A tight bound for the Delaunay triangulation of points on a polyhedron

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Abstract

We show that the Delaunay triangulation of a set of n points distributed nearly uniformly on a p-dimensional polyhedron (not necessarily convex) in d-dimensional Euclidean space is $O(n^{\frac{d-k+1}{p}})$, where $k = \lceil \frac{d+1}{p+1} \rceil$. This bound is tight, and improves on the prior upper bound for most values of p.

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1 Introduction

Overview. The Delaunay triangulation of a set of points is a fundamental geometric data structure, used in surface reconstruction, mesh generation, molecular modeling, geographic information systems, and many other areas of science and engineering. In higher dimensions, it is well-known [11] that the complexity of the Delaunay triangulation of n points is $O(n^{\lceil \frac{d}{2} \rceil})$ and that this bound is achieved by distributions of points along one-dimensional curves such as the moment curve. But points distributed uniformly in $\mathbb{R}^d$, for instance inside a d-dimensional ball, have Delaunay triangulations of complexity $O(n)$; the constant factor is exponential in the dimension, but the dependence on the number of points is linear. We are interested in filling in the picture in between these two extremes, that is, when the points are distributed on a manifold of dimension $2 \leq p \leq d-1$. In this paper we consider the easy case of a *p-dimensional polyhedron P*.

Main result. We consider a fixed *p-dimensional polyhedron P* in d-dimensional Euclidean space $\mathbb{R}^d$. Our point set S is a *sparse ε-sample* from P. Sparse ε-sampling requires the sampling to be neither too sparse nor too dense. Our sampling model also contains the restriction that *every* face of P must be ε-sampled, not just those of the highest dimension. This kind of sampling is used in low dimensions for mesh generation, eg. [4].

We consider the complexity of the Delaunay triangulation of S, as $\varepsilon \to 0$, while P remains fixed, so that $n = |S|$ goes to infinity. The main result in this paper is that the number of simplices of all dimensions is $O(n^{\frac{d-k+1}{p}})$ where $k = \lceil \frac{d+1}{p+1} \rceil$. The hidden constant factor depends, among other things, on the geometry of P, which is constant since P is fixed. This bound is tight. In an earlier abstract [1], we had an upper bound of $O(n^{(d-1)/p})$ (with a simpler proof), which is weaker for the smaller values of $1 < p \leq \frac{d-1}{2}$.

Prior work. The complexity of the Delaunay triangulation of a set of points on a two-manifold in $\mathbb{R}^3$ has received considerable attention, since such point sets arise in practice, and their Delaunay triangulations are found nearly always to have linear size. Golin and Na [8] proved that the Delaunay triangulation of a set of points distributed uniformly at random on the surface of a fixed convex polytope in $\mathbb{R}^3$ has expected size $O(n)$. They later [7] established an $O(n \lg^6 n)$ upper bound for the case in which the points are distributed uniformly at random on the surface of a non-convex polyhedron.

Attali and Boissonnat considered the problem using a sparse ε-sampling model similar to the one we use here, rather than a random distribution. For such a set of points distributed on a polygonal surface P, they showed that the size of the Delaunay triangulation is $O(n)$ [2]. In a subsequent paper with Lieutier [3] they considered "generic" smooth surfaces, and got an upper bound of $O(n \lg n)$. A "generic" surface is one for which each medial ball touches the surface in at most a constant number of points.

The genericity assumption is important. Erickson considered more general point distributions, which he characterized by the *spread*: the ratio of the largest inter-point distance to the smallest. The spread of a sparse ε-sample of n points from a two-dimensional manifold is $O(\sqrt{n})$. Erickson proved that the Delaunay triangulation of a set of points in $\mathbb{R}^3$ with spread Δ is $O(\Delta^3)$. Perhaps even more interestingly, he showed that this bound is tight for $\Delta = \sqrt{n}$, by giving an example of a sparse ε-sample of points from a cylinder that has a Delaunay triangulation of size $\Omega(n^{3/2})$ [6]. This surface is not generic and has a degenerate medial axis.

To the best of our knowledge, our earlier abstract [1] is the only prior result for $d > 3$.

Outline of the proof. We begin with the simple lower bound, which illustrates the importance of the quantity $k = \lceil \frac{d+1}{p+1} \rceil$ and motivates the upper bound.

At the coarsest level, the proof of the upper bound is similar to that of [1]: we map Delaunay simplices to the medial axis and then use a packing argument to count them. But the tight bound seems to require us to consider some additional phenomena which occur in higher dimensions, but not in the familiar settings of dimensions two and three. The most important of these is the observation that when $k = \lceil \frac{d+1}{p+1} \rceil > 2$, the vertices of any Delaunay simplex, which must span $\mathbb{R}^d$, have to be drawn from $k > 2$ faces of P. This allows us to relate Delaunay simplices to only the lower-dimensional parts of the medial axis, generated by k or more faces of P - the *k-medial axis*. This idea is embodied in Corollary 6.

Unfortunately, mapping Delaunay simplices to the k-medial axis does not seem to be straightforward. Instead, we define a related structure, the *annular k-medial axis*. Just as the medial axis consists of the centers of empty balls tangent to the boundary of P at more than one point, so the annular medial axis consists of the centers of annuli tangent to the boundary of P at more than one point. An annulus is tangent to the boundary when either the inner bounding sphere or the outer bounding sphere of the annulus is tangent.

The annular k-medial axis is typically unbounded, so to apply a packing argument we need to define a bounded subset of it, the *trimmed annular k-medial axis*. By definition, this object has dimension at most $d-k+1$ and we prove that its $(d-k+1)$-dimensional volume is bounded from above by a constant that does not depend on ε. It follows that we can construct an ε-sample M of the essential annular k-medial axis with $m = O(\varepsilon^{-(d-k+1)})$ points.

We then turn our attention to assigning Delaunay simplices to the samples in M. We define the cover of a point $z \in M$ as

$$\bigcup_{x \in \Pi(z)} B(x, 5d\varepsilon),$$

where $\Pi(z)$ is the set of all orthogonal projections of z onto the flats spanned by faces of P, and $B(x,r)$ is the ball centered at x with radius r. For $k = \lceil \frac{d+1}{p+1} \rceil$, we map each Delaunay simplex σ to a point $z \in M$ in such a way that the vertices of σ are contained in the cover of z; this is done by associating an annulus with each Delaunay simplex, and then "growing" the annulus to increase the number of its tangent points. Since the cover of each point $z \in M$ contains a constant number of points in S, each point $z \in M$ can only be charged for a constant number of Delaunay simplices. It follows that the size of the Delaunay triangulation is bounded from above by the size of M, which is $m = O(\varepsilon^{-(d-k+1)})$. Since our point set S is a sparse ε-sample from a p-dimensional polyhedron, its cardinality is $n = \Omega(\varepsilon^{-p})$. Eliminating ε gives the $O(n^{\frac{d-k+1}{p}})$ upper bound.

We carefully structure the argument so that we can avoid making any non-degeneracy assumptions on either P or the vertex set S of the Delaunay triangulation. Parts of the annular k-medial axis of a degenerate polyhedron may have dimension greater than $d-k+1$ (which is what one expects in the generic case). Nonetheless we can show that any simplex of the Delaunay triangulation can be mapped to a part of the annular k-medial axis which *does* have dimension at most $d-k+1$, by mapping the simplex to an annulus tangent to at least k *independent* faces (Definition 4). Lemma 5 shows that k independent faces generate a piece of the correct dimension, and Lemma 14 shows that a simplex can be mapped to an annulus tangent to k independent faces.

