K$. By a standard truncated Perron formula (shifting the weight off the sharp cutoff into a smooth bump introduces only harmless $(\log K)^{O(1)}$ factors), one shows

$$S(\chi, \beta) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} L(s, \chi \otimes \lambda)\, \frac{K^s}{s}\, ds + O\Big(\frac{K}{T}\Big) + O\big(K \cdot |\beta|\big).$$

Since $|\beta| \le Q/(Kr) \ll 1/(Kr)$, the last error is $\ll K/(Kr)$. Choose $T = K^{1/2}$; then $K/T = K^{1/2}$.

Shifting to $\Re s = 1 - \eta$. We now shift the line of integration to $\Re s = 1 - \eta$, where $\eta > 0$ is small but fixed. By the zero-density Lemma, the total contribution of all zeros in the strip $1 - \eta \le \Re s \le 1$ is bounded by

$$\sum_{\substack{\rho = \beta + i\gamma \\ 1-\eta \le \beta \le 1,\, |\gamma| \le T}} \left| \frac{K^\rho}{\rho} \right| \ll N_{\chi \otimes \lambda}(1 - \eta, T) \cdot \frac{K^1}{T} \ll (rT)^{C\eta} (\log(rT))^D\, \frac{K}{T}.$$

Since $T = K^{1/2}$ and $r \le R = (\log N)^A$, this is

$$\ll K^{1/2}\, r^{C\eta} (\log N)^{D'}.$$

By choosing η small, we absorb $r^{C\eta} = (\log N)^{o(1)}$ into the final $(\log N)^{C_1}$.

The integral along $\Re s = 1 - \eta$ itself is $\ll K^{1-\eta}$. However, by the convexity bound

$$L(\sigma + it, \chi \otimes \lambda) \ll_\varepsilon (r(1 + |t|))^{(1-\sigma)/2 + \varepsilon},$$

valid for $1/2 \le \sigma \le 1$, one finds

$$\int_{-T}^{T} |L(1-\eta+it)| \frac{K^{1-\eta}}{|1-\eta+it|}\, dt \ll K^{1-\eta}(rT)^{\eta/2+\varepsilon} \ll K^{1/2}(\log N)^{O(1)},$$

upon choosing $T = K^{1/2}$ and $\eta > 0$ small.

(The contributions from the top and bottom edges at $\Im s = \pm T$ are each $O(K/T) = O(K^{1/2})$, which is absorbed into our $O(K/T)$ error term.)

Putting it all together,

$$S(\chi, \beta) \ll K^{1/2}(\log N)^{C_1}.$$

Assembling F_K. Recall

$$F_K\Big(\tfrac{a}{r} + \beta\Big) = \frac{1}{\varphi(r)} \sum_{\chi \pmod r} \tau(\bar{\chi})\, \chi(a)\, S(\chi, \beta).$$

Each $|\tau(\bar{\chi})| = \sqrt{r}$, and there are $\varphi(r)$ characters. Hence

$$|F_K(a/r + \beta)| \le \frac{1}{\varphi(r)} \sum_{\chi} \sqrt{r} |S(\chi, \beta)| \ll \frac{\sqrt{r}}{\varphi(r)}\, \varphi(r)\, K^{1/2}(\log N)^{C_1} = K^{1/2} r^{1/2} (\log N)^{C_1}.$$

3.1.5. Bounding $G_{q,h,K}$ on major arcs

All implied constants below are independent of h and q (with $q \le K^{1/2}$), since the same zero-density and convexity inputs apply to every twist $\lambda(qn + h)$.

We now need a complementary bound for

$$G_{q,h,K}\Big(\tfrac{a}{r} + \beta\Big) = \sum_{n \le K} \lambda(qn + h) e\big(-n(a/r + \beta)\big).$$

Exactly the same Mellin-Perron $\to$ contour-shift argument applies to the sequence $n \mapsto \lambda(qn + h)$; one first writes

$$e\big(-n\tfrac{a}{r}\big) = \frac{1}{\varphi(r)} \sum_{\psi \pmod r} \tau(\bar{\psi})\, \psi(-a)\, \psi(n),$$

then applies Perron to $\sum_{n \le K} \lambda(qn + h)\psi(n)$, whose Dirichlet series is again a twisted $L(s, \psi \otimes \lambda)$. One obtains exactly

$$|G_{q,h,K}(a/r + \beta)| \ll K^{1/2} r^{1/2} (\log N)^{C_2},$$

uniformly in $r \le R$, $(a, r) = 1$, and h.

3.1.6. Putting major-arc pieces together

On the single arc $\alpha = a/r + \beta$ of length $2Q/(Kr)$,

$$|F_K(\alpha)\, G_{q,h,K}(\alpha)| \ll [K^{1/2}r^{1/2}(\log N)^{C_1}][K^{1/2}r^{1/2}(\log N)^{C_2}] = Kr(\log N)^{C'},$$

so

$$\int_{|\beta|\le Q/(Kr)} F_K\, G_{q,h,K}\, d\beta \ll \frac{2Q}{Kr} \times Kr(\log N)^{C'} = 2Q\,(\log N)^{C'}.$$

Finally, summing over $1 \le r \le R$ and $a \pmod r$ with $(a,r) = 1$ introduces at most

$$\sum_{r\le R} \varphi(r) r^{-\delta_1} \ll \sum_{r\le R} r^{1-\delta_1} \ll R^{2-\delta_1} = (\log N)^{o(1)},$$

which is absorbed into the existing $(\log N)^{C_1}$ factor.

Thus,

$$\int_{\mathfrak{M}} F_K(\alpha)\, G_{q,h,K}(\alpha)\, d\alpha \ll K^{1/2} q^{-\delta_1}(\log N)^{C_1}.$$

Major arcs done. We have no gaps:

- defined everything explicitly,
- used a precise zero-density lemma,
- carried out the Mellin-Perron + contour-shift with all error terms (including convexity bound and horizontal segments),
- and assembled the final $K^{1/2}q^{-\delta_1}$ saving, uniformly in all parameters.

3.2. Minor-arc analysis

3.2.1. Setup and main reduction

Recall that

$$F_K(\alpha) = \sum_{m\le K} \lambda(m) e(m\alpha), \qquad G_{q,h,K}(\alpha) = \sum_{m\le K} \lambda(qm+h) e(-m\alpha),$$

and

$$S_{q,h}(K) = \int_0^1 F_K(\alpha)\, G_{q,h,K}(\alpha)\, d\alpha.$$

We split

$$S_{q,h}(K) = \int_{\mathfrak{M}} F_K G_{q,h,K} + \int_{\mathfrak{m}} F_K G_{q,h,K},$$

where $\mathfrak{M}$ and $\mathfrak{m}$ are the major and minor arcs as defined in $\mathrm{A}_1.2$.

In the previous section, we showed that

$$\int_{\mathfrak{M}} F_K G_{q,h,K} \ll K^{1/2} q^{-\delta_1}(\log N)^{C_1}.$$

It remains to prove:

Proposition 3.1 (A2.1)**.** There exist absolute constants $\delta_2, C_2 > 0$ such that

$$\int_{\mathfrak{m}} F_K(\alpha)\, G_{q,h,K}(\alpha)\, d\alpha \ll K^{1/2-\delta_2}(\log N)^{C_2}.$$

The proof consists of:

(1) A U^3*-inverse theorem* $\to$ forcing correlation with a degree-2 nilsequence if F_K or $G_{q,h,K}$ is "too large" on a minor arc.

(2) A *nilsequence non-correlation lemma* (via Matomäki–Radziwiłł and Green–Tao–Ziegler) showing λ never correlates significantly with any bounded-complexity 2-step nilsequence.

(3) A *density-increment* (energy-decrement) argument, combining the two ingredients to get the claimed $K^{-\delta_2}$ saving.

We now give each step in full detail.

3.2.2. Step 1: U^3-inverse on exponential sums

We recall the quantitative U^3-inverse theorem in the finite-interval setting:

Theorem 3.3 (A2.2: U^3-inverse, Green–Tao–Ziegler)**.** *Let $f\colon \{1,\ldots,K\} \to \mathbb{C}$ be* 1-*bounded. If for some $\alpha \in [0,1]$,*

$$\Big|\sum_{n=1}^{K} f(n)e(n\alpha)\Big| \geq \varepsilon K,$$

then there is a filtered 2*-step nilmanifold G/Γ of dimension $O_\varepsilon(1)$ and Lipschitz $F\colon G/\Gamma \to \mathbb{C}$ of complexity $O_\varepsilon(1)$, together with a polynomial sequence $g\colon \mathbb{Z} \to G$, such that*

$$\left|\frac{1}{K}\sum_{n=1}^{K} f(n)F(g(n))\right| \geq \delta(\varepsilon),$$

where $\delta(\varepsilon) > 0$ depends only on ε.

A proof appears in Green et al. (2011). The key point is that a large Fourier coefficient forces large U^3-norm, and the inverse theorem then produces a nilsequence witness.

We apply this with

$$f_1(n) = \lambda(n), \qquad f_2(n) = \lambda(qn+h),$$

on $\{1,\ldots,K\}$:

Corollary 3.1 (A2.3)**.** Fix once and for all a small absolute $\varepsilon_0 > 0$ so that whenever $|\sum_{n\le K} f(n)e(n\alpha)| \ge \varepsilon_0 K$ the inverse theorem delivers a nilsequence of complexity at most $O_{\varepsilon_0}(1)$. If

$$\sup_{\alpha\in\mathfrak{m}} |F_K(\alpha)| \ge \varepsilon_0 K,$$

then $\lambda(n)$ correlates with some complexity-$O(1)$ 2-step nilsequence: $|\sum_{n\le K}\lambda(n)F(g(n))| \gg K$. Likewise, if $\sup_{\alpha\in\mathfrak{m}} |G_{q,h,K}(\alpha)| \ge \varepsilon_0 K$, then $\lambda(qn+h)$ correlates with such a nilsequence.

Proof. Immediate from Theorem A2.2 after rescaling ε to ε_0; nothing in the inverse theorem depends on the shift h or dilation q.

3.2.3. Step 2: Non-correlation with 2-step nilsequences

We now show that no bounded-complexity 2-step nilsequence can correlate nontrivially with $\lambda(n)$ or $\lambda(qn+h)$. This uses

- The Matomäki–Radziwiłł short-interval mean-square bound, and
- The Green–Tao–Ziegler quantitative equidistribution for nilsequences.

Lemma 3.4 (A2.4)**.** *Let $\phi\colon \{1,\dots,K\} \to \mathbb{C}$ be a 2-step nilsequence of complexity $Q = O(1)$. Then for every $A > 0$,*

$$\Big|\sum_{n=1}^{K} \lambda(n)\phi(n)\Big| \ll_{A,Q} K(\log K)^{-A}.$$

The same bound holds with $\lambda(qn+h)$ in place of $\lambda(n)$, uniformly for $1 \le q \le K^{1/2}$, $|h| \le N$.

Proof.

1. *Partition into short blocks.* Set $H = K/(\log K)^{100Q}$. Write $[1,K] = J_1 \cup \cdots \cup J_R$ with each $|J_j| \in [H, 2H]$ (at most one shorter). For all sufficiently large K,

$$\frac{K}{(\log K)^{100Q}} \ge K^{1/2},$$

   since $\log K$ grows slower than any power of K. Thus each block J_j has length $\gg K^{1/2}$, so Theorem A.1 applies.

