Shape reconstruction in high dimensions using Rips complexes

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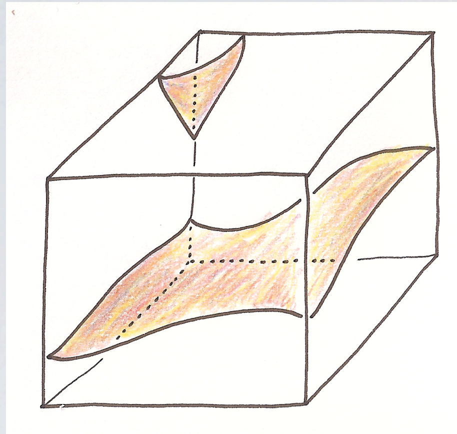
Topological data analysis and machine learning theory

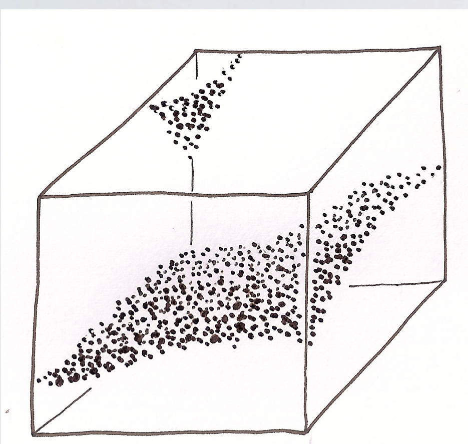
October 14 - 19, 2012

Shape



Approximation

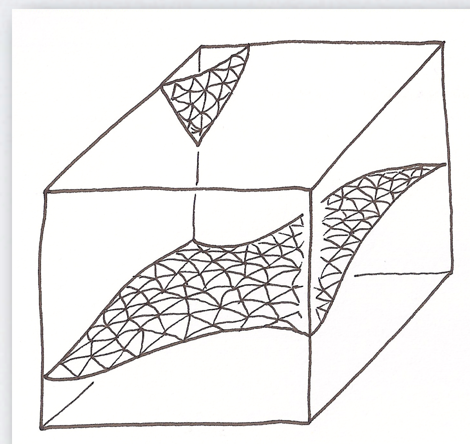
n points



Input

RECONSTRUCTION

Simplicial complex



Output

PROCESSING

Betti numbers

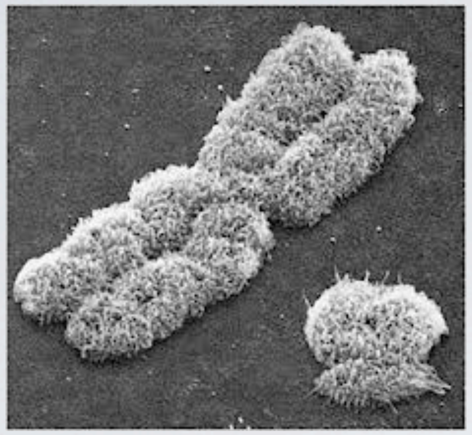
Volume

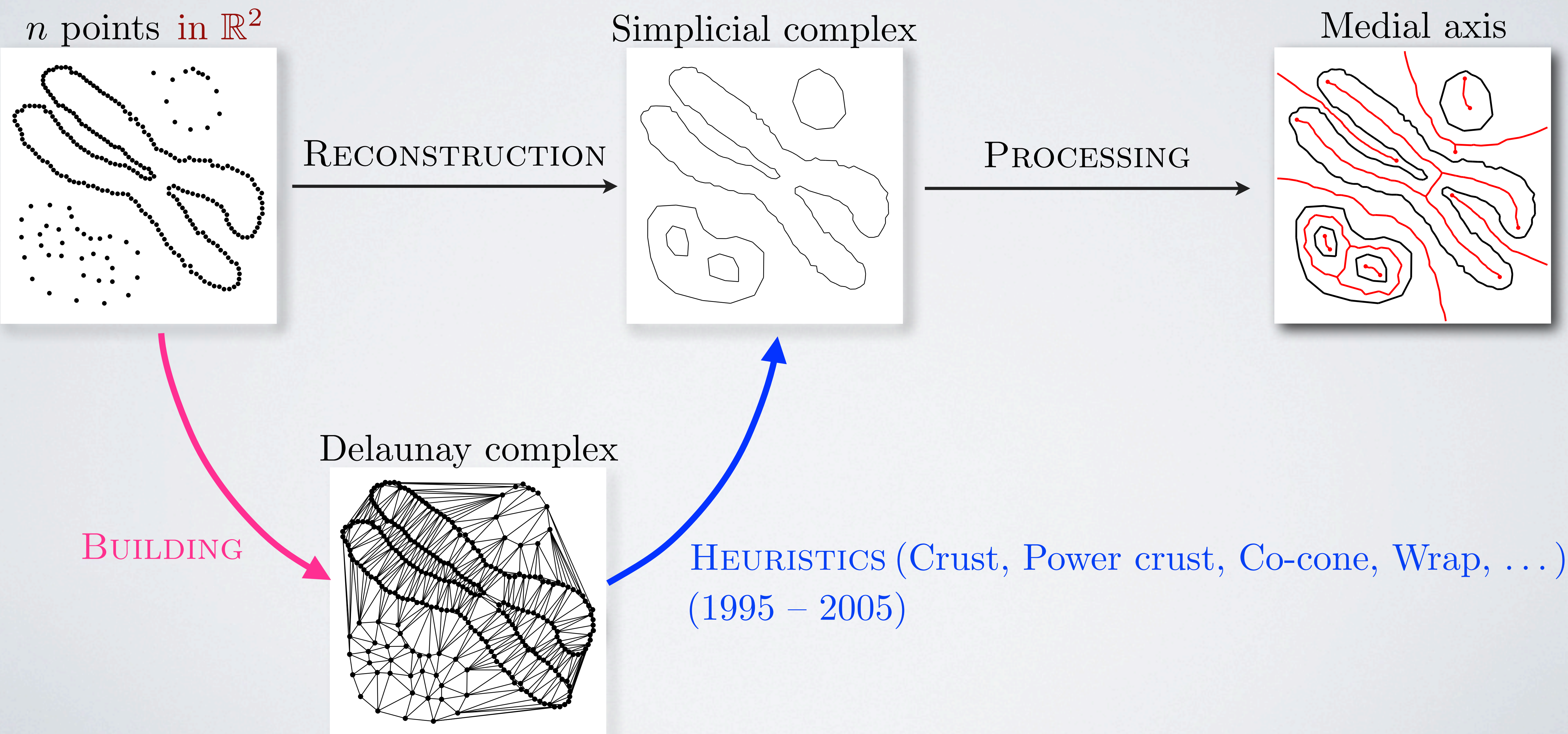
Medial axis

Signatures

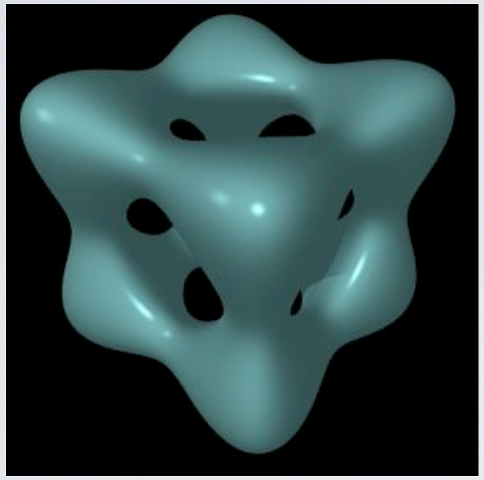
...

in 2D

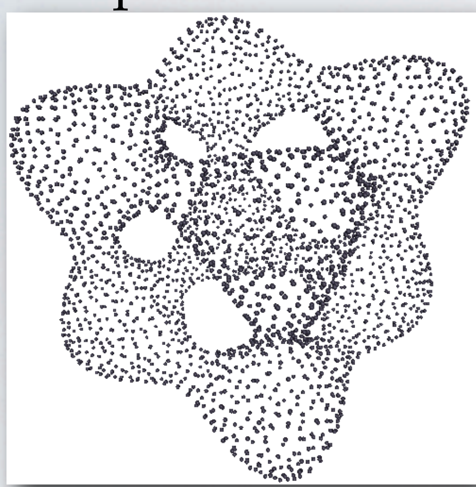



n points in $\mathbb{R}^2$
Simplicial complex
Medial axis
RECONSTRUCTION
PROCESSING
BUILDING
Delaunay complex
HEURISTICS (Crust, Power crust, Co-cone, Wrap, . . .)
(1995 – 2005)

in 3D

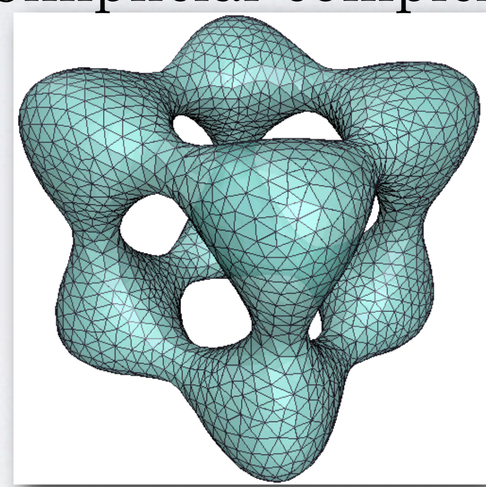


n points in $\mathbb{R}^3$



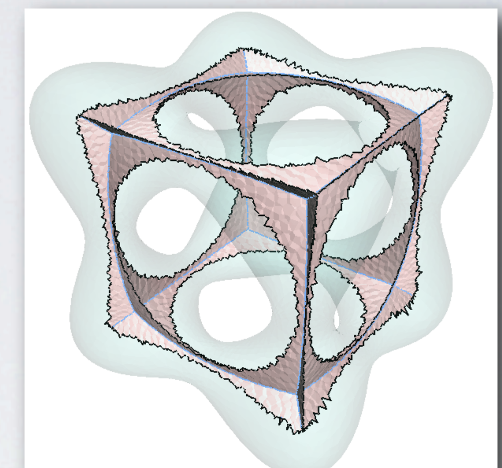
Reconstruction $\longrightarrow$

Simplicial complex



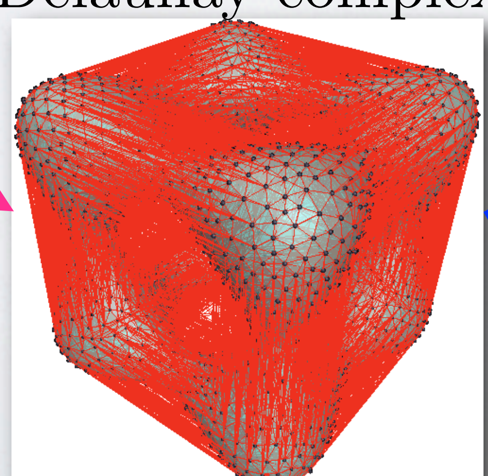
Processing $\longrightarrow$

Medial axis



Building

Delaunay complex

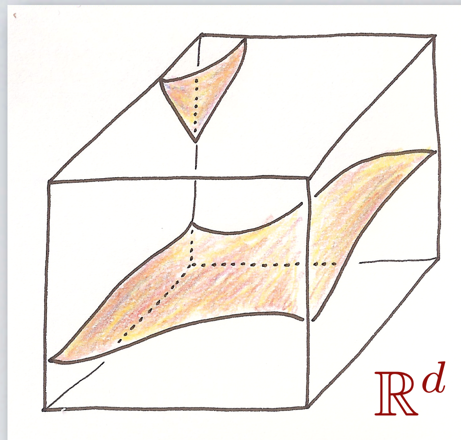


Heuristics (Crust, Power crust, Co-cone, Wrap, ...) (1995 – 2005)

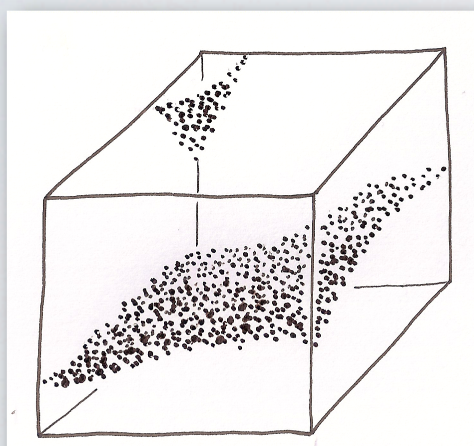
- ✴ In $\mathbb{R}^3$, has size $O(n^2)$
- ✴ In practice, has size $O(n)$

in dD

Shape

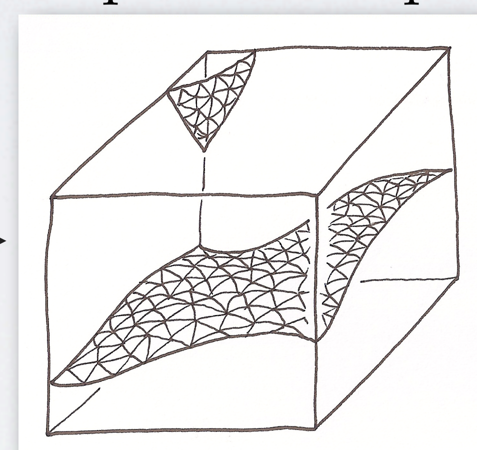


n points in $\mathbb{R}^d$



RECONSTRUCTION →

Simplicial complex



PROCESSING →

Betti numbers

Volume

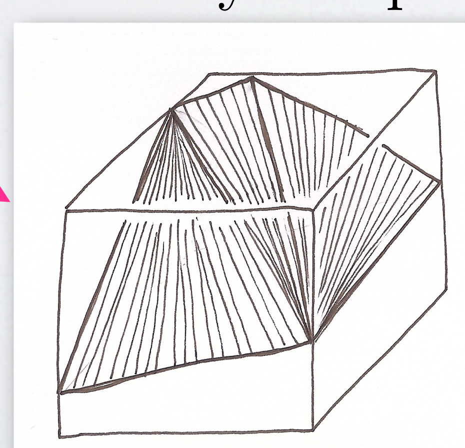
Medial axis

Signatures

$\ldots$

BUILDING

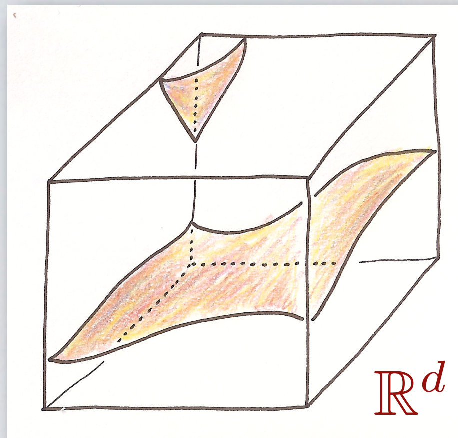
Delaunay complex



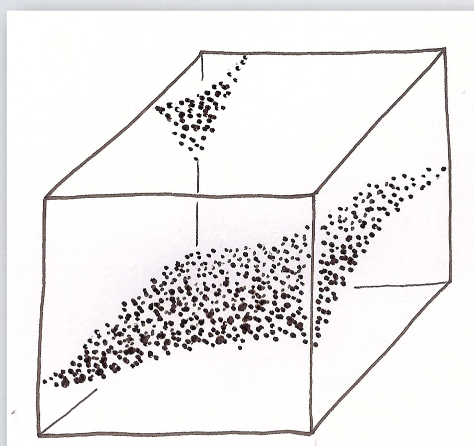
curse of dimensionality

in dD

Shape

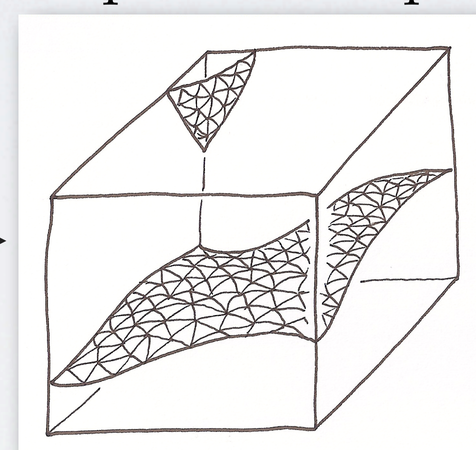


n points in $\mathbb{R}^d$



RECONSTRUCTION

Simplicial complex



PROCESSING

Betti numbers

Volume

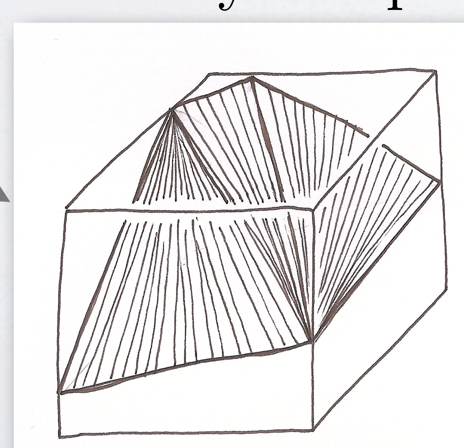
Medial axis

Signatures

...

~~BUILDING~~

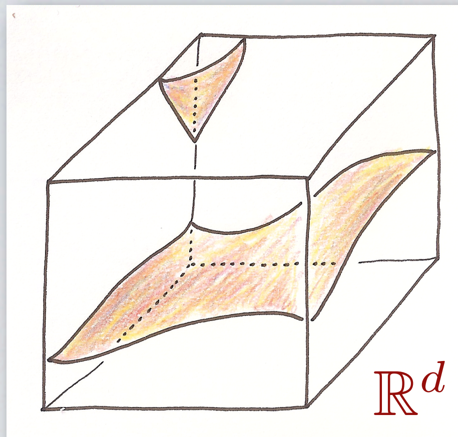
Delaunay complex



How to reconstruct without Delaunay?

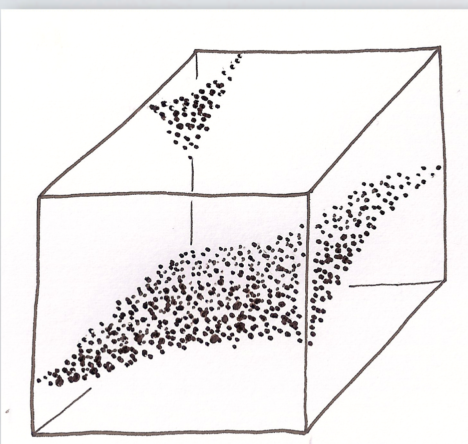
in dD

Shape



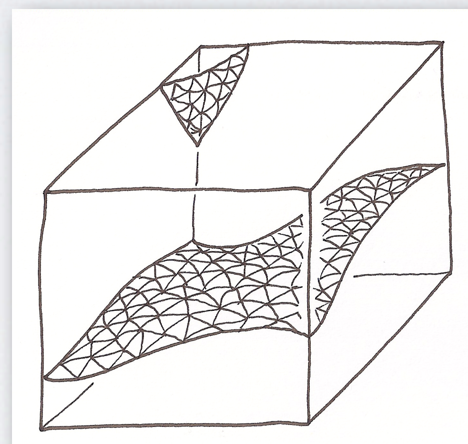
Guaranties on the result?

n points in $\mathbb{R}^d$



RECONSTRUCTION

Simplicial complex



PROCESSING

Betti numbers

Volume

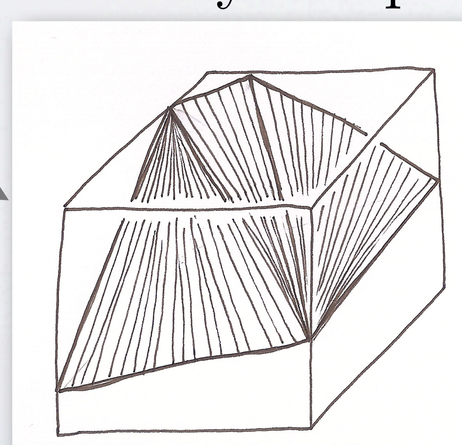
Medial axis

Signatures

...

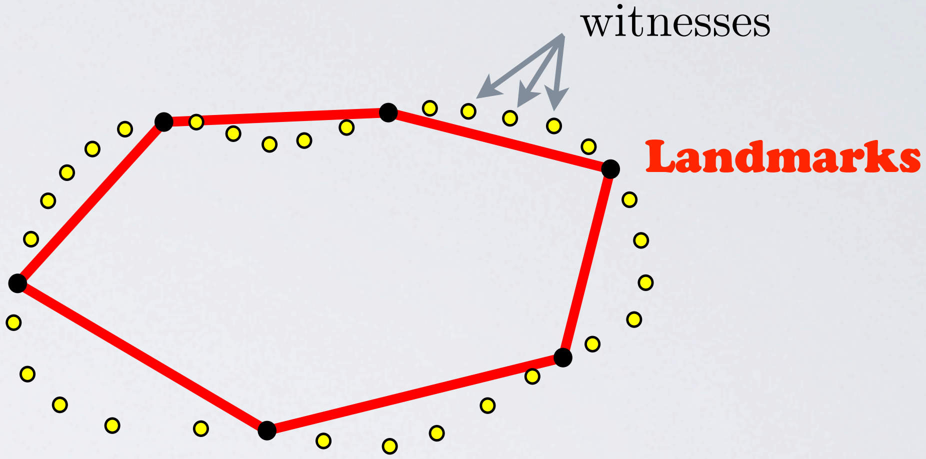
~~BUILDING~~

Delaunay complex

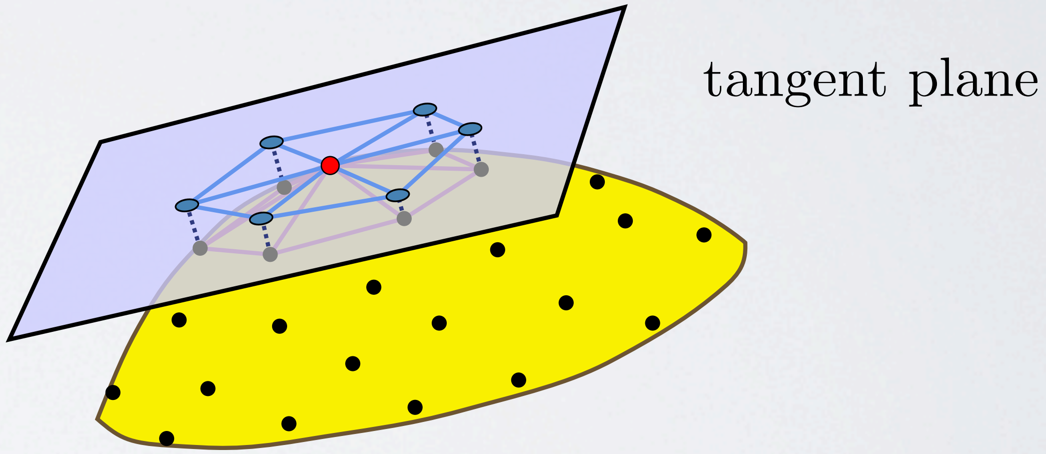


How to reconstruct without Delaunay?

How to reconstruct without building the whole Delaunay complex?



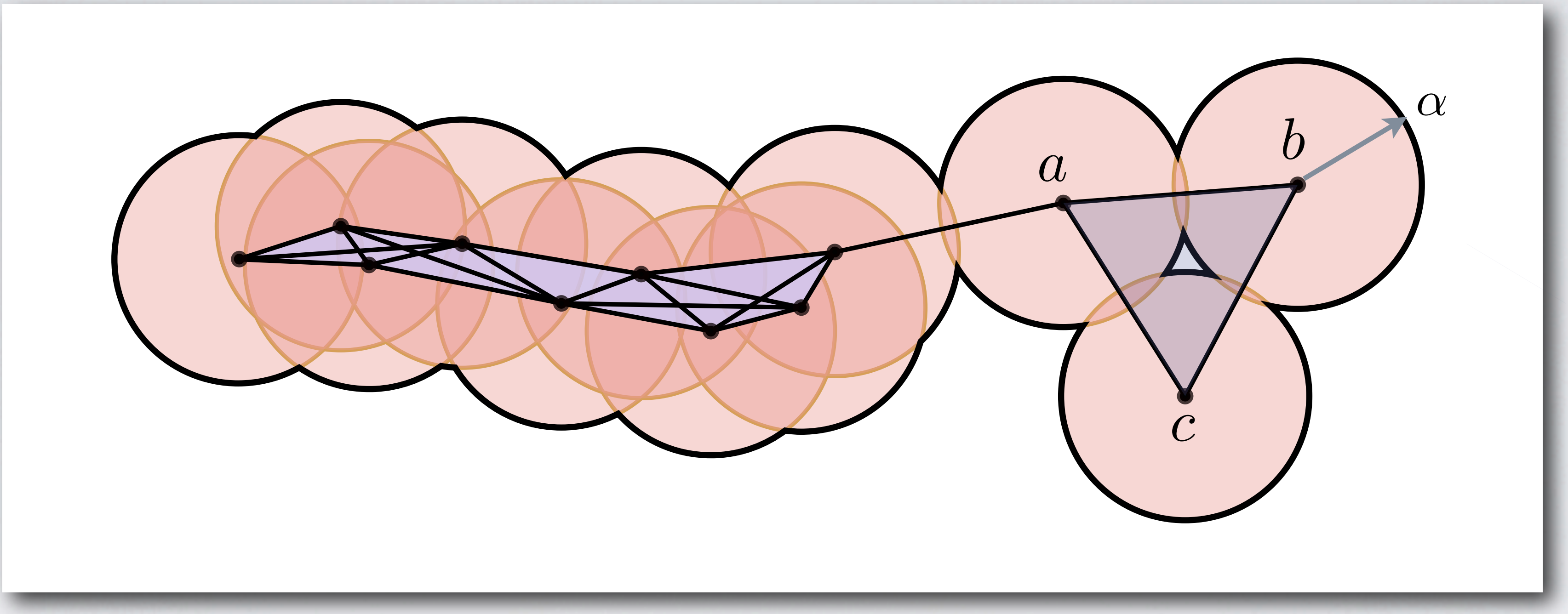
- weak Delaunay triangulation
  [de Silva 2008]



- tangential Delaunay complexes
  [Boissonnat & Ghosh 2010]

- Rips complexes
  our approach

Rips complexes



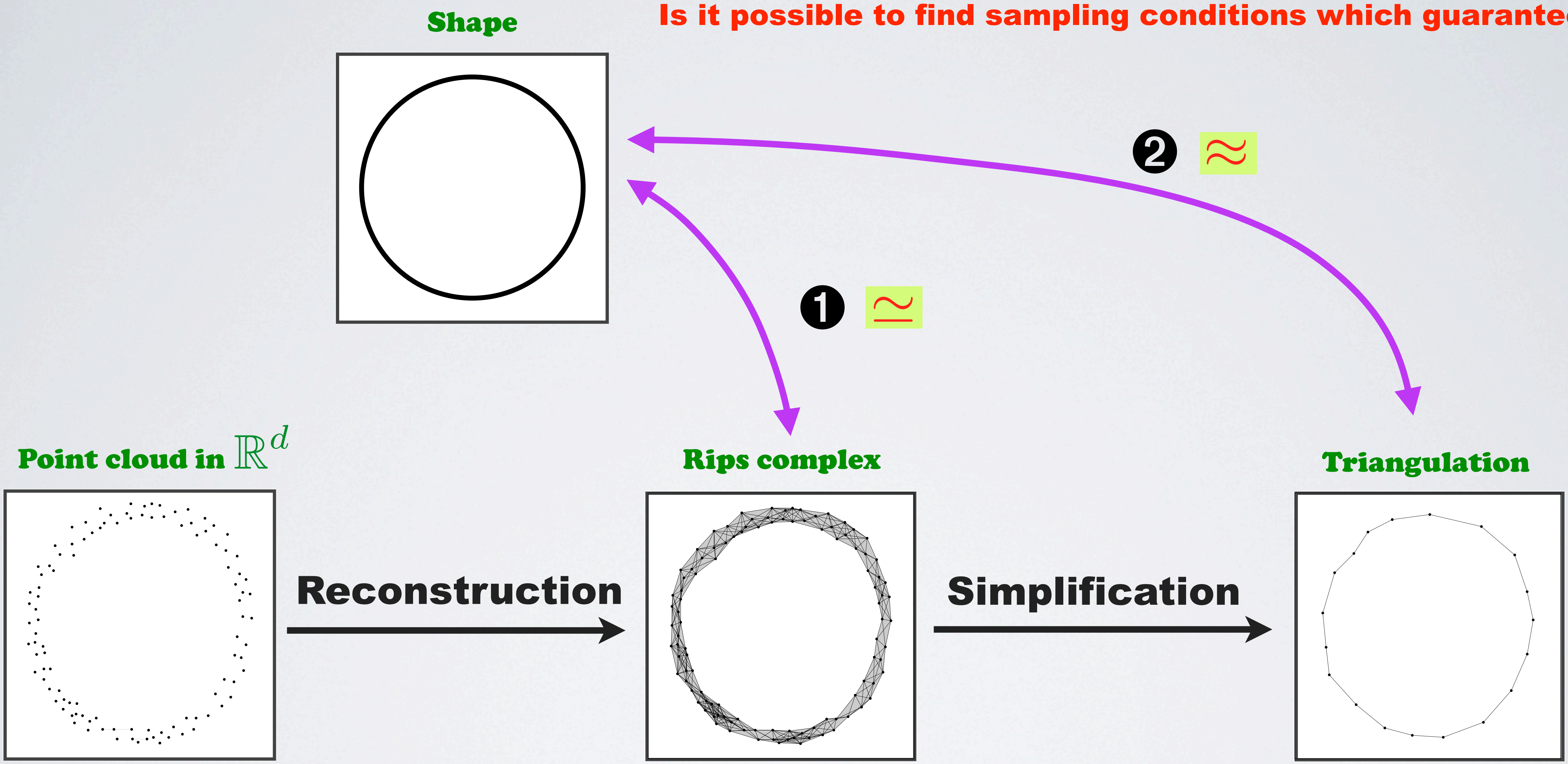
$$\mathrm{Rips}(P, \alpha) = \{\sigma \subset P \mid \mathrm{Diameter}(\sigma) \leq 2\alpha\}$$

$$\boxed{\mathrm{Rips}(P, \alpha) \supset \mathrm{Cech}(P, \alpha)}$$

- proximity graph G_α connects every pair of points within 2α
- $\mathrm{Rips}(P, \alpha) = \mathrm{Flag}\, G_\alpha$ [Flag G = largest complex whose 1-skeleton is G]
- compressed form of storage through the 1-skeleton
- easy to compute

Overview

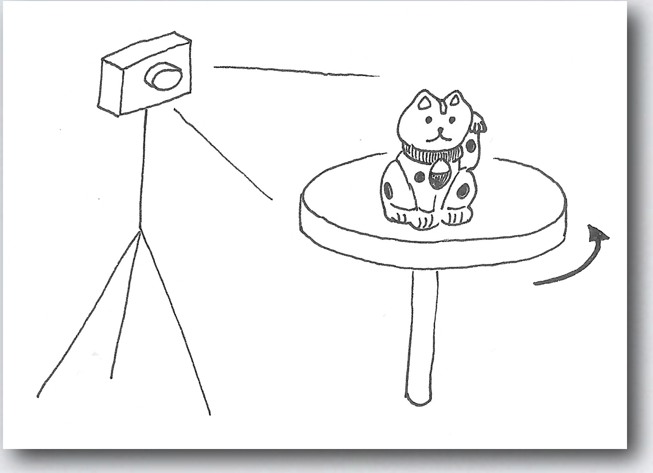
Is it possible to find sampling conditions which guarantee?



