UNIT 15 COLUMNS AND STRUTS

Structure

15.1 INTRODUCTION

Column or *strut* is a compression member in which the length is considerably larger compared to the cross-section dimension. *Column* denotes vertical member in compression. The terms *pillar* and *stanchion* are used for long vertical compression members. *Strut* is any member (including diagonal or horizontal) subjected to compression.

In case of long compression members, the load causes the column to bend and stresses are affected by the deflection produced. The stress due to direct compression is very small compared to stress due to bending. This phenomenon is known as *buckling.*

The derivation of Euler's buckling load is discussed in Section 15.2 along with concepts of *effective length* and *slenderness ratio*. The secant formula and laterally loaded columns are discussed in Section 15.3. Empirical formulae are discussed in Section 15.4.

Objectives

After studying this unit, you should be able to

- find the safe load a column can carry due to axial or eccentric loading,
- identify *slender* and *short* columns,
- find the maximum stresses in columns subjected to axial, with or without, transverse load, and
- design the column if the load and permissible stresses are known.

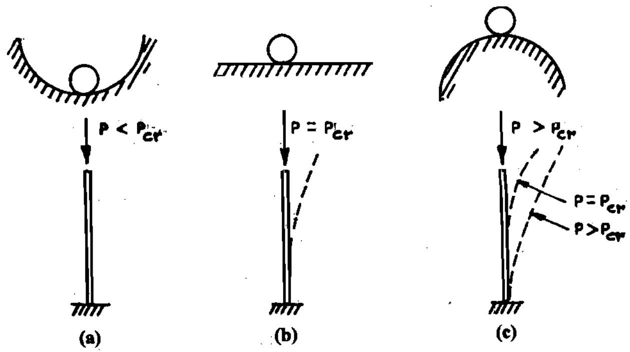
15.1.1 Stable and Unstable Equilibrium

From mechanics it is known that a body may be in three types of equilibrium, viz. *stable*, *neutral* or *unstable* as in Figure 15.1 (a), (b), and (c) respectively. These three conditions can be compared as under, by increasing the compressive load P on the column gradually.

(a) As shown in Figure 15.1 (a), when the ball resting on concave surface is disturbed slightly it will regain its original position, similarly the column is initially in a state of *stable equilibrium*. During this state if the column is perturbed by inducing small lateral deflections it will return to its straight configuration when the loads are removed.

(b) The ball resting on plane horizontal surface is in state of *neutral equilibrium* as shown in Figure 15.1 (b) which is the limiting condition between stable and unstable equilibrium. When the load on the column is increased further, a critical value is reached at which the column is on the verge of experiencing a lateral deflection, it will not return to its straight configuration. The load cannot be increased beyond this value unless the column is restrained laterally by lateral restraint.

(c) When the ball is resting on a convex surface, a negligible perturbation will cause *unstable equilibrium* as shown in Figure 15.1 (c). Similarly if the force P exceeds the critical load P_{cr}, the column becomes unstable. The column either collapses or undergoes large lateral deflection.



Figu~e 15.1 : (a) Stable (b) Neutral (c) Unstable Equilibrium

15.1.2 Buckling Phenomenon and Buckling Load

Buckling Phenomenon

When the length of the strut or column is large as compared to its lateral dimensions, failure generally occurs due to lateral deflection rather than by direct compression. This lateral deflection in a long column is termed as buckling. In contrast buckling is negligible in short columns. They fail due to crushing. In very long columns the effect of direct stresses is small as compared with bending stresses. Main causes of bending in the columns are as follows :

(a) lack of straightness and uniformity in the member itself,

(b) initial crookedness or curvature of the member,

(c) eccentricity of the applied load,

(d) non-homogeneity in the material of the member,

(e) minute flaws in the material, and

(f) casting of column may be out of plumb and load not being transmitted at the selected bearing (accidental eccentricity).

Buckling Load

Once a member shows signs of buckling, it will lead to the failure of the member. This load at which the member just buckles is called the *buckling load* or *critical load* or *crippling load*. The buckling load is less than the crushing load. The value of buckling load is low for long columns and relatively high for short columns. The value of the buckling load for a given member depends upon the length of the member and the least lateral dimension. It also depends upon the types of end-constraints of the column (hinged, fixed etc.). Thus, when an axially loaded compression member just buckles, it is said to develop an **elastic instability.**

15.2 THEORY OF CONCENTRICALLY LOADED COMPRESSION MEMBER

The first solution for the buckling of long slender columns was published in 1757 by the Swiss mathematician Euler (1707-1783). Although the results of this article can be used only for slender columns, the analysis, similar to that used by Euler, is mathematically revealing and helps in explanations of the behaviour of columns.