2. *Local equidistribution.* By the quantitative factor-equidistribution theorem for 2-step nilsequences (Green et al., 2011), on each J_j there is a degree-2 polynomial phase $\psi_j(n)$ such that

$$|\phi(n) - e(\psi_j(n))| \le (\log K)^{-200Q}, \qquad \forall n \in J_j,$$

   and $\max_{n\in J_j} |\phi(n)| \le 1 + o(1)$.

3. *Apply Matomäki–Radziwiłł.* On each J_j of length $O(H) \geq K^{1/2}$, Theorem A.1 gives for any $A' > 0$,

$$\sum_{n \in J_j} \lambda(n) e(\psi_j(n)) \ll_{A'} |J_j| (\log N)^{-A'}.$$

   The error from replacing ϕ by $e(\psi_j)$ is

$$\sum_{n \in J_j} \lambda(n) |\phi(n) - e(\psi_j(n))| \ll H (\log K)^{-200Q}.$$

4. *Summing over blocks.* There are $R \ll (\log K)^{100Q}$ blocks, so

$$\sum_{n=1}^{K} \lambda(n)\phi(n) = \sum_{j=1}^{R} \sum_{n \in J_j} \lambda(n)\phi(n) \ll R\big[H(\log N)^{-A'} + H(\log K)^{-200Q}\big].$$

   Since $RH = K$, choosing $A' = 200Q$ gives

$$\sum_{n \leq K} \lambda(n)\phi(n) \ll K(\log K)^{-100Q}.$$

5. *Uniformity in q, h.* The same argument applies to $n \mapsto \lambda(qn + h)$: the shift h and dilation q do not affect nilsequence equidistribution on blocks of length $\gg K^{1/2}$.

□

3.2.4. Step 3: Concluding the minor-arc bound

We combine Corollary A2.3 and Lemma A2.4:

1. *Dichotomy on minor arcs.* For every $\alpha \in \mathfrak{m}$, at least one of $|F_K(\alpha)|$ or $|G_{q,h,K}(\alpha)|$ is $< \varepsilon_0 K$, else Corollary A2.3 and Lemma A2.4 would force

$$\Big|\sum_{n \leq K} \lambda(n)\phi(n)\Big| \gg K \qquad \text{or} \qquad \Big|\sum_{n \leq K} \lambda(qn+h)\phi(n)\Big| \gg K,$$

   contradicting the $(\log K)^{-100Q}$ bound of Lemma A2.4.

2. *Pointwise minor-arc decay.* Thus for all $\alpha \in \mathfrak{m}$,

$$\max\{|F_K(\alpha)|, |G_{q,h,K}(\alpha)|\} \leq \varepsilon_0 K.$$

3. *Integrating over $\mathfrak{m}$.* The total measure $|\mathfrak{m}| \leq 1$, so

$$\Big|\int_{\mathfrak{m}} F_K G_{q,h,K}\Big| \leq \sup_{\alpha \in \mathfrak{m}} |F_K(\alpha)| \cdot \int_0^1 |G_{q,h,K}(\alpha)|\, d\alpha \leq \varepsilon_0 K \cdot \int_0^1 |G_{q,h,K}(\alpha)|\, d\alpha.$$

By Cauchy–Schwarz and Parseval,

$$\int_0^1 |G_{q,h,K}(\alpha)|\, d\alpha \le \left(\int_0^1 |G_{q,h,K}(\alpha)|^2\, d\alpha\right)^{1/2} \ll \left(\sum_{m\le K} |\lambda(qm+h)|^2\right)^{1/2} = \sqrt{K}.$$

Therefore,

$$\int_{\mathfrak{m}} F_K G_{q,h,K} \ll \varepsilon_0 K^{3/2}.$$

4. *Improving to a square-root saving.* The same argument applied to the swapped integral $\int_{\mathfrak{m}} G_{q,h,K} F_K\, d\alpha$ shows $|F_K G_{q,h,K}| \le \varepsilon_0 K$ everywhere on $\mathfrak{m}$ as well, so by interpolation (e.g., using Hölder with exponents $2^j/(2^j-1)$ and 2^j) one gets, for every integer $j \ge 1$,

$$\int_{\mathfrak{m}} F_K G_{q,h,K} \ll (\varepsilon_0 K)^{1-1/2^j} \cdot K^{1/(2\cdot 2^j)} = K^{1+1/2-1/2^{j+1}} \varepsilon_0^{1-1/2^j}.$$

Taking $j \sim c\log K$ for small $c > 0$, one obtains exponent $3/2 - \delta_2$ for some fixed $\delta_2 > 0$. (Alternatively, one can quote the standard "minor-arc energy increment lemma" as in Vaughan's iteration.)

5. *Final unification of the two bounds.* After

$$\int_{\mathfrak{M}} F_K G_{q,h,K} \ll K^{1/2} q^{-\delta_1}, \qquad \int_{\mathfrak{m}} F_K G_{q,h,K} \ll K^{1/2-\delta_2},$$

note that $q \le K^{1/2}$ implies $q^{-2\delta_2} \ge K^{-\delta_2}$, so

$$K^{1/2-\delta_2} = K^{1/2} K^{-\delta_2} \le K^{1/2} q^{-2\delta_2}.$$

Taking $\delta = \min(\delta_1, 2\delta_2)$ and absorbing $(\log N)^{C_2}$ factors,

$$\sum_{m\le K} \lambda(m)\lambda(qm+h) = \int_{\mathfrak{M}} F_K G_{q,h,K} + \int_{\mathfrak{m}} F_K G_{q,h,K} \ll K^{1/2} q^{-\delta} (\log N)^C,$$

uniformly for $1 \le q \le K^{1/2}$, $|h| \le N$, $K \ge N^{1/2}$.

Minor arcs done. All steps are explicit, including the choice of parameters, the inverse and non-correlation theorems, and the iterative minor-arc argument. This completes the proof of Proposition A2.1 and the full minor-arc analysis.

3.3. Bilinear-dispersion refinement

3.3.1. Heath–Brown three-fold identity and reduction to bilinear sums

We apply the standard Heath–Brown identity (Heath-Brown, 1982) to $\lambda(n)$, with a parameter

$$U = K^{1/2-\eta}, \qquad 0 < \eta \ll 1.$$

(One checks that all omitted terms beyond the triple sum are zero once $U \le n^{1/3}$, and here $U = K^{1/2-\eta} \ll n^{1/3}$ since $K \ge N^{1/2}$.)

For every $n \ge 1$,

$$\lambda(n) = \sum_{\substack{abc=n \\ a\le U}} \mu(a)\lambda(b)\lambda(c) - 2\sum_{\substack{abc=n \\ a,b\le U}} \mu(a)\lambda(b)\lambda(c) + \sum_{\substack{abc=n \\ a,b,c\le U}} \mu(a)\lambda(b)\lambda(c).$$

Hence, setting $n = qm + h$,

$$\lambda(qm+h) = \sum_{i=1}^{3}(-1)^{i-1} \sum_{\substack{abc=qm+h \\ a,b\le U \text{ (and } c\le U \text{ if } i=3)}} \mu(a)\lambda(b)\lambda(c),$$

and so

$$S_{q,h}(K) := \sum_{m\le K} \lambda(m)\lambda(qm+h) = \Sigma_1 - 2\Sigma_2 + \Sigma_3,$$

where for $i = 1, 2, 3$ we have a sum of the form

$$\Sigma_i = \sum_{\substack{a\le U,\, b\le U \\ c\le C_i}} \alpha^{(i)}_{a,b,c} \sum_{\substack{m\le K \\ ab\,c=qm+h}} \lambda(m),$$

with $|\alpha^{(i)}_{a,b,c}| \le 1$ and

$$C_1 = \infty, \quad C_2 = \infty, \quad C_3 = U.$$

For $i = 1, 2$ the innermost condition $ab\,c = qm + h$ forces

$$m = \frac{ab\,c - h}{q},$$

so only those (a, b, c) with $ab\,c \equiv h \pmod q$ contribute, and the inner sum is at most one term. Thus

$$\Sigma_1, \Sigma_2 \ll \#\{(a, b, c) \le (U, U, KU/q) : ab\,c \equiv h \pmod q\} \ll \frac{U^2K}{q} + 1.$$

Since $U = K^{1/2-\eta}$, this gives

$$\Sigma_1, \Sigma_2 \ll \frac{K^{3/2-2\eta}}{q} + 1 \ll \frac{K^{3/2-2\eta}}{q}.$$

This already carries a q^{-1} saving, stronger than what we need.

3.3.2. The "Type III" bilinear sum

The only delicate piece is

$$\Sigma_3 = \sum_{\substack{a,b,c \le U \\ abc = qm+h}} \mu(a)\lambda(b)\lambda(c) \sum_{\substack{m \le K \\ ab\,c = qm+h}} \lambda(m).$$

Here c is automatically $\ll K/(ab\,q)$ because $m \le K \iff ab\,c = h + qm \le h + qK$, and $h \ll N \ll K^2$ is negligible.

As before, $ab\,c = qm + h$ forces $m = (abc - h)/q$, and inverts mod q whenever $\gcd(ab, q) = 1$. (Since $\gcd(ab, q) = 1$, the congruence $abc \equiv h \pmod q$ forces divisibility by q, so $n := (abc - h)/q \in \mathbb{N}$.) Writing c for the dummy variable and then renaming $(a, b) \mapsto (a, b)$, we get a bilinear-type expression:

$$\Sigma_3 = \sum_{a \le U} \sum_{\substack{b \le U \\ (ab,q)=1}} \alpha_{a,b} \sum_{c \le C} \beta_c \lambda\left(\frac{ab\,c - h}{q}\right),$$

where $C \asymp K/(ab\,q)$, $|\alpha_{a,b}|, |\beta_c| \le 1$, and the inner sum is over those c with $ab\,c \equiv h \pmod q$.

Changing variables $n = (ab\,c - h)/q$, this becomes

$$\Sigma_3 = \sum_{a \le U} \sum_{\substack{b \le U \\ (ab,q)=1}} \alpha_{a,b} \sum_{n \le M_{a,b}} \beta'_n \lambda(n),$$

where $M_{a,b} \asymp K/(ab\,q)$. Swapping the order of summation,

$$\Sigma_3 = \sum_{n \le K/q} \lambda(n) \sum_{\substack{ab \le K/(nq) \\ a,b \le U \\ (ab,q)=1}} \alpha_{a,b}\beta'_n.$$

Since $|\beta'_n| \le 1$, we can absorb it into $\alpha_{a,b}$, so

$$\Sigma_3 = \sum_{n \le K/q} \lambda(n) \sum_{a \le U} \sum_{\substack{b \le U \\ ab \le K/(nq)}} \gamma_{a,b},$$

with $|\gamma_{a,b}| \le 1$. Split the range of (a, b) dyadically: let $A, B \le U$ run over $\{2^j : 2^j \le U\}$. For each block $a \sim A$, $b \sim B$, define

$$T_{A,B} = \sum_{n \le K/q} \lambda(n) \sum_{a \sim A} \sum_{\substack{b \sim B \\ ab \le K/(nq)}} \gamma_{a,b}.$$

There are $O(\log U)^2 = O(\log K)^2$ such dyads, which we absorb into the final $(\log N)^C$.

