Weak Restricted Delaunay Theorems

Attali, Edelsbrunner, Mileyko

I. Results

II. Preliminaries

III. Curves

IV. Surfaces

I. Results

I.1 WEAK WITNESSES

$L \subseteq \mathbb{R}^d$ ... finite set of landmarks

$\sigma \subseteq L$ ... simplex

DEF. A <u>weak</u> <u>witness</u> of σ is a point $x \in \mathbb{R}^d$ s.t.

$\|x-a\| \leq \|x-b\|$ for all $a \in \sigma$, $b \in L-\sigma$.

I.1 Weak Witnesses

$L \subseteq \mathbb{R}^d$ ... finite set of landmarks

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DEF. A strong ~~weak~~ witness of σ is a point $x \in \mathbb{R}^d$ s.t.

$\|x-a\| \le \|x-b\|$ for all $a \in \sigma$, $b \in L - \cancel{\sigma}$.

I.1 Weak Witnesses

$L \subseteq \mathbb{R}^d$ ... finite set of landmarks

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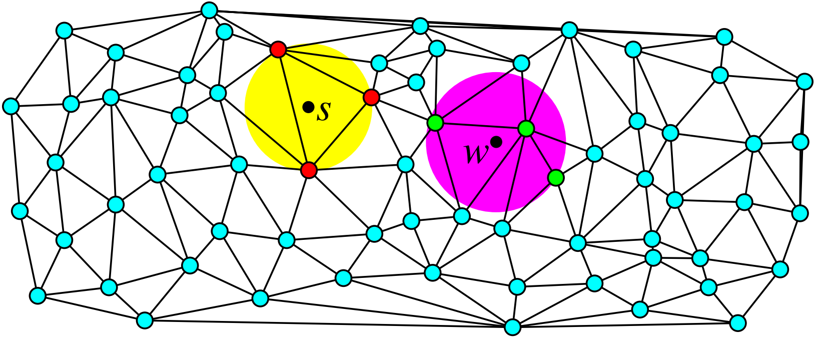
DEF. A ~~weak~~ strong witness of σ is a point $x \in \mathbb{R}^d$ s.t.

$\|x-a\| \leq \|x-b\|$ for all $a \in \sigma$, $b \in L - \cancel{\sigma}$.

1. $\exists$ strong witness $\implies \sigma \in \mathrm{Del}(L)$.
2. $\forall$ faces $\exists$ weak witness $\implies \sigma \in \mathrm{Del}(L)$

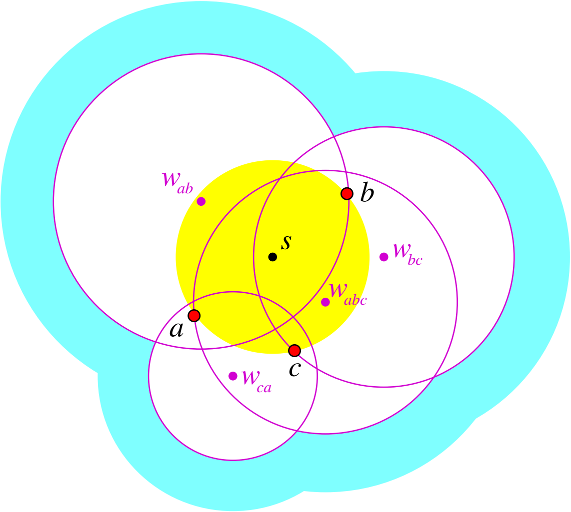
[de Silva 03]

Delaunay triangulation

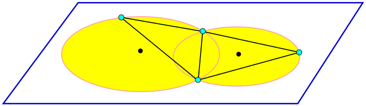


- s = strong witness
- w = weak witness

Theorem [de Silva 03]



Motivation



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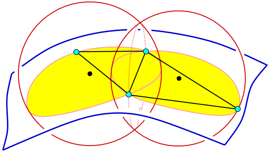
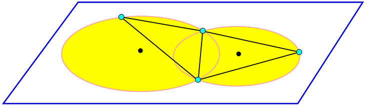


Restricted Delaunay triangulation

Motivation



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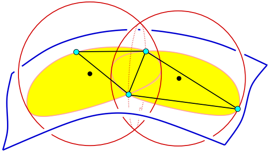
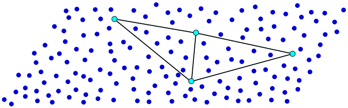
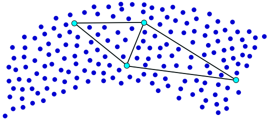


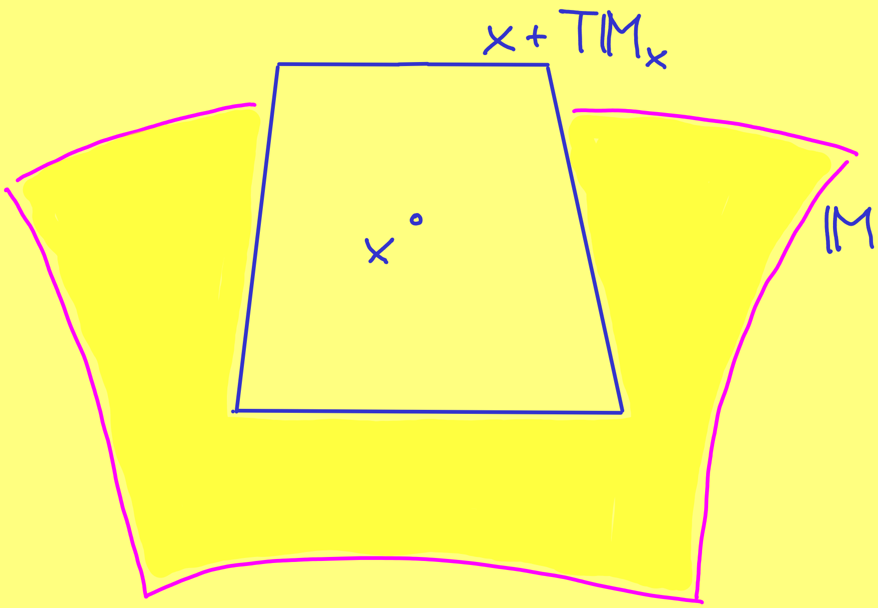
Restricted Delaunay triangulation





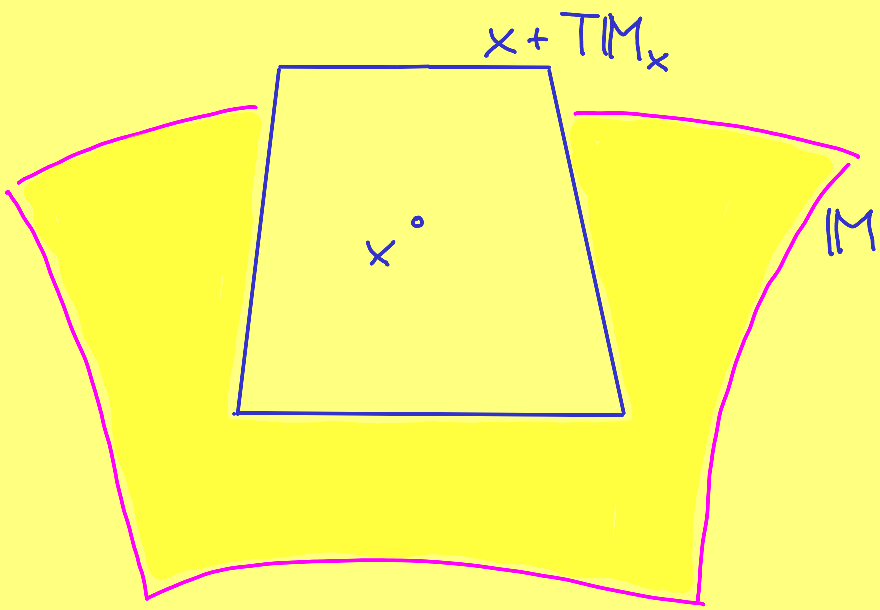
Witness complexes approximation of restricted Delaunay triangulation?