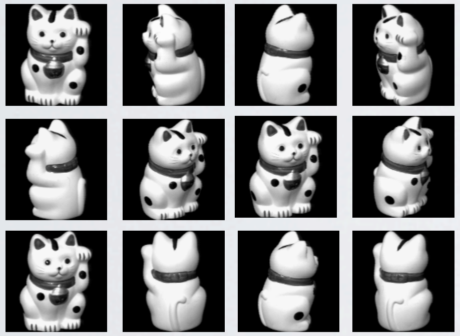
Can be high-dimensional!

Example

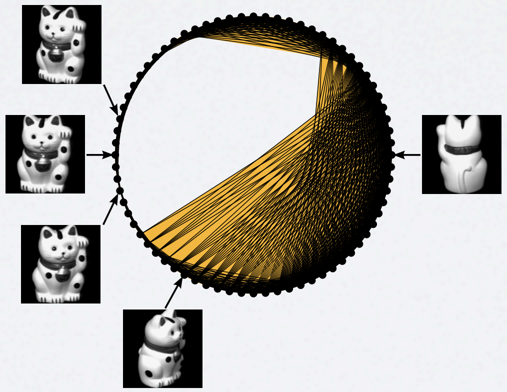
Physical system



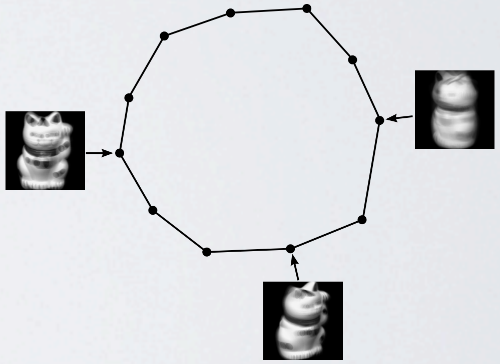
Point cloud in $\mathbb{R}^{128^2}$



Rips complex



Polygonal curve



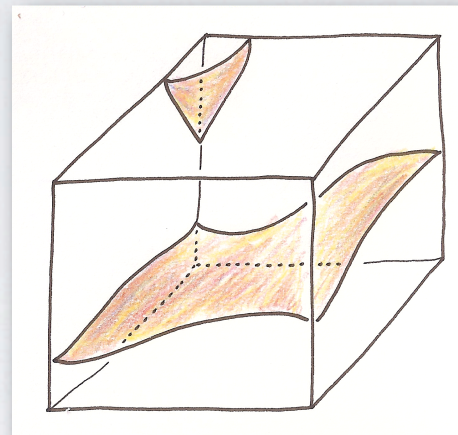
Correct homotopy type

Correct intrinsic dimension

Is high-dimensional!

Reconstruction Theorems

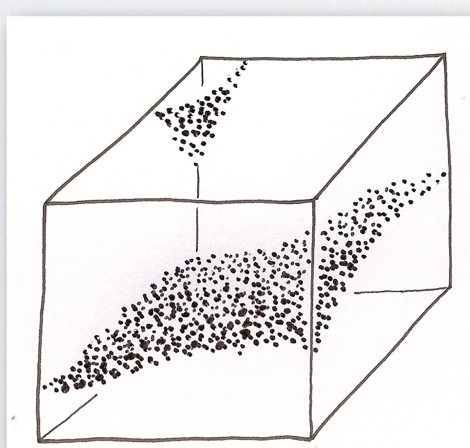
A



Sampling conditions: $\Longrightarrow \simeq$

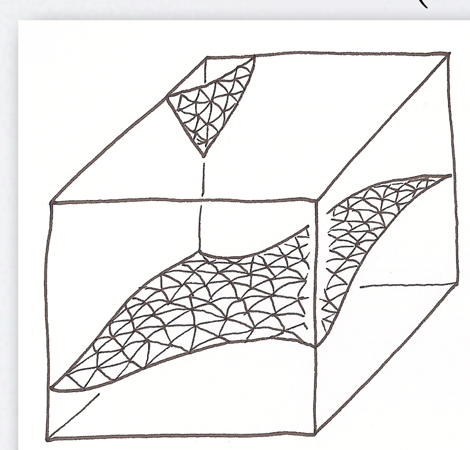
$$d_H(A, P) < \lambda\,\mathrm{feature\_size}(A)$$

P



Input

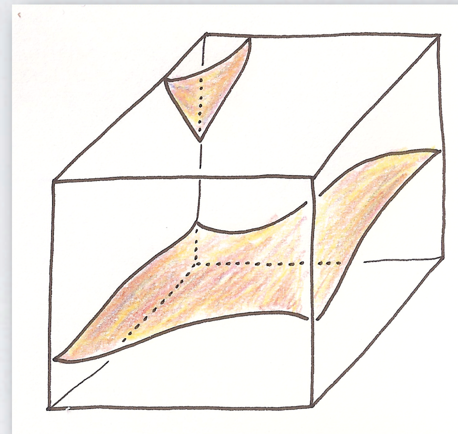
$\mathrm{Reconstruction}(P, \alpha)$



Output

Reconstruction Theorems

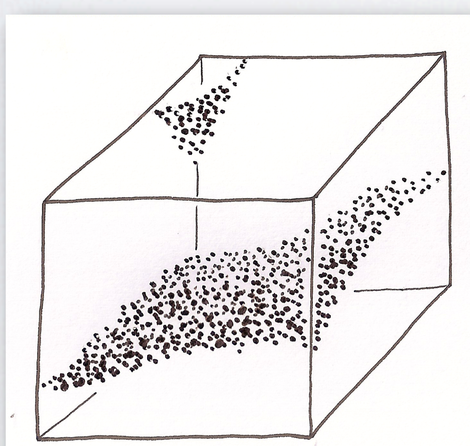
A



Sampling conditions:

$$d_H(A, P) < \lambda \, \mathrm{feature\_size}(A) \Longrightarrow \simeq$$

P



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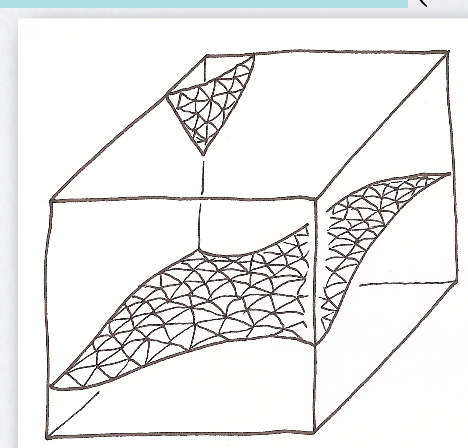
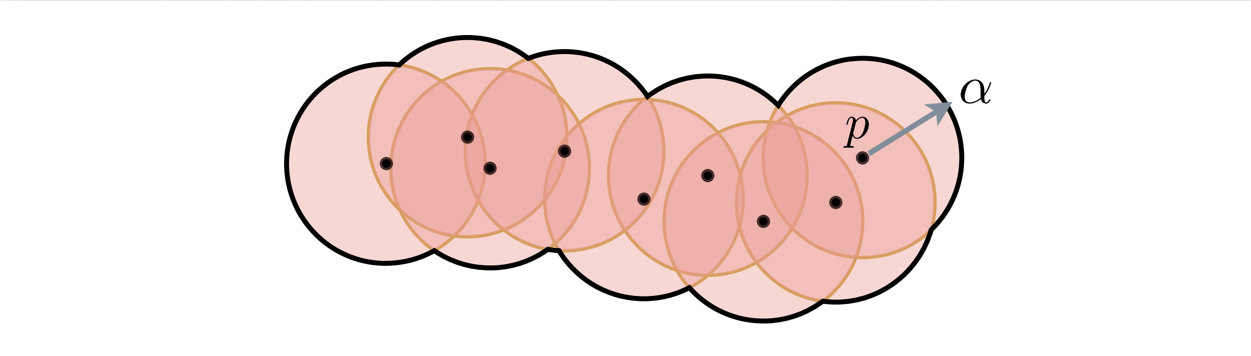
$\mathrm{Reconstruction}(P, \alpha)$



