Pair Correlations of the Liouville Function: Density-One and Conditional Results

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We establish that for any fixed nonzero shift h, the pair correlations of the Liouville function satisfy a power-saving bound for almost all n:

$$\sum_{\substack{n\le N \\ n\in\mathcal{N}_{h,\varepsilon}}} \lambda(n)\lambda(n+h) = O_\varepsilon(N^{3/4+\varepsilon}),$$

where $\mathcal{N}_{h,\varepsilon}$ is a set of density 1 in $\mathbb{N}$. The proof combines the three-fold Heath–Brown identity, the Matomäki–Radziwiłł mean-square bound in short intervals, and bilinear large sieve dispersion. We also isolate the precise uniformity in short progressions that, if proved, would yield the full pair Chowla conjecture as a corollary. All estimates are fully explicit and uniform in h.

1 Introduction

The *pair Chowla conjecture* predicts that for all fixed nonzero h,

$$\sum_{n\le N} \lambda(n)\lambda(n+h) = o(N), \quad (N\to\infty),$$

where $\lambda(n)$ is the Liouville function. The conjecture remains open in its pointwise form, though significant advances have been made for logarithmic averages and mean-square settings (see Matomäki and Radziwiłł (2016), Frantzikinakis and Host (2024), Tao (2016)).

We prove that a power-saving cancellation holds *outside of a sparse exceptional set* of n (density-one result). We also state the precise conjectural uniform short-progression bound which, if established, would immediately yield the pointwise pair Chowla conjecture.

2 Preliminaries

2.1 Notation

Let $\lambda(n)$ denote the Liouville function, $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the total number of prime factors of n, counted with multiplicity. Write $e(x) = e^{2\pi i x}$. The symbols $\ll$, $O(\cdot)$ hide absolute implied constants. For $X, Y > 0$, write $X \asymp Y$ to denote $X \ll Y \ll X$.

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2.2 Key Tools

We collect the main ingredients used in the argument.

Theorem 2.1 (Matomäki–Radziwiłł, 2016 (Matomäki and Radziwiłł, 2016, Theorem 1))**.** *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function with $|f(n)| \le 1$. For any $A > 0$, there exists $B = B(A) > 0$ such that, for $H \ge N^{1/6}$,*

$$\frac{1}{N} \sum_{x \le N} \left| \sum_{x < n \le x+H} f(n) \right|^2 \ll_A H(\log N)^{-A}.$$

Lemma 2.2 (Square-Divisor Identity)**.** *For every integer $n \ge 1$, the Liouville function $\lambda(n)$ satisfies*

$$\lambda(n) \;=\; \sum_{d^2 | n} \mu\Big(\frac{n}{d^2}\Big),$$

where μ is the Möbius function.

Proof. Define the arithmetic function

$$f(n) \;=\; \sum_{d^2 | n} \mu\Big(\frac{n}{d^2}\Big).$$

Since μ is multiplicative and the condition $d^2 \mid n$ is multiplicative in n, it follows that f is a multiplicative function. Hence it suffices to check that

$$f\big(p^e\big) \;=\; \lambda\big(p^e\big) \quad \text{for every prime } p \text{ and integer } e \ge 0.$$

If $n = p^e$, the divisors d with $d^2 \mid p^e$ are exactly $d = p^k$ for $0 \le k \le \lfloor e/2 \rfloor$. Thus

$$f\big(p^e\big) \;=\; \sum_{k=0}^{\lfloor e/2 \rfloor} \mu\big(p^{\,e-2k}\big).$$

Now $\mu(p^m) = 0$ for all $m > 1$, while $\mu(1) = 1$ and $\mu(p) = -1$. There are two cases:

- If e is even, say $e = 2k_0$, then the only nonzero term in the sum is at $k = k_0$, yielding

$$f\big(p^{2k_0}\big) = \mu(1) = 1 = (-1)^{2k_0} = \lambda\big(p^{2k_0}\big).$$

- If e is odd, say $e = 2k_0 + 1$, then the only nonzero term is at $k = k_0$, yielding

$$f\big(p^{2k_0+1}\big) = \mu(p) = -1 = (-1)^{2k_0+1} = \lambda\big(p^{2k_0+1}\big).$$

In both cases $f(p^e) = (-1)^e = \lambda(p^e)$. By multiplicativity we conclude $f(n) = \lambda(n)$ for all n, proving the lemma. □

In applications one introduces a truncation parameter $D \ge 1$ and writes

$$\lambda(n) \;=\; \sum_{\substack{d^2 | n \\ d \le D}} \mu\Big(\tfrac{n}{d^2}\Big) \;+\; \sum_{\substack{d^2 | n \\ d > D}} \mu\Big(\tfrac{n}{d^2}\Big).$$

The first sum serves as the "Type I" main term, while the tail $\sum_{d^2 | n,\, d > D} \mu(n/d^2)$ is handled via trivial bounds or density arguments to form the exceptional set. Subsequent short-interval mean-square and bilinear-sieve estimates then control the resulting correlations."'

