Representing & Simplifying Simplicial Complexes in High Dimensions

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Motivation

Reconstruction | Simplification | Analysis

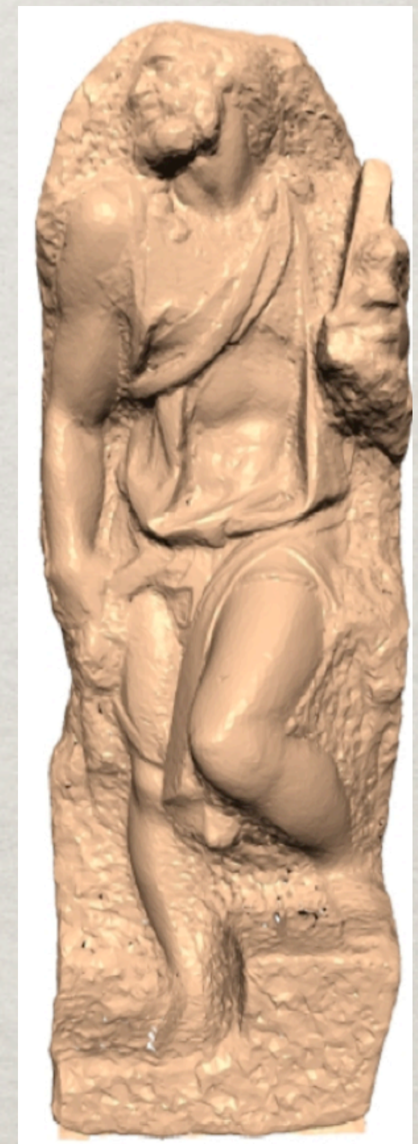
$\simeq$

3D scans

large data sets

$\approx 400,000,000$ triangles

gigantic simplicial complex

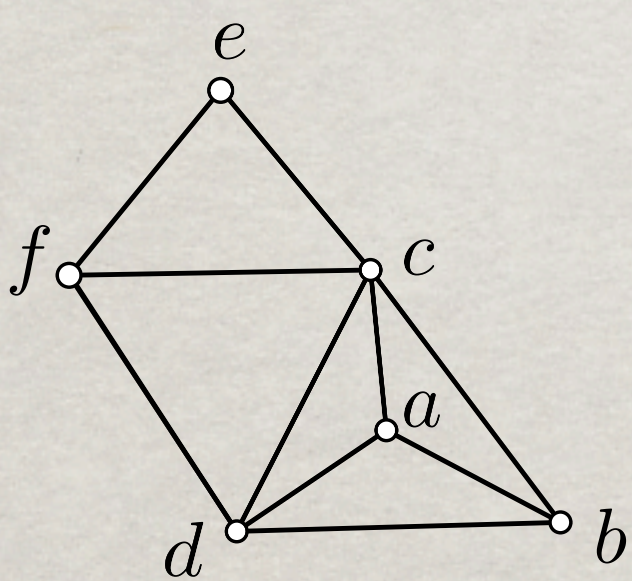


$\approx 100,000$ triangles

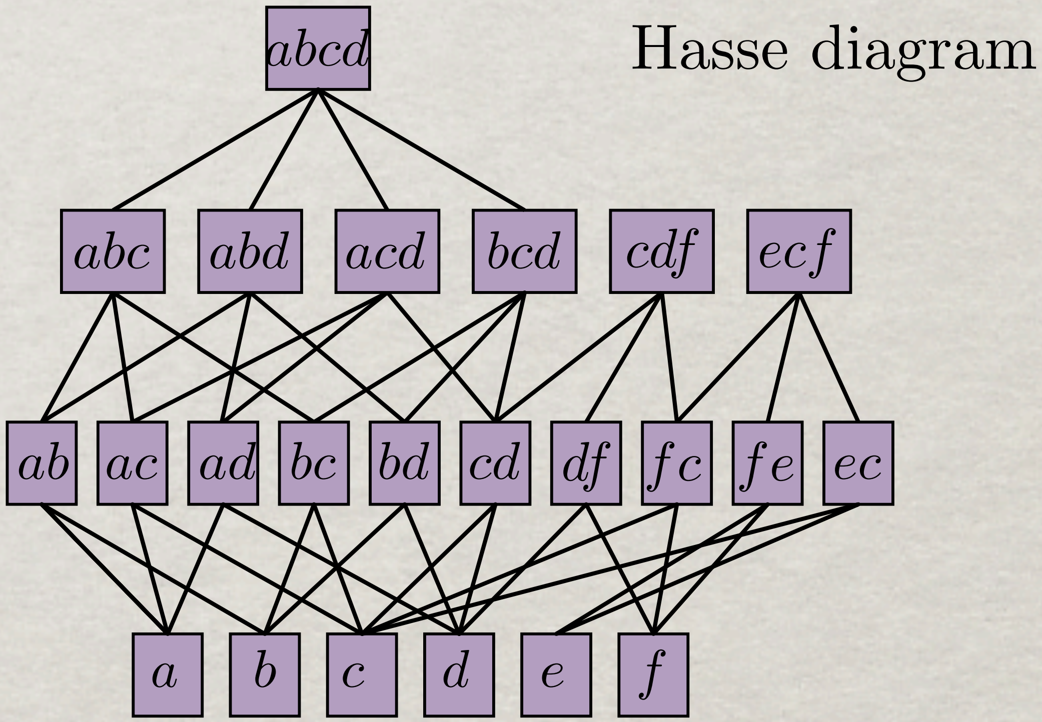
reduced simplicial complex

Flag Complexes

- Flag G = largest simplicial complex whose 1-skeleton is G.
- $\{v_0, \ldots, v_k\} \in \operatorname{Flag} G \iff v_i v_j \in G \;\; \forall i, j$



G



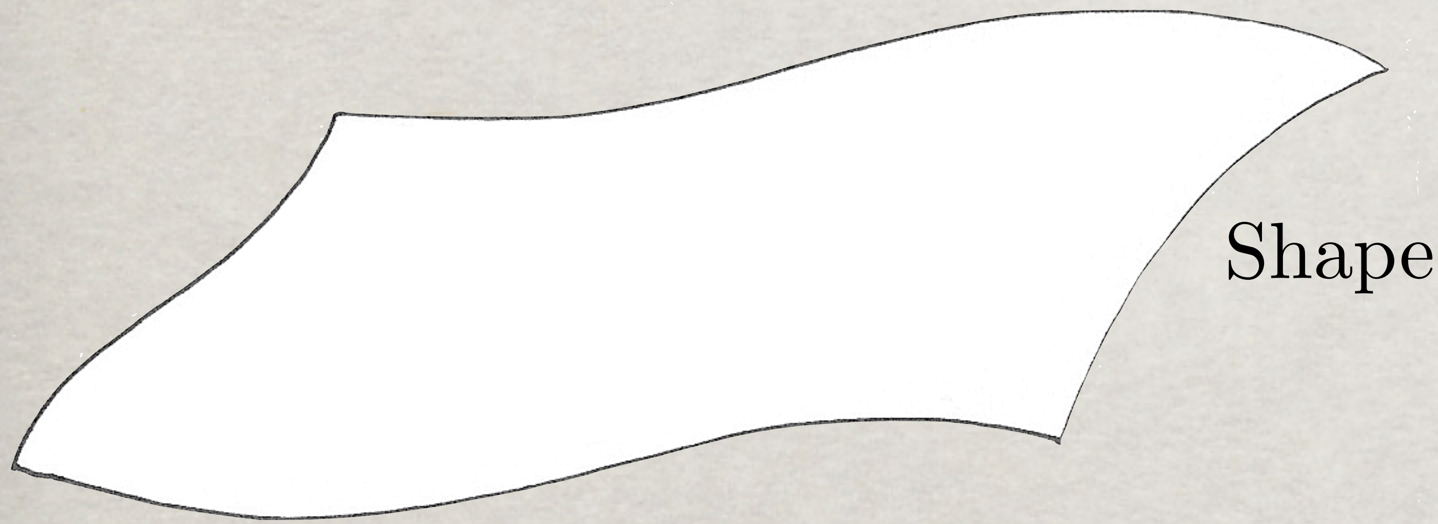
Flag G

Flag complexes have a compressed form of storage

Shape Reconstruction

INPUT

Point cloud P



$G_\alpha(P)$ = proximity graph

$$pq \in G_\alpha(P) \iff d(p,q) \le 2\alpha$$

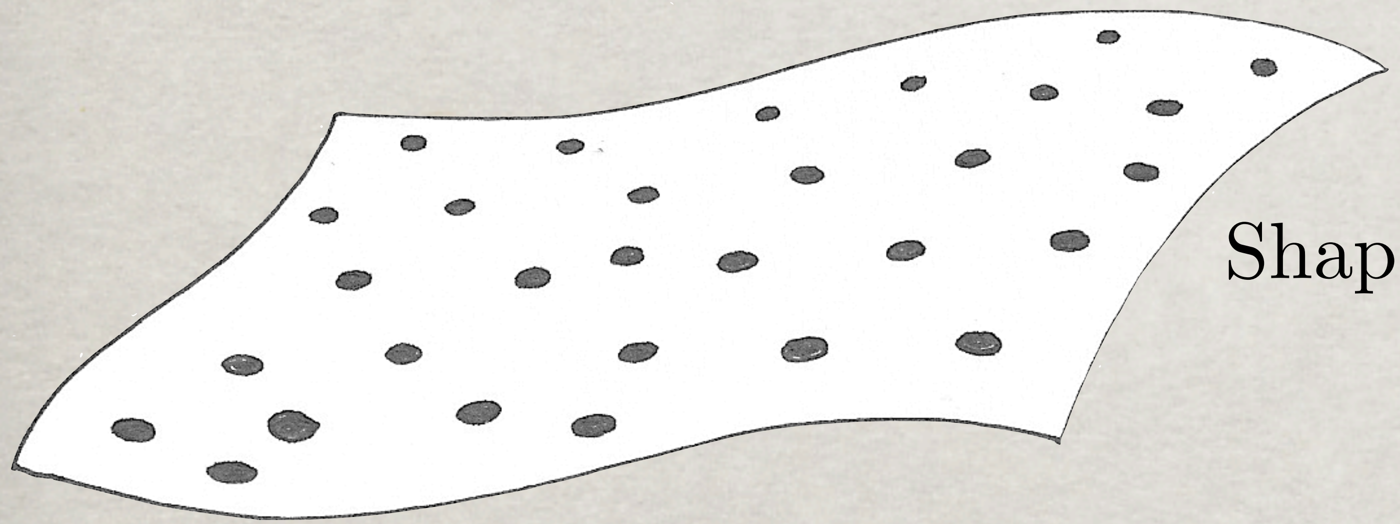
OUTPUT

Flag $G_\alpha(P)$ = Rips complex

Shape Reconstruction

INPUT

Point cloud P



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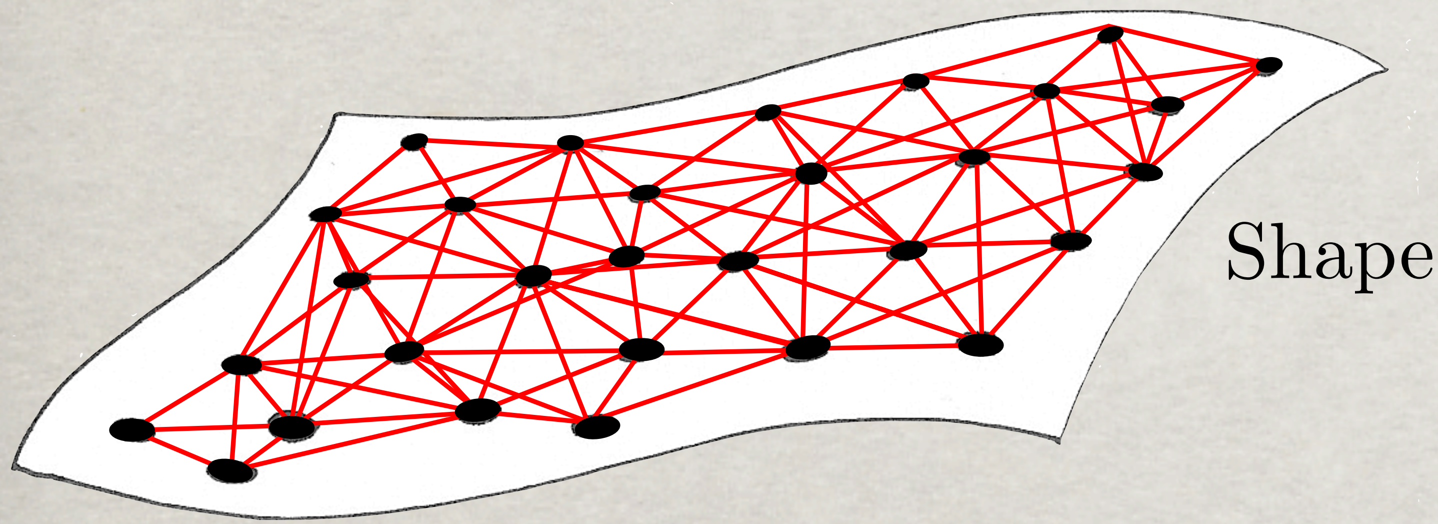

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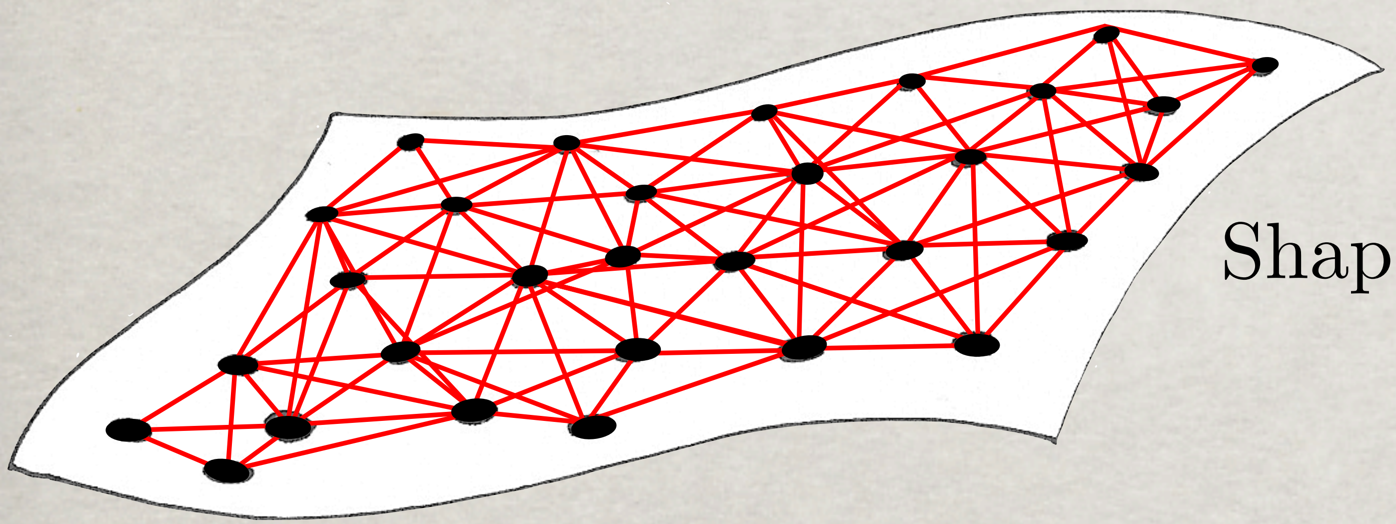


$G_\alpha(P)$ = proximity graph

$pq \in G_\alpha(P) \iff d(p,q) \leq 2\alpha$

Shape

$\simeq$

OUTPUT

Flag $G_\alpha(P)$ = Rips complex

for sampling conditions stated in

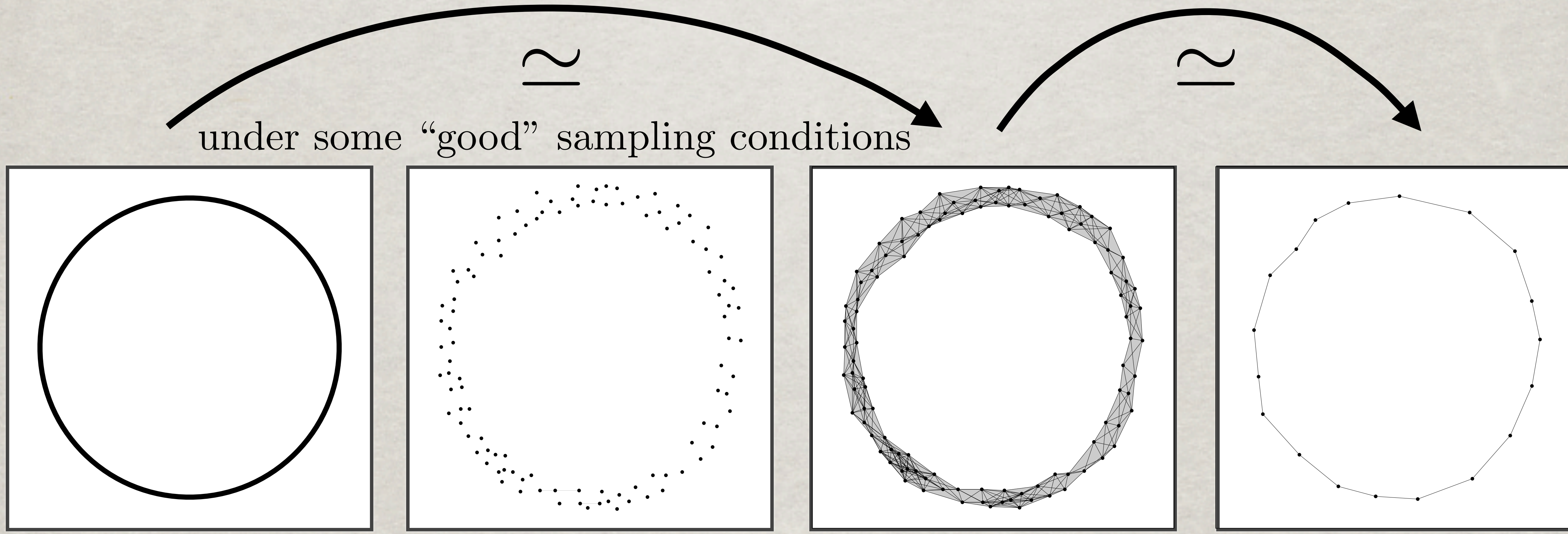
[AL 2010] when $d = d_\infty$

[ALS 2011] when $d = d_2$

Overview

Part I:
Guarantees

Part II:
Simplification



Shape

Point cloud

Flag complex

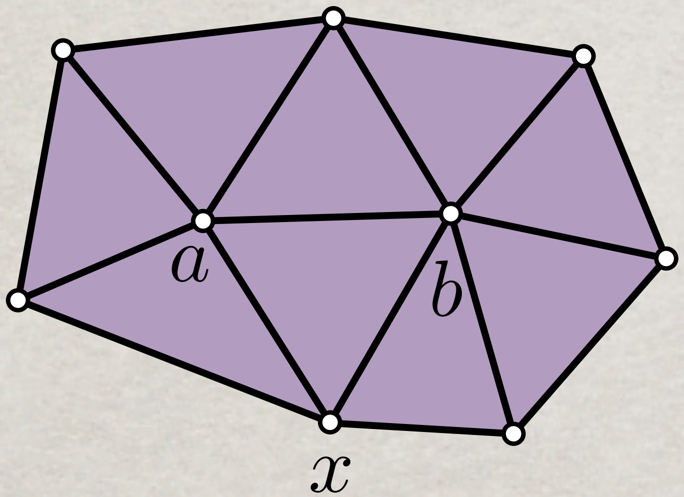
can be high dimensional !

Edge contraction

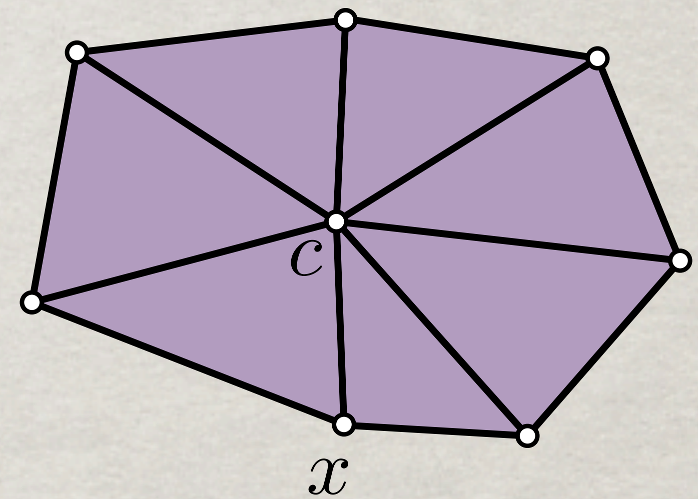
operation that identifies vertices a and b to vertex c

$$K \xrightarrow{ab \mapsto c} K' = \{f(\sigma) \mid \sigma \in K\}$$

$$f(v) = \begin{cases} c & \text{if } v \in \{a, b\} \\ v & \text{if } v \notin \{a, b\} \end{cases}$$



$\{\cdots, a, b, x, ab, ax, bx, abx, \cdots\}$



$\{\cdots, c, x, cx, \cdots\}$

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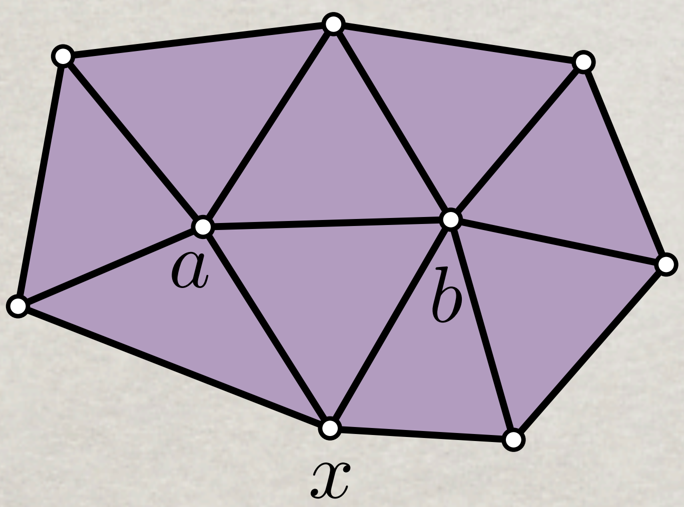

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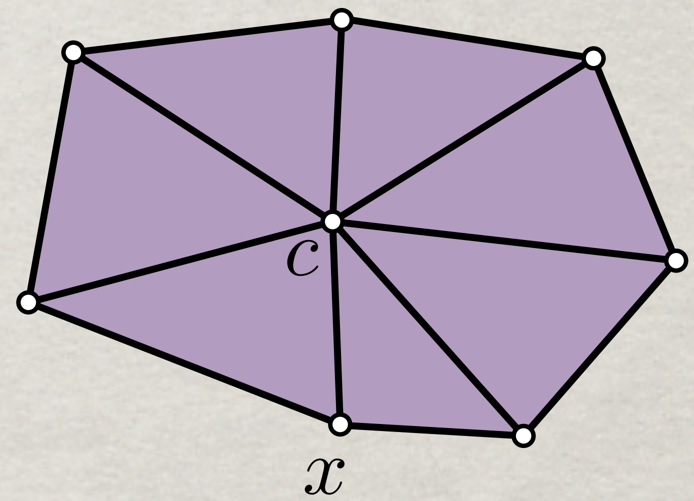
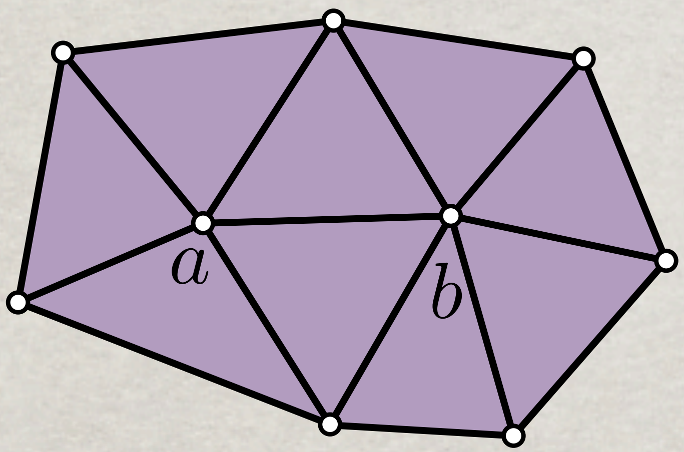


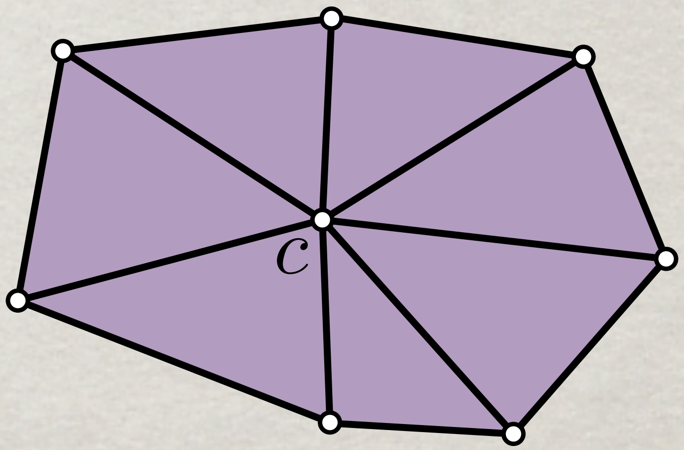
$\{\cdots, c, x, cx, \cdots\}$

- What if the result is not a flag complex?
- How to preserve homotopy type?