2 Statement of Theorem

To formally state the theorem, we must first define the sampling condition.

2.1 Sampling

Let $P \subseteq \mathbb{R}$ denote our input polyhedron, not necessarily convex or connected [1]. We assume that P itself spans $\mathbb{R}^d$, since otherwise we can consider the Delaunay triangulation in the subspace which is spanned by P. The flat *spanned* at a point x is the flat (affine subspace) H of largest dimension passing through x, such that the intersection of a neighborhood of x with H is contained in P.

An *i-face* F of P is a maximal collection of points sharing the same spanning i-flat. Notice that under this definition, faces are relatively open, faces are not necessarily connected, and every point $x \in P$ belongs to a unique face that we denote by F_x. The

[1] More formally, a p-dimensional polyhedron can be defined as the underlying space of any geometric simplicial complex of dimension p.

0-faces are the *vertices* of P.

We say that a set of points $S \subseteq P$ is a λ-*sparse* ε-*sample* of P iff it satisfies the following two conditions:

Density: Every point x in P is at distance ε or less to a point in S lying on the closure of F_x. In other words,

$$\forall x \in P,\ \exists s \in S \cap \mathrm{cl}(F_x),\ \|x - s\| \leq \varepsilon;$$

Sparsity: Every closed d-ball with radius $5d\varepsilon$ contains at most λ points of S.

The density condition implies that all faces, of all dimensions, are nearly uniformly sampled, not just the faces with maximal dimension. We treat λ as a constant. As ε goes to zero, the number n of points in a λ-sparse ε-sample of a p-dimensional polyhedron is related to ε by $n = \Theta(\varepsilon^{-p})$, i.e. n tends to infinity.

2.2 Main Theorem

We are now ready to state our main result:

Theorem 1 *Let S be a λ-sparse ε-sample of a p-dimensional polyhedron P in $\mathbb{R}^d$, with $n = |S|$. In the worst case, the Delaunay triangulation of S has size $\Theta(n^{\frac{d-k+1}{p}})$ where $k = \lceil \frac{d+1}{p+1} \rceil$.*

3 The lower bound

We begin with the lower bound, which is comparatively simple and conveys the intuition as to why $\Theta(n^{\frac{d-k+1}{p}})$ is the correct bound.

Proof of Lower Bound for Theorem 1: We will construct a polyhedron P, and examine the complexity of the Delaunay triangulation of a λ-sparse ε-sample on P. We first select a set of $d+1$ affinely independent points, and partition them into groups $Q_1, \ldots, Q_k$ so that groups $Q_1 \ldots Q_{k-1}$ contain $p+1$ points each, and Q_k contains the remainder. Thus it must be that $k = \lceil \frac{d+1}{p+1} \rceil$.

Let C_i be the convex hull of Q_i, and let c_i be the dimension of C_i. For all $1 \leq i < k$, $c_i = p$, and $c_k \leq p$. Let $P = \bigcup_{i=1}^{k} C_i$; note that P indeed has dimension p. Now let us consider a λ-sparse ε-sample of P; let us call it S. Any choice of sample points $s_i \in S \cap C_i$, for $1 \leq i \leq k$ is linearly independent, since each s_i is a convex combination of points which are all independent of each of the spaces spanned by the other C_j, so it defines a $(k-1)$-simplex σ.

Also, for any choice of the s_i, there is a unique $(d-1)$-sphere Σ tangent to P at the s_i for $1 \leq i \leq k$. To see this, notice that being tangent to C_i at s_i is equivalent to passing through $c_i + 1$ coincident points; and

$$\sum_{i=1}^{k} (c_i + 1) = d + 1. \tag{1}$$

Σ encloses no sample point of S in its interior, since it is tangent to each of the C_i at a single point. Thus σ must be a Delaunay simplex.

Since C_i contains $\Omega(\varepsilon^{-c_i})$ points of S, the number of distinct Delaunay $(k-1)$-simplices that we can construct is at least

$$\Omega(\varepsilon^{-c_1} \times \cdots \times \varepsilon^{-c_k}) = \Omega(\varepsilon^{-(d-k+1)}) = \Omega(n^{\frac{d-k+1}{p}}).$$

and this is a lower bound on the overall complexity of the Delaunay triangulation. $\square$

4 The annular medial axis

The *annular medial axis* $\mathcal{AM}$ is the key geometric object in our proof of the upper bound. The annular medial axis is formed by tangent annuli rather than tangent spheres. Setting $k = \lceil \frac{d+1}{p+1} \rceil$, we will be particularly interested in $\mathcal{AM}^k$, the part of the annular medial axis generated by at least k faces of P. If the faces of P were in general position, it would be fairly simple to establish that $\mathcal{AM}^k$ had dimension at most $d - k + 1$ everywhere, but when P is degenerate this is not the case. Instead, we identify the subset of $\mathcal{AM}^k$ which is guaranteed to have dimension at most $d - k + 1$.

4.1 Definitions

It is clear what it means for a sphere to be tangent in the interior of an open face F of P; let us introduce a definition which also handles the boundaries. We denote the closure of F by $\mathrm{cl}(F)$ and write $\mathrm{Aff}(F)$ for the affine space spanned by F. We say that a $(d-1)$-sphere Σ is *tangent* to F at point x if both $\mathrm{cl}(F)$ and $\mathrm{Aff}(F)$ intersect Σ in a unique point x. Since faces are relatively open, this means that x might be a limit point of F, so that $\Sigma \cap F = \emptyset$. Note also that a sphere can be tangent to several faces at x, only one of which is the face F_x containing x (the one of smallest dimension).

An annulus with center z, inner radius r and outer radius R is the set of points x whose distance to the center satisfies $r \leq \|x - z\| \leq R$. The boundary of an annulus consists of two $(d-1)$-spheres and we call the smaller one the *inner sphere* and the larger one the *outer sphere*. We say that an annulus A is *tangent* to F at x if one of the two spheres bounding A is tangent to F at x (see Figure 1, or Figure 6 for a more intricate example). Point x is called a *tangency point* of A. An annulus is *hollow* if its inner sphere bounds a d-ball whose interior does not intersect P; notice that P might intersect the annulus itself. The *width* of an annulus is the difference between the outer and inner radii $R - r$. The *penetration* of an annulus is the maximum radius of the intersection of the annulus with a flat, such that the open ball bounded by the inner sphere remains empty. Specifically, the penetration is $\sqrt{R^2 - r^2}$. We will mostly be concerned with annuli of penetration ε; such an annulus has width $\frac{\varepsilon^2}{R+r}$.

Definition 2 *The* annular k-medial axis $\mathcal{AM}^k$ *of P is the set of points $z \in \mathbb{R}^d$ for which the largest hollow annulus of penetration ε centered at z is tangent to at least k faces of P.*

We will write $A(z, \varepsilon)$ for the largest hollow annulus of penetration ε centered at z. Figure 1 pictures an example of the annular 2-medial axis in $\mathbb{R}^2$. Observe that this is a *superset* of the medial axis, even for $\varepsilon = 0$: the medial axis of the polyhedron is the set of points $z \in \mathbb{R}^d$ for which $A(z, 0)$ touches the polyhedron in two points or more, while the annular 2-medial axis with $\varepsilon = 0$ is the set of points z for which $A(z, 0)$ is tangent to two faces of P or more (possibly at the same point).