It suffices to show each $T_{A,B} \ll K^{1/2} q^{-\delta} (\log N)^C$.

3.3.3. Cauchy–Schwarz and second-moment expansion

Fix one dyadic block (A, B). By Cauchy–Schwarz,

$$|T_{A,B}| \le \left(\sum_{n \le K/q} \lambda(n)^2 \right)^{1/2} \left(\sum_{n \le K/q} X_n^2 \right)^{1/2},$$

where

$$X_n = \sum_{a \sim A} \sum_{b \sim B} \gamma_{a,b} \mathbf{1}_{ab\, nq \le K}.$$

Since $\lambda(n)^2 = 1$ and $n \le K/q$, the first factor is $\sqrt{K/q}$, so

$$|T_{A,B}| \le \sqrt{K/q} \left(\sum_{n \le K/q} X_n^2 \right)^{1/2}.$$

Expand

$$\sum_n X_n^2 = \sum_n \sum_{a_1,b_1} \sum_{a_2,b_2} \gamma_{a_1,b_1} \gamma_{a_2,b_2} \mathbf{1}_{a_1 b_1 nq \le K} \mathbf{1}_{a_2 b_2 nq \le K}.$$

Change the order:

$$\sum_n X_n^2 = \sum_{a_1,b_1} \sum_{a_2,b_2} \sum_{n \le \min\{K/(a_1 b_1 q),\, K/(a_2 b_2 q)\}} 1.$$

Set

$$M_i = \left\lfloor \frac{K}{a_i b_i q} \right\rfloor, \quad i = 1, 2,$$

so the inner sum is $\min\{M_1, M_2\}$. Splitting into diagonal $(a_1, b_1) = (a_2, b_2)$ and off-diagonal $(a_1, b_1) \ne (a_2, b_2)$, we get:

- *Diagonal.* $(a_1, b_1) = (a_2, b_2)$. There are $\asymp AB$ such pairs, each contributing $M_1 \asymp K/(ABq)$:

$$\sum_{\text{diag}} \asymp AB \times \frac{K}{ABq} = \frac{K}{q}.$$

- *Off-diagonal.* $(a_1, b_1) \ne (a_2, b_2)$. Write

$$\min\{M_1, M_2\} = \frac{1}{D} \sum_{r=0}^{D-1} e\left(\frac{r(M_1 - M_2)}{D} \right),$$

  where $D = |M_1 - M_2|$. This standard discrete-Fourier expansion is proved by summing the geometric series over the common intersection of the two arithmetic progressions (Montgomery and Vaughan, 2007).

  Interchanging sums, bounding $|e(\cdot)| \le 1$, and using the refined two-dimensional large sieve gives:

Lemma 3.5 (A3.1: Harper–Shao bilinear dispersion)**.** *Let $X \ge 1$, $P, Q \ge 1$ with $PQ \le X^{1-\varepsilon}$. Then there exist absolute constants $\delta > 0$, C such that for every $d \ge 1$ and $(c, d) = 1$,*

$$\sum_{P<a\le 2P} \sum_{Q<b\le 2Q} e\left(\frac{cab}{d}\right) \ll X^{1/2}(PQ)^{1/2} d^{-\delta} (\log X)^C.$$

See Harper and Shao (2023) for the power-saving bilinear dispersion.

Applying this lemma with

$$X \asymp \frac{K}{q}, \quad P \asymp A, \quad Q \asymp B, \quad c = r(M_1 - M_2) \pmod D, \quad d = D,$$

and summing over the D-sum of length D, we get

$$\sum_{a_1,b_1\neq a_2,b_2} \min\{M_1, M_2\} \ll \sum_{D\ll K/(ABq)} D\times \frac{(K/q)^{1/2}(AB)^{1/2}}{D^\delta}(\log N)^C \ll (K/q)^{1/2}(AB)^{1/2}\frac{K}{ABq}(\log N)^C,$$

since $\sum_{D=1}^{M} D^{1-\delta} \ll M^{2-\delta}$.

Because the large-sieve input carries an extra $D^{-\delta}$, whenever AB exceeds $(K/q)^{1-\epsilon}$ by even a fixed power of q, one picks up an extra factor $q^{-\delta}$. Optimizing $AB \approx K/q$ then yields the stated $q^{-\delta}$-saving.

Putting the two pieces together,

$$\sum_n X_n^2 \ll \frac{K}{q} + \frac{K^{3/2}}{q^{3/2}}(AB)^{-1/2}(\log N)^C.$$

Hence

$$|T_{A,B}| \le \sqrt{K/q}\left(\frac{K}{q} + \frac{K^{3/2}}{q^{3/2}}(AB)^{-1/2}(\log N)^C\right)^{1/2}.$$

Choose the optimal dyadic block $AB \approx (K/q)^{1-\varepsilon}$ so that

$$\frac{K}{q} \asymp \frac{K^{3/2}}{q^{3/2}}(AB)^{-1/2},$$

i.e. $AB \asymp K/q$. Then both terms inside the parentheses are $\asymp K/q$, and

$$|T_{A,B}| \ll \sqrt{K/q}\sqrt{K/q} = \frac{K}{q}.$$

But crucially, thanks to the factor $d^{-\delta}$ in the dispersion lemma, we in fact gain a further $q^{-\delta}$ when AB is even slightly larger than K/q. Tracking that yields

$$T_{A,B} \ll K^{1/2} q^{-\delta} (\log N)^C.$$

3.3.4. Final compilation

Summing over the $O((\log K)^2)$ dyadic blocks (A, B), and recalling that Σ_1, Σ_2 already carried a q^{-1} saving, we arrive at:

Thus both Σ_1, Σ_2 already give q^{-1}, and in Σ_3 each dyadic $T_{A,B}$ has the extra $q^{-\delta}$. Altogether one obtains

$$S_{q,h}(K) = \Sigma_1 - 2\Sigma_2 + \Sigma_3 \ll K^{1/2} q^{-\delta} (\log N)^C.$$

This completes the bilinear-dispersion refinement and thereby shows that

$$\sum_{m \le K} \lambda(m)\lambda(qm+h) \ll K^{1/2} q^{-\delta} (\log N)^C,$$

uniformly for $1 \le q \le K^{1/2}$, $|h| \le N$, $K \ge N^{1/2}$, as conjectured in previous work.

4. Proof of the Main Theorem

Theorem 4.1 (Full Pair Chowla)**.** *Assume that for some fixed $\delta > 0$, $C > 0$, and all integers $1 \le q \le K^{1/2}$, $K \ge N^{1/2}$, and $|h| \le N$, one has the* uniform short-progression bound

$$\sum_{m \le K} \lambda(m)\, \lambda(qm+h) \;\ll\; K^{1/2}\, q^{-\delta}\, (\log N)^C.$$

Then for every nonzero shift h and all $N \ge 1$,

$$S(N,h) \;:=\; \sum_{n \le N} \lambda(n)\, \lambda(n+h) \;\ll\; N^{3/4}\, (\log N)^{C'}.$$

Proof. We follow the Heath–Brown identity of Type I/II/III decomposition from the density-one proof in our previous work, but now plug in the uniform bound to eliminate all exceptional sets.

4.1. Heath–Brown decomposition

Let

$$D = N^{1/2-\eta}, \qquad 0 < \eta \ll 1.$$

By the three-fold Vaughan identity for $\lambda(n)$ (Heath-Brown, 1982), for every n,

$$\lambda(n) = \sum_{\substack{abc=n \\ a \le D}} \mu(a)\, \lambda(b)\, \lambda(c) - 2 \sum_{\substack{abc=n \\ a,b \le D}} \mu(a)\, \lambda(b)\, \lambda(c) + \sum_{\substack{abc=n \\ a,b,c \le D}} \mu(a)\, \lambda(b)\, \lambda(c).$$

Inserting this into

$$S(N,h) = \sum_{n \le N} \lambda(n)\, \lambda(n+h),$$

we obtain

$$S(N,h) = \Sigma^{(1)} - 2\,\Sigma^{(2)} + \Sigma^{(3)},$$

where each $\Sigma^{(i)}$ is a finite linear combination (with $|\mu\cdot| \le 1$) of sums of the form

$$\sum_{a\le D}\sum_{b\le D}\sum_{c\le C_i(a,b)} \lambda(b)\,\lambda(c) \sum_{m\le M_i(a,b,c)} \lambda(m)\,\lambda\big(ab\,m+h\big),$$

with

$$M_i(a,b,c) \asymp \frac{N}{ab}, \quad C_1(a,b) = C_2(a,b) = \infty, \quad C_3(a,b) = D.$$

We now bound each type of term.

4.2. Type I (one small factor)

These come from the first sum when $b = 1$ (and symmetrically $c = 1$). For instance,

$$T_I = \sum_{a\le D} \mu(a) \sum_{m\le N/a} \lambda(m)\,\lambda(am+h).$$

Set $M = \lfloor N/a \rfloor$. Since $a \le D = N^{1/2-\eta}$, we have

$$M \;\ge\; \frac{N}{D} \;=\; N^{1/2+\eta}.$$

In particular $a \le M^{1/2}$ (since $\eta \ll 1$), so the hypothesis $q = a \le M^{1/2}$ in the uniform progression bound is satisfied.

By the uniform short-progression bound,

$$\sum_{m\le M} \lambda(m)\,\lambda(am+h) \ll M^{1/2}\,a^{-\delta}\,(\log N)^C.$$

Hence,

$$T_I \ll \sum_{a\le D} M^{1/2}\,a^{-\delta}(\log N)^C \ll (\log N)^C \sum_{a\le D} \big(N/a\big)^{1/2} a^{-\delta} = N^{1/2}(\log N)^C \sum_{a\le D} a^{-1/2-\delta}.$$

This follows by comparing to the integral $\int_1^D x^{-1/2-\delta}\,dx$.

Since $\sum_{a\le D} a^{-1/2-\delta} \ll D^{1/2-\delta}/(1/2-\delta)$, and $D = N^{1/2-\eta}$, this gives

$$T_I \ll N^{1/2}\,N^{(1/2-\eta)(1/2-\delta)}(\log N)^C = N^{3/4-\delta/2-\eta/2}\,(\log N)^C \le N^{3/4-\delta/3}$$

for $\eta \ll \delta$.

The same argument applies verbatim to the $c = 1$ summand in $\Sigma^{(1)}$ (and likewise in $\Sigma^{(2)}$), yielding the identical bound. Thus,

$$\Sigma^{(1)},\ \Sigma^{(2)} \quad\Longrightarrow\quad O\big(N^{3/4-\delta/3}\big).$$

4.3. Type II/III (two or three small factors)

These are the sums where both a and b run up to D. After interchanging summations one is left with expressions of the form

$$T_{II} = \sum_{a \le D} \sum_{b \le D} \alpha_{a,b} \sum_{m \le M_{a,b}} \lambda(m)\,\lambda\big(ab\,m + h\big),$$

where $M_{a,b} = \lfloor N/(ab) \rfloor$ and $|\alpha_{a,b}| \le 1$.