Edge contraction

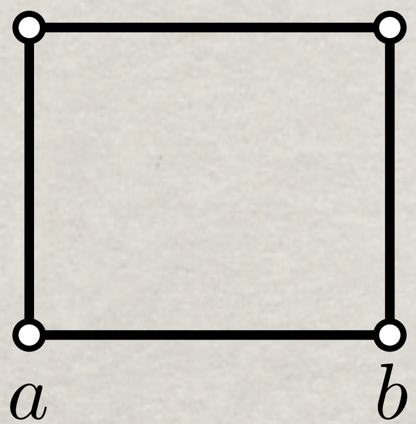
$K = \operatorname{Flag} K^{(1)} \xrightarrow{ab \mapsto c} K' = \operatorname{Flag} K'^{(1)}$

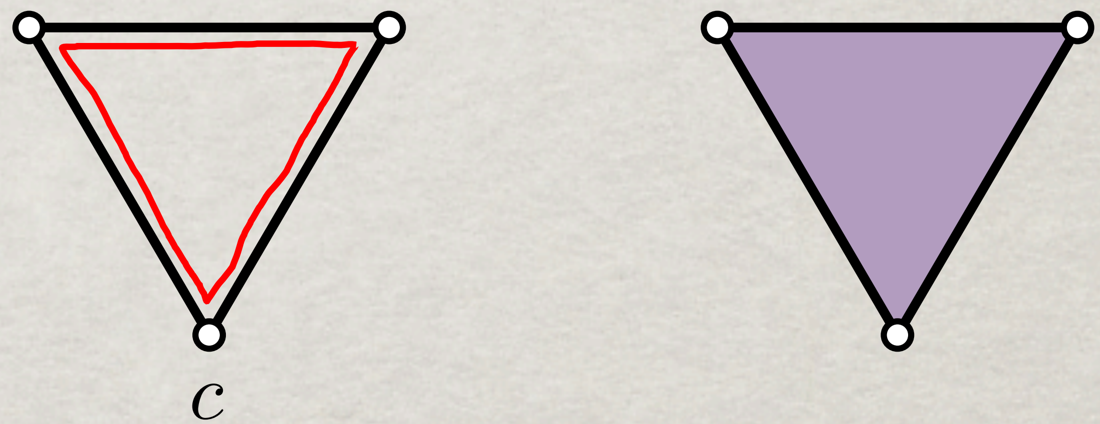




Edge contraction

$K = \operatorname{Flag} K^{(1)} \xrightarrow{ab \mapsto c} K' \neq \operatorname{Flag} K'^{(1)}$





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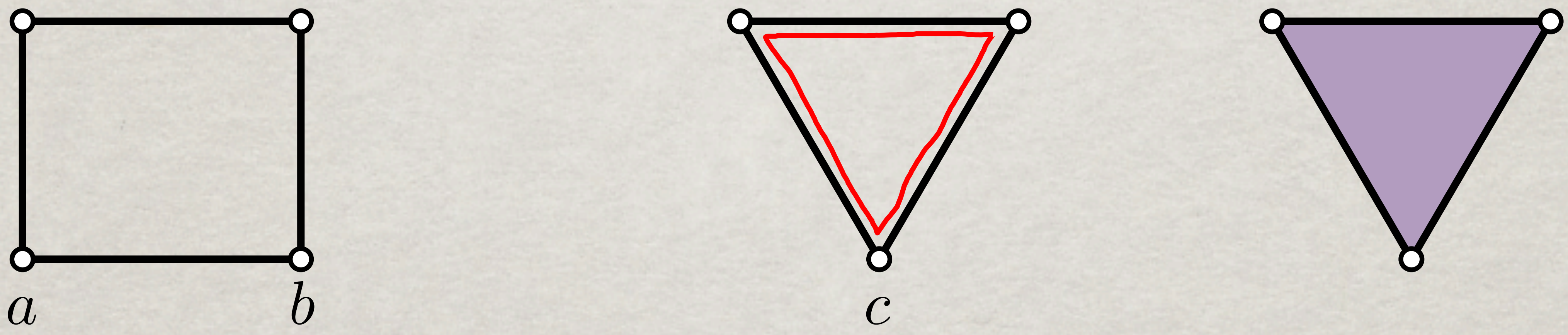
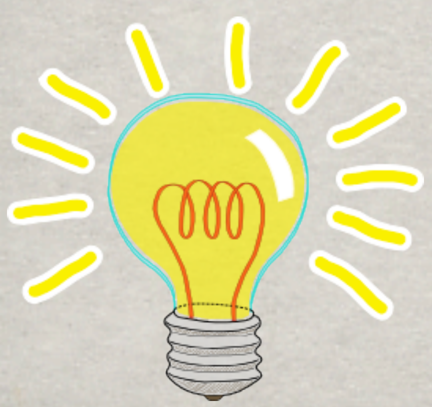




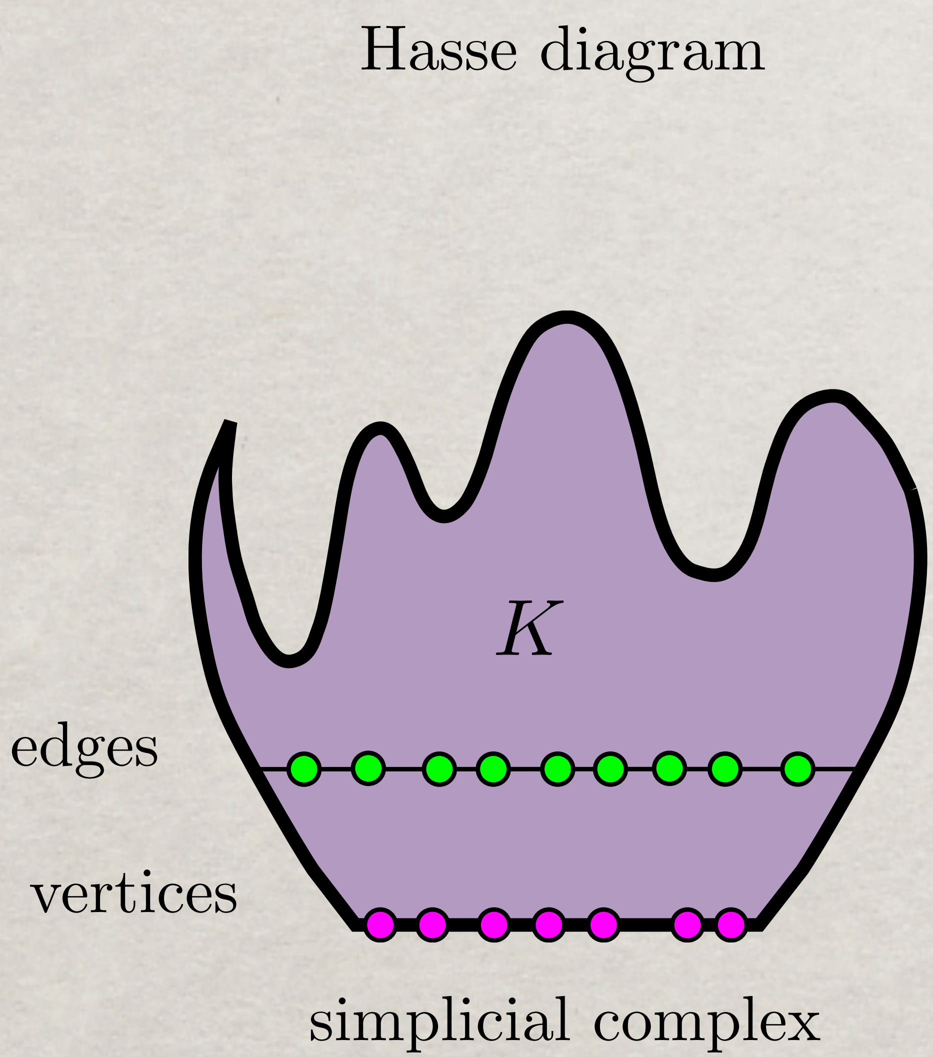
Encode a simplicial complex K by storing the pair:

$$(\; K^{(1)}, \operatorname{Blockers}(K) \;)$$

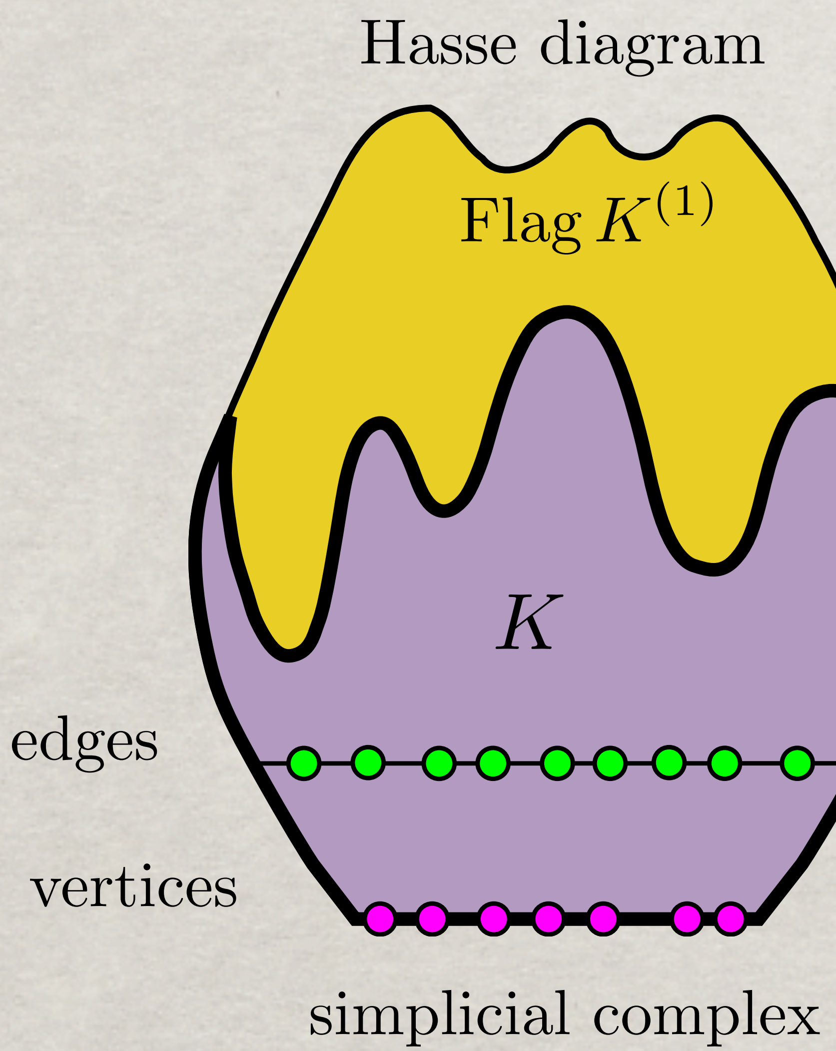
vertices and edges

indicates how much K differs from $\operatorname{Flag} K^{(1)}$

Data Structure for Simplicial complexes



Data Structure for Simplicial complexes



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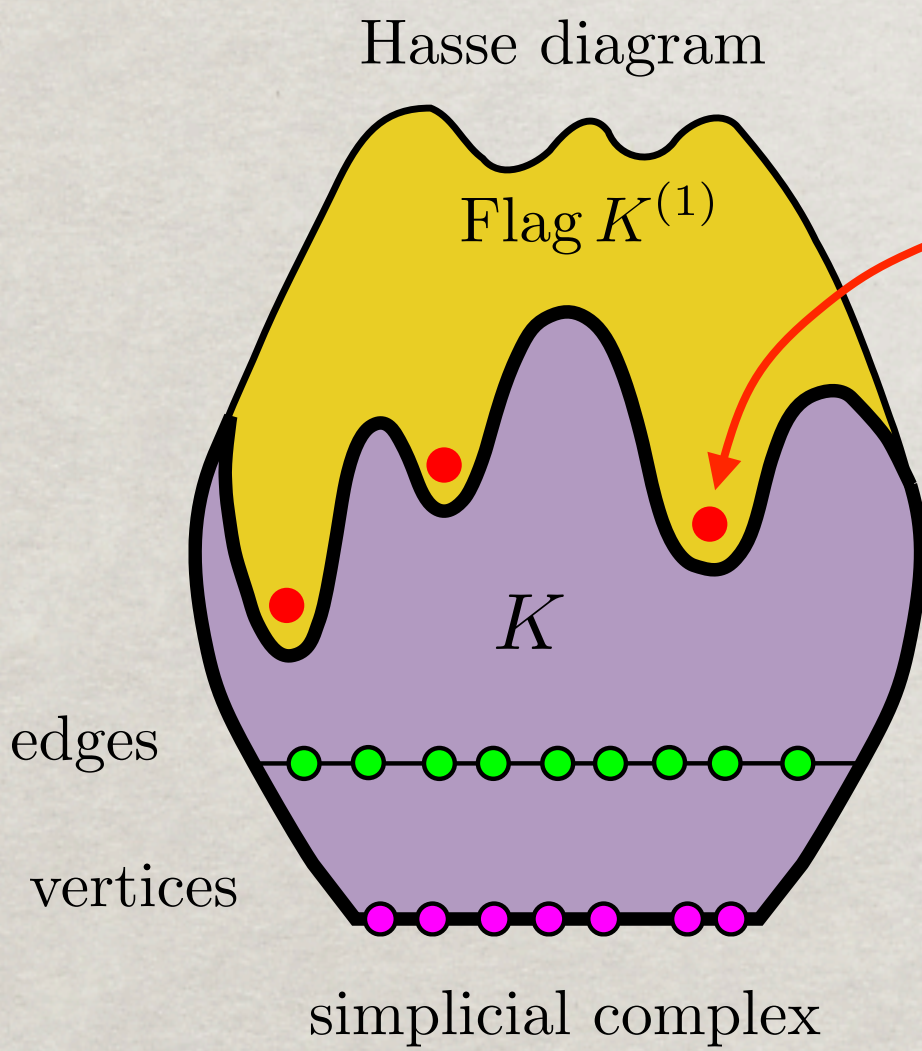
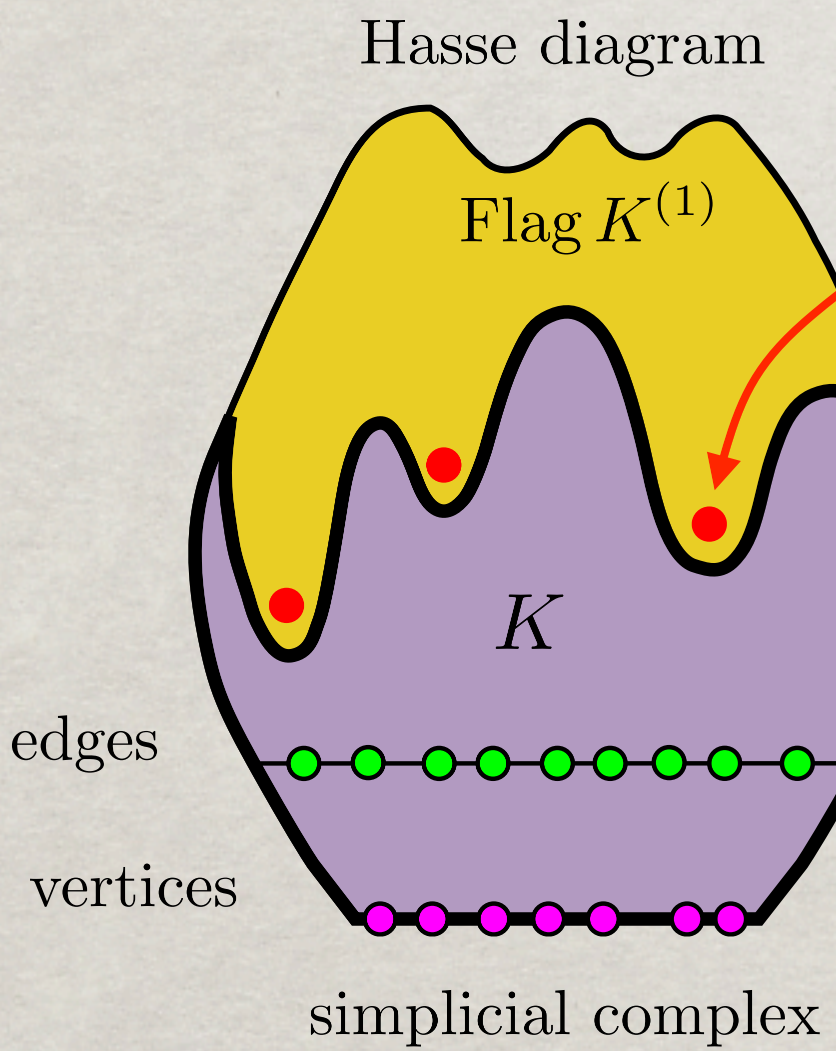


Blockers of K are inclusion-minimal simplices of $\mathrm{Flag}\, K^{(1)} \setminus K$

Data Structure for Simplicial complexes



Blockers of K are inclusion-minimal simplices of $\text{Flag}\, K^{(1)} \setminus K$