I.2 Curvature



$$\dim \mathbb{M} = k = \dim T\mathbb{M}_x$$

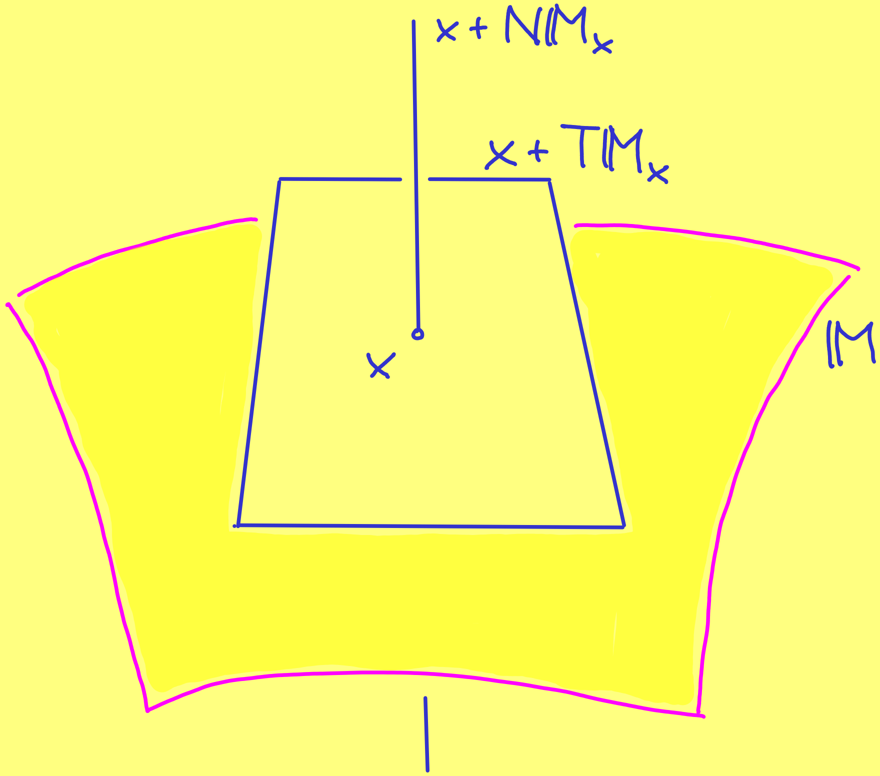
I.2 CURVATURE



$\dim \mathbb{M} = k = \dim \mathbb{T}M_x$

$K = \max_{x \in \mathbb{M}} \max_{v \in \mathbb{T}M_x} \kappa(x, v)$ is <u>max</u>. <u>abs</u>. <u>sect</u>. <u>curvature</u>

I.2 Curvature and Reach



$\dim \mathbb{M} = k = \dim T\mathbb{M}_x$

$d-k = \dim N\mathbb{M}_x$

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$\varrho = \min_{x \in \mathbb{M}} \min_{u \in N\mathbb{M}_x} \varrho(x,u)$ is (global) reach

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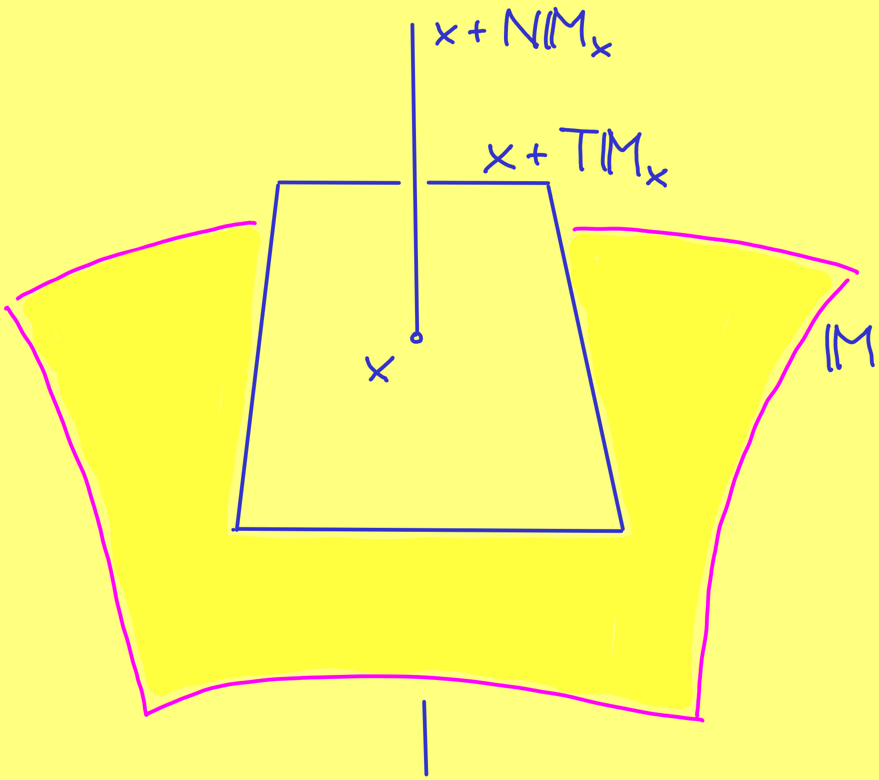

$$\dim \mathbb{M} = k = \dim T\mathbb{M}_x$$

$$d - k = \dim N\mathbb{M}_x$$

$$\kappa \cdot \varrho \leq 1$$

$$\kappa = \max_{x \in \mathbb{M}} \max_{v \in T\mathbb{M}_x} \kappa(x, v)$$ is <u>max</u>. <u>abs</u>. <u>sect</u>. <u>curvature</u>

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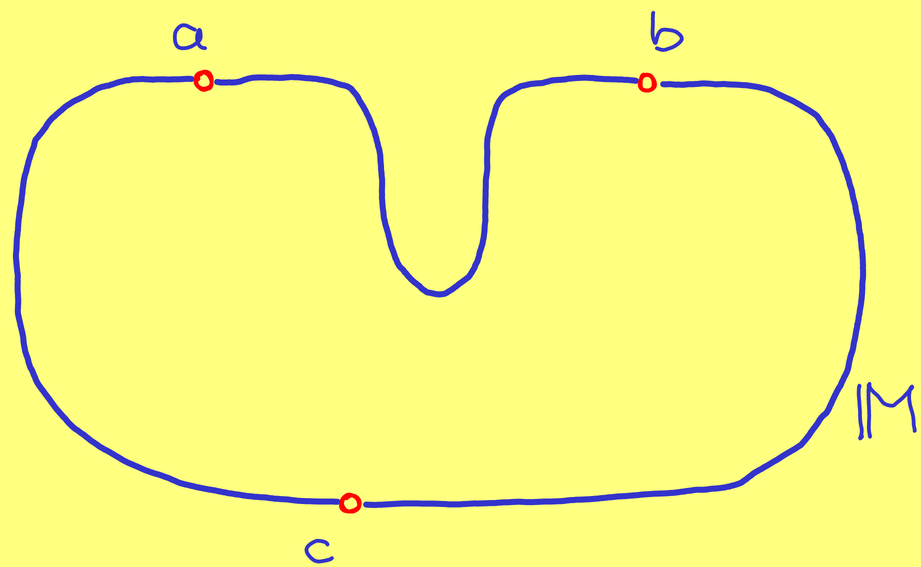
II.2 COUNTEREXAMPLE



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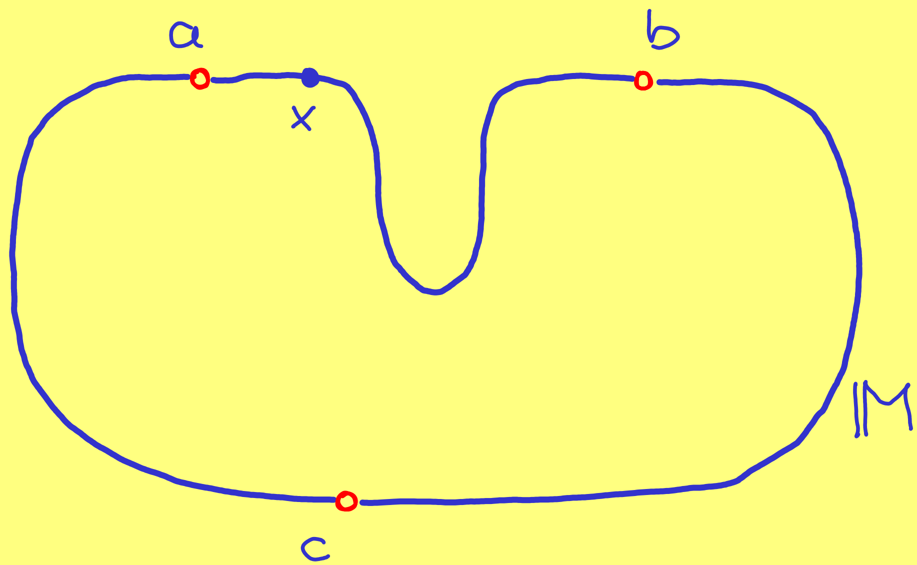

$x \in \mathbb{M}$ is weak witness of ab

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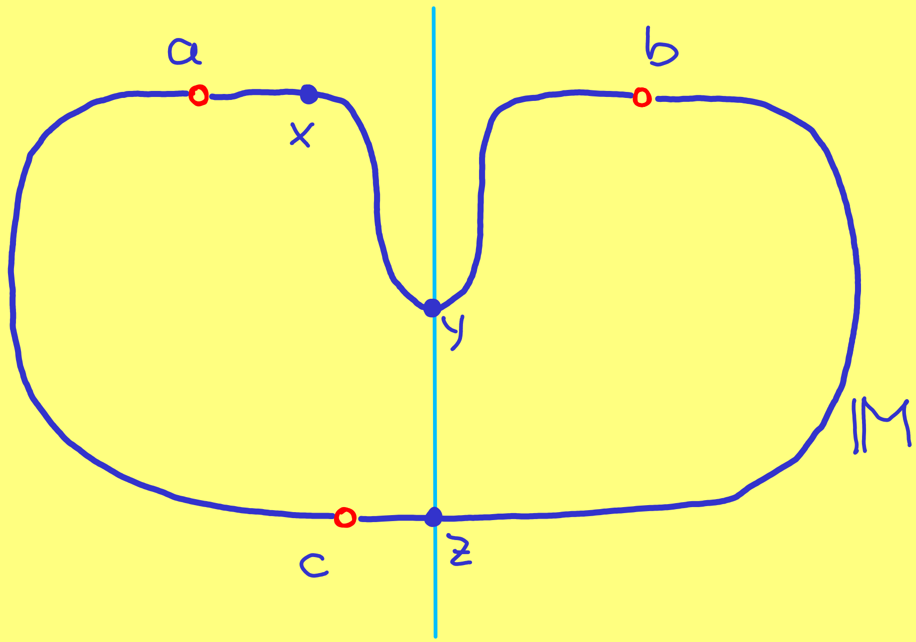

$x \in \mathbb{M}$ is weak witness of ab

$y, z \in \mathbb{M}$ are not strong witnesses of ab

I.3 Sampling Assumption

$\dim \mathbb{M} = k$

Def. An ε-sample is a subset $L \subseteq \mathbb{M}$ s.t. every point $x \in \mathbb{M}$ has at least $k+1$ points $a \in L$ with $\|x-a\| < \varepsilon \cdot \varrho$.