Output

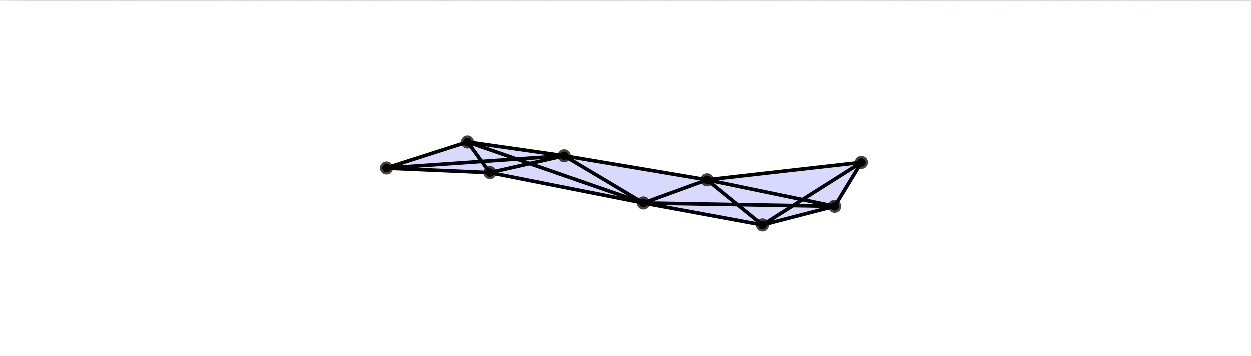
Cech complex

$$P^\alpha = \bigcup_{p \in P} B(p, \alpha)$$ α-offset of P



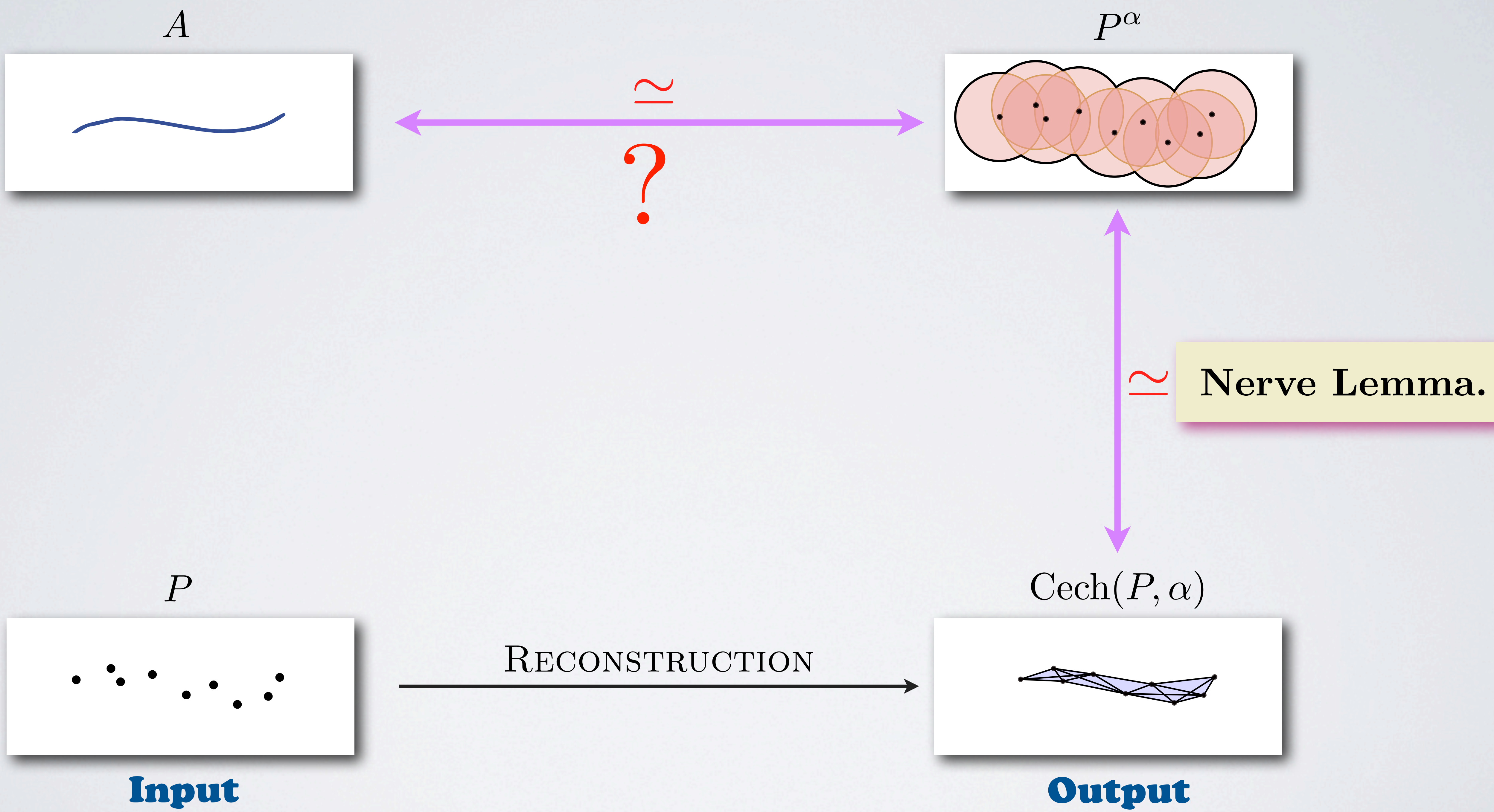
$\simeq$ **Nerve Lemma.**

$$\mathrm{Cech}(P, \alpha) = \mathrm{Nerve}\{B(p, \alpha) \mid p \in P\}$$

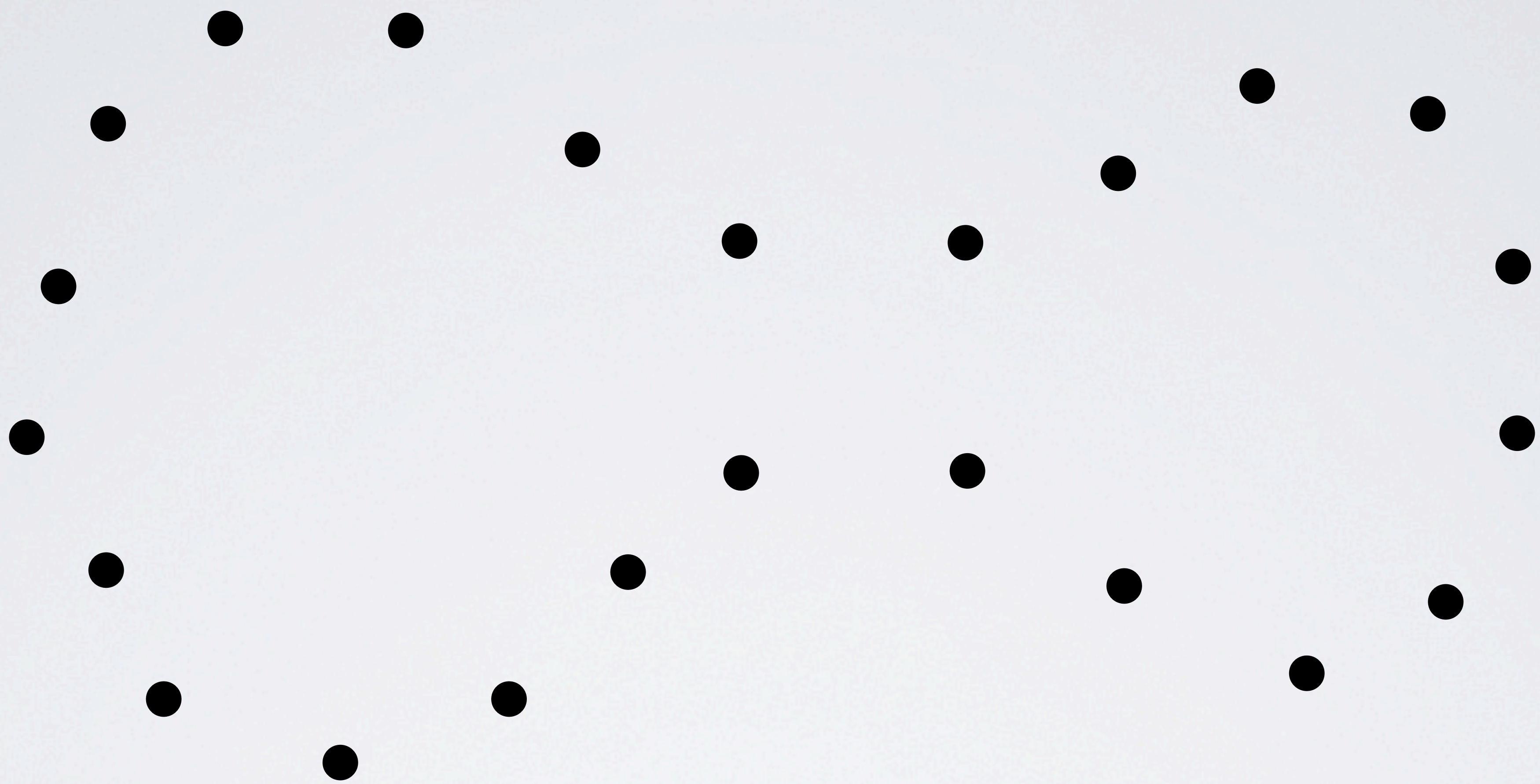


Can be
high-dimensional!
&
expensive to compute

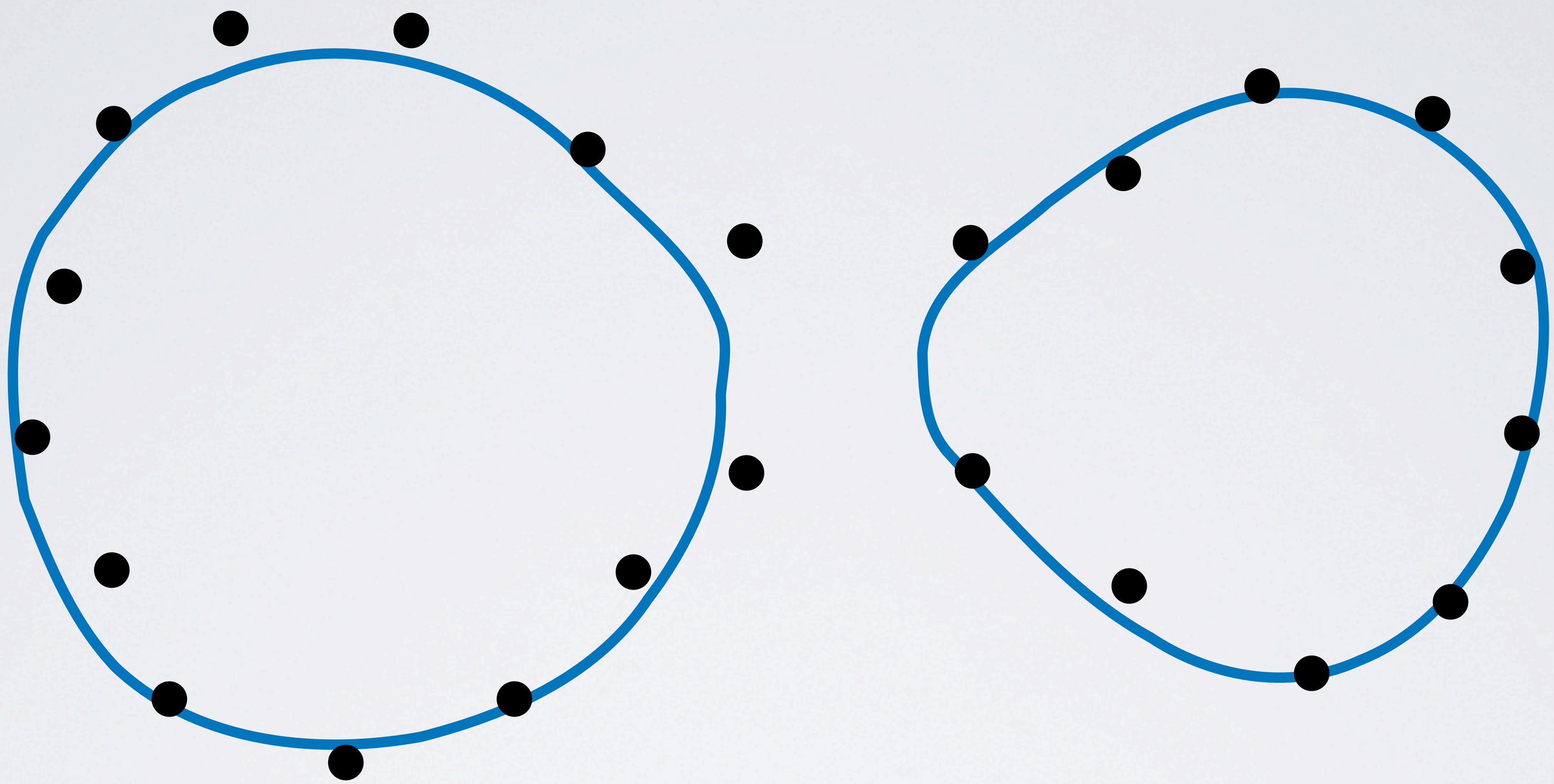
Cech complex



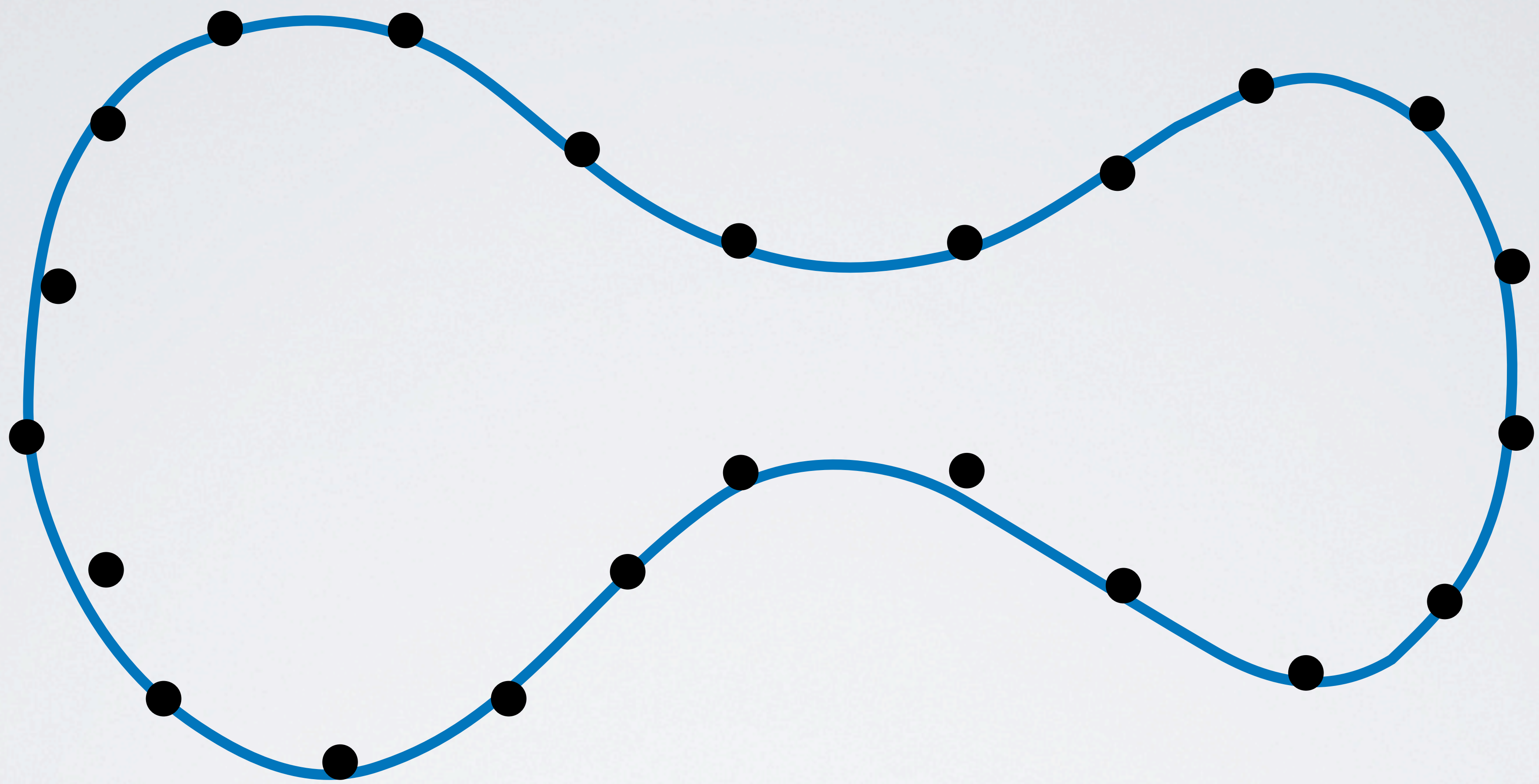
Shapes and Reach



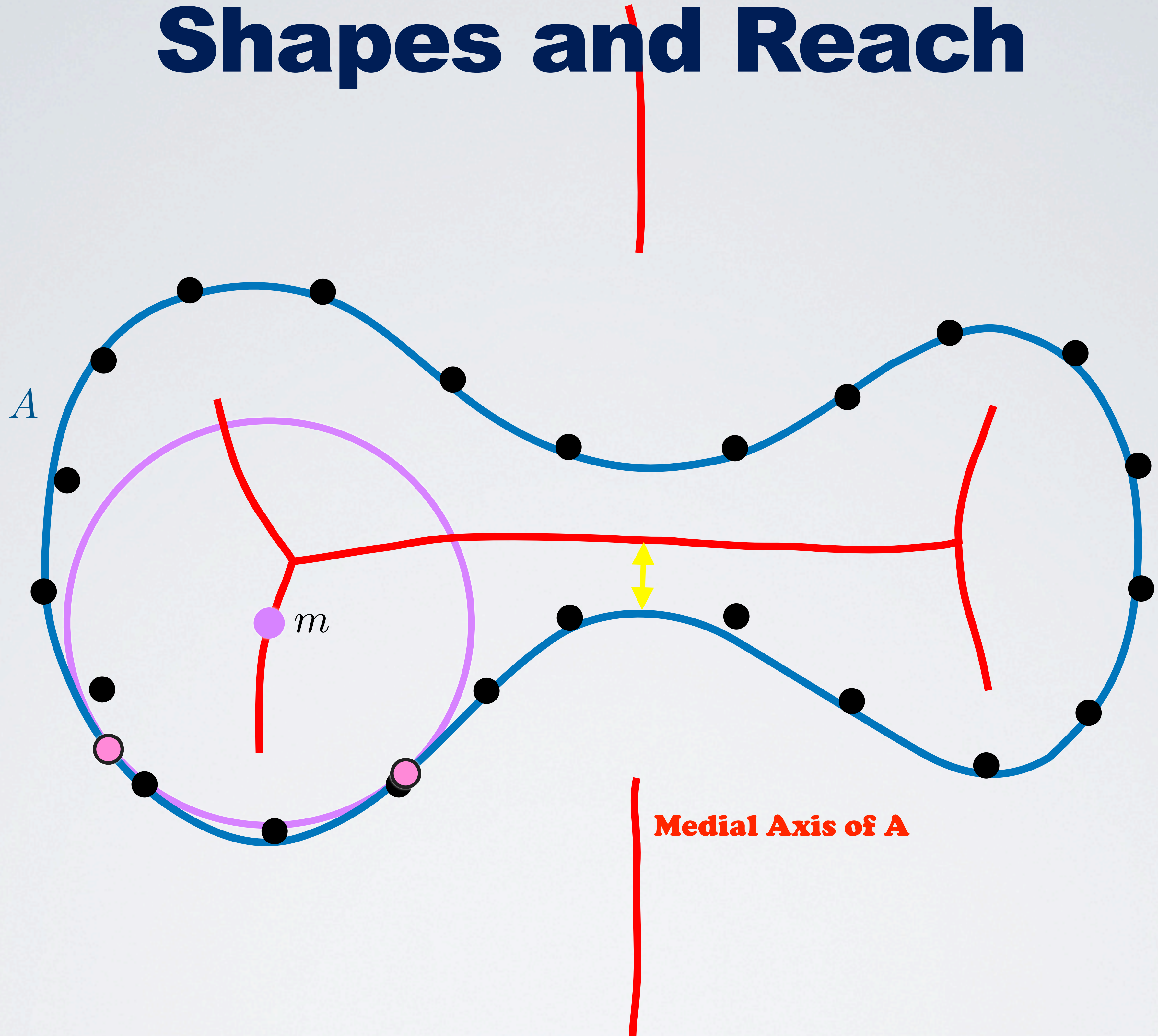
Shapes and Reach



Shapes and Reach



Shapes and Reach

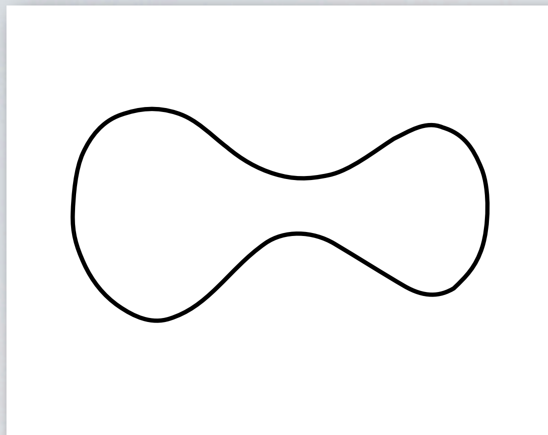


$$\text{MedialAxis}(A) = \{m \in \mathbb{R}^d \mid \ m \text{ has at least two closest points in } A \ \}$$

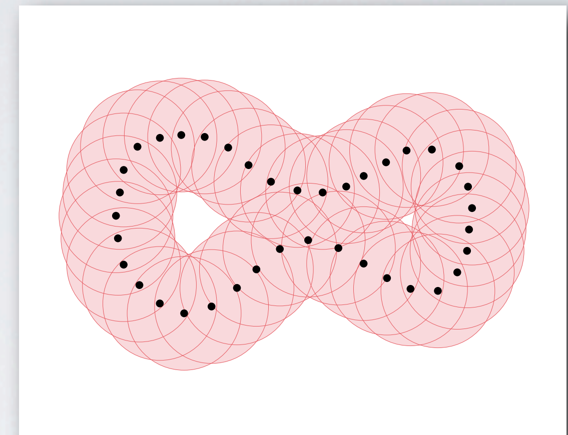
$$\text{Reach } A = d(A, \text{MedialAxis}(A))$$

Cech complex

A



P^α



[Niyogi Smale Weinberger 2004]

deformation retracts to

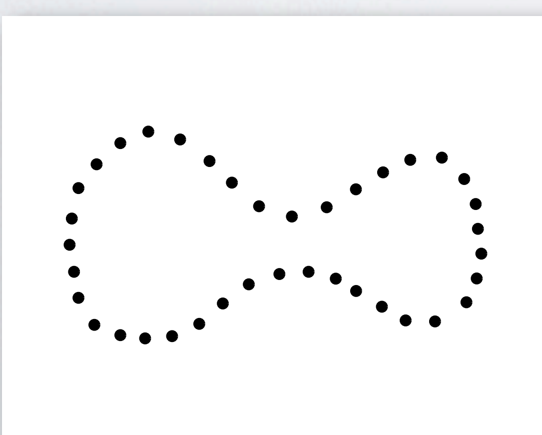
if

$$d_H(A, P) \leq \varepsilon < (3 - \sqrt{8})\,\text{Reach}\,A$$

$$\alpha = (2 + \sqrt{2})\varepsilon$$

$\simeq$ **Nerve Lemma.**

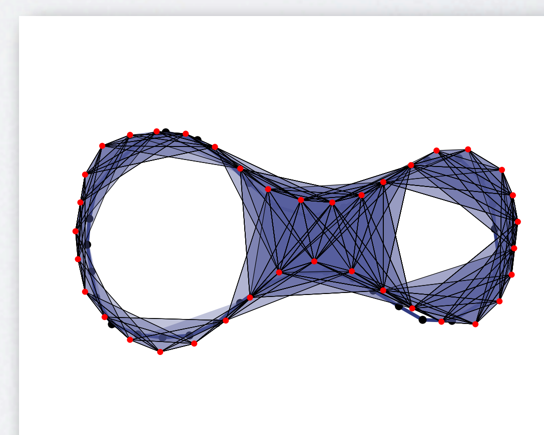
P



Input

Reconstruction

$\text{Cech}(P, \alpha)$



Output

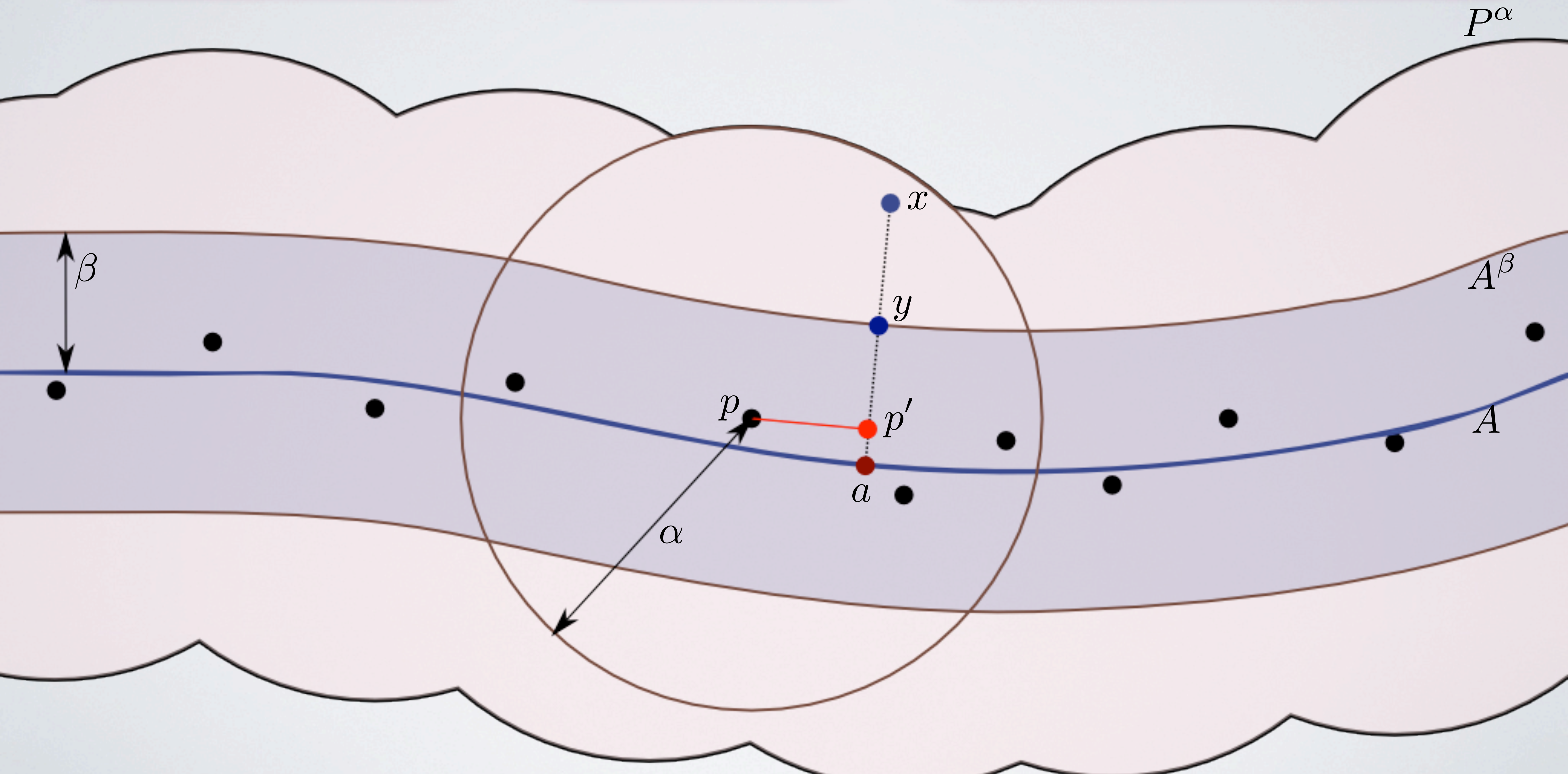
$R = \text{Reach } A$

$\beta = \sqrt{R - (R-\varepsilon)^2 - \alpha^2}$

Short proof

$$\left.\begin{array}{l} \varepsilon < (3-\sqrt{8})R \\ \alpha = (2+\sqrt{2})\varepsilon \end{array}\right\} \implies \left.\begin{array}{l} \alpha < R - \varepsilon \\ \beta < \alpha - \varepsilon \end{array}\right\} \implies$$

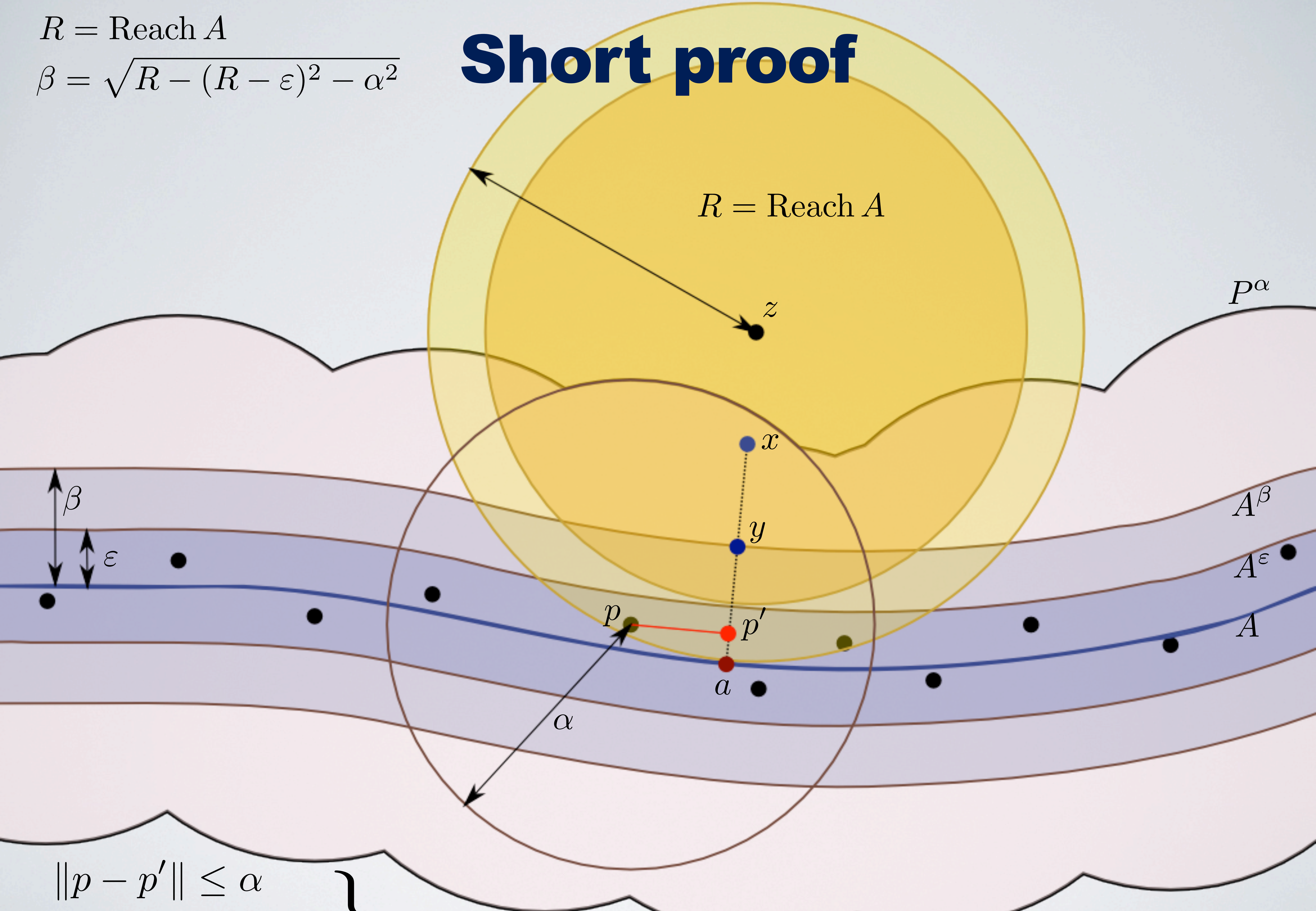
P^α deformation retracts to A^β



prove that $\|a - p'\| \leq \beta \implies y$ lies between x and p'

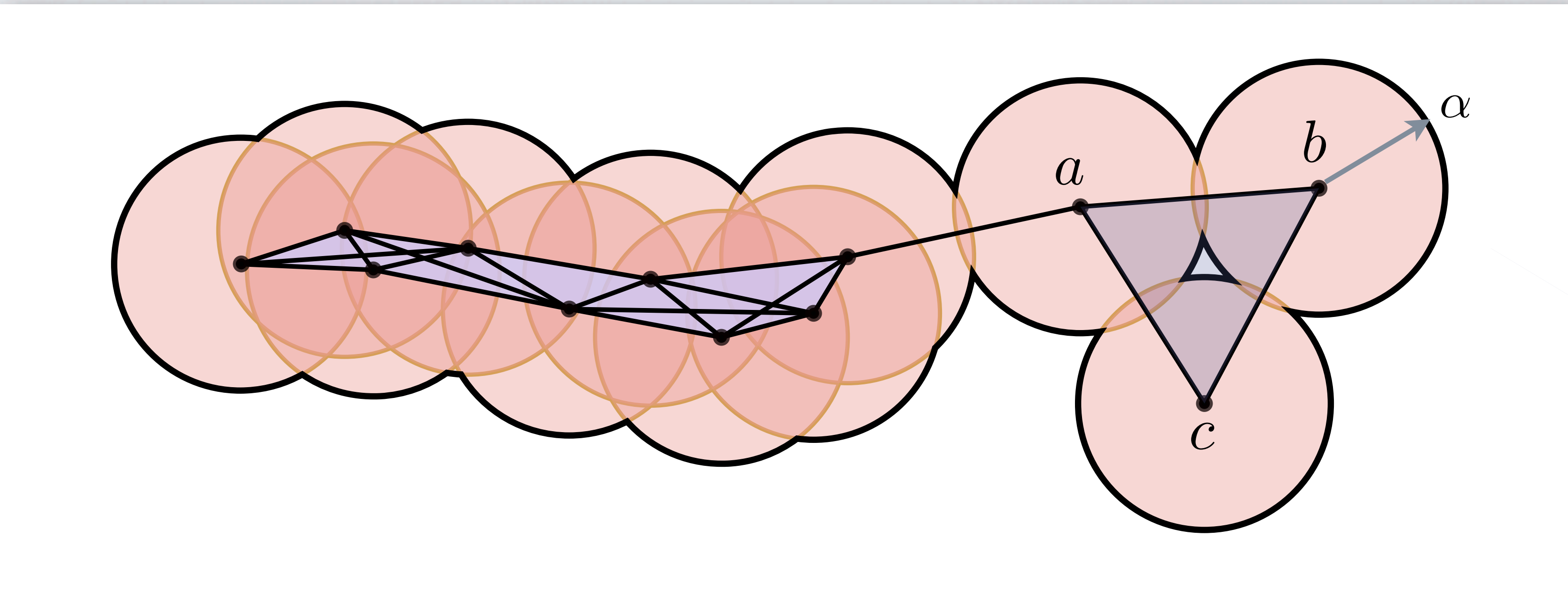
$$R = \text{Reach } A$$
$$\beta = \sqrt{R - (R-\varepsilon)^2 - \alpha^2}$$

Short proof



$$\left.\begin{array}{l}\|p - p'\| \le \alpha \\ \|z - p\| \ge R - \varepsilon\end{array}\right\} \rightarrow \|a - p'\| = R - \sqrt{\|z-p\|^2 - \|p-p'\|^2} \le \beta$$

Rips complexes



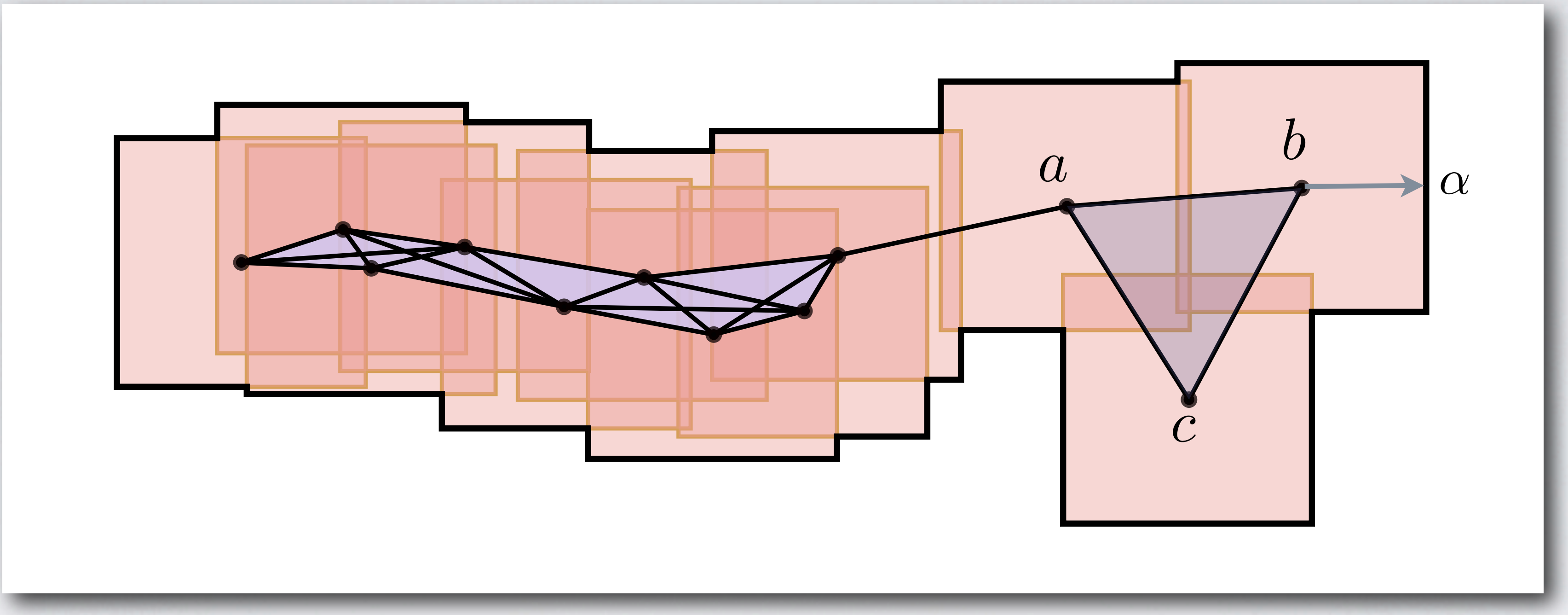
$$\mathrm{Rips}(P,\alpha) = \{\sigma \subset P \mid \mathrm{Diameter}(\sigma) \leq 2\alpha\}$$

$$\boxed{\mathrm{Rips}(P,\alpha) \supset \mathrm{Cech}(P,\alpha)}$$

- **proximity graph G_α connects every pair of points within 2α**
- $\mathrm{Rips}(P,\alpha) = \mathrm{Flag}\, G_\alpha$ [Flag G = largest complex whose 1-skeleton is G]
- **compressed form of storage through the 1-skeleton**
- **easy to compute**

Rips complexes with L_∞

When distances are measured using L_∞



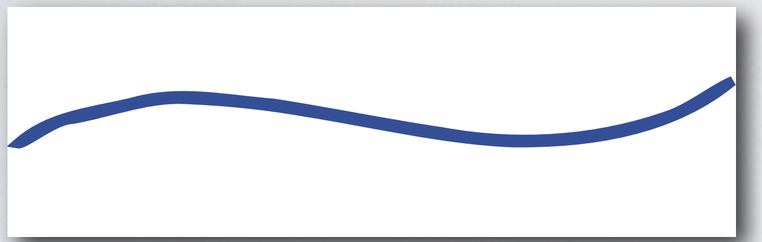
$$\mathrm{Rips}(P, \alpha) = \{\sigma \subset P \mid \mathrm{Diameter}(\sigma) \leq 2\alpha\}$$

$$\boxed{\mathrm{Rips}(P, \alpha) = \mathrm{Cech}(P, \alpha)}$$

- **proximity graph** G_α **connects every pair of points within** 2α
- $\mathrm{Rips}(P, \alpha) = \mathrm{Flag}\, G_\alpha$ [Flag G = largest complex whose 1-skeleton is G]
- **compressed form of storage through the 1-skeleton**
- **easy to compute**

Rips complexes with L_∞

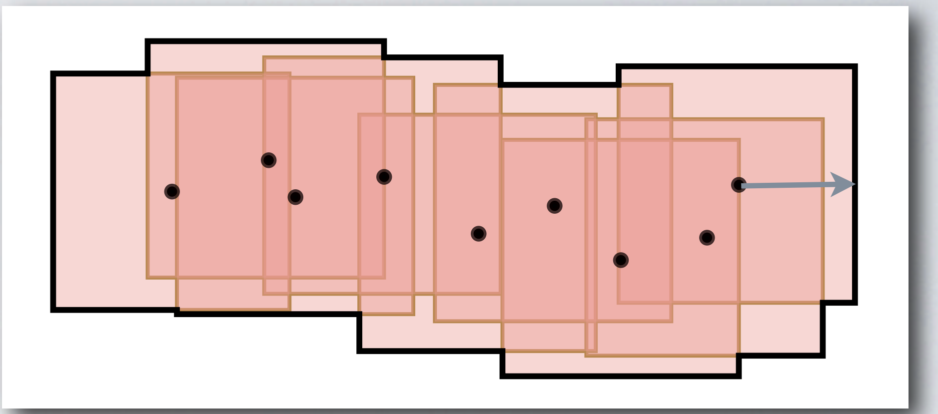
A



$\simeq$

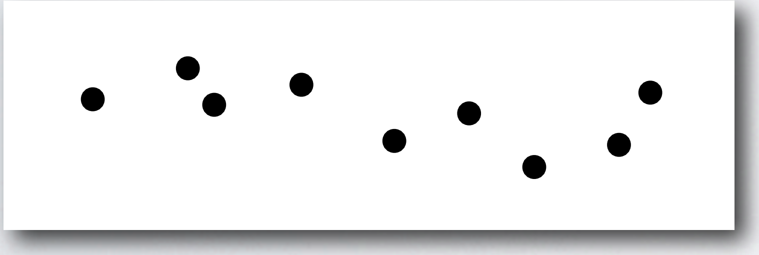
?

$P + \alpha\, B^\infty(0,1)$



$\simeq$ **Nerve Lemma.**

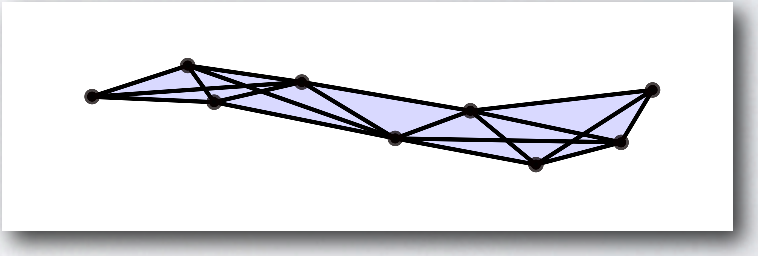
P



Input

RECONSTRUCTION

$\mathrm{Rips}(P,\alpha) = \mathrm{Cech}(P,\alpha)$

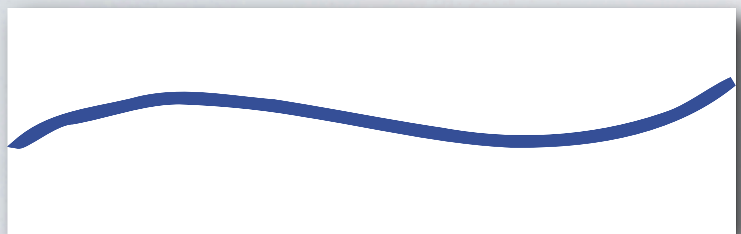


Output

easy to compute

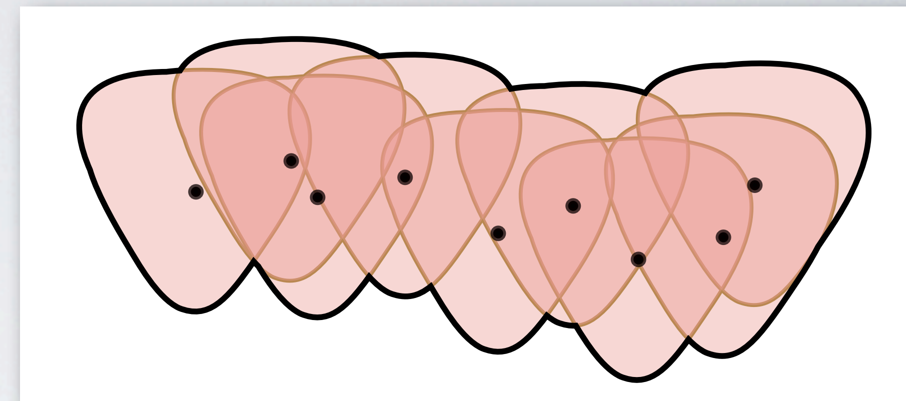
Minkowski sum

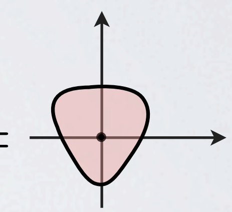
A



$\overset{?}{\simeq}$

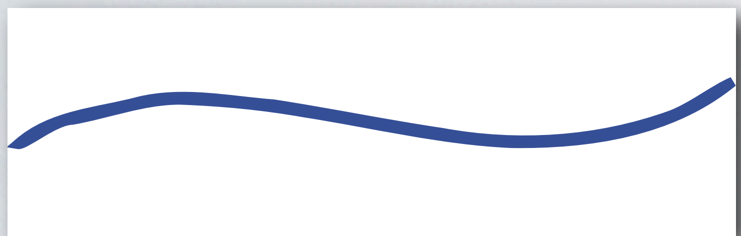
$P + \alpha C$



where $C =$  compact convex set

Minkowski sum

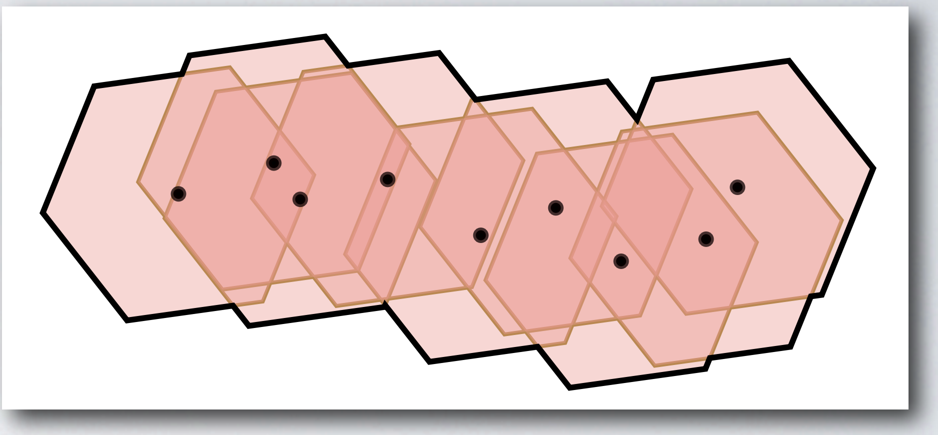
A

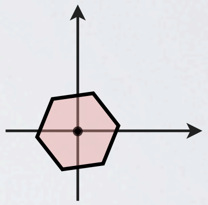


$\underline{\simeq}$

?