σ blocker of K

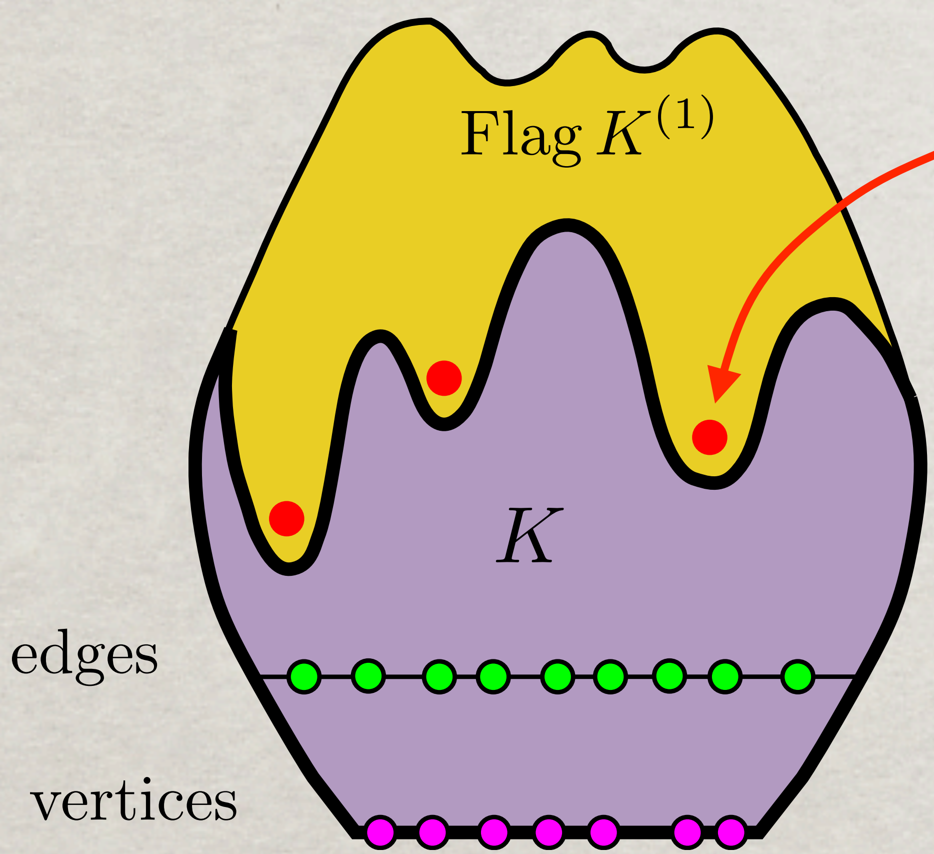
$\Longleftrightarrow$

$\dim \sigma \geq 2$

$\sigma \notin K$

$\forall \tau \subsetneq \sigma,\ \tau \in K$

Data Structure for Simplicial complexes



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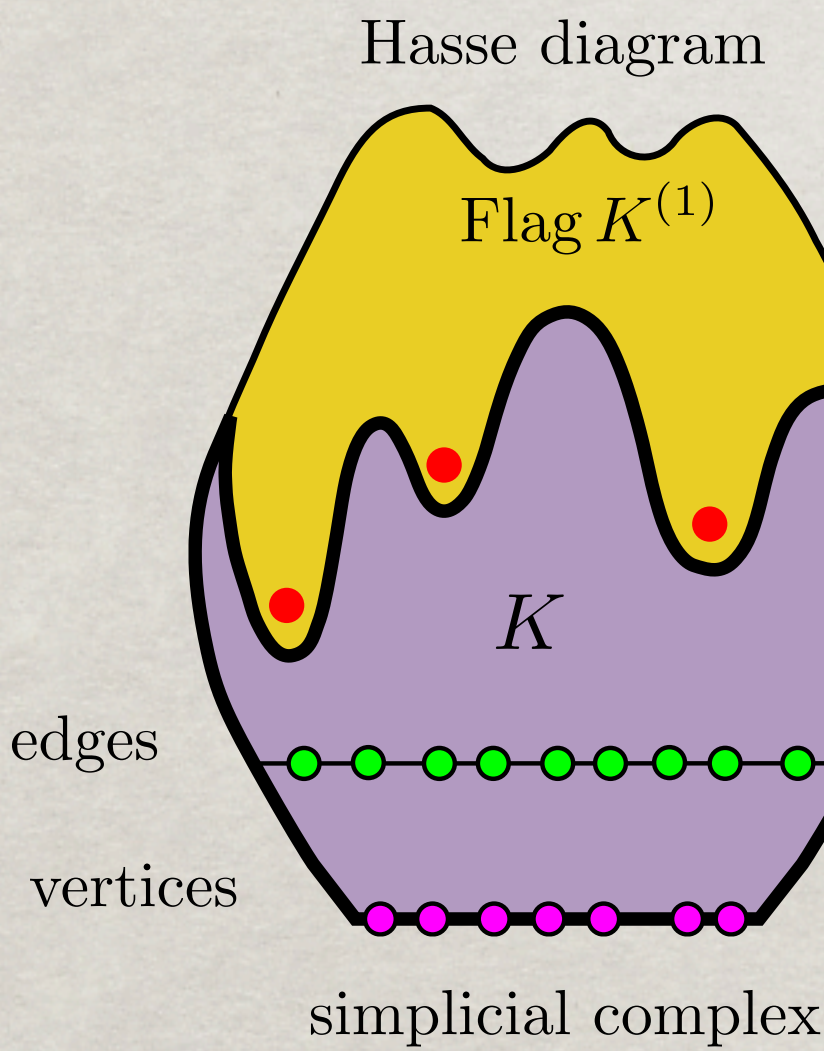
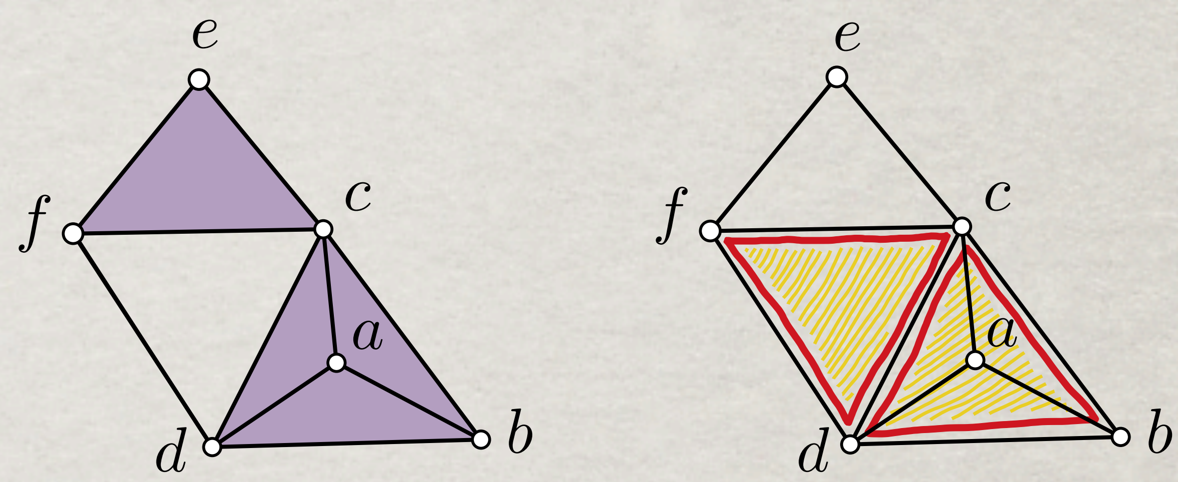
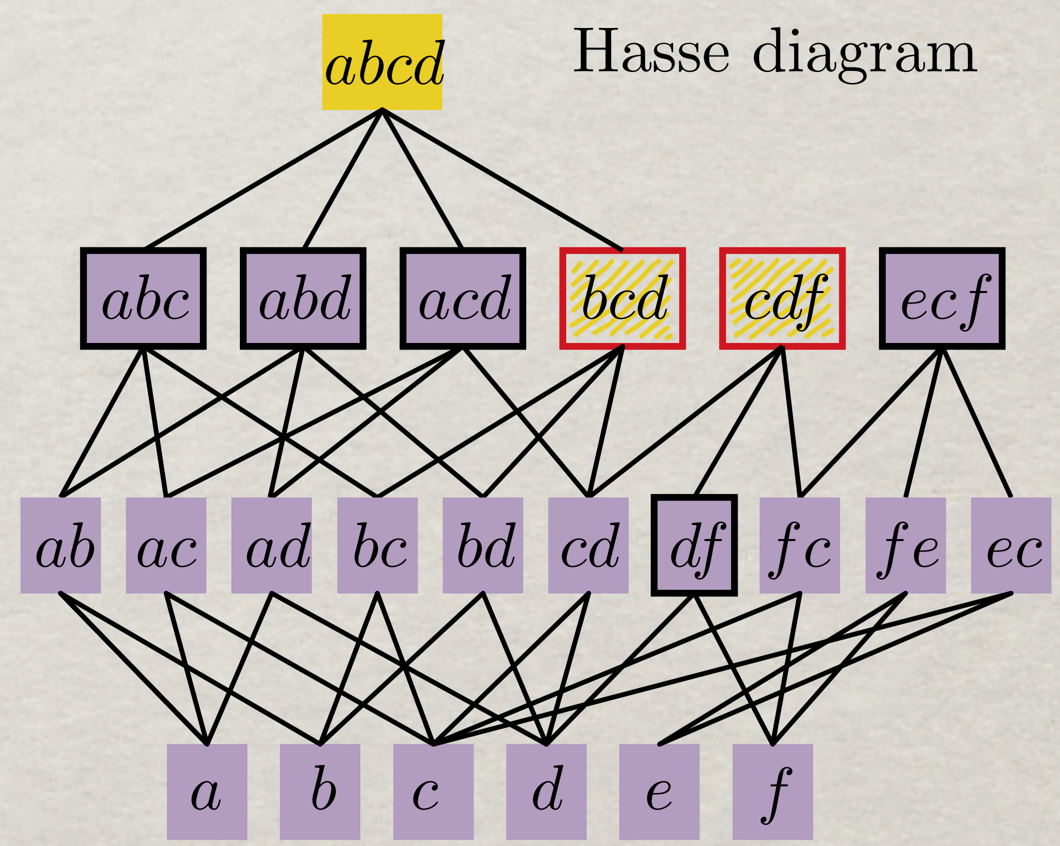
Encode a simplicial complex K by storing the pair:

$(\ K^{(1)}, \mathrm{Blockers}(K)\ )$

A small example

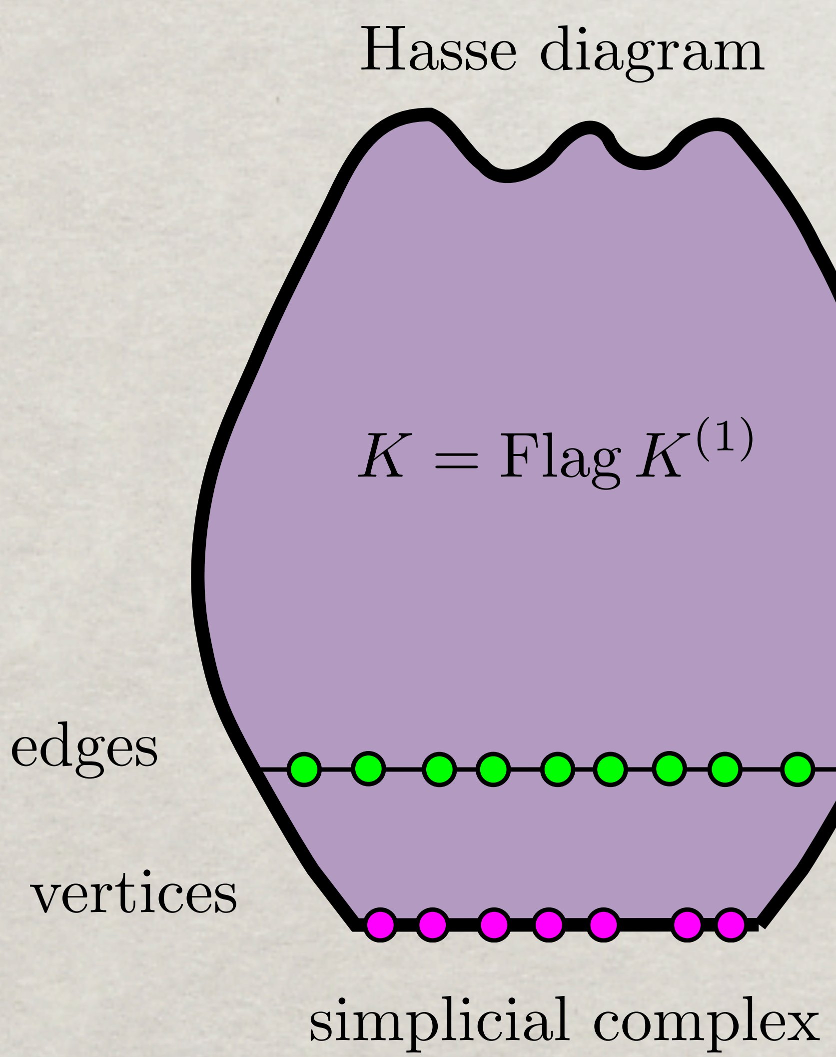






$$\text{Blockers}(K) = \{bcd, cdf\}$$

Data Structure for Simplicial complexes



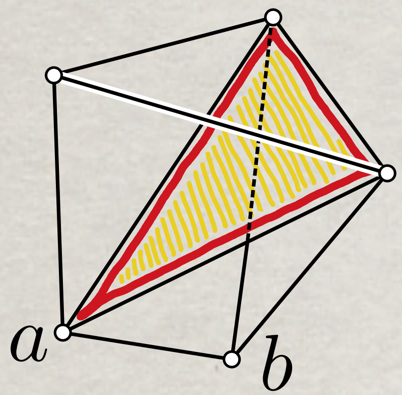
Blockers of K are inclusion-minimal simplices of $\operatorname{Flag} K^{(1)} \setminus K$

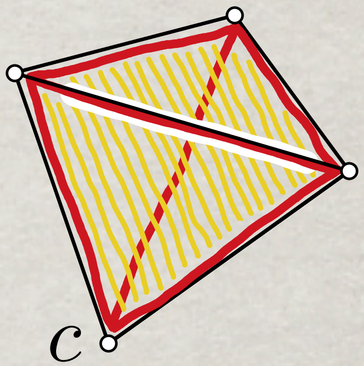
If K is a flag complex
$\text{Blockers}(K) = \emptyset$

For a flag complex K, the pair reduces to:

$$(\, K^{(1)} \, , \, \emptyset \,)$$

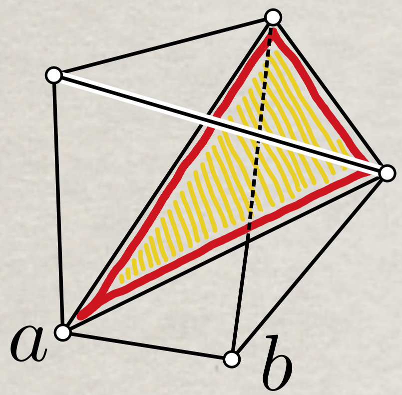
Updating Data Structure





$$K = (\, K^{(1)} \,,\, \text{Blockers}(K)\,) \xrightarrow{\quad ab \mapsto c \quad} K' = (\, K'^{(1)} \,,\, \text{Blockers}(K')\,)$$

Updating Data Structure



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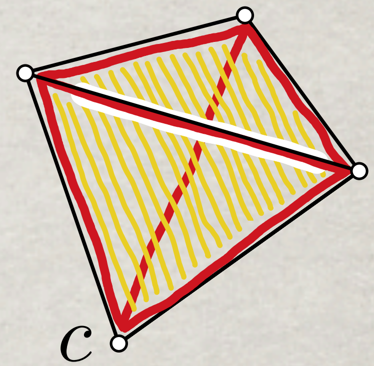


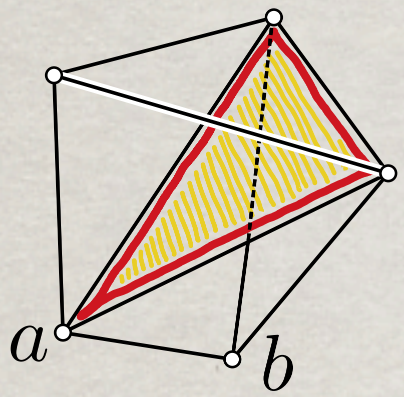
Lemma 1. $c\sigma \in \mathrm{Blockers}\,(K')$ *with* $\sigma \subset \mathrm{Vert}(K) \setminus \{a, b\}$ *and* $\dim \sigma \geq 1$ *iff:*

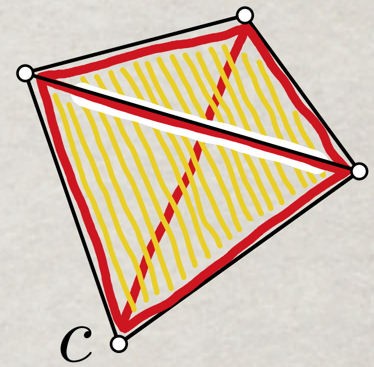
(i) $\sigma \in K$*; for all* $\tau \subsetneq \sigma$, $\tau \in \mathrm{Lk}(a) \cup \mathrm{Lk}(b)$*;*

(ii) $\sigma = \alpha\beta$ *with* $a\beta \in \mathrm{Blockers}_0(K)$ *and* $b\alpha \in \mathrm{Blockers}_0(K)$,

where $\mathrm{Blockers}_0(K) = \mathrm{Blockers}\,(K) \cup$ *complement of* $K^{(1)}$

Updating Data Structure





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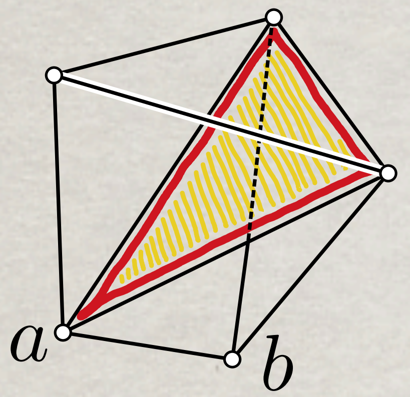
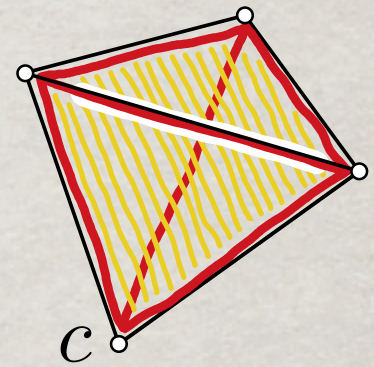
$$\mathrm{Lk}\,\sigma \;= ((\mathrm{Lk}\,\sigma)^{(1)}, \mathrm{Blockers}(\mathrm{Lk}\,\sigma))$$

Updating Data Structure





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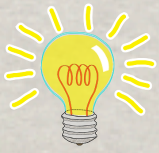
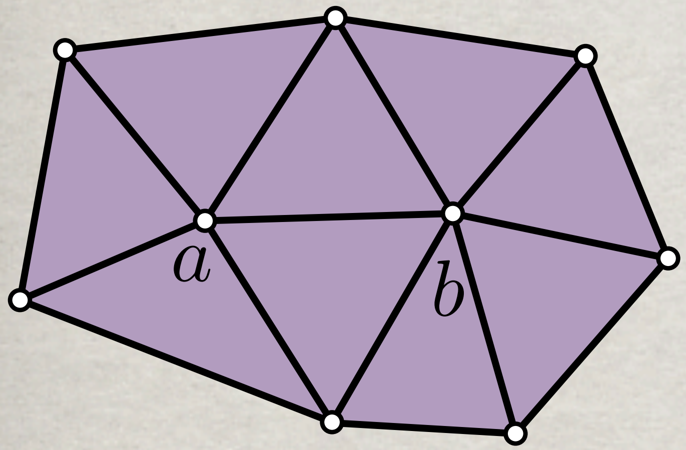


$$\mathrm{Lk}\,\sigma \;= ((\mathrm{Lk}\,\sigma)^{(1)}, \mathrm{Blockers}(\mathrm{Lk}\,\sigma))$$

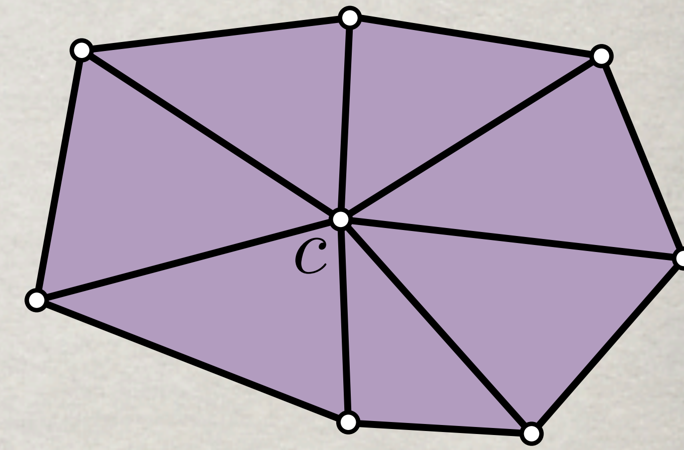
If no blockers "around" a and b, costs in $\tilde{O}(\sharp\mathrm{neighbors}(a) \times \sharp\mathrm{neighbors}(b))$

Edge contraction

How to preserve homotopy type?



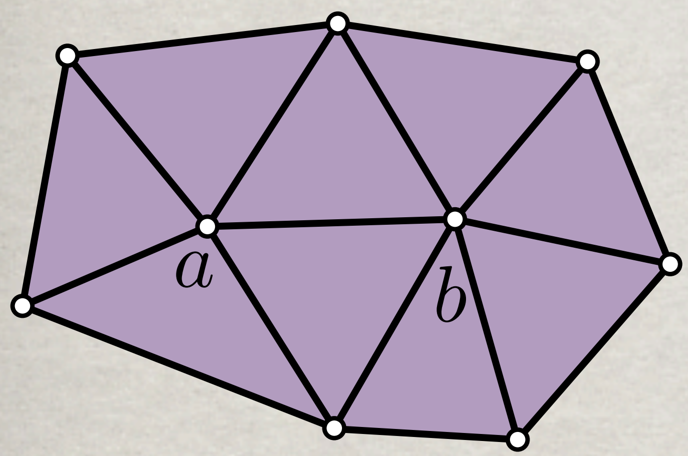
$$K \xrightarrow{ab \mapsto c} K'$$



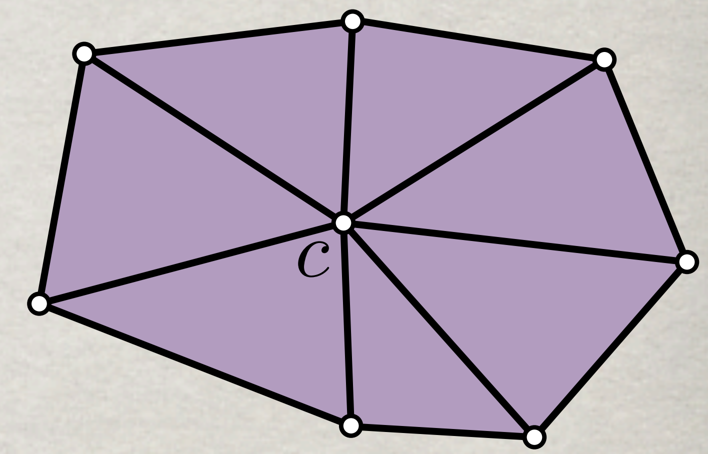
Edge contraction

[Dey, Edelsbrunner, Guha & Nekhayev 1999]

If K triangulates a 2- or 3-manifold



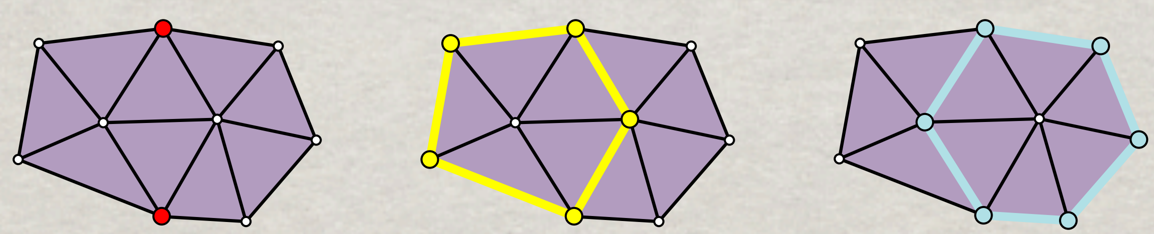
$$K \xrightarrow{ab \mapsto c} K'$$



$\approx$

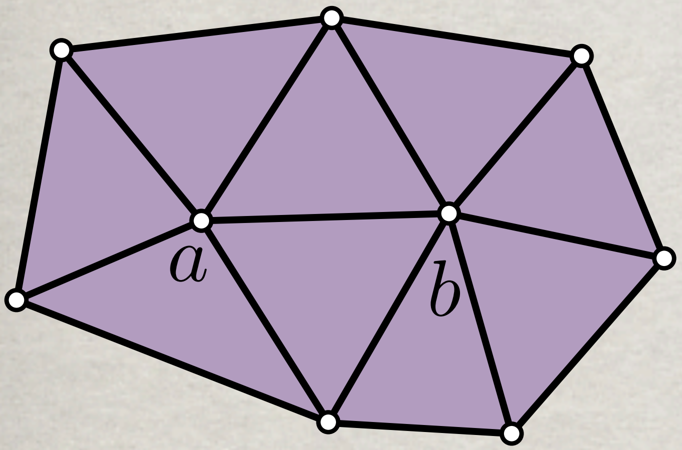
iff

$$\boxed{\operatorname{Lk} ab = \operatorname{Lk} a \cap \operatorname{Lk} b}$$



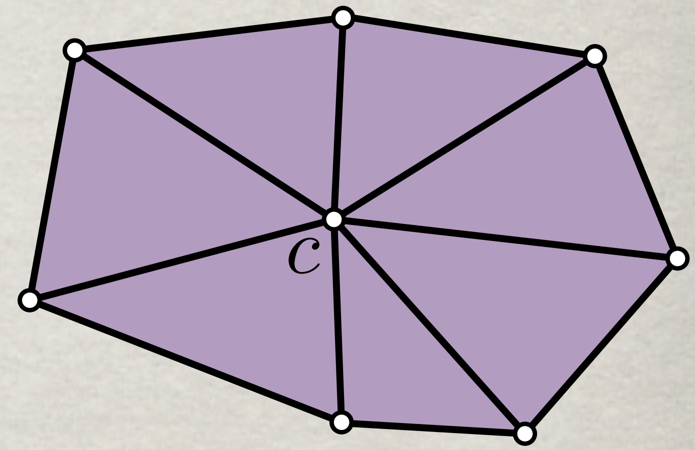
Edge contraction

For arbitrary simplicial complexes, we established that:



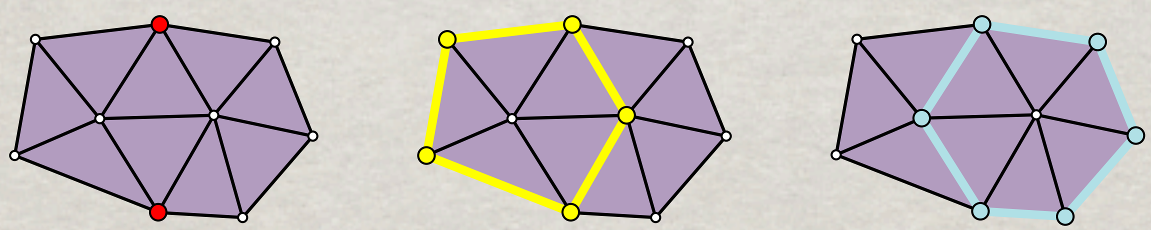
$$K \xrightarrow{ab \mapsto c} K'$$

$$\simeq$$



whenever

$$\boxed{\operatorname{Lk} ab = \operatorname{Lk} a \cap \operatorname{Lk} b}$$



Edge contraction

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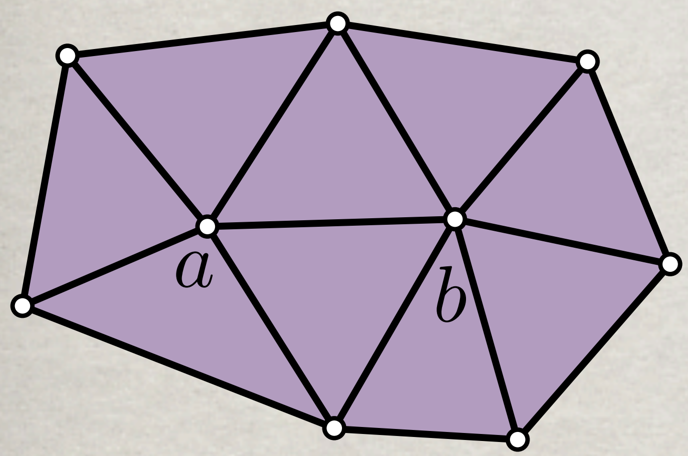
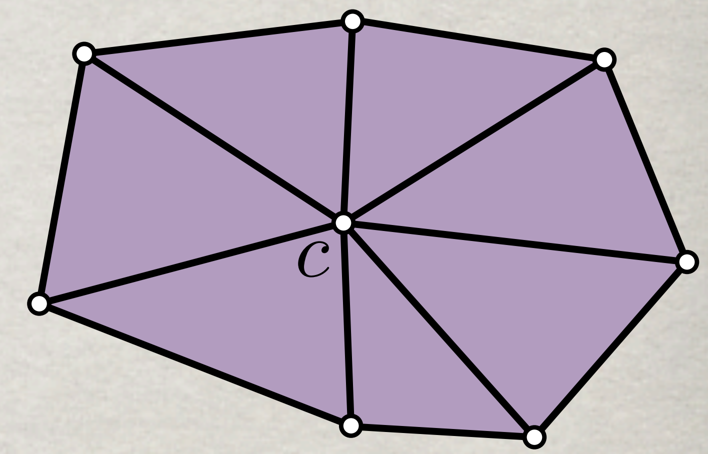
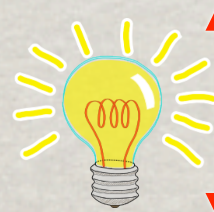


$$K \xrightarrow[\simeq]{ab \mapsto c} K'$$



whenever

$$\mathrm{Lk}\, ab = \mathrm{Lk}\, a \cap \mathrm{Lk}\, b$$

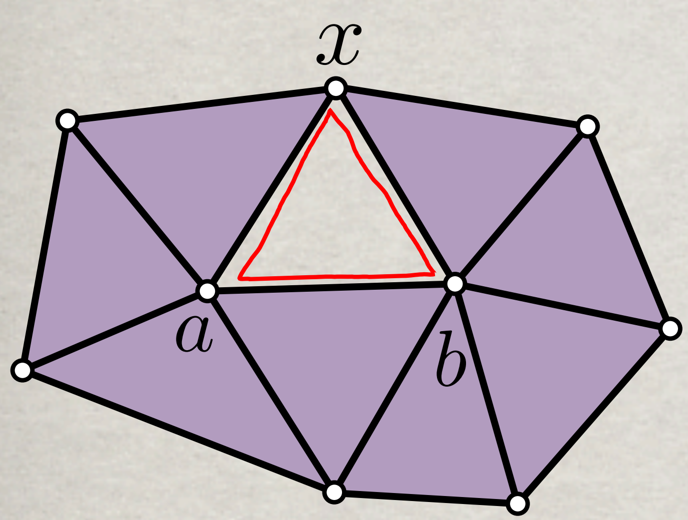


$$\Updownarrow$$

No blocker of K contains ab

Edge contraction

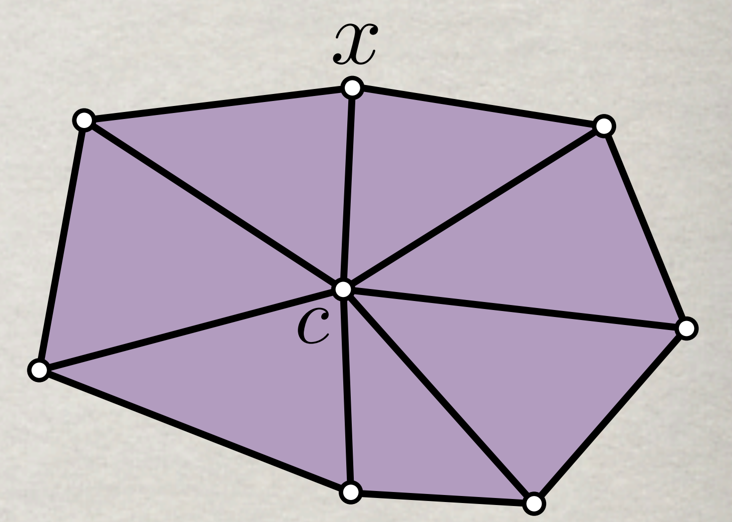
For arbitrary simplicial complexes, we established that:



$$K \xrightarrow{ab \mapsto c} K'$$

$\not\cong$

implies that

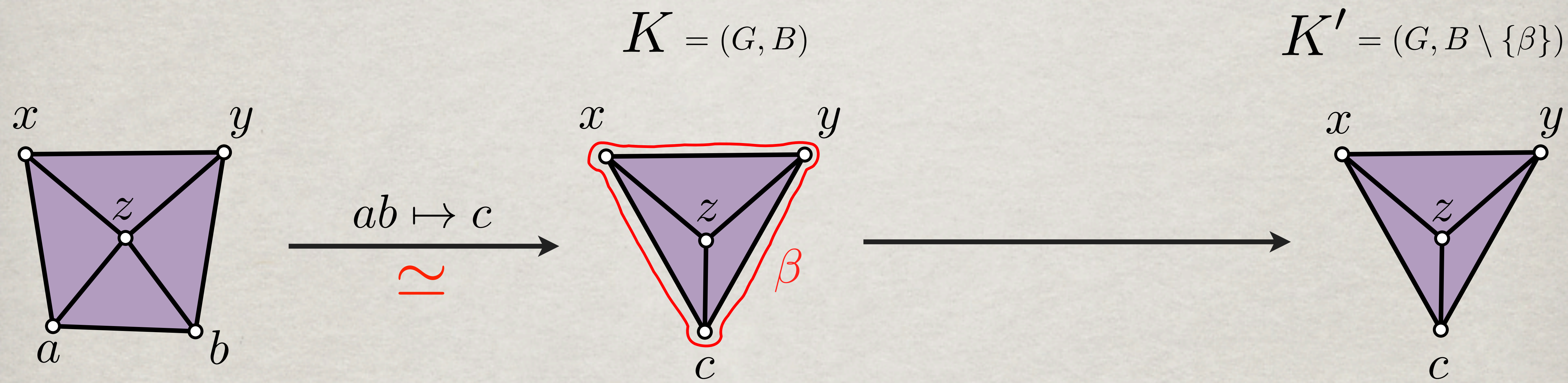


$$\mathrm{Lk}\, ab \neq \mathrm{Lk}\, a \cap \mathrm{Lk}\, b$$

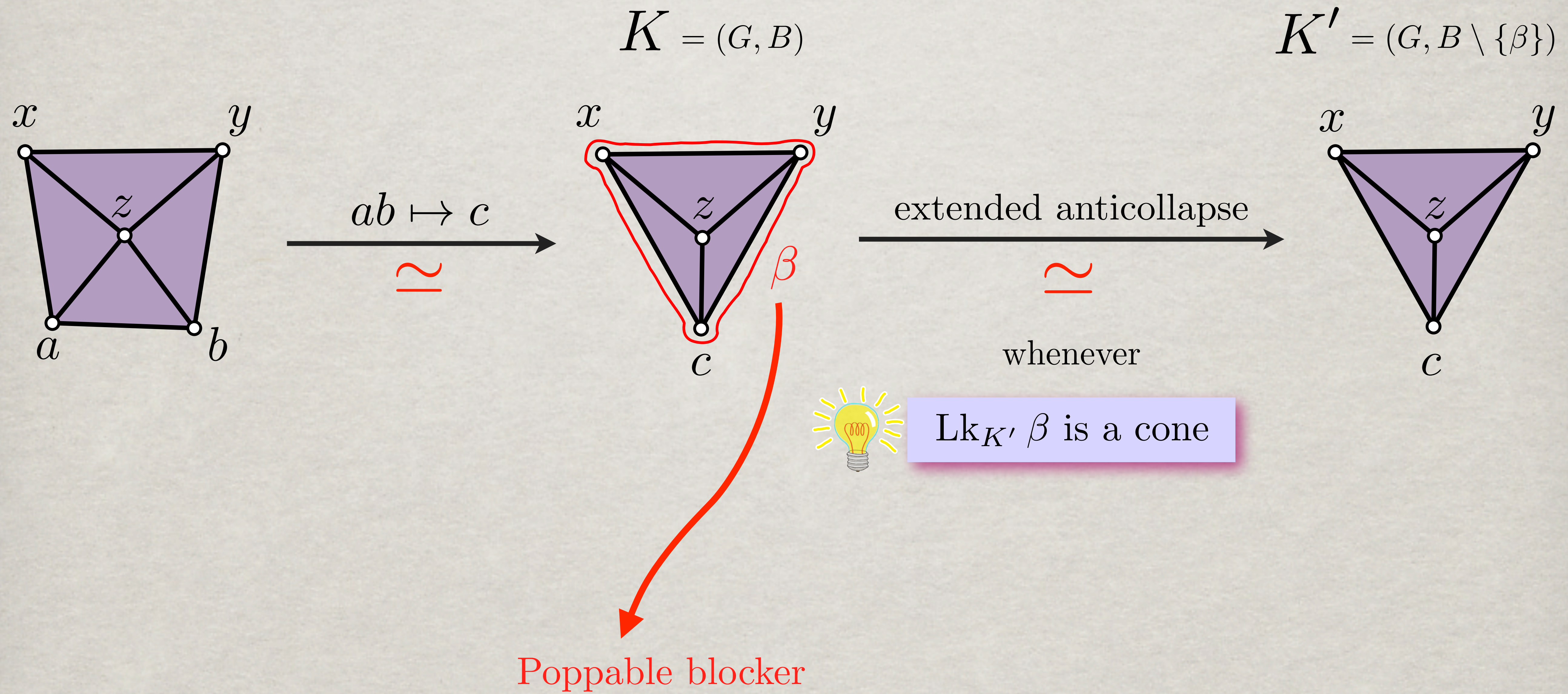
$$\Updownarrow$$

$\exists$ a blocker of K containing ab

Poppable blockers

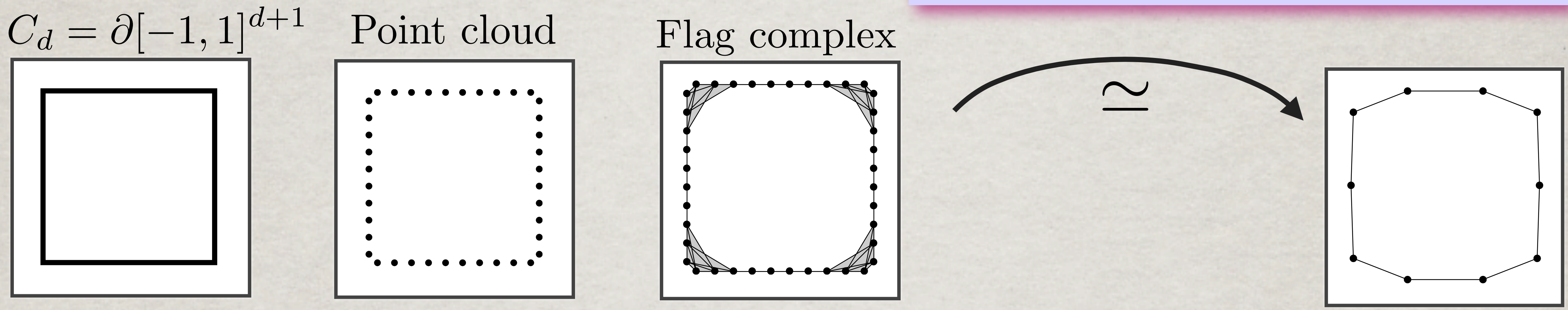


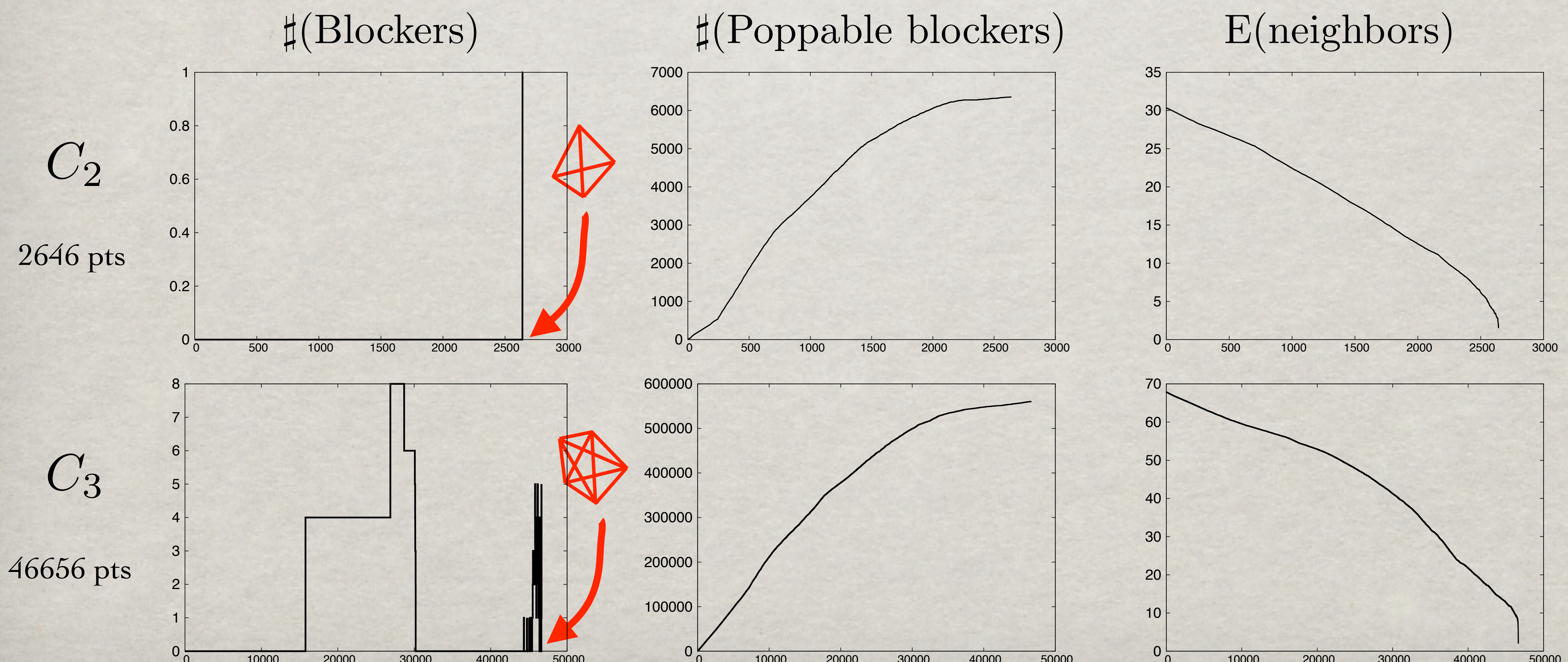
Poppable blockers



Experiments

We keep contracting shortest edge with no blocker through it and remove poppable blockers

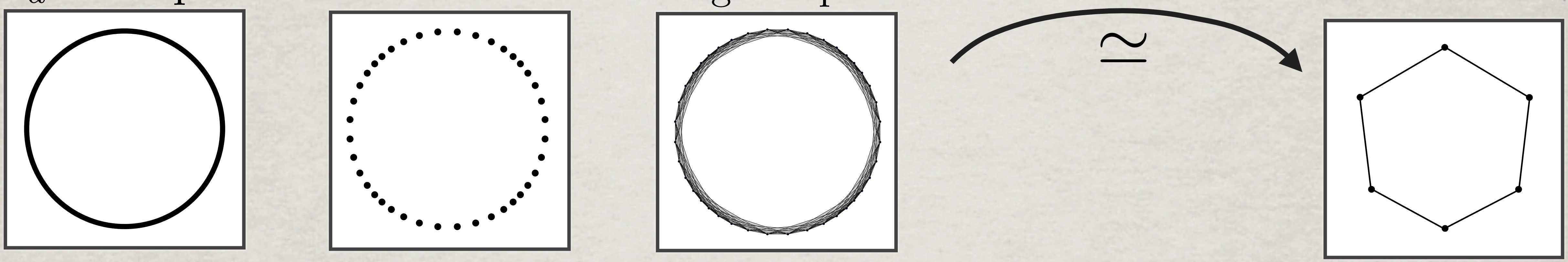




EXPERIMENTS

We keep contracting shortest edge with no blocker through it and remove poppable blockers

S_d = d-sphere — Point cloud — Flag complex



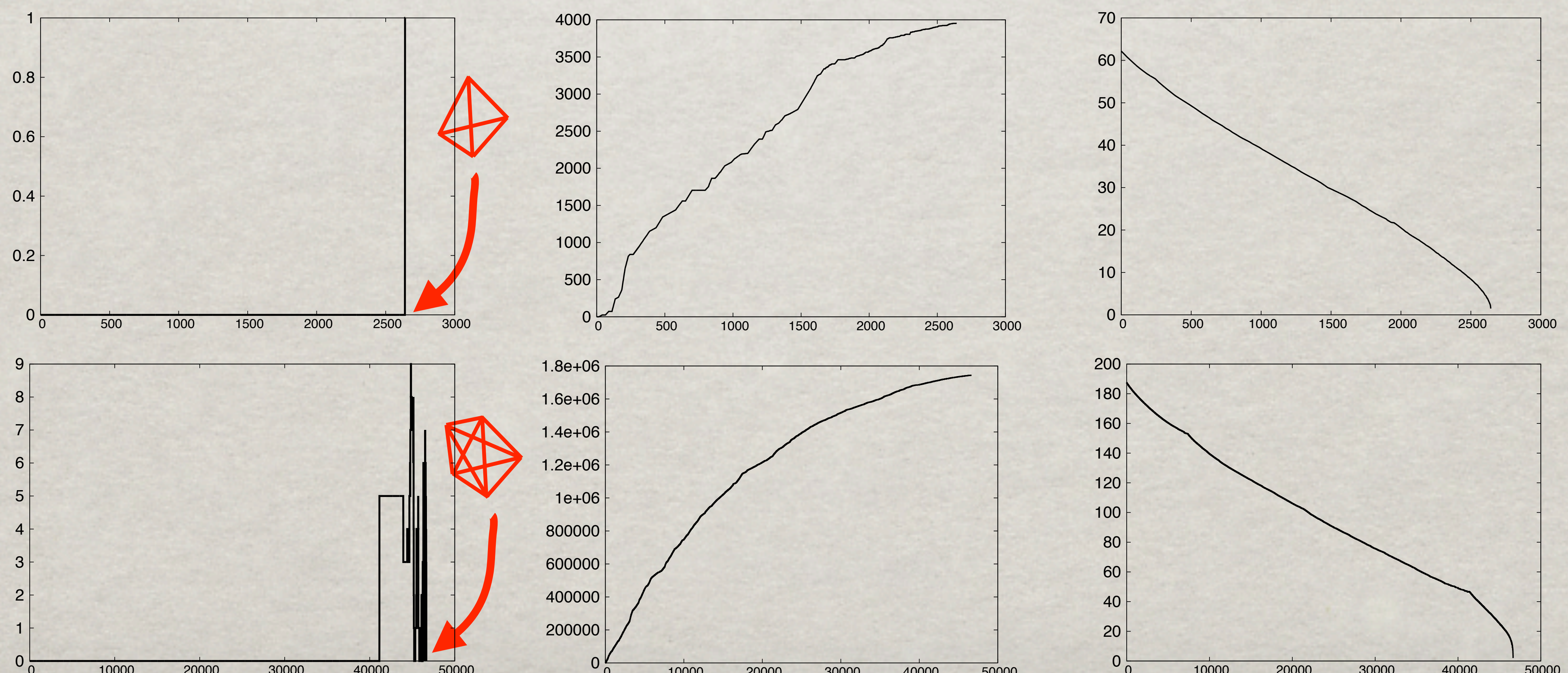
$\sharp$(Blockers) — $\sharp$(Poppable blockers) — E(neighbors)