I.4 Main Result

$\dim \mathbb{M} = k$, ε-sample $L_\varepsilon \subseteq \mathbb{M}$

<u>weak</u> <u>Delaunay</u> <u>constant</u> is $\varepsilon_k = \sup_{\varepsilon}$ s.t.

$\forall \sigma \subseteq L_\varepsilon$ $\forall$ faces τ of σ $\exists$ weak witness of τ in $\mathbb{M}$

$\Rightarrow$ $\exists$ strong witness of σ in $\mathbb{M}$

Theorem. $\varepsilon_1 = \sqrt{3}$,

$$\frac{1}{\sqrt{5}} \leq \varepsilon_2 \leq \sqrt{2}.$$

II. Preliminaries

II.1 EUCLIDEAN SPACE

$\mathbb{M} = \mathbb{R}^d$, $\quad L \subseteq \mathbb{R}^d$

THEOREM [de Silva 03]

If every face of $\sigma \subseteq L$ has a weak witness in $\mathbb{R}^d$ then σ has a strong witness in $\mathbb{R}^d$.

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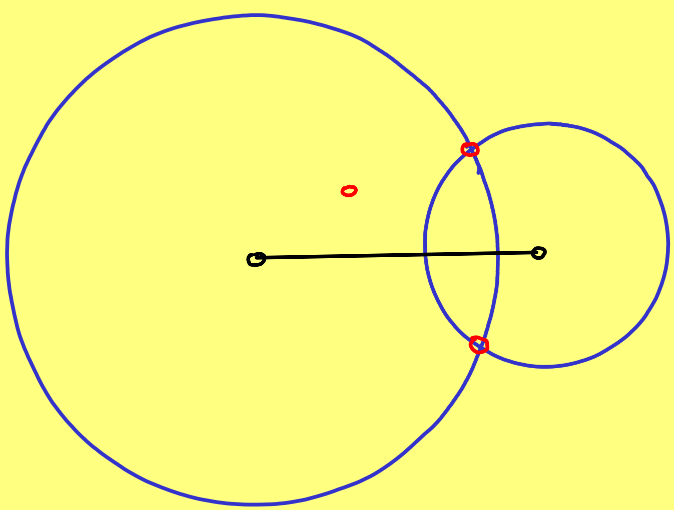
PROOF



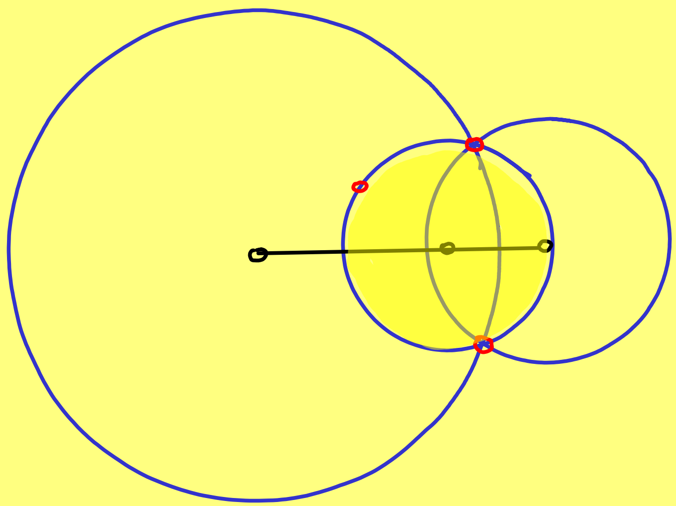
II.1 Euclidean Space

$\mathbb{M} = \mathbb{R}^d, \quad L \subseteq \mathbb{R}^d$

Theorem [de Silva 03]

If every face of $\sigma \subseteq L$ has a weak witness in $\mathbb{R}^d$ then σ has a strong witness in $\mathbb{R}^d$.

Proof



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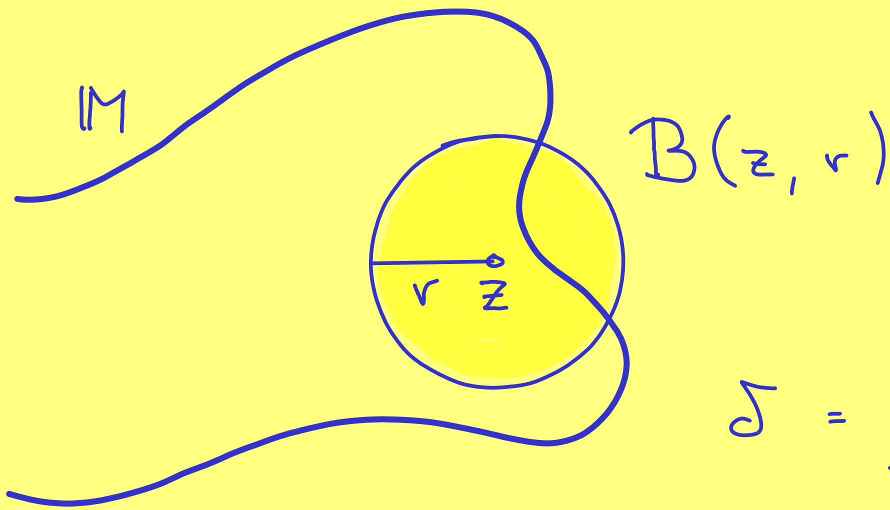
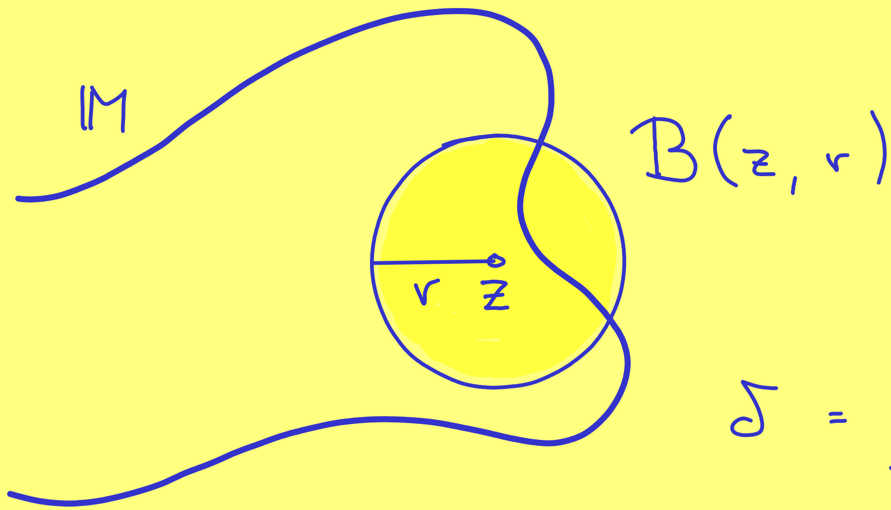
II.3 Key Topological Lemma



$$\delta = \min_{x \in \mathbb{M}} \| x - z \|$$

II.3 Key Topological Lemma



$$\delta = \min_{x \in \mathbb{M}} \|x - z\|$$

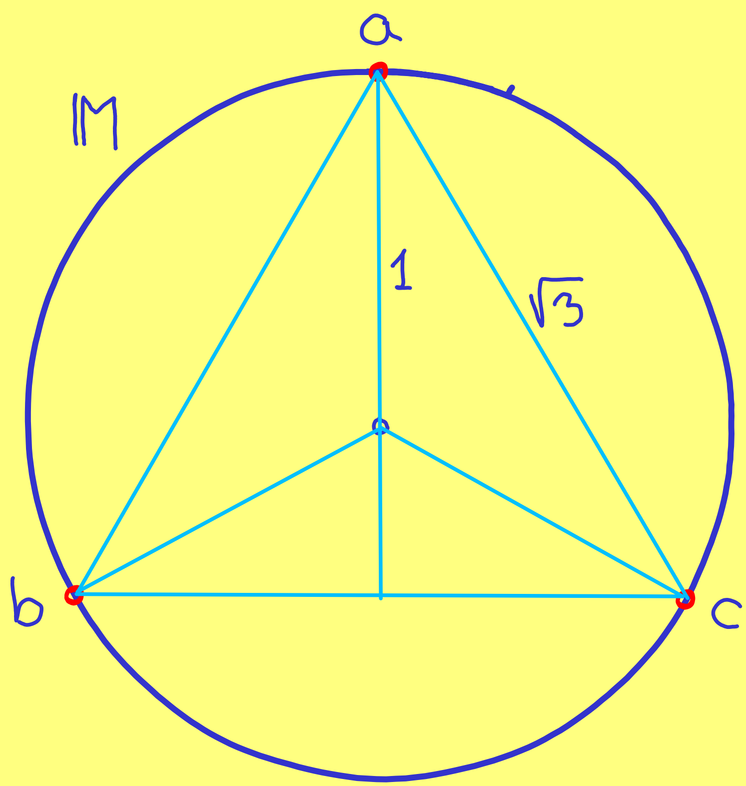
Reach Lemma.