Assumptions Made in Euler's Theory

(a) The column is initially perfectly straight and is axially loaded.

(b) Column section is uniform and material is perfectly elastic, homogeneous, isotropic and obeys Hooke's law.

(c) Length of the column is very large as compared to the lateral dimensions.

(d) The direct stress is very small compared with the bending stress corresponding to the buckling condition.

(e) Self weight of the column is negligible and the column will fail by buckling only.

(f) Joints are frictionless.

15.2.1 Derivation of Euler's Buckling Load Formula for Pin-end Condition

The purpose of this analysis is to determine the minimum axial compressive load for which a column will experience lateral deflections. A straight, slender, pivot-ended column having uniform section is concentrically loaded by axial compressive force P at each end, is shown in Figure 15.2 (a). A pivot-ended column is supported such that the bending moment and lateral movement are zero at the ends.

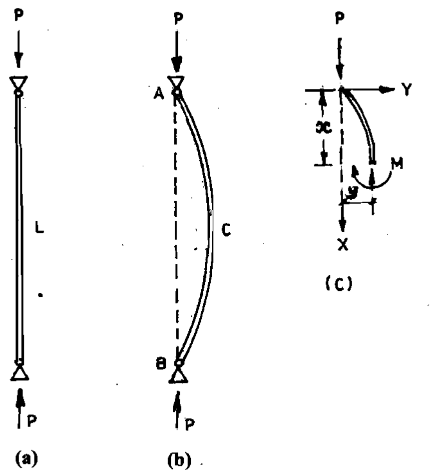


Figure 15.2 : Pin-ended Column : Euler Load Determination

Due to crippling load the column will deflect into a curved shape ACB shown in Figure 15.2 (b). Consider a cross-section at a distance x from the end A. Let y be the lateral deflection at this section.

The bending moment due to crippling load, $M = Py$

$$\frac{EI\, d^2y}{dx^2} = -M = -Py \tag{15.1}$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

$$\frac{d^2y}{dx^2} + k^2y = 0 \qquad \text{where, } k = \sqrt{\left(\frac{P}{EI}\right)}$$

The solution of this differential equation is

$$y = A \sin kx + B \cos kx \tag{15.2}$$

where, A and B are constants of integration which are evaluated by putting the end conditions, namely at

$$x = 0,\ y = 0,\ \text{ i.e. } B = 0$$

and at

$$x = l,\ y = 0,\ \text{ i.e. } 0 = A \sin kl. \tag{15.3}$$

From Eq. (15.3), it is seen that either $A = 0$ or $\sin kl = 0$.

As $B = 0$, then if A is also equal to zero, then from Eq. (15.2), we get $y = 0$, i.e. column is not deflecting at all, which is trivial solution.

Therefore, $$\sin kl = 0, \text{ or } kl = 0, \pi, 2\pi, \ldots, n\pi$$

i.e. $$k = 0, \frac{\pi}{l}, \frac{2\pi}{l}, \ldots, \frac{n\pi}{l}$$

Since $k = \left(\frac{P}{EI}\right)^{1/2}$

$$P = k^2 EI$$

$$P = \frac{\pi^2 EI}{l^2}, \frac{4\pi^2 EI}{l^2}, \ldots, \frac{n^2\pi^2 EI}{l^2} \qquad (15.4)$$

The values given by Eq. (15.4) are required values of critical loads. The lowest critical load is the most significant and which is as follows :

$$P_{cr} = \pi^2 \frac{EI}{l^2} \qquad (15.5)$$

This load is termed as Euler's load and is denoted by P_E. Eq. (15.5) is known as Euler's formula.

It can be seen that the column will have a tendency to bend or buckle in that plane about which flexural rigidity EI is least. Therefore in the above equation minimum moment of inertia should be used. It can be seen that critical load is proportional to flexural rigidity and inversely proportional to length does not depend upon permissible and stress of material from which the column is made.