S_2

2646 pts

S_3

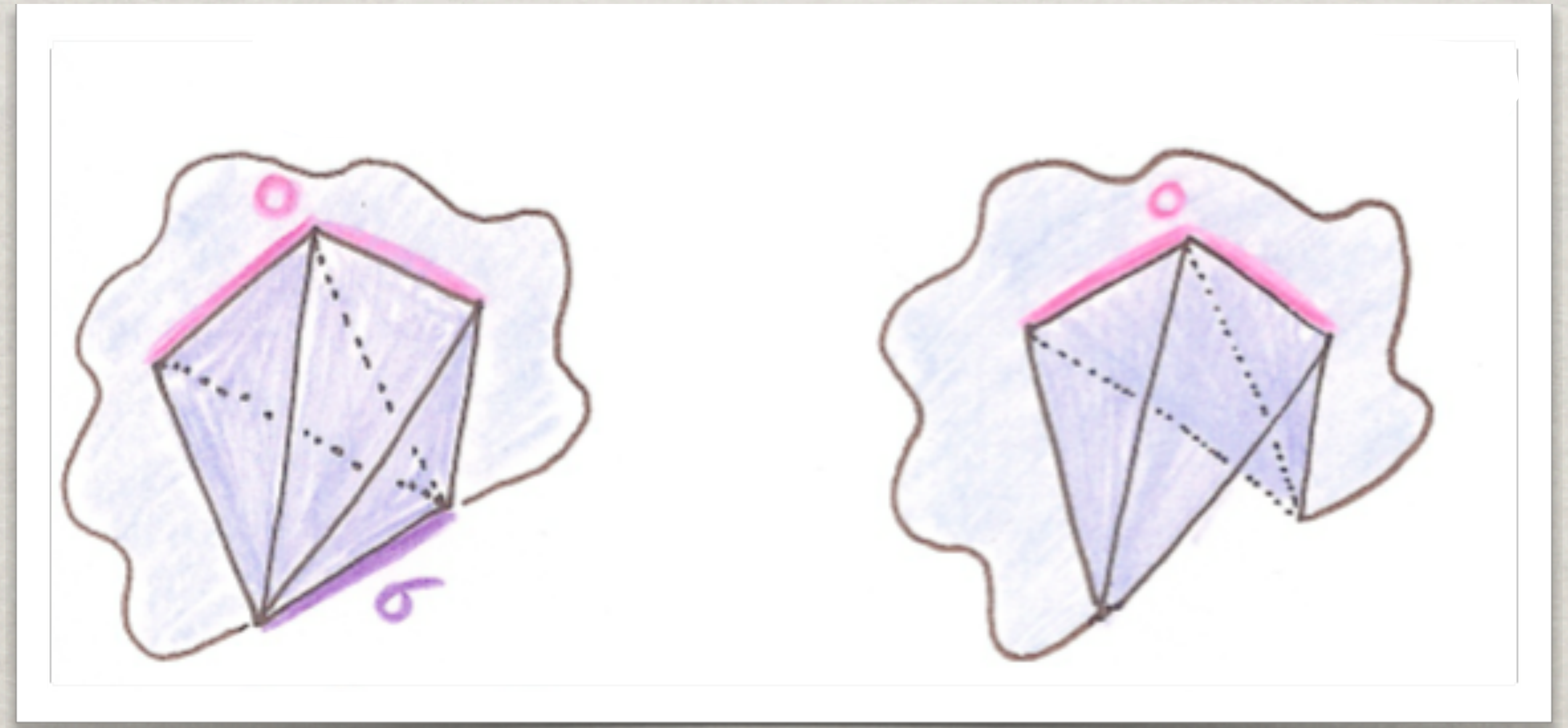
46656 pts



Extended Collapse

operation that removes the cofaces of σ

$$K \longrightarrow K'$$



$\simeq$

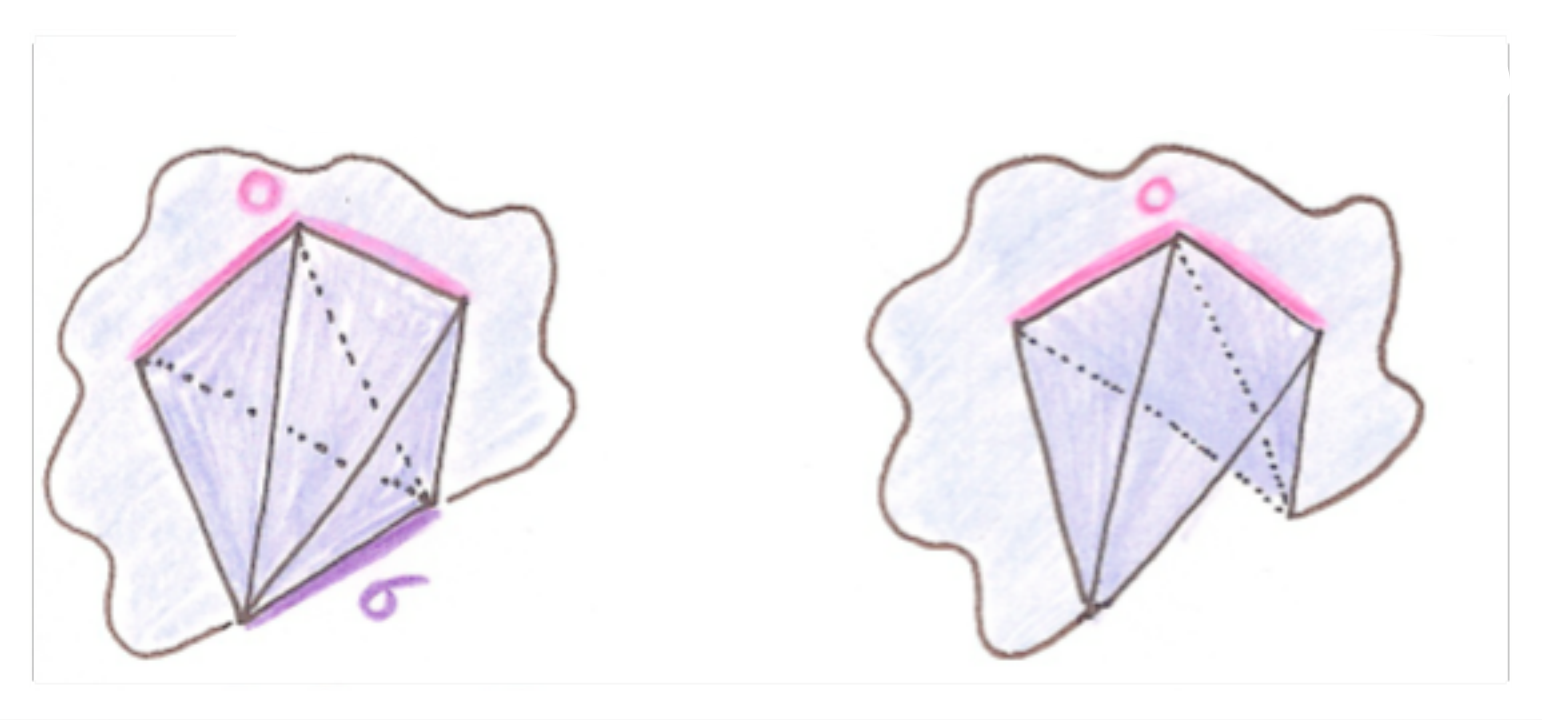
whenever

$\mathrm{Lk}_K\, \sigma$ is a cone

Extended Collapse

operation that removes the cofaces of σ

$$K = \operatorname{Flag} K^{(1)} \xrightarrow{\text{If } \sigma \text{ vertex or edge}} K' = \operatorname{Flag} K'^{(1)}$$

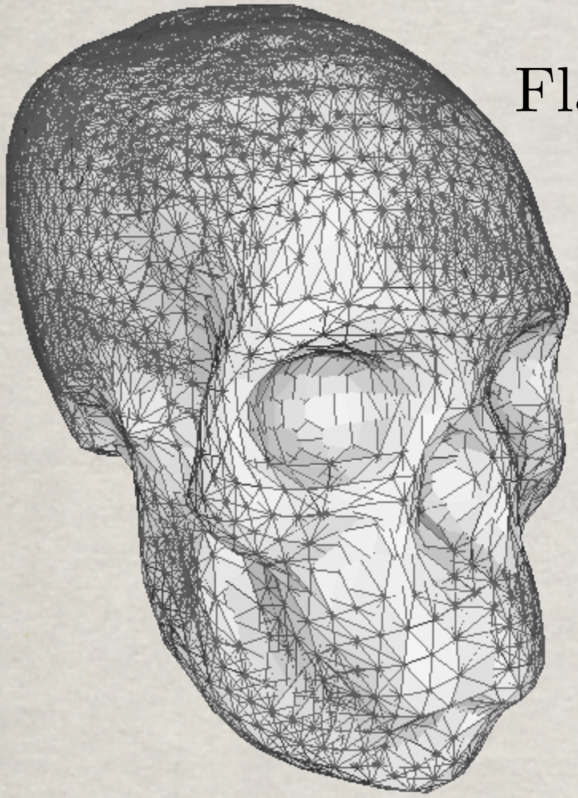


$\simeq$

whenever

$\mathrm{Lk}_K\, \sigma$ is a cone

Experiments

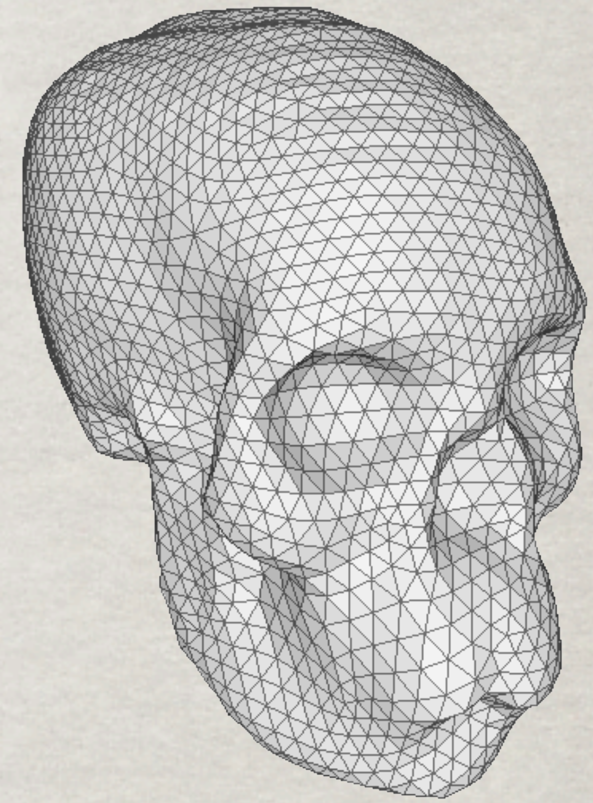


Flag complex of a point cloud
that samples a surface

$\simeq$

We keep collapsing the longest edge
whose link is a cone.
Then, we collapse vertices.

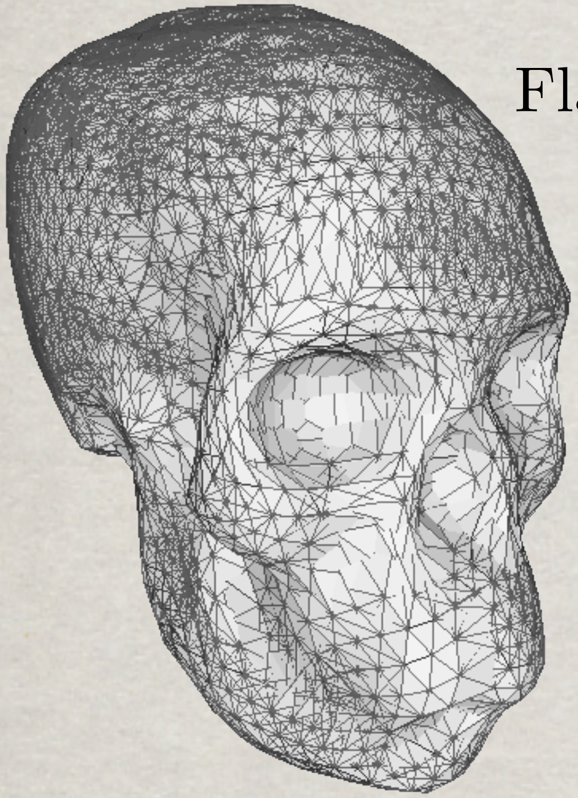
Triangulated surface



How to explain the "good" behaviors
of this simplification process?

Experiments

Triangulated surface



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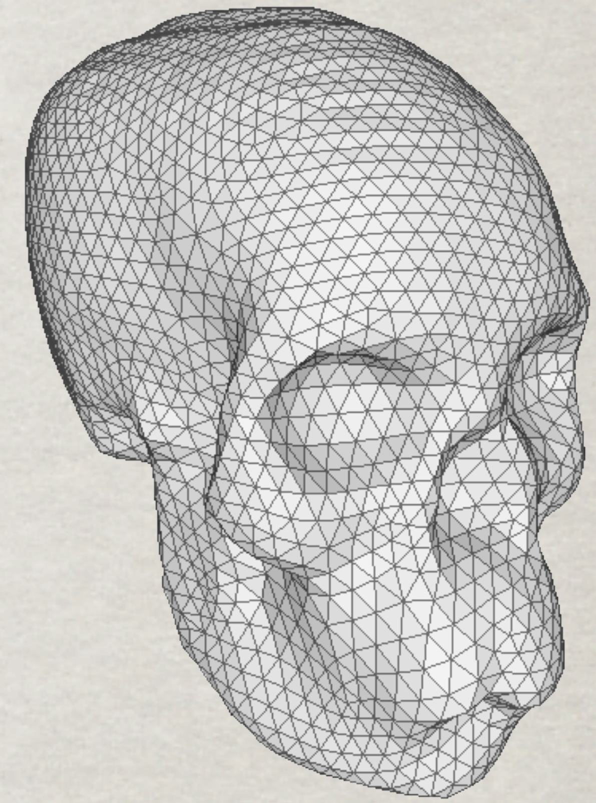

How to explain the "good" behaviors
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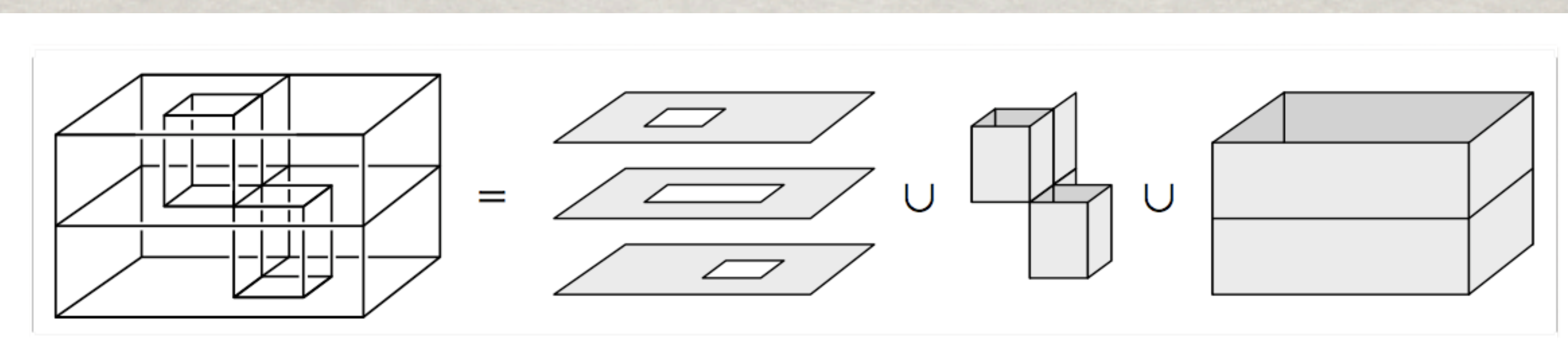
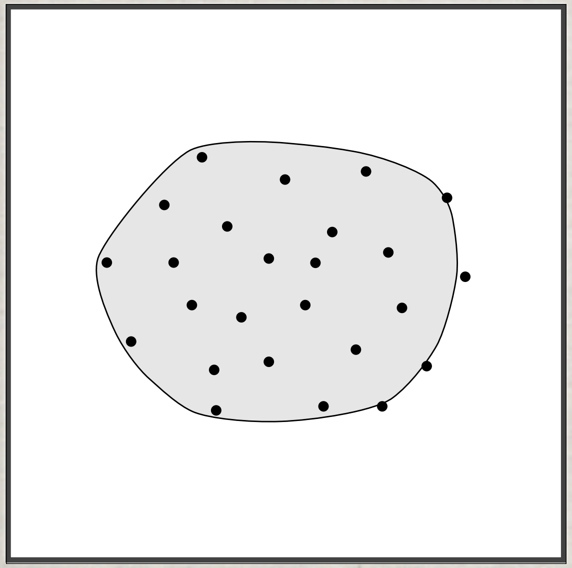


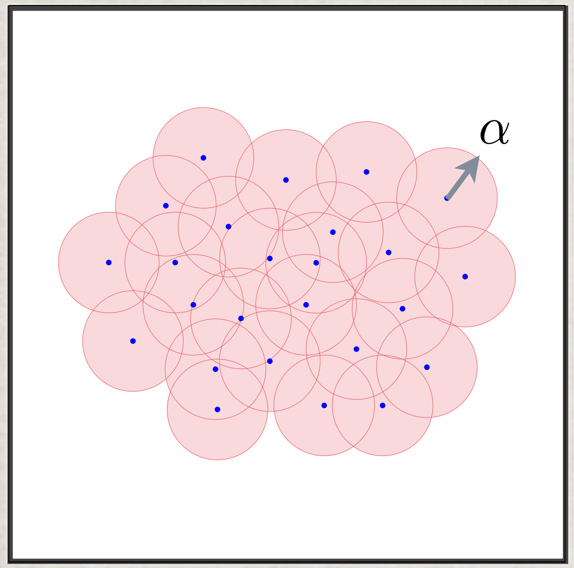
A triangulated Bing's house is contractible but is not collapsible.

A simpler Framework

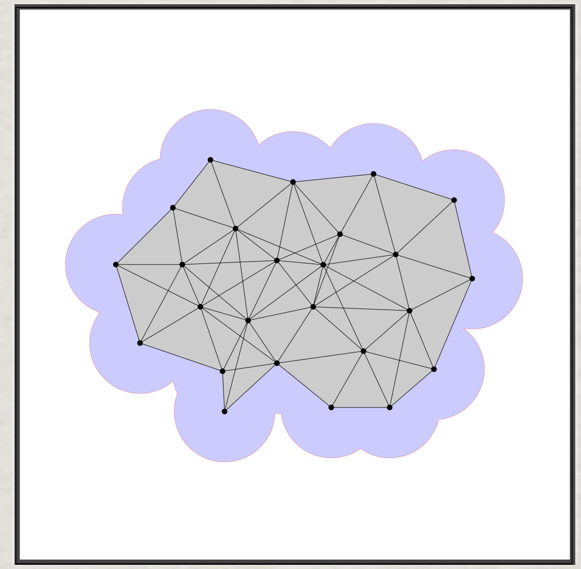
Point cloud P



α-offset P^α



Rips complex $\mathcal{R}(P, \alpha)$



Is it collapsible?

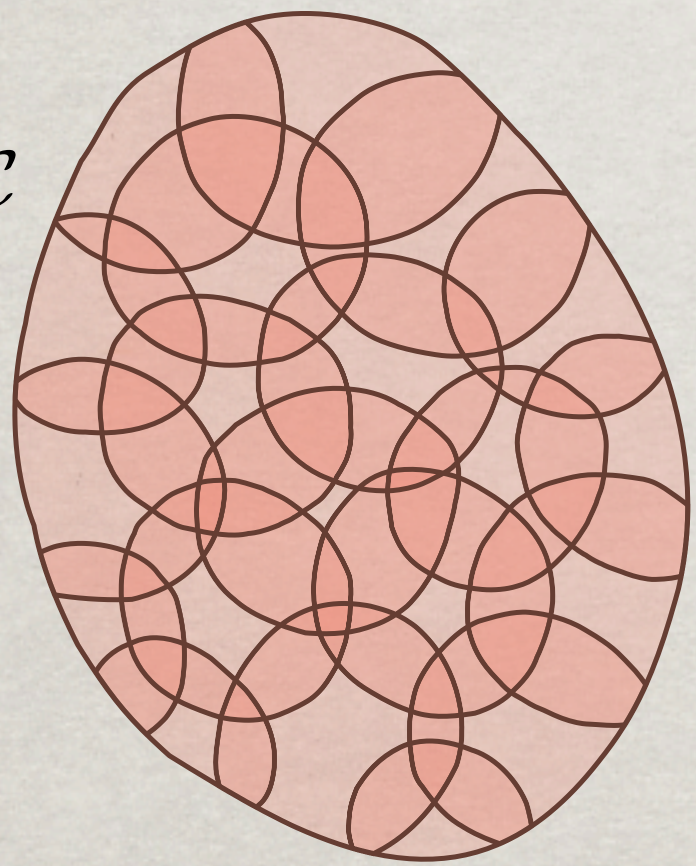
that samples a convex set A

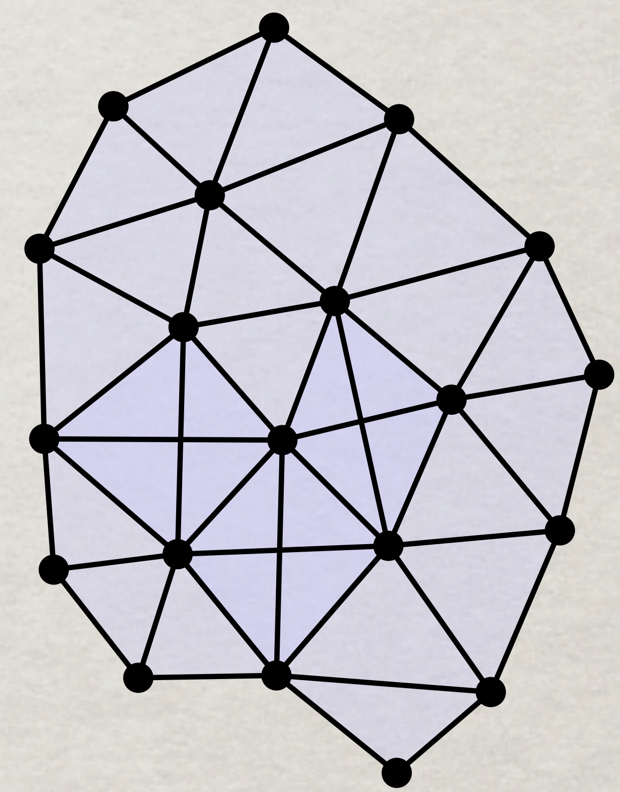
Our results:

- If $\mathrm{Hull}(P) \subset P^\alpha$, then $\mathcal{C}(P, \alpha)$ is collapsible. true whenever $\alpha \geq 2d_H(P, A)$
- If $\mathrm{Hull}(P) \subset P^{(2-\sqrt{3})\alpha}$, then $\mathcal{R}(P, \alpha)$ is collapsible. true whenever $\alpha \geq \frac{2d_H(P,A)}{2-\sqrt{3}}$

Geometry Driven Collapses

$\bigcup \mathcal{C}$



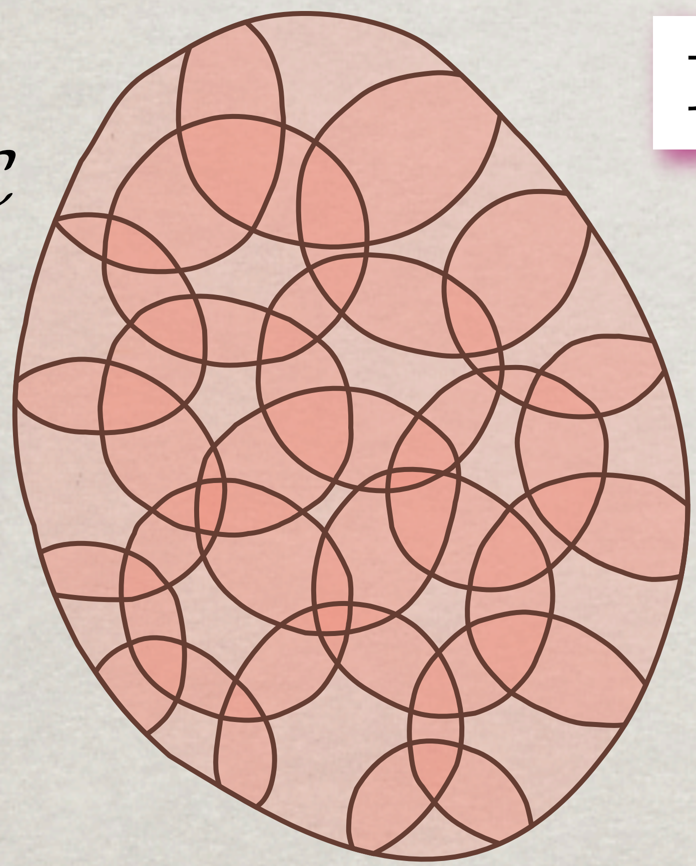


$\mathcal{C}$ a finite collection of compact convex sets in $\mathbb{R}^d$ whose union is convex

$$\mathrm{Nerve}\, C = \{\eta \subset \mathcal{C} \mid \bigcap \eta \neq \emptyset\}$$

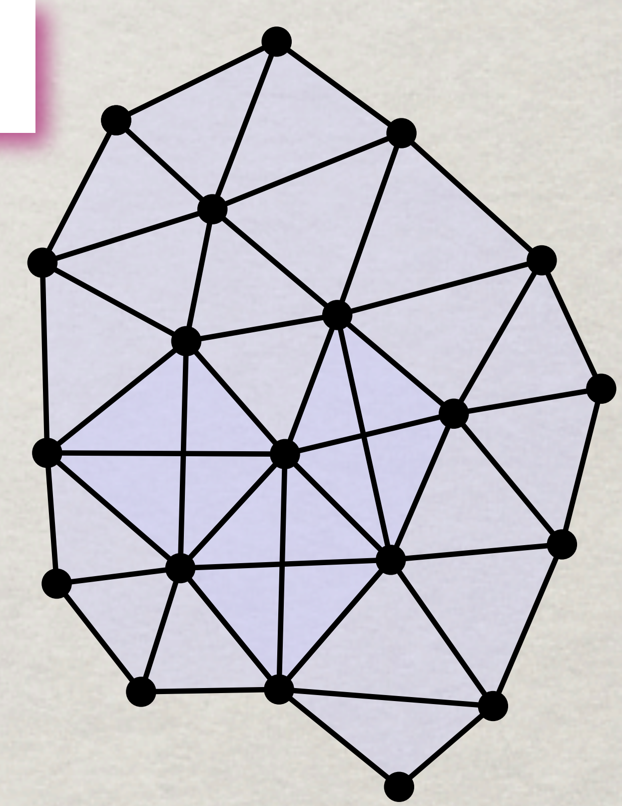
Geometry Driven Collapses

$\bigcup \mathcal{C}$



Nerve Lemma.