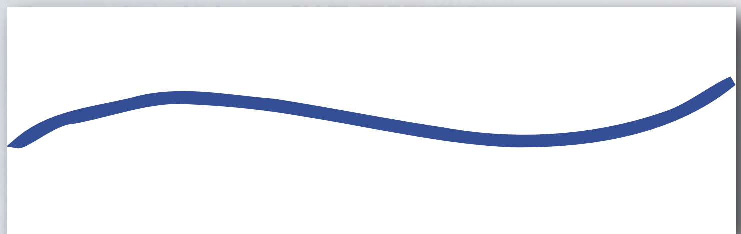
$P + \alpha C$



where $C =$  compact convex set

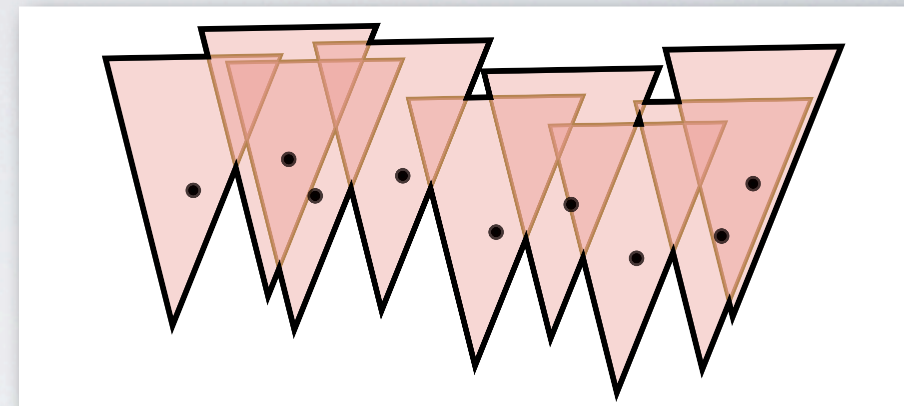
Minkowski sum

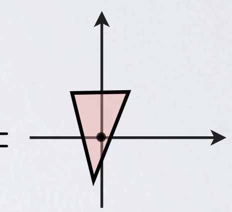
A



$\overset{\simeq}{\underset{?}{\longleftrightarrow}}$

$P + \alpha C$



where $C =$  compact convex set

Minkowski sum

A $\xleftrightarrow[\text{if}]{\text{inclusion homotopy equivalence}}$ $P + \alpha C$

$P \subset A^{\varepsilon}$ and $A \subset P + \varepsilon C$ and $\frac{\varepsilon}{\text{Reach } A}$ small enough and $\frac{\alpha}{\varepsilon} = \frac{4}{1-\xi}$

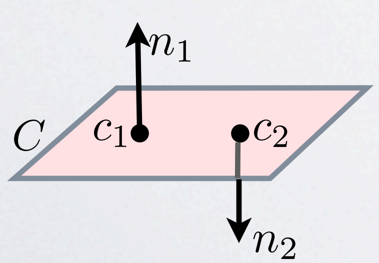
where C compact convex set that satisfies:

(i) $B(0,1) \subset C \subset \delta B(0,1)$ for some $\delta \geq 1$; ("distortion" to unit ball)

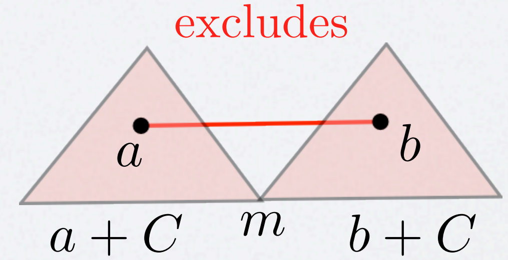
(ii) C is $(\theta, \varkappa)$-round for $\theta = \arccos(-\frac{1}{d})$ and $\varkappa > 0$; ("curvature")

(iii) C is ξ-eccentric for $\xi < 1$. ("distance" between $\bigcap_{q \in Q}(q + C)$ and $\mathrm{Hull}(Q)$)

excludes



excludes



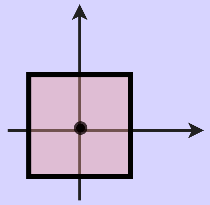
Minkowski sum

A $\xleftrightarrow[\text{if}]{\text{inclusion homotopy equivalence}}$ $P + \alpha C$

$P \subset A^\varepsilon$ and $A \subset P + \varepsilon C$ and $\frac{\varepsilon}{\text{Reach } A}$ small enough and $\frac{\alpha}{\varepsilon} = \frac{4}{1-\xi}$

❶ d-balls satisfy (i) (ii) and (iii) for $\delta = 1$, $\varkappa = 1$ and $\xi = 0$.

❷ d-cubes satisfy (i) (ii) and (iii) for $\delta = \sqrt{d}$



$$\varkappa = \begin{cases} \frac{1}{2\sqrt{2}}\left(\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\right) & \text{if } d = 2, \\ \frac{1}{\sqrt{6}} & \text{if } d = 3, \\ \frac{1}{(d-2)\sqrt{d}} & \text{if } d \geq 4, \end{cases}$$

$\xi = 1 - \frac{2}{d}$

Minkowski sum

A $\xleftrightarrow[\text{if}]{\text{inclusion homotopy equivalence}}$ $P + \alpha C$

$P \subset A^{\varepsilon}$ and $A \subset P + \varepsilon C$ and $\dfrac{\varepsilon}{\text{Reach } A} < \lambda$ and $\dfrac{\alpha}{\varepsilon} = \eta$

Admissible values of ε and α are solutions of a system of equations that depends upon $(\delta, \varkappa, \xi)$.

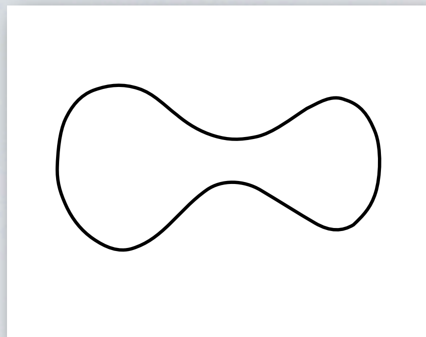
C	d	λ	η
d-ball with [NSW04]	$\forall d$	3 - $\sqrt{8} \approx 0.17$	2 + $\sqrt{2} \approx 3.41$
d-ball with this proof	$\forall d$	0.077	3.96
d-cube [Rips(P, α) with ℓ_∞]	2	0.04	4.04
	3	0.01	6.14
	4	0.004	8.18
	5	0.002	10.2
	10	0.0002	20.23
	100	0.0000002	200.23

What now?

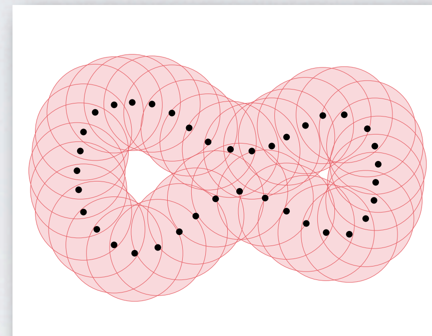
- The largest ratio $\frac{\varepsilon}{\text{Reach } A}$ that we get for $\text{Rips}(P, \alpha)$ with ℓ_∞:
    - Decreases quickly with d
    - Is it tight?
- $\ell_\infty \Rightarrow \ell_2$

Rips complexes with L_2

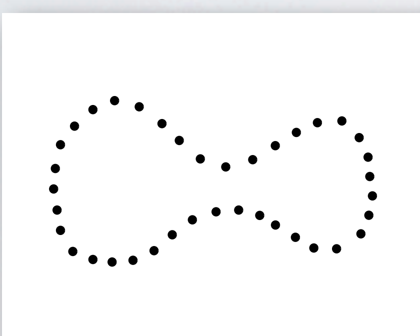
A



P^α

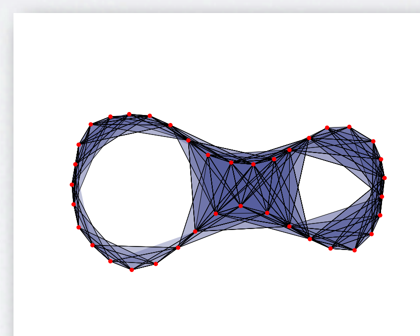


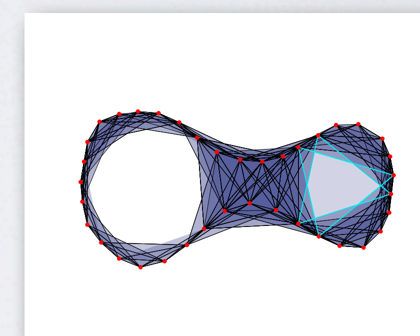
P



Input

$\text{Cech}(P, \alpha) \quad \subset \quad \text{Rips}(P, \alpha)$



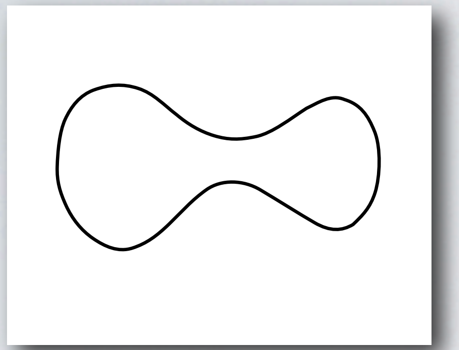


Output

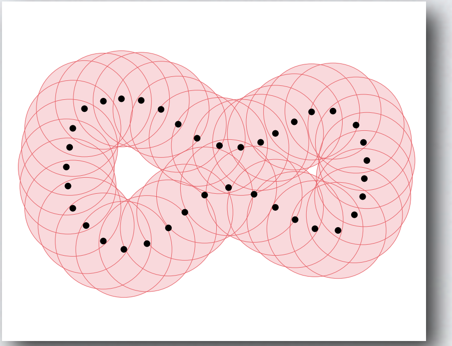
easy to compute

Rips complexes with L_2

A

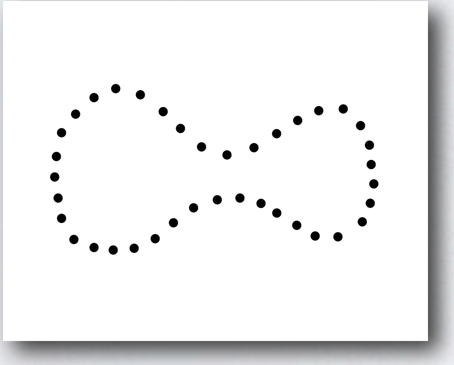


P^α



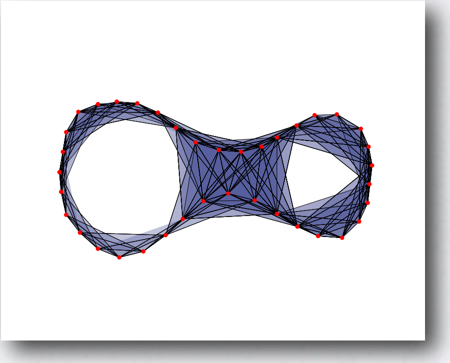
$\simeq$ **Nerve Lemma**

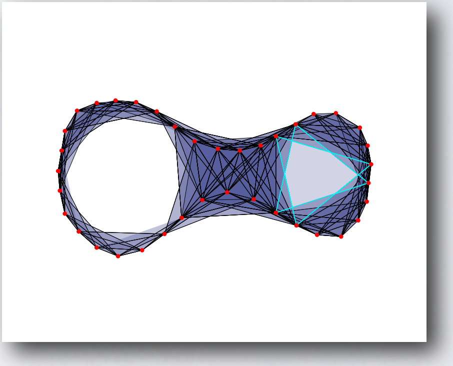
P



Input

$\mathrm{Cech}(P, \alpha) \subset \mathrm{Rips}(P, \alpha)$



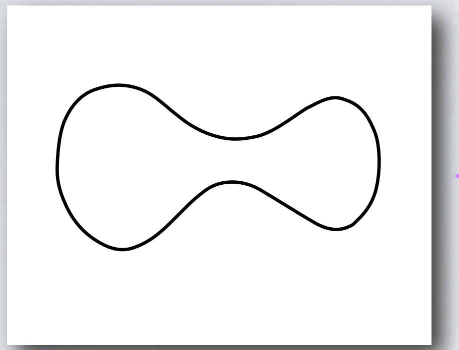


Output

easy to compute

Rips complexes with L_2

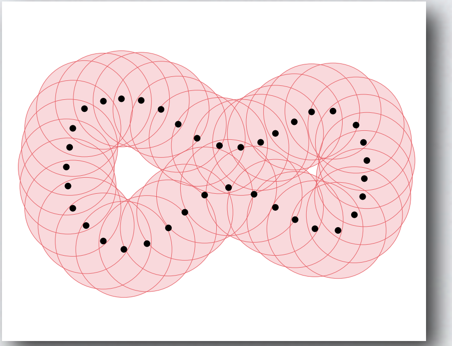
A



deformation retracts to

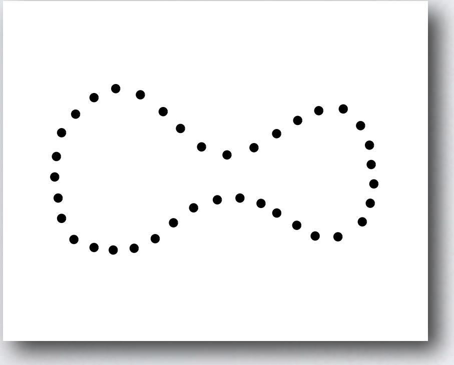
[NSW 04]

P^α



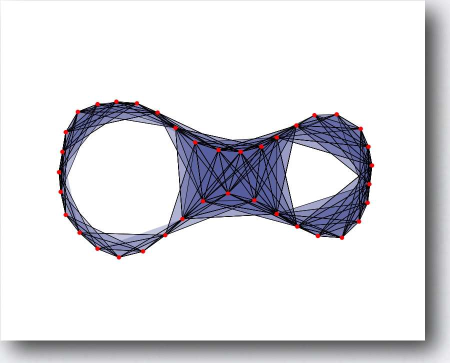
$\simeq$ **Nerve Lemma**

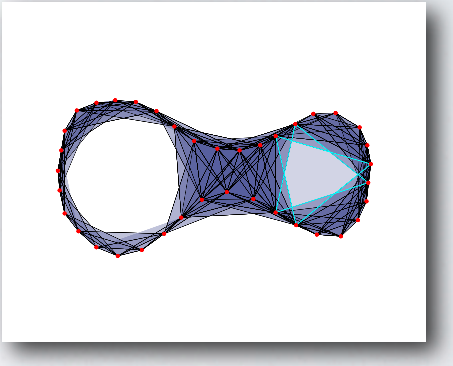
P



Input

$\mathrm{Cech}(P, \alpha) \subset \mathrm{Rips}(P, \alpha)$



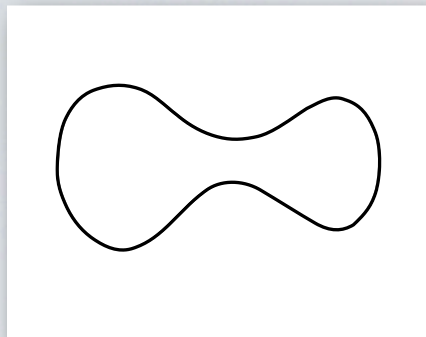


Output

easy to compute

Rips complexes with L_2

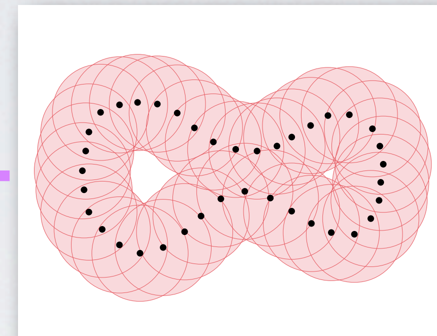
A



deformation retracts to

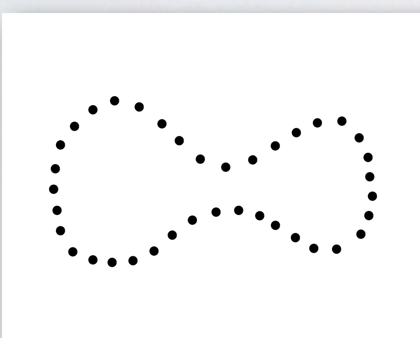
[NSW 04]

P^{α}



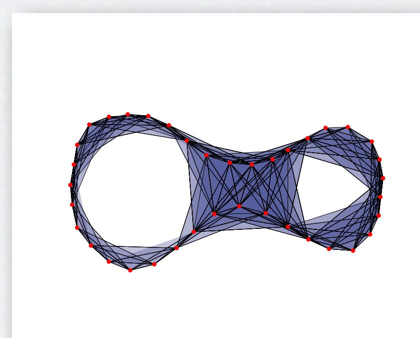
$\simeq$ **Nerve Lemma**

P



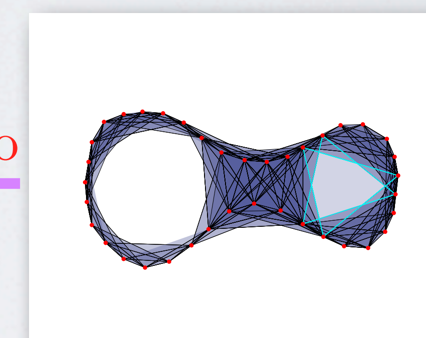
Input

$\mathrm{Cech}(P, \alpha)$ $\subset$ $\mathrm{Rips}(P, \alpha)$



deformation retracts to

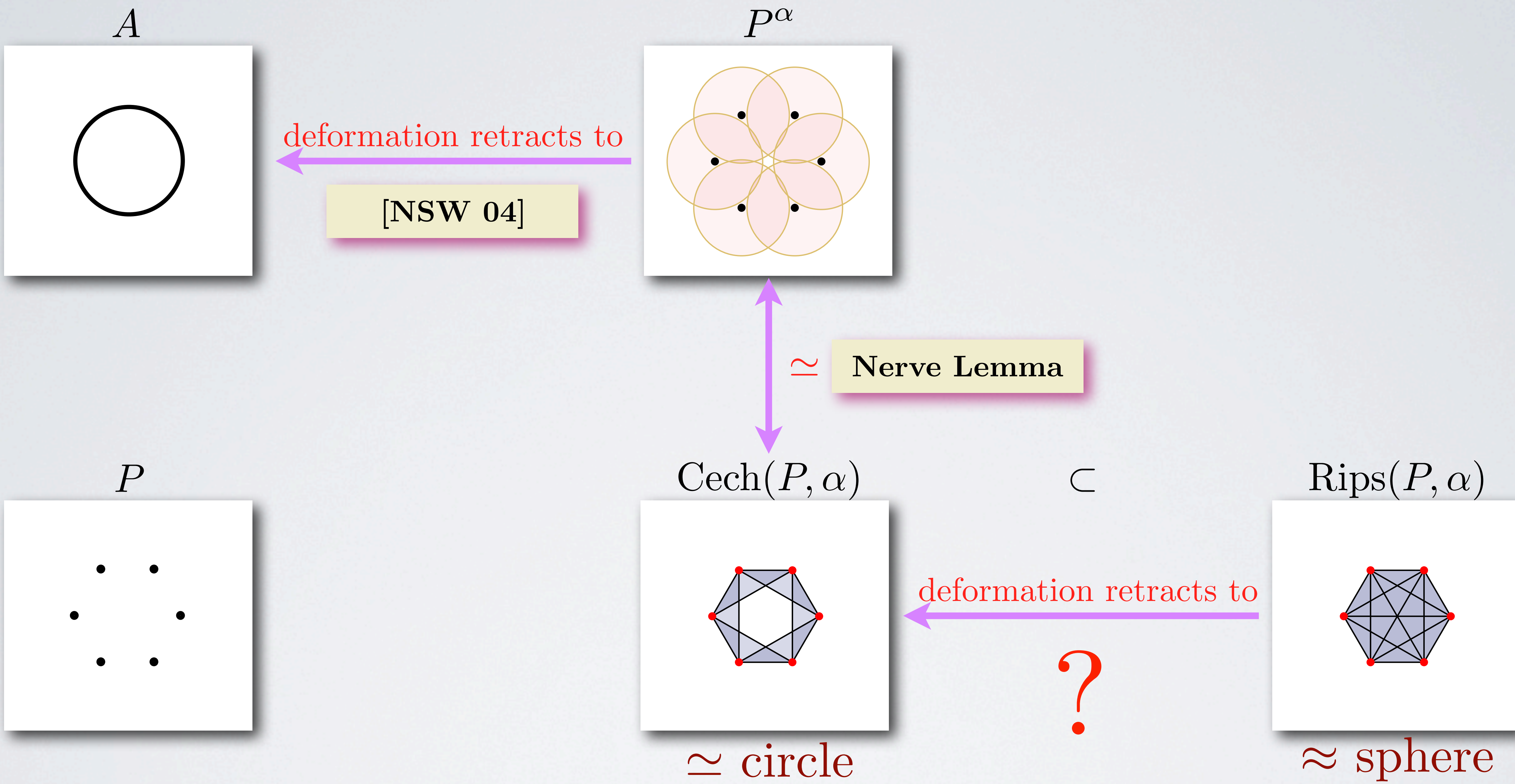
?



Output

easy to compute

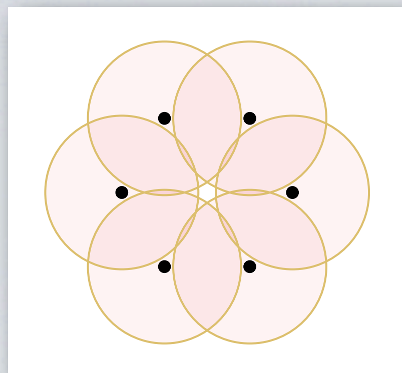
Rips complexes with L_2



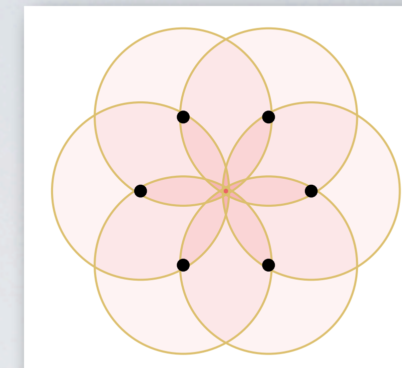
Rips and Cech complexes generally don't share the same topology, but ...

Roadmap

P^{α}

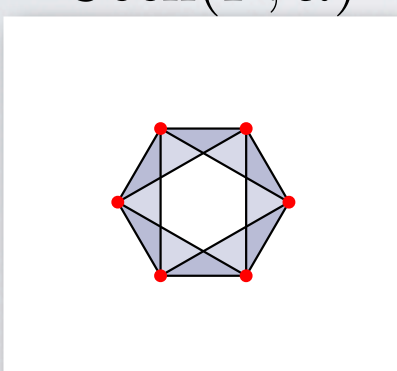


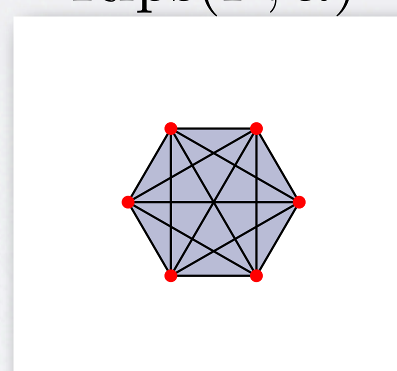
$P^{\vartheta_d \alpha}$

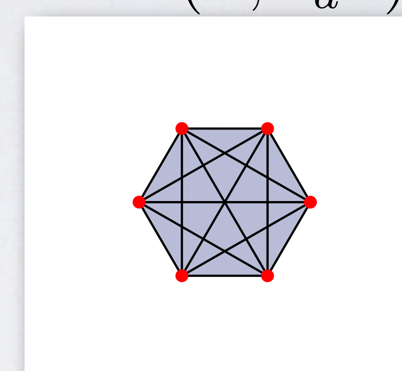


$\simeq$ $\simeq$

$\mathrm{Cech}(P, \alpha)$ $\subset$ $\mathrm{Rips}(P, \alpha)$ $\subset$ $\mathrm{Cech}(P, \vartheta_d \alpha)$





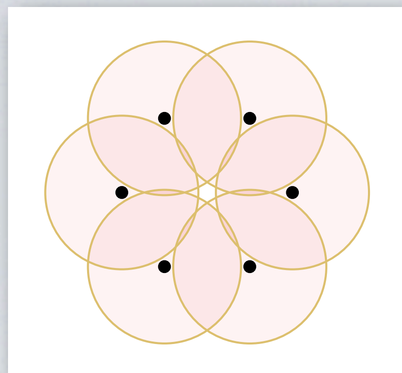


$\simeq$ circle $\approx$ sphere $\approx$ 5-ball

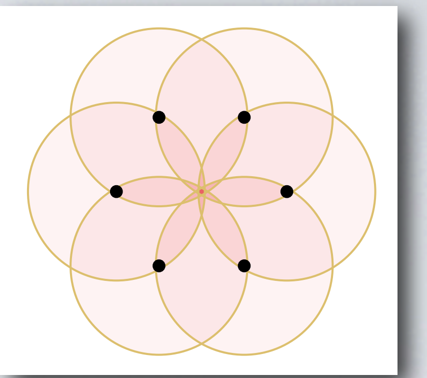
for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

Roadmap

P^{α}



$P^{\vartheta_d \alpha}$



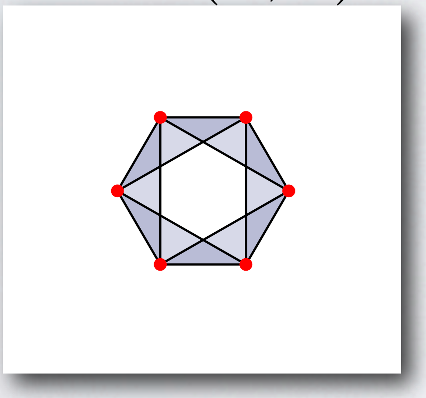
❶ Find a condition under which the topology of $\{\,\mathrm{Cech}(P,t)\,\}_{\alpha \le t \le \vartheta_d \alpha}$ is "stable"

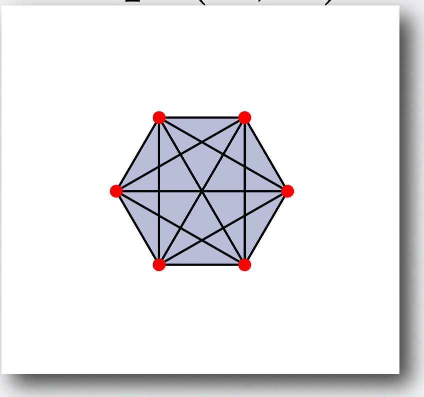
$\simeq$

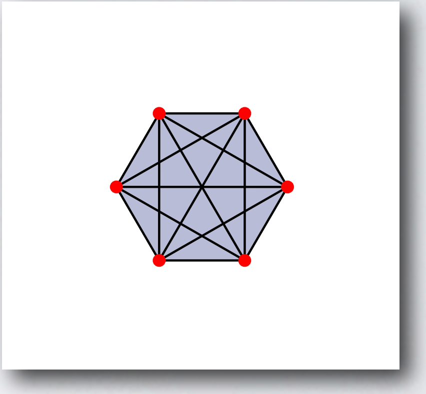
$\simeq$

sequence of collapses ?