15.2.2 Concepts of Effective Length, Slenderness Ratio, Critical Stress, Short and Long Columns

The moment of inertia, I, refers to the axis about which bending occurs. Putting $I = Ar^2$, where, r is the radius of gyration about the axis of bending,

$$P_{cr} = \frac{\pi^2 EAr^2}{L^2}$$

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} = \frac{\pi^2 E}{\rho^2} = \sigma_{cr} \qquad (15.6)$$

where, ρ is slenderness ratio, σ_{cr} is called critical stress.

The quantity $\left(\frac{L}{r} = \rho\right)$, ratio of effective length to radius of gyration, is the *slenderness ratio* and is determined for the axis about which bending tends to occur. For a pivot-ended concentrically loaded column with no intermediate bracing to restrain lateral motion, bending occurs about the axis of minimum moment of inertia. Therefore, r, radius of gyration is taken as minimum.

The Euler buckling load as given by Eq. (15.5) agrees well with experiment only if the slenderness ratio is large, whereas short compression members can be analyzed easily considering direct stress $\sigma = P/A$. Many columns lie between these extremes in which neither of these solutions is applicable. These intermediate length columns are analyzed by empirical formulae described in later sections.

Limitations of Euler's Formula

To check the validity of Euler's formula consider Eq. (15.5) which implies that if the slenderness ratio is small, the stress at the failure σ_{cr} will be large. Let σ_p be the crushing strength of column material. If $\sigma_{cr} > \sigma_p$ the failure of column will be due to crushing and not due to buckling. Hence, the Euler's formula will not be applicable for smaller slenderness ratio.

For validity of Euler's formula,

$$\sigma_{cr} \leq \sigma_p$$

$$\frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} \leq \sigma_p$$

$$\therefore \quad \left(\frac{L}{r}\right) \geq \pi \sqrt{\left(\frac{E}{\sigma_p}\right)}$$

Let us consider two materials, namely cast iron and mild steel. For mild steel, yield point stress = 330 MPa and $E = 2.1 \times 10^5$ N/mm^2. Equating the crippling stress to yield stress, the corresponding minimum value of slenderness ratio is around 80. Figure 15.3 shows variation of critical stress, σ_{cr}, for these two materials as a function of slenderness ratio. The curves representing Eq. (15.5) are hyperbola.

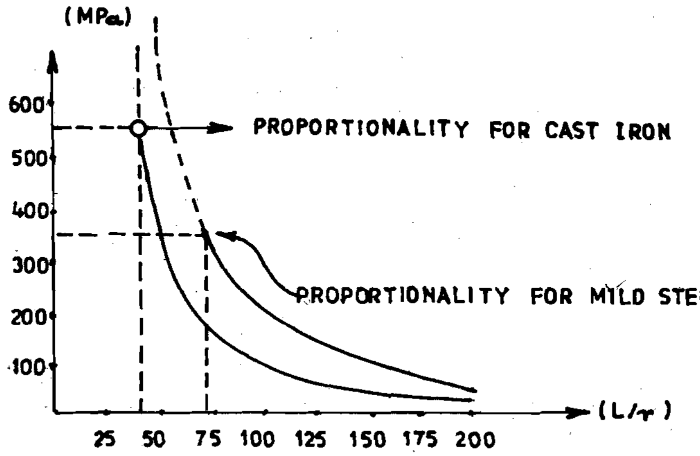


Figure 15.3 : Variation of Critical Stress with Slenderness Ratio

A column, whether short or long, is determined by the numerical values of slenderness ratios. Smaller the slenderness ratio, lesser will be the tendency to deflect and higher will be the buckling load.

Example 15.1

A steel bar of rectangular cross section 30 × 50 mm pinned at each end is 2 m long. Determine the buckling load when it is subjected to axial compression and also calculate axial stress using Euler's expression. Determine the minimum length for which Euler's equation may be valid.

Take proportionality limit as 250 MPa and $E = 200$ GPa $(2 \times 10^5$ N/mm$^2)$.