4.2 Dimension and degeneracy

Given a subset $X \subseteq \mathbb{R}^d$, a stratification of X is a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_j = X$$

by subspaces such that the set difference $X_i \setminus X_{i-1}$ is an open i-dimensional manifold (possibly not connected), called the i-dimensional *stratum* S_i of X. For example, the Voronoi diagram of a point set in $\mathbb{R}^2$ admits a stratification into its cells, edges and

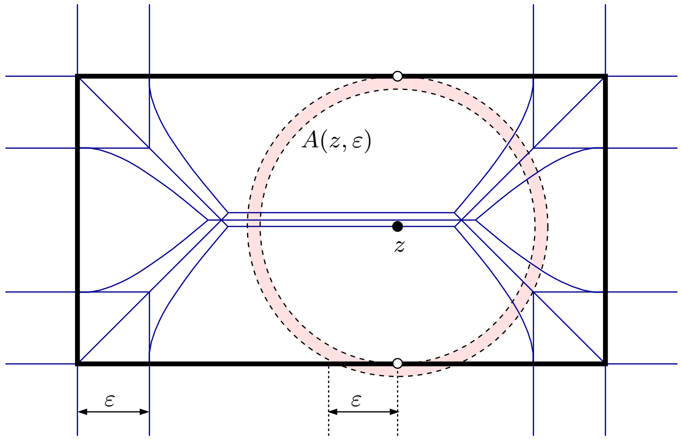

Figure 1: A rectangle and its annular 2-medial axis, composed of 16 half-lines, 7 segments and 8 pieces of hyperbolas. The annulus $A(z, \varepsilon)$ is tangent to the rectangle at the two hollow dots.

vertices. Semi-algebraic sets admit a stratification [9], and since the annular k-medial axes of polyhedra are piecewise semi-algebraic, they also admit a stratification. In this section, we give conditions under which a point $z \in \mathcal{AM}^k$ belongs to a stratum of dimension $d - k + 1$ or less.

If the faces of P were in general position, this would be easy: any z belonging to the stratum of dimension $d-k+1$ would be the center of an annulus tangent to k faces. But in degenerate situations an annulus tangent to k faces might belong to a stratum of dimension greater than $d-k+1$, as illustrated in Figure 2. So we need to make a more careful argument.

Recall that $\mathcal{AM}^k$ is defined as the set of points for which $A(z, \varepsilon)$ is tangent to *at least* k faces. Let us concentrate instead in this section on the subset $\mathcal{C}^k$, defined as the set of points z such that annulus $A(z, \varepsilon)$ is tangent to *exactly* k faces of P. When does $\mathcal{C}^k$ form a $(d - k + 1)$-dimensional stratum?

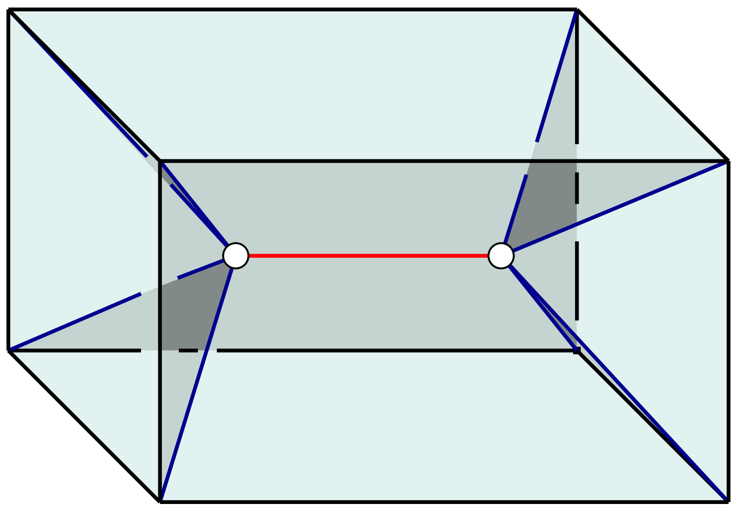

Figure 2: The box is P, and we take $\varepsilon = 0$. The set $\mathcal{C}^4 \subset \mathcal{AM}^4$, consisting of the centers of annuli tangent to four faces is the segment connecting the hollow dots. It has dimension one, rather than dimension zero as we would expect in the absence of degeneracy.

We start by writing down some equations that determine $\mathcal{C}^k$ locally around z. Since $A(z,\varepsilon)$ is tangent to exactly k faces $F_1,\ldots,F_k$, there exists $\delta > 0$ such that every face of the polyhedron not in $\{F_1,\ldots,F_k\}$ is at distance at least δ to the boundary of $A(z,\varepsilon)$. Using a compactness argument as in [10], it follows that for a point y close enough to z, the only faces that might possibly be tangent to annulus $A(y)$ are $F_1,\ldots,F_k$. We set $e_i = -\varepsilon^2$ if F_i is tangent to the outer sphere of $A(z,\varepsilon)$ and $e_i = 0$ if F_i is tangent to the inner sphere of $A(z,\varepsilon)$. In a small neighborhood of z, $\mathcal{C}^k$ is thus determined by the following $k-1$ equations:

$$g^i(y) = d(y,F_i)^2 - d(y,F_k)^2 + e_i - e_k = 0,$$

for $0 < i < k$. Each $g^i(y)$ is a polynomial of degree two, so every neighborhood in $\mathcal{C}^k$ is a subset of the intersection of $k-1$ quadrics. In general, $k-1$ hypersurfaces meet at point z in a $(d-k+1)$-manifold, but this might not hold in the presence of degeneracies.

Lemma 3 *Suppose $z \in \mathcal{C}^k$ is the center of an annulus $A(z,\varepsilon)$ tangent to the polyhedron at k affinely independent points $x_1,\dots,x_k$. Then the neighborhood of z in $\mathcal{C}^k$ is a $(d-k+1)$-manifold. Furthermore, the tangent space to $\mathcal{C}^k$ at z is the set of vectors orthogonal to the affine space spanned by $x_1,\dots,x_k$.*

PROOF. In a small neighborhood of z, $\mathcal{C}^k$ coincides with the zero-set of the functions $g^i(y)$. Let $F_i = F_{x_i}$ be the face to which x_i belongs. One can check that the gradient of $g^i(y)$ is $2(x_k - x_i)$. Since the x_i are affinely independent for $1 \leq i \leq k$, we have that the Jacobian of the map $g = (g^1(y),\dots,g^{k-1}(y))$ has rank $k-1$, so the implicit function theorem implies that its zero-set $g^{-1}(0)$ is a $(d-k+1)$-dimensional manifold in a neighborhood of z. Furthermore, the tangent space of $g^{-1}(0)$ at z is the null space of the derivative $Dg(z)$, which is the set of vectors orthogonal to the affine space spanned by $x_1,\dots,x_k$. $\square$

While the assumption that the tangency points of $A(z,\varepsilon)$ are independent is sufficient to show that z belongs to the stratum of dimension at most $d-k+1$, this condition is not necessary. Again, this is a phenomenon that occurs only in high dimension; let's consider the following example in $\mathbb{R}^6$. P consists of segments of four lines, represented parametrically by:

$$(0,0,s,0,0,0)$$

$$(1,0,0,t,0,0)$$

$$(0,1,0,0,u,0)$$

$$(1,1,0,0,0,v)$$

Let us take $\varepsilon = 0$, and consider the part of $\mathcal{C}^4$, consisting of points z with $A(z,\varepsilon)$ tangent to the interiors of all four line segments. This is a three-dimensional manifold, as one would expect, and at most points z in this portion of $\mathcal{AM}^3$ the four points of tangency span a three-dimensional subspace. But the point $z = (1/2, 1/2, 0, 0, 0, 0)$ also belongs to this part of $\mathcal{C}^4$ and its points of tangency all lie in a common 2-flat. Nonetheless z is a regular point of this manifold, and $\mathcal{C}^k$ has dimension three in the neighborhood of z. Hence we need to consider not just the local points of tangency in the neighborhood of z but also the tangent faces.