$\delta < \rho$ and $\delta < r < 2\rho - \delta$

$\Longrightarrow$ $B(z,r) \cap \mathbb{M}$ is a topological k-ball.

III. Curves

III.1 Upper Bound : $\varepsilon_1 \leq \sqrt{3}$



ε - sample for $\varepsilon > \sqrt{3}$

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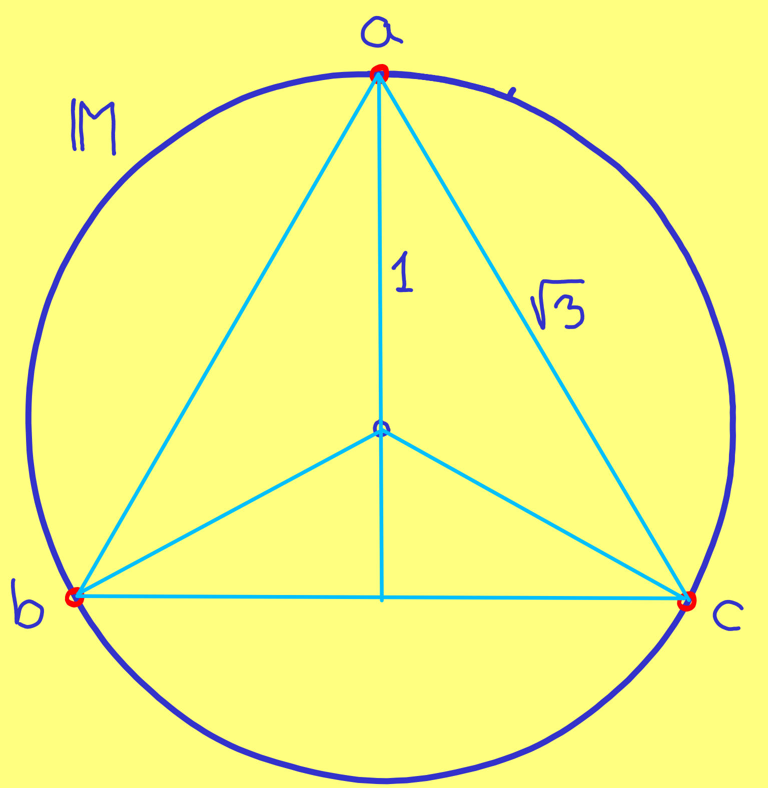
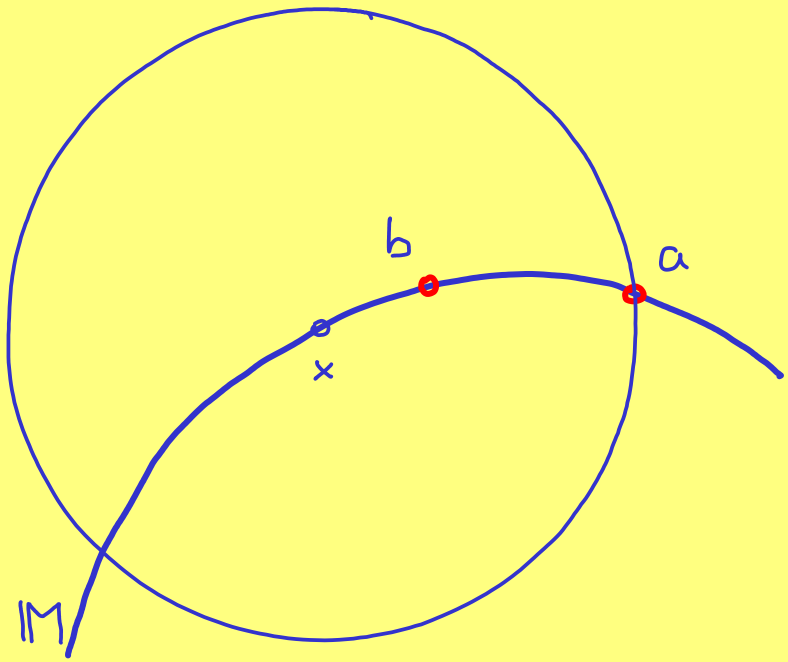


ε - sample for $\varepsilon > \sqrt{3}$

every face of abc has a weak witness,

but abc has no strong witness

III.2 EDGES



$B(x, r)$ with $r < \sqrt{3}\varrho$

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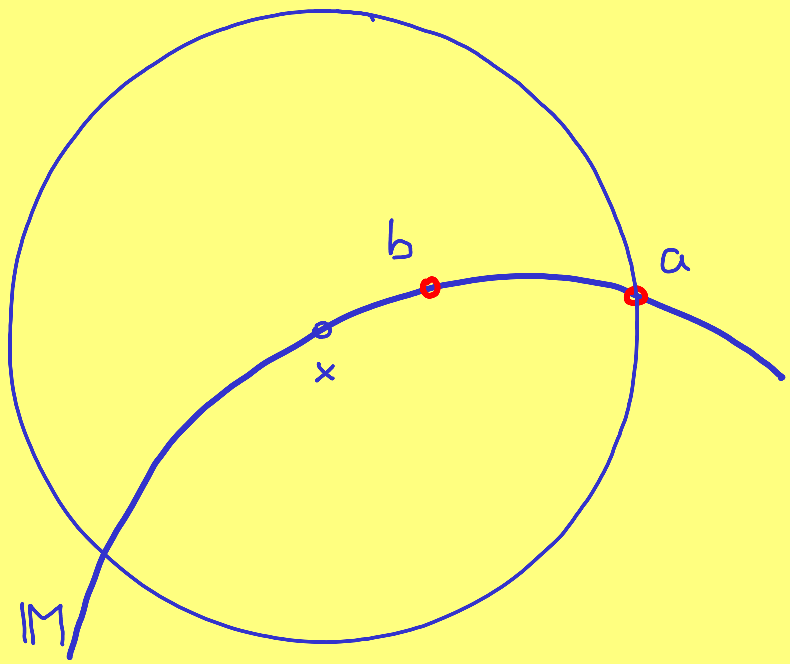

$B(x, r)$ with $r < \sqrt{3}\rho$

$\overset{\text{R.L.}}{\Longrightarrow}$ $B(x,r) \cap \mathbb{M}$ is topological interval

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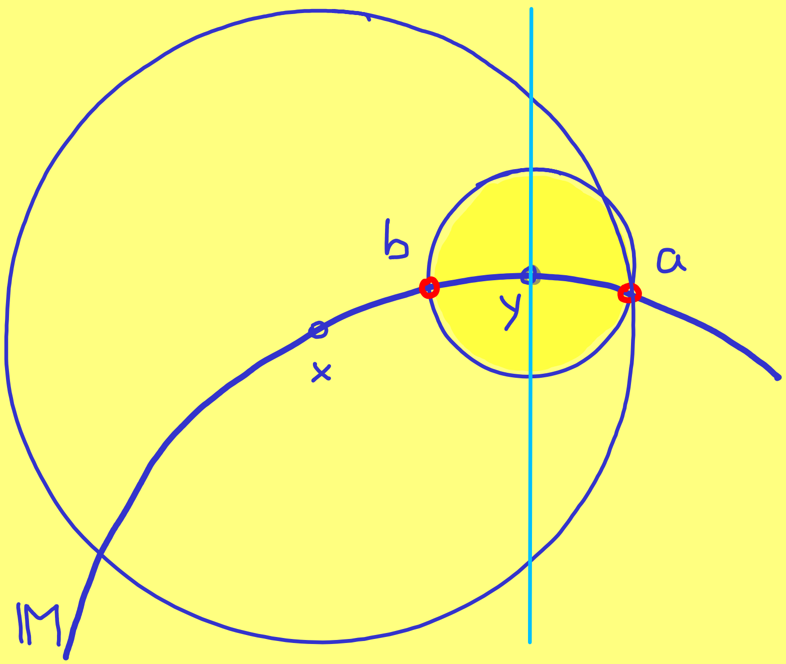


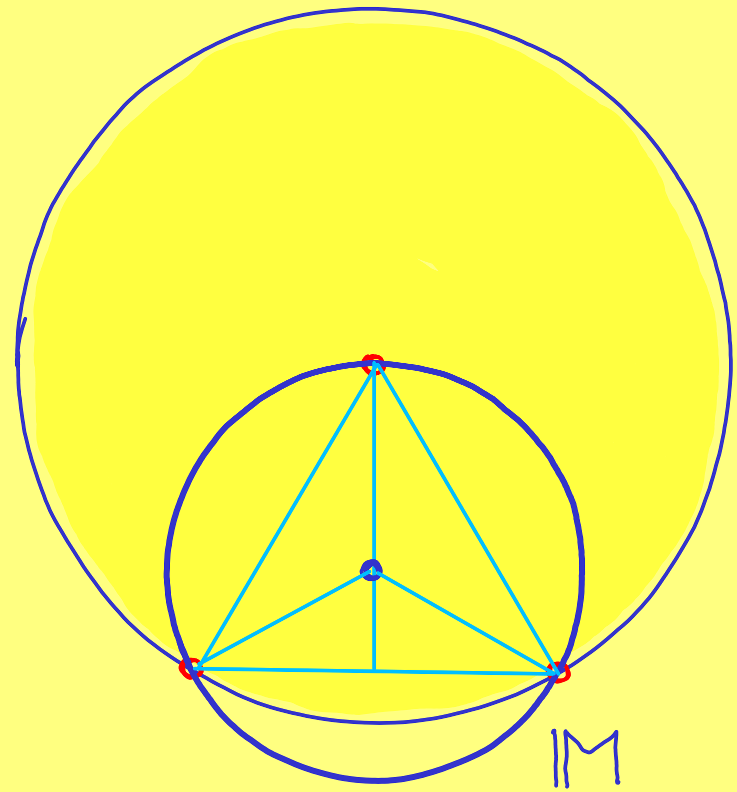
$B(x, r)$ with $r < \sqrt{3}\,\varrho$

R.L.