Solution

Here, we can calculate I_{min} as follows :

$$I_{min} = \frac{50 \times 30^3}{12} = 112.5 \times 10^3 \text{ mm}^4, \quad L = 2000 \text{ mm}$$

Using Eq. (15.5), we get

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{(\pi^2 \times 2 \times 10^5 \times 112.5 \times 10^3)}{(2000 \times 2000)}$$

$$P_{cr} = 55.51 \times 10^3 \text{ N} \quad \text{and} \quad \sigma_{cr} = \frac{P_{cr}}{A} = 37.01 \text{ N/mm}^2$$

$$r_{min} = \sqrt{\left(\frac{I_{min}}{A}\right)} = 8.66 \text{ mm} \quad \text{and} \quad \frac{L}{r_{min}} = 230.94$$

Or, using Eq. (15.6)

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} = 37.01 \text{ N/mm}^2$$

Since $\sigma_{cr} < \sigma_p$,

$$\frac{L}{r} > \pi \sqrt{\left(\frac{E}{\sigma_p}\right)} > 88.86$$

As $\sigma_{cr} < \sigma_p$ (i.e. $\frac{L}{r_{min}} > 88.86$) for $L = 2$ m.

Euler's expression is valid for limiting value of minimum length

$$\frac{L_{min}}{r_{min}} = 88.85$$

$L = 769.5 \approx 0.77$ m which is minimum length.

15.2.3 Partial End Constraints

Effect of Different Idealized End Conditions

The Euler buckling formula, as expressed by Eq. (15.5) was derived for a column with pivoted or hinged ends. The Euler's equation changes for columns with different end conditions. Four such common cases are shown in Figure 15.4.

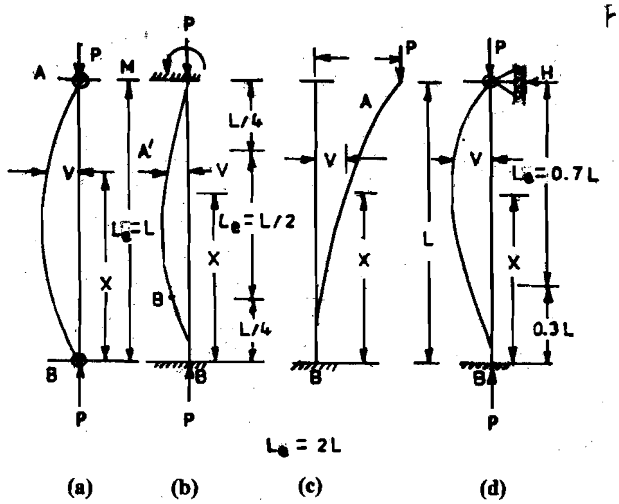


Figure 15.4 : Effective Length of Column with Different Restraints

It is possible to set up the differential equations with the appropriate boundary conditions to determine the Euler's equation for each new case, by using the concept of effective length of column. The pivot ended column, by definition has zero bending moments at each end. The length L in the Euler equation, therefore, is the distance between successive points of zero bending moment [Figure 15.4 (a)]. All that is needed to modify the Euler column formula for use with othei end conditions is to replace L by L_e where, L_e is defined as the effective length of the column (the distance between two successive inflection points or points of zero moment).

The ends A and B of the column in Figure 15.4 (b) are built-in or fixed, since the deflection curve is symmetrical, the distance between successive points of zero moment (inflection points) A' and B' is half the length of the column. Thus, the effective length, L_e, of a fixed ended column for use in the Euler's formula is half the actual length ($L_e = 0.5\ L$). The column in Figure 15.4 (c), being fixed at one end B and free at the other end A, has zero moment only at the free end. If a mirror image of this column is imaginary visualized below the fixed end B, A' will be the point of mirror image of A, the effective length between points of zero moment would be twice the actual length of the column AB ($L_e = 2L$). The column in Figure 15.4 (d) is fixed at one end B and pinned at the other end. The effective length of this column cannot be determined by inspection, as could be done in the previous two cases. Therefore, it is necessary to solve the differential equation to determine the effective length. This procedure yields

$$L_e = \frac{L}{\sqrt{2}} \approx 0.7\,L$$

SAQ 1

Referring to Figure 15.4 (b) to (d), considering bottom end B as origin and equilibrium of portion above the section, derive Euler's crippling load expression.

SAQ 2

Calculate safe compressive load on a hollow cast iron column with one end hinged and other rigidly fixed. The external and internal diameters are 120 mm and 90 mm respectively and length of the column is 9 m.

Take factor of safety as 3 and E = 95 GPa. Also calculate critical axial stress.

Example 15.2

A beam is fixed at both ends is loaded transversely by total uniformly distributed load of 32 kN. It is found that deflection at centre is 1/325 of span. Now if this transverse load is removed and beam is loaded axially, find out safe axial load for the condition such that it fails by buckling only.