$\mathrm{Cech}(P,\alpha) \quad \subset \quad \mathrm{Rips}(P,\alpha) \quad \subset \quad \mathrm{Cech}(P,\vartheta_d \alpha)$



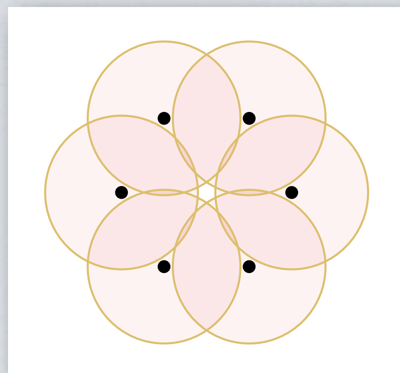




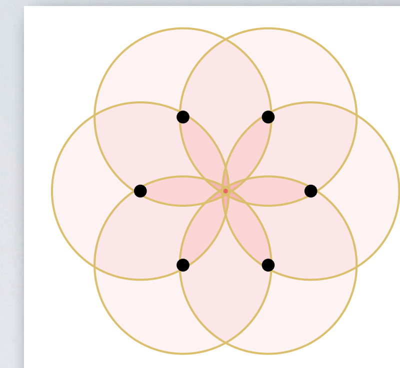
for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

Roadmap

P^{α}



$P^{\vartheta_d \alpha}$



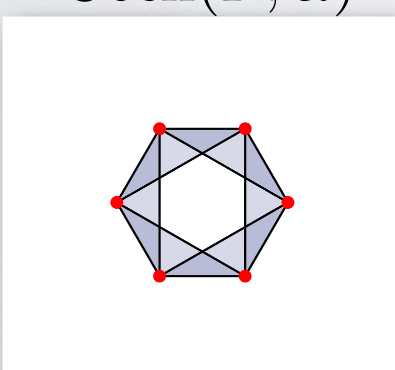
❶ Find a condition under which the topology of $\{\,\mathrm{Cech}(P,t)\,\}_{\alpha \leq t \leq \vartheta_d \alpha}$ is "stable"

$\simeq$

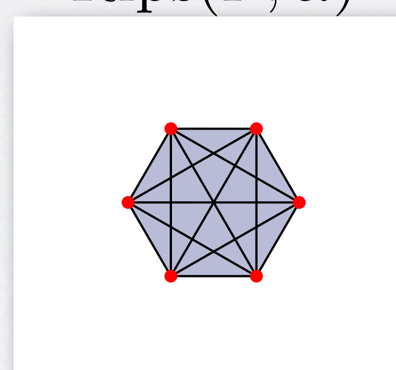
sequence of collapses ?

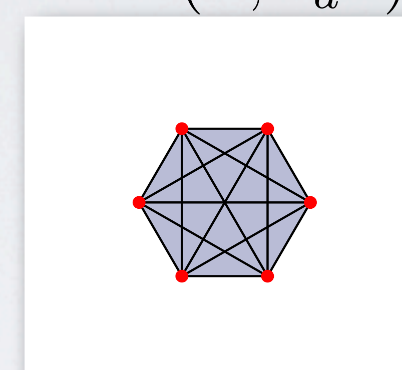
$\simeq$

$\mathrm{Cech}(P,\alpha) \quad \subset \quad \mathrm{Rips}(P,\alpha) \quad \subset \quad \mathrm{Cech}(P,\vartheta_d \alpha)$



sequence of collapses ?



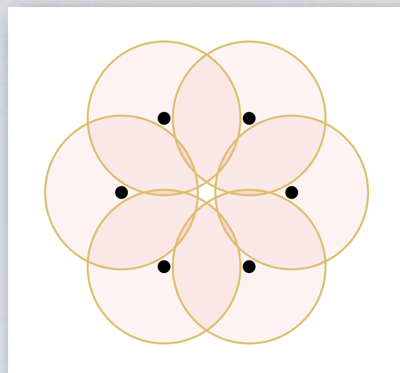


for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

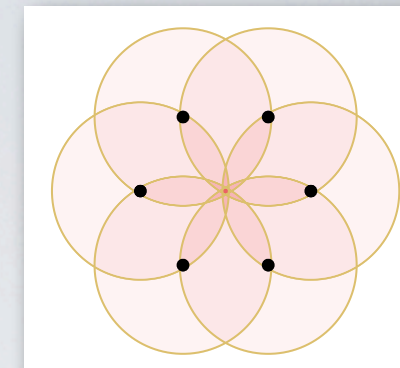
❷ Deduce a condition under which the topology of $\{\,\mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha)\,\}_{\alpha \leq t \leq \vartheta_d \alpha}$ is "stable"

Roadmap

P^{α}



$P^{\vartheta_d \alpha}$



deformation retracts to

?

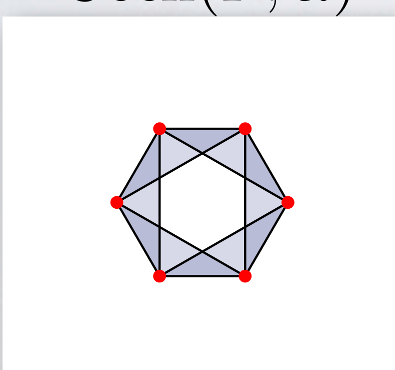
$\simeq$

Find a condition under which the topology of

$\{\, \mathrm{Cech}(P, t) \,\}_{\alpha \le t \le \vartheta_d \alpha}$ is "stable"

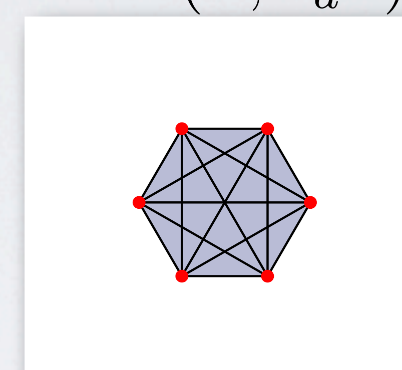
$\simeq$

$\mathrm{Cech}(P, \alpha)$



$\simeq$ circle

$\mathrm{Cech}(P, \vartheta_d \alpha)$



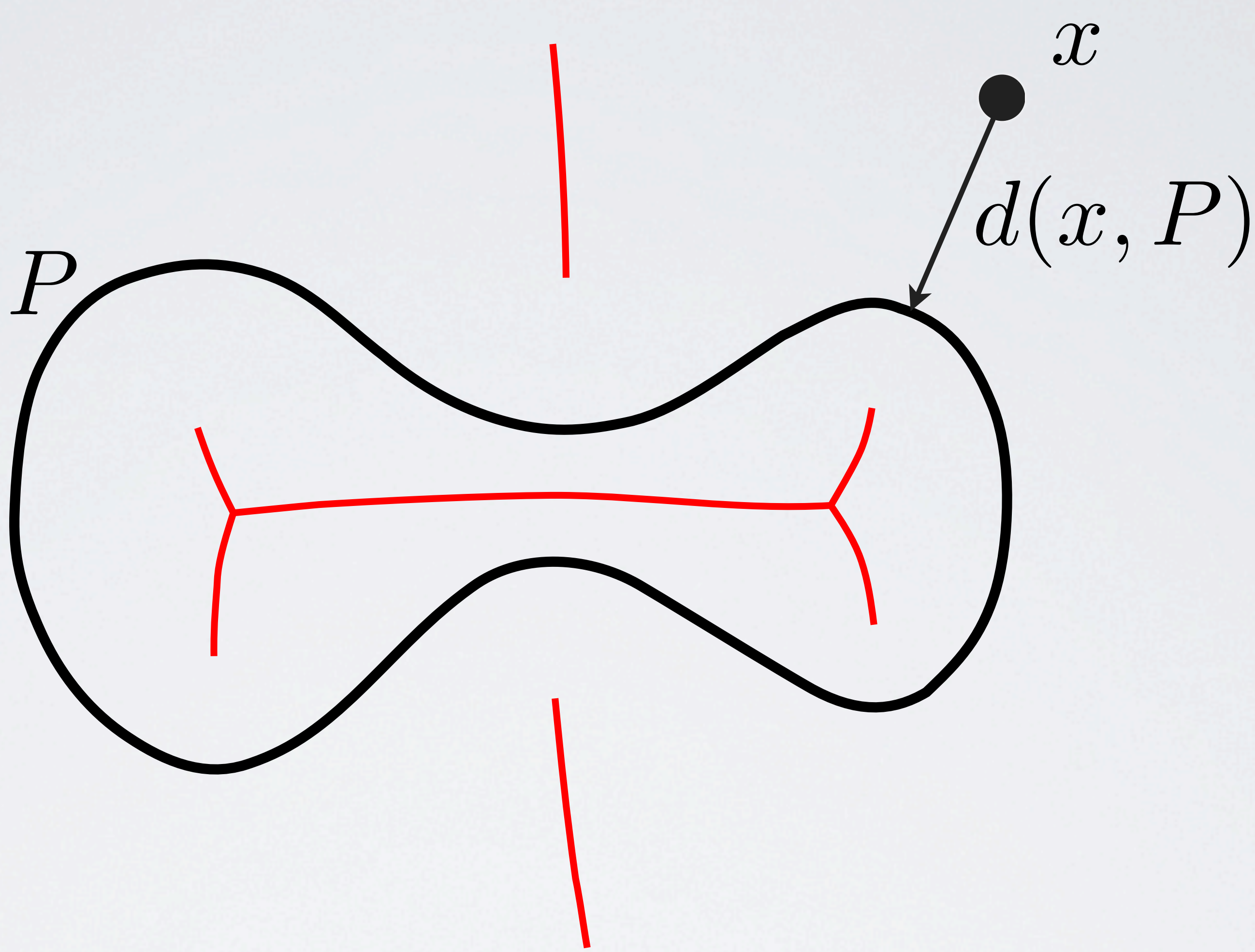
$\approx$ 5-ball

sequence of collapses

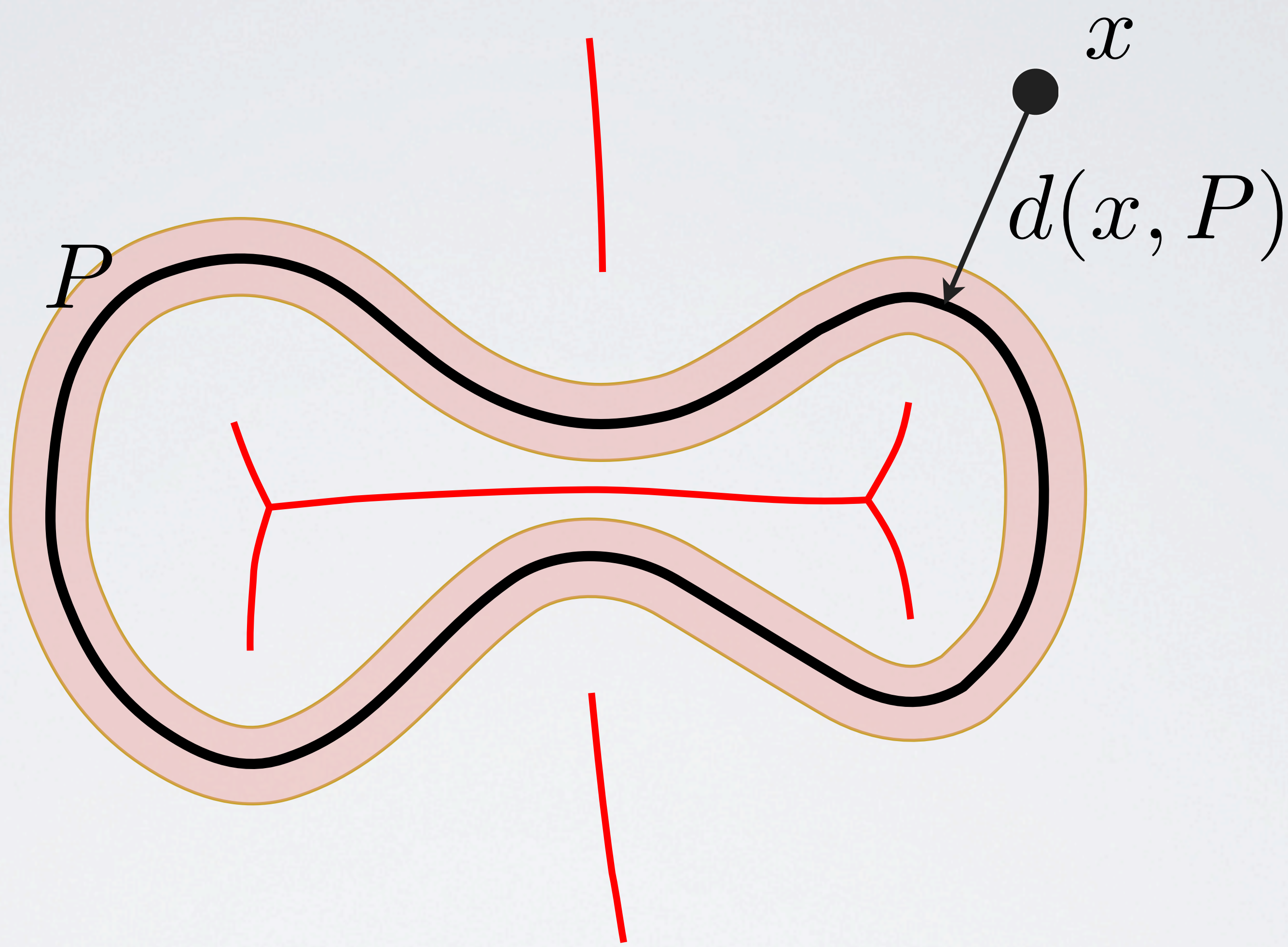
?

for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

Distance function

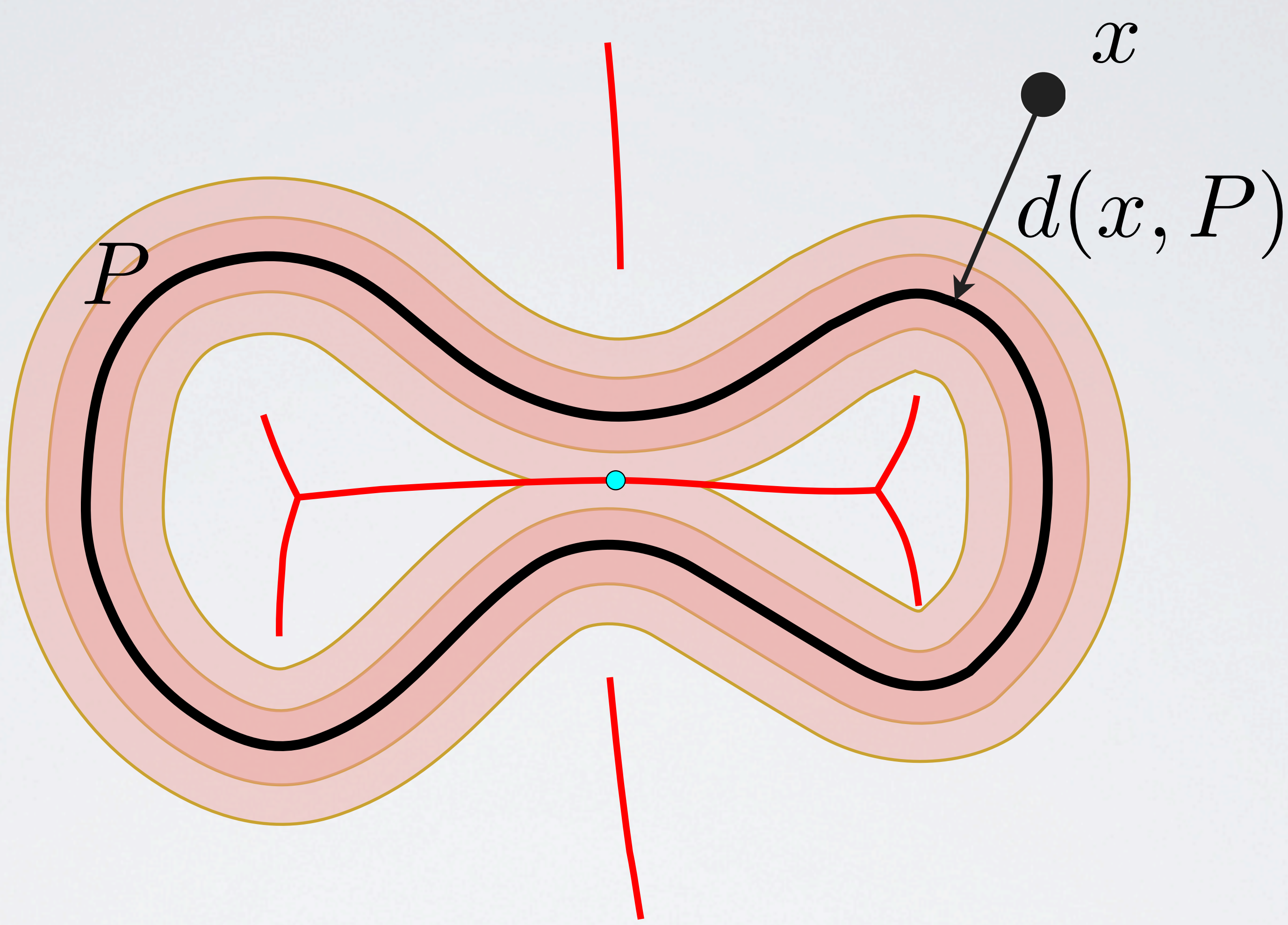


Distance function



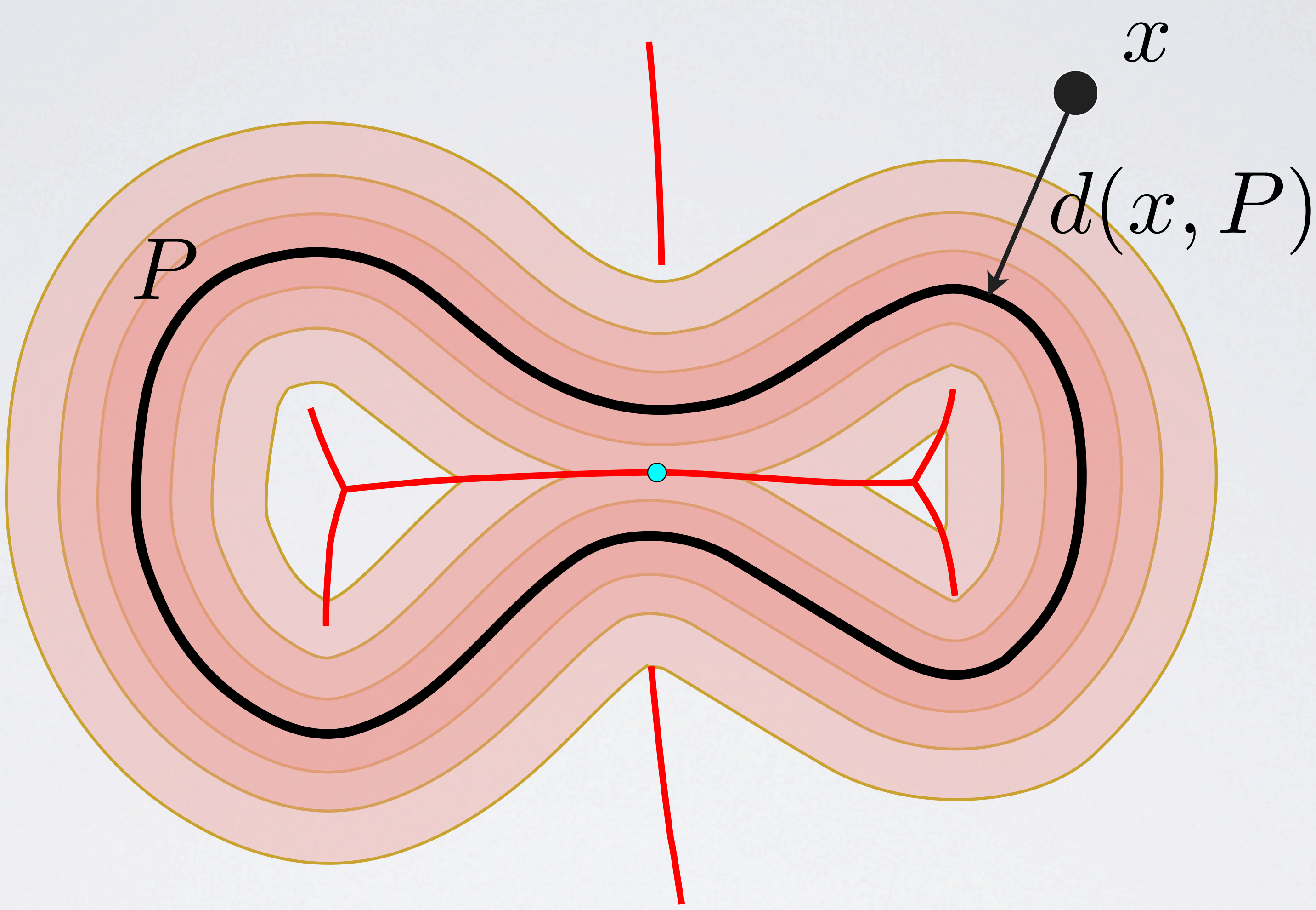
- Sublevel sets of $d(\cdot, P)$ are offsets of P.

Distance function



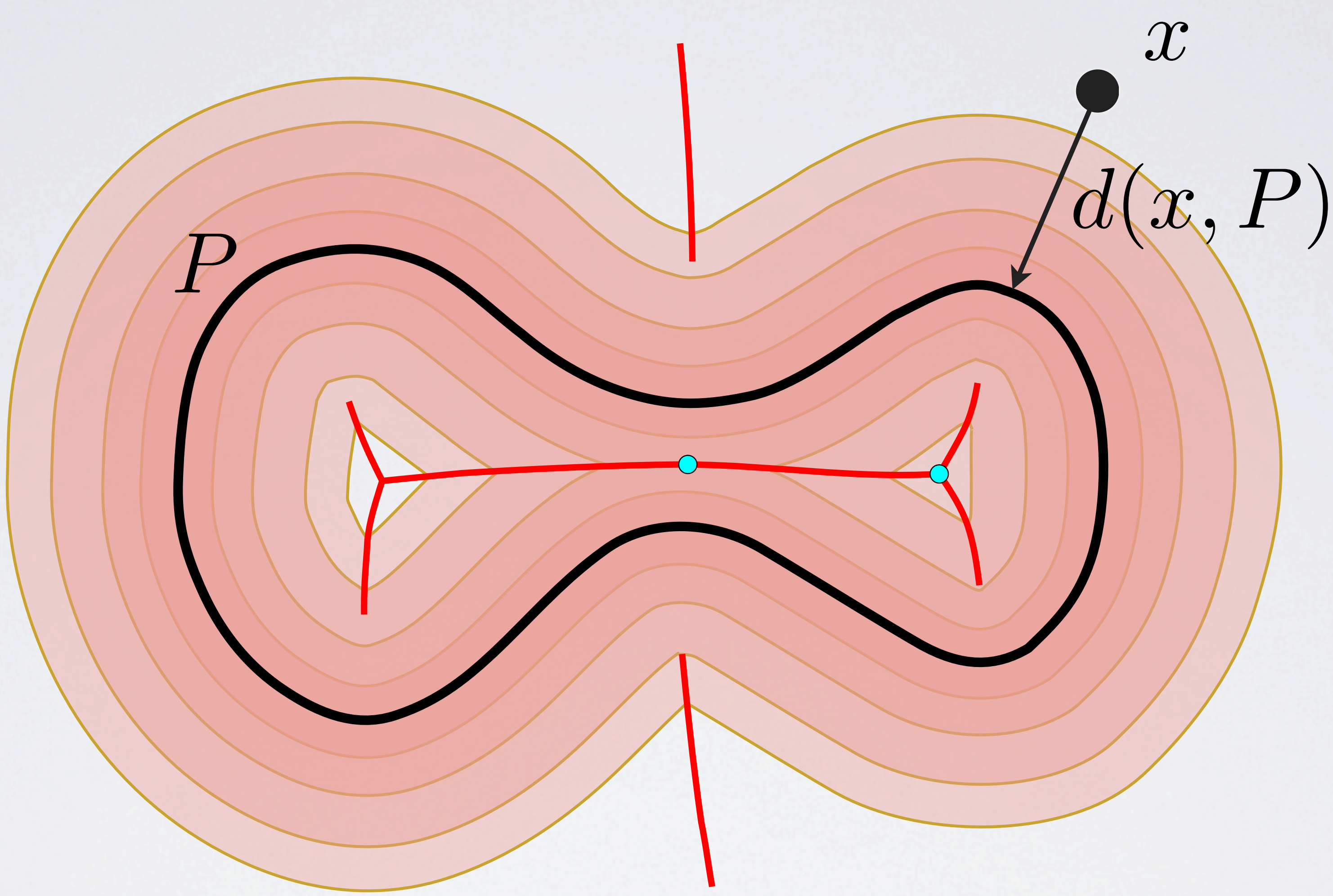
- Sublevel sets of $d(\cdot, P)$ are offsets of P.
- Topology of sublevel sets changes at critical values t_0.

Distance function



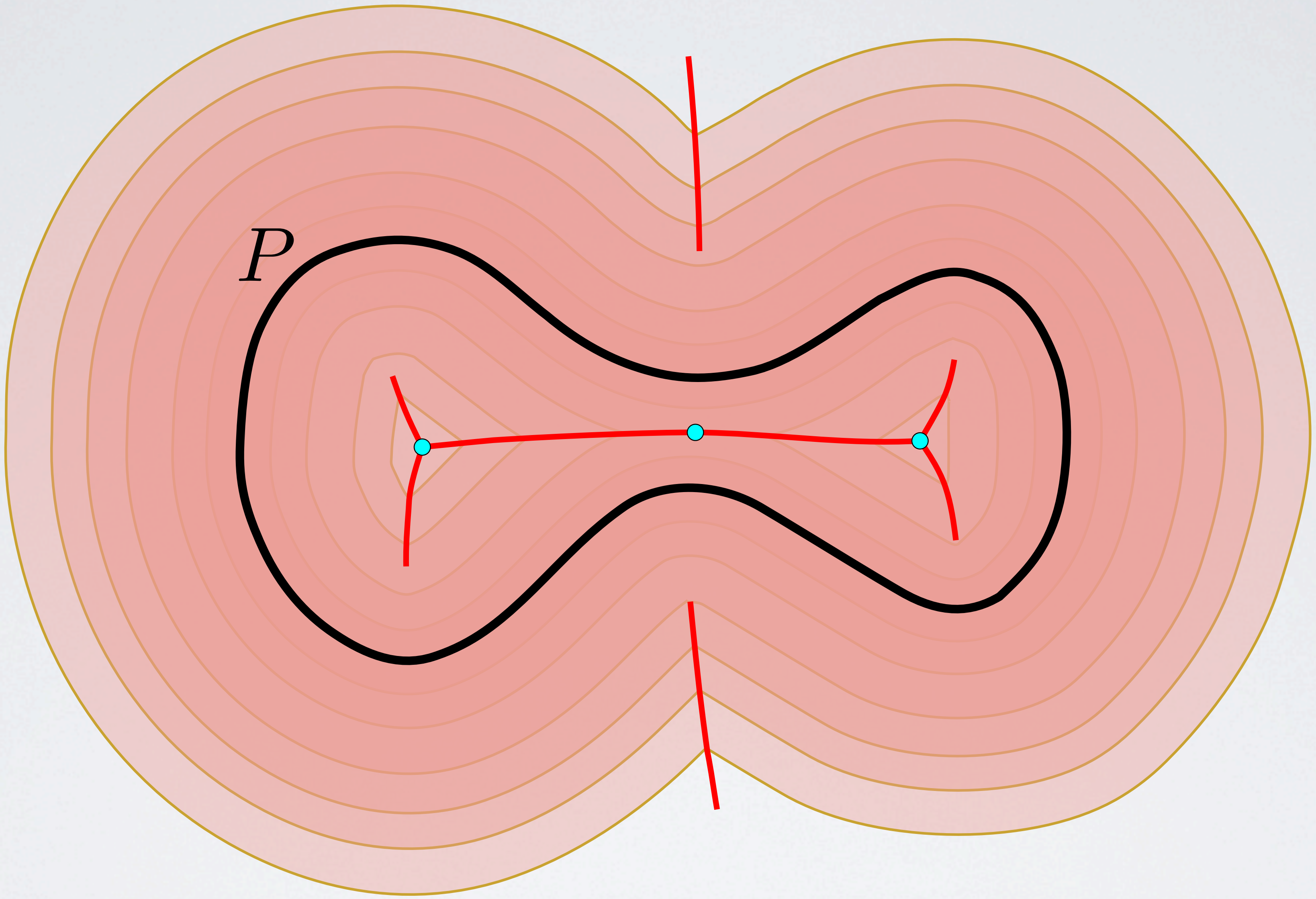
- Sublevel sets of $d(\cdot, P)$ are offsets of P.
- Topology of sublevel sets changes at critical values t_0.