Definition 4 *We say that k faces $F_1, \ldots, F_k$ are* independent *if none of them is contained in the affine space spanned by the union of the others, that is for $1 \leq i \leq k$,*

$$F_i \quad \not\subseteq \quad \mathrm{Aff}(F_1 \cup \cdots \cup \widehat{F_i} \cup \cdots \cup F_k),$$

where the symbol $\widehat{\ }$ over F_i indicates that it is omitted in the union.

For example, the four segments in $\mathbb{R}^6$ which we just considered are independent.

Lemma 5 *Suppose that $A(z,\varepsilon)$ is tangent to exactly k faces. If those k faces are independent, then $\mathcal{C}^k$ is a manifold of dimension at most $d-k+1$ in a neighborhood of z.*

PROOF. Recall that $\mathcal{C}^k$ is the set of points $z \in \mathbb{R}^d$ such that $A(z, \varepsilon)$ is tangent to exactly k faces of P. We partition $\mathcal{C}^k$ into k pieces (some of which might be empty), as follows. We write $\mathcal{S}_i$ for the set of points $y \in \mathcal{C}^k$ whose tangency points span a space of dimension i. Thus we have $\mathcal{C}^k = \bigcup_i \mathcal{S}_i$, for $0 \leq i \leq k-1$. All we need to prove is that in the neighborhood of any $z \in \mathcal{S}_i$, $\mathcal{S}_i$ has dimension at most $d-k+1$, for all $0 \leq i \leq k-1$. By Lemma 3, we already know that $\mathcal{S}_{k-1}$ is a $(d-k+1)$-dimensional manifold, so let us consider $i < k-1$. The idea of this proof is to define $\mathcal{S}_i$, in the neighborhood of z, as the intersection of a family of surfaces, and to compute the dimension of the intersection.

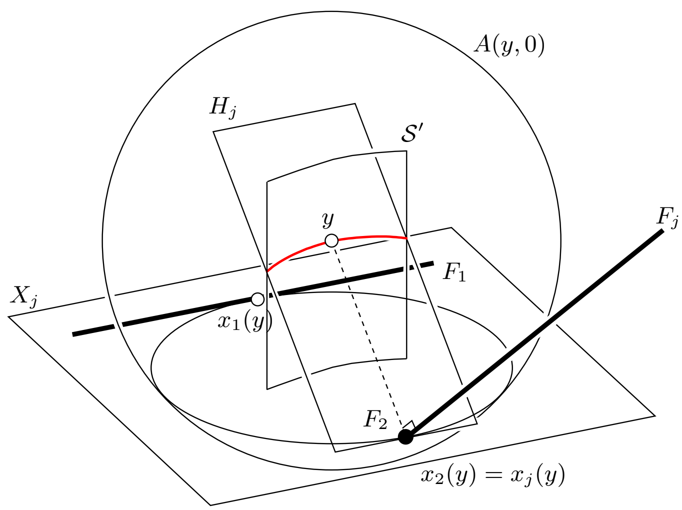

Figure 3: Notations for the proof of Lemma 5. $A(y, 0)$ is tangent to F_1, F_2 and F_j.

Let $F_1, \ldots, F_k$ be the k faces tangent to $A(z, \varepsilon)$. Using the same compactness argument as before, there exists a small neighborhood U of z in $\mathbb{R}^d$ such that for every point in $w \in \mathcal{S}_i \cap U$, the only faces possibly tangent to the annulus $A(w)$ are $F_1, \ldots, F_k$.

For $y \in \mathcal{S}_i \cap U$, let $x_i(y)$ be the orthogonal projection of y onto $\mathrm{Aff}(F_i)$ (see Figure

3). We can always reduce the size of U such that the tangency points $x_1(y), \ldots, x_k(y)$ span an affine space of dimension i; without loss of generality, let us assume that the first i tangency points $x_1(y), \ldots, x_i(y)$ are affinely independent. The first surface we will construct is locus of annulus centers tangent just to the faces $F_1, \ldots, F_i$. Specifically, let $P' = \mathrm{cl}(F_1) \cup \cdots \cup \mathrm{cl}(F_i)$ and write $\mathcal{S}'$ for the set of points which are the center of a hollow annulus (with respect to P') of penetration ε tangent to the i faces $F_1, \ldots, F_i$. By Lemma 3, $\mathcal{S}'$ is a $(d-i+1)$-manifold in a neighborhood of y. Clearly $\mathcal{S}_i \subseteq \mathcal{S}'$.

Now, for each of the other F_j, we will construct another surface containing y. For $i < j \leq k$, the point $x_j(y)$ is an affine combination of $x_1(y), \ldots, x_i(y)$ and therefore belongs to

$$X_j \quad = \quad \mathrm{Aff}(F_1 \cup \cdots \cup \widehat{F_j} \cup \cdots F_{k-1}).$$

Since $F_j \cap X_j$ contains the tangency point $x_j(y)$, it is not empty. We define H_j as the set of points $w \in \mathbb{R}^d$ such that the nearest point to w on $\mathrm{Aff}(F_j)$ lies in $\mathrm{Aff}(F_j \cap X_j)$, that is, the set of points that, like y, fall into $\mathrm{Aff}(F_j \cap X_j)$ when projected to $\mathrm{Aff}(F_j)$. H_j is an affine space, orthogonal to F_j, of dimension $d - \dim F_j + \dim(F_j \cap X_j)$.

So all y in the neighborhood of z lie in $\mathcal{S}' \cap H_{i+1} \cap H_{i+2} \cdots \cap H_k$. Let us prove that this intersection has dimension at most $d - k + 1$. In Lemma 3, we observed that the normal space to $\mathcal{S}'$ at y is spanned by the $i-1$ vectors $v_2 = x_1(y) - x_i(y), \ldots, v_i = x_{i-1}(y) - x_i(y)$. For $i + 1 \leq j \leq k$, we can always find a vector v_j in the normal space to H_j, by choosing v_j in the tangent plane to F_j and orthogonal to $F_j \cap X_j$. Because the F_j are independent faces, the vectors $v_2, \ldots, v_k$ are linearly independent and all belong to the normal space of the intersection $\mathcal{S}' \cap H_{i+1} \cap H_{i+2} \cap \cdots \cap H_k$. It follows that the intersection is a manifold of dimension at most $d-k+1$, and that $\mathcal{S}_i$ is a stratified space of dimension at most $d-k+1$. ◻

We deduce immediately the following corollary:

Corollary 6 *Let $z \in \mathcal{AM}^k$ and suppose that $A(z,\varepsilon)$ is tangent to j faces amongst which k faces are independent. Then, z lies on a i-dimensional stratum of $\mathcal{AM}^k$ with $i \leq d-k+1$.*

5 Trimmed annular medial axis

Although the subset of $\mathcal{AM}^k$ induced by independent faces has the "right" dimension $d-k+1$, it may not have finite volume. In this section we define a bounded $(d-k+1)$-dimensional subset of $\mathcal{AM}^k$, which we call the *trimmed annular k-medial axis* $\mathcal{TAM}^k$. We end this section by proving that the volume of $\mathcal{TAM}^k$ is bounded by a constant that does not depend on ε.