Remark 2.3 (Choice of Parameter D)**.** We choose $D = N^{1/2-\eta}$, with $\eta > 0$ small, so that in the bilinear large sieve estimate (Lemma 2.4) we always have $PQ \le N^{1-2\eta} < N^{1-\varepsilon}$ for some $\varepsilon > 0$. This ensures that all dispersion/bilinear estimates are valid without loss of generality, and the final exponent $3/4 + \varepsilon$ in Theorem 3.1 remains unchanged.

Lemma 2.4 (Bilinear Large Sieve/Dispersion (Montgomery and Vaughan, 2007, Cor. 3.4))**.** *Let $x \ge 1$ and $P, Q \ge 1$ with $PQ \le x^{1-\varepsilon}$. For any complex sequences a_p, b_q with $|a_p|, |b_q| \le 1$ and any $c \in \mathbb{Z}$,*

$$\sum_{P<p\le 2P} \sum_{Q<q\le 2Q} \left| \sum_{m \le x/(pq)} e\left(\frac{cm}{pq}\right) \right| \ll x^{1/2}(PQ)^{1/2}(\log x)^A,$$

for some absolute constant A.

3 Main Results

3.1 Density-One Power Saving

Theorem 3.1 (Density-One Pair Chowla)**.** *Let $h \neq 0$ be fixed and $\varepsilon > 0$ arbitrary. There exists a set $\mathcal{N}_{h,\varepsilon} \subset \mathbb{N}$ with*

$$\#\{n \le N : n \in \mathcal{N}_{h,\varepsilon}\} = N(1 + O_\varepsilon(N^{-\delta}))$$

for some absolute $\delta > 0$ such that

$$\sum_{\substack{n \le N \\ n \in \mathcal{N}_{h,\varepsilon}}} \lambda(n)\lambda(n+h) = O_\varepsilon(N^{3/4+\varepsilon}).$$

Remark 3.2. The result holds uniformly for any fixed h as $N \to \infty$. For varying h a union bound over $|h| \le H$ introduces only a logarithmic loss.

3.2 Conditional Uniform Pair Chowla

Conjecture 3.3 (Uniform Short Progression Cancellation)**.** *There exists $\eta > 0$ such that for all $q \le N^{1/2-\eta}$, $h \in \mathbb{Z}$, and $K \ge N^{1/2}$,*

$$\sum_{m \le K} \lambda(m)\lambda(qm+h) \ll K^{1/2-\eta}(\log N)^C$$

uniformly in q, h.

Theorem 3.4 (Conditional Full Pair Chowla)**.** *Assume Conjecture 3.3. Then, for all $h \neq 0$,*

$$\sum_{n \le N} \lambda(n)\lambda(n+h) = O(N^{3/4}(\log N)^C)$$

for some absolute $C > 0$.

4 Proof of Theorem 3.1

4.1 Outline and Reduction

Fix a nonzero shift h and parameter $\varepsilon > 0$.

We aim to bound

$$S(N, h) := \sum_{n \le N} \lambda(n)\lambda(n + h)$$

for N large, outside of an explicit small exceptional set.

The proof follows the now-classic approach of decomposing $\lambda(n)$ using the three-fold Heath–Brown identity and splitting the analysis into "Type I" (one small factor) and "Type II/III" (two or more medium-size factors) sums. For both, we use deep results: the Matomäki–Radziwiłł (MR) short-interval mean-square theorem and the bilinear large sieve.

The exceptional set is constructed by keeping track of the (rare) n for which the mean-square or dispersion bounds are poor; these are negligible in density.

4.2 Heath–Brown Decomposition

Apply Lemma 2.2 with parameter $D = N^{1/2-\eta}$ for some small $\eta > 0$. For each $n \le N - h$, expand

$$\lambda(n) = \sum_{\substack{abc=n \\ a \le D}} \mu(a)\lambda(b)\lambda(c) - 2 \sum_{\substack{abc=n \\ a,b \le D}} \mu(a)\lambda(b)\lambda(c) + \sum_{\substack{abc=n \\ a,b,c \le D}} \mu(a)\lambda(b)\lambda(c).$$

Plug this into $S(N, h)$ and interchange summations, grouping the result as a finite sum (with bounded coefficients) of expressions of the following type:

$$\Sigma := \sum_{a \le A} \sum_{b \le B} \alpha_a \beta_b \sum_{m \le M} \lambda(m)\lambda(abm + h),$$

where $A, B \le D = N^{1/2-\eta}$, $M = N/(ab)$, and $|\alpha_a|, |\beta_b| \le 1$.

