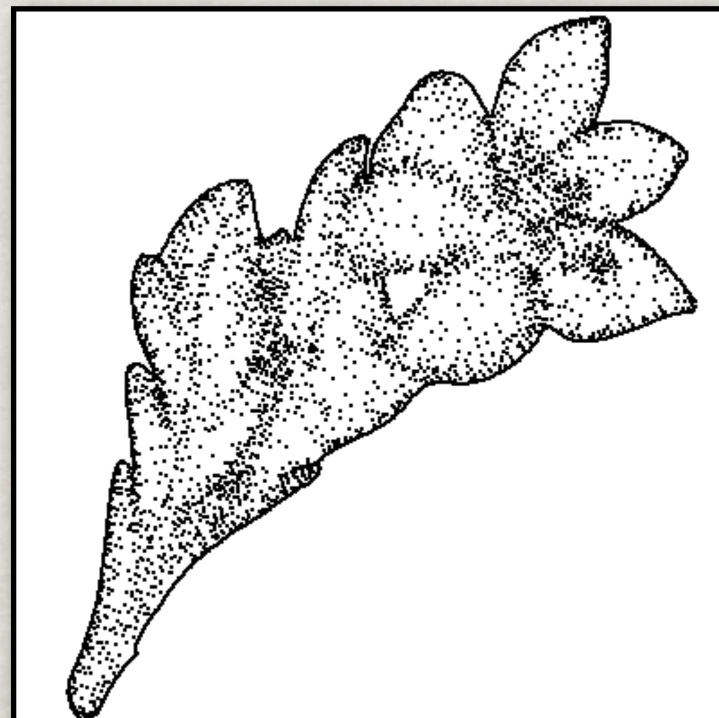
Reconstructing shapes with Guarantees by unions of Convex Sets

D. Attali & A. Lieutier

Shape Reconstruction

Shape

INPUT



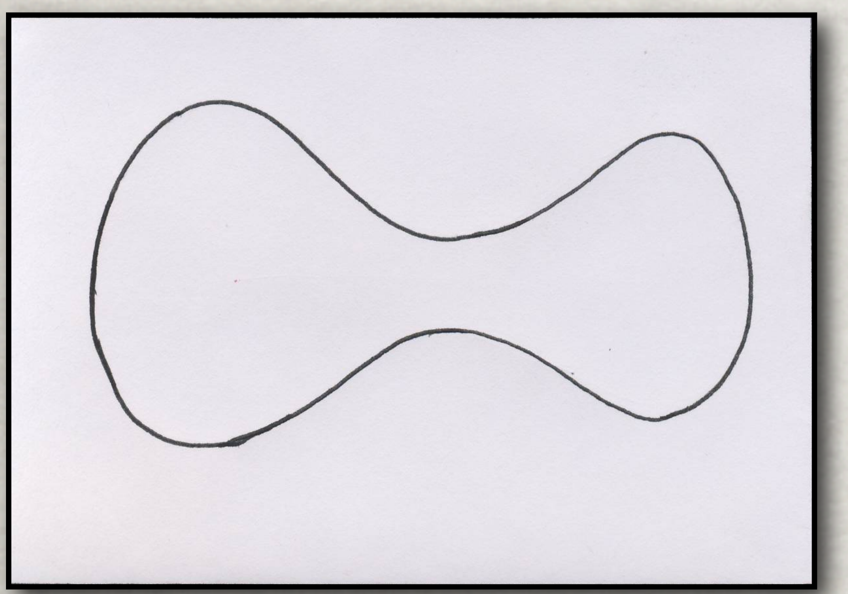
Sample

OUTPUT

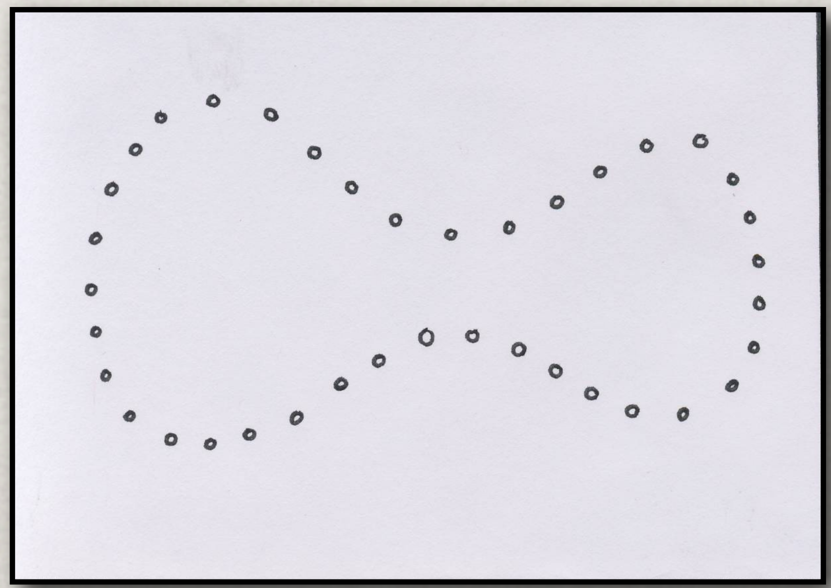
- **geometrically accurate**
- **topologically correct**

A Simple Algorithm

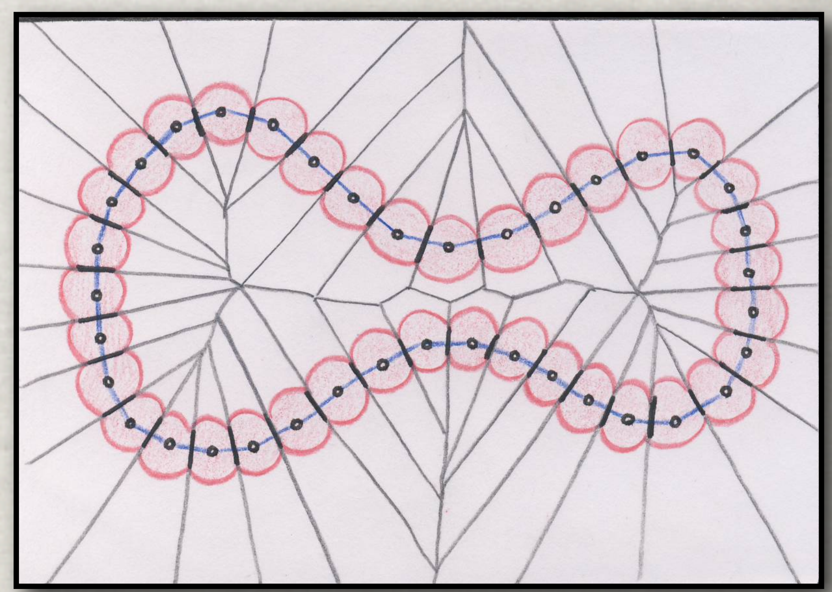
INPUT **OUTPUT**



Shape



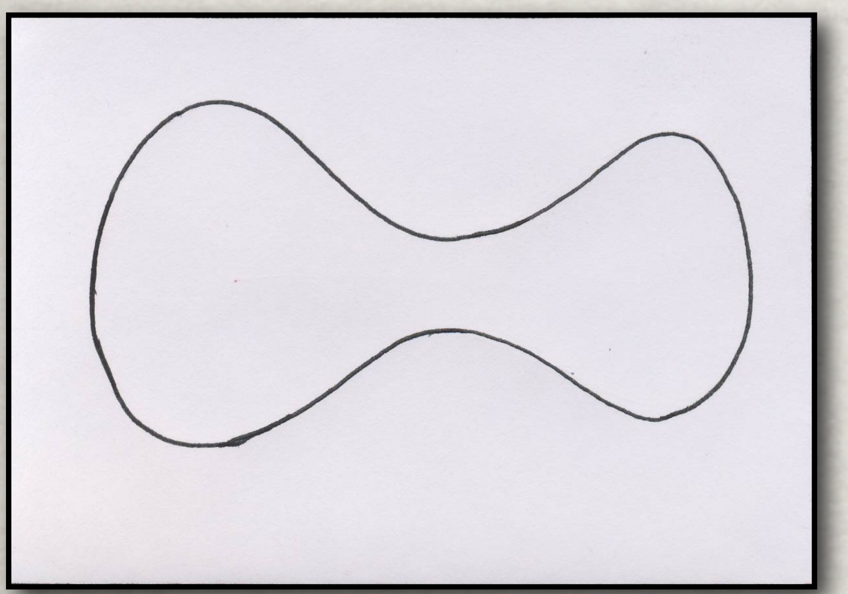
Sample



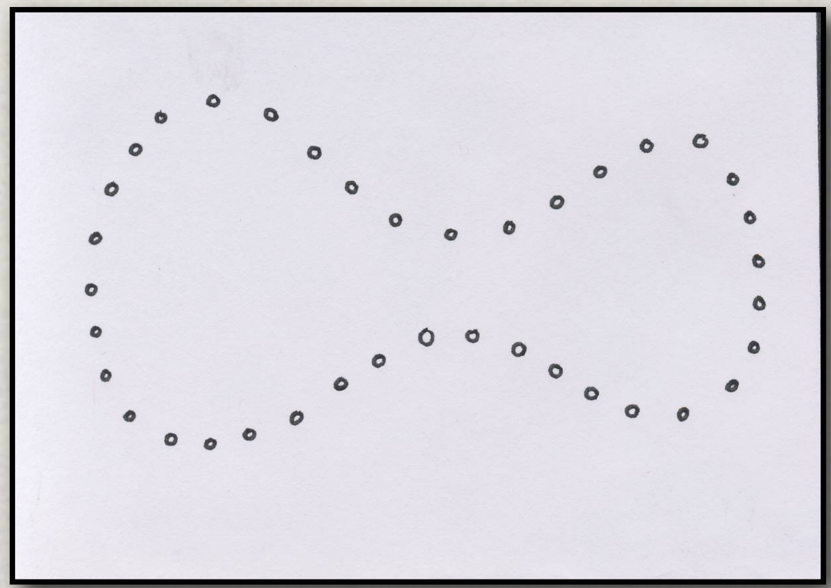
α-offset = union of balls with radius α centered on the sample

A Simple Algorithm

INPUT **OUTPUT**



Shape



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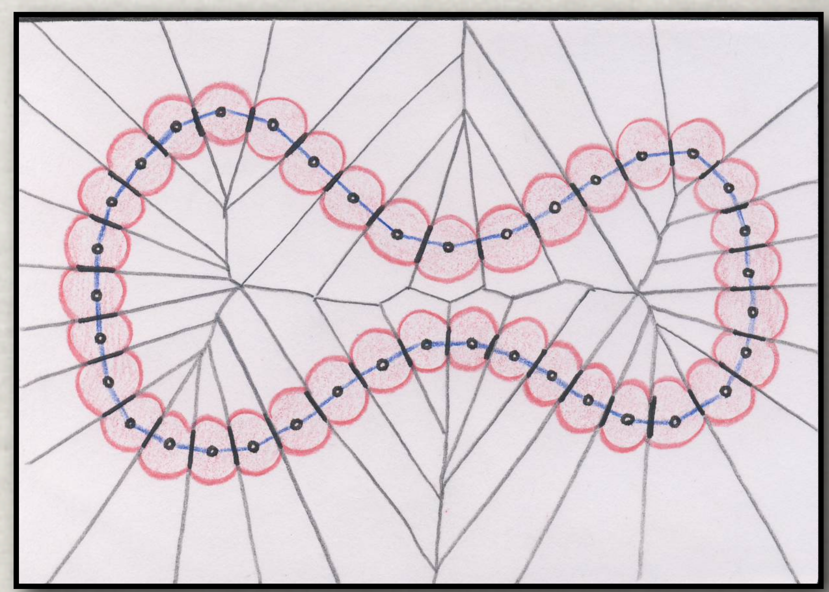
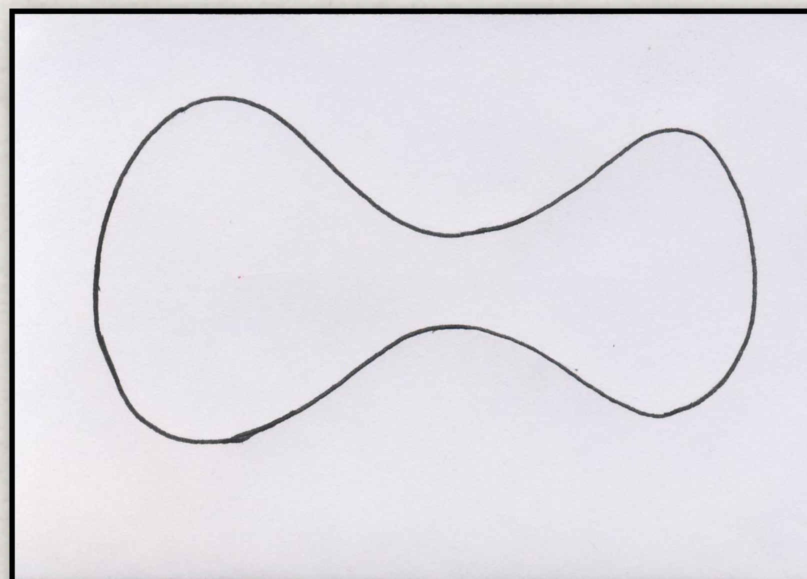
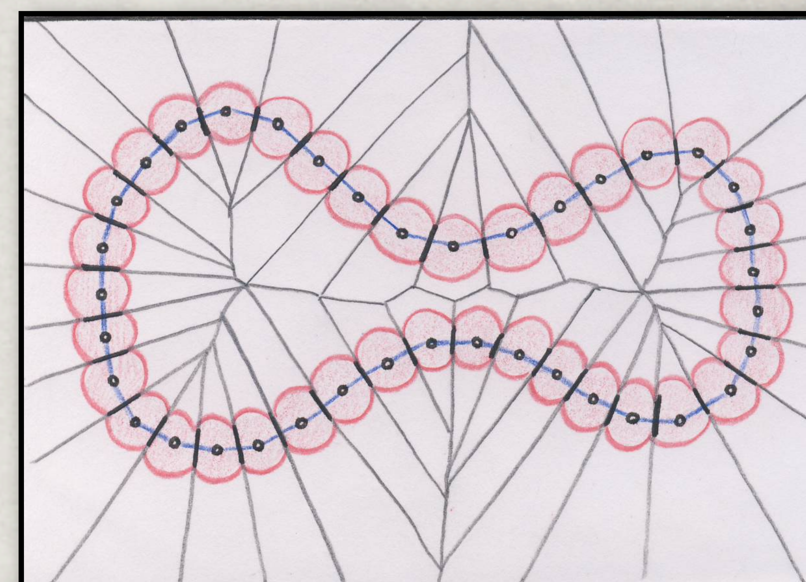


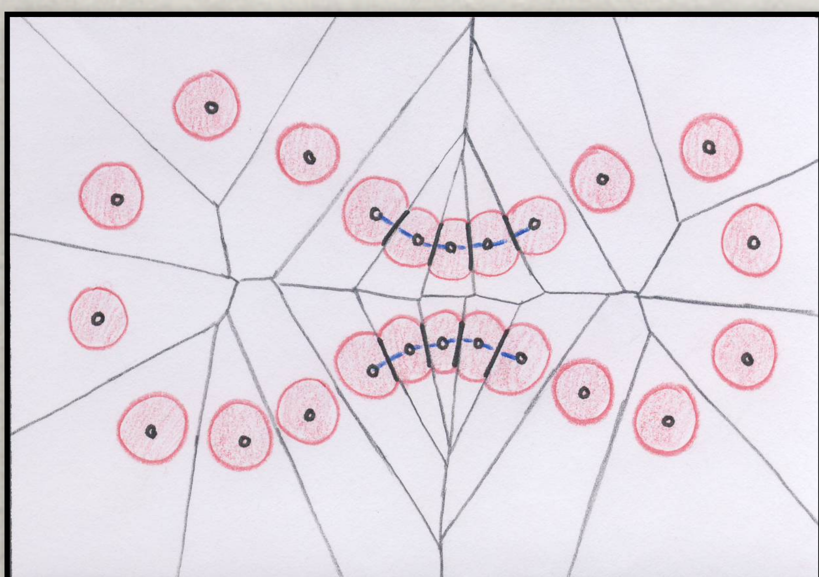
α**-offset** $\simeq$ α**-complex**

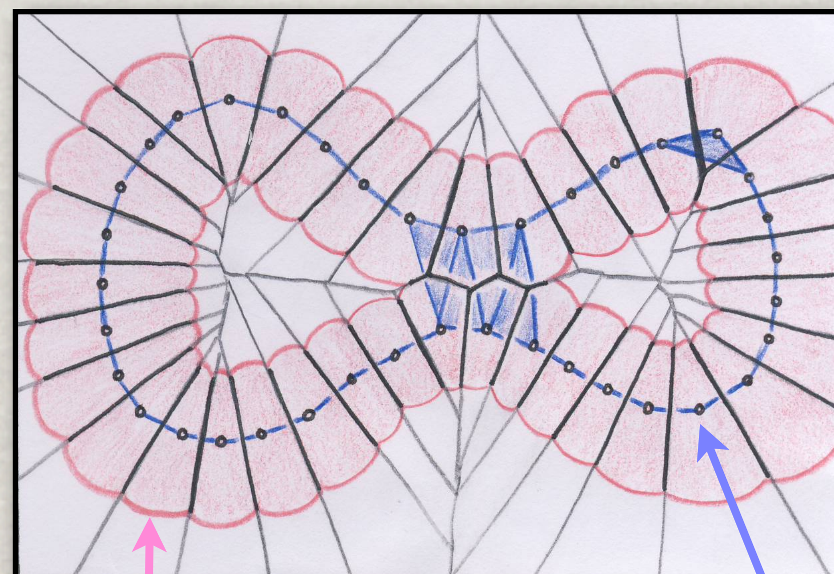
A Simple Algorithm

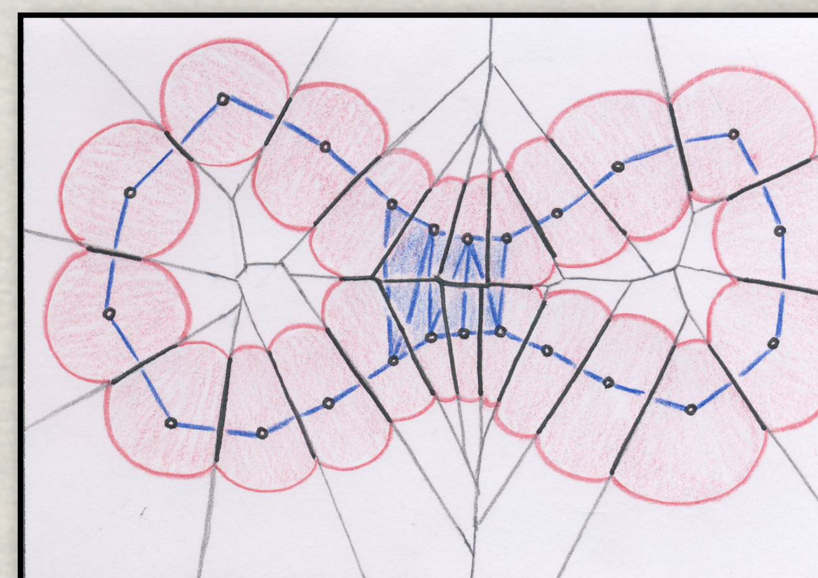
Shape









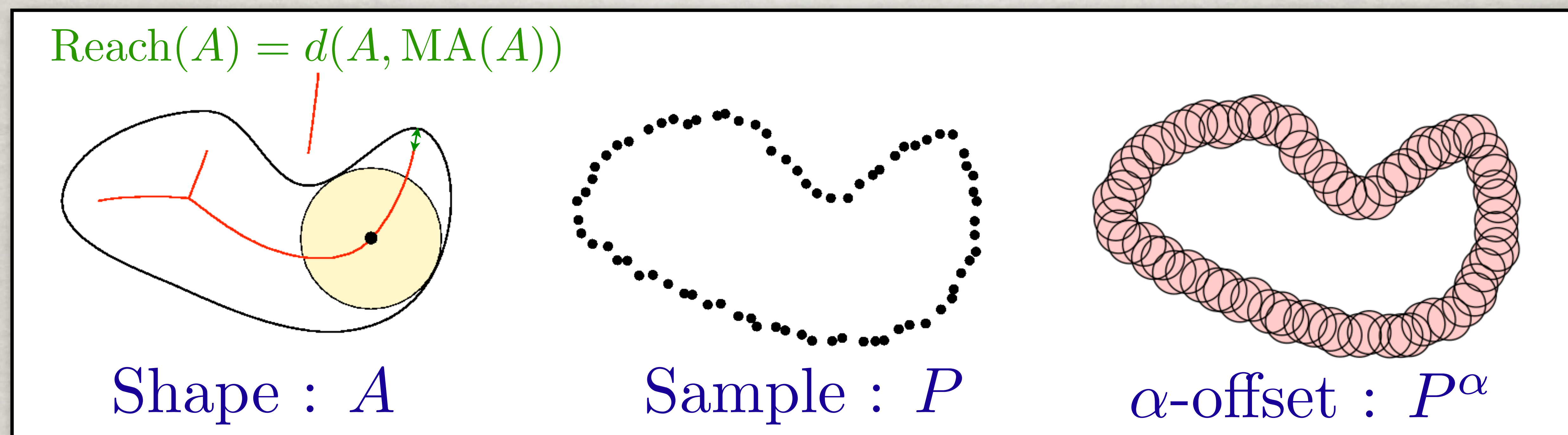


OUTPUT: $\boxed{\alpha\text{-offset}}$ $\simeq$ $\boxed{\alpha\text{-complex}}$

A Simple Algorithm

- [Niyogi Smale Weinberger 2004][Lieutier Chazal 2005]
  Sampling conditions:

P^α deformation retracts to A

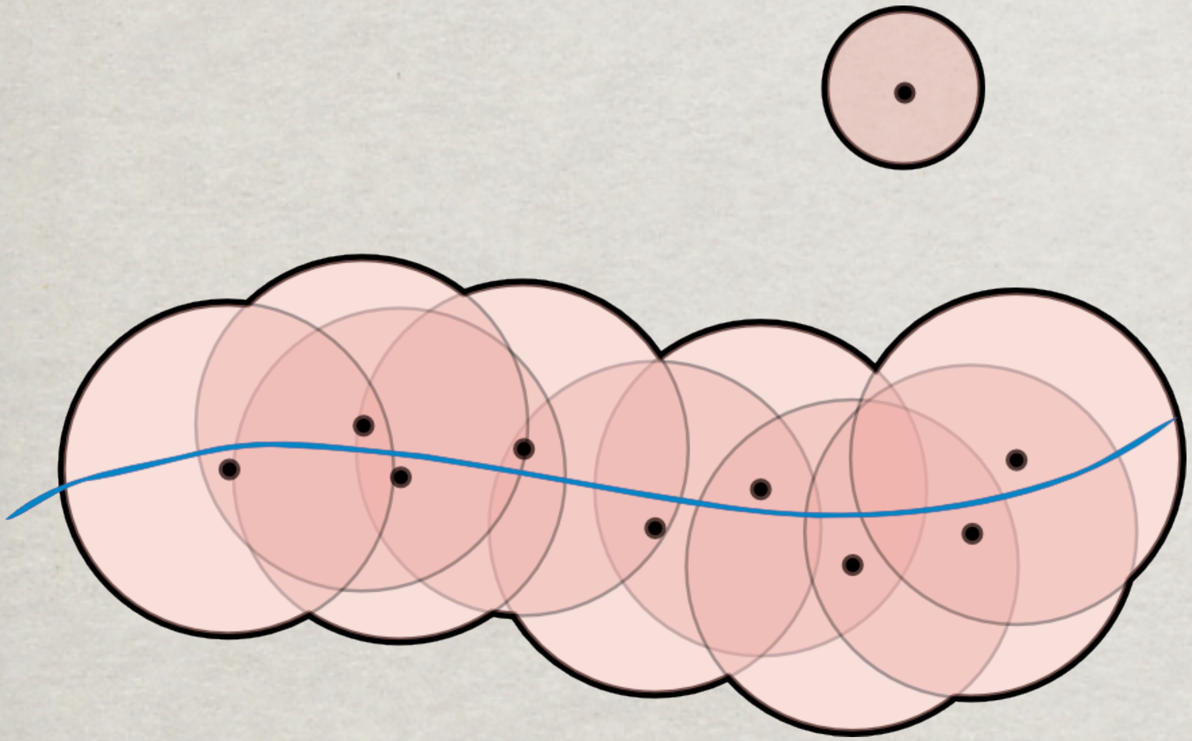


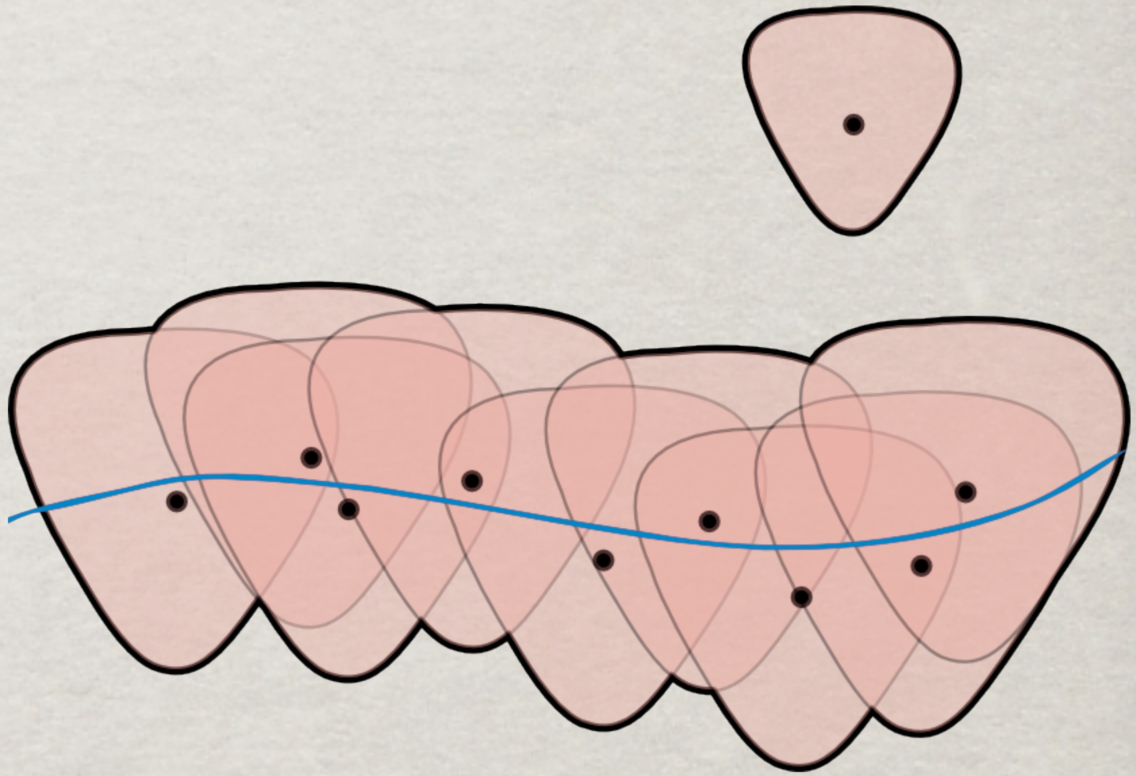
$$d_H(A, P) < (3 - \sqrt{8})\,\mathrm{Reach}(A) \approx 0.17\,\mathrm{Reach}(A)$$

$$\alpha = (2 + \sqrt{2})\,d_H(A, P)$$

Goal of this Work

- Understand how this result extends:



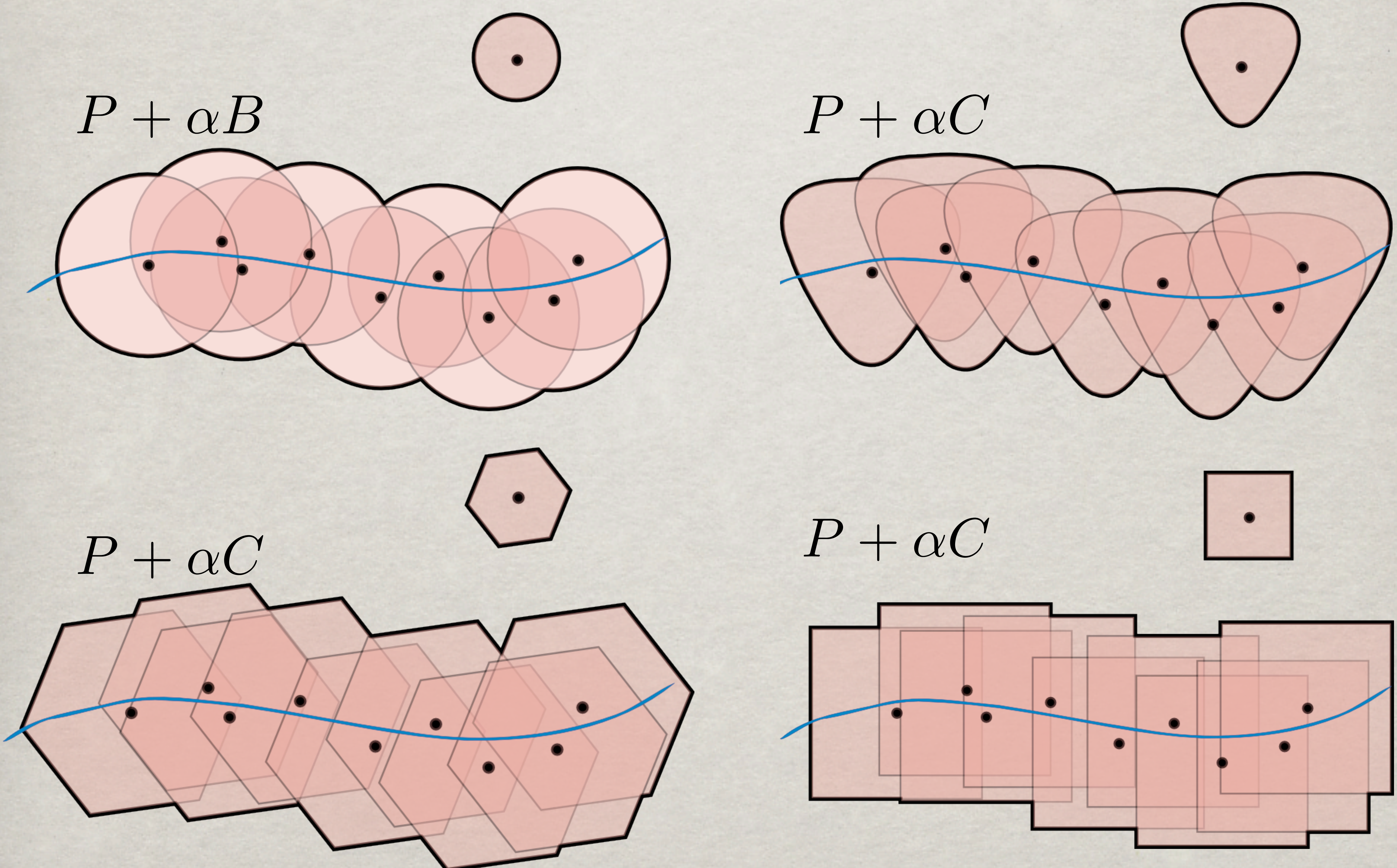


$\boxed{\alpha\text{-offset}} \longrightarrow \boxed{\textbf{Minkowski sum}}$

$$P^\alpha = P + \alpha B \qquad\qquad P + \alpha C$$

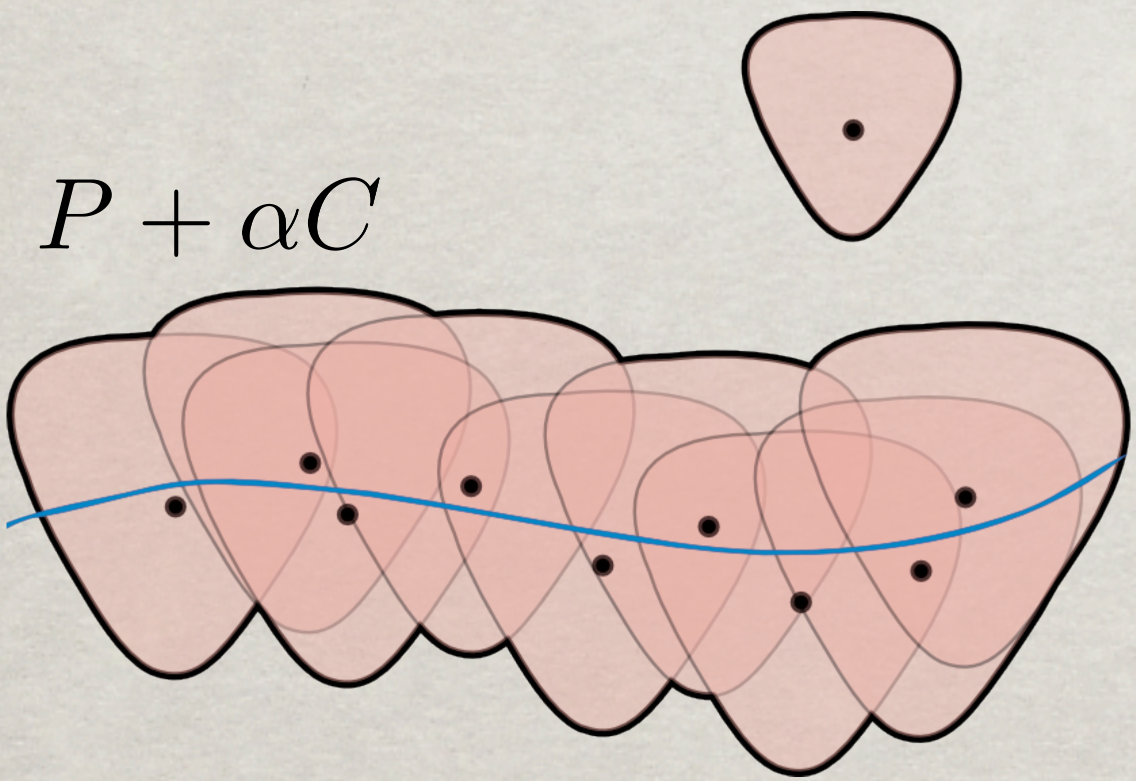
B: Euclidean ball $\qquad$ C: Convex set

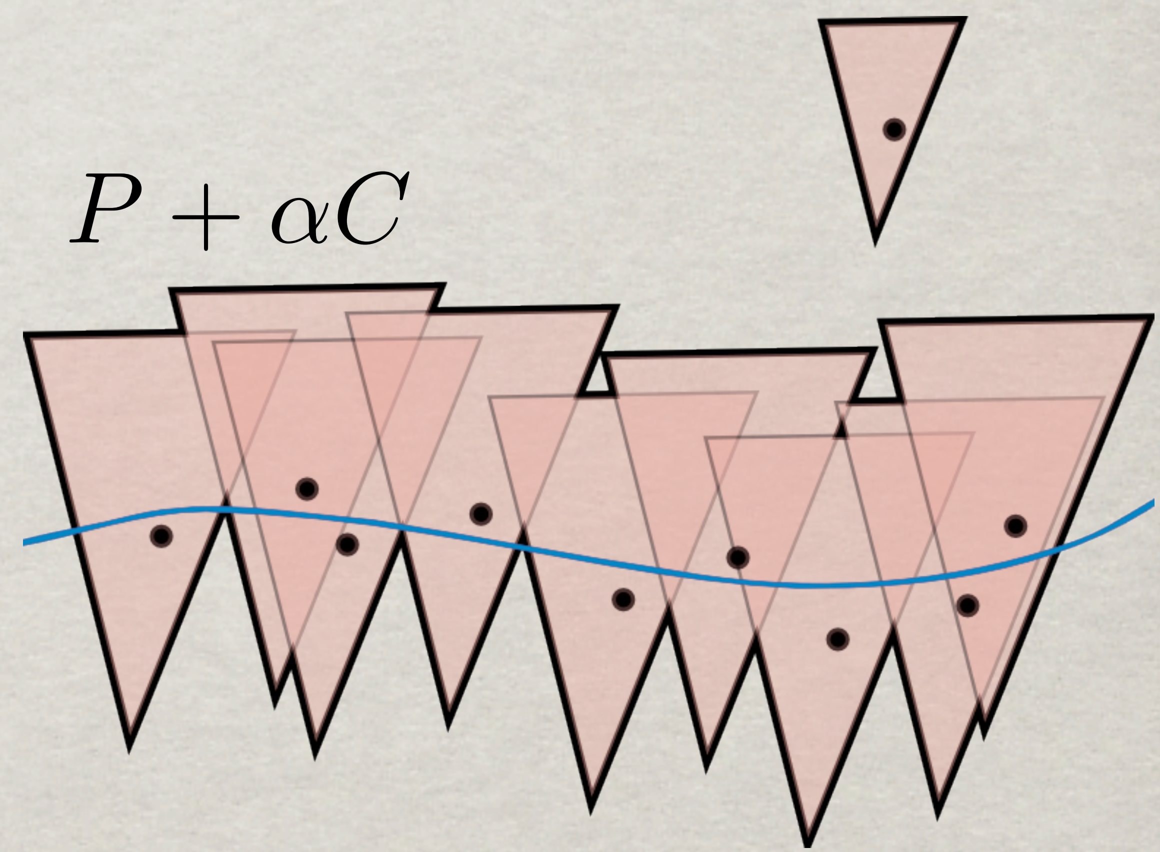
Minkowski Sums



Minkowski Sums

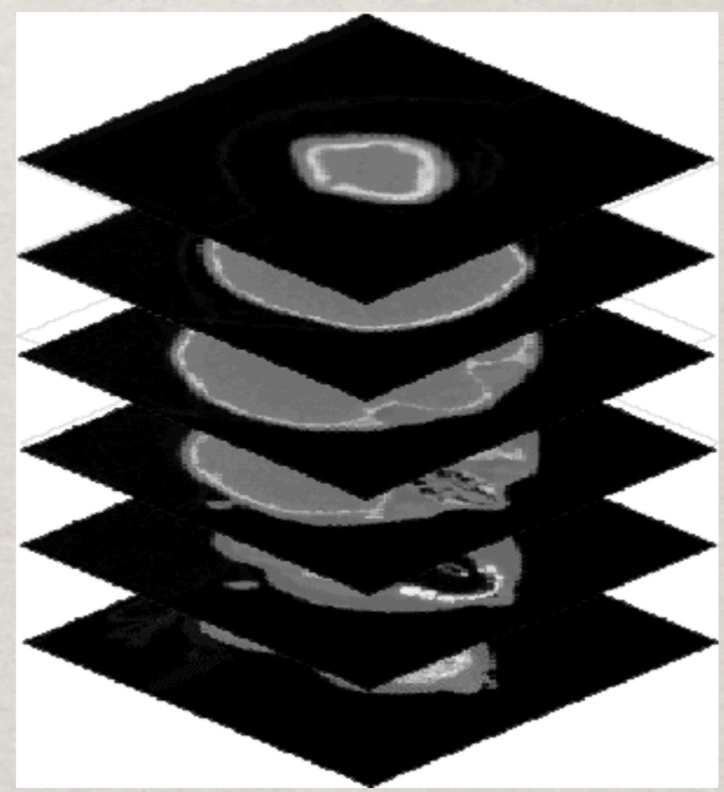
When do we retrieve
the topology of the shape?