We will need some definitions. We say that a hyperplane *supports* $X \subseteq \mathbb{R}^d$ if it has non-empty intersection with the boundary of X and empty intersection with the interior of X.

Definition 7 *A point $z \in \mathcal{AM}^k$ is* non-essential *if there exists a hyperplane supporting the convex hull of P and containing all faces tangent to $A(z,\varepsilon)$. The point z is* essential *if it is not non-essential.*

An annulus $A(z,\varepsilon)$ may be essential because it is tangent to a face F not on the boundary of the convex hull, and/or because the union of faces tangent to $A(z,\varepsilon)$ spans

$\mathbb{R}^d$. The set of essential points is non-empty if and only if $\mathrm{Aff}(P) = \mathbb{R}^d$.

Definition 8 *The* trimmed annular k-medial axis*, $\mathcal{TAM}^k$, is the set of essential points lying on the i-dimensional strata of the annular k-medial axis for $i \leq d - k + 1$.*

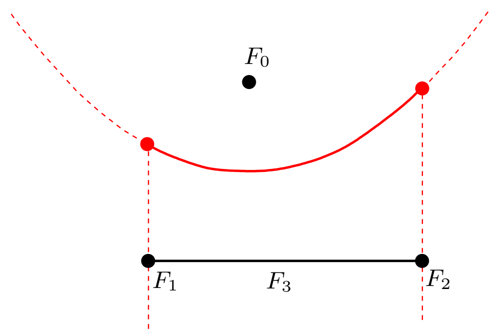

Figure 4: A polyhedron formed of four faces and its annular 2-medial axis, taking ε to be zero. The set of essential points forms the closed piece of parabola consisting of points equidistant to F_0 and F_3. The two endpoints are essential because the spheres centered at the endpoints are also tangent to F_3.

We will begin with two simple, and similar, technical lemmata.

Lemma 9 *Let A be an annulus tangent to a hyperplane H at point x and whose inner sphere does not enclose point q, with q and the center of A on the same side of H. Let R and r be respectively the outer and inner radii of A. Suppose that there exist two scalars D and $h > 0$ such that $d(q, H) \geq h$, $\|x - q\| \leq D$ and $R - r \leq \frac{h}{2}$. Then, the inner radius of A satisfies $r \leq \frac{D^2}{h}$.*

Proof. We first consider what happens when the inner sphere of A passes through q and point x lies on the outer sphere of A (see Figure 5). Let y be the intersection of the inner sphere of A with the segment connecting x to the center z of A. Let c be the

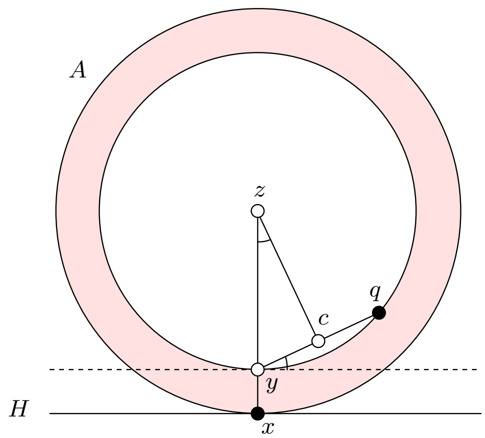

Figure 5: Notations for the proof of Lemma 9.

midpoint of the segment yq. Since the angle between the two vectors $c - z$ and $q - z$ is equal to the angle between the vector $q - y$ and the hyperplane H, we have

$$\frac{\|q - y\|}{2r} = \frac{d(q, H) - (R - r)}{\|q - y\|}$$

The bound on r follows immediately.

For the case in which x lies on the inner sphere of A, apply the same argument, assuming that $r = R$ and $x = y$. $\square$

Lemma 10 *Let $A(z, \varepsilon)$ be tangent to two hyperplanes H_i, H_j, at points x_i, x_j such that their normal vectors v_i, v_j, pointing towards z, have angle difference $\angle(v_i, v_j) > \alpha$. Let D be the diameter of P. Then the inner radius r of A is at most $\frac{D}{\sin \alpha/2}$.*

PROOF. If x_i is tangent to the larger sphere of $A(z, \varepsilon)$, we let y_i be the intersection point of segment x_i, z with the inner sphere; similarly we define y_j if x_j is tangent to the outer sphere. In any case, we consider the distance D' between the two points on the inner sphere, and note that $D' \leq D$. We have $\sin \alpha/2 = D'/2$, so that $r = d(y,z) \leq \frac{D}{2 \sin \alpha/2}$. ◻

We plan to show that the radius of any essential annulus can be bounded using one or the other of these technical lemmata. Let $\mathcal{F} = \{F_i\}_{i \in I}$ be the set of faces of P tangent to $A(z, \varepsilon)$. Each F_i is tangent to $A(z, \varepsilon)$ at a point x_i, and we define the normal v_i to $A(z, \varepsilon)$ at x_i to be the unit vector with direction of $z - x_i$, i.e. $v_i = (z - x_i)/||z - x_i||$.

We will choose a constant α later; it will depend only on P and not on z or ε. If there is a pair of normals v_i, v_j such that $\angle(v_i, v_j) > \alpha$, we bound the radius of $A(z, \varepsilon)$ using Lemma 10, with D taken to be the diameter of P. So it only remains to bound the size of an essential annulus, for which every pair of normals differs by at most α, which is done in Lemma 12.

We will begin by characterizing an annulus as essential or non-essential using only its set $\mathcal{F}$ of tangent faces, irrespective of z and ε. Let $\mathbb{S}^{d-1} = \{v \in \mathbb{R}^d \mid \|v\| = 1\}$ be the space of direction vectors in $\mathbb{R}^d$. To each face F_i of the polyhedron P, we associate the function $\delta_{F_i} : \mathbb{S}^{d-1} \to \mathbb{R}$ which maps every unit vector $v \in \mathbb{S}^{d-1}$ to

$$\delta_{F_i}(v) = \max\{\langle q - x, v \rangle \mid \forall x \in \mathrm{cl}(F_i),\, \forall q \in P\}.$$

Roughly speaking, $\delta_{F_i}(v)$ represents the distance between an extreme point in direction v on P and an extreme point in direction $-v$ on the closure of F_i. Notice that because $F_i \subset P$, $\delta_{F_i}(v) \geq 0$ for all v, since zero is achieved for any $x \in F_i$ by choosing $q = x$. In addition, we have $\delta_{F_i}(v) = 0$ exactly when every point $x \in F_i$ is an extreme point of P in direction v, that is, when there is a supporting plane of the convex hull of P

containing F_i. Finally, we observe that $\delta_{F_i}(v)$ is continuous, and, since it is defined on the compact space $\mathbb{S}^{d-1}$, it is uniformly continuous (by the Heine - Cantor Theorem).

We now consider any set $\mathcal{F} = \{F_i\}_{i \in I}$ of faces. Let

$$\delta_{\mathcal{F}}(v) = \max \sum_{i \in I} \delta_{F_i}(v).$$

As the maximum of a set of continuous non-negative functions, this is again a continuous non-negative function.