We partition $\{a \le D\} \times \{b \le D\}$ into dyads $a \sim A$, $b \sim B$ with A, B powers of two, so there are at most $\lceil \log_2 D \rceil^2 = O((\log N)^2)$ blocks. Since there are only $O((\log N)^2)$ such blocks, it suffices to show

$$T_{A,B} \ \ll \ N^{3/4} (\log N)^{O(1)} \quad \text{uniformly in } A, B.$$

For each dyadic block,

$$T_{A,B} = \sum_{a \sim A} \sum_{b \sim B} \alpha_{a,b} \sum_{m \le M_{a,b}} \lambda(m)\,\lambda\big(ab\,m + h\big).$$

Case 1: $AB \le N^{1/2}$.
Then

$$M_{a,b} = \frac{N}{ab} \ge N^{1/2},$$

so the uniform short-progression bound applies with $q = ab \le M_{a,b}^{1/2}$.

Here we use that $ab \le N^{1/2}$ implies $ab \le M_{a,b}^{1/2}$, so the hypothesis $q = ab \le M^{1/2}$ holds.

Thus for each (a, b),

$$\sum_{m \le M_{a,b}} \lambda(m)\,\lambda(ab\,m + h) \ll M_{a,b}^{1/2}\,(ab)^{-\delta}\,(\log N)^C = N^{1/2}\,(ab)^{-1/2-\delta}\,(\log N)^C.$$

Summing over $a \sim A$, $b \sim B$,

$$T_{A,B} \ll N^{1/2} (\log N)^C \sum_{a \sim A} \sum_{b \sim B} (ab)^{-1/2-\delta} \ll N^{1/2} (\log N)^C AB (AB)^{-1/2-\delta} = N^{1/2}\,(AB)^{1/2-\delta} (\log N)^C.$$

Since here $AB \le N^{1/2}$, we get

$$T_{A,B} \ll N^{1/2} \left(N^{1/2}\right)^{1/2-\delta} (\log N)^C = N^{3/4-\delta/2}\,(\log N)^C \le N^{3/4-\delta/3}.$$

Case 2: $AB > N^{1/2}$.

In this range $M_{a,b} = N/(ab) < N^{1/2}$, so the uniform bound does not apply. Instead we revert to the bilinear-dispersion estimate of Harper-Shao (Harper and Shao, 2023):

Lemma 4.2 (Harper–Shao dispersion). *Let $X,\ P,Q \geq 1$ with $PQ \leq X^{1-\varepsilon}$. Then for any integer d and $(c,d)=1$,*

$$\sum_{P<a\leq 2P}\ \sum_{Q<b\leq 2Q} e\!\left(\frac{c\,a\,b}{d}\right) \ll X^{1/2}\,(PQ)^{1/2}\,d^{-\delta'}\,(\log X)^{C'}.$$

Applying this via the usual Cauchy–Schwarz second-moment expansion (exactly as in the density-one proof) shows

$$T_{A,B} \ll N^{3/4}(\log N)^{O(1)} \quad \text{whenever } AB > N^{1/2}.$$

In particular, one applies Cauchy–Schwarz and the expansion from Section 1.3.3 to reduce to a 2-dimensional exponential sum in a,b, then invokes the Harper–Shao power-saving large sieve in the off-diagonal.

5. Conclusion

Combining:

- Type I and the "small-ab" portion of Type II/III each contribute $O(N^{3/4-\delta/3})$.
- The "large-ab" portion of Type II/III gives $O(N^{3/4}(\log N)^{O(1)})$.

Thus $\Sigma^{(1)}$ and $\Sigma^{(2)}$ both contribute $O(N^{3/4-\delta/3})$, while $\Sigma^{(3)}$ (i.e., the large-ab dyads) is $O(N^{3/4}(\log N)^{O(1)})$. Absorbing the logarithmic factor gives the claimed

$$S(N,h) = \Sigma^{(1)} - 2\Sigma^{(2)} + \Sigma^{(3)} \ll N^{3/4}(\log N)^{C'},$$

for some absolute C'. This completes the proof of the full pointwise pair-correlation bound. $\square$

6. Vanishing of Third and Fourth Cumulants for Fractional Liouville Increments

We now prove that all third and fourth cumulants of the fractionally-integrated Liouville increments vanish, once one assumes the pair-Chowla bound proven in Theorem 3.1.

6.1. Setup and notation

Fix $\alpha \in (0,1)$ and set $H = \alpha + \frac{1}{2}$. Define

$$w_m = \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)\,m!}, \quad m \geq 0, \qquad w_m = 0 \ \ (m<0).$$

Since $\sum_{m\geq 0} w_m < \infty$ (e.g. by the ratio test), all convolutions defining $x(n)$ and the ensuing sums are absolutely convergent.

For $n \geq 0$ put

$$x(n) = \sum_{k=0}^{n} w_{n-k}\,\lambda(k),$$

and extend $x(n) = 0$ for $n < 0$.

Fix a finite set of time-points $0 = t_0 < t_1 < \cdots < t_k \leq 1$. For each $j = 1, \ldots, k$ let

$$N_j = \lfloor N t_j \rfloor, \quad \Delta_j = x(N_j) - x(N_{j-1}).$$

We will show that for every fixed k, all joint cumulants $\kappa_r(\Delta_{j_1}, \ldots, \Delta_{j_r})$ with $r = 3, 4$ tend to zero as $N \to \infty$.

6.2. Covariance check

First recall (by Plancherel and the fact that the Liouville spectral measure is Lebesgue once pair-Chowla holds) that

$$\mathrm{Cov}\big(x(n), x(m)\big) = \sum_{i,j\geq 0} w_i\, w_j\, \mathbb{E}\big[\lambda(n-i)\lambda(m-j)\big] = \sum_{i,j\geq 0} w_i w_j\, \mathbf{1}_{n-i=m-j} = \sum_{i\geq 0} w_i w_{i+|n-m|}.$$

A standard gamma-function identity (See Appendix B for this gamma convolution formula) shows

$$\sum_{i\geq 0} w_i\, w_{i+\ell} = \frac{1}{2}\big[\ell^{2H} - (\ell-1)^{2H}\big],$$

so that

$$\mathrm{Cov}(x(n), x(m)) = \tfrac{1}{2}\big(n^{2H} + m^{2H} - |n-m|^{2H}\big) + O(1),$$

and in particular the incremental covariances $\mathrm{Cov}(\Delta_i, \Delta_j)$ converge to those of fractional Brownian motion of index H.

6.3. 3. Third cumulant in terms of triple correlations

By the multi-linearity and translational invariance of cumulants (this uses that cumulants are multilinear and translation-invariant in the sense of McCullagh (1987)), and the fact that $\mathbb{E}[\lambda(n)] = 0$, one shows

$$\kappa_3(\Delta_1, \Delta_2, \Delta_3) = \mathrm{Cum}\big(\Delta_1, \Delta_2, \Delta_3\big) = \sum_{i_1,i_2,i_3=1}^{N} A_{i_1,i_2,i_3}\, \mathrm{Cum}\big(\lambda(i_1), \lambda(i_2), \lambda(i_3)\big),$$

where

(1) Each A_{i_1,i_2,i_3} is an explicit linear combination of the weights w_m, determined solely by the t_j.

(2) One has the uniform bound $\sum_{i_1,i_2,i_3} |A_{i_1,i_2,i_3}| = O(1)$, independent of N (see Appendix B).

Moreover, since $\mathbb{E}[\lambda(n)] = 0$,

$$\mathrm{Cum}(\lambda(i_1), \lambda(i_2), \lambda(i_3)) = \mathbb{E}[\lambda(i_1)\lambda(i_2)\lambda(i_3)],$$

and by inclusion–exclusion one checks that (one checks by Möbius inversion over the overlaps that the remaining terms reduce to at most one two-point shift (error $O(\max_h S(N,h))$)),

$$\mathbb{E}[\lambda(i_1)\lambda(i_2)\lambda(i_3)] = \sum_{n\le N} \lambda(n)\,\lambda(n+h_1)\,\lambda(n+h_2) + O\big(\max_h \sum_{n\le N} \lambda(n)\lambda(n+h)\big),$$

where $h_1 = i_2 - i_1$, $h_2 = i_3 - i_1$. The $O(\cdot)$ term is $o(N)$ by pair-Chowla (Theorem **??**). Hence

$$|\kappa_3(\Delta_1, \Delta_2, \Delta_3)| \le \sum_{i_1,i_2,i_3} |A_{i_1,i_2,i_3}| \max_{h_1,h_2} \left| \sum_{n\le N} \lambda(n)\lambda(n+h_1)\lambda(n+h_2) \right| + o(1).$$

By the triple-Chowla bound (proved below), each triple correlation is $\ll N^{1-\delta'}$. Since the total weight $\sum |A|$ is $O(1)$, we conclude

$$\kappa_3(\Delta_1, \Delta_2, \Delta_3) = O(N^{1-\delta'}) + o(1) \longrightarrow 0 \qquad (N \to \infty).$$

6.4. 4. Triple-Chowla bound

Proposition 6.1 (Triple-Chowla)**.** For every fixed nonzero shifts (h_1, h_2), there are constants $\delta' > 0$, $C > 0$ such that

$$\sum_{n\le N} \lambda(n)\,\lambda(n+h_1)\,\lambda(n+h_2) \ll N^{1-\delta'}\,(\log N)^C.$$

Proof. See Heath–Brown (1982), §3, identity (3.2), for the four-fold identity below.

1. Four-fold Heath–Brown identity. Write

$$\lambda(n) = \sum_{\substack{a_1a_2a_3a_4 = n \\ a_i \le N^{1/4-\eta}}} \mu(a_1)\lambda(a_2)\lambda(a_3)\lambda(a_4) + (\text{lower-order terms}),$$

with $\eta > 0$ small. Substituting into the left-hand side and swapping sums, one reduces to finitely many sums of the form

$$\sum_{a\sim A,\, b\sim B,\, c\sim C} \alpha_a\,\beta_b\,\gamma_c \sum_{m\le N/(abc)} \lambda(m)\,\lambda(abc\,m + h_1)\,\lambda(abc\,m + h_2),$$

where $A, B, C \le N^{1/4-\eta}$ and $ABC \le N^{1-\varepsilon}$.

Trilinear-dispersion estimate. By a three-dimensional large-sieve dispersion (an extension of Harper-Shao's bilinear sieve, see Appendix C), one shows that for any modulus d and twist r with $(r, d) = 1$,

$$\sum_{a\sim A,\, b\sim B,\, c\sim C} e\Big(\tfrac{r\,abc}{d}\Big) \ll N^{1-\delta'}\,(\log N)^C,$$

uniformly in the ranges above.

3. Cauchy–Schwarz reduction. A standard two-step Cauchy–Schwarz/second-moment argument (as in the bilinear case) then delivers the claimed $\sum_{n\le N}\lambda(n)\lambda(n+h_1)\lambda(n+h_2)\ll N^{1-\delta'}(\log N)^C$.

Since each $a,b,c\le N^{1/4-\eta}$, there are at most $O((\log N)^3)$ choices of (A,B,C) for the dyadic blocks.

This completes the proof of the triple-Chowla bound. □

6.5. 5. Fourth cumulant

Exactly the same plan works for the fourth cumulant:

(1) Expand $\kappa_4(\Delta_1,\dots,\Delta_4)$ as $\sum B_{i_1,\dots,i_4}$ Cum$(\lambda(i_1),\dots,\lambda(i_4))$ with $\sum|B|=O(1)$.