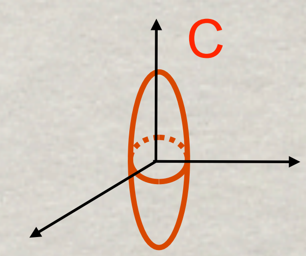




Motivations (1/2)

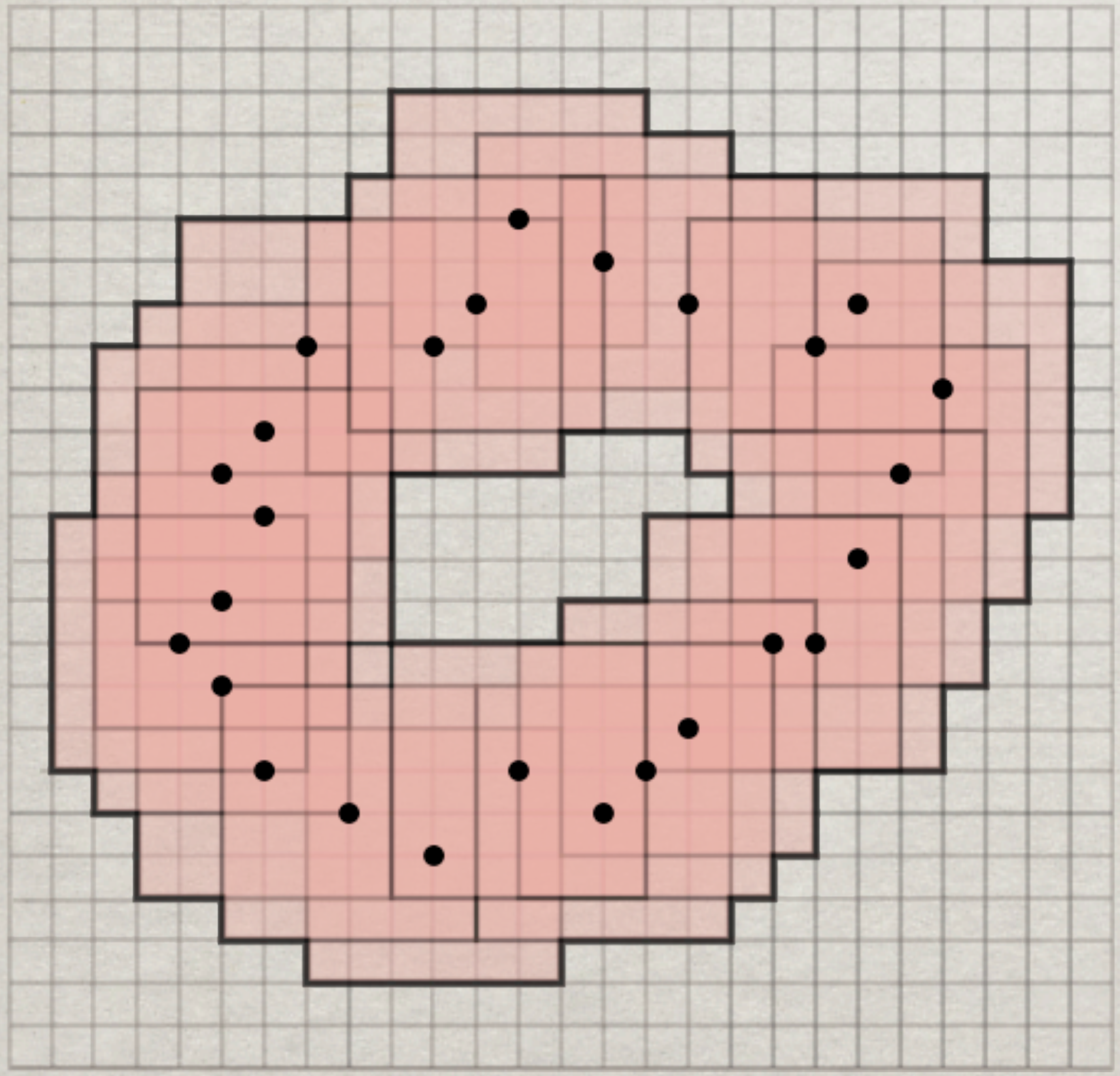
- Recover the topology of a shape with a convex set which takes into account the anisotropy of the measurement device.

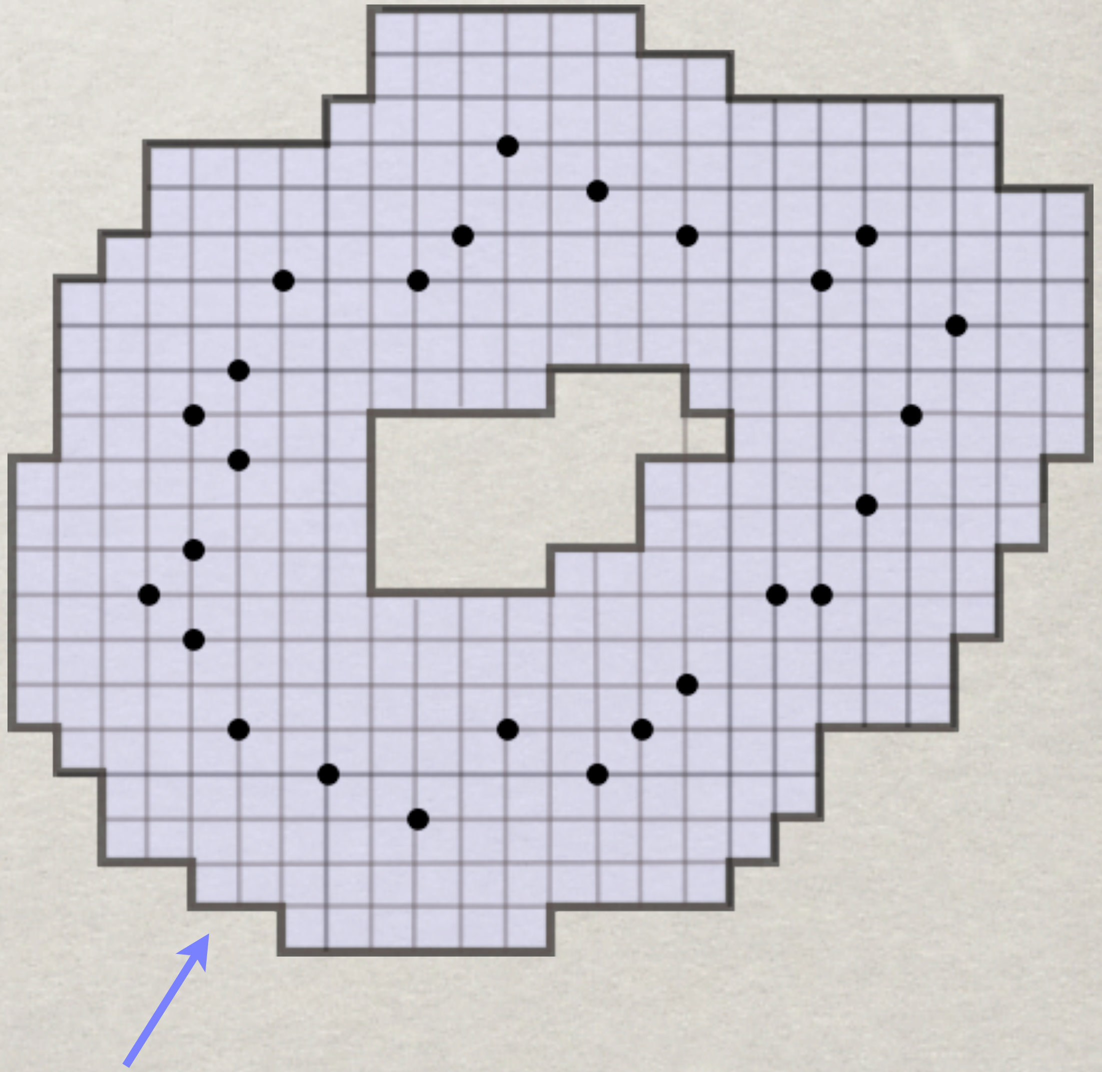




Motivations (2/2)

Corollary 1. *Let* $d_N = \frac{1}{40N^3\lceil\sqrt{N}\rceil}$. *For all compact sets* $A \subset \mathbb{R}^N$ *with reach greater than* $\rho > 0$, *there exists a* $(d_N\rho)$*-cubical set* X *such that* $A \subset X \subset A^\rho$ *and the inclusion maps* $A \hookrightarrow X$ *and* $X \hookrightarrow A^\rho$ *are homotopy equivalences.*

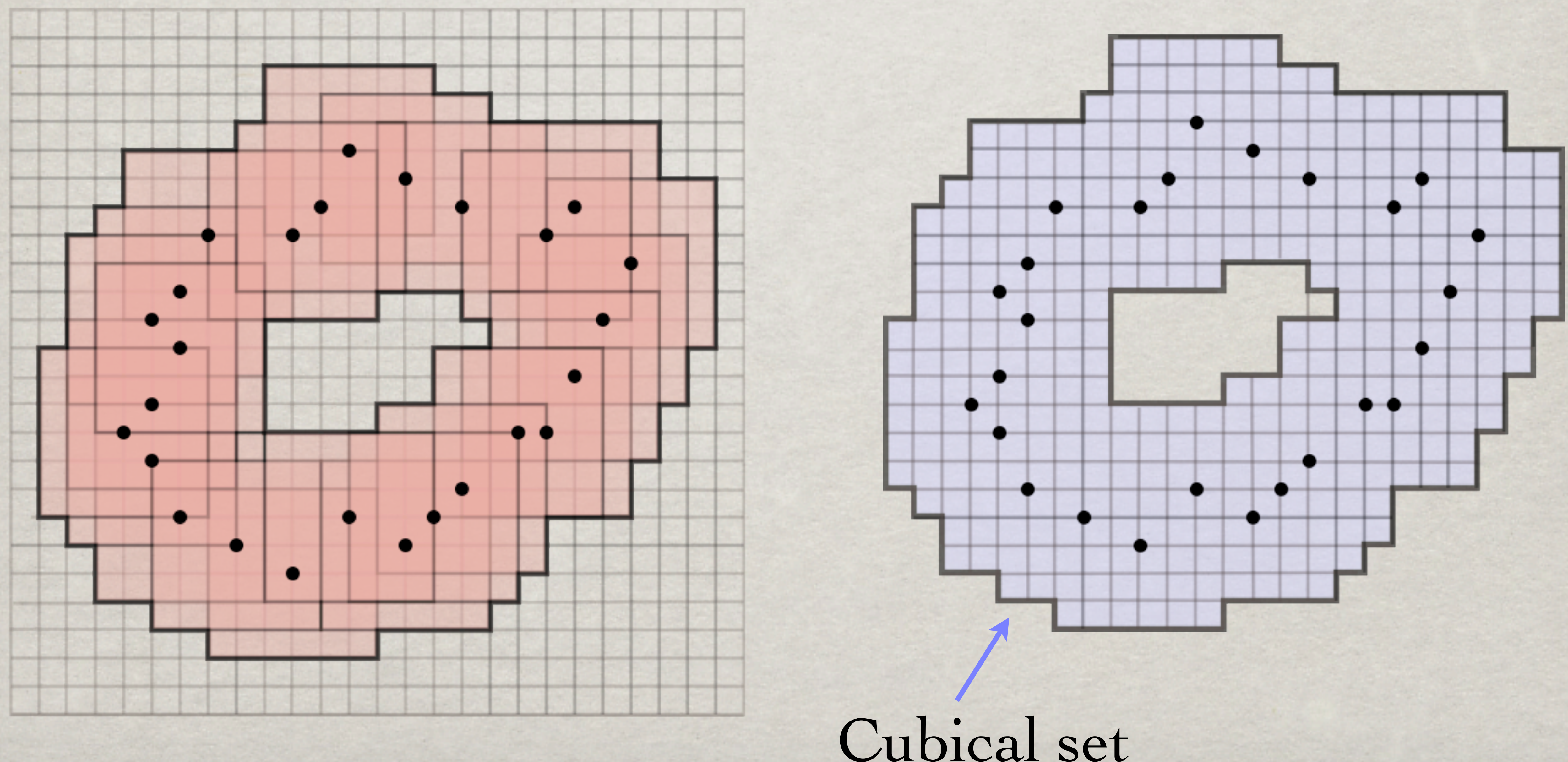




Cubical set

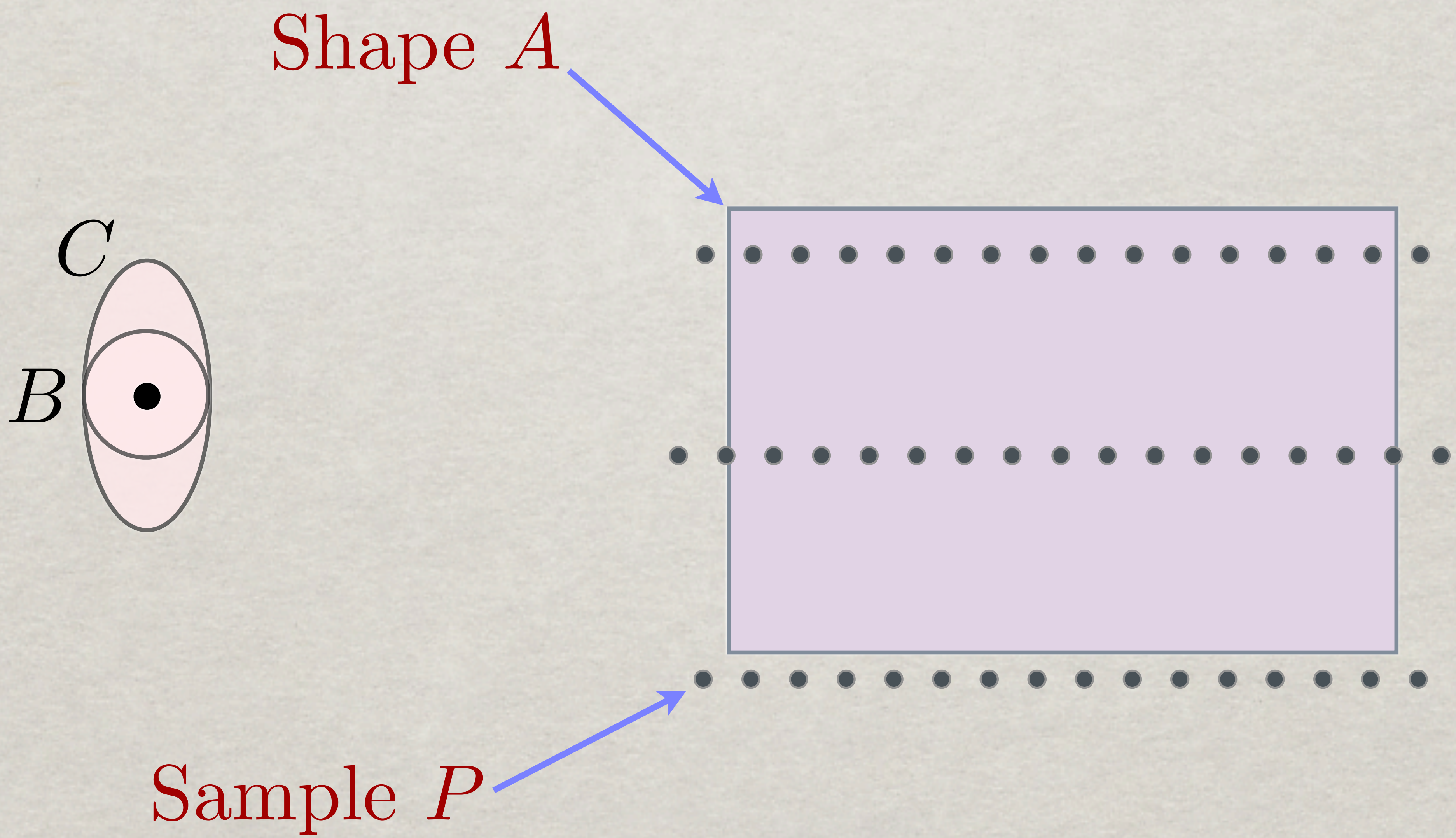
Motivations (2/2)

- Cubical complexes might be convenient for topological computations in high dimensions



MAIN THEOREM

Definition 1. *Given a non-negative real number ε and a subset $C \subset \mathbb{R}^N$, we say that $P \subset \mathbb{R}^N$ is an (ε, C)-*sample *of $A \subset \mathbb{R}^N$ if $A \subset P + \varepsilon C$ and $P \subset A + \varepsilon B$.*



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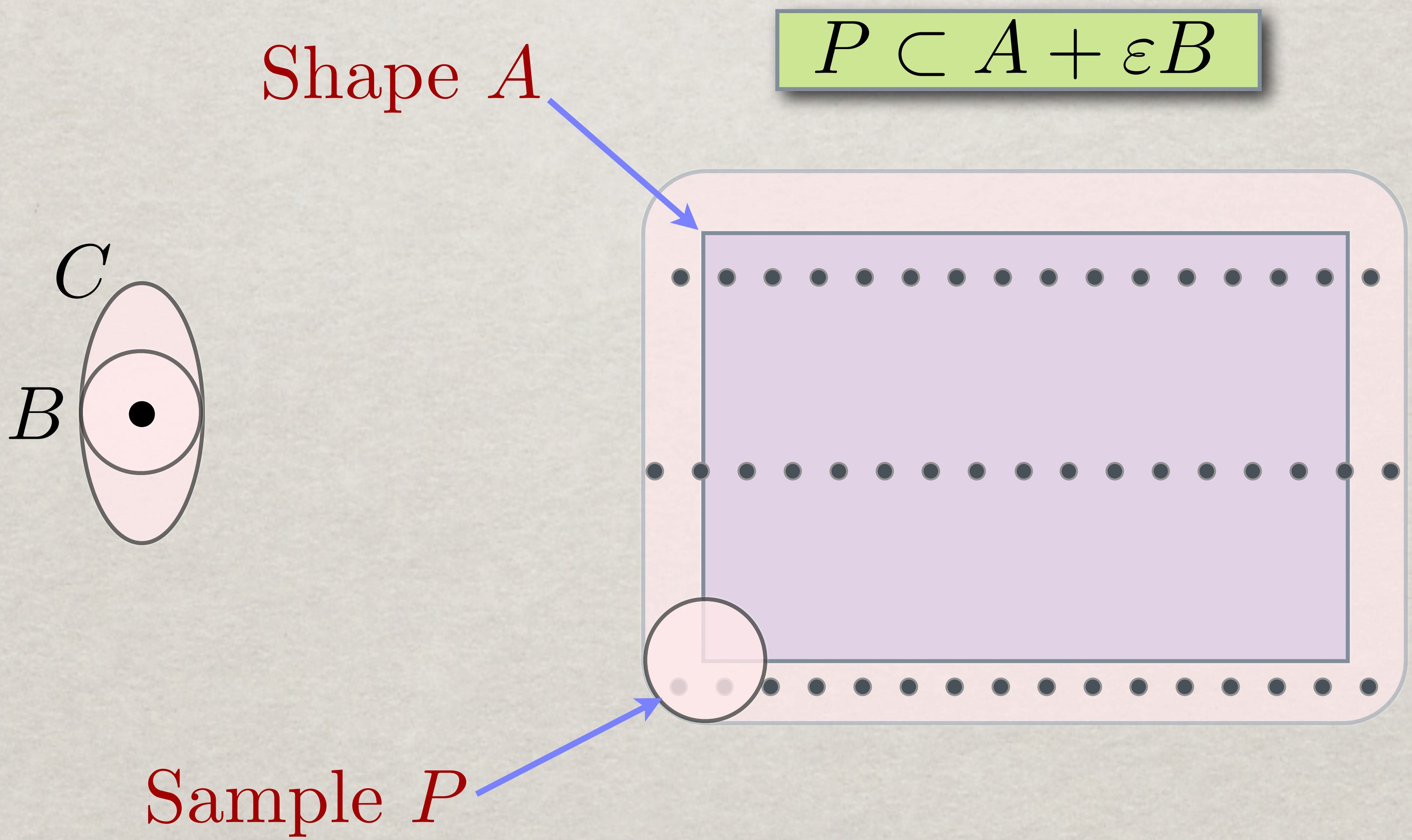