$\simeq$

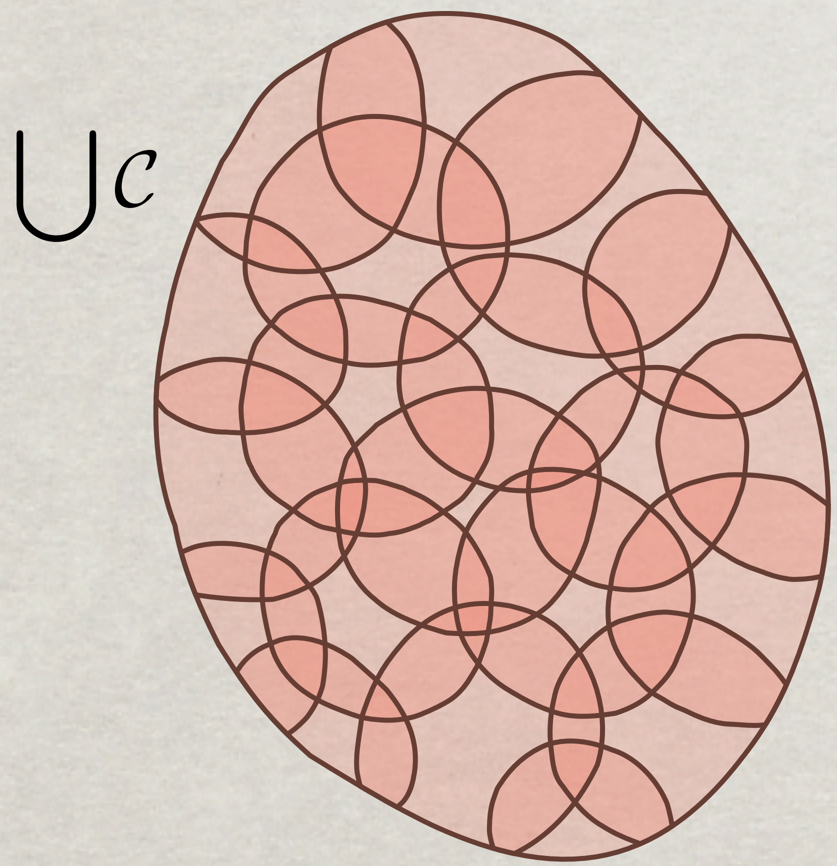


$\mathcal{C}$ a finite collection of compact convex sets in $\mathbb{R}^d$ whose union is convex

$\mathrm{Nerve}\, C = \{\eta \subset \mathcal{C} \mid \bigcap \eta \neq \emptyset\}$

$\bigcup \mathcal{C}$ contractible $\Longrightarrow$ $\boxed{\mathrm{Nerve}\, \mathcal{C} \text{ contractible}}$

Geometry Driven Collapses



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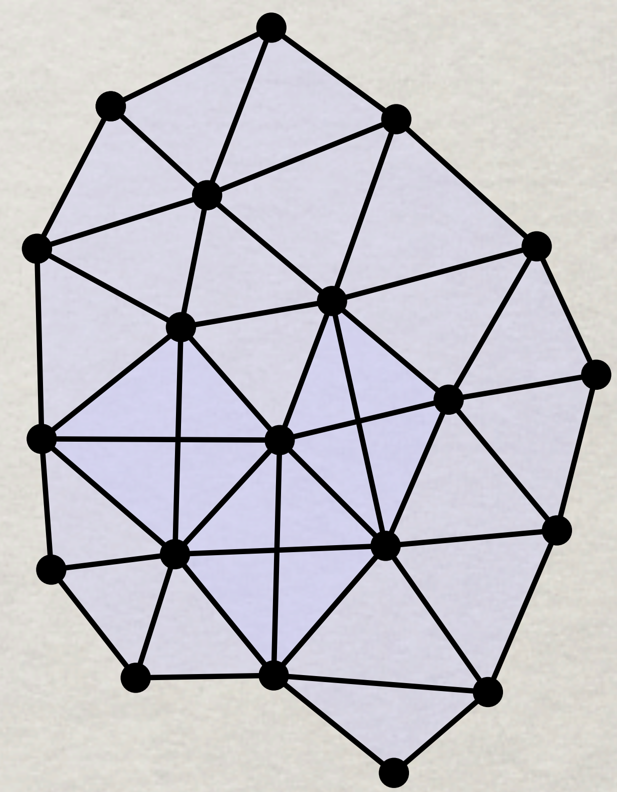


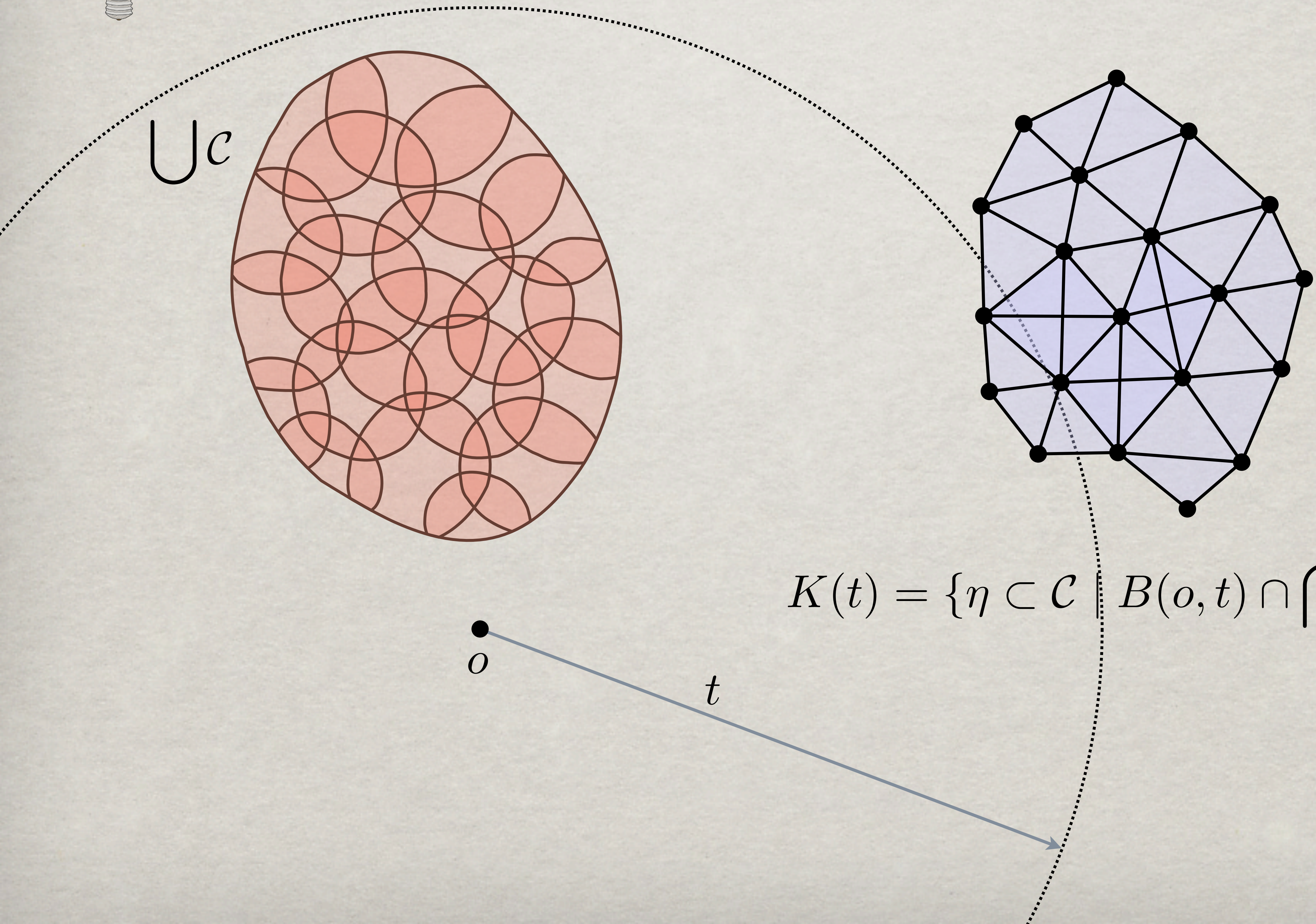
$\mathrm{Nerve}\, C = \{\eta \subset \mathcal{C} \mid \bigcap \eta \neq \emptyset\}$

Our first result.

Nerve $\mathcal{C}$ collapsible

Geometry Driven Collapses

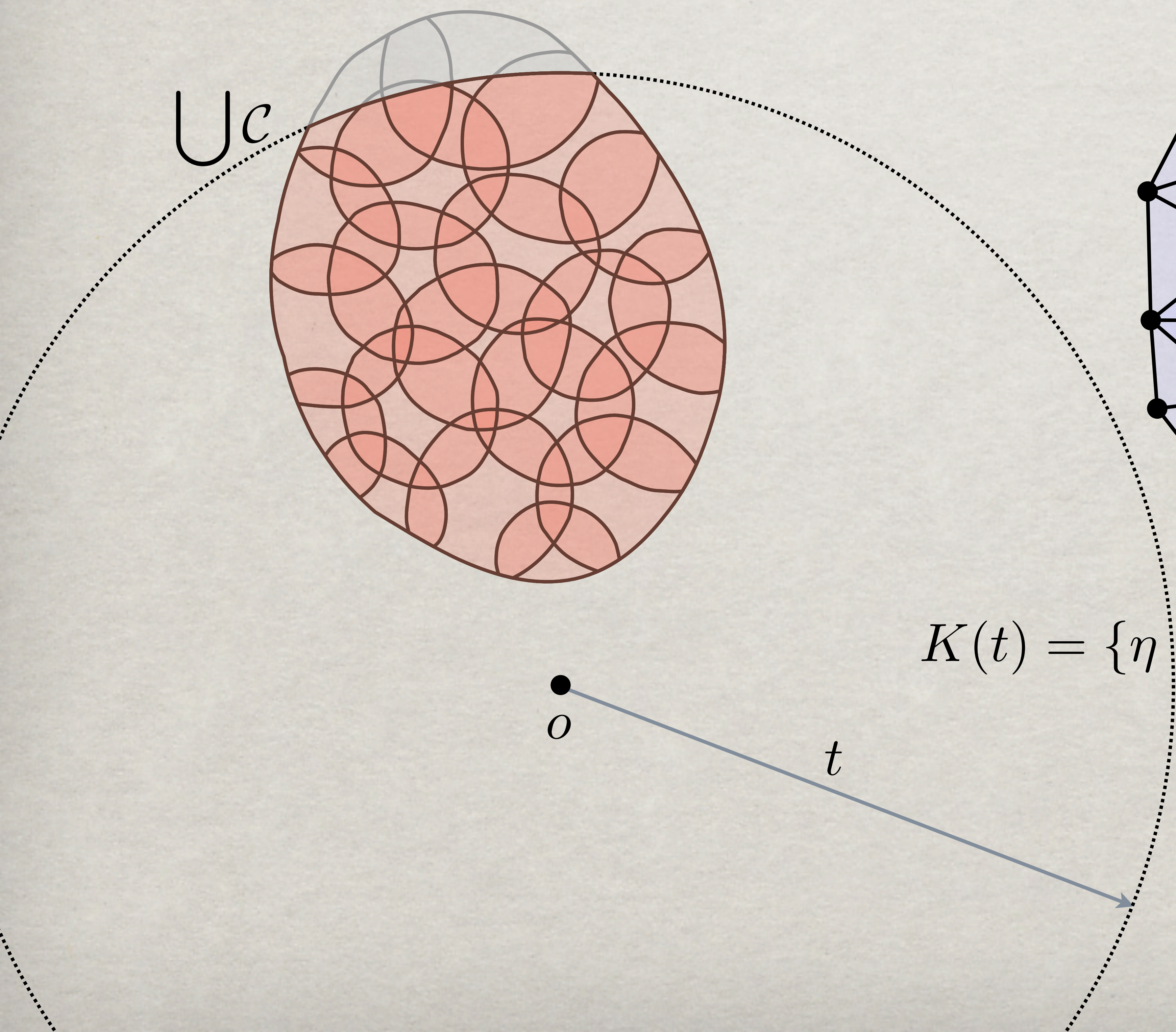
Sweep space with a sphere centered at o whose radius t decreases.

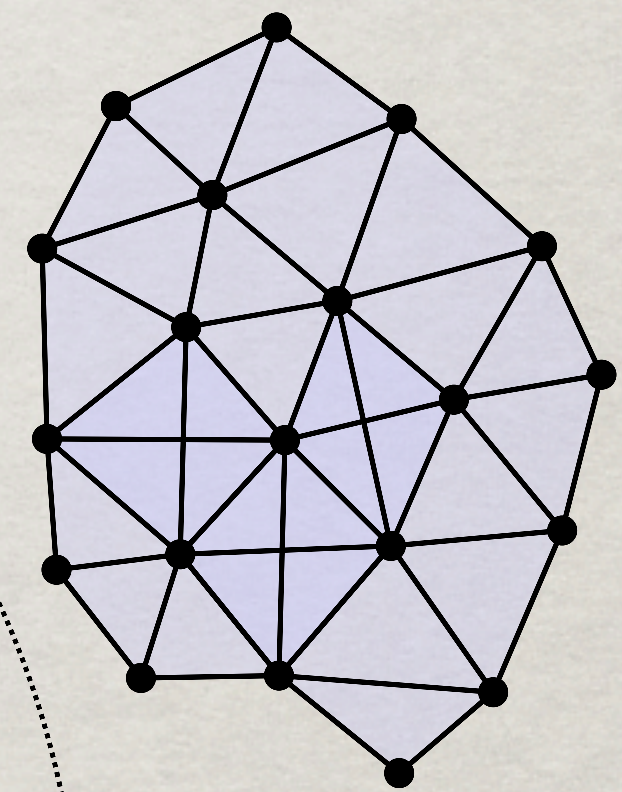


$$K(t) = \{\eta \subset \mathcal{C} \mid B(o,t) \cap \bigcap \eta \neq \emptyset\}$$

Geometry Driven Collapses

Sweep space with a sphere centered at o whose radius t decreases.





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Geometry Driven Collapses

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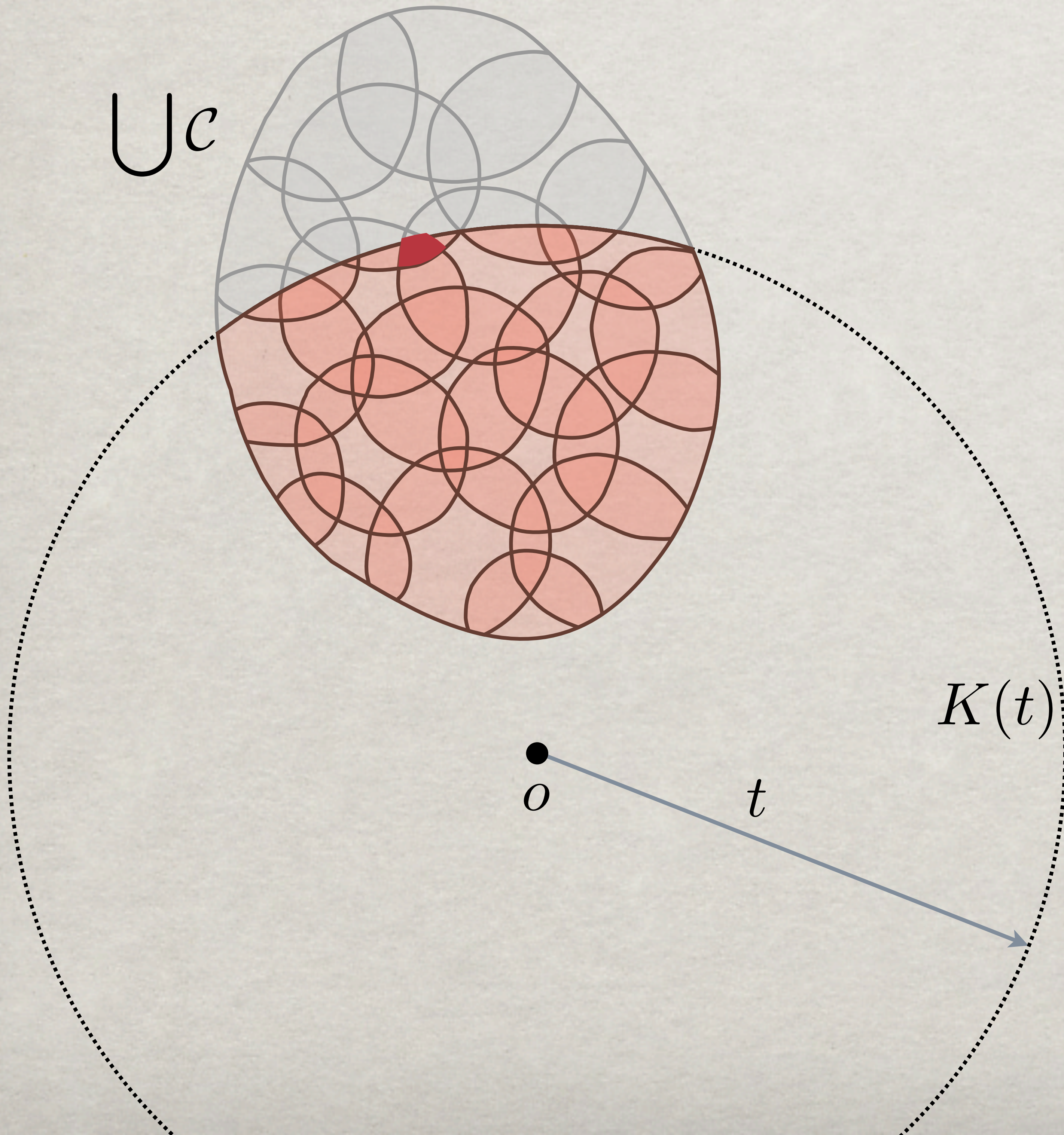

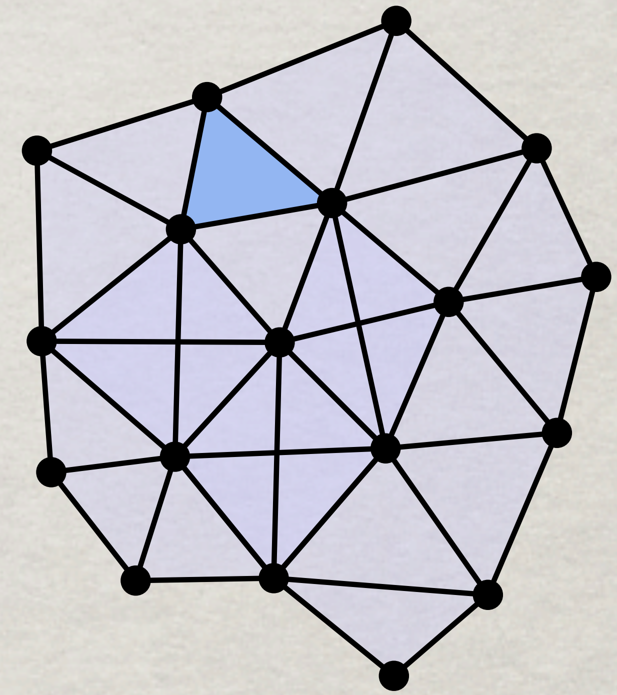

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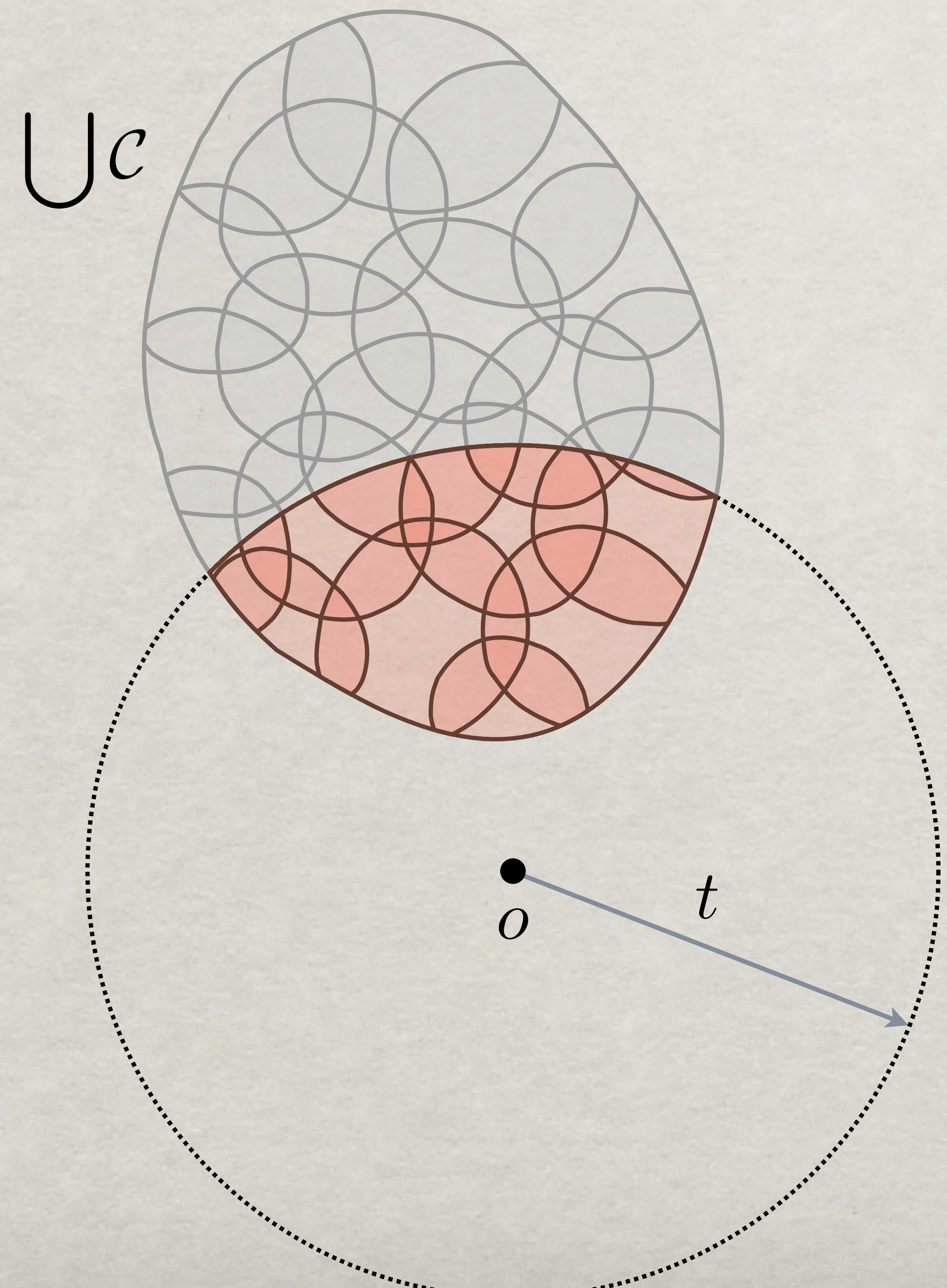
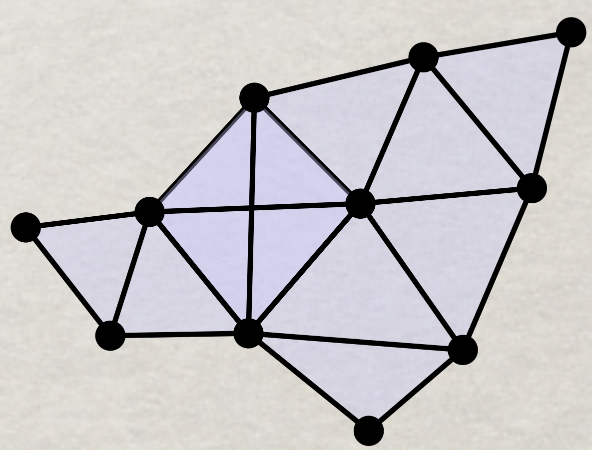
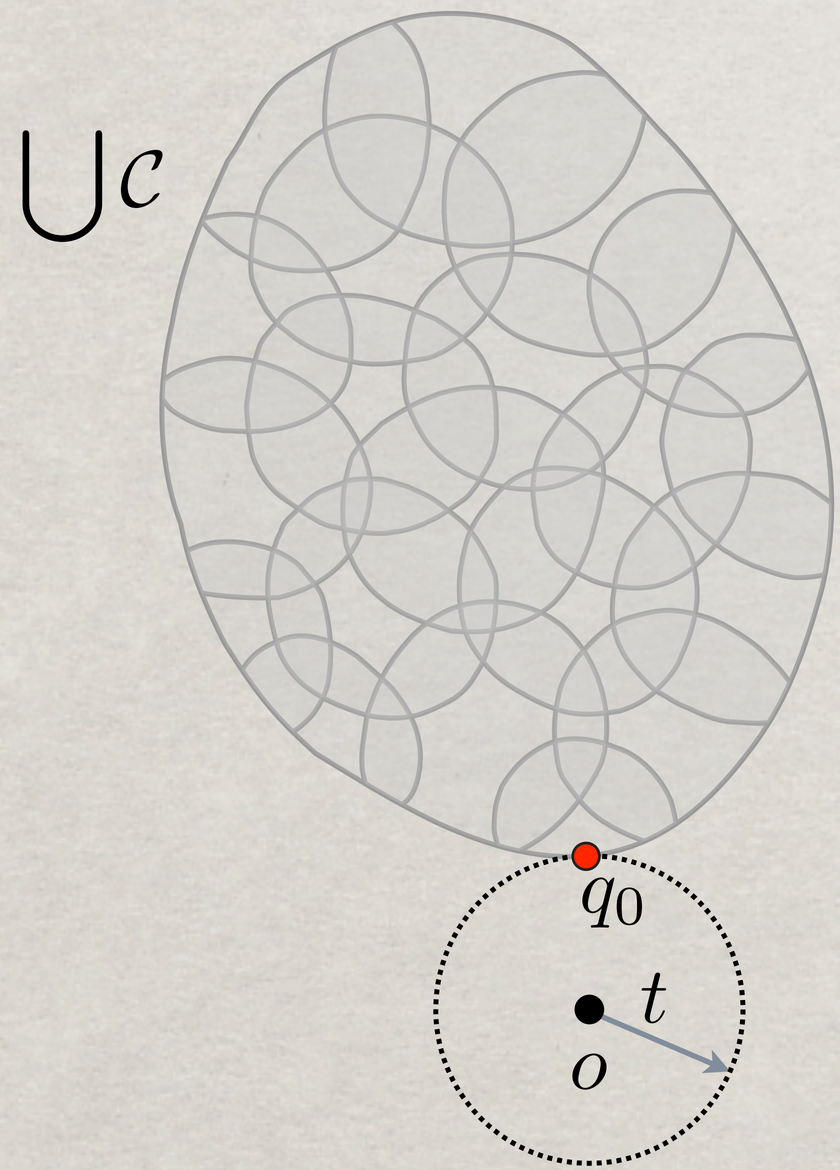




$$K(t) = \{\eta \subset \mathcal{C} \mid B(o,t) \cap \bigcap \eta \neq \emptyset\}$$