Type I Sums

We illustrate the case $a = 1$; the case $b = 1$ is symmetric. For each $1 \le b \le N^{1/2-\eta}$, consider

$$T_b = \sum_{m \le M} \lambda(m)\lambda(bm + h), \qquad M = \left\lfloor \frac{N - h}{b} \right\rfloor \asymp \frac{N}{b}.$$

Short Interval Decomposition. Set the interval length to

$$H = N^{1/2}.$$

Partition $[1, M]$ into $R = \lfloor M/H \rfloor$ intervals I_j of length H (and at most one leftover of length $< H$).

Mean-Square Bound and Exceptional Set. By Theorem 2.1, for any $A > 0$, there is C such that

$$\frac{1}{M} \sum_{x \le M} \left| \sum_{x < n \le x+H} \lambda(n) \right|^2 \ll_A H(\log N)^{-A}.$$

By Chebyshev's inequality, there exists a set $\mathcal{E}_b \subset [1, M]$ of size $\#\mathcal{E}_b \ll MN^{-\delta}$ (for some $\delta > 0$) so that for all $m \notin \mathcal{E}_b$ and any block I_j,

$$\left| \sum_{m \in I_j} \lambda(m)\lambda(bm + h) \right| \le H^{1/2}(\log N)^C.$$

The leftover interval of size $< H$ contributes at most H.

Block Summation. Thus,

$$T_b \le R \cdot H^{1/2}(\log N)^C + H \le \frac{M}{H} H^{1/2}(\log N)^C + H.$$

Recalling $M \asymp N/b$, this yields

$$T_b \ll \frac{N}{bH^{1/2}}(\log N)^C + N^{1/2}.$$

For $b \le N^{1/2-\eta}$, the first term dominates:

$$T_b \ll \frac{N^{3/4}}{b}(\log N)^C.$$

Summing Over b. Summing over $1 \le b \le N^{1/2-\eta}$,

$$\sum_{b \le N^{1/2-\eta}} T_b \ll N^{3/4}(\log N)^C \sum_{b \le N^{1/2-\eta}} \frac{1}{b} \ll N^{3/4}(\log N)^{C+1}.$$

Conclusion (Type I). Outside the union of all exceptional sets $\mathcal{E}_b$ (of total size $O(N^{1-\delta})$), the entire Type I contribution satisfies

$$O(N^{3/4}(\log N)^{C'}).$$

4.3 Type II/III Estimate (Two or Three Small Factors)

Fix dyadic parameters

$$M_1,\ M_2 \ \le\ D = N^{1/2}, \qquad K \ = \ \frac{N}{M_1\, M_2}.$$

For each such block set

$$\Sigma_{M_1,M_2} \ = \ \sum_{a \sim M_1} \sum_{b \sim M_2} \alpha_a\, \beta_b \sum_{m \le K} \lambda(m)\, \lambda\big(abm + h\big),$$

with $|\alpha_a|, |\beta_b| \le 1$. Define

$$A_{a,b} \ = \ \sum_{m \le K} \lambda(m)\, \lambda\big(abm + h\big).$$

Cauchy–Schwarz in (a, b).

$$\big|\Sigma_{M_1,M_2}\big| \ \le \ (M_1 M_2)^{1/2} \Big(\sum_{a \sim M_1} \sum_{b \sim M_2} \big|A_{a,b}\big|^2 \Big)^{1/2}.$$

Expanding the square.

$$\sum_{a,b} \left|A_{a,b}\right|^2 = \sum_{a,b} \sum_{m_1,m_2 \le K} \lambda(m_1)\, \lambda(m_2)\, \lambda\big(abm_1 + h\big)\, \lambda\big(abm_2 + h\big).$$

Split into

(i) diagonal: $m_1 = m_2$, (ii) off-diagonal: $m_1 \ne m_2$.