Observation 11 *The annulus $A(z, \varepsilon)$ with tangent face set $\mathcal{F}$ is non-essential if and only if there exists a unit vector v such that $\delta_{\mathcal{F}}(v) = 0$.*

PROOF. We have $\delta_{\mathcal{F}}(v) = 0$ if and only if $\delta_{F_i}(v) = 0$ for all i. This happens when the tangent plane to P in direction v contains all of the $F_i \in \mathcal{F}$. This means, by definition, that $A(z, \varepsilon)$ is non-essential. ◻

Thus, since $A(z, \varepsilon)$ is essential, we must have $\delta_{\mathcal{F}}(v) > 0$ for all unit vectors v. Since the map $\delta_{\mathcal{F}}$ is continuous and defined on a compact set, it attains a global minimum and this minimum is positive. We define

$$2h = \min_{\mathcal{F}} \min_{v} \delta_{\mathcal{F}}(v),$$

where v ranges over all unit vectors and $\mathcal{F}$ ranges over all subsets of faces such that $\mathcal{F}$ is not contained in a hyperplane supporting the convex hull of P. We have $h > 0$. We use this constant, dependent on P, in the following theorem.

Lemma 12 *There exists a constant α, dependent on the geometry of P but not on ε, such that for any essential annulus $A(z, \varepsilon)$ for which every pair v_i, v_j of its normal* *vectors has angle distance $\angle(v_i, v_j) < \alpha$, the distance $d(z, P) < \frac{D^2}{h}$, where D is the diameter of P.*

PROOF. From the fact that $A(z, \varepsilon)$ is essential, we can conclude that for any of the $F_i \in \mathcal{F}$, there is some v_j such that $\delta_{F_i}(v_j) > 2h$. If $\delta_{F_i}(v_i) > 2h$, we are done, but this is not guaranteed to be the case. So now we use the fact that we assume that the v_j are all similar. Since δ_{F_i} is uniformly continuous, we can select an α_{F_i} (again depending on F and not on z or ε), such that

$$\angle(v_i, v_j) < \alpha_{F_i} \implies |\delta_{F_i}(v_i) - \delta_{F_i}(v_j)| < h.$$

We define $\alpha = \min_F \alpha_F$ over all faces F of the polyhedron.

With this choice of α, the fact that $\delta_{F_i}(v_j) > 2h$ implies that $\delta_{F_i}(v_i) > 2h - h = h$. This in turn implies that there exists a point $q \in P$ such that $d(q, H_i) \geq h$, where H_i is the hyperplane containing F_i with normal v_i. So we can apply Lemma 9 to bound the inner radius of $A(z, \varepsilon)$. $\square$

Combining Lemma 10 and Lemma 12, we conclude that the set of essential points is bounded and at distance at most $\max\left\{\frac{D}{\sin \alpha/2}, \frac{D^2}{h}\right\}$ from P. Let $B(P)$ be the smallest ball containing the set of essential points. The $\mathcal{TAM}^k$ is a stratified set, of dimension at most $d - k + 1$, in $B(P)$. Its i-dimensional stratum, for $1 \leq i \leq d - k + 1$, is the union of pieces of semi-algebraic sets, each of dimension i, formed by the intersection of at least k polynomials, each of degree at most two. The number of such pieces is bounded by a function of P (corresponding to the choices of at least k faces producing the polynomials), that is independent of ε. Each piece is also trimmed by $B(P)$. The i-dimensional volume of the intersection of a ball of radius R with an algebraic variety of dimension i formed by m polynomials of bounded degree is bounded by a function of i, m and R (see [5], Lemma 7.4). So the overall i-dimensional volume of the stratum is at most the sum of these trimmed varieties, and hence is itself bounded by some function of the geometry of P, independent of ε.

6 Mapping Delaunay spheres to $\mathcal{TAM}^k$

The goal of this section is to assign every Delaunay sphere Σ to some point z on the trimmed annular k-medial axis. We begin with a quick geometric lemma.

Lemma 13 *A Delaunay sphere Σ with center z_1 is contained in the annulus $A(z_1, \varepsilon)$.*

PROOF. Let x be a point in P with minimum distance to z_1 and let s be the sample point in $S \cap \mathrm{cl}(F_x)$ closest to x. The distance $\|x - s\| \leq \varepsilon$ and therefore $s \in A(z_1, \varepsilon)$. Because Σ encloses no sample point, $\Sigma \subseteq A(z_1, \varepsilon)$. ◻

It is this lemma which necessitates the restrictive sampling requirement that the faces of all dimension must be sampled; if the lower-dimensional subfaces of some face f were not sampled, it is possible that a hollow annulus with center z_1 could contain a large portion of the boundary of f, extending far away from the projection x of z_1 onto a subface of f.

We use the following incremental construction to associate an annulus called $\mathrm{Expansion}(\Sigma)$ with each Delaunay sphere Σ; we will then prove that the center z of $\mathrm{Expansion}(\Sigma)$ lies in $\mathcal{TAM}^k$, and that $\Sigma \cap P$ is contained in a set that we shall call the cover of z, the definition of which depends on d, p and ε.

Recall that we assume that P spans $\mathbb{R}^d$, and that p is the dimension of P. Initially, we let z_1 be the center of Σ. The inner sphere $B(z_1, r)$ of the annulus $A(z_1, \varepsilon)$ is tangent to P in at least one point x, contained in a face $F_1 = F_x$. At each step j of the construction we will perturb the annulus to find another tangent face, and we increase the dimension of the subspace spanned by $\{F_1 \dots F_j\}$ by at least one.

At an arbitrary step of the construction, we have a set $\{F_1, \dots, F_j\}$ of j faces tangent to $A(z_j, \varepsilon)$. If the dimension of $\mathrm{Aff}(\{F_1, \dots, F_j\}) < d$, we can find a hyperplane H that contains their union $\bigcup_{i=1}^{j} F_i$ (possibly there are many such H; we can pick one arbitrarily). Let L be the line orthogonal to H and containing the current z_j. Consider any annulus $A(y, \varepsilon)$ centered on L in the neighborhood of z_j. Since $A(y, \varepsilon)$ also has penetration ε, its intersection $A(y, \varepsilon) \cap H = A(z_j, \varepsilon) \cap H$, and $A(y, \varepsilon)$ remains tangent to $F_1, \dots, F_j$. Consider the y, nearest to z_j, such that $A(y, \varepsilon)$ is tangent to an additional face F_{j+1}, not contained in H. Such a y must exist, since P spans $\mathbb{R}^d$. We set $z_{j+1} = y$.

When there is a set of faces $F_1, \dots, F_j$ tangent to $A(z_j, \varepsilon)$ that spans $\mathbb{R}^d$, we stop. Clearly this will occur after at most d steps. We associate Σ with the final annulus: define $\mathrm{Expansion}(\Sigma) = A(z_j, \varepsilon)$, for the final z_j.

Lemma 14 *For every Delaunay sphere Σ, the center z of the annulus* $\mathrm{Expansion}(\Sigma)$ *belongs to the trimmed annular k-medial axis $\mathcal{TAM}^k$.*

PROOF. Since we know that there is a set of faces $\{F_1 \dots F_j\}$ tangent to $A(z, \varepsilon) = \mathrm{Expansion}(\Sigma)$ that span $\mathbb{R}^d$, we know that they cannot all be contained in a hyperplane supporting the convex hull, and this implies that z is essential. The crux of the proof is showing that $A(z, \varepsilon)$ is tangent to at least k independent faces. This will establish that

z belongs to the $(d-k+1)$-dimensional stratum of $\mathcal{AM}^k$, and hence to $\mathcal{TAM}^k$.