Distance function



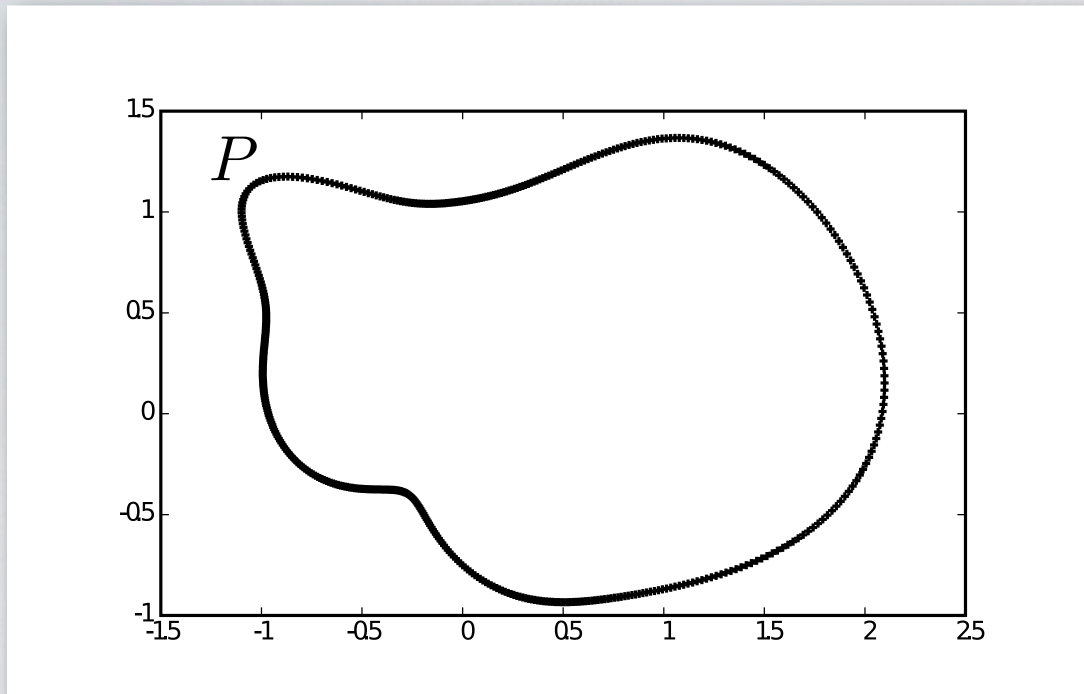
- Sublevel sets of $d(\cdot, P)$ are offsets of P.
- Topology of sublevel sets changes at critical values t_0.

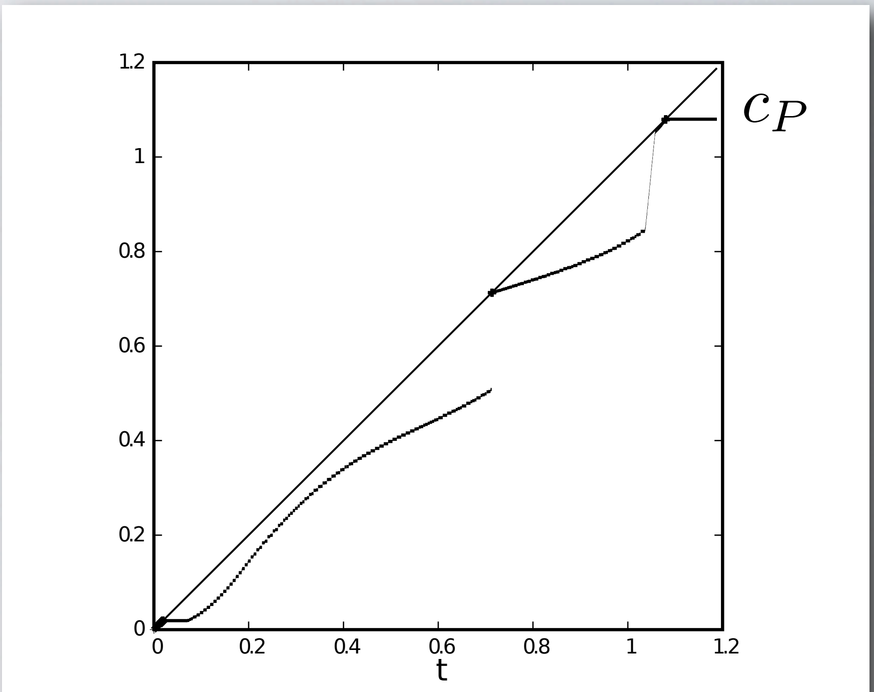
Distance function



- Sublevel sets of $d(\cdot, P)$ are offsets of P.
- Topology of sublevel sets changes at critical values t_0.
- t_0 critical value $\iff$ $c_P(t_0) = t_0$

Convexity defects



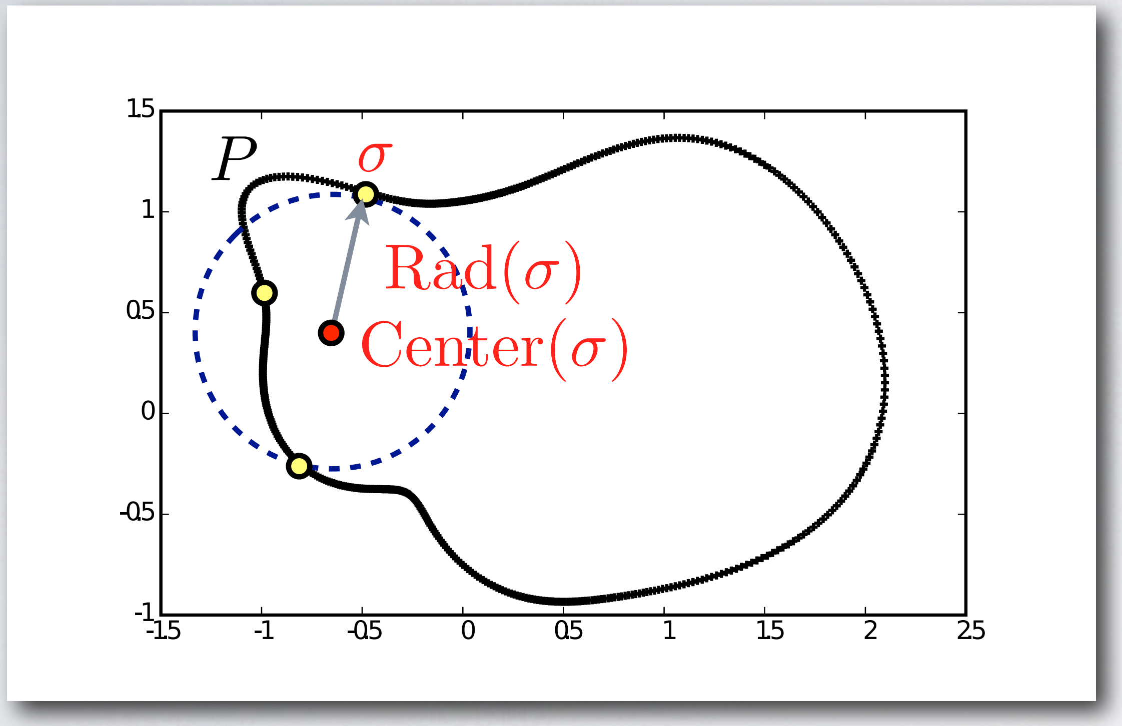


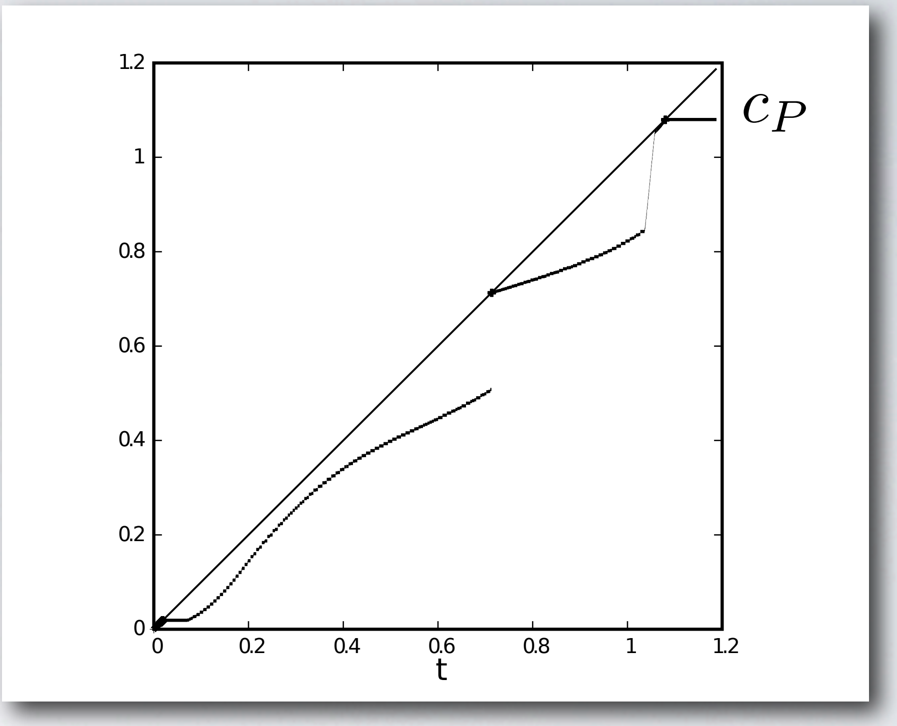
$$\mathrm{Centers}(P,t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \mathrm{Rad}(\sigma) \leq t}} \{\mathrm{Center}(\sigma)\}.$$

$$c_P(t) = d_H(\mathrm{Centers}(P,t) \mid P)$$

- For a compact set P: P convex $\iff c_P = 0$
- c_P non decreasing
- $c_P(t) \leq t$
- $c_P(t) = t \iff t$ critical value $d(\cdot, P)$

Convexity defects





$$\mathrm{Centers}(P,t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \mathrm{Rad}(\sigma) \leq t}} \{\mathrm{Center}(\sigma)\}.$$

$$c_P(t) = d_H(\mathrm{Centers}(P,t) \mid P)$$

- For a compact set P: P convex $\iff c_P = 0$
- c_P non decreasing
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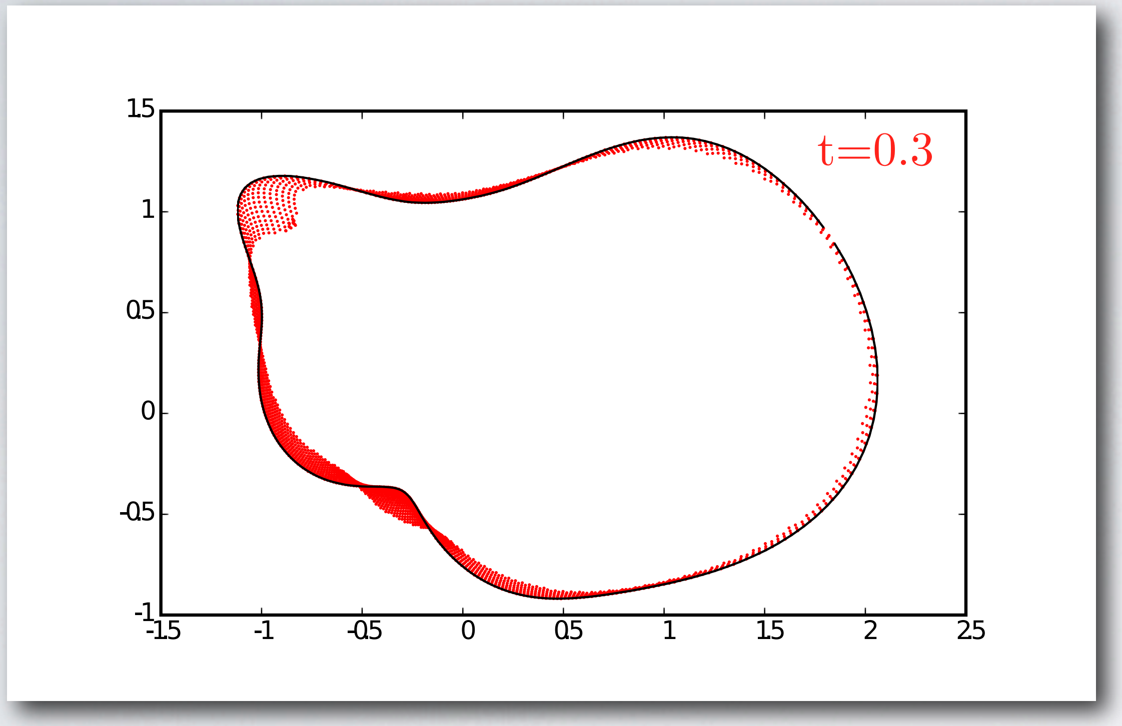

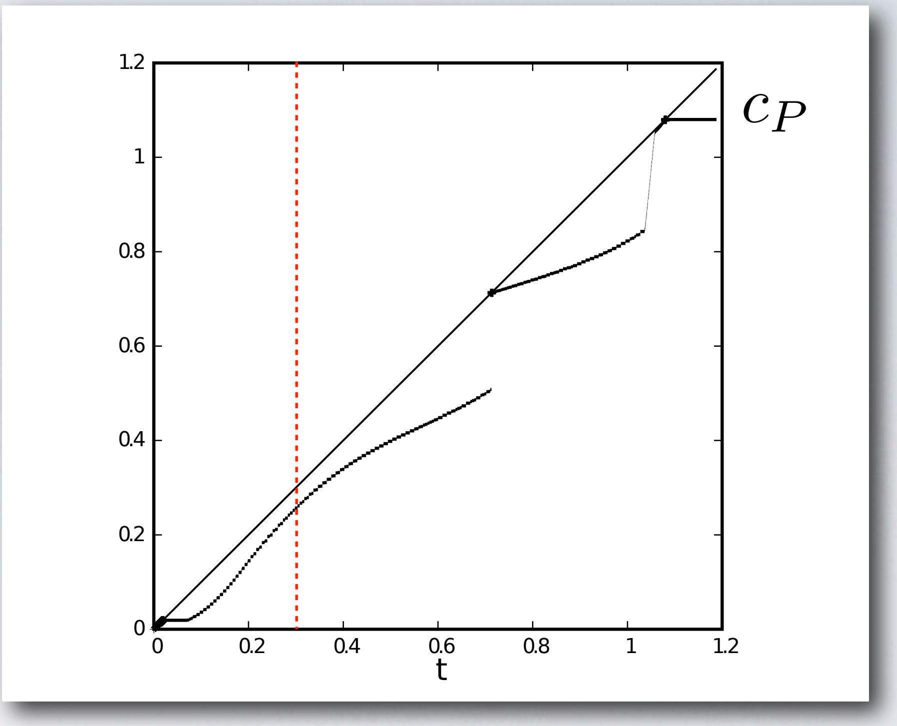

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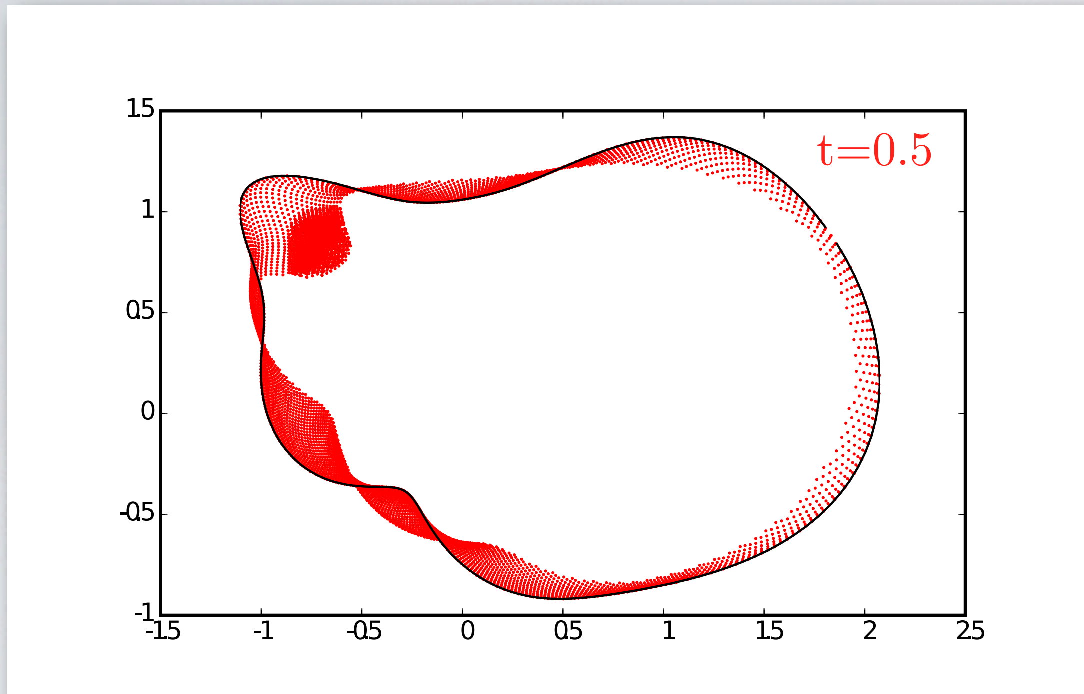

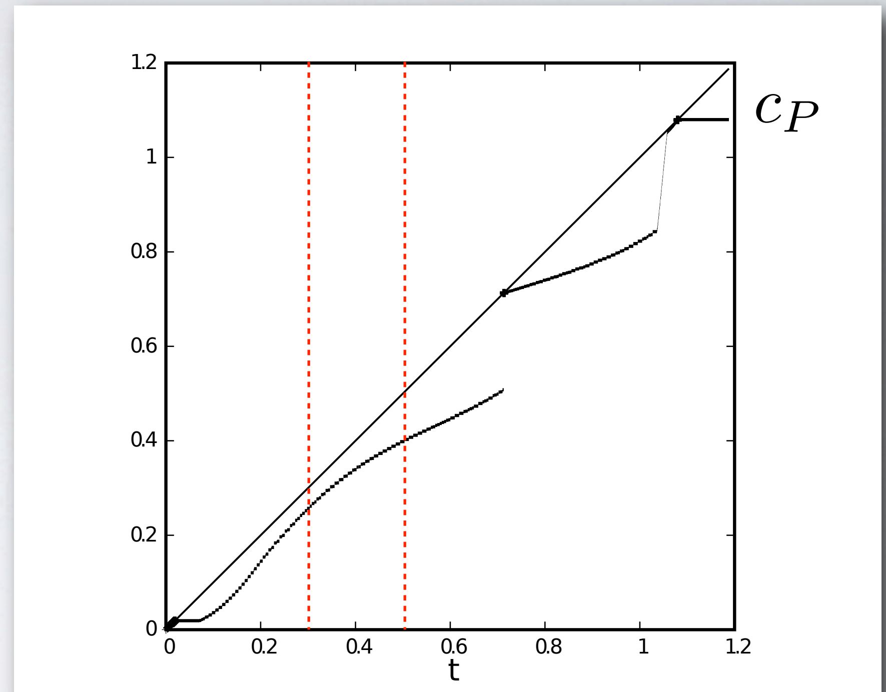

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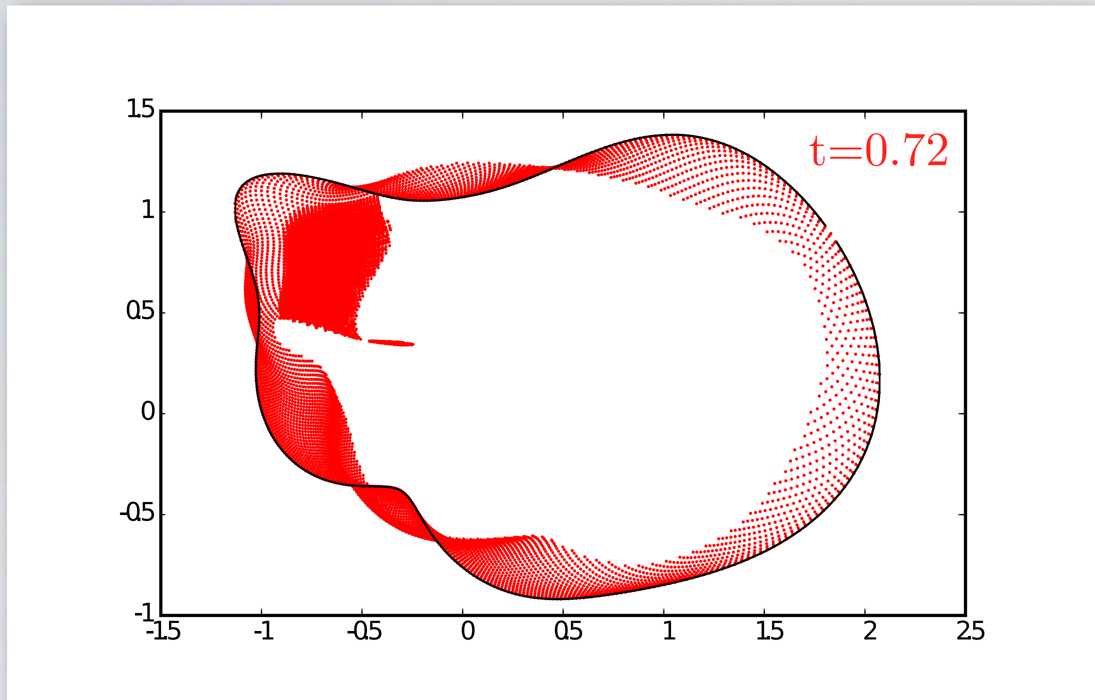

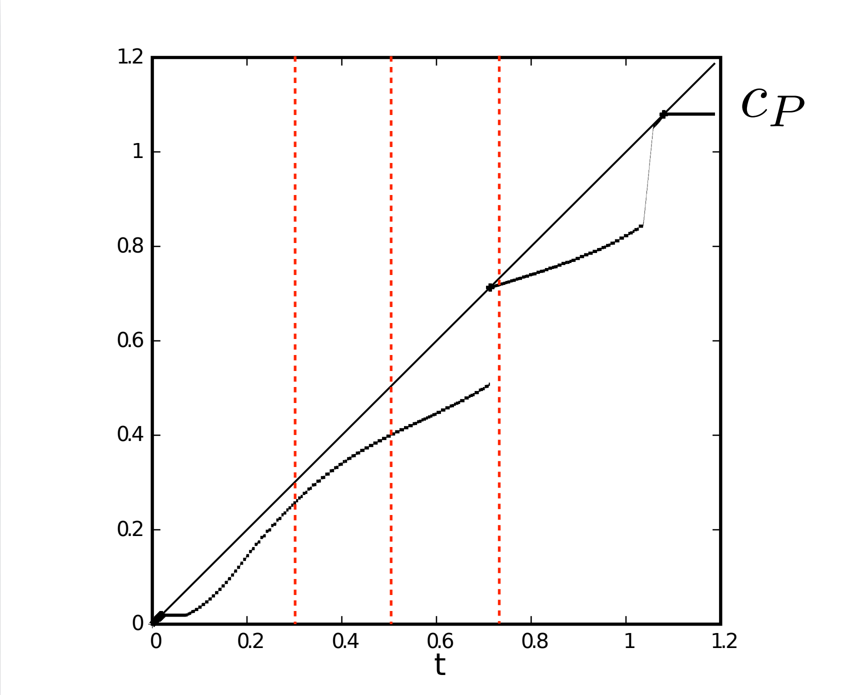

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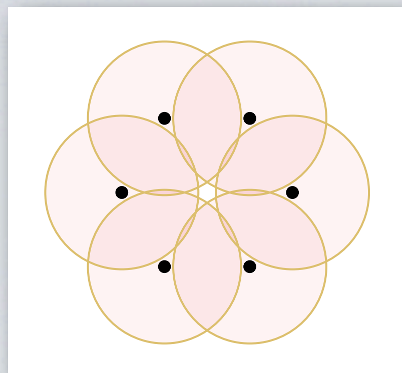
Roadmap

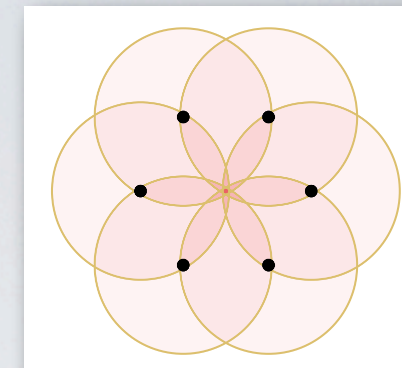
P^{α}



deformation retracts to

$c_P(t) < t, \quad \forall t \in [\alpha, \vartheta_d \alpha]$

$P^{\vartheta_d \alpha}$



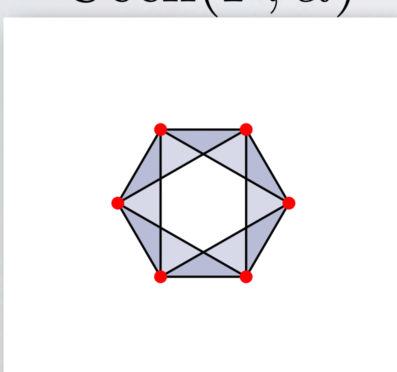
$\simeq$

sequence of collapses

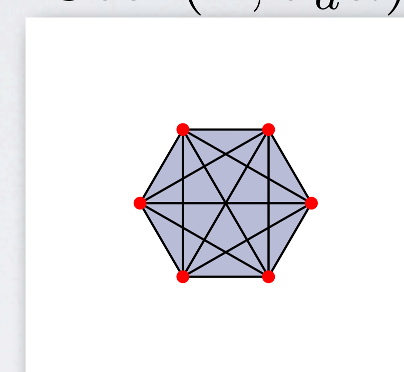
$\{\, \mathrm{Cech}(P,t) \,\}_{\alpha \le t \le \vartheta_d \alpha}$