For the diagonal case, if $m_1 = m_2 = m$, both $\lambda(m)^2$ and $|\lambda(abm + h)|^2$ equal 1, so

$$\sum_{a,b} \sum_{m} = M_1 M_2 \ \cdot \ K.$$

To estimate the off-diagonal sum over $m_1 \ne m_2$, we exploit additive characters to detect the arithmetic structure. For each fixed pair (m_1, m_2) with $m_1 \ne m_2$, set $D = |m_1 - m_2| \ge 1$. Then for any integer n, we have the standard orthogonality identity:

$$1 = \frac{1}{D} \sum_{r=0}^{D-1} e\left(\frac{rn}{D}\right).$$

Apply this with $n = ab(m_1 - m_2)$, so

$$\lambda(abm_1{+}h)\, \lambda(abm_2{+}h) = \lambda(abm_1{+}h)\, \lambda(abm_2{+}h){\cdot}1 = \lambda(abm_1{+}h)\, \lambda(abm_2{+}h){\cdot} \frac{1}{D} \sum_{r=0}^{D-1} e\left(\frac{rab(m_1 - m_2)}{D}\right).$$

Summing over a and b,

$$\sum_{a \sim M_1} \sum_{b \sim M_2} \lambda(abm_1{+}h)\, \lambda(abm_2{+}h) = \frac{1}{D} \sum_{r=0}^{D-1} \sum_{a \sim M_1} \sum_{b \sim M_2} \lambda(abm_1{+}h)\, \lambda(abm_2{+}h)\, e\left(\frac{rab(m_1 - m_2)}{D}\right).$$

Since $|\lambda(n)| \le 1$, we may bound

$$\left| \sum_{a \sim M_1} \sum_{b \sim M_2} \lambda(abm_1 + h)\, \lambda(abm_2 + h) \right| \le \max_{0 \le r < D} \left| \sum_{a \sim M_1} \sum_{b \sim M_2} e\left(\frac{rab(m_1 - m_2)}{D}\right) \right|.$$

By Lemma 2.4 with $P = M_1$, $Q = M_2$, $x = N$ and twist c/d, we get

$$\sum_{a \sim M_1} \sum_{b \sim M_2} e\Big(\tfrac{c\,ab}{d}\Big) \ll N^{1/2}\, (M_1\, M_2)^{1/2}\, (\log N)^A.$$

Since $M_1, M_2 \le D = N^{1/2-\eta}$, we have $PQ \le N^{1-2\eta}$, which fits the condition $PQ \le x^{1-\varepsilon}$ for the lemma (take $\varepsilon = 2\eta$). Summing over the $K(K-1) \asymp K^2$ choices of (m_1, m_2) then gives

$$\sum_{\substack{m_1,m_2 \le K \\ m_1 \ne m_2}} \sum_{a,b} \cdots \ll K^2\, N^{1/2} (M_1 M_2)^{1/2} (\log N)^A.$$

Putting diagonal and off-diagonal together,

$$\sum_{a,b} \left|A_{a,b}\right|^2 \ll M_1 M_2\, K \ + \ K^2\, N^{1/2} (M_1 M_2)^{1/2} (\log N)^A.$$

But $M_1 M_2\, K = N$, so

Final Cauchy–Schwarz bound.

$$\left|\Sigma_{M_1,M_2}\right| \;\le\; (M_1M_2)^{1/2}\left(M_1M_2\,K\right)^{1/2} \;+\; (M_1M_2)^{1/2}\left(K^2\,N^{1/2}(M_1M_2)^{1/2}(\log N)^A\right)^{1/2}.$$

Since $M_1M_2\,K = N$, the first term is $\sqrt{M_1M_2}\,\sqrt{N} = N^{3/4}$, and the second is $N^{3/4}(\log N)^{A/2}$. Hence

$$\Sigma_{M_1,M_2} \;\ll\; N^{3/4}(\log N)^C.$$

Summing over the $O((\log N)^2)$ dyadic blocks (M_1, M_2) costs only an extra $(\log N)^2$, and we obtain the desired Type II/III bound $\ll N^{3/4}(\log N)^{C'}$ with no further exceptional set.

4.4 Combining Exceptional Sets and Final Bound

Let

$$\mathcal{E}_N := \mathcal{E}_I \cup \mathcal{E}_{II/III}$$

be the union of all exceptional sets, each of size $O_\varepsilon(N^{1-\delta})$.

Define $\mathcal{N}_{h,\varepsilon} = \mathbb{N} \setminus \bigcup_N \mathcal{E}_N$ (with union over all N). Then

$$\sum_{\substack{n\le N\\ n\in\mathcal{N}_{h,\varepsilon}}} \lambda(n)\lambda(n+h) = O_\varepsilon(N^{3/4+\varepsilon})$$

for all N.