$\Longrightarrow$ $B(x,r) \cap \mathbb{M}$ is topological interval

$\Longrightarrow$ y is strong witness of ab

III.3 TRIANGLES

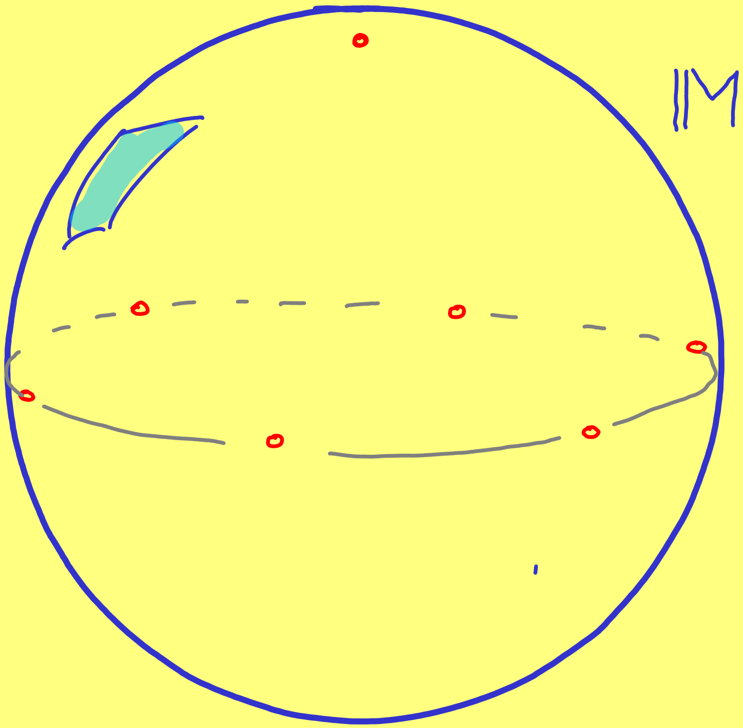


$$\text{length}(\mathbb{M}) \geq 2\pi\rho$$

$$\text{distance between landmarks} < \frac{2\pi}{3}\rho$$

IV. Surfaces

IV.1 UPPER BOUND : $\varepsilon_2 \leq \sqrt{2}$



ε - sample for $\varepsilon > \sqrt{2}$

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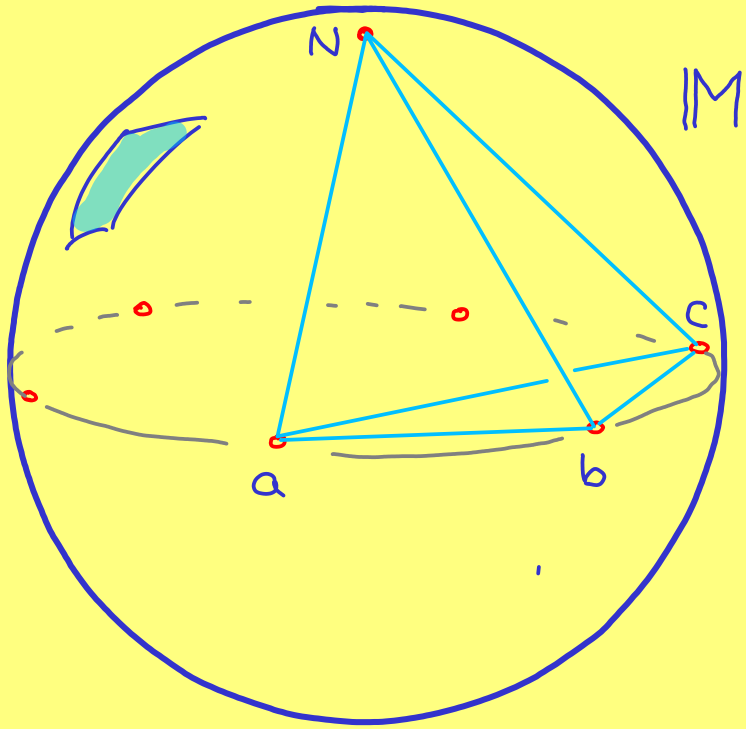

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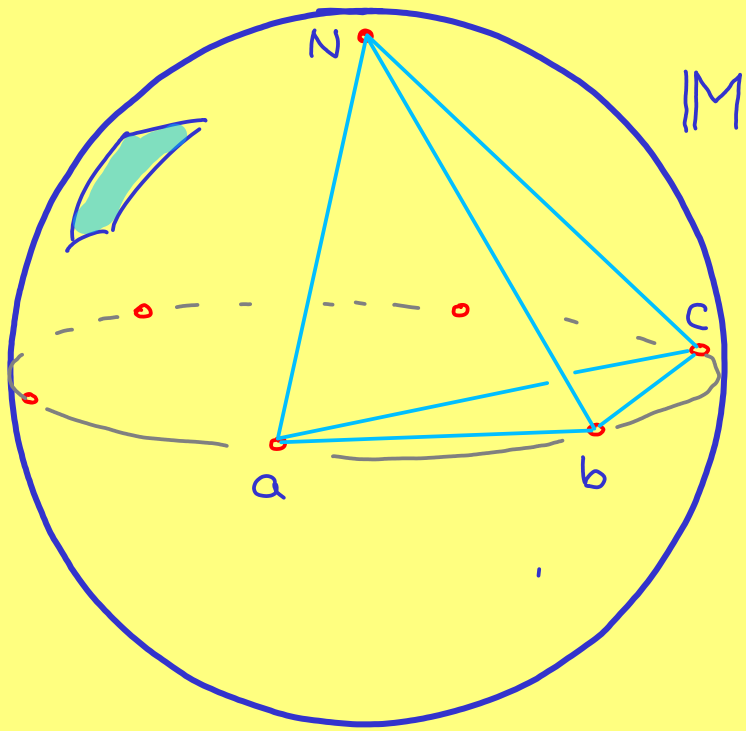

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every face of Nabc has a weak witness,

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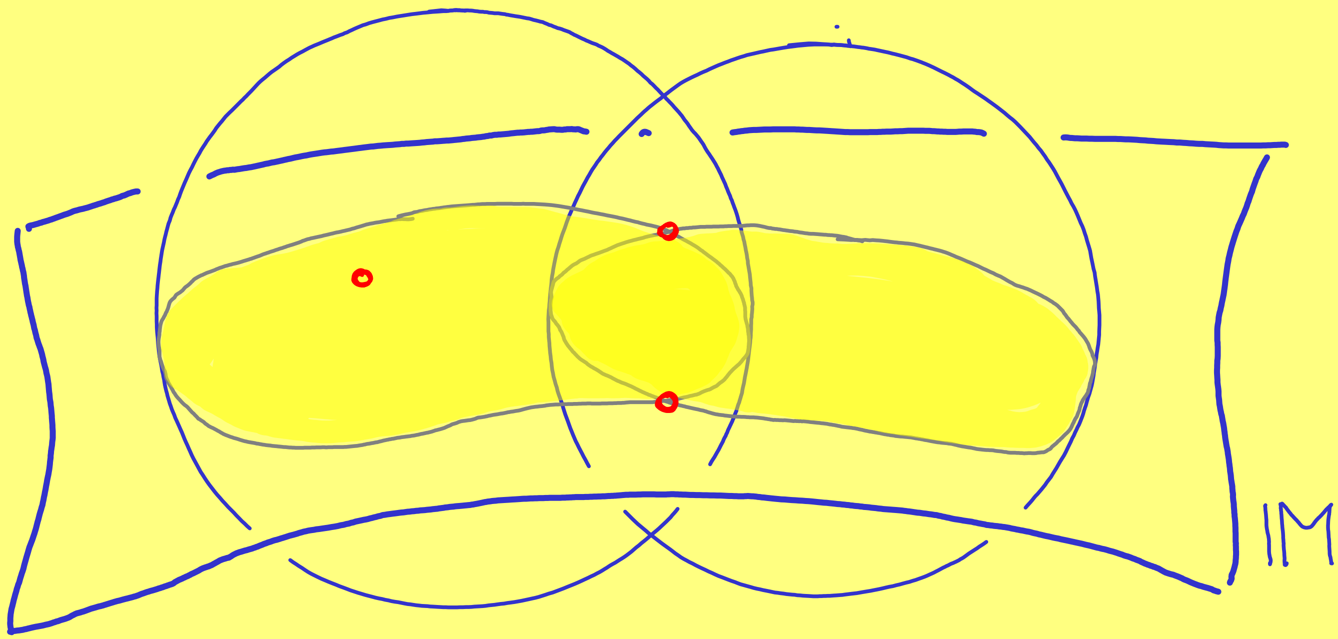
IV.2 TRIANGLES