$\simeq$

$\mathrm{Cech}(P,\alpha)$

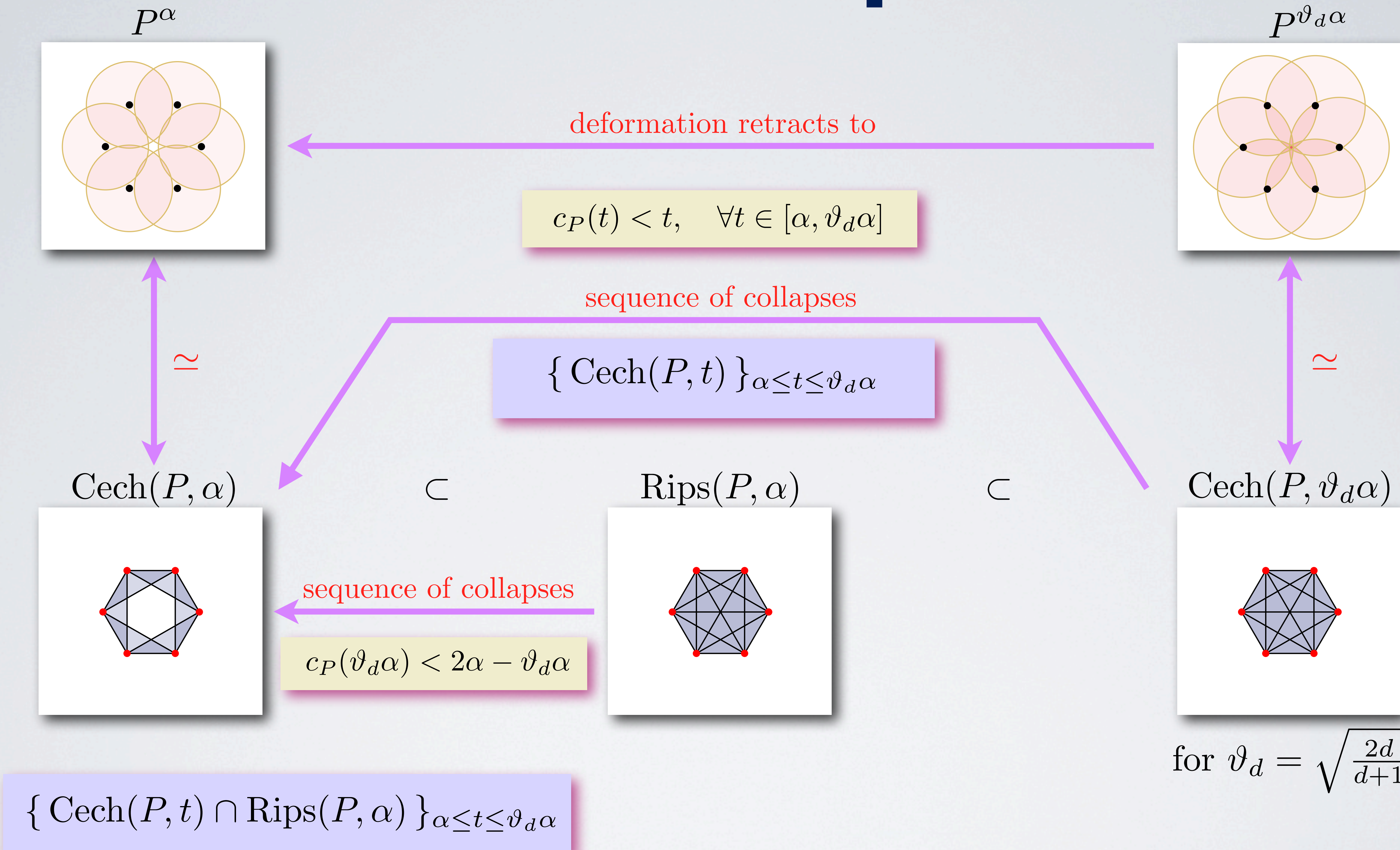


$\mathrm{Cech}(P,\vartheta_d \alpha)$



for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

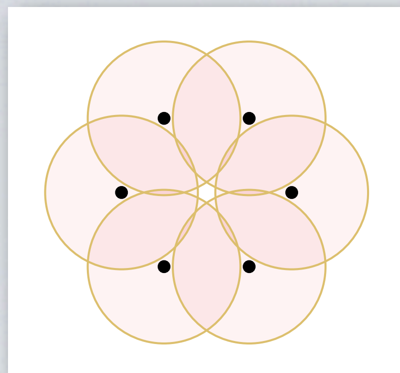
Roadmap



P^{α}
$P^{\vartheta_d \alpha}$
deformation retracts to
$c_P(t) < t, \quad \forall t \in [\alpha, \vartheta_d \alpha]$
$\simeq$
$\simeq$
sequence of collapses
$\{\, \text{Cech}(P,t) \,\}_{\alpha \le t \le \vartheta_d \alpha}$
$\text{Cech}(P,\alpha)$
$\subset$
$\text{Rips}(P,\alpha)$
$\subset$
$\text{Cech}(P,\vartheta_d \alpha)$
sequence of collapses
$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$
for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$
$\{\, \text{Cech}(P,t) \cap \text{Rips}(P,\alpha) \,\}_{\alpha \le t \le \vartheta_d \alpha}$

Roadmap

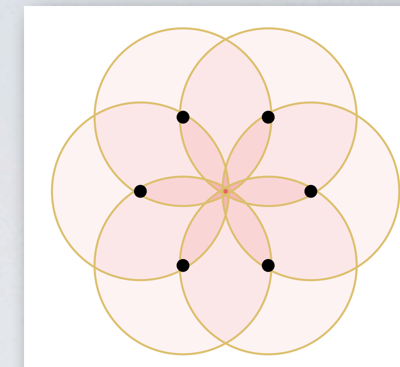
P^{α}



deformation retracts to

$c_P(t) < t, \quad \forall t \in [\alpha, \vartheta_d \alpha]$

$P^{\vartheta_d \alpha}$



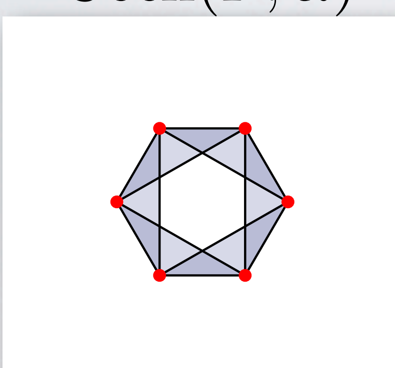
$\simeq$

sequence of collapses

$\{\, \mathrm{Cech}(P,t) \,\}_{\alpha \le t \le \vartheta_d \alpha}$

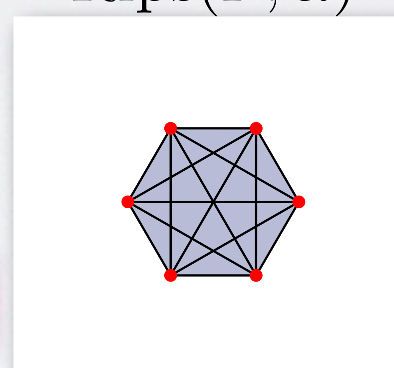
$\simeq$

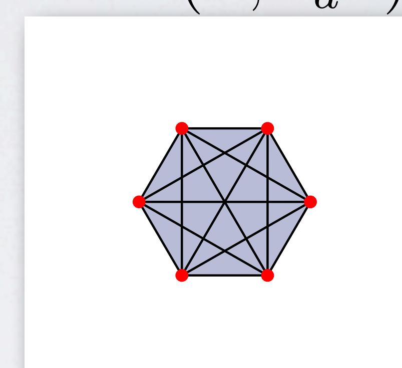
$\mathrm{Cech}(P,\alpha) \quad \subset \quad \mathrm{Rips}(P,\alpha) \quad \subset \quad \mathrm{Cech}(P,\vartheta_d \alpha)$



sequence of collapses

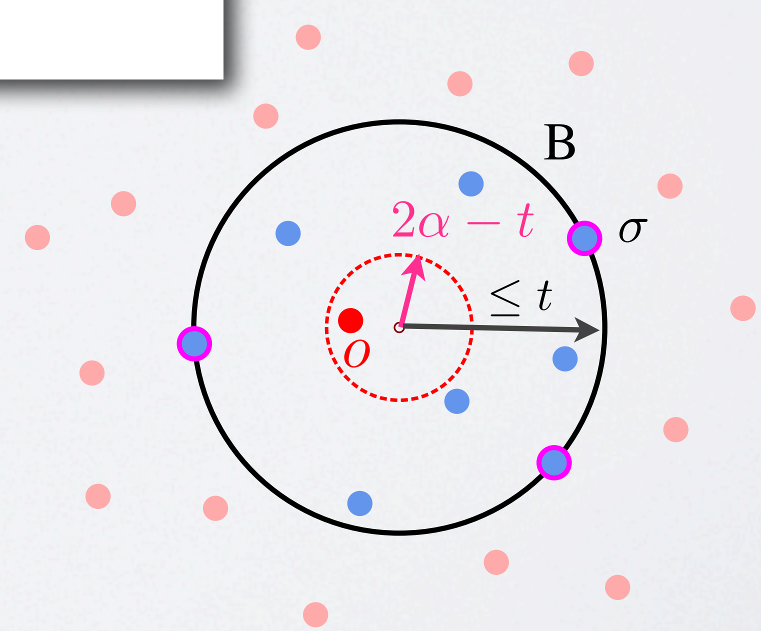
$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$





for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

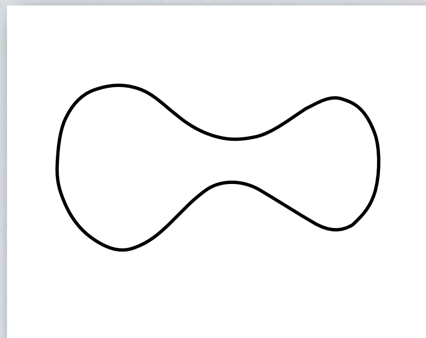
$\{\, \mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha) \,\}_{\alpha \le t \le \vartheta_d \alpha}$



$\forall t \in [\alpha, \vartheta_d \alpha], \forall \sigma \in \mathrm{Cech}(P,t)$:
the link of $\sigma \in \mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha)$ is a cone.

Rips complexes with L_2

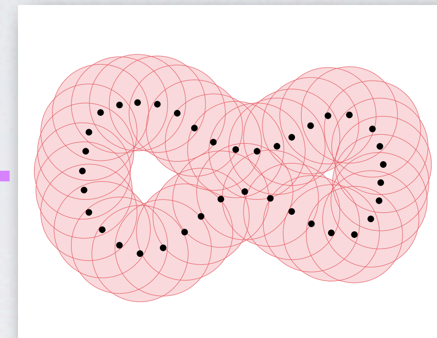
A



deformation retracts to

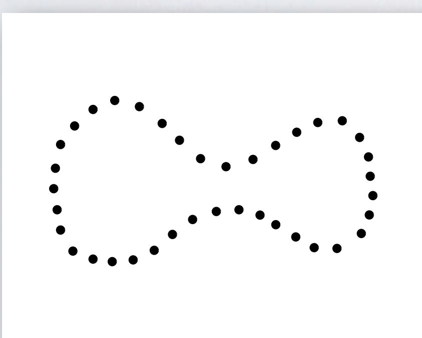
[NSW 04]

P^α

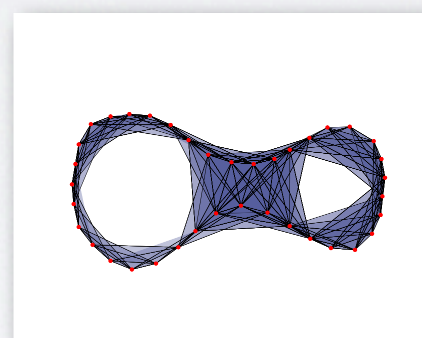


$\simeq$ **Nerve Lemma**

P

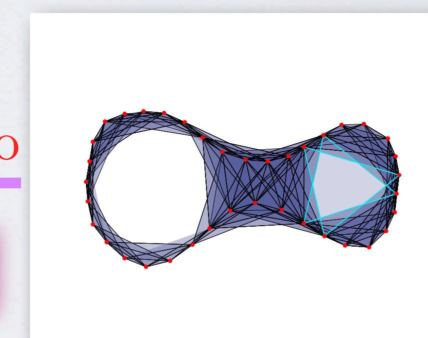


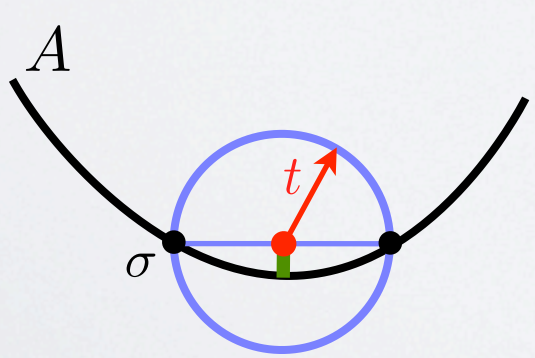
$\mathrm{Cech}(P, \alpha)$ $\subset$ $\mathrm{Rips}(P, \alpha)$



deformation retracts to

$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$





if $d_H(A, P) \leq \varepsilon$, then for $t < \mathrm{Reach}\,(A) - \varepsilon$

$$c_P(t) \leq \mathrm{Reach}\,(A) - \sqrt{\mathrm{Reach}\,(A)^2 - (t+\varepsilon)^2} + 2\varepsilon$$

Shapes with a positive reach

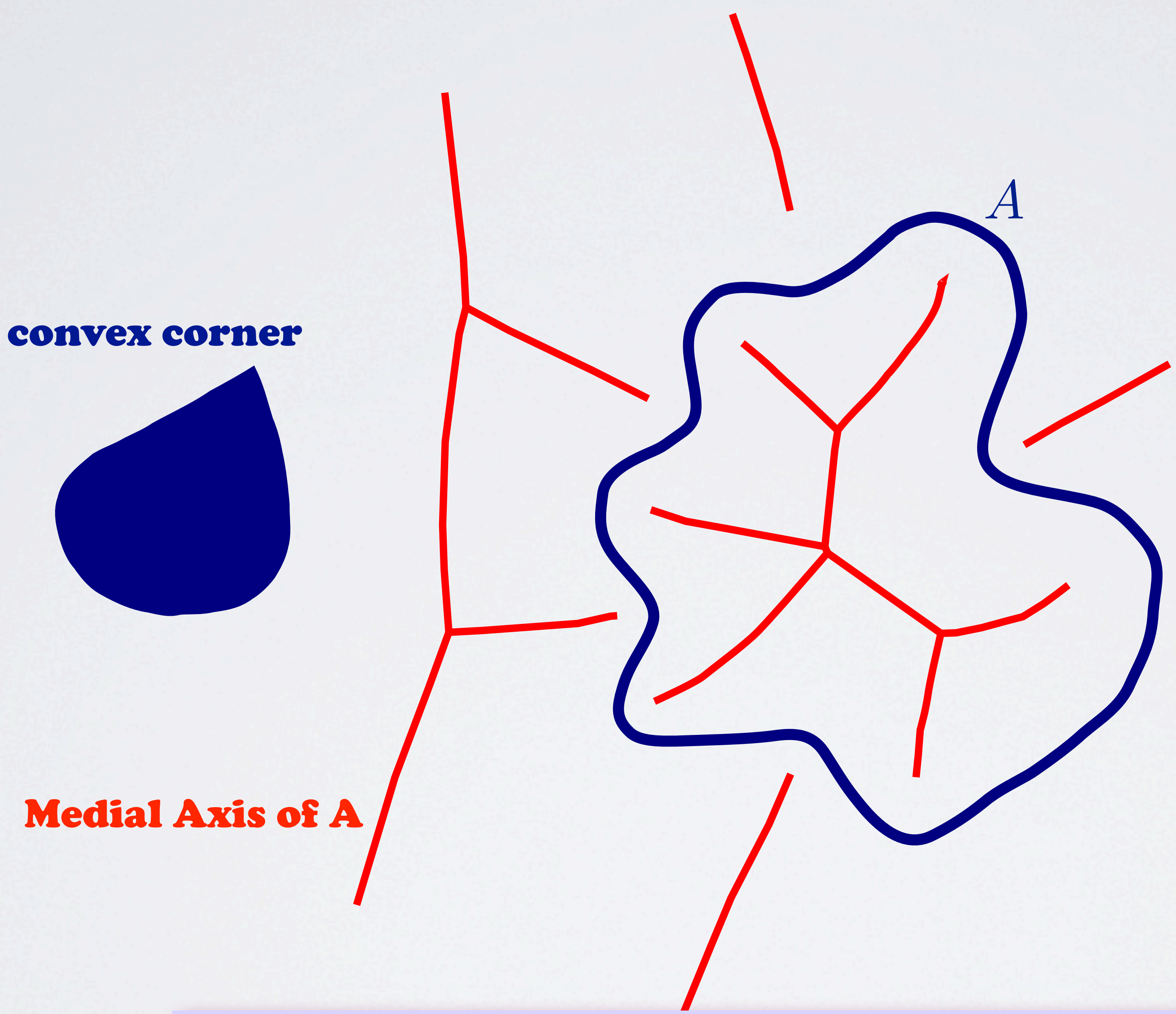
A $\xleftrightarrow{\simeq}$ $\mathrm{Rips}(P, \alpha)$ with ℓ_2

if $d_H(A, P) \leq \varepsilon$ and $\frac{\varepsilon}{\mathrm{Reach}\, A} < \lambda$ and $\frac{\alpha}{\varepsilon} = \eta$

Reconstruction	d	λ	η
P^α with [NSW04]	$\forall d$	$3 - \sqrt{8} \approx 0.17$	$2 + \sqrt{2} \approx 3.41$
$\mathrm{Rips}\,(P, \alpha)$	2	0.063	5.00
	3	0.055	5.46
	4	0.050	5.76
	5	0.047	5.97
	10	0.041	6.50
	100	0.035	7.22
	$+\infty$	$\frac{2\sqrt{2-\sqrt{2}}-\sqrt{2}}{2+\sqrt{2}} \approx 0.0340$	7.22

What now?

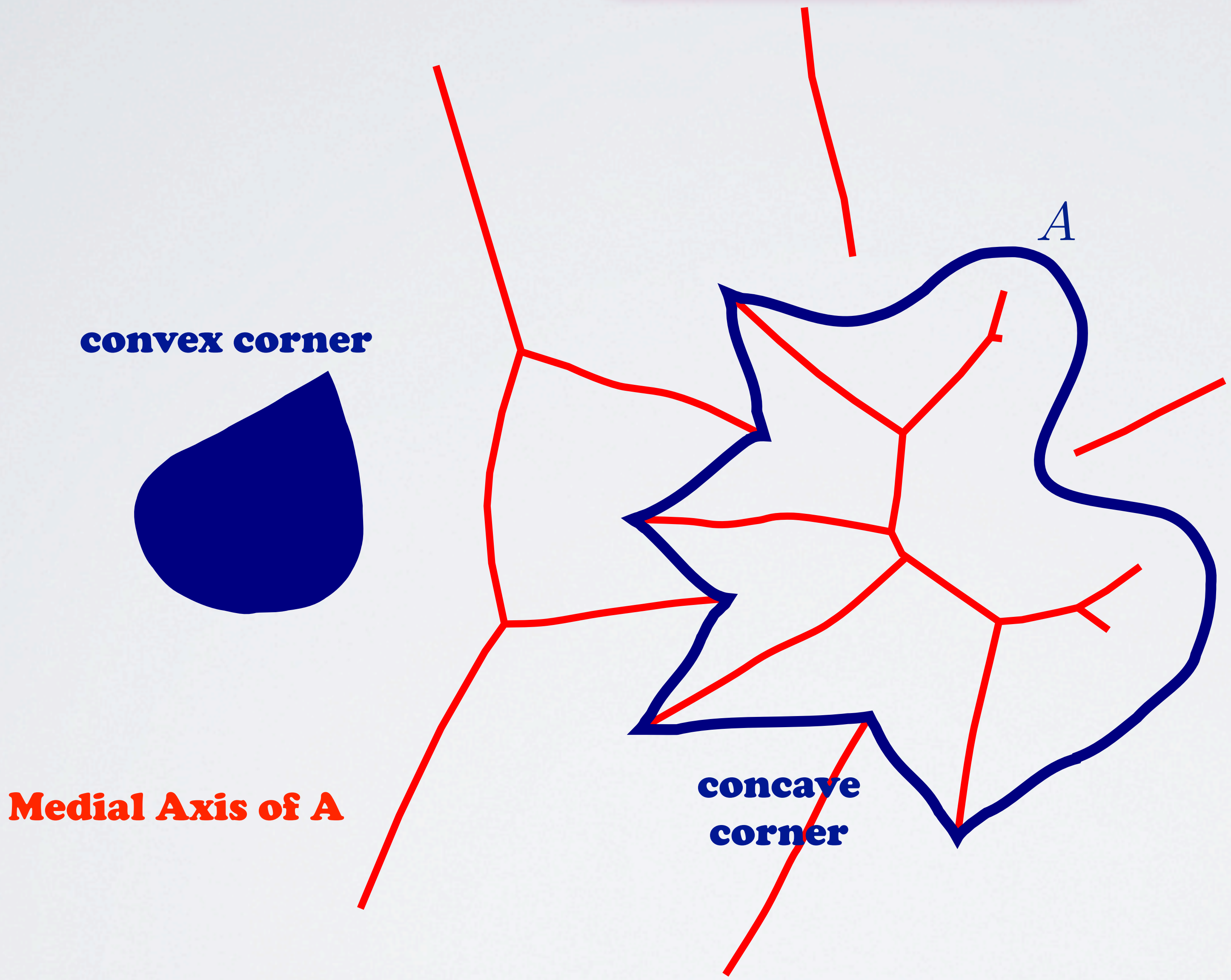
We assumed: $\text{Reach } A > 0$



Can we enlarge the class of shapes for which this is possible?

Shapes with a positive μ-reach

Here: $\text{Reach } A = 0$



But:

$\exists \mu \in (0,1]$ such that

$\mu\text{-Reach}(A) > 0$

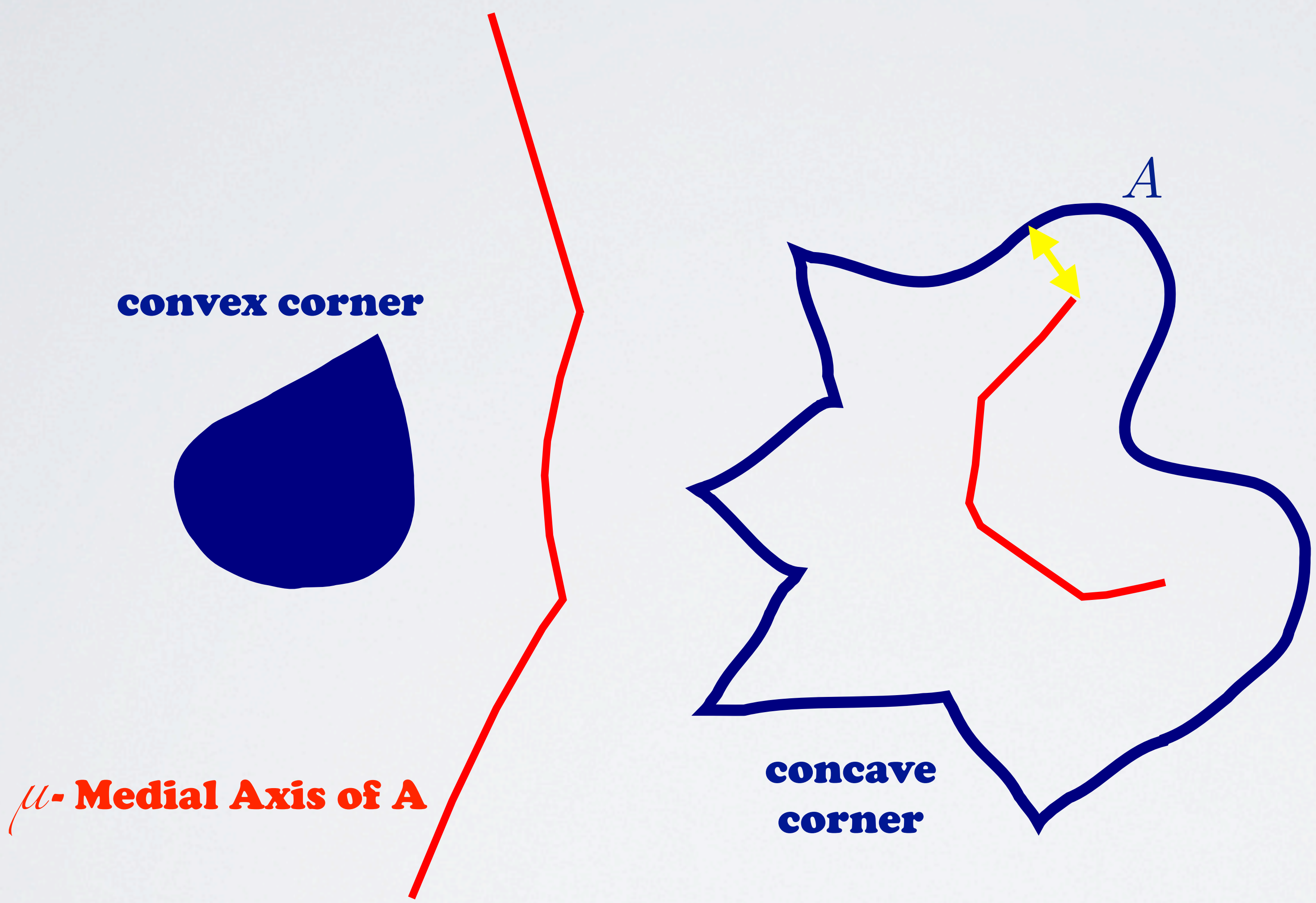
Introduced in [CCL 09]:

$$\mu\text{-MedialAxis}(A) = \{m \in \mathbb{R}^d \mid \|\nabla m\| < \mu\}$$

$$\mu\text{-Reach}(A) = d(A, \mu\text{-MedialAxis}(A))$$

Shapes with a positive μ-reach

Here: $\text{Reach } A = 0$



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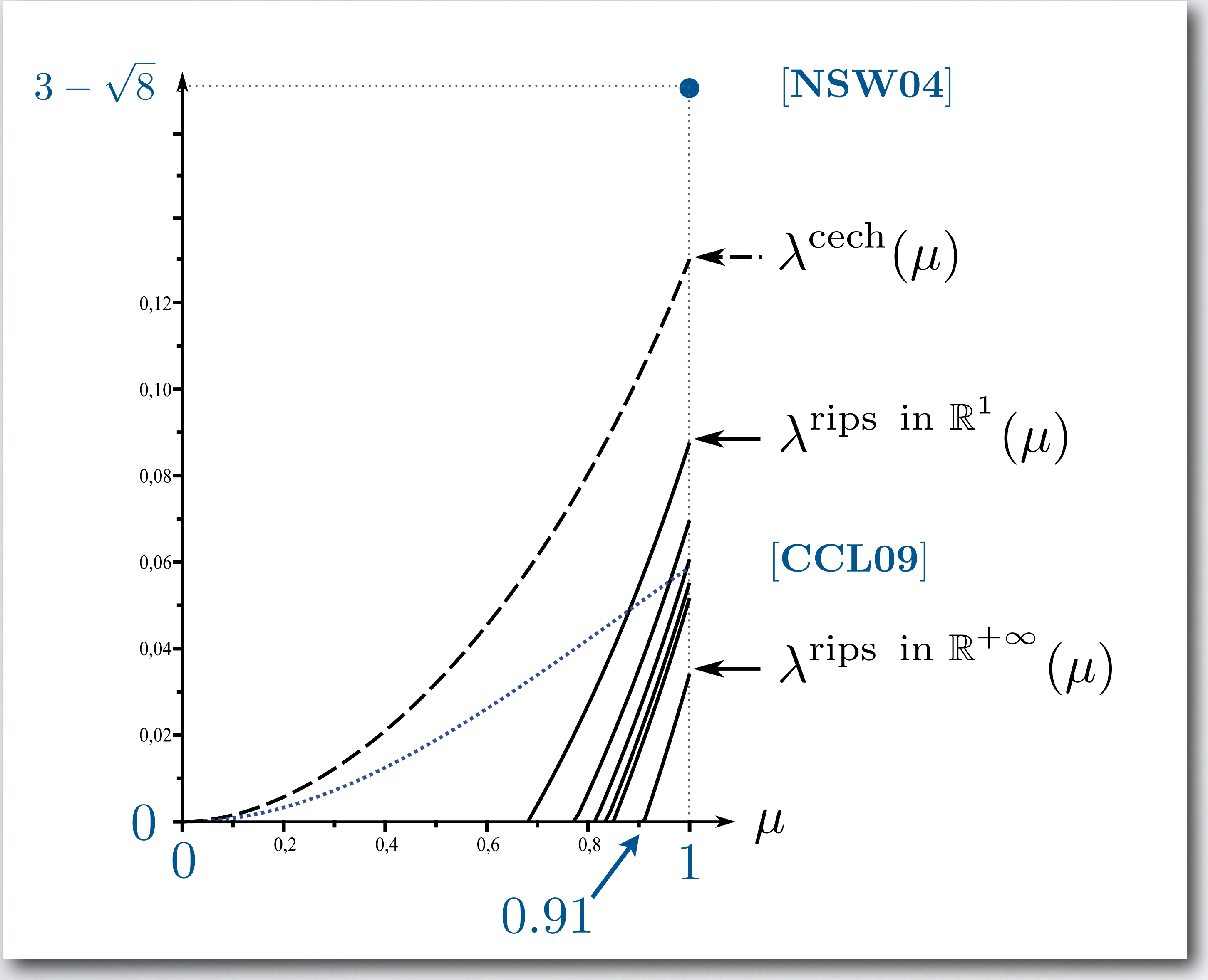
Shapes with a positive μ-reach