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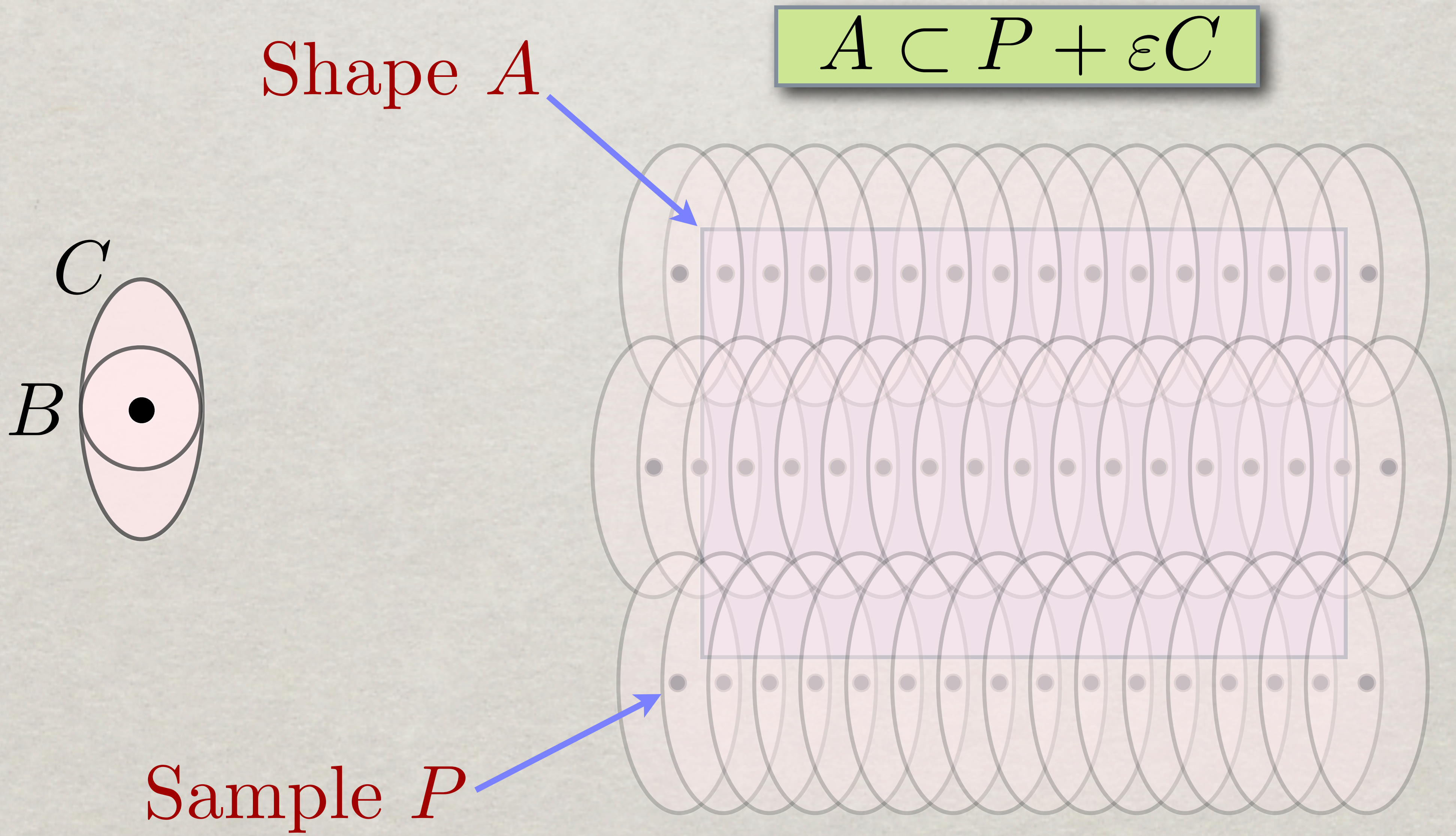

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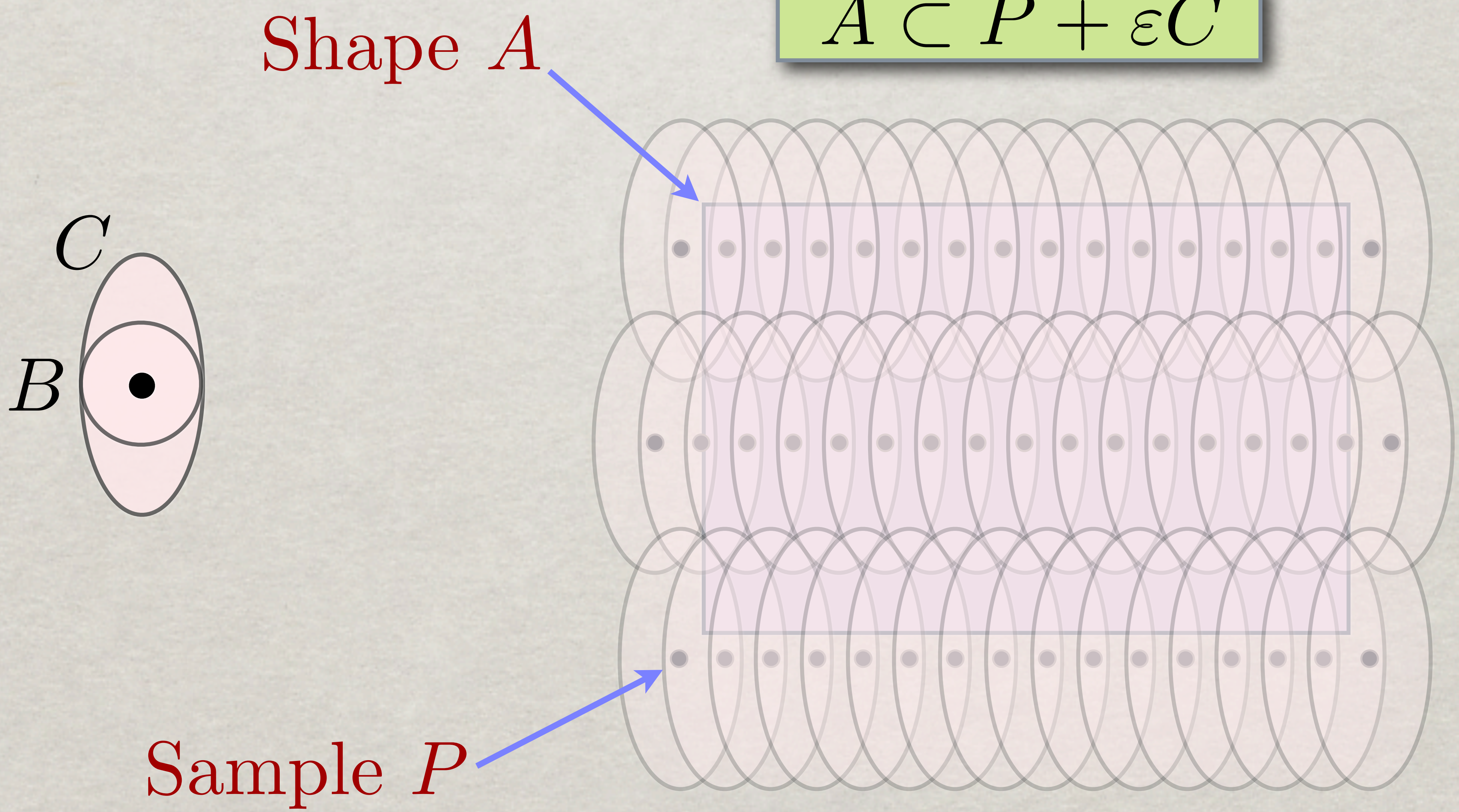
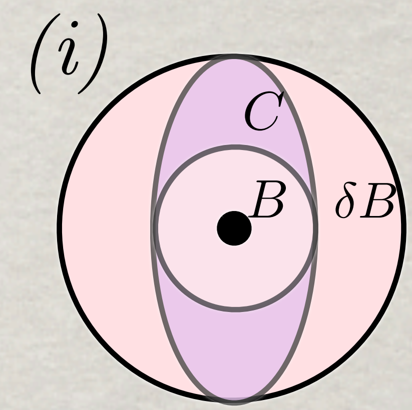
$$\boxed{A \subset P + \varepsilon C}$$



P (ε, B)-sample of A iff $d_H(A, P) \leq \varepsilon$

Main Theorem

Theorem 1. *Let A be a compact subset of $\mathbb{R}^N$ with positive reach R. Let C be a compact convex set of $\mathbb{R}^N$ satisfying conditions:*



(i) $B \subset C \subset \delta B$ *for some* $\delta \geq 1$;

(ii) C *is* $(\theta, \varkappa)$*-round for* $\theta \leq \theta_N = \arccos(-\frac{1}{N})$ *and* $\varkappa > 0$;

(iii) C *is* ξ*-eccentric for* $\xi < 1$.

Let P be a finite (ε, C)-sample of A. Then, the inclusion $A \hookrightarrow P + rC$ is a homotopy equivalence for all positive real numbers r and ε such that:

(1) $(\delta - 1)r < \min\{R - r, R_{r/\varkappa}\}$,

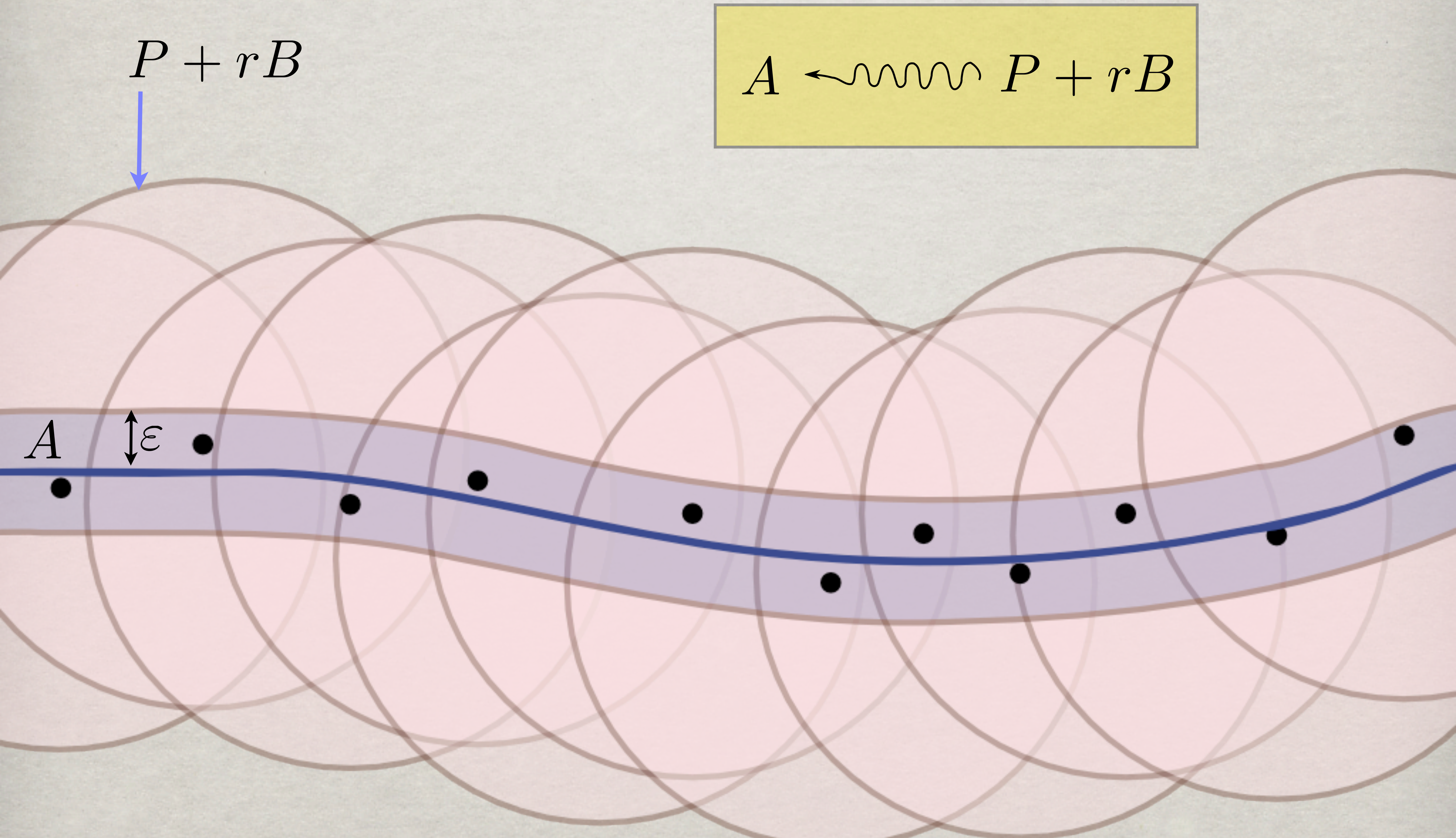
(2) $\delta r < R - \varepsilon$,

(3) $\delta(r + \alpha_0) < R$

(4) $2R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2} - \sqrt{R^2 - \delta^2(r+\alpha_0)^2} < (1-\xi)r - \varepsilon$,

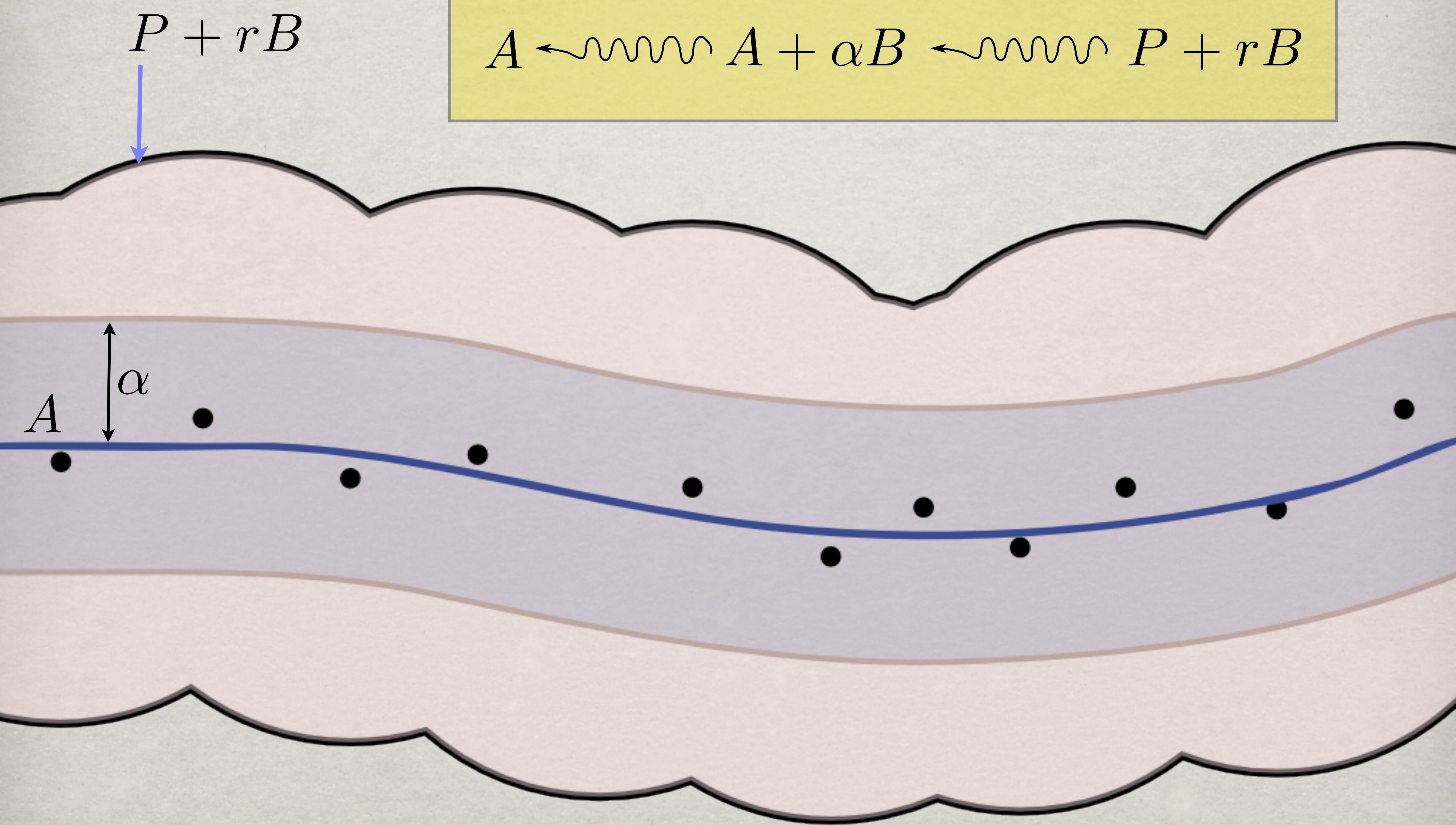
where $\alpha_0 = \xi r + R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2}$ *and* $R_r = R - \frac{r}{4} - \sqrt{\frac{r}{4}\left(2R + \frac{r}{4}\right)}$.