Take factor of safety = 4, and E = 200 GPa.

Solution

(a) Load on beam, W = 32 kN, and deflection, $\delta = \frac{L}{325}$,

where L is the span of the beam.

Deflection in case of a fixed beam subjected to u.d.l.

$$\delta = \frac{WL^3}{192\,EI}$$

$$\frac{L}{325} = \frac{WL^3}{192\,EI}$$

$$\frac{EI}{L^2} = \frac{W \times 325}{192}$$

$$= \frac{32 \times 10^3 \times 325}{192} = 54166.66 \text{ N}$$

(b) When beam is loaded axially it will acts as column,

$$\frac{EI}{L^2} = 54166.66$$

As the column is to fail by buckling only, it should be considered as long column, therefore using Euler's formula for crippling load

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

We know for both ends fixed,

$$L_e = \frac{L}{2}$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = 4 \times \pi^2 \times 54166.66 = 2138414 \text{ N}$$

Thus, $$P = \text{Safe Load} = \frac{\text{Crippling load}}{\text{Factor of safety}}$$

$$= \frac{2138414}{4} = 534603 \text{ N}$$

$$P = 534.6 \text{ kN}$$

SAQ 3

A bar when used as simply supported beam and subjected to total load of 96 kN which is uniformly distributed over the whole span, the deflection at centre is 1/100 of span. Determine the crippling load when it is used as a column with both ends hinged.

Example 15.3

A truss member which is having length equal to 2 m and acting as a tension member for normal loading, is to be designed to take up tensile load equal to 100 kN. But due to wind load it is subjected to compressive load equal to 46 kN. Assume factor of safety for compression equal to 2 and allowable stresses in tension σ_{st} equal to 150 MPa. Find outer diameter when $D_o = 1.2\, D_i$ where, D_o = outer diameter, D_i = inner diameter.

Solution

(a) When acting as tension member

Given : Load=100 kN, Allowable stress, $\sigma_{st} = 150$ MPa

$$\text{Stress, } \sigma_{st} = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}, \quad A = \frac{P}{\sigma_{st}}$$

$$\frac{\pi\,(D_o^2 - D_i^2)}{4} = \frac{P}{\sigma_{st}}$$

$$\frac{\pi\,(1.2^2\, D_i^2 - D_i^2)}{4} = \frac{P}{\sigma_{st}}$$

$$D_i^2 = \frac{4P}{\pi\sigma_{st}}\,(1.44 - 1)$$

$$D_i^2 = 1929.15 \text{ mm}^2$$

$$D_i = 43.9 \text{ mm}$$

$$D_o = 1.2 \times 43.9 = 52.7 \text{ mm}$$

(b) When acting as compression member

Given : working load $P = 46$ kN, $E = 2 \times 10^5$ MPa, $L = 2$ m $= 2000$ mm, and Factor of safety (F. O. S) = 2.

As it is acting as truss member, both ends are considered as hinged.

Hence, $$L_{eff} = L$$

Critical load = Working load × F.O.S

$$\therefore \ P_{cr} = 46 \times 2 = 92 \text{ kN}$$

Using Euler's equation,

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$92 \times 10^3 = \left(\frac{\pi^2 \times 2 \times 10^5}{2000^2}\right) \times \frac{\pi\left(D_o^4 - D_i^4\right)}{64}$$

Putting $D_o = 1.2\ D_i$

$$D_i^4 = 3537575 \text{ mm}^4$$

$$D_i = 43.36 \text{ mm}$$

$$D_o = 1.2 \times 43.36 = 52.03 \text{ mm}$$

Selecting higher values from the two cases, we get

D_i = 43.944 mm, and

D_o = 52.753 mm.

Example 15.4

A solid column of diameter 50 mm is required to be replaced by hollow column whose external diameter is 1.25 times internal diameter. The column is long enough to fail by buckling only. Compute percent saving in material.