A^δ $\xleftrightarrow{\simeq}$ $K \in \{\mathrm{Cech}(P,\alpha), \mathrm{Rips}(P,\alpha)\}$

δ arbitrarily small

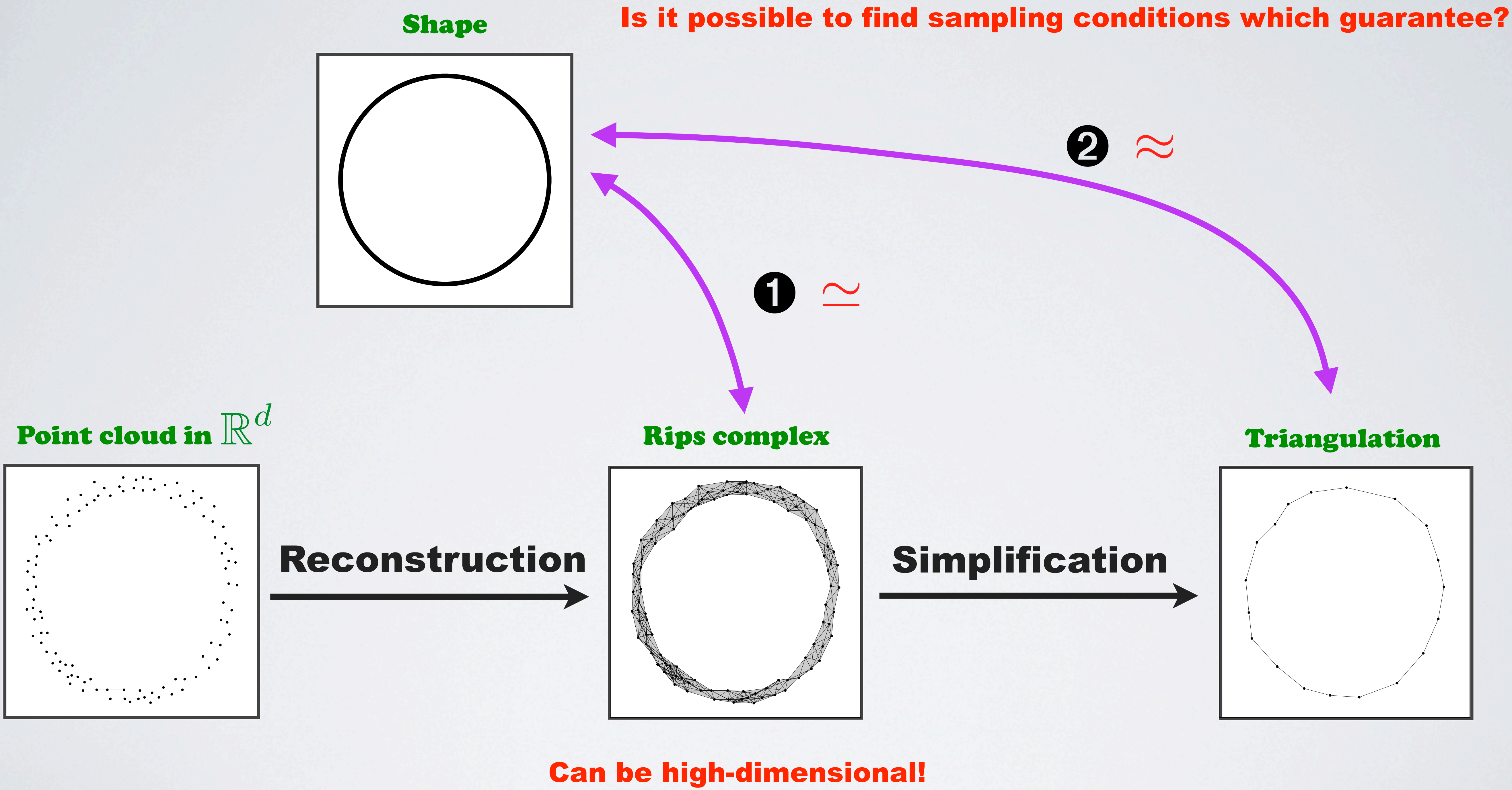
if $d_H(A,P) \leq \varepsilon$ and $\dfrac{\varepsilon}{\mu\text{-Reach}(A)} < \lambda^K(\mu)$

for



and a well chosen α

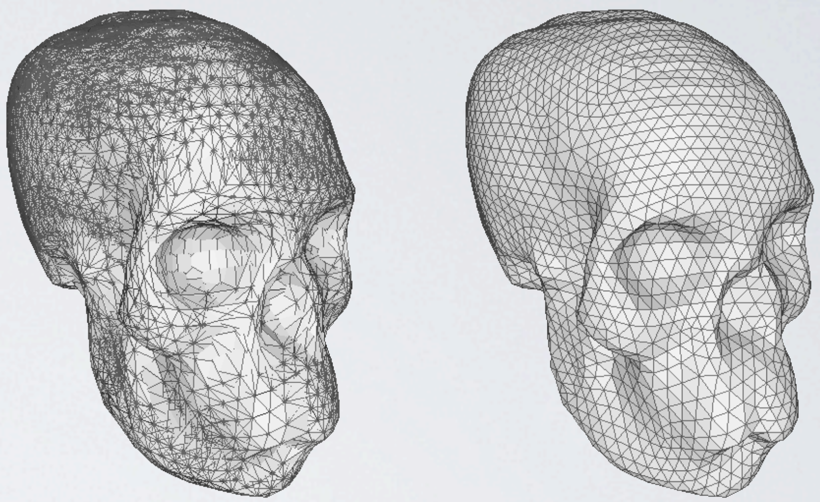
Overview


Shape
Is it possible to find sampling conditions which guarantee?
❷ $\approx$
❶ $\simeq$
Point cloud in $\mathbb{R}^d$
Rips complex
Triangulation
Reconstruction
Simplification
Can be high-dimensional!

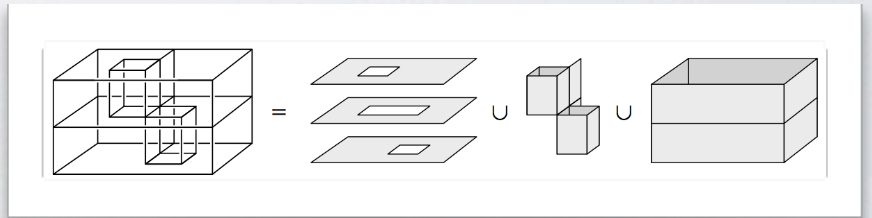
Simplification

How to get an object with the right dimension?

- Different strategies:
  - Edge contractions;
  - Vertex and edge collapses …
  - Seems to work well in practice



- And yet, not all reductions are possible nor easy:
  - It is algorithmically unsolvable to decide whether a simplicial complex is contractible.
  - A triangulated Bing's house is contractible but not collapsible
  - Collapsibility of 3-complexes is NP-hard [Martin Tancer 2012]
  - Geometry has to play a key role.



Ongoing work

Shape A

$\approx$

Triangulation of A

$\mathrm{Rips}(P, \alpha)$

Ongoing work

Shape A

$\approx$

Triangulation of A

$\mathrm{Cech}(P, \alpha)$ $\xleftarrow{\text{sequence of collapses}}$ $\mathrm{Rips}(P, \alpha)$

$\mathrm{Nerve}\{B(p, \alpha) \mid p \in P\}$

$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$

Ongoing work

Shape A

$\approx$

Triangulation of A

$\mathrm{Cech}_A(P,\alpha)$ $\xleftarrow{\text{sequence of collapses}}$ $\mathrm{Cech}(P,\alpha)$ $\xleftarrow{\text{sequence of collapses}}$ $\mathrm{Rips}(P,\alpha)$

$\mathrm{Nerve}\{A \cap B(p,\alpha) \mid p \in P\}$

$\mathrm{Nerve}\{B(p,\alpha) \mid p \in P\}$

$d_H(A,P) \leq \varepsilon < (3-\sqrt{8})\,\mathrm{Reach}\,A$

$\alpha = (2+\sqrt{2})\varepsilon$

$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$

Ongoing work

Shape A

$\approx$

$\exists$? α-Nice Triangulation of A

$\text{Nerve}\{A \cap \text{Hull}_\alpha(\text{Cell}(v)) \mid v \in V\}$

with $A = \bigcup_{v \in V} \text{Cell(v)}$ and $\text{Cell}(v) \subset B(p, \alpha)$ for some $p \in P$

sequence of collapses

$\alpha < \text{Reach } A$

$\text{Cech}_A(P, \alpha) \xleftarrow{\text{sequence of collapses}} \text{Cech}(P, \alpha) \xleftarrow{\text{sequence of collapses}} \text{Rips}(P, \alpha)$

$\text{Nerve}\{A \cap B(p, \alpha) \mid p \in P\}$

$\text{Nerve}\{B(p, \alpha) \mid p \in P\}$

$d_H(A, P) \leq \varepsilon < (3 - \sqrt{8}) \text{ Reach } A$

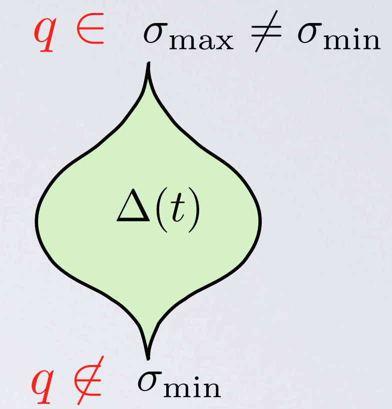
$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$

$\alpha = (2 + \sqrt{2})\varepsilon$

Restricting the Cech complex

$K(t) = \text{Nerve}\{A^t \cap B(p, \alpha) \mid p \in P\}$ as t goes from $+\infty$ to 0

$\Delta(t) =$ set of simplices that disappear at time t



Does the operation that removes $\Delta(t)$ from $K(t)$ a collapse?

If $t \geq \beta = \sqrt{R - (R-\varepsilon)^2 - \alpha^2}$, then $\Delta(t) = \emptyset$

If $t < \beta$ such that $\Delta(t) \neq \emptyset$

(1) Generically, $\Delta(t)$ has a unique minimal element $\sigma_{\min}$

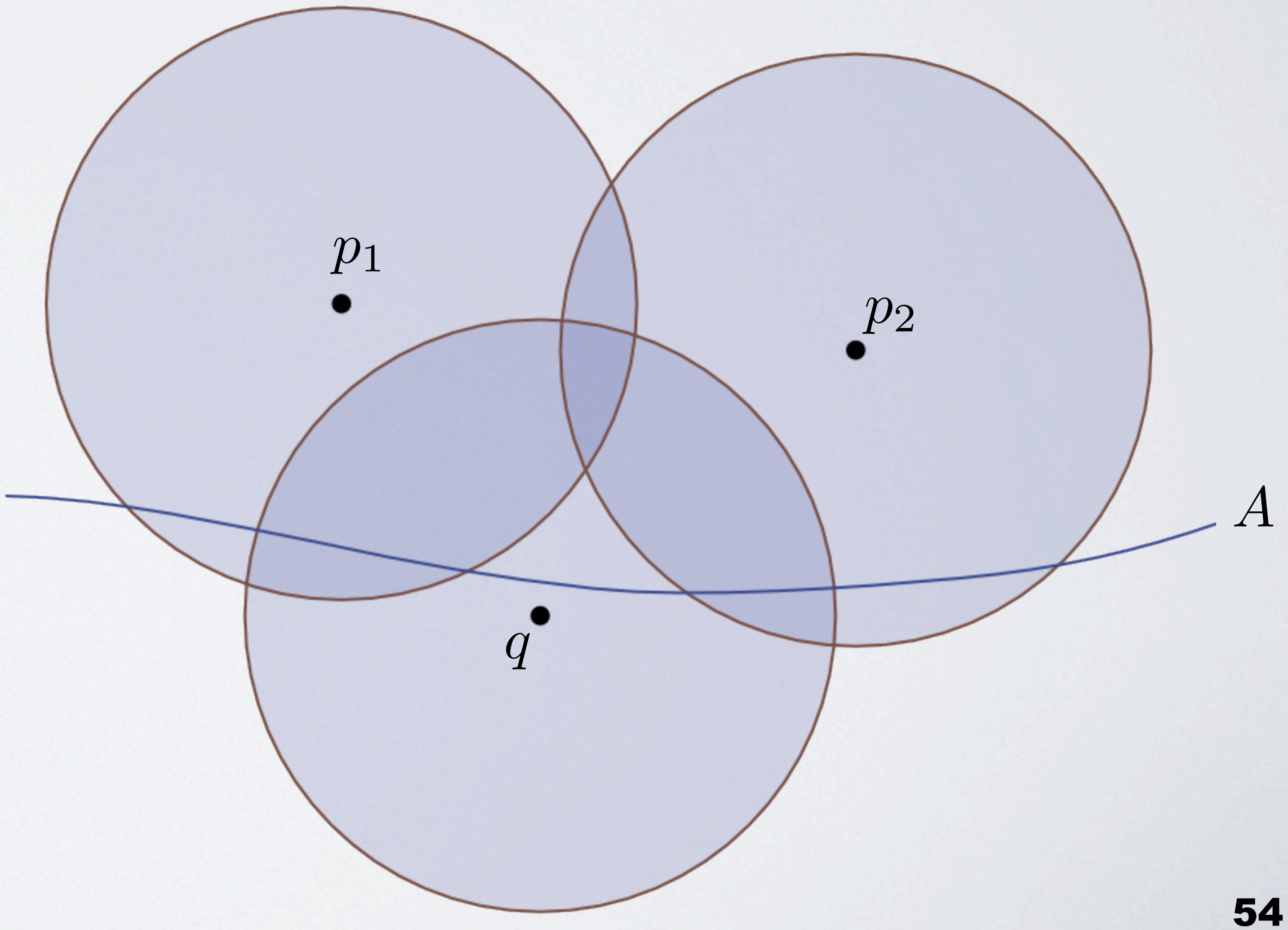
(2) $\bigcap_{p \in \sigma_{\min}} C_p(t) = \{y\}$

(3) $\sigma_{\max} = \{p \in P \mid y \in B(p, \alpha)\}$

(4) $y \in \partial C_p(t), \quad \forall p \in \sigma_{\min}$

(5) $\exists q \in P$ such that $y \in C_p(t)^\circ$

(6) $\sigma_{\min} \neq \sigma_{\max} \implies$ removing $\Delta(t)$ is a collapse



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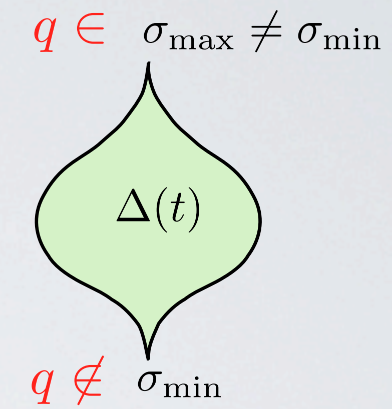

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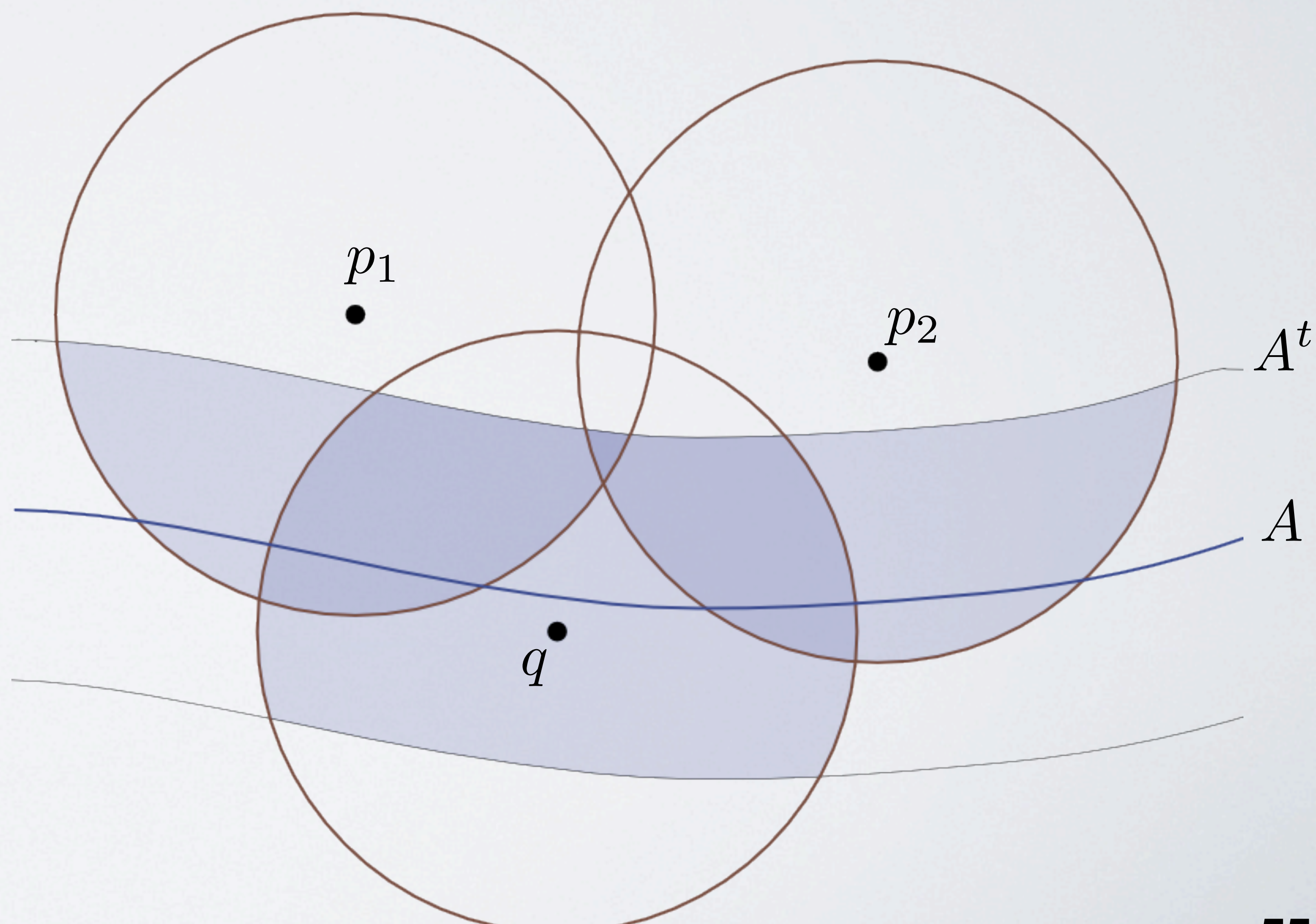

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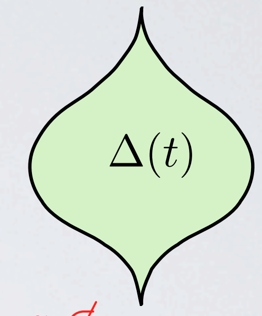

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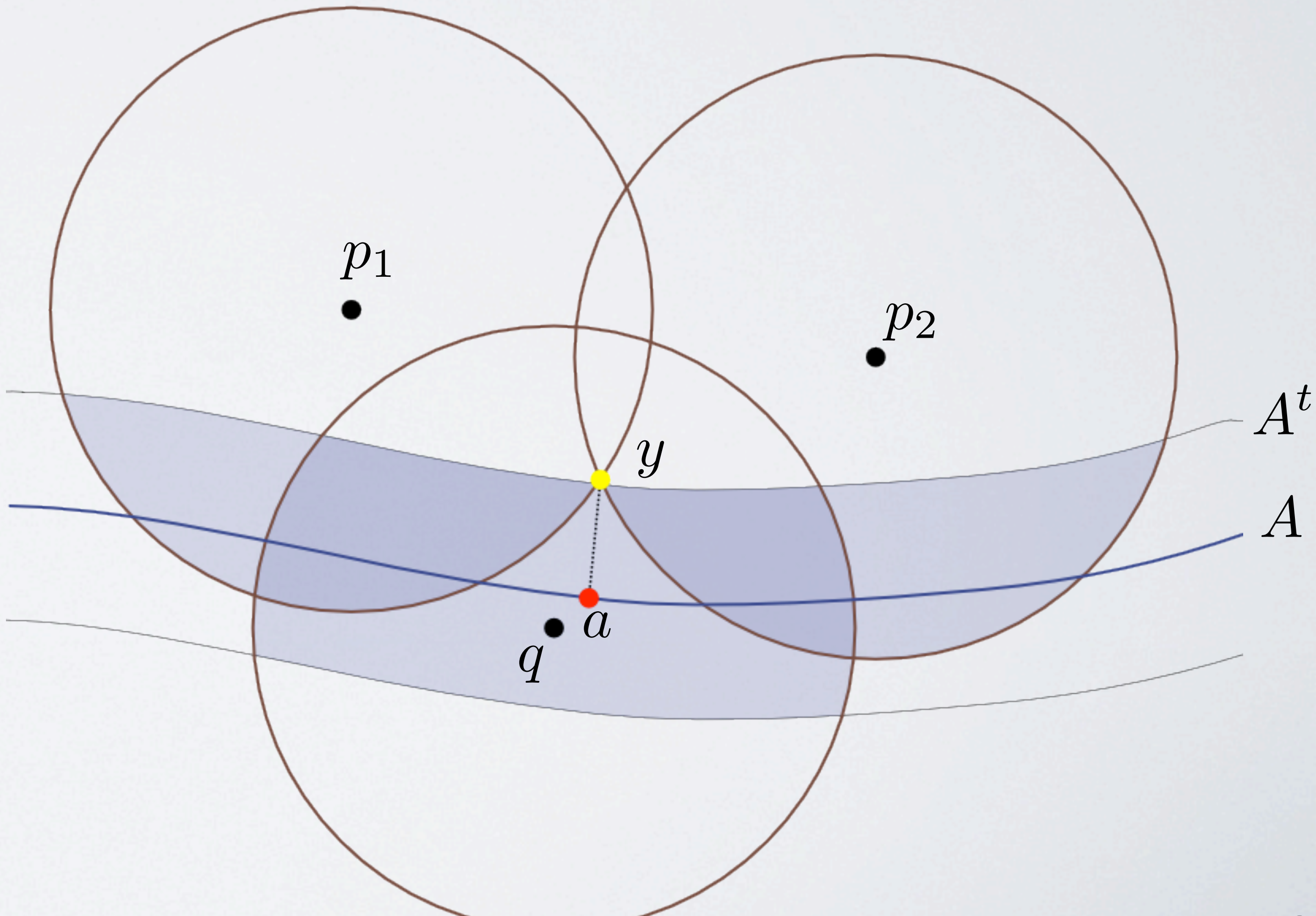

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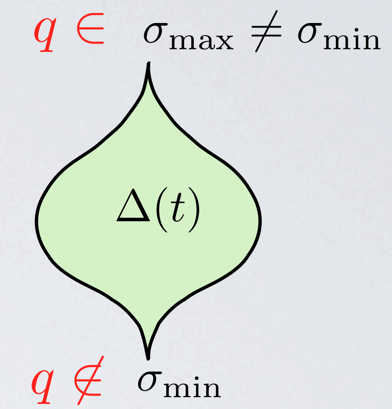

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If $t < \beta$ such that $\Delta(t) \neq \emptyset$

(1) Generically, $\Delta(t)$ has a unique minimal element $\sigma_{\min}$

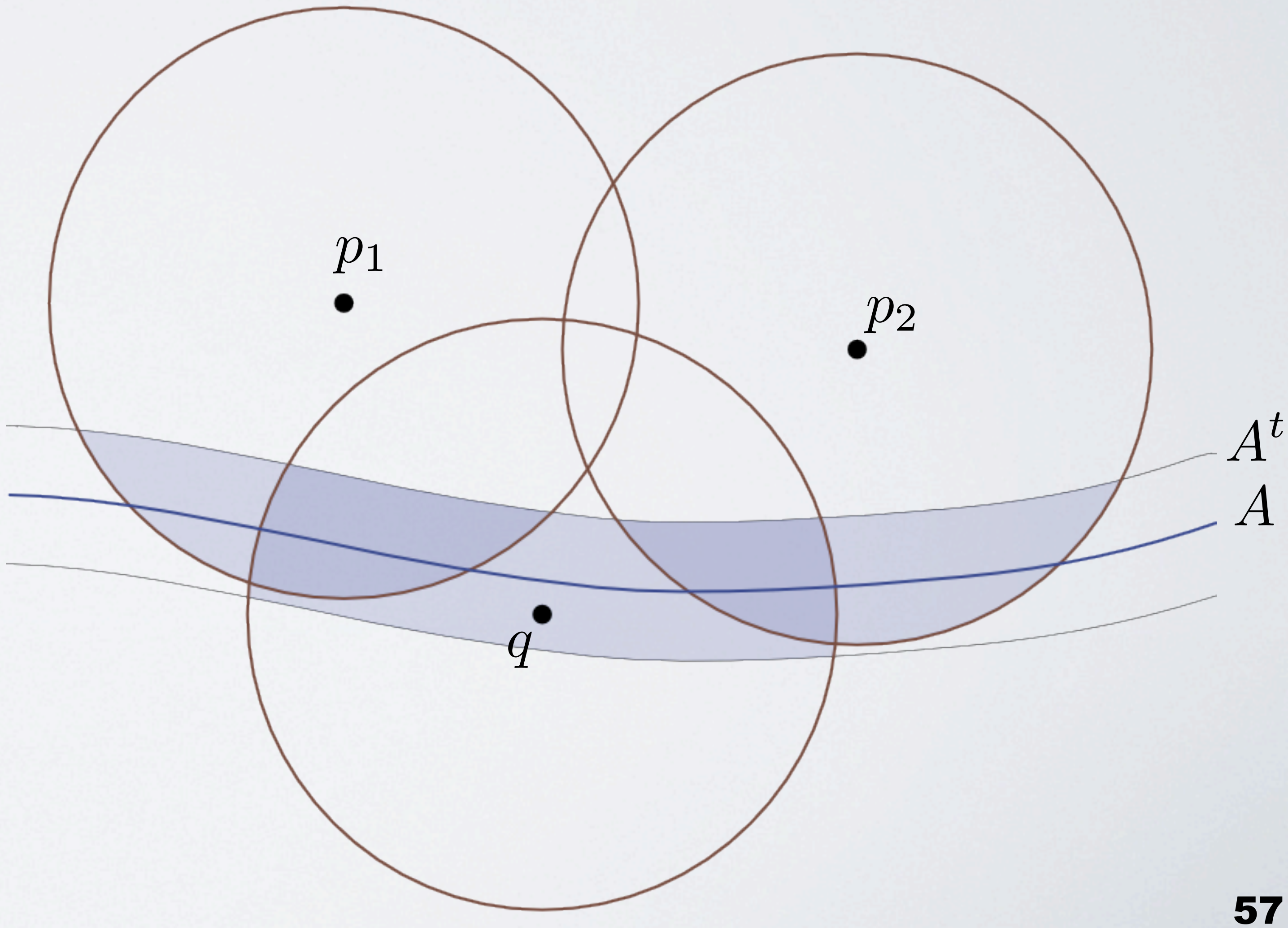
(2) $\bigcap_{p \in \sigma_{\min}} C_p(t) = \{y\}$

(3) $\sigma_{\max} = \{p \in P \mid y \in B(p,\alpha)\}$

(4) $y \in \partial C_p(t), \quad \forall p \in \sigma_{\min}$

(5) $\exists q \in P$ such that $y \in C_p(t)^\circ$

(6) $\sigma_{\min} \neq \sigma_{\max} \implies$ removing $\Delta(t)$ is a collapse



Future work

- Identify surfaces with α-nice triangulations.
- Turn this into a practical algorithm.

References

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