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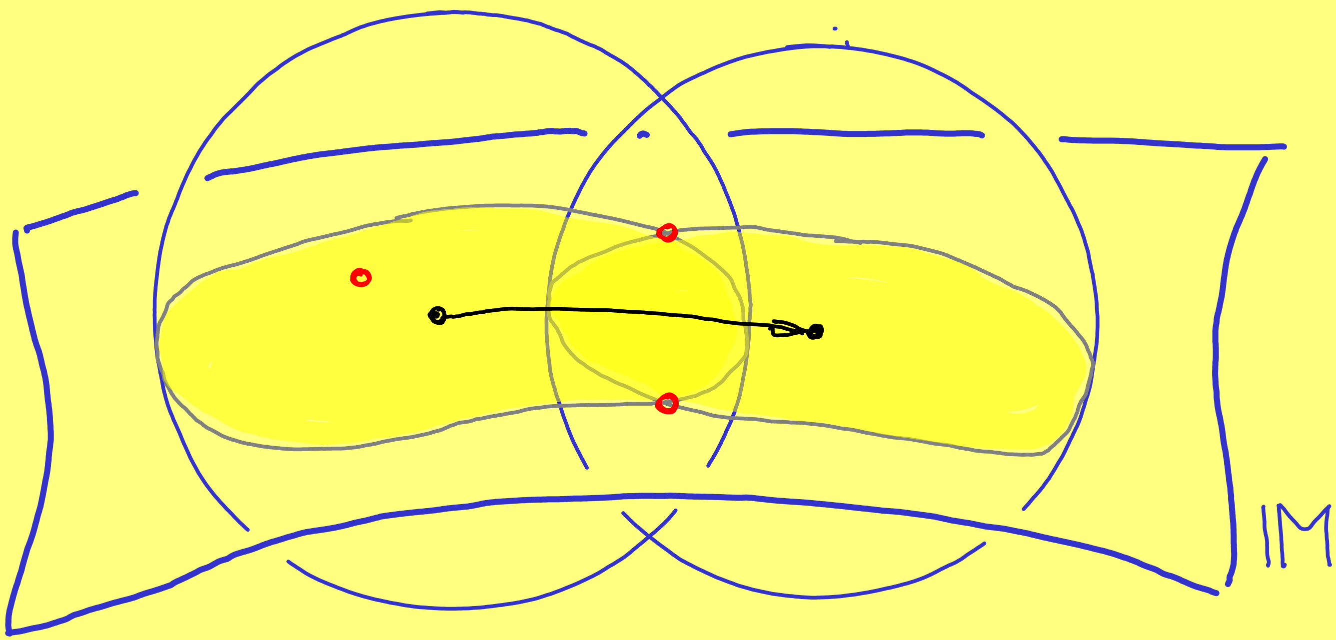
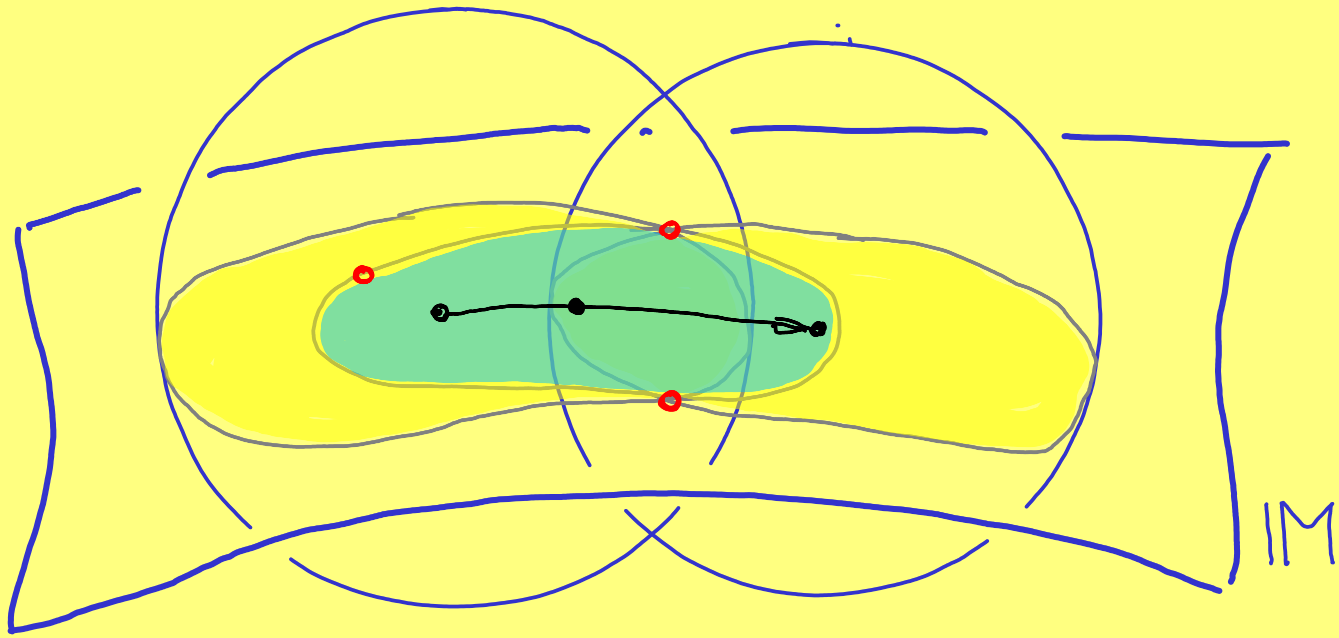


construct $\alpha : [0,1] \to \mathbb{M}$ by intersecting $\mathbb{M}$ with normal diameter disk