Solution

Given, D_s = 50 mm (solid column)

$D_o = 1.25\ D_i$ (hollow column)

Here, we know,

Critical load for solid column = Critical load for hollow column

$$P_s = P_h$$

$$\frac{\pi^2 EI_s}{L^2} = \frac{\pi^2 EI_h}{L^2}$$

$$I_s = I_h$$

$$\frac{\pi D_s^4}{64} = \frac{\pi\,(D_o^4 - D_i^4)}{64}$$

$$50^4 = (1.25 \times D_i)^4 - D_i^4$$

$$D_i^4 = 4336043.5$$

$$\therefore \ D_i = 45.6 \text{ mm}$$

Thus, $D_o = 1.25 \times D_i = 57.04$ mm

Further,

$$\text{Percentage saving} = \frac{A_{\text{solid}} - A_{\text{hollow}}}{A_{\text{solid}}} \times 100$$

$$= \frac{(\pi/4)\,D_s^2 - (\pi/4)\,(D_o^2 - D_i^2)}{\pi/4 \times D_s^2} \times 100$$

$$= \frac{50^2 - (57.04^2 - 45.6^2)}{50^2} \times 100$$

$$= 53.03\%$$

Example 15.5

A long column of 2 m length is hinged at both ends. Yielding on outer fibres starts when the central deflection is equal to 15 mm. Determine the breadth and depth of rectangular section with $\frac{b}{d} = 0.4$.

Take Yield stress, $f_y = 250$ MPa, and $E = 2 \times 10^5$ MPa

Solution

Given, $f_y = 250$ MPa, and $E = 2 \times 10^5$ MPa

$$L = L_{eff} = 2 \text{ m} = 2000 \text{ mm, and } \delta = 15 \text{ mm} = 0.015 \text{ m}$$

At yielding stage, $M = P \times \delta$

We know, $$M = f_y z = f_y \frac{bd^2}{6} = f_y \times 0.4\ d \times \frac{d^2}{6} = f_y \frac{d^3}{15} = 250 \frac{d^3}{15}$$

As $$I_{min} = (0.4\ d)^3 \times \frac{d}{12}$$

$$P = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2 \times 2 \times 10^5 \times d^4}{187.5 \times 2000^2} = \frac{d^4}{379.95}$$

Substituting and simplifying,

$$250 \frac{d^3}{15} = \frac{d^4}{379.95} \times 15$$

It gives, $d = 422.17$ mm, i.e. $b = 0.4 \times d = 168.8$ mm.

SAQ 4

Compare the buckling loads of two columns hinged at ends

(a) of rectangular section 3 cm × 12 cm, and

(b) other of square section having same cross-section area 6 cm × 6 cm as that of rectangular column and of same length and made up of same material. Use Euler's formula.

SAQ 5

For the mild steel bar with yield stress at 3410 MPa and $E = 215$ GPa, deduce the value of slenderness ratio upto and beyond which Euler's formula is valid. Consider (i) both ends hinged (ii) both fixed.

15.3 THEORY OF ECCENTRICALLY LOADED COMPRESSION MEMBER

Although a column will support the maximum load when the load is applied concentrically, but many times the column is subjected to eccentric loads. For example, a beam in a building may be connected to side of a column by welding or riveting, the column is then subjected to a bending moment due to the connection of beam and column. The eccentricity of the load, or their bending moment will increase the stress in the column and reduce its load carrying capacity.

15.3.1 Derivation of Secant Formula

Many times column do not behave as predicted by the Euler's formula because of imperfections in the alignment of the loading. In this section, the effect of imperfect alignment is examined by considering an eccentric loading. The pivot (pin) ended column shown in Figure 15.5 (a) is subjected to compressive forces acting at a distance e from the centerline of the undeformed column. As the loading is increased, the column deflects laterally, as shown in Figure 15.5 (b). If axes are chosen with the origin at the top, the bending moment at a section x from A would be as follows :

$$M = Py \tag{15.7}$$

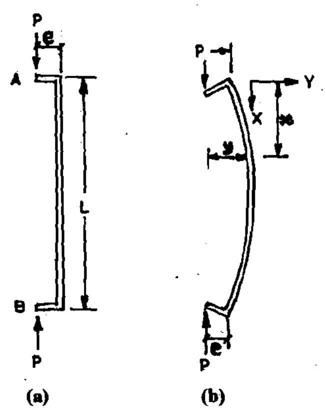


Figure 15.5 : Eccentrically-Loaded Compression Member (Secant Formula)

If the stress doesn't exceed the proportional limit and deflections are small, the differential equation for the elastic curve becomes

$$\frac{EI\,d^2y}{dx^2} = -M = -Py$$

or,

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0 \quad \text{i.e.} \quad \frac{d^2y}{dx^2} + k^2y = 0 \tag{15.8}$$

where,

$$k = \sqrt{\frac{P}{EI}}$$

This is the same differential equation as Eq. (15.1) by Euler's method, the solution of this equation is

$$y = A \sin kx + B \cos kx + 0 \tag{15.9}$$

where, $k^2 = \frac{P}{EI}$, A and B are constants to be determined from the boundary conditions.