Initially, since any single face is independent, the set $\{F_1\}$ is independent. The dimension of $\mathrm{Aff}(\{F_1 \ldots F_{j+1}\})$ is at least one greater than the dimension of $\mathrm{Aff}(\{F_1 \ldots F_j\})$, since $\mathrm{Aff}(F_1 \ldots F_j) \subset H$, and F_{j+1} is not contained in H. This also implies that at each step F_{j+1} is independent of $F_1 \ldots F_j$. However, it is possible that some F_i in $\{F_1 \ldots F_j\}$ may be spanned by some subset of $\{F_1 \ldots F_{j+1}\}$, for instance if $F_i \subseteq F_{j+1}$. Any such F_i can be removed, with $\mathrm{Aff}(\{F_1 \ldots F_{j+1}\} - F_i) = \mathrm{Aff}(\{F_1 \ldots F_{j+1}\})$. Let $\mathcal{F}$ be the remaining set of independent faces.

We claim that at the end of the construction, there are at least k independent faces in $\mathcal{F}$, for $k = \lceil \frac{d+1}{p+1} \rceil$. Consider adding each F_i to the $\mathcal{F}$ in turn. Each face has dimension at most p, so that adding F_i increases the dimension of the union $\bigcup\{F_1 \ldots F_{i-1}\}$ by at most $p+1$ (adding to the subspace basis a vector from the previous subspace $\mathrm{Aff}(F_1 \ldots F_{i-1})$ to some point in F_i, and then a set of $\leq p$ vectors spanning F_i). So the number of independent faces required to span all of d-dimensional space is at least $k = \lceil \frac{d+1}{p+1} \rceil$.

This establishes that the final $A(z, \varepsilon)$ is tangent to at least k independent faces. $\square$

7 Covering Delaunay simplices

Now we would like to relate the intersection $\mathrm{Expansion}(\Sigma) \cap P$ back to the original Delaunay sphere Σ. We do this by defining the *cover* of a point z (roughly its region of influence on P), and then showing that the intersection $\Sigma \cap P$ is contained in the cover of the annulus $\mathrm{Expansion}(\Sigma)$ centered at a point $z \in \mathcal{TAM}^k$.

So lets begin by defining the cover of a point z. Write $\pi_x(z)$ for the orthogonal projection of z onto the tangent plane of $x \in P$ and let $\Pi(z) = \{\pi_x(z) \mid x \in P\}$ be the set of orthogonal projections of z onto the planes supporting faces of P. For any non-negative number w, called the *radius* of the cover, we define the w-cover:

$$\mathrm{Cover}(z, w) = \bigcup_{x \in \Pi(z)} B(x, w).$$

We say that x is an anchor point of z if $x \in A(z, \varepsilon)$ and $\pi_x(z) = x$. Thus, anchor points of z form a subset of $\Pi(z)$ that contains the tangency points of the annulus with the polyhedron and possibly other points of $P \cap A(z, \varepsilon)$. Now we establish that any point in $P \cap A(z, \varepsilon)$ must be close to an anchor point of z.

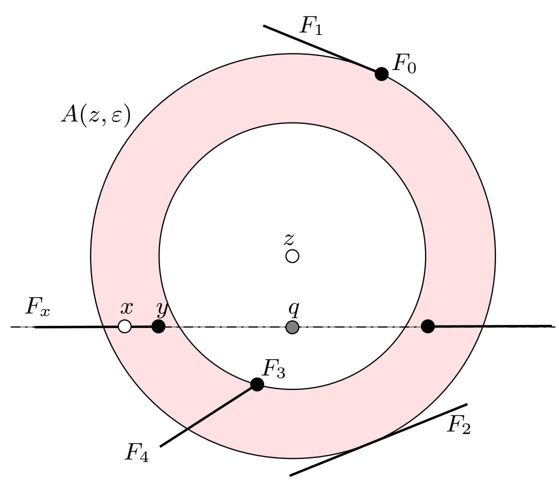

Figure 6: The annulus is tangent to the four faces F_0, F_1, F_2 and F_3. Notations for the proof of Lemma 15.

Lemma 15 *For every point $x \in P \cap A(z, \varepsilon)$ there exists an anchor point y of z in the closure of F_x such that*

$$\|x - y\| \leq (\dim F_x - \dim F_y + 1)\, \varepsilon.$$

PROOF. The proof is by induction over the dimension $d_x = \dim F_x$ of the face F_x containing x. If $d_x = 0$, the result holds for $y = x$. Suppose $d_x > 0$ and let $q = \pi_x(z)$ be the orthogonal projection of z onto $\mathrm{Aff}(F_x)$. We distinguish two cases: (1) if $q \in F_x$, the segment xq lies inside $A(z, \varepsilon)$ and therefore $\|x - q\| \leq \varepsilon$; and the result holds for $y = q$. (2) If $q \notin F_x$, we consider the point $y \in P$ on the segment xq, which is closest to x on the boundary of F_x (as in Figure 6). Since the segment xy is contained in $A(z, \varepsilon)$, this implies $\|x - y\| \leq \varepsilon$. Furthermore, since y belongs to the boundary of F_x, the dimension $d_y < d_x$. We apply our induction hypothesis: $\mathrm{cl}(F_y)$ contains an anchor point w of z, with $\|y - w\| \leq (\dim F_y - \dim F_w + 1)\, \varepsilon$, so $\|x - w\| \leq \varepsilon + (\dim F_y - \dim F_w + 1)\, \varepsilon$, which gives the result. ◻

The preceding Lemma shows that any point in $P \cap A(z, \varepsilon)$ is at distance at most $(p+1)\varepsilon$ to an anchor point of z. Thus, $P \cap A(z, \varepsilon) \subseteq \mathrm{Cover}(z, (p+1)\varepsilon)$. From there, we show how to cover a Delaunay sphere using an annulus centered on the trimmed medial axis.

Lemma 16 *For every Delaunay sphere Σ, we have $\Sigma \cap P \subseteq \mathrm{Cover}(z, 4d\varepsilon)$ where z is the center of* $\mathrm{Expansion}(\Sigma)$.

PROOF. Again, this follows from the construction of $A(z_j, \varepsilon) = \mathrm{Expansion}(\Sigma)$. Initially, z_1 is the center of Σ. By Lemma 13, $\Sigma \cap P \subset A(z_1, \varepsilon)$, and by Lemma 15, this implies that for every point $x_0 \in \Sigma \cap P$, there exists an anchor point x_1 of z such that $\|x_0 - x_1\| \leq (\dim F_{x_0} - \dim F_{x_1} + 1)\varepsilon$. In particular, $\Sigma \cap P \subseteq \mathrm{Cover}(z_1, (p+1)\varepsilon)$. Now we need to show that as we proceed from z_i to z_{i+1}, $\Sigma \cap P$ remains covered for $1 \leq i < j$. So we will construct a sequence of j points $x_1, x_2, \ldots, x_j$ such that x_{i+1}

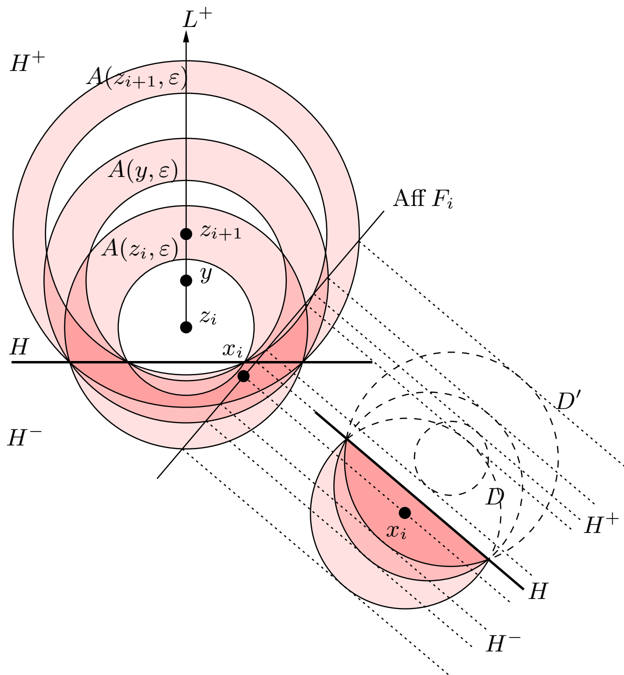

Figure 7: On the upper left, three annuli centered at z_i, y and z_{i+1} that share the same intersection with a hyperplane H. On the lower right, intersection of the three annuli with the tangent space passing through x_i. The restriction of those intersections to H^- are nested.