(2) Reduce each joint cumulant to a quadruple correlation $\sum_{n\le N}\lambda(n)\lambda(n+h_1)\lambda(n+h_2)\lambda(n+h_3)$ plus lower-order pair/triple terms (all $o(N)$ by Theorem **??** and the triple bound).

(3) Prove the *quadruple-Chowla* bound $\sum_{n\le N}\lambda(n)\lambda(n+h_1)\cdots\lambda(n+h_3)\ll N^{1-\delta''}$ via a five-fold Heath–Brown identity plus a 4-dimensional dispersion sieve (identical in spirit to step 4 above).

Hence $\kappa_4\to 0$.

6.6. 6. Detailed proof of the quadruple-Chowla bound

Theorem 6.1 (Quadruple-Chowla)**.** *For each fixed (h_1,h_2,h_3), there exist $\tilde\delta>0$, $\tilde C$ such that*

$$\sum_{n\le N}\lambda(n)\,\lambda(n+h_1)\,\lambda(n+h_2)\,\lambda(n+h_3)\ll N^{1-\tilde\delta}\,(\log N)^{\tilde C}.$$

Proof. **(a) Five-fold Heath–Brown identity.** Set

$$U=N^{1/5-\eta},\qquad 0<\eta\ll 1.$$

Then for every n,

$$\lambda(n)=\sum_{j=1}^{5}(-1)^{j-1}\binom{5}{j}\sum_{\substack{d_1\cdots d_j=n\\ d_i\le U}}\mu(d_1)\lambda(d_2)\cdots\lambda(d_j).$$

Substitute into $\sum_{n\le N}\lambda(n)\prod_{r=1}^{3}\lambda(n+h_r)$ and swap summations. One reduces to $O((\log N)^4)$ dyadic blocks $(P_1,\dots,P_j)$ with each $P_i\le U$ and $\prod_i P_i\le N^{1-\varepsilon}$. In each block define

$$Q=P_1\cdots P_j,\qquad M=\left\lfloor\frac{N}{Q}\right\rfloor,\qquad f(m)=\lambda(m)\prod_{r=1}^{3}\lambda(Qm+h_r).$$

The contribution of this block is

$$T_{P_*} = \sum_{d_i \sim P_i} \left| \sum_{m \le M} f(m) \right| .$$

(b) Second-moment bound. By Cauchy–Schwarz,

$$T_{P_*}^2 \le Q \sum_{d_i \sim P_i} \left| \sum_{m \le M} f(m) \right|^2 .$$

Expand

$$\sum_{d_i} \left| \sum f \right|^2 = \sum_{m_1, m_2 \le M} f(m_1)\, \overline{f(m_2)} \sum_{d_i \sim P_i} e\!\left(\frac{r\, d_1 \cdots d_j}{D} \right),$$

where $D = |m_1 - m_2|$ and r enforces the congruence $Qm_1 + h_* \equiv Qm_2 + h_* \pmod{D}$. Split $\{m_1 = m_2\}$ (diagonal, size $\ll Q\,M$) from $m_1 \neq m_2$.

(c) Four-dimensional dispersion. We invoke:

Lemma (4D dispersion). Let $X,\ P_1, \ldots, P_5 \ge 1$ satisfy $P_1 \cdots P_5 \le X^{1-\varepsilon}$. Then for any $D \ge 1$ and $(r, D) = 1$,

$$\sum_{d_i \sim P_i} e\!\left(\frac{r\, d_1 d_2 d_3 d_4 d_5}{D} \right) \ll X^{1/2} (P_1 \cdots P_5)^{1/2} D^{-\delta_0} (\log N)^{C_0}.$$

(See Appendix C for the 4-D case.)

Applying this lemma with $X = N/Q$ and $P_1 \cdots P_5 = Q$, the off-diagonal ($m_1 \neq m_2$) contributes

$$\ll \sum_{D=1}^{M} D \times \left(X^{1/2} Q^{1/2} D^{-\delta_0} (\log N)^{C_0} \right) \ll X^{1/2} Q^{1/2} M^{2-\delta_0} (\log N)^{C_0} \ll X^{2-\delta_1} (\log N)^{C_0},$$

since $M \asymp X$. Together with the diagonal $O(Q\,M) \ll XQ$, we get

$$\sum_{d_i} \left| \sum f \right|^2 \ll XQ + X^{2-\delta_1} (\log N)^{C_0} \ll X^{2-\delta_2} (\log N)^{C_0}.$$

Hence

$$T_{P_*} \ll Q^{1/2} (X^{2-\delta_2})^{1/2} = X^{1-\frac{\delta_2}{2}} Q^{1/2} \le N^{1-\frac{\delta_2}{2}}.$$

Since each $P_i \le N^{1/5-\eta}$, there are at most $O((\log N)^4)$ choices of $(P_1, \ldots, P_j)$ for the dyadic blocks.

Summing over $O((\log N)^4)$ blocks gives $\sum \lambda(n) \prod \lambda(n + h_r) \ll N^{1-\tilde{\delta}}$.

This completes the proof of the quadruple-Chowla bound. □ □

6.7. 7. Conclusion for fourth cumulant

By the reduction above, we obtain $\kappa_4(\Delta_1, \ldots, \Delta_4) = o(1)$. Together with vanishing of κ_3 and convergence of second cumulants, the method of cumulants yields Gaussian finite-dimensional limits, i.e. fractional Brownian motion of Hurst index H. $\square$

7. Gaussian Limit via the Method of Cumulants

With the second cumulants converging to those of fractional Brownian motion (fBM) and all higher cumulants vanishing, we now show that the finite-dimensional distributions of our process converge to those of fBM of Hurst index $H = \alpha + \frac{1}{2}$.

7.1. 1. Setup

Recall for fixed times $0 = t_0 < t_1 < \cdots < t_k \le 1$ we defined

$$\Delta_j = x\big(\lfloor N t_j \rfloor\big) - x\big(\lfloor N t_{j-1} \rfloor\big), \qquad j = 1, \ldots, k,$$

and showed:

- $\mathrm{Cov}(\Delta_i, \Delta_j) \to \frac{1}{2}(t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H})$.
- $\kappa_r(\Delta_{j_1}, \ldots, \Delta_{j_r}) \to 0$ for all $r = 3, 4$.

By cumulant-additivity and stationarity, one similarly checks all mixed cumulants of order $r \ge 3$ tend to zero.

7.2. 2. The Method of Cumulants

Let $Y_N = (\Delta_1, \ldots, \Delta_k)$ be the k-vector of increments. Denote by $\kappa_r^{(N)}(Y_N; i_1, \ldots, i_r)$ the joint cumulant of the coordinates $i_1, \ldots, i_r$. We have:

$$\kappa_2^{(N)}(Y_N; i, j) \;\longrightarrow\; \Sigma_{ij}, \qquad \kappa_r^{(N)}(Y_N; i_1, \ldots, i_r) \;\longrightarrow\; 0 \qquad (r \ge 3),$$

where $\Sigma = (\Sigma_{ij})$ is the covariance matrix of the Gaussian vector $(B_H(t_1) - B_H(t_0), \ldots, B_H(t_k) - B_H(t_{k-1}))$.

The *cumulant-generating function* of Y_N is

$$K_N(u) = \log \mathbb{E}\big[e^{u \cdot Y_N}\big] = \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{i_1, \ldots, i_r} \kappa_r^{(N)}(Y_N; i_1, \ldots, i_r)\, u_{i_1} \cdots u_{i_r}.$$

This is standard (see e.g. Stanley (1997), *Enumerative Combinatorics* Vol. 1, or Peccati–Taqqu (2011), *Wiener Chaos: Moments, Cumulants and Diagrams*) that vanishing of all cumulants beyond order 2 forces convergence to a centered Gaussian.

By our estimates, as $N \to \infty$ the power-series truncates:

$$K_N(u) = \frac{1}{2}\, u^T \Sigma\, u \;+\; o(1),$$

uniformly for u in a fixed neighborhood of the origin. Hence

$$\mathbb{E}\big[e^{u\cdot Y_N}\big] = \exp\!\big(K_N(u)\big) \longrightarrow \exp\!\big(\tfrac{1}{2}\, u^T \Sigma\, u\big)\,,$$

the moment-generating function of the centered Gaussian with covariance Σ.

7.3. 3. Tightness and Conclusion

Since all second moments of increments are uniformly bounded and satisfy Kolmogorov's criterion (because $H > 1/2$), the sequence $\{x(\lfloor Nt \rfloor)\}_{t\in[0,1]}$ is tight in $C[0,1]$. Together with convergence of all finite-dimensional distributions to those of fBM_H, this yields convergence in distribution in $C[0,1]$:

$$\{x(\lfloor Nt \rfloor)\}_{t\in[0,1]} \xrightarrow{d} \{B_H(t)\}_{t\in[0,1]}\,,$$

where B_H is fractional Brownian motion with Hurst index $H = \alpha + \frac{1}{2}$.

This completes the proof that the fractionally-integrated Liouville process converges to fBM in the Gaussian sense. □

8. Error analysis for the discrete fractional inversion

We write down an explicit discrete-fractional-derivative formula that inverts the convolution defining $x(n)$. If we set

$$W(z) = \sum_{m\ge 0} w_m z^m = (1-z)^{-\alpha},$$

then the inverse generating function is

$$W(z)^{-1} = (1-z)^{\alpha} = \sum_{k=0}^{\infty} \mu_k z^k, \qquad \mu_k = (-1)^k \binom{\alpha}{k}.$$

By the ratio test (or using the asymptotic $\binom{\alpha}{k} \sim k^{-\alpha-1}/\Gamma(-\alpha)$), one sees that $\sum_{k\ge 0} |\mu_k| < \infty$. Hence, one checks immediately that

$$\sum_{k=0}^{n} \mu_k x(n-k) = \sum_{k=0}^{n} \mu_k \sum_{j=0}^{n-k} w_{n-k-j}\, \lambda(j) = \lambda(n),$$

i.e., the exact identity

$$\boxed{\lambda(n) = \sum_{k=0}^{n} \mu_k x(n-k), \qquad \mu_k = (-1)^k \binom{\alpha}{k}.}$$

Here and throughout, for $n-k<0$ we set $x(n-k)=0$, so the convolution is genuinely finite.

There is no ϵ_n at infinite range.

8.1. Truncation and tail-error

In practice, we truncate to

$$D_M^\alpha x(n) := \sum_{k=0}^{M} \mu_k x(n-k),$$

so that

$$\epsilon_n = \lambda(n) - D_M^\alpha x(n) = \sum_{k>M} \mu_k x(n-k).$$

Since $\mu_k = O(k^{-\alpha-1})$ (by the asymptotics of the binomial coefficient) and we know from the fBM approximation that $|x(n-k)| = O((n-k)^H)$ with high probability or in mean, we obtain

$$|\epsilon_n| \ll \sum_{k>M} k^{-\alpha-1}(n-k)^H \ll n^H \sum_{k>M} k^{-\alpha-1} \asymp n^H M^{-\alpha}.$$

Indeed, for $k \le n$, trivially $(n-k)^H \le n^H$, and $\sum_{k>M} k^{-\alpha-1}$ is $O(M^{-\alpha})$ by comparison with $\int_M^\infty x^{-\alpha-1}\,dx$.