Proof when B=C



Proof when B=C

$$A \leftsquigarrow A + \alpha B \leftsquigarrow P + rB$$



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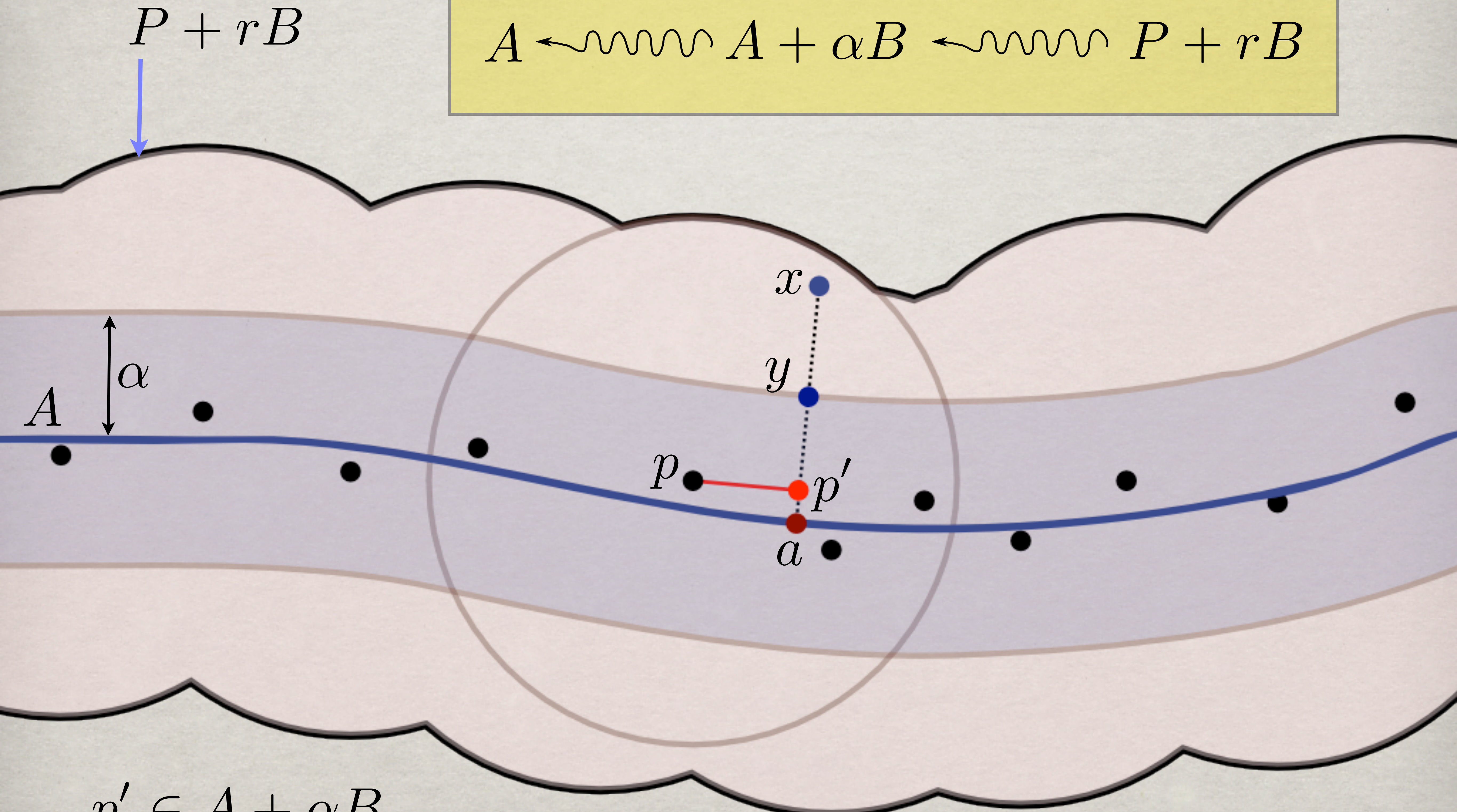
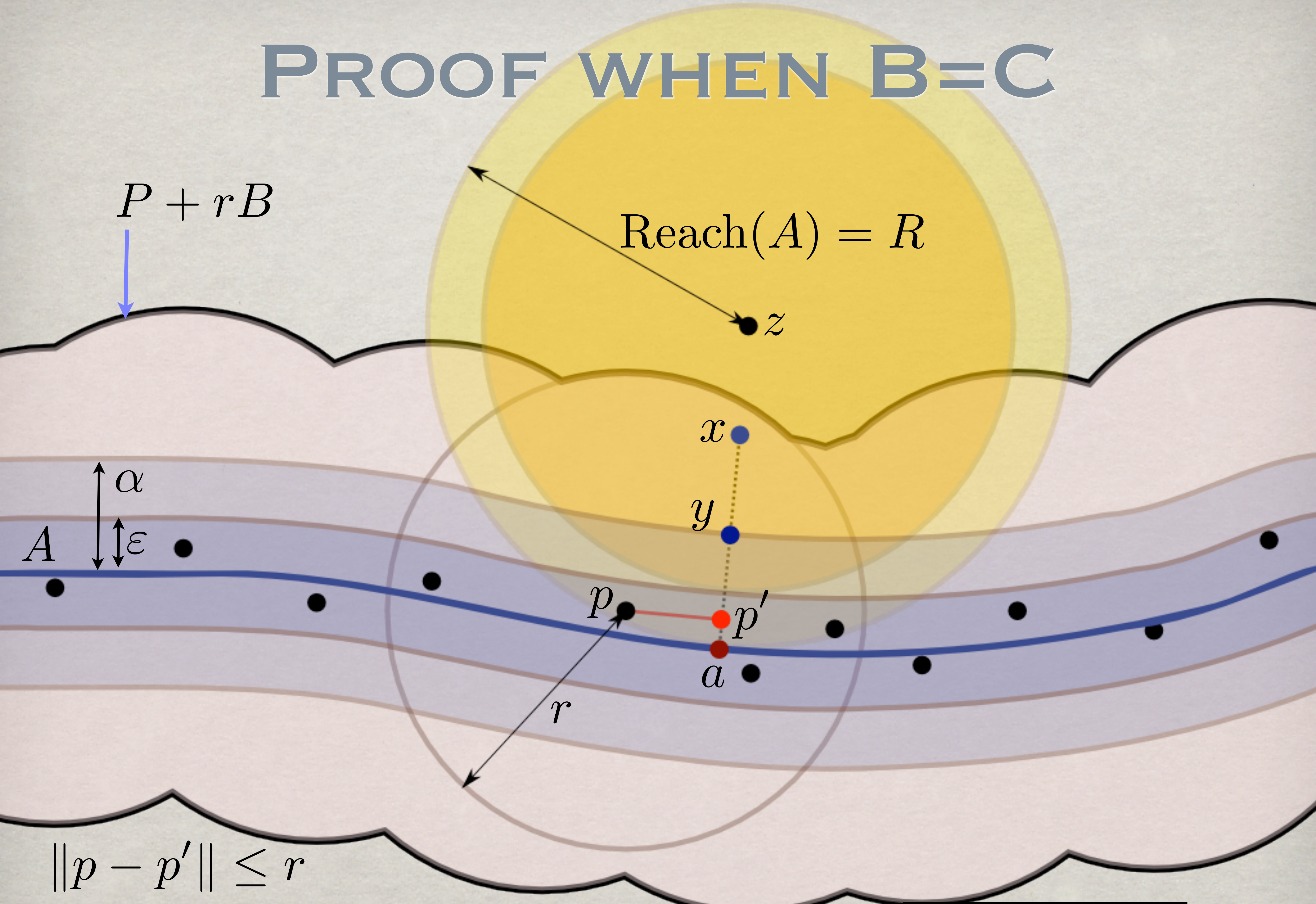


$p' \in A + \alpha B$

Proof when B=C

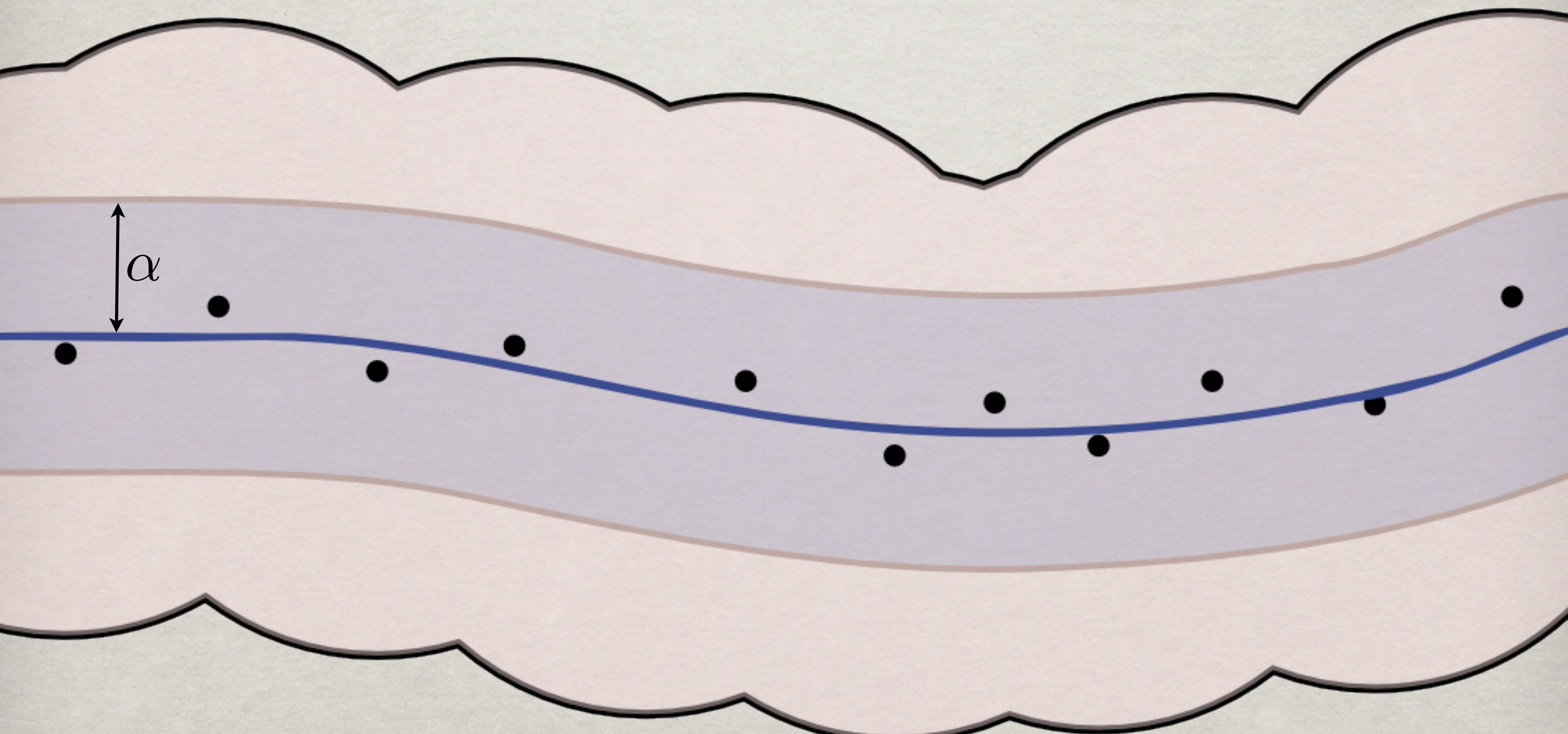


$$\|p - p'\| \leq r$$

$$\|p - z\| \geq R - \varepsilon \implies \|a - p'\| \leq R - \sqrt{(R-\varepsilon)^2 - r^2} = \alpha$$

Extending the Proof

$$A \leftsquigarrow A + \alpha B \leftsquigarrow P + rB$$



Extending the Proof

$$A \leftsquigarrow A + \alpha B \leftsquigarrow P + rB$$

$$A \overset{?}{\leftsquigarrow} A + \alpha C \overset{?}{\leftsquigarrow} P + rC$$

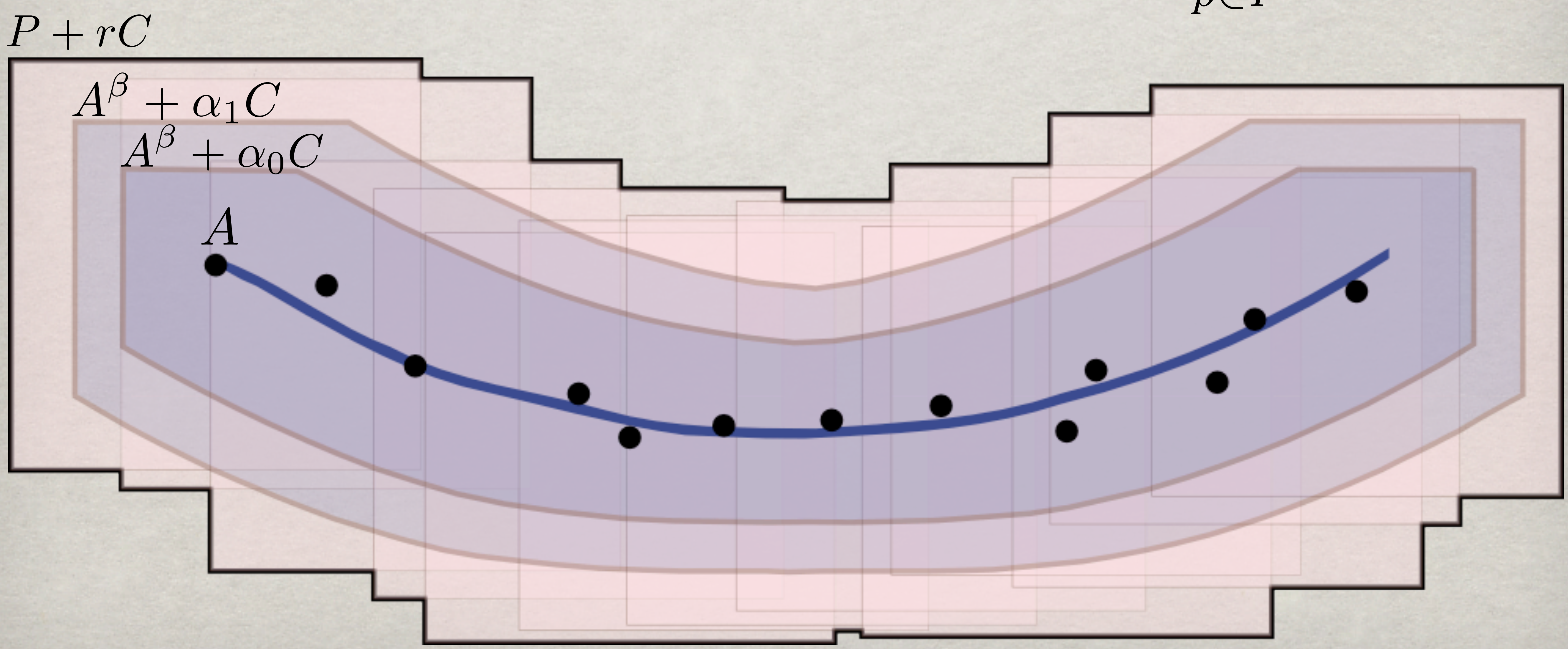
$$\begin{tikzcd}
A \arrow[rrrr, bend left, "\text{Inc. Homotopy eq.}"] \arrow[rrrrrr, bend right=40, "\text{Inc. Homotopy eq.}"'] & \subset & \mathcal{H}(\alpha_0) \arrow[rrrr, bend right, "\text{Inc. Homotopy eq.}"'] & \subset & A^\beta + \alpha_1 C & \subset & P + rC
\end{tikzcd}$$

Step 1:

?

$$A \quad \subset \quad \mathcal{H}(\alpha_0) \quad \subset \quad A^\beta + \alpha_1 C \quad \subset \quad P + rC$$

$$P + rC = \bigcup_{p \in P} (p + rC)$$



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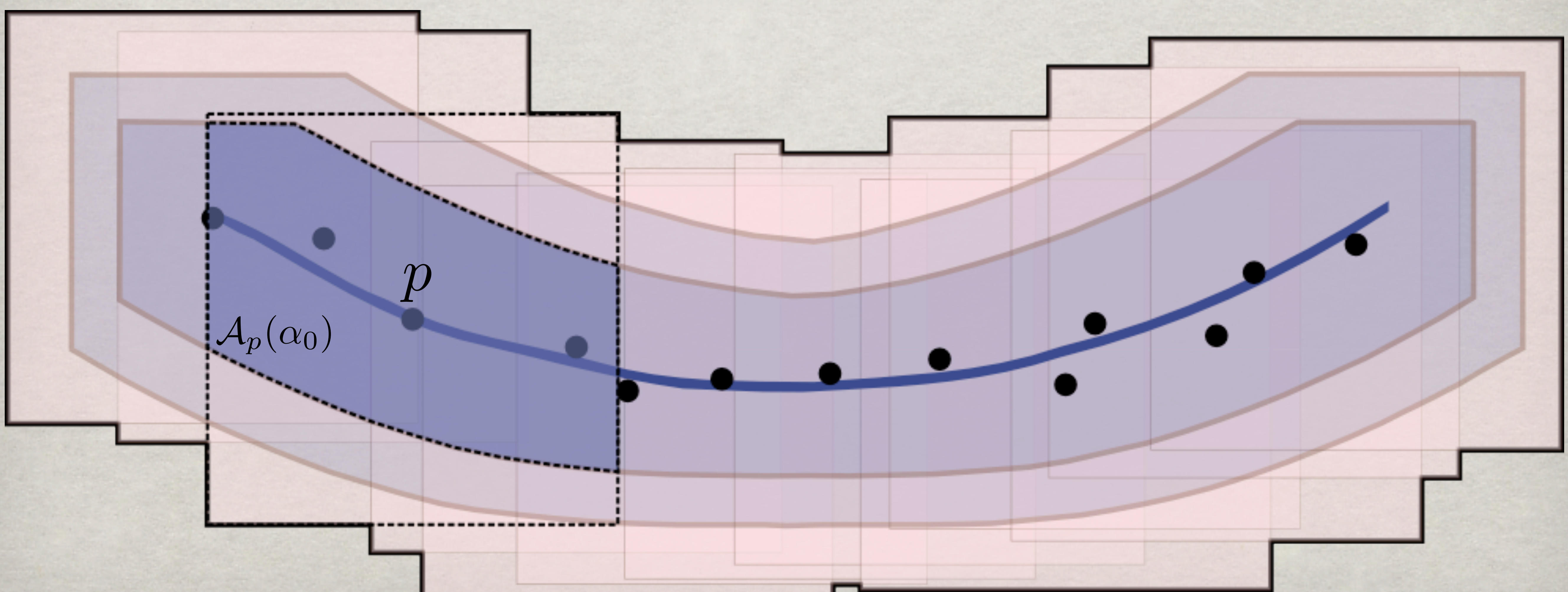

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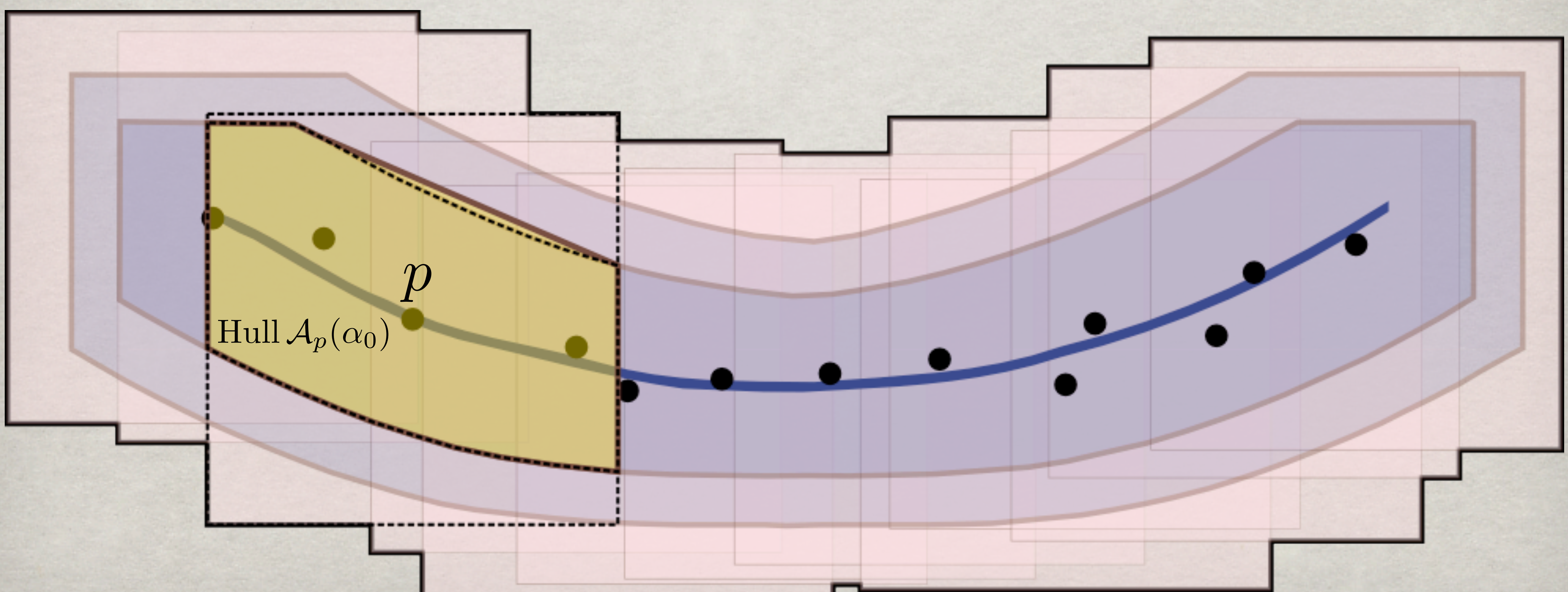

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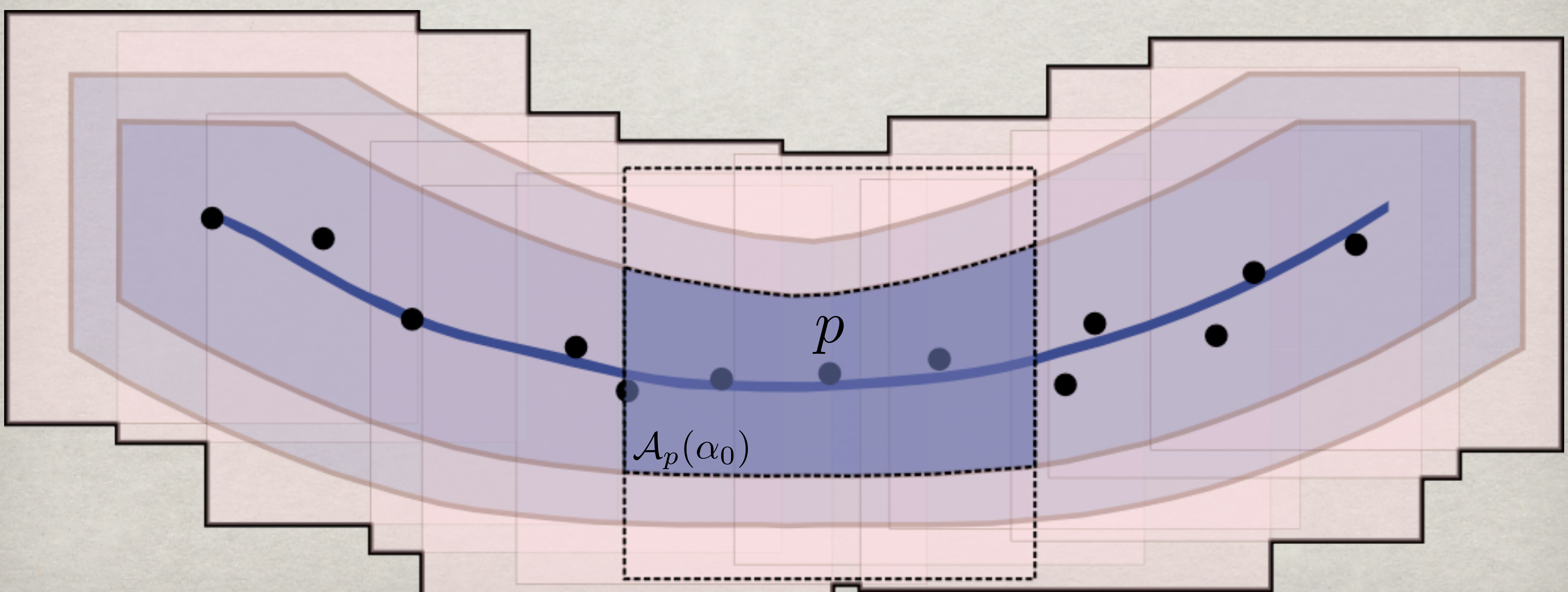