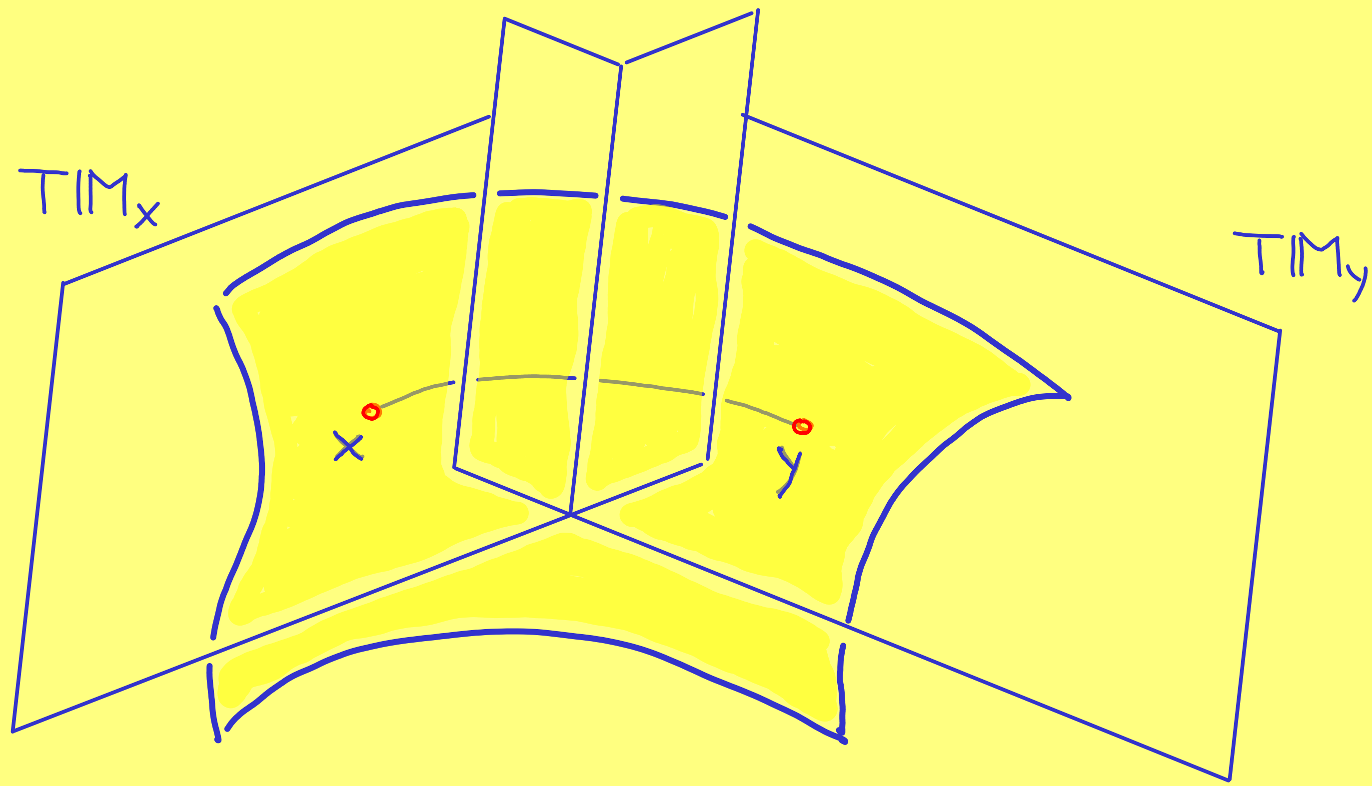
IV.2 Triangles



construct $\alpha : [0,1] \to \mathbb{M}$

move along α to find strong witness

II.3 INEQUALITIES



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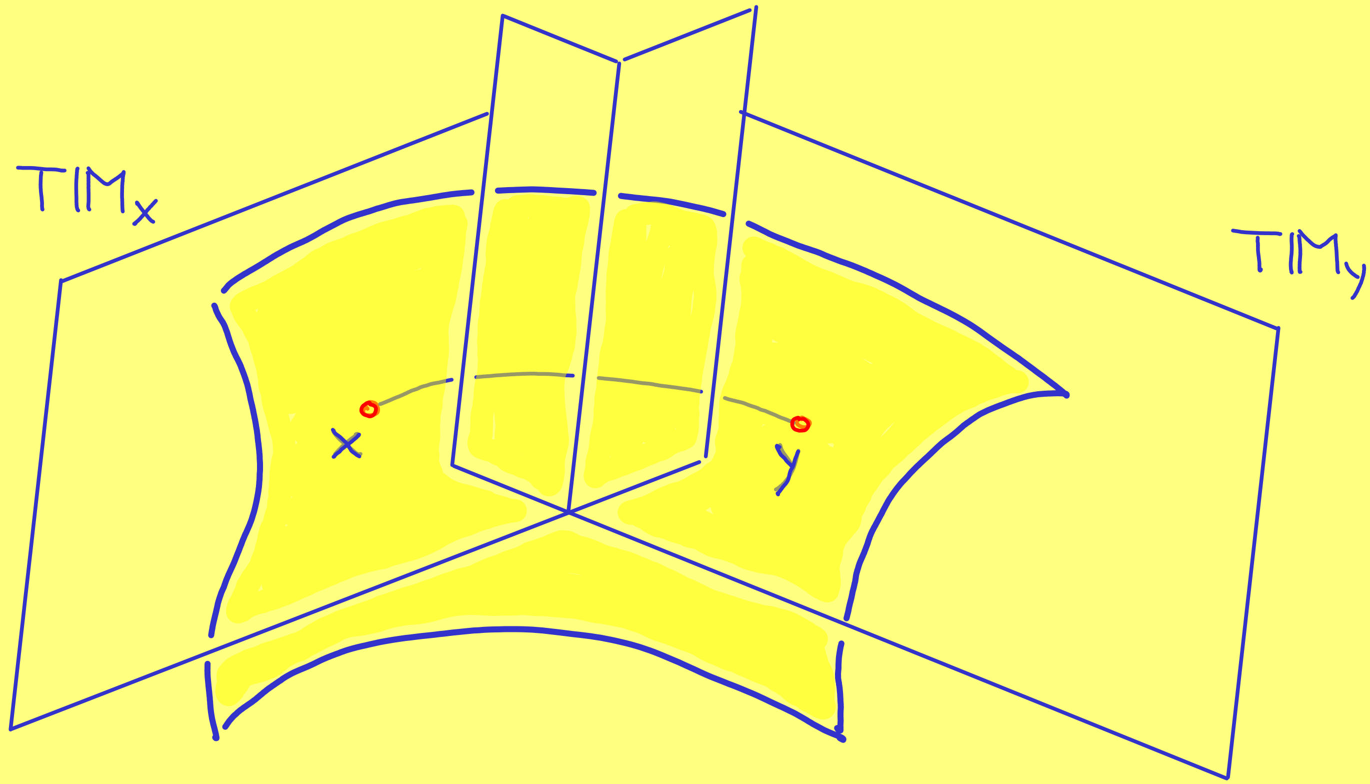

PROP. I $\quad d(x,y) \le \frac{2}{\kappa} \arcsin\left(\frac{\kappa}{2}\|x-y\|\right)$ if $\|x-y\| \le \frac{2}{\kappa}$.

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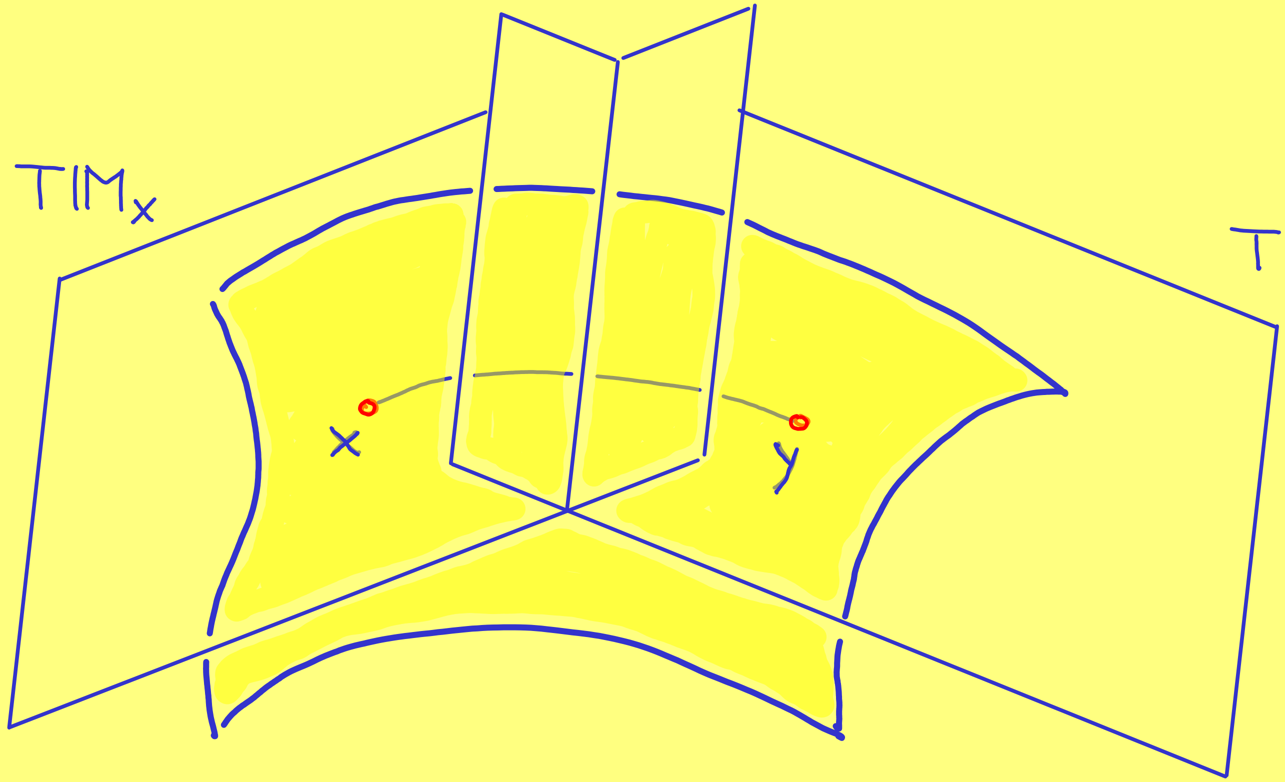

Prop. I $\quad d(x,y) \le \frac{2}{\kappa}\arcsin\left(\frac{\kappa}{2}\|x-y\|\right)$ if $\|x-y\| \le \frac{2}{\kappa}$.

Prop. II $\quad \measuredangle\, T_xM, T_yM \le \kappa\, d(x,y)$.

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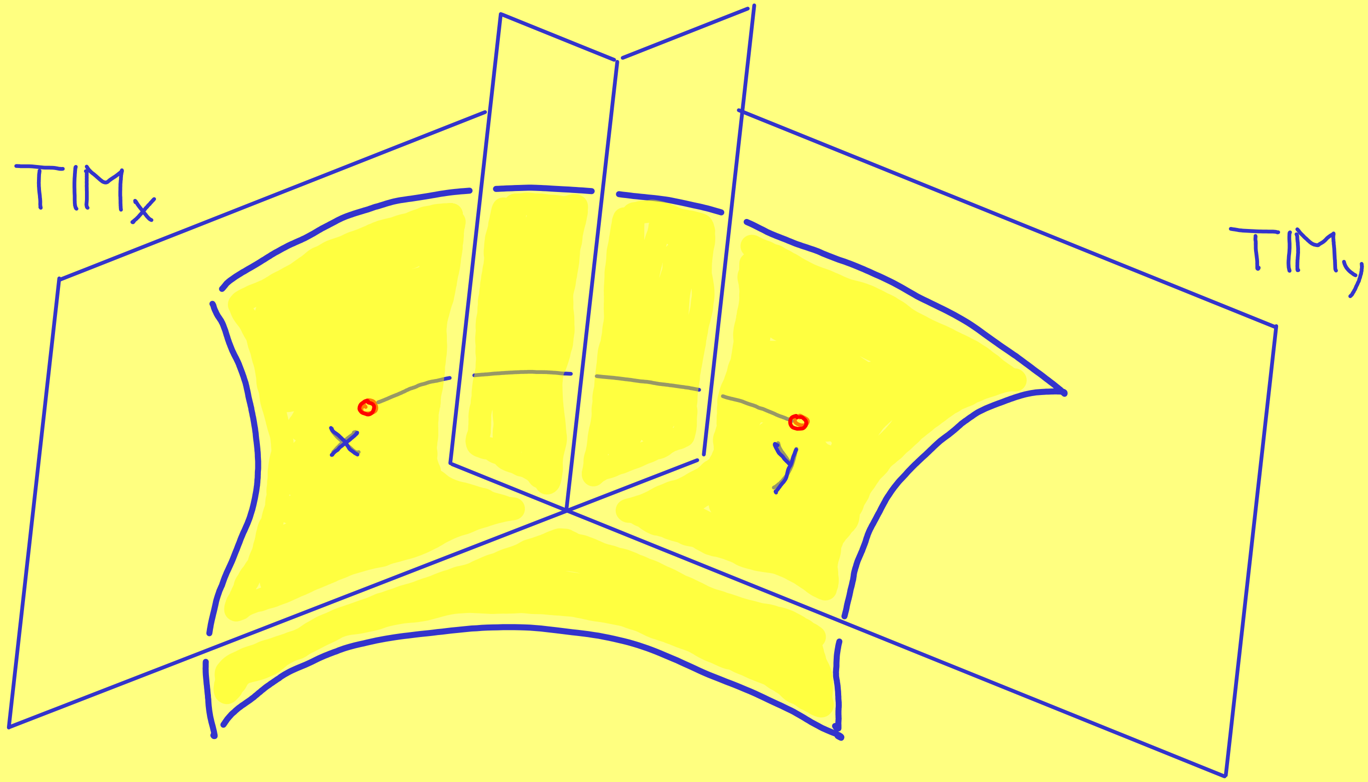