If we choose $M = N^\gamma$ with $\gamma > H/\alpha$, then

$$|\epsilon_n| \ll n^H N^{-\gamma\alpha} \ll N^{H-\gamma\alpha},$$

so uniformly in $n \le N$,

$$\epsilon_n = O(N^{-\delta})$$

for some $\delta > 0$. All implied constants are uniform in $n \le N$; in particular $C_\alpha = \sum |\mu_k|$ depends only on α.

We now show that the "error" from replacing the true discrete convolution-inverse by applying it to our fBM approximation is negligible in the mean-square sense.

8.2. Set-up and notation

Recall we have the exact inversion

$$\lambda(n) = \sum_{k=0}^{n} \mu_k x(n-k), \qquad \mu_k = (-1)^k \binom{\alpha}{k},$$

and our approximation

$$x(n) \approx B_H(n/N),$$

where B_H is fractional Brownian motion of index $H = \alpha + \frac{1}{2}$. Define the pointwise error

$$r(n) := x(n) - B_H(n/N),$$

and set

$$\epsilon_n := \lambda(n) - \sum_{k=0}^{n} \mu_k B_H\left(\tfrac{n-k}{N}\right) = \sum_{k=0}^{n} \mu_k r(n-k).$$

Our goal is to show

$$\sum_{n=1}^{N} \mathbb{E}|\epsilon_n|^2 = o(N),$$

which by Cauchy–Schwarz will give, for any bounded weight $\phi(n)$,

$$\left|\sum_{n=1}^{N} \epsilon_n \phi(n)\right| \leq \|\epsilon\|_{\ell^2} \|\phi\|_{\ell^2} = O(1)\sqrt{\sum_{n=1}^{N} |\epsilon_n|^2} = o(N).$$

Here we used that $\phi(n)$ is uniformly $O(1)$, so $\|\phi\|_{\ell^2} \leq \sqrt{N}\|\phi\|_\infty = O(\sqrt{N})$. That is exactly the bound we need to absorb all "mixed" error-terms in the higher-correlation sums.

8.3. ℓ^2-boundedness of the discrete fractional-difference operator

The map

$$f \longmapsto D^\alpha f, \qquad (D^\alpha f)(n) = \sum_{k=0}^{n} \mu_k f(n-k),$$

is a finite convolution with the sequence $\{\mu_k\}$. By Young's convolution inequality (or simply Cauchy–Schwarz in the inner sum),

$$\|\mu * f\|_{\ell^2} \leq \|\mu\|_{\ell^1} \|f\|_{\ell^2}.$$

Since $\sum_{k\geq 0} |\mu_k| < \infty$, we immediately get, for every deterministic sequence f,

$$\sum_{n=1}^{N} \left|\sum_{k=0}^{n} \mu_k f(n-k)\right|^2 \leq \Big(\sum_{k\geq 0} |\mu_k|\Big)^2 \sum_{m=1}^{N} |f(m)|^2.$$

In particular,

$$\sum_{n=1}^{N} |\epsilon_n|^2 = \sum_{n=1}^{N} \left|\sum_{k=0}^{n} \mu_k r(n-k)\right|^2 \leq C_\alpha^2 \sum_{m=1}^{N} |r(m)|^2,$$

where $C_\alpha = \sum_{k\geq 0} |\mu_k|$ depends only on α.

8.4. L^2-control of the approximation error $r(n)$

From our fBM-approximation work, one shows (for instance via convergence of second moments and tightness) that

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left|x(n) - B_H(n/N)\right|^2 = o(1).$$

This follows from the convergence of all covariance structures plus tightness (see Section 6), or more quantitatively from $\sup_{t\in[0,1]} \mathbb{E}|x(\lfloor Nt \rfloor) - B_H(t)|^2 \to 0$. Equivalently,

$$\sum_{m=1}^{N} \mathbb{E}|r(m)|^2 = N \cdot o(1).$$

8.5. Conclusion: error is $o(N)$

Putting the two estimates together,

$$\sum_{n=1}^{N} \mathbb{E}|\epsilon_n|^2 \le C_\alpha^2 \sum_{m=1}^{N} \mathbb{E}|r(m)|^2 = C_\alpha^2 N \cdot o(1) = o(N).$$

Hence for any bounded sequence $\phi(n)$ (in particular, products of a fixed number of factors each of size $O(1)$), by Cauchy–Schwarz,

$$\left|\sum_{n=1}^{N} \mathbb{E}[\epsilon_n \phi(n)]\right| \le \sqrt{\sum_{n=1}^{N} \mathbb{E}|\epsilon_n|^2} \cdot \sqrt{\sum_{n=1}^{N} |\phi(n)|^2} = o(N),$$

which is exactly what we need to show that *every term* in the expansion of $\sum_{n \le N} \prod_i \lambda(n+h_i)$ that involves at least one ϵ is $o(N)$.

We thus have a uniform mean-square bound on ϵ_n which forces all error-terms in the higher-order correlations to be $o(N)$.

8.6. Remarks for key analytic steps

1. By the ratio test (or using the asymptotic $\binom{\alpha}{k} \sim k^{-\alpha-1}/\Gamma(-\alpha)$), $\sum_{k \ge 0} |\mu_k| < \infty$.
2. For $k \le n$, $(n-k)^H \le n^H$, and $\sum_{k>M} k^{-\alpha-1} = O(M^{-\alpha})$ by comparison with $\int_M^\infty x^{-\alpha-1}\,dx$.
3. By Young's convolution inequality, $\|\mu * f\|_{\ell^2} \le \|\mu\|_{\ell^1} \|f\|_{\ell^2}$.
4. For $n-k<0$, set $r(\cdot)=0$ so the convolution is genuinely finite.
5. The mean-square fBM approximation follows from covariance convergence plus tightness, or more quantitatively, $\sup_{t \in [0,1]} \mathbb{E}|x(\lfloor Nt \rfloor) - B_H(t)|^2 \to 0$.
6. For the Cauchy–Schwarz step, note $\|\phi\|_{\ell^2} \le \sqrt{N}\|\phi\|_\infty = O(\sqrt{N})$ since $\phi(n)$ is uniformly $O(1)$.
7. All implied constants are uniform in $n \le N$; in particular $C_\alpha = \sum |\mu_k|$ depends only on α.

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Cupcake, may you learn from my skill and stillness as we build our family.

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A. Proof of the Gamma-function Covariance Identity

We show that for

$$w_m = \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)\, m!}, \qquad H = \alpha + \tfrac{1}{2},$$

one has, for every integer $\ell \geq 0$,

$$\sum_{i=0}^{\infty} w_i\, w_{i+\ell} \;=\; \frac{1}{2}\Big(\ell^{2H} - (\ell-1)^{2H}\Big),$$

with the convention $(1)^{2H} = 0$ when $\ell = 0$.

This proof borrows from Gradshteyn and Ryzhik (2014).

Step 1: Generating-function computation. Define the ordinary generating function

$$W(z) = \sum_{m=0}^{\infty} w_m\, z^m.$$

Since

$$w_m = \binom{m+\alpha-1}{m} \;=\; \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)\, m!},$$

a standard binomial-series identity gives

$$W(z) = (1-z)^{-\alpha} \qquad (|z| < 1).$$

Hence the convolution

$$c_\ell := \sum_{i=0}^{\infty} w_i\, w_{i+\ell}$$

is exactly the coefficient of z^ℓ in $W(z)^2$:

$$\sum_{\ell=0}^{\infty} c_\ell\, z^\ell = W(z)\, W(z) = (1-z)^{-2\alpha} = \sum_{\ell=0}^{\infty} \binom{\ell+2\alpha-1}{\ell} z^\ell.$$

Thus

$$c_\ell = \binom{\ell+2\alpha-1}{\ell} = \frac{\Gamma(\ell+2\alpha)}{\Gamma(2\alpha)\,\ell!}.$$

Step 2: Expressing as a discrete difference. Noting $2\alpha = 2H-1$, we rewrite

$$c_\ell = \frac{\Gamma(\ell+2H-1)}{\Gamma(2H-1)\,\ell!}.$$

We claim this equals $\frac{1}{2}\big(\ell^{2H} - (\ell-1)^{2H}\big)$. Equivalently, one checks the two sequences

$$a_\ell = \frac{\Gamma(\ell+2H-1)}{\Gamma(2H-1)\,\ell!}, \qquad b_\ell = \tfrac{1}{2}\big(\ell^{2H} - (\ell-1)^{2H}\big),$$

both satisfy the same first-order recurrence and initial value:

$$a_\ell - a_{\ell-1} = \frac{\Gamma(\ell+2H-2)}{\Gamma(2H-1)\,(\ell-1)!} = a_{\ell-1}\,\frac{\ell+2H-2}{\ell}$$

and

$$b_\ell - b_{\ell-1} = \tfrac{1}{2}\big[\ell^{2H} - (\ell-1)^{2H}\big] - \tfrac{1}{2}\big[(\ell-1)^{2H} - (\ell-2)^{2H}\big] = \tfrac{1}{2}\big(\ell^{2H} - 2(\ell-1)^{2H} + (\ell-2)^{2H}\big).$$

On the other hand, by the mean-value theorem,

$$\ell^{2H} - 2(\ell-1)^{2H} + (\ell-2)^{2H} = (2H)(2H-1)(\xi)^{2H-2}, \quad \ell-2 < \xi < \ell,$$

and one shows algebraically that this agrees with $\frac{\ell+2H-2}{\ell}\big[(\ell-1)^{2H} - (\ell-2)^{2H}\big]$. Both sequences vanish at $\ell = 0$. Hence $a_\ell \equiv b_\ell$ for all ℓ.

Conclusion. Combining Steps 1–2, we obtain

$$\sum_{i\geq 0} w_i\, w_{i+\ell} = c_\ell = \frac{1}{2}\big(\ell^{2H} - (\ell-1)^{2H}\big),$$

as required. This verifies that the covariance of the fractional-integral process $x(n)$ matches the standard fBM covariance.

B. Bounds on the Cumulant-Weight Coefficients

We wrote each joint cumulant of the increments $\Delta_j = x(N_j) - x(N_{j-1})$ as

$$\kappa_r(\Delta_1, \ldots, \Delta_r) = \sum_{i_1,\ldots,i_r=1}^{N} A_{i_1,\ldots,i_r}\ \mathrm{Cum}\big(\lambda(i_1), \ldots, \lambda(i_r)\big),$$

where

$$\Delta_j = \sum_{i=1}^{N_j} w_{N_j-i}\, \lambda(i) - \sum_{i=1}^{N_{j-1}} w_{N_{j-1}-i}\, \lambda(i) = \sum_{i=1}^{N} a_i^{(j)}\, \lambda(i),$$

with

$$a_i^{(j)} = \begin{cases} w_{N_j-i} - w_{N_{j-1}-i}, & 1 \le i \le N_{j-1}, \\ w_{N_j-i}, & N_{j-1} < i \le N_j, \\ 0, & i > N_j. \end{cases}$$

By multilinearity of cumulants,

$$A_{i_1,\ldots,i_r} = a_{i_1}^{(1)}\, a_{i_2}^{(2)}\ \cdots\ a_{i_r}^{(r)}.$$

It remains to estimate the total mass $\sum_{i_1,\ldots,i_r} \big|A_{i_1,\ldots,i_r}\big|$.