Geometry Driven Collapses

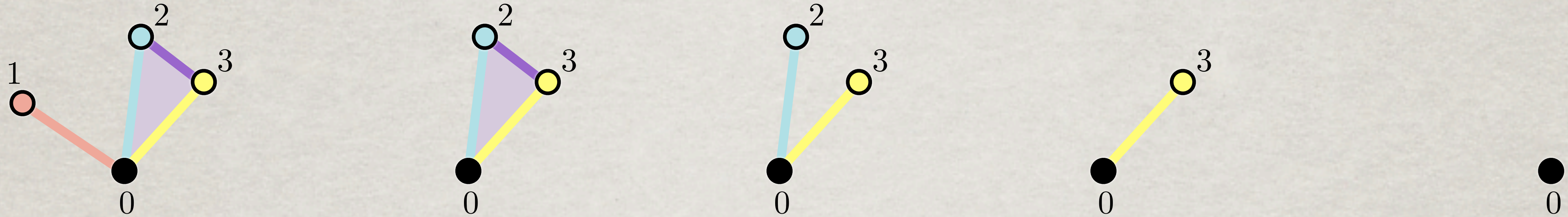
Stop when $t = d(o, \bigcup \mathcal{C})$.

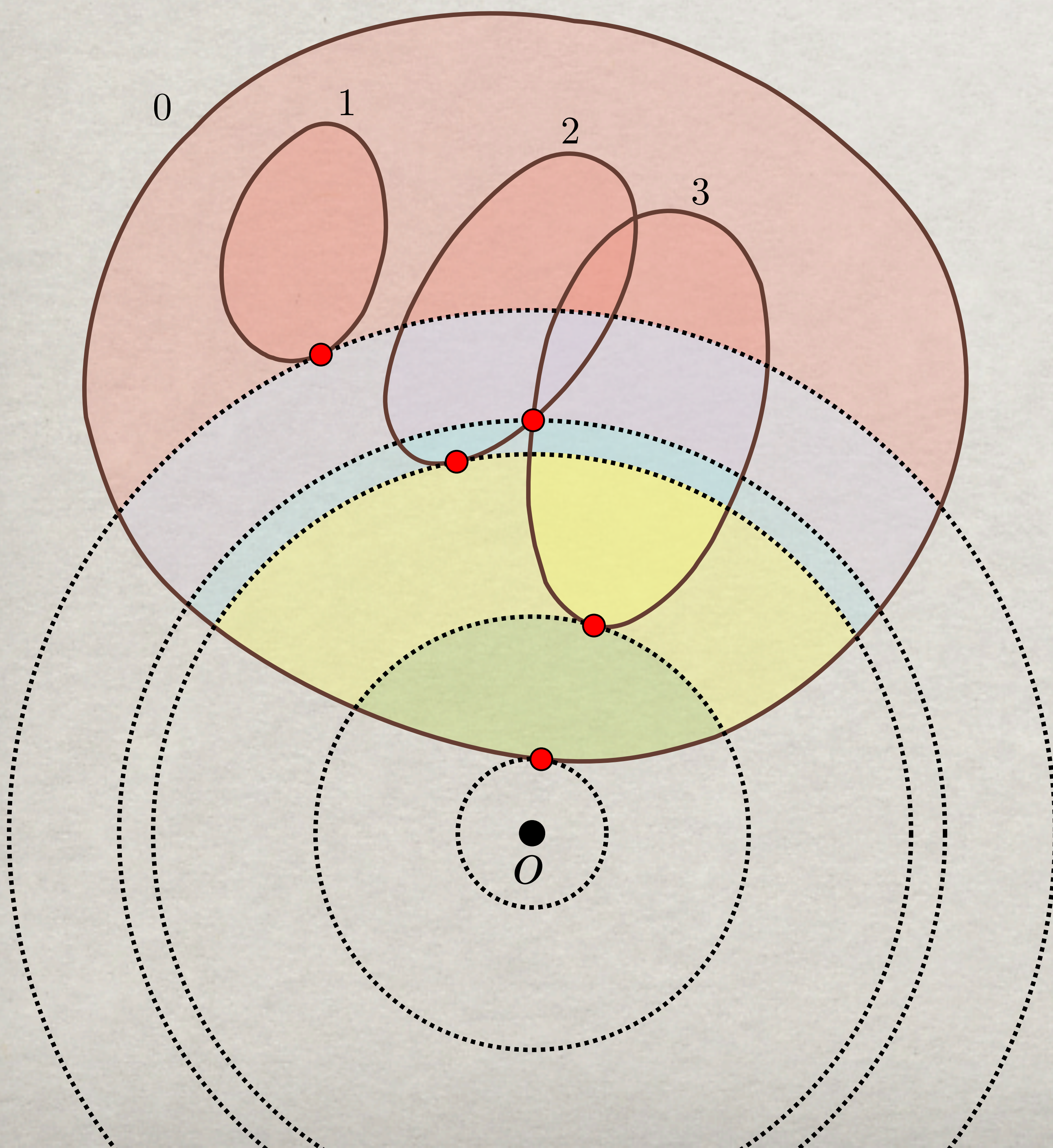


$$K_0 = \{\eta \in \mathcal{C} \mid q_0 \in \bigcap \eta\}$$

K_0 is collapsible

$$\mathrm{Nerve}\,\mathcal{C} = K(t_m) \longrightarrow K(t_i) \xrightarrow{\Delta(t_i)} K(t_{i-1}) \longrightarrow K(t_1) \longrightarrow K(t_0) = K_0$$

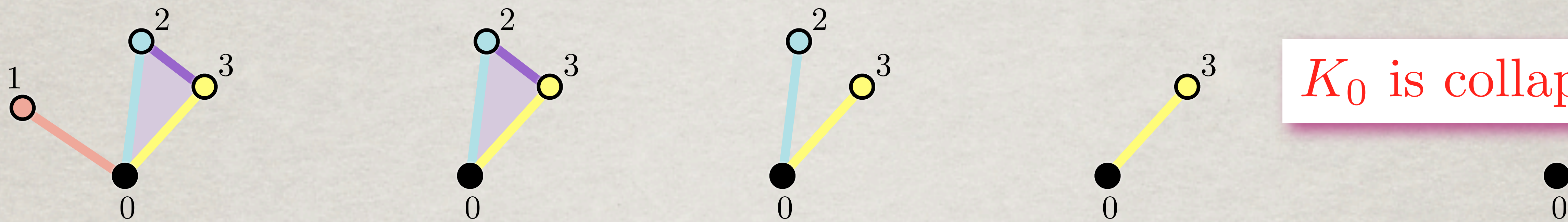




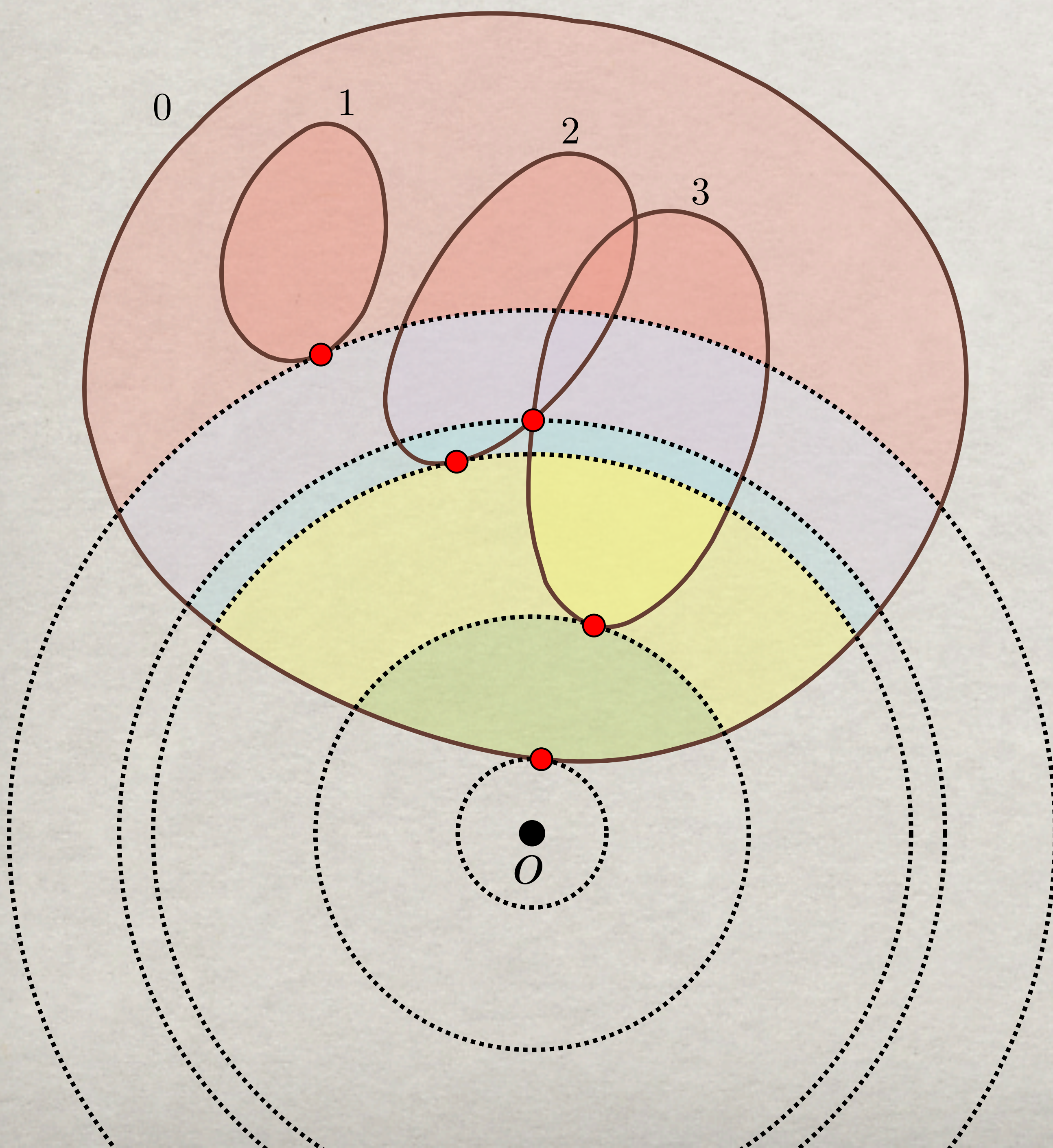
$K(t)$ changes at t when

$$\Delta(t) = \{\eta \subset \mathcal{C} \mid d(o, \bigcap \eta) = t\} \neq \emptyset$$

$$\mathrm{Nerve}\,\mathcal{C} = K(t_m) \longrightarrow K(t_i) \xrightarrow{\Delta(t_i)} K(t_{i-1}) \longrightarrow K(t_1) \longrightarrow K(t_0) = K_0$$



K_0 is collapsible



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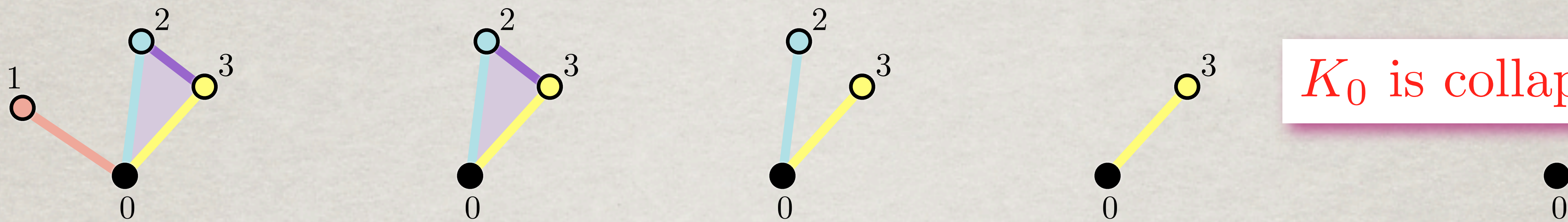
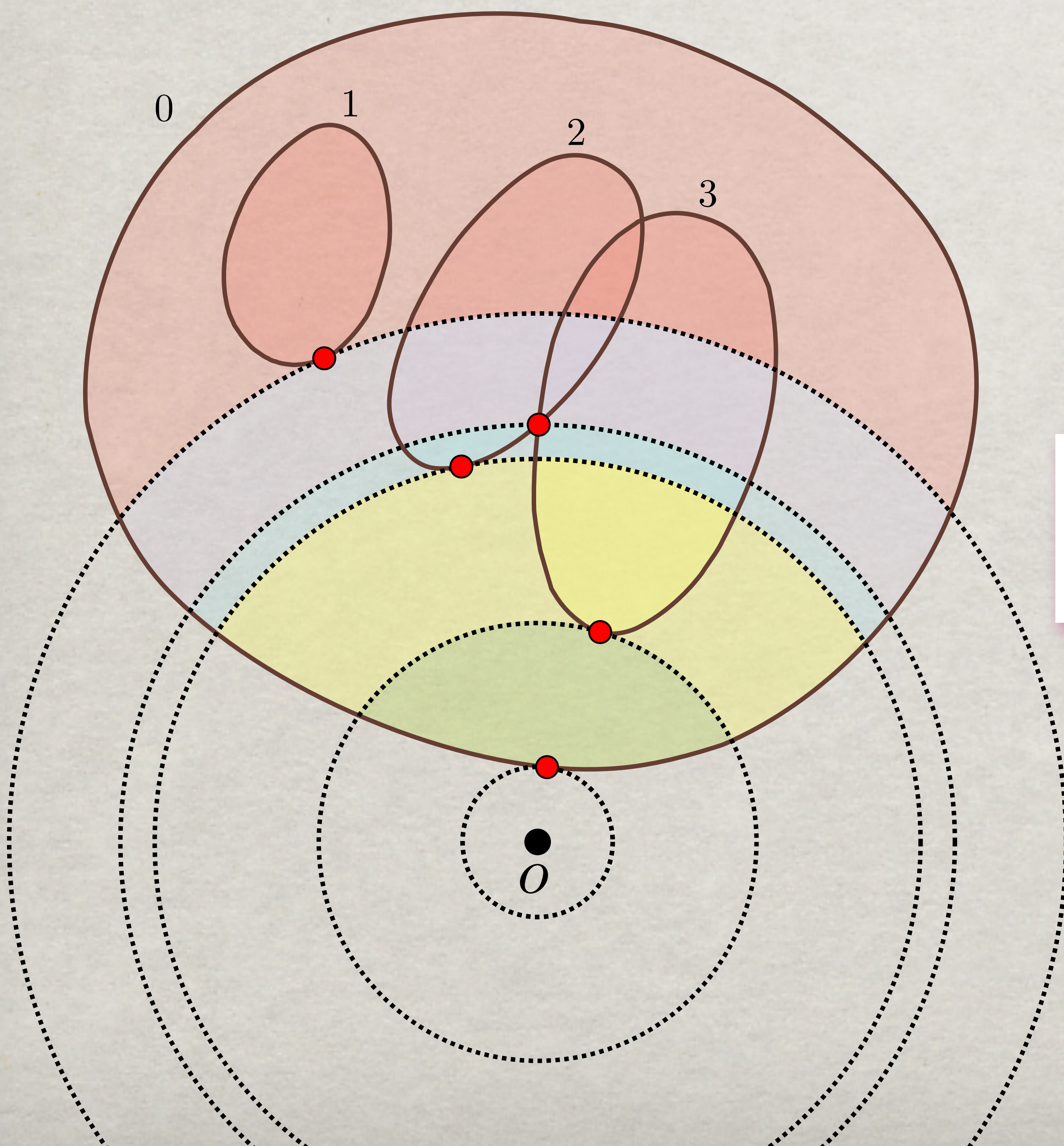


K_0 is collapsible

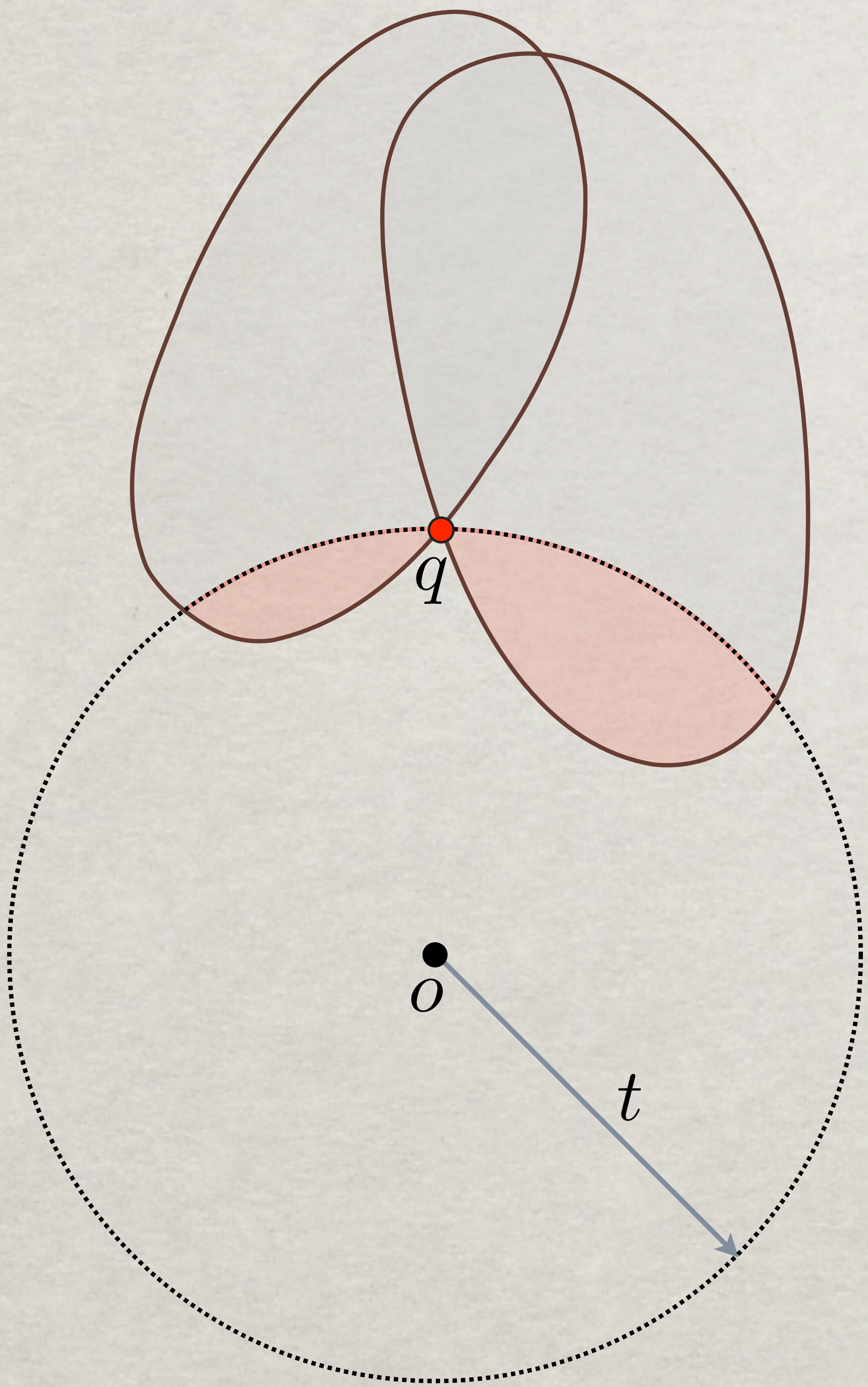


$K(t)$ changes at t when

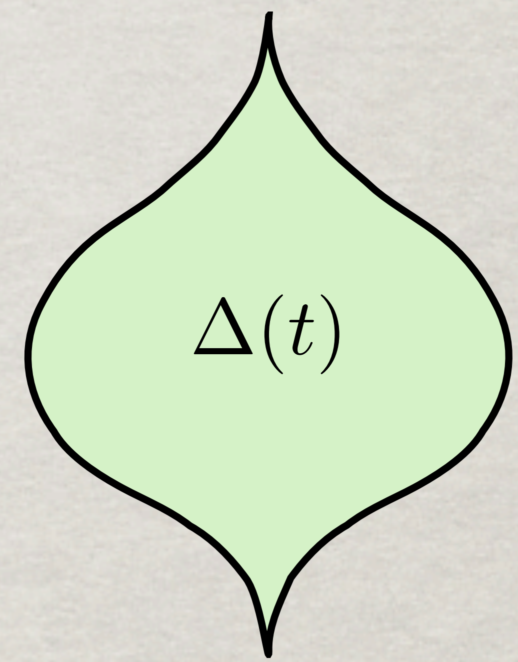
$$\Delta(t) = \{\eta \subset \mathcal{C} \mid d(o, \bigcap \eta) = t\} \neq \emptyset$$

Does the operation that removes $\Delta(t)$ from $K(t)$ a collapse?