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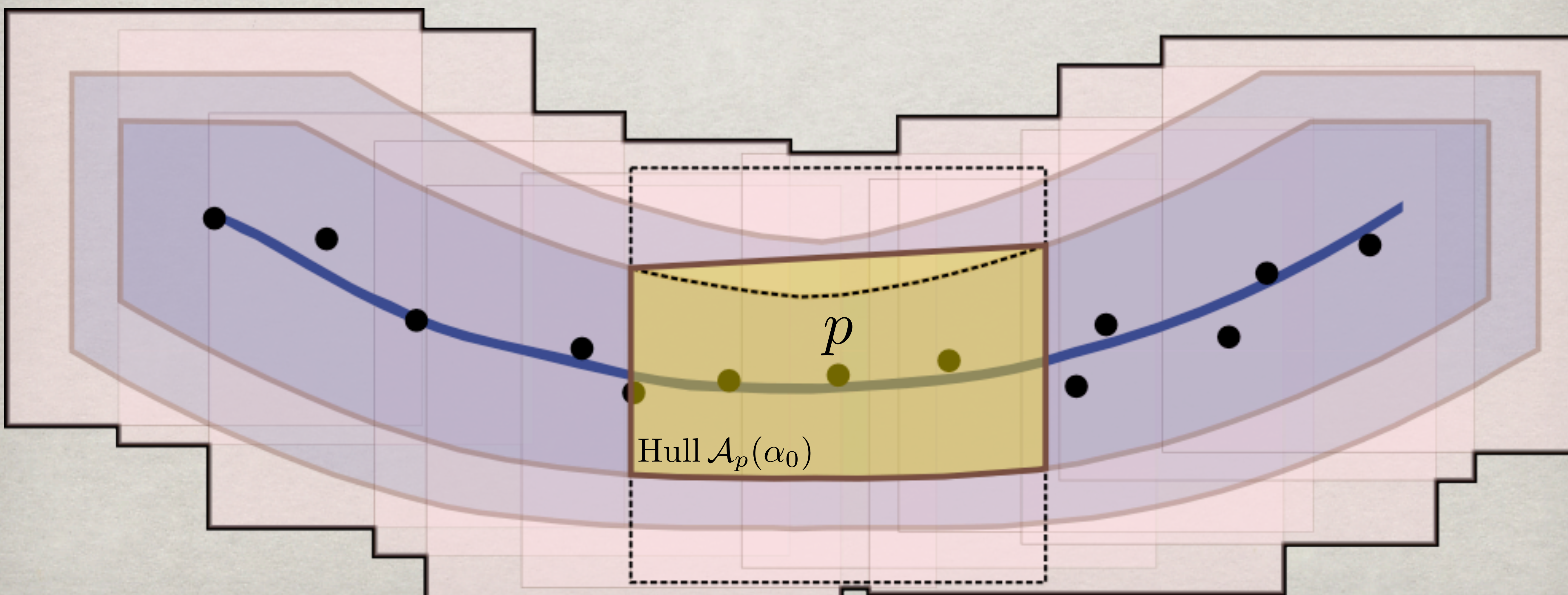

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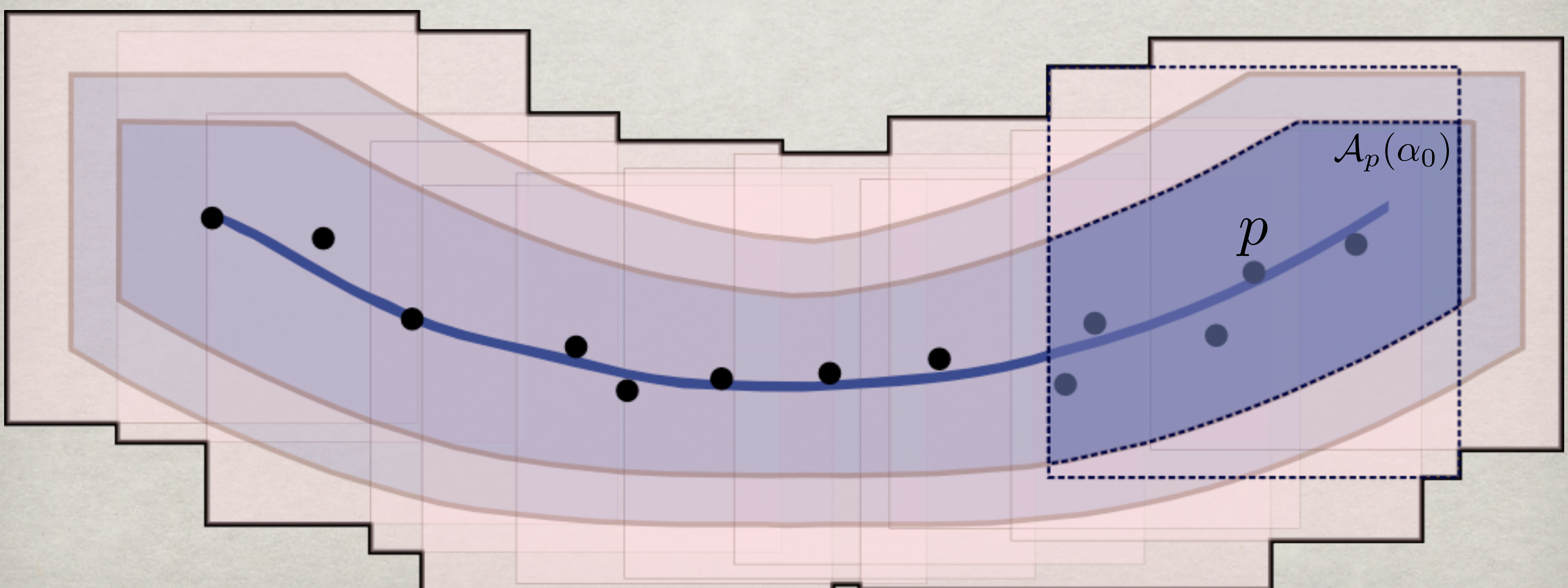

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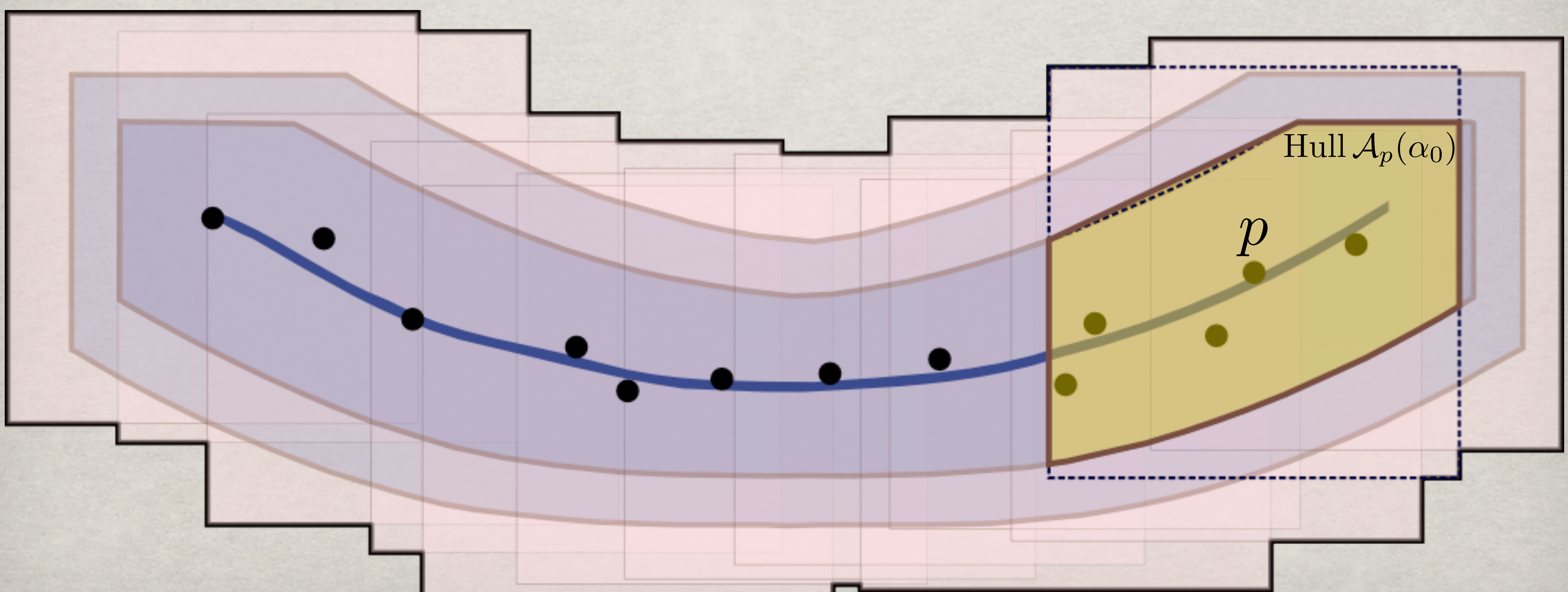

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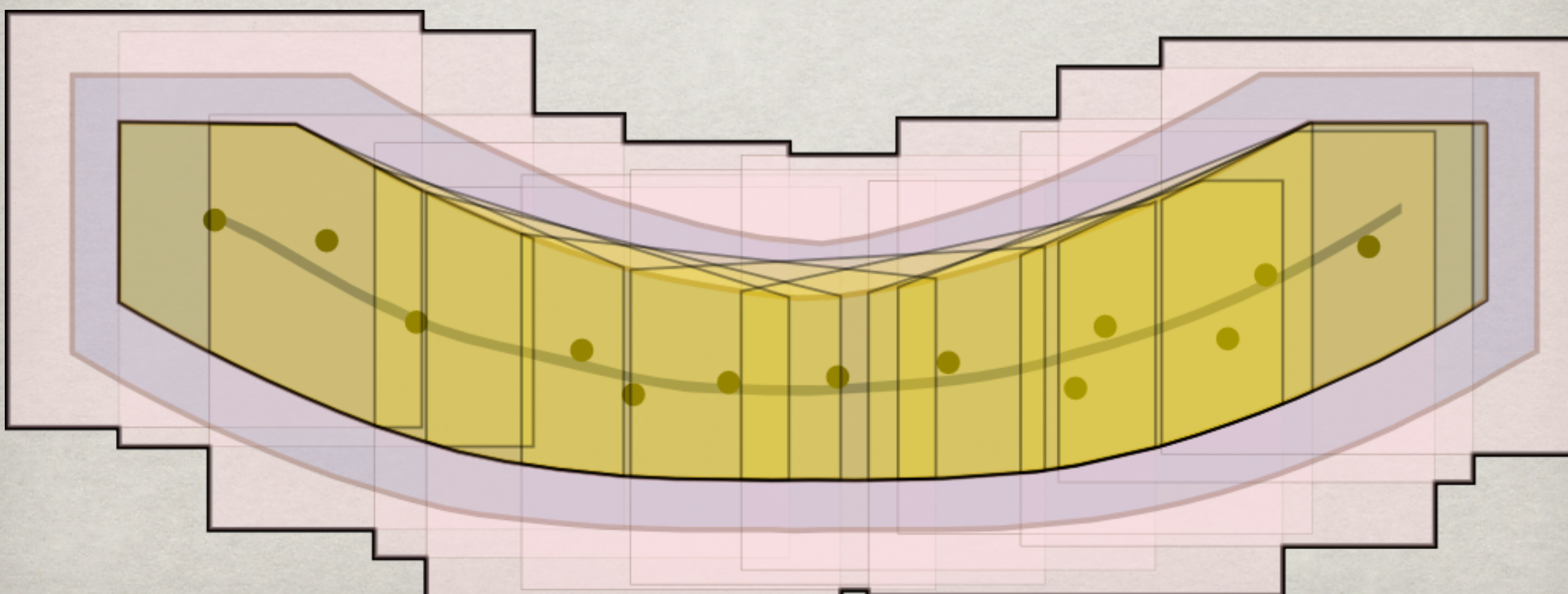
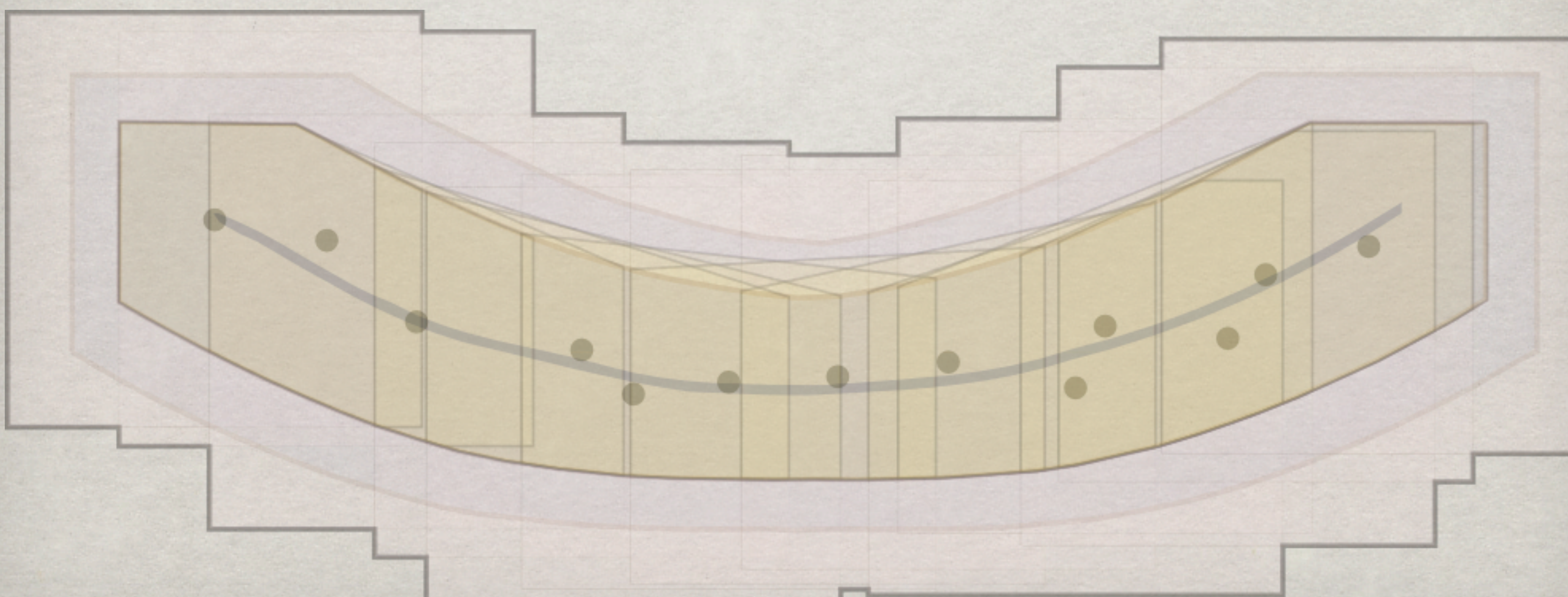
$$\begin{array}{ccccccc} A & \subset & \mathcal{H}(\alpha_0) & \subset & A^\beta + \alpha_1 C & \subset & P + rC \\ & & \| & & & & \| \\ & & \displaystyle\bigcup_{p\in P} \mathrm{Hull}\,\mathcal{A}_p(\alpha_0) & & & & \displaystyle\bigcup_{p\in P} (p + rC) \end{array}$$



Step 1:

$$\begin{tikzcd} A & \subset & \mathcal{H}(\alpha_0) \arrow[rrrr, bend left, "?"] \arrow[d, equal] & \subset & A^{\beta} + \alpha_1 C & \subset & P + rC \arrow[d, equal] \\ & & \bigcup_{p \in P} \mathrm{Hull}\, \mathcal{A}_p(\alpha_0) & & & & \bigcup_{p \in P} (p + rC) \end{tikzcd}$$



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$$\begin{tikzcd}
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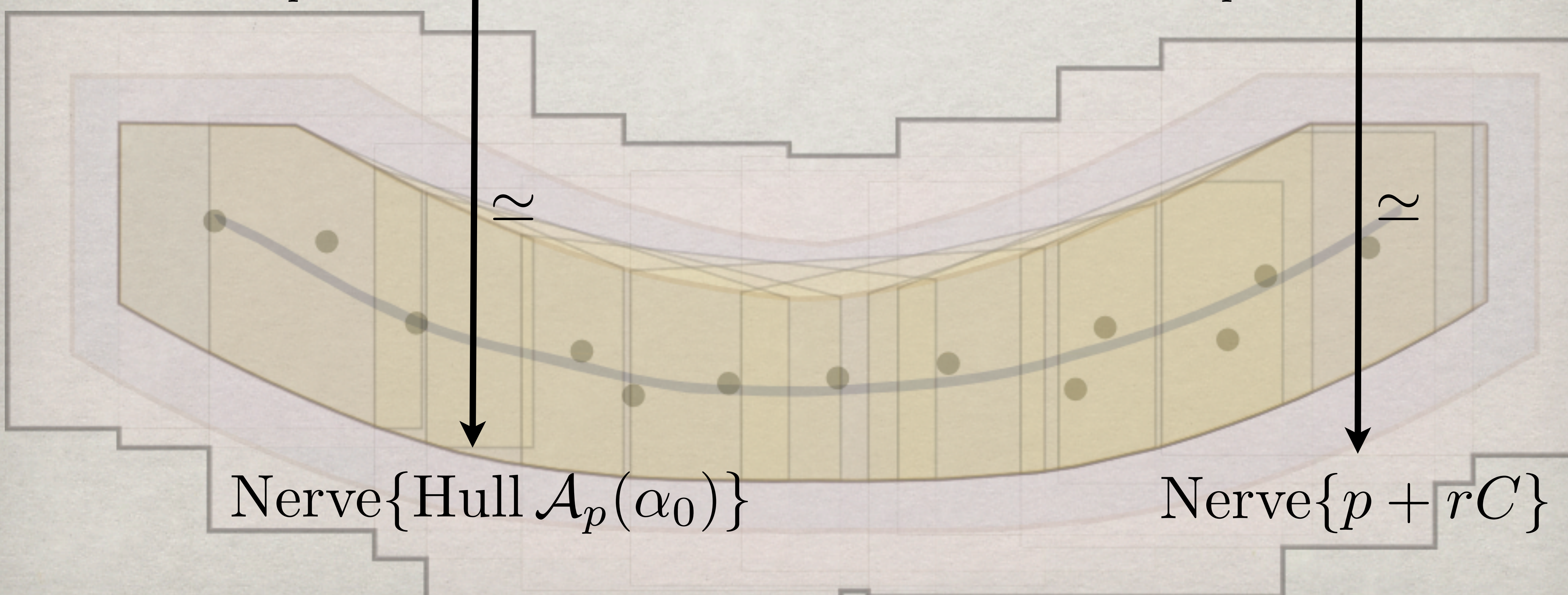
& \bigcup_{p \in P} \mathrm{Hull}\, \mathcal{A}_p(\alpha_0) \arrow[d, leftrightarrow, "\simeq"] & & \bigcup_{p \in P} (p + rC) \arrow[d, leftrightarrow, "\simeq"] \\
& \mathrm{Nerve}\{\mathrm{Hull}\, \mathcal{A}_p(\alpha_0)\} & & \mathrm{Nerve}\{p + rC\}
\end{tikzcd}$$