At upper end A, $\quad x = 0,\ y = e \quad$ i.e. $B = e$

and Eq. (15.9) becomes

$$y = A \sin kx + e \cos kx$$

$$\frac{dy}{dx} = Ak \cos kx - ek \sin kx$$

Due to symmetry, $\quad \frac{dy}{dx} = 0 \quad$ at $x = \frac{l}{2}$

On putting the value of x, $\quad 0 = Ak \cos\left(\frac{kl}{2}\right) - ek \sin\left(\frac{kl}{2}\right)$

Thus, we get

$$A = \frac{e \sin\left(\frac{kl}{2}\right)}{\cos\left(\frac{kl}{2}\right)}$$

Hence, the equation of elastic

$$y = e\left[\frac{\sin\left(\frac{kl}{2}\right)}{\cos\left(\frac{kl}{2}\right)}\sin kx + \cos kx\right] \qquad (15.10)$$

The maximum deflection occurs at $x = \frac{l}{2}$. Putting these values

$$y_{max} = e\left[\frac{\sin^2\left(\frac{kl}{2}\right)}{\cos\left(\frac{kl}{2}\right)} + \cos\left(\frac{kl}{2}\right)\right] \qquad (15.11)$$

$$y_{max} = e\ \sec\left(\frac{kl}{2}\right)$$

Since, it has been assumed that the stresses do not exeeed the proportional limit, the maximum compressive stress can be obtained by superposition of the axial stress and the maximum bending stress. The maximum bending stress occurs on a section at the midspan of the column where the bending moment is assumed to be the largest value, $M_{max} = Py_{max}$. Thus, the maximum stress is,

$$\sigma_{max} = \frac{P}{A} + M_{max}\frac{c}{I} = \frac{P}{A} + Py_{max}\frac{c}{Ar^2} \qquad (15.12)$$

where, c is distance of extreme fibre from neutral plane, r is the radius of gyration of the column cross-section, and A is area of cross-section.

Therefore, maximum bending moment

$$M_{max} = Py_{max} = Pe\sec\left(\frac{kL}{2}\right)$$

$$\sigma_{max} = \left(\frac{P}{A}\right)\left[1 + \left(\frac{ec}{r^2}\right)\left(\sec\left(\frac{kL}{2}\right)\right)\right]$$

Substituting the value of k, we get,

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \left(\frac{ec}{r^2}\right)\sec\left[\left(\frac{L}{r}\right)\sqrt{\left(\frac{P}{4AE}\right)}\right]} \qquad (15.13)$$

where, σ_{max} is the allowable stress.

Eq. (15.13) is known as the secant formula and relates the average stress P/A to the dimensions of the column, the properties of the column material, and the eccentricity e. The term L/r is the same slenderness ratio found in the Euler buckling formula. For columns with different end conditions, the effective slenderness ratio L_e/r should be used accordingly. The quantity ec/r^2 is called the eccentricity ratio and is seen to depend on the eccentricity of the load and the dimensions of the column.

If the column was loaded axially, e would presumably be zero, and the maximum stress would be equal to P/A. It is virtually impossible however, to eliminate all eccentricity that might result from various factors such as the initial crookedness of the column, minute flaws in the material and a lack of uniformity of the cross-section, as well as accidental eccentricity of the load.

An extensive study of the results of many column tests indicated that a value of 0.2 to 0.25 for ec/r^2 would give results with the secant formula that would be in good agreement with experimental test on axially loaded columns of structural steel in ordinary structural sizes.

If σ_y is yield stress of material to be equated to σ_{max}, the corresponding value of direct stress will be $m\sigma_c = \frac{mP}{A}$ where, m is factor of safety, σ_c being permissible stress = P/A.