$$K(t) \xrightarrow{\text{remove } \Delta(t)} K(t')$$



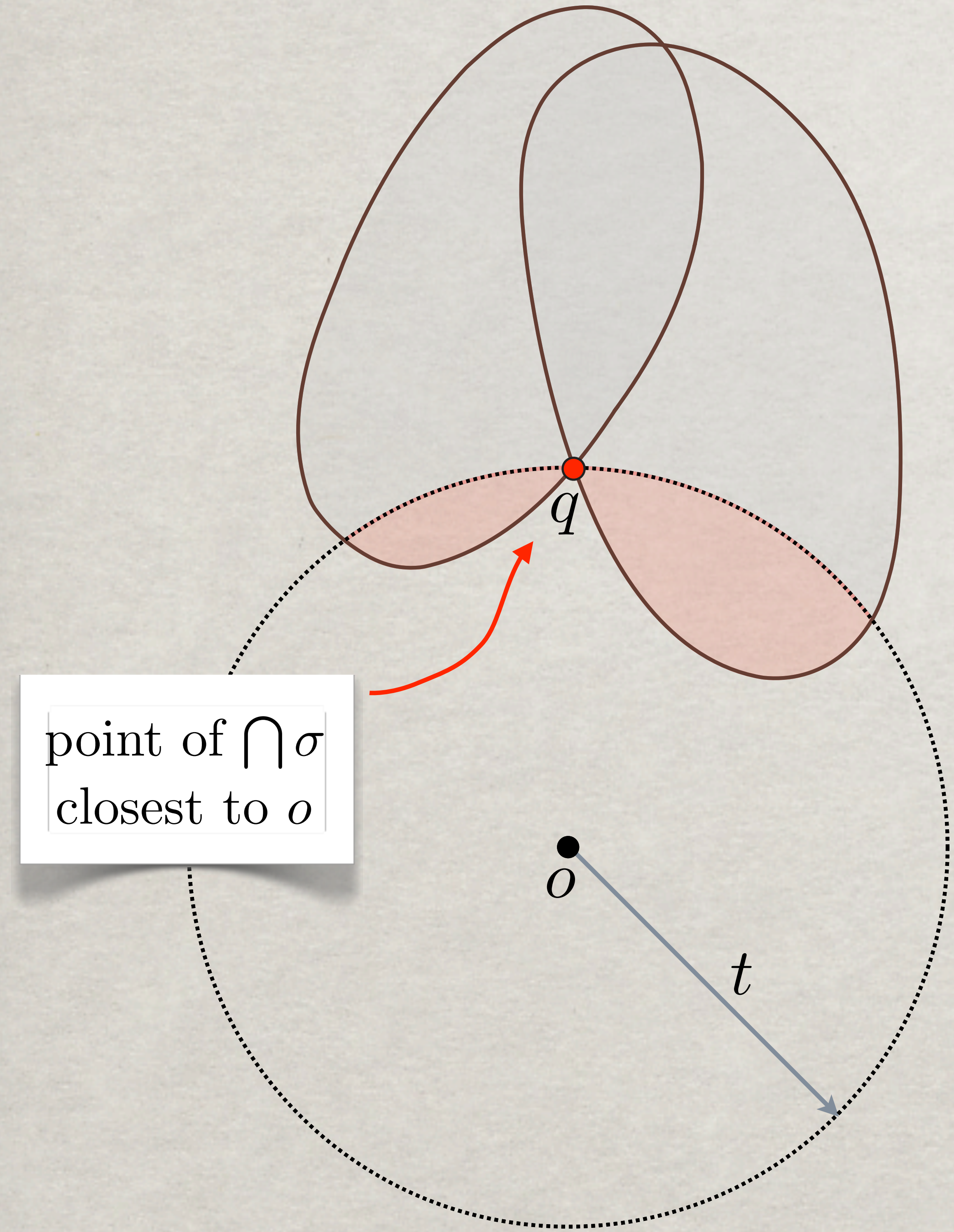
$$\tau = \{C \in \mathcal{C} \mid q \in C\}$$



$(\star)$ σ

$(\star)$ Generically, unique inclusion-minimal element

$$K(t) \xrightarrow{\text{remove } \Delta(t)} K(t')$$



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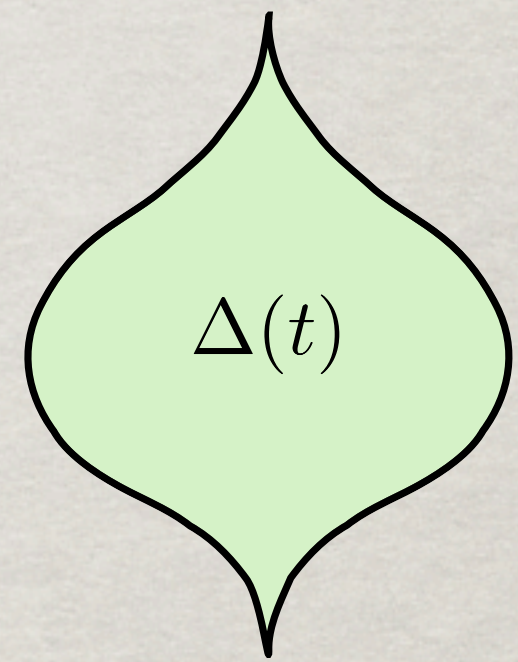


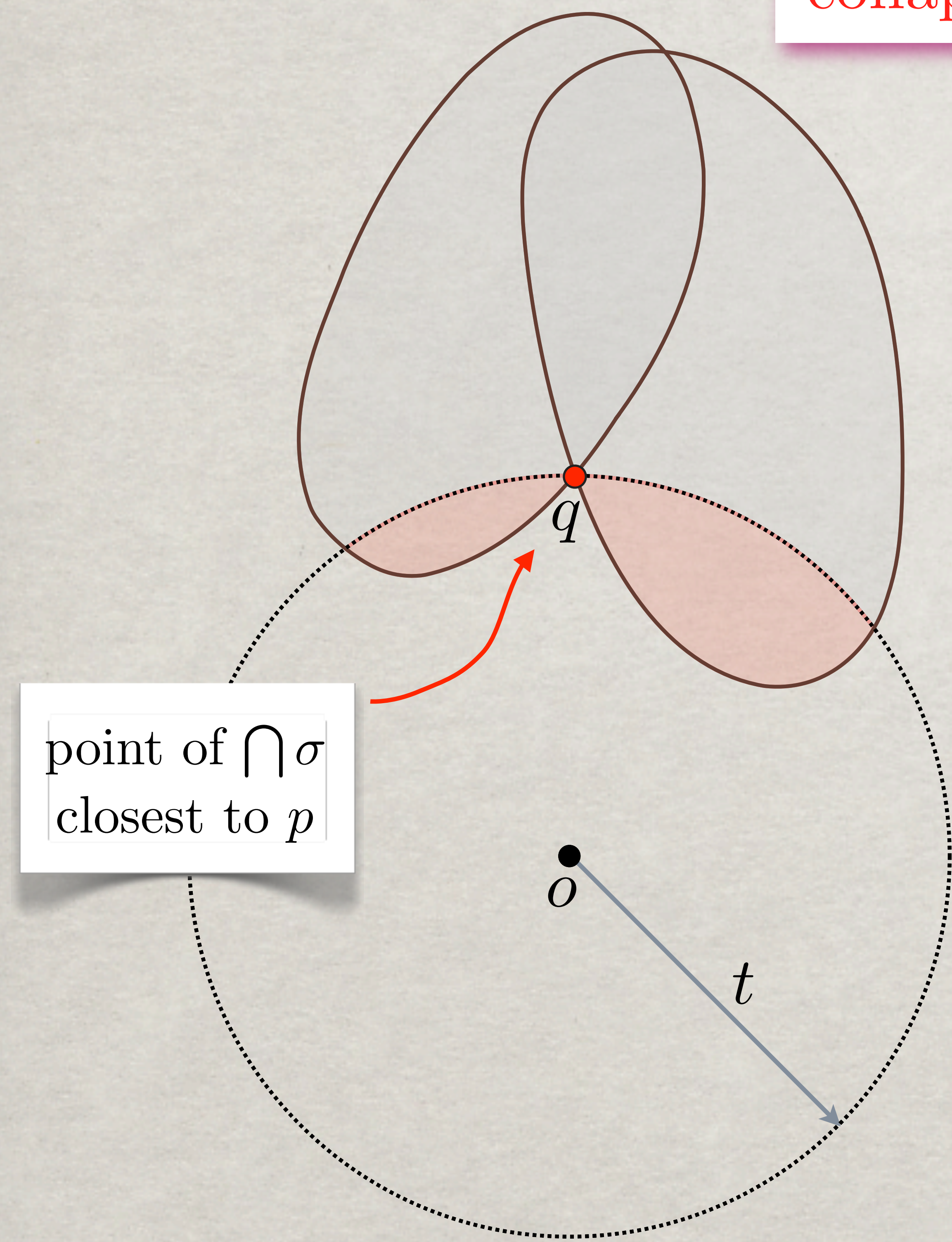
$(\star)\ \sigma$

$\forall C \in \sigma,\ q \in \partial C$

$(\star)$ Generically, unique inclusion-minimal element

$$K(t) \xrightarrow{\text{remove } \Delta(t)} K(t')$$

collapse iff $\sigma \neq \tau$



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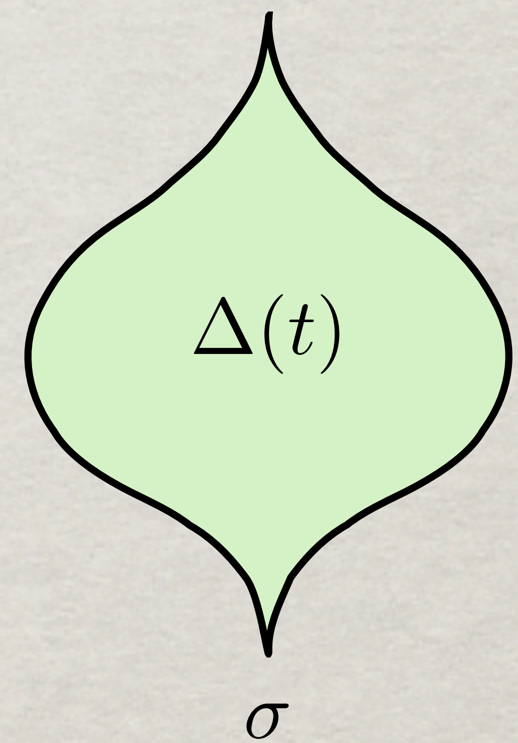
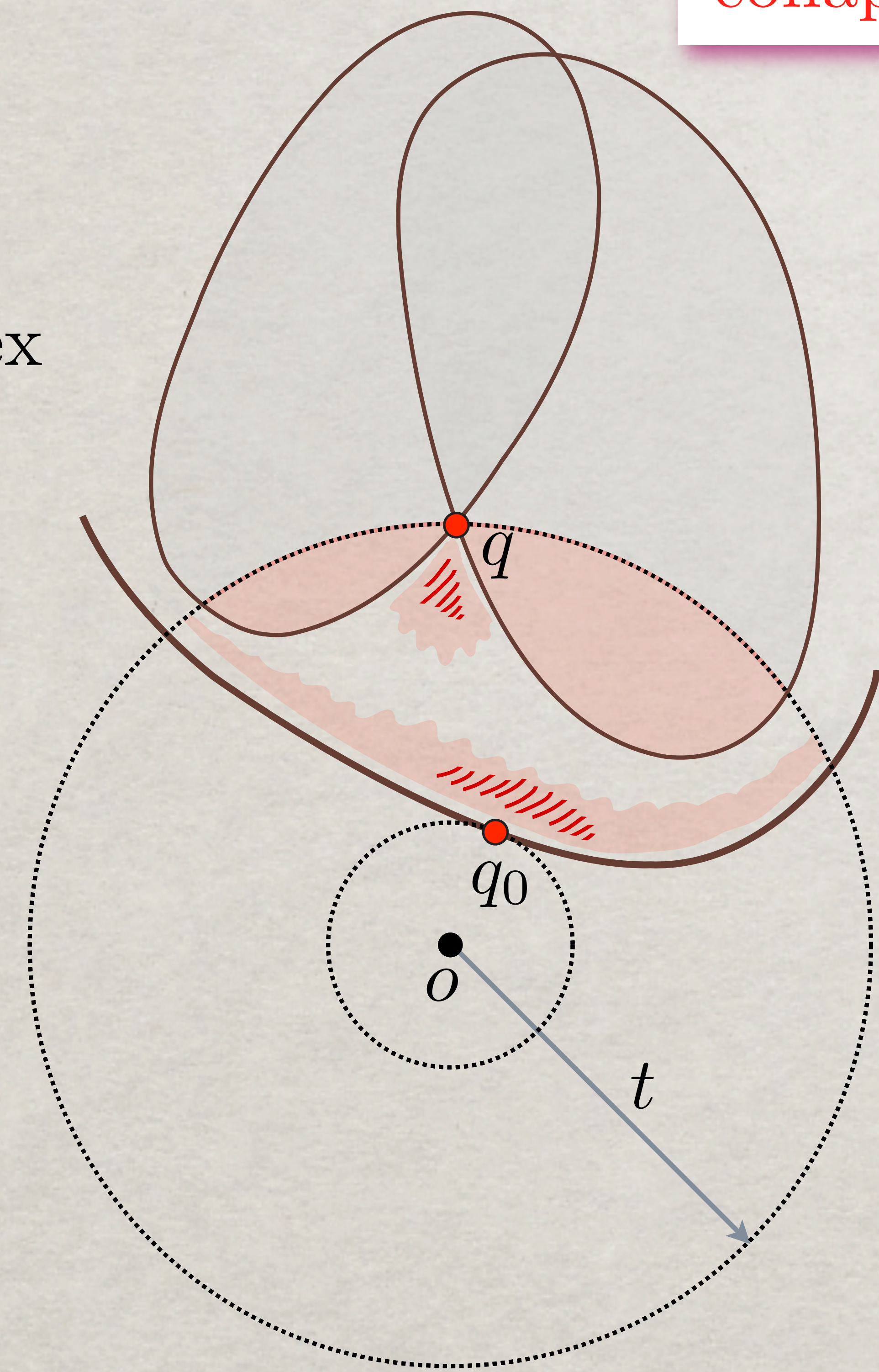


$\forall C \in \sigma,\ q \in \partial C$

$$K(t) \xrightarrow{\text{remove } \Delta(t)} K(t')$$

collapse iff $\sigma \neq \tau$

$\bigcup \mathcal{C}$
convex



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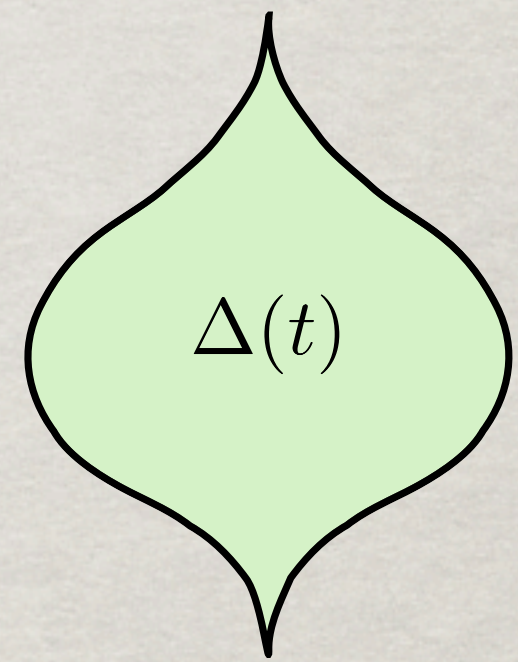


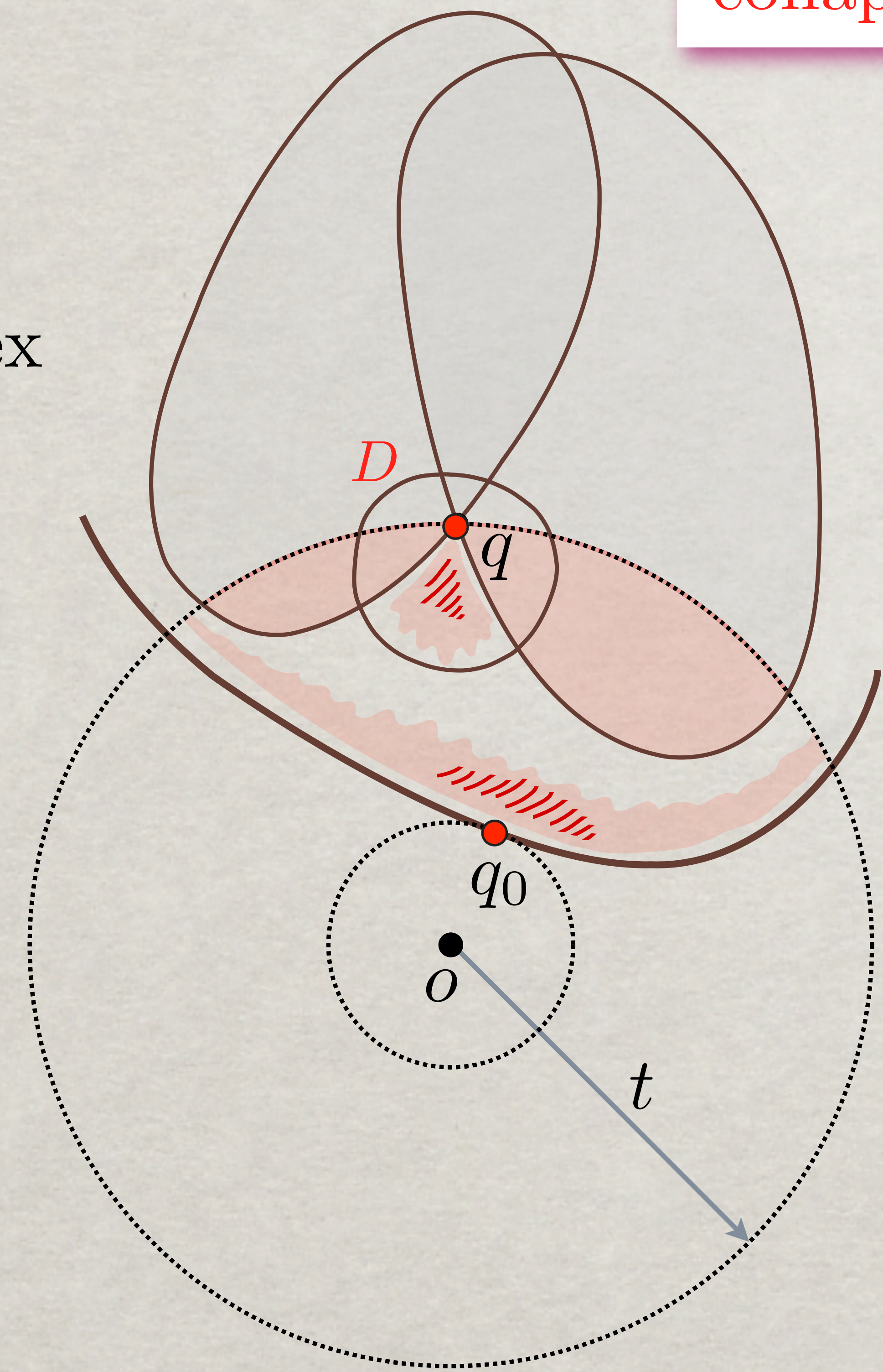
σ

$\forall C \in \sigma,\ q \in \partial C$

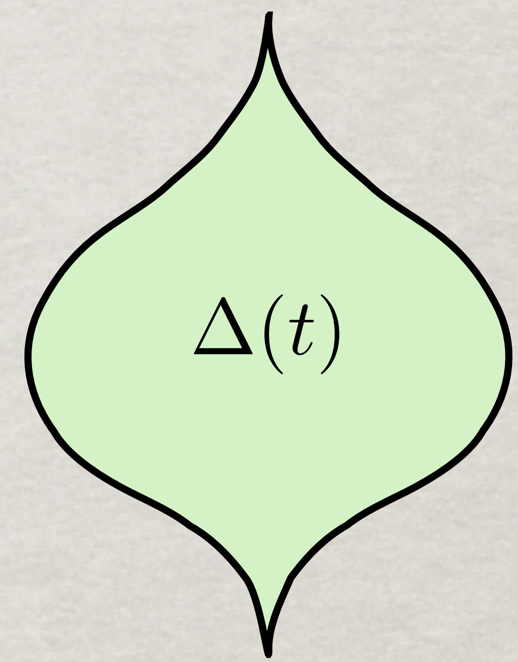
$$K(t) \xrightarrow{\text{remove } \Delta(t)} K(t')$$

collapse iff $\sigma \neq \tau$

$\bigcup \mathcal{C}$
convex



$D \in \tau = \{C \in \mathcal{C} \mid q \in C\}$

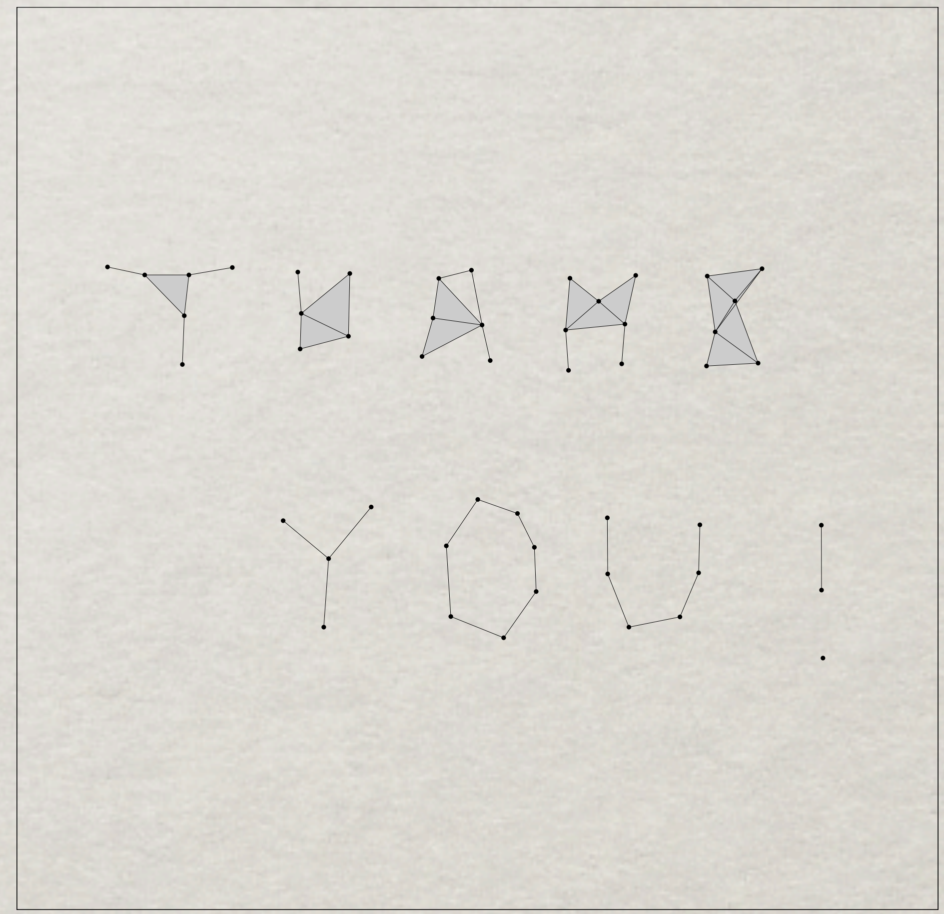


$D \notin \sigma$

$\forall C \in \sigma,\ q \in \partial C$

Conclusion

- Promising data structure for encoding high dimensional simplicial complexes.
- Need further invistigations to understand and improve results obtained in practice:
  - As a first step, we gave conditions under which the Cech complex and the Rips complex are collapsible when vertices sample a convex set.
  - Analogous techniques could be used to derive simplicial complexes with the correct intrinsec dimension.


THANK
YOU!

THANK
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