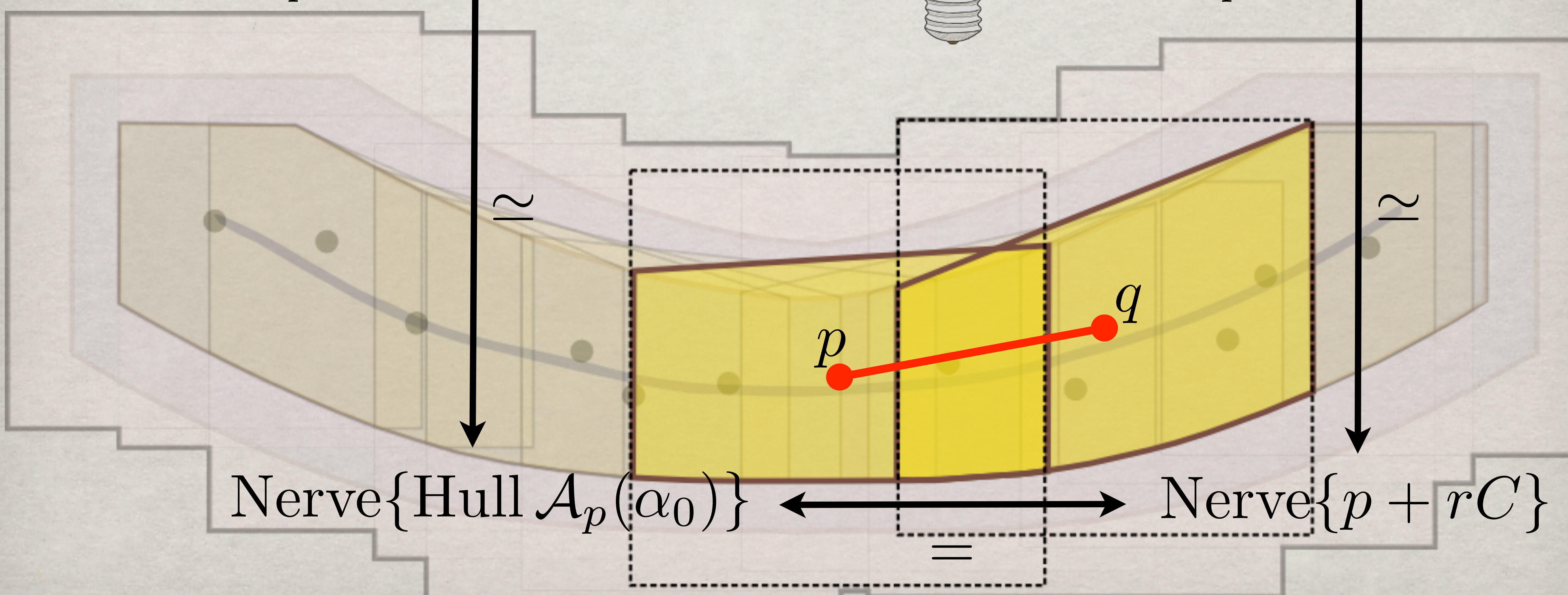
Step 1:

$\simeq$ Inc. Homotopy eq.

$$A \quad \subset \quad \mathcal{H}(\alpha_0) \quad \subset \quad A^{\beta} + \alpha_1 C \quad \subset \quad P + rC$$

$$\mathcal{H}(\alpha_0) = \bigcup_{p \in P} \mathrm{Hull}\, \mathcal{A}_p(\alpha_0) \qquad\qquad P + rC = \bigcup_{p \in P} (p + rC)$$

$$\bigcup_{p \in P} \mathrm{Hull}\, \mathcal{A}_p(\alpha_0) \simeq \mathrm{Nerve}\{\mathrm{Hull}\, \mathcal{A}_p(\alpha_0)\} \qquad\qquad \bigcup_{p \in P} (p + rC) \simeq \mathrm{Nerve}\{p + rC\}$$



$$\mathrm{Nerve}\{\mathrm{Hull}\, \mathcal{A}_p(\alpha_0)\} \longleftrightarrow \mathrm{Nerve}\{p + rC\}$$

$=$

Step 1:

$$
\begin{tikzcd}
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& \operatorname{Nerve}\{\operatorname{Hull} \mathcal{A}_p(\alpha_0)\} \arrow[rr, leftrightarrow, "="] & & \operatorname{Nerve}\{p + rC\}
\end{tikzcd}
$$

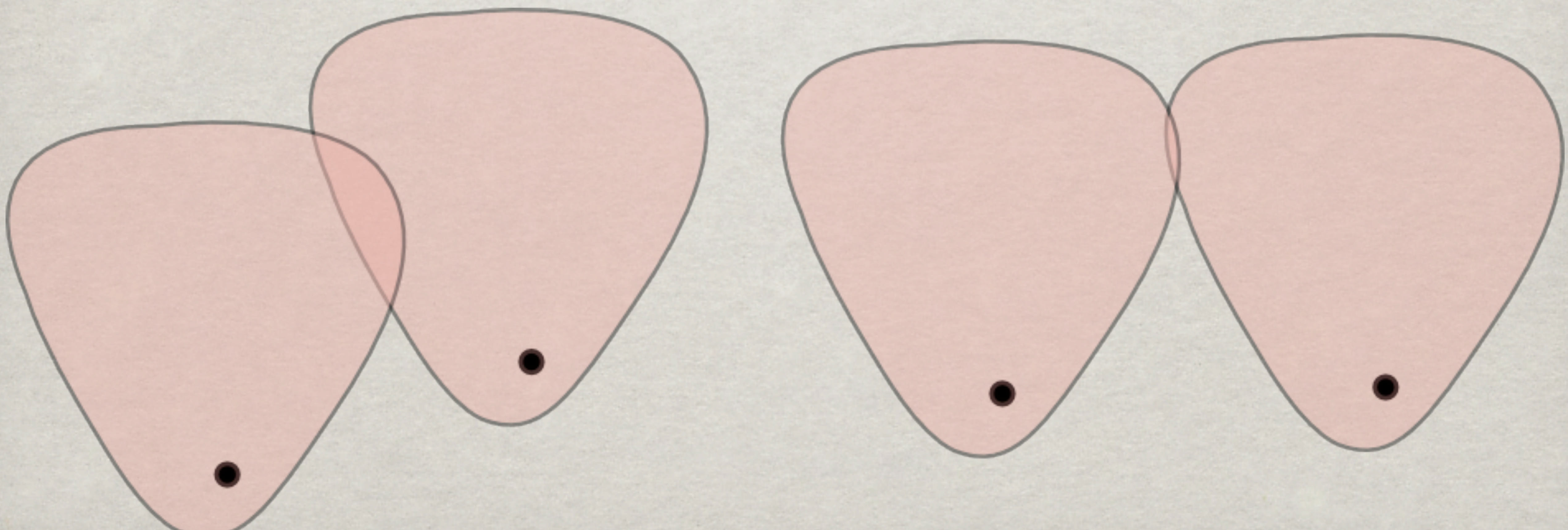
If C is ξ-eccentric for $\xi < 1$

Eccentricity

Definition 2. *Given $\xi \geq 0$, we say that C is ξ-*eccentric *if for all compact Q:*

$$\bigcap_{q \in Q} (q + C) \neq \emptyset \implies \left(\bigcap_{q \in Q} (q + C) \right) \cap (\mathrm{Hull}(Q) + \xi C) \neq \emptyset.$$

The eccentricity *of C is the infimum of $\xi \geq 0$ such that C is ξ-eccentric.*

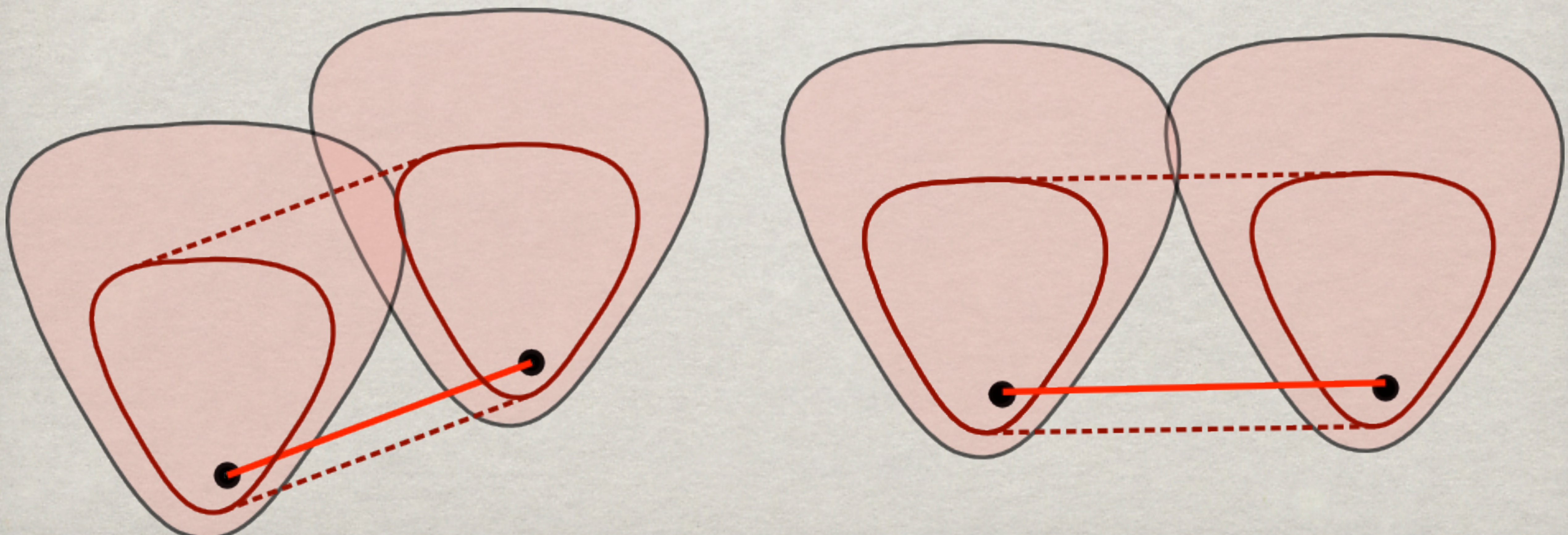


Eccentricity

Definition 2. *Given $\xi \geq 0$, we say that C is ξ*-eccentric *if for all compact Q:*

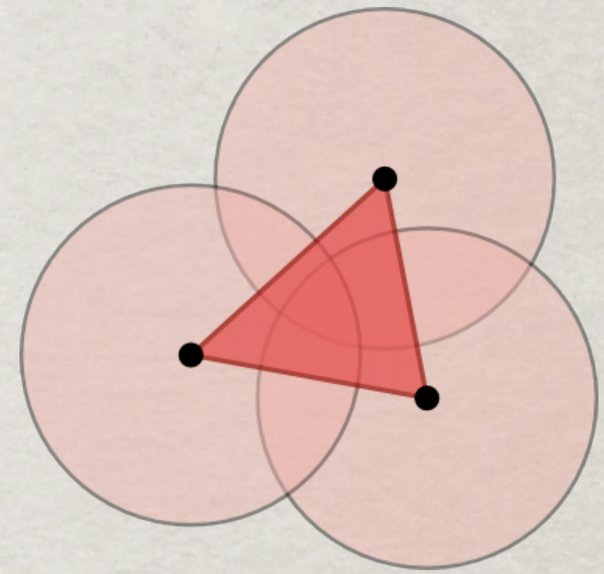
$$\bigcap_{q \in Q} (q + C) \neq \emptyset \implies \left(\bigcap_{q \in Q} (q + C) \right) \cap (\mathrm{Hull}(Q) + \xi C) \neq \emptyset.$$

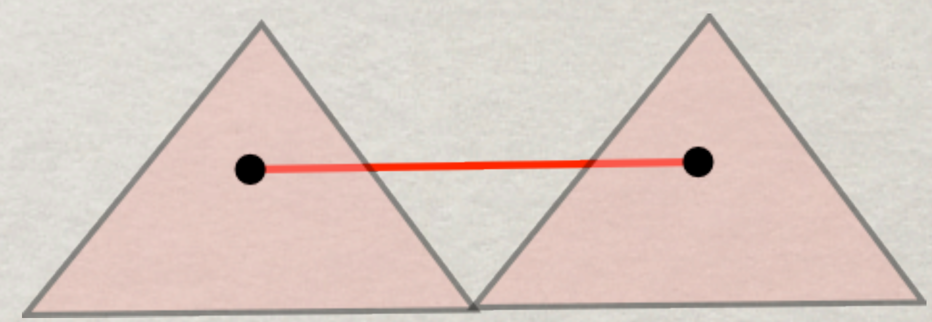
The eccentricity *of C is the infimum of $\xi \geq 0$ such that C is ξ-eccentric.*

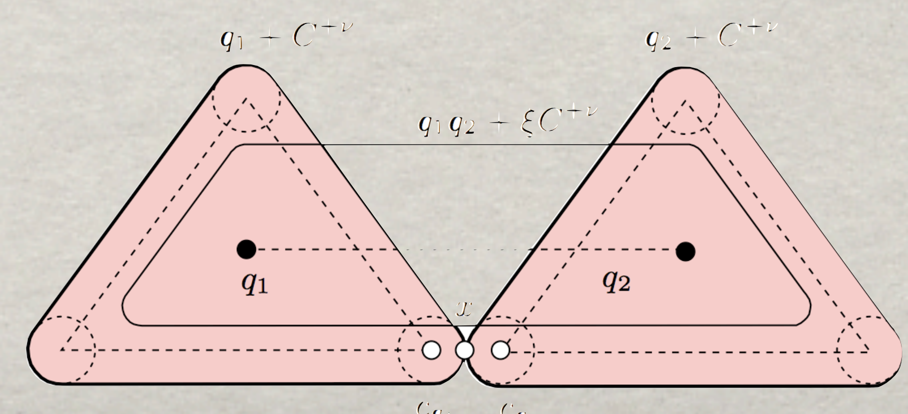


Eccentricity

- real number in [0,1]
- Eccentricity(B) = 0
- If C planar symmetric convex set, Eccentricity(C) = 0
- Eccentricity(triangle) = 1
- Eccentricity(N-dimensional cube) = 1 - 2/N
- Eccentricity(offsets) < 1

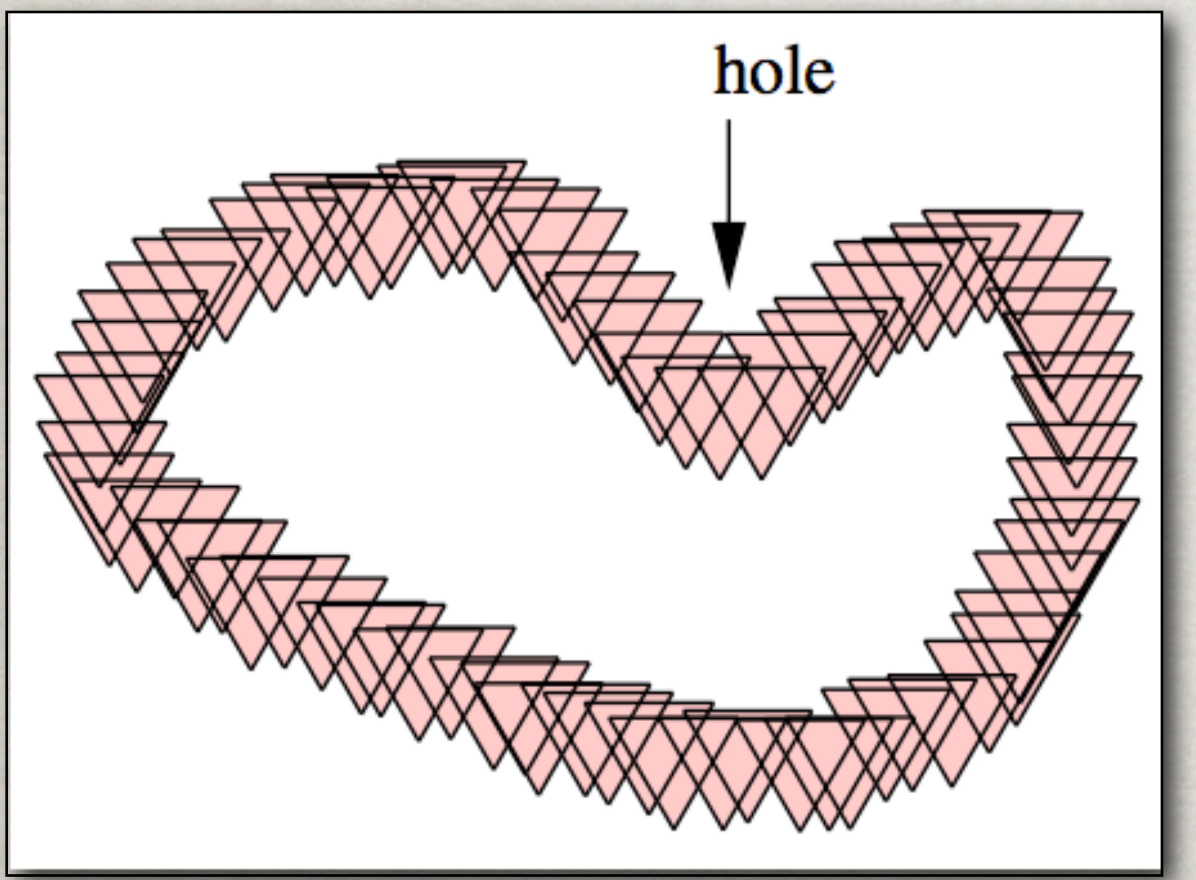


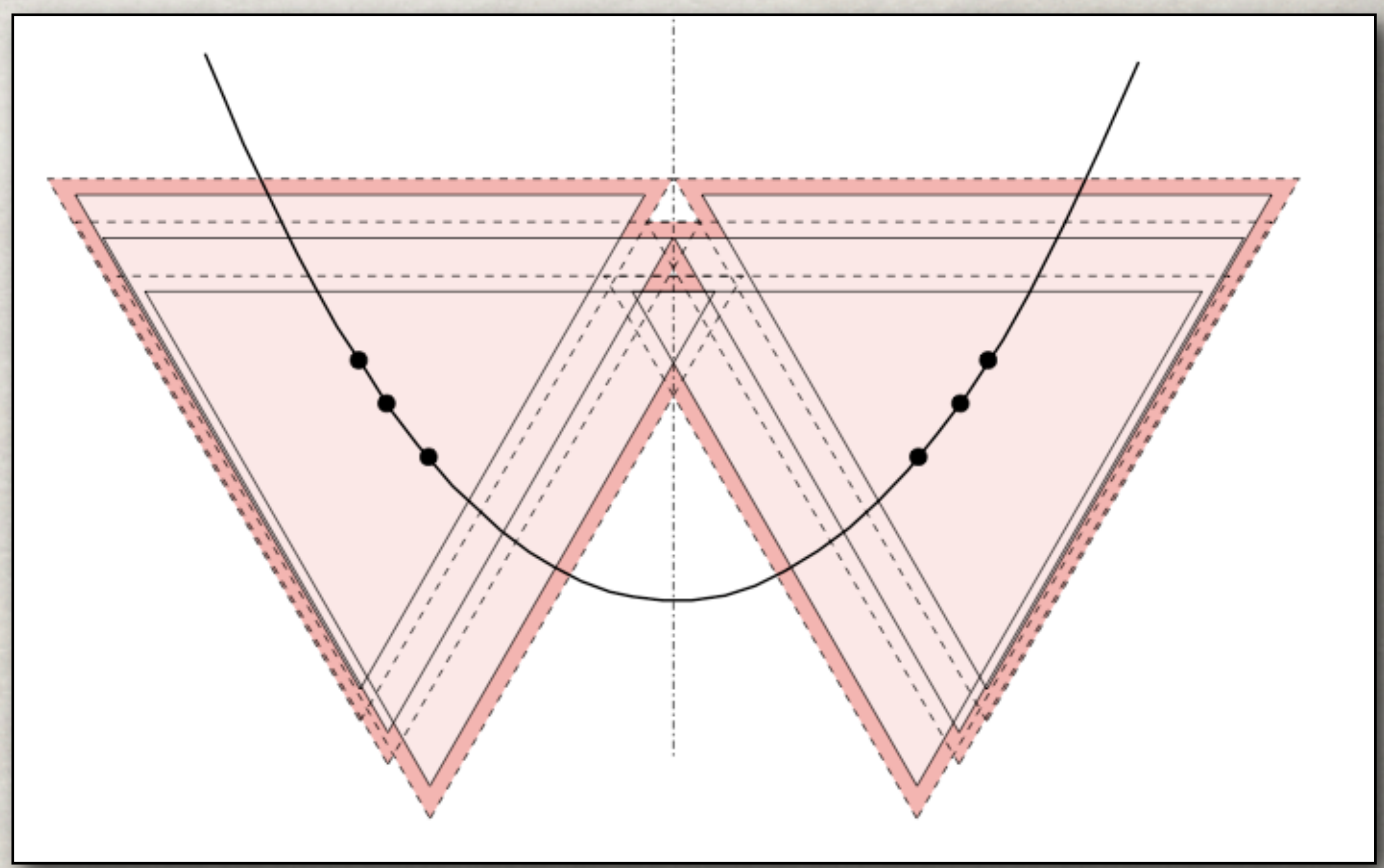




Eccentricity

- For Main Theorem, we need Eccentricity(C) < 1
- Condition is necessary:





Holes in $P + rC$ for all $r \in [\varepsilon, R - \varepsilon]$

Proof of Main Theorem

?