is an anchor point of z_{i+1} and

$$\|x_i - x_{i+1}\| \leq (\dim F_{x_i} - \dim F_{x_{i+1}} + 2)\varepsilon,$$

for all $1 \leq i < j$. For simplicity, we write $F_i = F_{x_i}$ and $d_i = \dim F_i$. Let H be the hyperplane used during the construction of z_{i+1} and which contains the union $\bigcup_{l=1}^{i} F_l$. Let H^+ be the closed half-space bounded by H in the direction $z_{i+1} - z_i$, with H^- being the open half space on the other side. We consider two cases:

1. The original anchor point $x_i \in H^+$, including the case in which $x_i \in H$. Let R_i be the outer radius of $A(z_i, \varepsilon)$. The intersection $H^+ \cap B(z_i, R_i) \subseteq H^+ \cap$

$B(z_{i+1}, R_{i+1})$, and x_i is of course outside both inner balls, so x_i remains in the annulus $A(z_{i+1}, \varepsilon)$. It may no longer be an anchor point of z_{i+1}, but by Lemma 15, there exists an anchor point x_{i+1} of z_{i+1} on the closure of F_i such that $\|x_i - x_{i+1}\| \leq (d_i - d_{i+1} + 1)\,\varepsilon$.

2. The case in which $x_i \in H^-$ is illustrated in Figure 7. Let us begin by considering the intersections $D = \mathrm{Aff}(F_i) \cap A(z_i, \varepsilon)$ and $D' = \mathrm{Aff}(F_i) \cap A(z_{i+1}, \varepsilon)$. Since x_i is an anchor point contained in $A(z_i, \varepsilon)$, the intersection D is a d_i-dimensional ball of radius at most ε, with center x_i. The annulus is shrinking on this side of H, implying $D' \cap H^- \subseteq D \cap H^-$. It is not possible for $D' \cap \mathrm{cl}(F_i)$ to be empty, since that would require there has been some last point at which the annulus $A(y, \varepsilon)$, with y on the segment connecting z_i and z_{i+1}, was in contact with $\mathrm{cl}(F_i)$, contradicting the choice of z_{i+1} as the annulus with the first new tangent point. Since $D' \cap \mathrm{cl}(F_i)$ is non-empty, it must contain some point p and by Lemma 15, there exists an anchor point x_{i+1} of z_{i+1} on the closure of F_i such that $\|p - x_{i+1}\| \leq (d_i - d_{i+1} + 1)\,\varepsilon$. Since $p \in D' \subseteq D$ and D is a ball centered at x_i with radius at most ε, we have $\|x_i - p\| \leq \varepsilon$. Triangular inequality yields the desired inequality.

To summarize, for every point $x_0 \in \Sigma \cap P$, we have constructed a sequence of points $x_1, x_2, \ldots, x_j$ such that x_i is an anchor point of z_i for all $1 \leq i \leq j$, $\|x_0 - x_1\| \leq (d_0 - d_1 + 1)\varepsilon$ and $\|x_i - x_{i-1}\| \leq (d_i - d_{i-1} + 2)\varepsilon$. Thus, $\|x_0 - x_j\| \leq (d_0 - d_j + 1 + 2(j-1))\varepsilon$. Since the number of steps in the construction is bounded by d, i.e. $j - 1 \leq d$, we get $\|x_0 - x_j\| \leq (3d + 1)\varepsilon$ yielding the result. ◻

8 Size of Delaunay triangulation

To establish the upper bound on the number of Delaunay simplices, it remains only to combine what we know about the size of $\mathcal{TAM}^k$ with our method of mapping Delaunay simplices to $\mathcal{TAM}^k$.

We first consider a sample M of the trimmed annular k-medial axis $\mathcal{TAM}^k$, such that every point $x \in \mathcal{TAM}^k$ has a sample within distance ε, and such that the number of samples in a ball of radius $O(\varepsilon)$ centered at x is at most λ (unlike our original sample S of P, it is not necessary to sample the lower-dimensional strata of $\mathcal{TAM}^k$). The size $m = |M|$ is $O(\varepsilon^{-(d-k+1)})$, with $k = \lceil \frac{d+1}{p+1} \rceil$. This follows from the fact that the dimension of $\mathcal{TAM}^k$ is $d - k + 1$, and the results in Section 5, which established that the volume of $\mathcal{TAM}^k$ is bounded by a constant that does not depend on ε.

Next, we map each Delaunay simplex $\sigma \in \mathrm{Del}(S)$ to a point $z \in M$. Consider the Delaunay sphere Σ passing through the vertices of σ. Lemma 14 and Lemma 16 tell us that the Delaunay sphere Σ associated with σ belongs to $\mathrm{Cover}(z, 4d\varepsilon)$ for some point $z \in \mathcal{TAM}^k$, where z is an arbitrary point not belonging to M. But there must be some $z' \in M$ at distance at most ε from z.

Lemma 17 *For every points z and z' with $\|z - z'\| \leq \varepsilon$:*

$$\mathrm{Cover}^{w\varepsilon}(z) \subseteq \mathrm{Cover}^{(w+1)\varepsilon}(z').$$

PROOF. Recalling that $\pi_x(z)$ is the orthogonal projection of z onto the tangent plane to P at x, we have $\|\pi_x(z) - \pi_x(z')\| \leq \|z - z'\| \leq \varepsilon$. The claim follows immediately. $\square$

Using Lemma 17 and Lemma16, we get that for $d \geq 1$ there exists a point $z' \in M$

such that

$$\Sigma \cap P \subseteq \mathrm{Cover}(z', 5d\varepsilon)$$

The cover of z' is a union of d-balls, each with radius $(5d\varepsilon$, at most one for each face of the polyhedron, and therefore it contains a constant number of points of S. So the number of simplices that we can form by picking points in the cover of z' is also a constant. This means that only a constant number of Delaunay simplices can be charged to each point $z' \in M$, and the size of the Delaunay triangulation is proportional to the size m of M.

Recall that the number of points in a λ-sparse ε-sample S of a p-dimensional polyhedron P is $n = \Theta(\epsilon^{-p})$ and that the i-faces of P have $\Theta(\varepsilon^{-i})$ points of S. Using $n = \Omega(\varepsilon^{-p})$, we get that the number of Delaunay simplices is

$$O(m) = O(\varepsilon^{-(d-k+1)}) = O(n^{\frac{d-k+1}{p}}),$$

where $k = \lceil \frac{d+1}{p+1} \rceil$.

9 Conclusion

This paper answers only the first of many possible questions about the complexity of the Delaunay triangulations of points distributed nearly uniformly on manifolds. Similar bounds for smooth surfaces rather than polyhedra would be of more practical interest. The proof in this paper seems to rely critically on some properties specific to polyhedra, particularly that sample points on k faces are needed to form a simplex, so other techniques will be needed for the cases of more general manifolds of dimension $1 < p < d$.

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