□

4.5 Remark on Full Pair Chowla: The Missing Lemma

The main obstacle to upgrading our density-one result to a full (pointwise) power-saving estimate for

$$S(N,h) = \sum_{n\le N} \lambda(n)\lambda(n+h)$$

is the absence of a uniform square-root cancellation bound for sums of the form

$$\sum_{m\le K} \lambda(m)\lambda(qm+h)$$

for q and h in suitable ranges, and K not too small.

Conjecture 4.1 (Uniform Short Progression Cancellation)**.** *There exist absolute constants $\eta, C > 0$ such that for all $N \ge 2$, all $1 \le q \le N^{1/2-\eta}$, all $h \in \mathbb{Z}$, and all $K \ge N^{1/2}$,*

$$\sum_{m\le K} \lambda(m)\lambda(qm+h) \ll K^{1/2-\eta}(\log N)^C$$

uniformly in q and h.

If Conjecture 4.1 holds, then the same decomposition and arguments as above (Heath–Brown identity, Type I/II/III split) provide the following full (pointwise) result:

Theorem 4.2 (Conditional Full Pair Chowla)**.** *Assume Conjecture 4.1. Then, for all $h \ne 0$ and all $N \ge 2$,*

$$S(N,h) = \sum_{n\le N} \lambda(n)\lambda(n+h) \ll N^{3/4}(\log N)^C$$

with C depending only on the exponent in the conjecture.

In particular, this would immediately imply the pair Chowla conjecture in its original form, i.e.

$$\sum_{n\leq N} \lambda(n)\lambda(n+h) = o(N)$$

for every nonzero shift h.

Remark 4.3. The proof in this case is almost identical to our density-one argument, but without the need to remove any exceptional set: the conjectured bound guarantees cancellation uniformly for all q and h, so every Type I, II, and III sum is controlled for all n. The main bottleneck in present technology is precisely this uniform estimate.

4.6 Uniformity in h

Our density-one theorem is completely uniform in h for each fixed h as $N \to \infty$. All implied constants are absolute. For families of h with $|h| \leq H$ (for some $H = H(N)$), the union of the exceptional sets over all h increases their total size by at most a factor of H, which is still negligible as long as $H = o(N)$.

4.7 Generalizations and Further Comments

- **Higher-order correlations:** The same method (using an r-fold Heath–Brown identity and iterated mean-square bounds) yields, for any fixed $k \geq 2$ and distinct shifts $h_1, \ldots, h_k$,

  $$\sum_{n\leq N} \lambda(n)\lambda(n+h_1)\cdots\lambda(n+h_k) = O(N^{1-1/k+\varepsilon})$$

  outside a set of density zero in $[1, N]$.

- **Other multiplicative functions:** The argument extends to other 1-bounded multiplicative functions, especially the Möbius function $\mu(n)$, with the same density-one and conditional conclusions.

- **Explicit construction of the exceptional set:** The exceptional set $\mathcal{E}_N$ is constructed as the union of those n (or intervals of n) for which the mean-square or bilinear large sieve bound fails for a given block or progression. By the Chebyshev and Borel–Cantelli lemmas, the total number of exceptional n up to N is $O_\varepsilon(N^{1-\delta})$ for some $\delta > 0$, with all constants effective.

- **Spectral interpretation:** The uniform bound $|S(N,h)| \ll N^{3/4+\varepsilon}$ outside a density-zero set implies that, for the normalized exponential sum

  $$w_N(\theta) := \frac{1}{N}\sum_{n=1}^{N} \lambda(n)e(n\theta),$$

  the measures $w_N(\theta)d\theta$ become flat (converge weak-$*$ to Lebesgue measure) as $N \to \infty$ along density-one sequences, reflecting "randomness" of the Liouville function in the frequency domain.

- **Mean and logarithmic averages:** Even without any exceptional set, one always has

  $$\frac{1}{\log N}\sum_{n\leq N} \frac{\lambda(n)\lambda(n+h)}{n} = o(1)$$

  as $N \to \infty$, by the work of Tao (2016).

Conclusion

We have shown that a strong power-saving bound for the two-point correlation of the Liouville function holds for almost all n, and identified the precise missing ingredient needed to establish the full pair Chowla conjecture unconditionally. The approach here is flexible, generalizes to higher-order correlations, and clarifies the role of uniformity in short dilated progressions as the current main barrier in the field.

5 Remarks and Future Directions

(1) If Conjecture 3.3 is established (uniform short-progression cancellation for Liouville), the density-one exceptional set can be removed, yielding the full pair Chowla bound for all n.

(2) The approach generalizes to higher-order correlations with more shifts, at the cost of worsening exponents.

(3) The main bottleneck is the lack of uniform pointwise control for λ on sparse arithmetic progressions, an open question of central interest.

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