Inc. Homotopy eq.

$$A \quad \subset \quad \mathcal{H}(\alpha_0) \quad \subset \quad A^{\beta} + \alpha_1 C \quad \subset \quad P + rC$$

Inc. Homotopy eq.

If C is ξ-eccentric for $\xi < 1$

Inc. Homotopy eq.

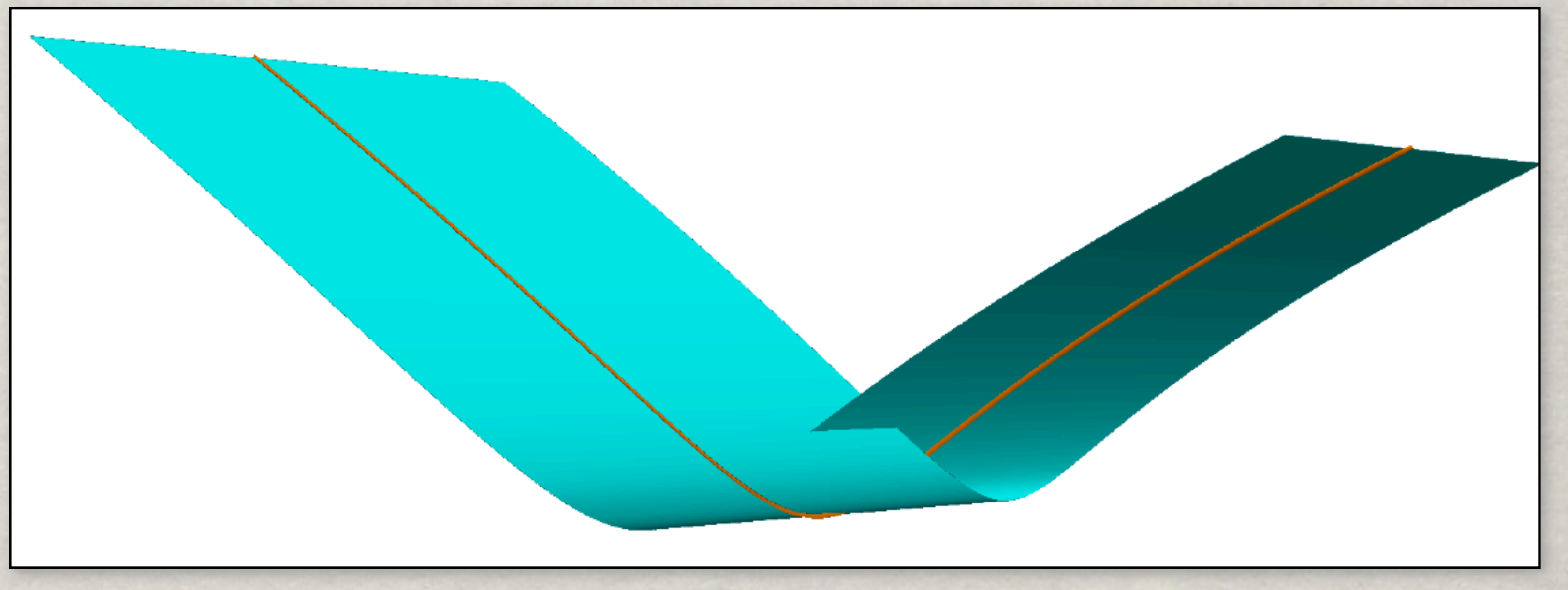
Proof of Main Theorem

$$\overbrace{A \quad \subset \quad \mathcal{H}(\alpha_0) \quad \subset \quad A^\beta}^{?} + \alpha_1 C \quad \subset \quad P + rC$$

Counterexample:

A: moment curve

C: segment



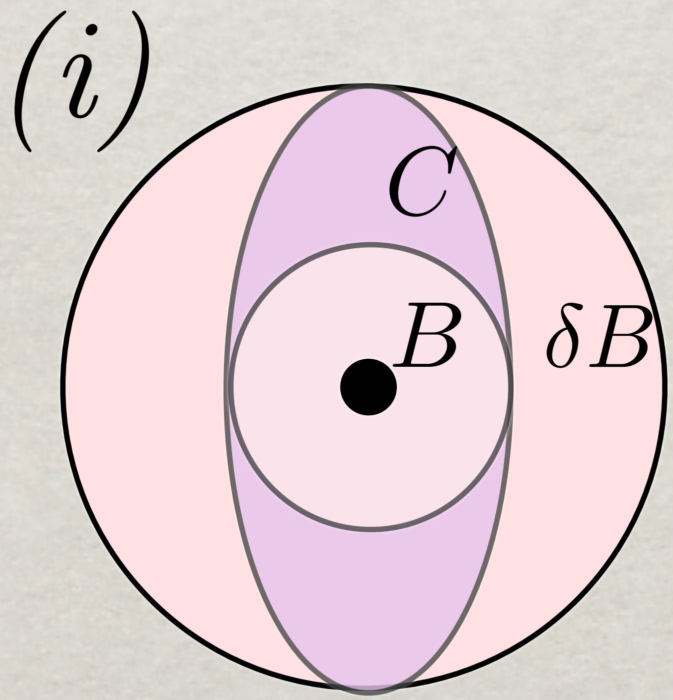
Proof of Main Theorem

Deformation retracts

$$A \subset A^{\beta} + C \subset A^{\beta'} \subset A^{\beta''} + C$$

Deformation retracts

Deformation retracts



Proof of Main Theorem

Offsets of A

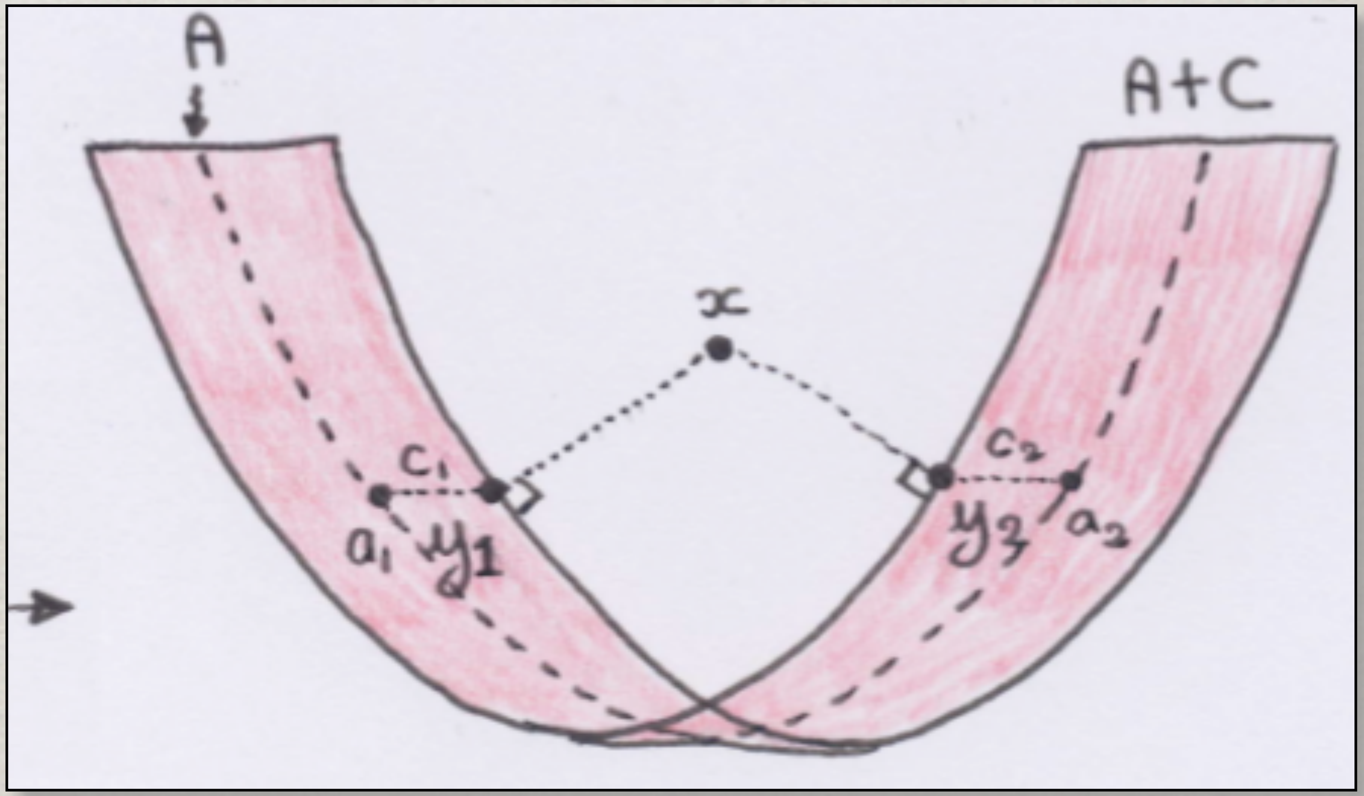
$$A \quad \subset \quad A^{\beta} + C \quad \subset \quad A^{\beta'} \quad \subset \quad A^{\beta''} + C$$

Deformation retracts

Offsets of A+C

C is (θ_N,κ)–round

d(.,A+C) has no critical points in $(0,R_{1/\kappa})$

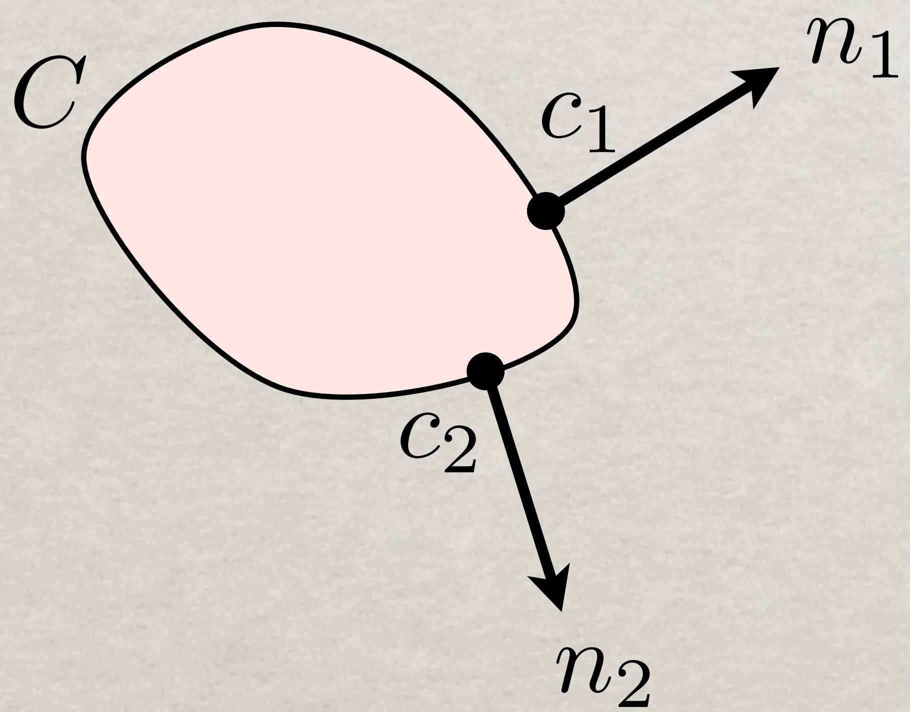


Roundness

Definition 3. *Let $\theta \in [0, \pi]$ and $\varkappa \geq 0$. We say that the compact convex set C is $(\theta, \varkappa)$-*round *if for all points $c_1, c_2 \in C$ and all vectors $n_1 \in \mathcal{N}(c_1)$ and $n_2 \in \mathcal{N}(c_2)$:*

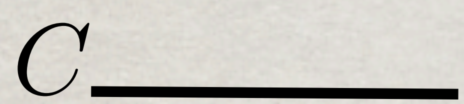
$$\angle(n_1, n_2) \geq \theta \implies (c_1 - c_2) \cdot (n_1 - n_2) \geq \varkappa \|c_1 - c_2\|^2.$$

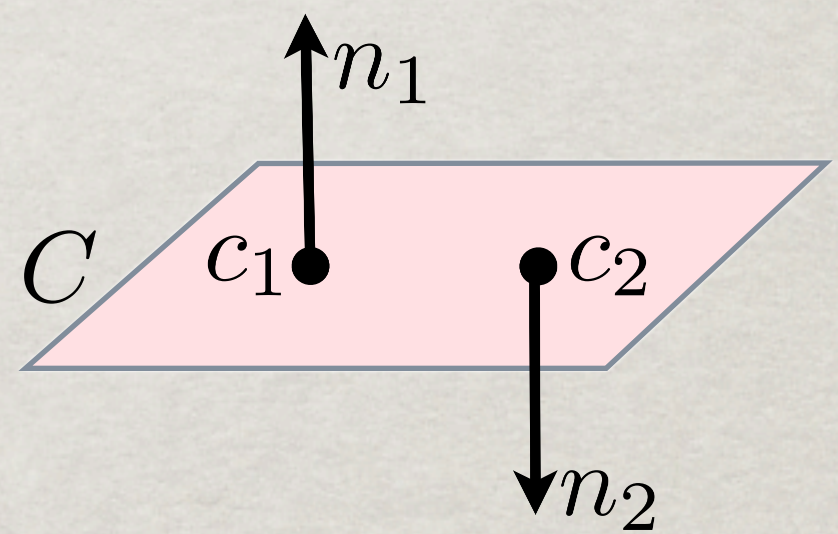
*The θ-*roundness *of C is the supremum of $\varkappa \geq 0$ such that C is $(\theta, \varkappa)$-round.*



Roundness

- If ∂C C^2-smooth, 0-roundness$(C) = \kappa_{\min}(\partial C)$
- If $C \subset$ affine space of dimension $i < N$,
  π-roundness$(C) = 0$



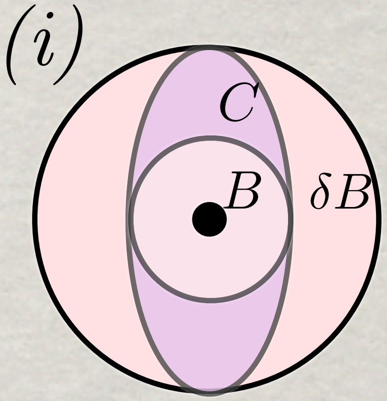


- If C N-dimensional cube,

$$\varkappa(B_\infty) \quad = \quad \begin{cases} \frac{1}{2\sqrt{2}}\left(\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\right) & \text{if } N = 2, \\ \frac{1}{\sqrt{6}} & \text{if } N = 3, \\ \frac{1}{(N-2)\sqrt{N}} & \text{if } N \geq 4, \end{cases}$$

Main Theorem

Theorem 1. *Let A be a compact subset of $\mathbb{R}^N$ with positive reach R. Let C be a compact convex set of $\mathbb{R}^N$ satisfying conditions:*



(i) $B \subset C \subset \delta B$ *for some* $\delta \geq 1$*;*

(ii) C *is* $(\theta, \varkappa)$*-round for* $\theta \leq \theta_N = \arccos(-\frac{1}{N})$ *and* $\varkappa > 0$*;*

(iii) C *is* ξ*-eccentric for* $\xi < 1$*.*

Let P be a finite (ε, C)-sample of A. Then, the inclusion $A \hookrightarrow P + rC$ is a homotopy equivalence for all positive real numbers r and ε such that:

(1) $(\delta - 1)r < \min\{R - r, R_{r/\varkappa}\}$,

(2) $\delta r < R - \varepsilon$,

(3) $\delta(r + \alpha_0) < R$

(4) $2R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2} - \sqrt{R^2 - \delta^2(r+\alpha_0)^2} < (1-\xi)r - \varepsilon$,

where $\alpha_0 = \xi r + R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2}$ *and* $R_r = R - \frac{r}{4} - \sqrt{\frac{r}{4}\left(2R + \frac{r}{4}\right)}$.

Giving constants

$$\frac{1}{3-\sqrt{8}} \approx 5.82$$

convex set C	δ	θ	$\varkappa$	ξ	R/ε	r/ε
Euclidean ball $B \subset \mathbb{R}^N$	1	0	1	0	12.9781	3.95723
cube B_∞ in $\mathbb{R}^N$	$\sqrt{N}$	$\arccos(-\frac{1}{N})$	$\varkappa(B_\infty)$	$1-\frac{2}{N}$		
cube B_∞ in $\mathbb{R}^2$	$\sqrt{2}$	$\frac{2\pi}{3}$	0.65974	0	24.9973	4.04227
cube B_∞ in $\mathbb{R}^3$	$\sqrt{3}$	$0.608\,\pi$	$\frac{1}{\sqrt{6}}$	$1/3$	96.4687	6.14485
cube B_∞ in $\mathbb{R}^4$	$\sqrt{4}$	$0.5804\,\pi$	$1/4$	$1/2$	247.528	8.1826
cube B_∞ in $\mathbb{R}^5$	$\sqrt{5}$	$0.5641\,\pi$	0.149071	$3/5$	508.183	10.2006
cube B_∞ in $\mathbb{R}^{10}$	$\sqrt{10}$	$0.5319\,\pi$	0.03953	$4/5$	4505.44	20.2264
cube B_∞ in $\mathbb{R}^{100}$	10	$0.503183\,\pi$	0.0010204	$49/50$	4948245	200.232
p-gon $\mathcal{P}_p$ in $\mathbb{R}^2$ (p even)	$\frac{1}{\cos\frac{\pi}{p}}$	$\frac{2\pi}{3}$	$\varkappa(\mathcal{P}_p)$	0		
square in $\mathbb{R}^2$	$\sqrt{2}$	$\frac{2\pi}{3}$	0.65974	0	24.9973	4.04227
hexagon in $\mathbb{R}^2$	1.1547	$\frac{2\pi}{3}$	0.69936	0	16.9858	3.99837
octagon in $\mathbb{R}^2$	1.08239	$\frac{2\pi}{3}$	0.793353	0	15.04119	3.98101
dodecagon in $\mathbb{R}^2$	1.03528	$\frac{2\pi}{3}$	0.8660254	0	13.84148	3.968
36-gon in $\mathbb{R}^2$	1.00382	$\frac{2\pi}{3}$	0.951917	0	13.07011	3.95844
360-gon in $\mathbb{R}^2$	1.00004	$\frac{2\pi}{3}$	0.9949868	0	12.97897	3.95724

Questions

- We require P to be finite. Can we relax this condition?
- Eccentric and θ-Roundness when δ close to 1?
- Applications in high dimensions?
- Are the bounds optimal for cubes? Can we do better?