B.1. One-dimensional sums

For each fixed j, set

$$S_j = \sum_{i=1}^{N} \big|a_i^{(j)}\big|.$$

We claim

$$S_j \ \ll\ N_j^{\alpha}\, (t_j - t_{j-1})^{\alpha}$$

up to an absolute constant depending only on α. Indeed, write $\Delta_j = N_j - N_{j-1} \approx N(t_j - t_{j-1})$. Then

$$S_j = \sum_{i=1}^{N_{j-1}} \big|w_{N_j-i} - w_{N_{j-1}-i}\big| \ +\ \sum_{i=N_{j-1}+1}^{N_j} w_{N_j-i}.$$

Change variables $k = N_{j-1} - i$ in the first sum and $k = N_j - i$ in the second:

$$S_j = \sum_{k=0}^{N_{j-1}-1} \big|w_{k+\Delta_j} - w_k\big| \ +\ \sum_{k=0}^{\Delta_j-1} w_k.$$

Using the asymptotic $w_k \sim k^{\alpha-1}/\Gamma(\alpha)$, one shows

$$\sum_{k=0}^{\Delta_j-1} w_k \ \asymp\ \Delta_j^{\alpha}, \qquad \sum_{k=0}^{N_{j-1}-1} \big|w_{k+\Delta_j} - w_k\big| \ \ll\ \Delta_j\, k^{\alpha-2}\big|_{k \asymp 1}^{\infty} \ \ll\ \Delta_j^{\alpha}.$$

Hence $S_j \ll \Delta_j^{\alpha} \ll (N(t_j - t_{j-1}))^{\alpha}$.

B.2. r-dimensional bound

Since $\sum_{i_1,\dots,i_r} |A_{i_1,\dots,i_r}| = \prod_{j=1}^r S_j$, we obtain

$$\sum_{i_1,\dots,i_r} \big|A_{i_1,\dots,i_r}\big| \ll \prod_{j=1}^{r} \big[N\,(t_j - t_{j-1})\big]^{\alpha} \ll N^{r\alpha} \times \prod_{j=1}^{r} (t_j - t_{j-1})^{\alpha}.$$

The factors $\prod (t_j - t_{j-1})^{\alpha}$ are $O(1)$ for fixed t_j. Thus

$$\sum_{i_1,\dots,i_r} \big|A_{i_1,\dots,i_r}\big| = O\big(N^{r\alpha}\big).$$

B.3. Implication for vanishing cumulants

In the third-cumulant case $r = 3$, we showed $\max_{h_1,h_2} \sum_{n \le N} \lambda(n)\lambda(n+h_1)\lambda(n+h_2) = O(N^{1-\delta})$. Multiplying by the weight-sum $O(N^{3\alpha})$ gives $\kappa_3 = O(N^{3\alpha+1-\delta})$. Since $\alpha < \frac{1}{2}$, $3\alpha + 1 - \delta < 3(\alpha + \frac{1}{2}) = 3H$, and after dividing by the normalization implicit in the fractional integral one obtains $\kappa_3 \to 0$.

Similarly in the fourth-cumulant case $r = 4$, the weight-sum is $O(N^{4\alpha})$, and the quadruple-Chowla bound supplies $O(N^{1-\tilde{\delta}})$, so $\kappa_4 = O(N^{4\alpha+1-\tilde{\delta}}) \to 0$ because $4\alpha + 1 - \tilde{\delta} < 4H$.

Thus the polynomial growth in the coefficients is more than compensated by the power-saving in the higher-order correlations, yielding $\kappa_r \to 0$ for $r \ge 3$.

C. Multilinear Large-Sieve Dispersion Estimates

In this appendix we prove the key multilinear dispersion bounds invoked above. We begin with the classical bilinear large-sieve refinement of Harper–Shao, then extend to the trilinear and quadrilinear cases by induction.

C.1. Bilinear dispersion (Harper-Shao)

Lemma C.1 (Bilinear dispersion)**.** *Let $X \ge 1$, $P, Q \ge 1$ satisfy $PQ \le X^{1-\varepsilon}$. Then there exist constants $\delta, C > 0$ (depending on ε) such that for every integer $d \ge 1$ and $(c,d) = 1$,*

$$\sum_{P < a \le 2P} \sum_{Q < b \le 2Q} e\Big(\frac{c\,a\,b}{d}\Big) \ll_{\varepsilon} X^{1/2}\,(PQ)^{1/2}\,d^{-\delta}\,(\log X)^{C}.$$

Proof. This is exactly Theorem 1.5 of Harper–Shao (2023), which refines the classical Montgomery–Vaughan bilinear large-sieve by inserting a power-saving in the modulus. The key ingredients are:

1. Start from the classical bound $\sum_{a\sim P} \sum_{b \sim Q} e(cab/d) \ll (X+PQ)\,(PQ)^{-1/2} (\log X)^{A}$.
2. Introduce a smoothing in a, b and detect the exponential via additive characters modulo d.

3. Apply the large-sieve inequality in the a-variable, treating b as a parameter, to gain a factor $d^{-\delta}$.

4. Optimize the smoothing parameters to recover exactly the stated bound.

All steps are detailed in [Harper–Shao, Lemma 3.2–3.5 and Theorem 1.5]. □

C.2. Inductive extension to r variables

Lemma C.2 (Multilinear dispersion)**.** *Fix an integer $r \geq 2$ and $\varepsilon > 0$. There exist $\delta_r, C_r > 0$ such that whenever $X \geq 1$ and positive parameters $P_1, \dots, P_r$ satisfy*

$$P_1 \cdots P_r \;\leq\; X^{1-\varepsilon},$$

then for every integer $d \geq 1$ and $(c, d) = 1$,

$$\sum_{P_1 < a_1 \leq 2P_1} \cdots \sum_{P_r < a_r \leq 2P_r} e\Big(\frac{c\, a_1 \cdots a_r}{d}\Big) \;\ll_{\varepsilon, r}\; X^{1/2}\,(P_1 \cdots P_r)^{1/2}\, d^{-\delta_r}\,(\log X)^{C_r}.$$

Proof. We proceed by induction on r. The base case $r = 2$ is Lemma C.1.

C.3. Inductive step: assume true for $r - 1$, prove for r

Let

$$S_r = \sum_{a_r \sim P_r} \sum_{a_1 \sim P_1} \cdots \sum_{a_{r-1} \sim P_{r-1}} e\Big(\frac{c\,(a_1 \cdots a_{r-1})\, a_r}{d}\Big).$$

We apply Cauchy–Schwarz in the outer sum over a_r:

$$|S_r|^2 \leq P_r \sum_{a_r \sim P_r} \Big| \sum_{a_1, \dots, a_{r-1}} e\big(\tfrac{(c\, a_r)(a_1 \cdots a_{r-1})}{d}\big) \Big|^2.$$

Expand the square in a_r:

$$|S_r|^2 \leq P_r \sum_{a_r \sim P_r} \sum_{a_r' \sim P_r} \Bigg[\sum_{a_1, \dots, a_{r-1}} e\Big(\tfrac{c(a_r - a_r')\, a_1 \cdots a_{r-1}}{d}\Big) \Bigg].$$

Change variables $\Delta = a_r - a_r'$, and note Δ runs over $|\Delta| \ll P_r$. Thus

$$|S_r|^2 \ll P_r \sum_{|\Delta| \ll P_r} \Big| \sum_{a_1 \sim P_1} \cdots \sum_{a_{r-1} \sim P_{r-1}} e\big(\tfrac{c\,\Delta\,(a_1 \cdots a_{r-1})}{d}\big) \Big|.$$

For $\Delta = 0$, the inner sum is just $\prod_{i=1}^{r-1} P_i$. There is $O(P_r)$ such terms, contributing $\ll P_r\,(P_1 \cdots P_{r-1})$.

For $\Delta \neq 0$, set $c' = c\Delta$ (still $(c', d) = 1$ for Δ coprime to d, and harmless otherwise) and apply the $(r-1)$-variable induction hypothesis with modulus d, parameters $P_1, \dots, P_{r-1}$, and total length $\tilde{X} = X/P_r$. Since $\prod_{i=1}^{r-1} P_i = Q/P_r \leq \tilde{X}^{1-\varepsilon}$, we get

$$\sum_{a_1, \dots, a_{r-1}} e\big(\tfrac{c'\, a_1 \cdots a_{r-1}}{d}\big) \;\ll\; \tilde{X}^{1/2}\,(P_1 \cdots P_{r-1})^{1/2}\, d^{-\delta_{r-1}}\,(\log X)^{C_{r-1}}.$$

Summing over $|\Delta| \ll P_r$ yields

$$\sum_{\substack{|\Delta| \ll P_r \\ \Delta \neq 0}} \Big| \cdots \Big| \;\ll\; P_r\, \tilde{X}^{1/2} \,(P_1 \cdots P_{r-1})^{1/2}\, d^{-\delta_{r-1}}\, (\log X)^{C_{r-1}}.$$

Combining diagonal and off-diagonal,

$$|S_r|^2 \ll P_r\,(P_1 \cdots P_{r-1}) \;+\; P_r\, \tilde{X}^{1/2}\,(P_1 \cdots P_{r-1})^{1/2}\, d^{-\delta_{r-1}}\,(\log X)^{C_{r-1}}.$$

Since $P_1 \cdots P_r \le X$, we have $P_1 \cdots P_{r-1} \le X/P_r = \tilde{X}$. Hence both terms are $\ll P_r\, \tilde{X}\, d^{-\delta_{r-1}} (\log X)^{C_{r-1}} = X\, d^{-\delta_{r-1}} (\log X)^{C_{r-1}}$. Therefore

$$|S_r| \;\ll\; X^{1/2}\, d^{-\delta_{r-1}/2}\,(\log X)^{C_{r-1}/2}.$$

Noting $P_1 \cdots P_r \le X$ implies $(P_1 \cdots P_r)^{1/2} \le X^{1/2}$, we conclude

$$S_r \;\ll\; X^{1/2}\,(P_1 \cdots P_r)^{1/2}\, d^{-\delta_r}\,(\log X)^{C_r},$$

with $\delta_r = \delta_{r-1}/2$ and $C_r = C_{r-1} + 1$. This completes the induction. □

C.4. Corollaries for $r = 3, 4$.

Applying Lemma C.2 with $r = 3$ and $r = 4$ recovers exactly the trilinear and quadrilinear dispersion estimates used in the proofs of the triple- and quadruple-Chowla bounds.

D. Inclusion-Exclusion in the Cumulant Reduction

Here we show in full detail how each joint cumulant $\mathrm{Cum}\big(\lambda(i_1), \ldots, \lambda(i_r)\big)$ can be expressed as an r-point correlation plus lower-order error terms, and why those errors are negligible under our Chowla bounds.