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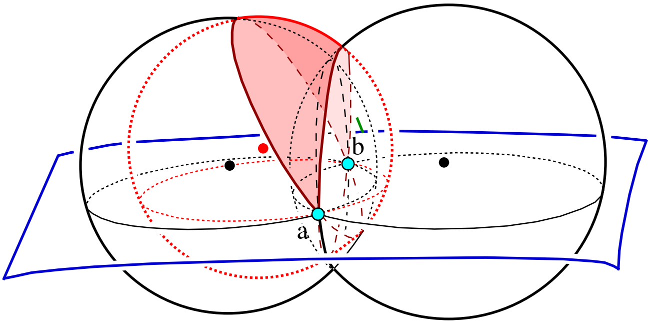
Prop. II $\quad \measuredangle\, TIM_x TIM_y \le \kappa\, d(x,y)$.

Prop. III $\quad \measuredangle (y-x) TIM_x \le \frac{\kappa}{2} d(x,y)$ if $d(x,y) \le \frac{\pi}{2\kappa}$.

Triangles



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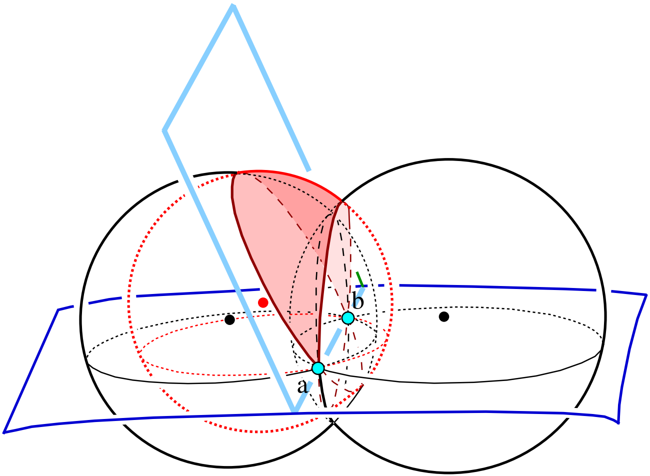

IV.3 TETRAHEDRA

4 nodes

12 arcs

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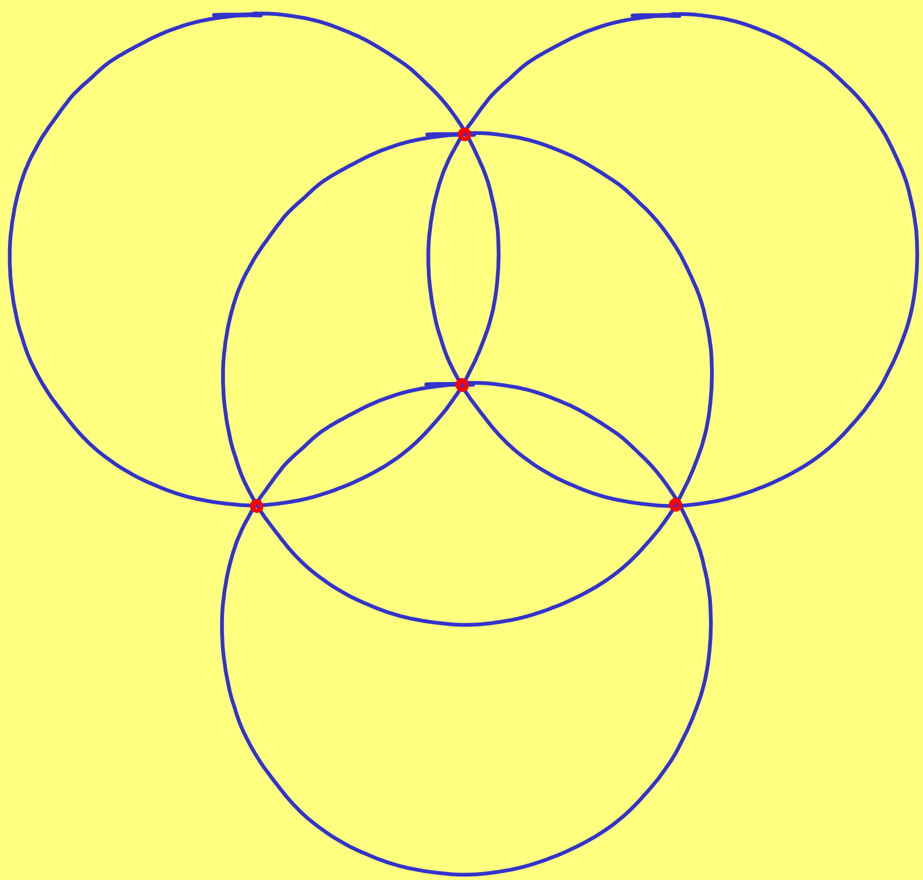

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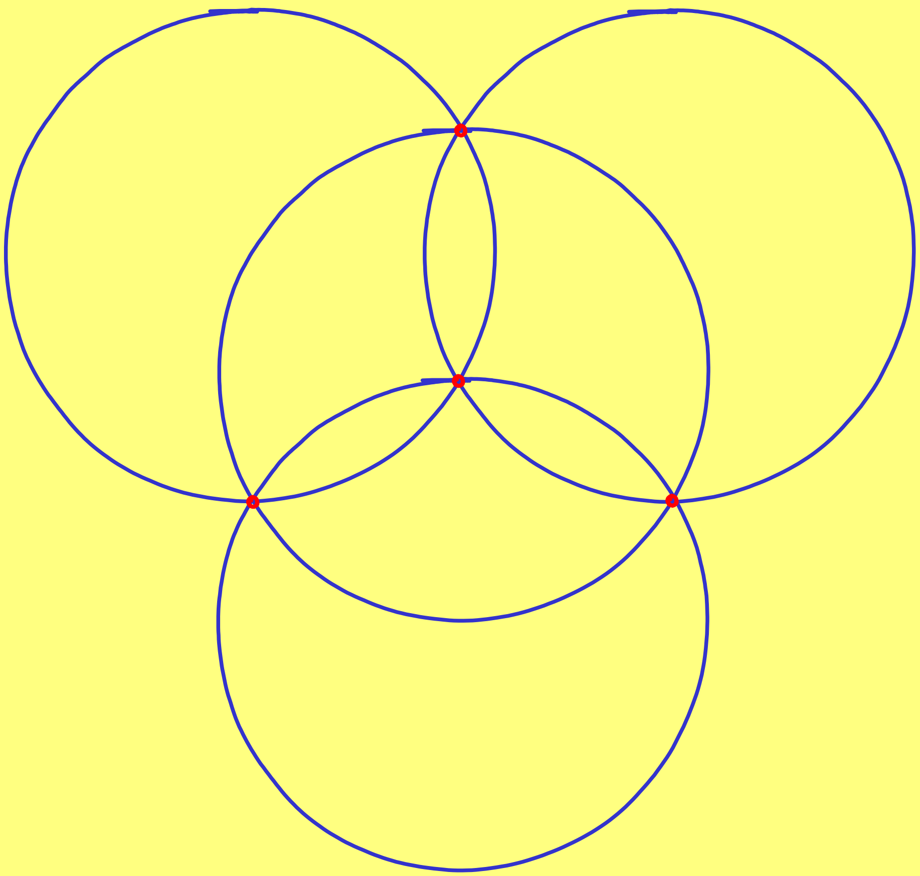

4 nodes

12 arcs

planar

$\Longrightarrow$ node inside circle

$\Longrightarrow$ all circles the same

Conclusion

- Witness complexes approximate restricted Delaunay triangulations for curves and surfaces.
  - $\varepsilon_1 = \sqrt{3}$.
  - $\frac{1}{\sqrt{5}} \leq \varepsilon_2 \leq \sqrt{2}$.

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Conclusion

- Witness complexes approximate restricted Delaunay triangulations for curves and surfaces.
  - $\varepsilon_1 = \sqrt{3}$.
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- For k-manifolds with $k \geq 3$, situation more complicated:
  - $\varepsilon_k = 0$ for $k \geq 3$ $\rightarrow$ counterexample by Oudot uses slivers
  - Boissonnat et al. assign weights to landmarks to eliminate slivers

THANK YOU

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