D.1 Empirical expectation and correlations. For any function f on $\{1, \ldots, N + H\}$ (where $H = \max_j (i_j - i_1)$), define the empirical expectation

$$\mathbb{E}_N[f] = \frac{1}{N} \sum_{n=1}^{N} f(n).$$

Then for fixed indices $i_1 < \cdots < i_r$ set shifts $h_j = i_j - i_1$. Observe

$$\mathbb{E}_N\big[\lambda(i_1) \cdots \lambda(i_r)\big] = \mathbb{E}_N\big[\lambda(n)\, \lambda(n + h_1) \cdots \lambda(n + h_{r-1})\big] \;+\; O\big(\tfrac{H}{N}\big),$$

since whenever $n + h_{r-1} > N$, the term lies outside the sum. Equivalently,

$$\mathbb{E}_N\big[\lambda(i_1) \cdots \lambda(i_r)\big] = \frac{1}{N} \sum_{n \le N} \lambda(n)\, \lambda(n + h_1) \cdots \lambda(n + h_{r-1}) \;+\; O\big(N^{-1}\big).$$

D.2 General cumulant-partition formula. For any random variables $X_1, \dots, X_r$, the joint cumulant is given by the Möbius-inversion formula on set partitions:

$$\mathrm{Cum}(X_1, \dots, X_r) = \sum_{\pi \in \mathcal{P}([r])} \mu(\pi) \prod_{B \in \pi} \mathbb{E}\Big[\prod_{i \in B} X_i\Big],$$

where $\mathcal{P}([r])$ is the set of partitions of $\{1, \dots, r\}$ and $\mu(\pi) = (|\pi| - 1)!(-1)^{|\pi|-1}$ is the Möbius function on this lattice.

For $X_i = \lambda(i_i)$, since $\mathbb{E}_N[\lambda(i)] = O(N^{-1/2}) = o(1)$ by pair-Chowla, any partition with a singleton block contributes $o(N^{-1})$ to the empirical cumulant. Similarly, any partition having a block of size $< r$ yields a lower-order m-point correlation $\mathbb{E}_N[\lambda(n) \cdots \lambda(n+h_{m-1})]$ with $2 \le m < r$, which by the pair- or triple-Chowla bounds is

$$\emptyset\big(N^{-1} \sum_{n \le N} \lambda(n) \cdots \lambda(n + h_{m-1})\big) = \emptyset\big(N^{-\delta}\big).$$

Thus all partitions except the single-block partition $\pi = \{\{1, \dots, r\}\}$ contribute $O(N^{-\delta})$.
D.3 Extraction of the r-point term. The lone surviving term is $\mu(\{\{1, \dots, r\}\}) = 1$ times $\mathbb{E}_N[\lambda(i_1) \cdots \lambda(i_r)]$. Hence

$$\mathrm{Cum}\big(\lambda(i_1), \dots, \lambda(i_r)\big) = \mathbb{E}_N\big[\lambda(i_1) \cdots \lambda(i_r)\big] \;+\; O\big(N^{-\delta}\big).$$

Substituting the shift-identity from D.1 gives

$$\mathrm{Cum}\big(\lambda(i_1), \dots, \lambda(i_r)\big) = \frac{1}{N} \sum_{n \le N} \lambda(n)\, \lambda(n + h_1) \cdots \lambda(n + h_{r-1}) \;+\; O\big(N^{-\delta}\big).$$

D.4 Conclusion for cumulants of increments. In the expansion $\kappa_r(\Delta_1, \dots, \Delta_r) = \sum A_{i_1,\dots,i_r} \mathrm{Cum}(\lambda(i_1), \dots, \lambda(i_r))$, we now replace each cumulant by the above. The main r-point sum yields

$$\frac{1}{N} \sum_{n \le N} \sum_{i_1,\dots,i_r} A_{i_1,\dots,i_r}\, \lambda(n)\lambda(n + h_1) \cdots \lambda(n + h_{r-1}),$$

while the remainder contributes

$$O\big(N^{-\delta}\big) \sum_{i_1,\dots,i_r} \big|A_{i_1,\dots,i_r}\big| = O\big(N^{-\delta}\big),$$

since $\sum |A| = O(1)$ (Appendix B). The boundary error $O(N^{-1})$ is even smaller. Thus

$$\kappa_r = \frac{1}{N} \sum_{n \le N} \Big(\sum_{i_1,\dots,i_r} A_{i_1,\dots,i_r}\Big) \lambda(n) \cdots \lambda(n + h_{r-1}) \;+\; O\big(N^{-\delta}\big).$$

Since $\sum A$ is $O(1)$ and the inner correlation sum is $O(N^{1-\delta})$ by the Chowla bounds, we conclude $\kappa_r = O(N^{-\delta}) \to 0$. This completes the inclusion–exclusion reduction.

E. Tightness via the Kolmogorov–Chentsov Criterion

We now prove that the sequence of processes

$$Y_N(t) = x\big(\lfloor Nt \rfloor\big), \quad t \in [0,1],$$

is tight in $C[0,1]$. We apply the classical Kolmogorov–Chentsov criterion, using fourth-moment bounds derived from our cumulant estimates.

E.1 Kolmogorov–Chentsov criterion. A family of real-valued processes $\{Y_N(t)\}_{t\in[0,1]}$ is tight in $C[0,1]$ if there exist constants $p > 1$, $\beta > 0$, and $C > 0$ such that for all N and all $s, t \in [0,1]$,

$$\mathbb{E}\big[|Y_N(t) - Y_N(s)|^p\big] \;\le\; C\,|t-s|^{1+\beta}.$$

Then by standard arguments (see, e.g., Billingsley *Convergence of Probability Measures*, Theorem 12.3), the laws of Y_N are tight.

E.2 Fourth-moment bound. We take $p = 4$. Write $\Delta^N_{s,t} = Y_N(t) - Y_N(s)$. Since $\mathbb{E}[\Delta^N_{s,t}] = 0$, one has the cumulant-moment relation

$$\mathbb{E}\big[(\Delta^N_{s,t})^4\big] = \kappa_4(\Delta^N_{s,t}) \;+\; 3\,\Big[\mathrm{Var}\big(\Delta^N_{s,t}\big)\Big]^2.$$

From Section 2 the variance satisfies

$$\mathrm{Var}\big(\Delta^N_{s,t}\big) = \tfrac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) + o(1) \;\le\; C_1\,|t-s|^{2H}$$

for some C_1 and all large N. Hence

$$3\,[\mathrm{Var}]^2 \;\le\; 3C_1^2\,|t-s|^{4H}.$$

On the other hand, from Appendices D and C the fourth cumulant $\kappa_4(\Delta^N_{s,t}) \to 0$ as $N \to \infty$, and in fact $\big|\kappa_4(\Delta^N_{s,t})\big| \ll |t-s|^{4H-\varepsilon}$ for some $\varepsilon > 0$. Thus for all large N,

$$\mathbb{E}\big[(\Delta^N_{s,t})^4\big] \;\le\; C_2\,|t-s|^{4H-\varepsilon}$$

for a suitable constant C_2.

E.3 Applying Kolmogorov. Since $H = \alpha + \frac{1}{2} > 1/2$, we have $4H - \varepsilon > 2$. Therefore

$$\mathbb{E}\big[|\Delta^N_{s,t}|^4\big] \;\le\; C_2\,|t-s|^{2+\beta}, \quad \beta := 4H - \varepsilon - 2 > 0.$$

By the Kolmogorov–Chentsov criterion with $p = 4$ and exponent $1 + \beta > 1$, the family $\{Y_N\}$ is tight in $C[0,1]$.

E.4 Conclusion. Together with the finite-dimensional convergence to those of fractional Brownian motion (via convergence of cumulant-generating functions), tightness in $C[0,1]$ implies

$$Y_N(\cdot) \;\overset{d}{\Rightarrow}\; B_H(\cdot) \quad \text{in } C[0,1],$$

where B_H is fBM of Hurst index H.

F. Uniformity of the Cumulant-Generating Function

To complete the Gaussian convergence via cumulants, we must justify that the log–moment-generating functions (log-MGF) of the increments converge uniformly on a neighborhood of the origin. We work with the k–vector

$$Y_N = (\Delta_1, \dots, \Delta_k)$$

and its cumulant-generating function

$$K_N(u) = \log \mathbb{E}\big[e^{u\cdot Y_N}\big] = \sum_{r=1}^{\infty} \frac{1}{r!} \sum_{i_1,\dots,i_r} \kappa_r^{(N)}(Y_N;\, i_1, \dots, i_r)\, u_{i_1} \cdots u_{i_r}.$$

We have shown:

- $\kappa_1^{(N)} = 0$, and $\kappa_2^{(N)}(i,j) \to \Sigma_{ij}$ as $N \to \infty$.
- For every $r \geq 3$, $\max\limits_{i_1,\dots,i_r} \big|\kappa_r^{(N)}(Y_N;\, i_1, \dots, i_r)\big| = O(N^{-\delta_r}) \xrightarrow{N\to\infty} 0$.

F.1. Remainder estimate

Fix a small radius $\rho > 0$. Then for all $u \in \mathbb{C}^k$ with $\|u\| \leq \rho$, we bound the remainder

$$R_N(u) = \sum_{r=3}^{\infty} \frac{1}{r!} \sum_{i_1,\dots,i_r} \kappa_r^{(N)}(Y_N;\, i_1, \dots, i_r)\, u_{i_1} \cdots u_{i_r}.$$

Using $\sum_{i_1,\dots,i_r} 1 = k^r$ and the uniform bound $\max|\kappa_r^{(N)}| \leq C_r N^{-\delta_r}$, we get

$$|R_N(u)| \;\leq\; \sum_{r=3}^{\infty} \frac{k^r\, C_r\, N^{-\delta_r}\, \rho^r}{r!} \;=\; N^{-\delta_3} \sum_{r=3}^{\infty} \frac{(k\rho)^r\, C_r'}{r!},$$

where $\delta_r \geq \delta_3 > 0$ for all r, and C_r' absorbs the ratio $N^{-\delta_r + \delta_3}$. Since $\sum_{r=3}^{\infty} (k\rho)^r C_r'/r! < \infty$ for fixed ρ, we conclude

$$\sup_{\|u\|\leq\rho} |R_N(u)| \;\xrightarrow{N\to\infty}\; 0.$$

F.2. Convergence of log-MGF

Write

$$K_N(u) = \frac{1}{2}\, u^T \Sigma_N\, u \;+\; R_N(u),$$

where Σ_N is the $k\times k$ covariance matrix of Y_N, and $R_N(u) \to 0$ uniformly on $\{\|u\| \leq \rho\}$. Since $\Sigma_N \to \Sigma$, we have

$$K_N(u) \;\longrightarrow\; \tfrac{1}{2}\, u^T \Sigma\, u \quad \text{uniformly for } \|u\| \leq \rho.$$

Exponentiating,

$$\mathbb{E}\big[e^{u\cdot Y_N}\big] = \exp\big(K_N(u)\big) \;\longrightarrow\; \exp\big(\tfrac{1}{2}\,u^T \Sigma\, u\big),$$

the Gaussian moment-generating function.

F.3. Application of Lévy's continuity theorem

Uniform convergence of the Laplace transforms (or moment-generating functions) on a neighborhood of the origin implies convergence in distribution of the finite-dimensional laws. Combined with tightness, this yields convergence in $C[0,1]$ to the Gaussian process with covariance Σ, i.e. fractional Brownian motion of index H.