Thus,

$$\sigma_y = m\sigma_c \left[1 + \left(\frac{ec}{r^2} \right) \sec \left(\frac{l}{r} \right) \sqrt{\left(\frac{m\,\sigma_c}{4E} \right)} \right] \qquad (15.14)$$

In order to make effective use of Eq. (15.13), curves showing P/A and L/r can be drawn for various values of ec/r^2 for any given material. Figure 15.6 is such a set of curves for a material with σ_y = 280 MPa and E = 210 GPa. For these material properties, solutions can be obtained from Figure 15.6. Digital computers can also be programmed to directly solve the Secant Formula using iterative techniques.

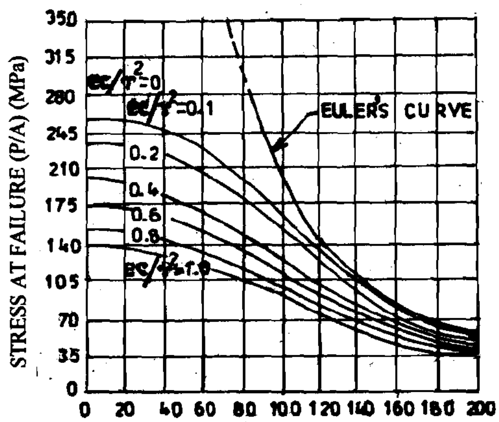


Figure 15.6 : Relation between P/A and l/r

The outer envelope of Figure 15.6, consisting of the horizontal line σ_y = 280 MPa and the Euler curve, corresponds to zero eccentricity. The Euler curve is truncated at 280 MPa. Since this is the maximum allowable stress for the material with Young's modulus equal to E = 210 GPa, the truncation occurs at L/r = 86. From the data presented in Figure 15.6, it is seen that eccentricity of loading plays a significant role in reducing the working load (the maximum safe load) in the short and intermediate ranges (slenderness ratio less than 150 for the steel in Figure 15.6). For the large slenderness ratios, the curves for the various eccentricity ratios tend to merge with the Euler curve.

Example 15.6

A column 4 meters long of circular section made of cast iron with 200 mm external diameter and 20 mm thick is used as a column. Both ends of the column are fixed. The column carries a load of 150 kN at an eccentricity of 25 mm from the axis of the column.

(a) Find the extreme stress on the column section, and

(b) Find also the maximum eccentricity in order to have no tension anywhere on the section. Take $E = 9.4 \times 10^4$ MPa.

Solution

Area of the column

$$A = \frac{\pi}{4} (200^2 - 160^2) = 11310 \text{ mm}^2$$

Moment of inertia

$$I = \frac{\pi}{64} (200^4 - 160^4) = \frac{\pi}{64} (200^2 - 160^2)(200^2 + 160^2)$$

$$I = 4637 \times 10^4 \text{ mm}^4$$

Section modulus, $Z = \dfrac{I}{y}$

$$\frac{I}{y} = \frac{4637 \times 10^4}{(200/2)} = 463.7 \times 10^3 \text{ mm}^3$$

Effective length of the column $L_e = \dfrac{L}{2} = \dfrac{4}{2} = 2$ m

Thus, $L_e = 2000$ mm

(a) Maximum moment, $M_{max} = Py_{max}$

$$M = P \times e \times \sec\left(\frac{L}{2}\right)\sqrt{\left(\frac{P}{EI}\right)}$$

where,

$$\sec\left(\frac{L}{2}\right)\left(\frac{P}{EI}\right) = \sec\left(\frac{2000}{2}\right)\sqrt{\left[\frac{150 \times 10^3}{(9.4 \times 10^4 \times 463.7 \times 10^4)}\right]}$$

$$= \sec 0.1856 \text{ radian}$$

$$= \sec\ 10^\circ 40' = 1.017$$

$$M_{max} = 150 \times 25 \times 1.017 = 3814 \text{ kN mm}$$

Maximum compresive stress = $\sigma_{max} = \dfrac{P}{A} + \dfrac{M}{Z}$

$$\sigma_{max} = \left(\frac{150 \times 10^3}{11310}\right) + \left(\frac{3814 \times 10^3}{463.7 \times 10^3}\right) = 21.5 \text{ MPa}$$

The maximum stress on the column is 21.5 MPa.

(b) If tension is just to be avoided corresponding to the maximum eccentricity

$$\frac{dP}{A} = \frac{M}{Z}$$

$$\frac{P}{A} = \frac{Pe\ \sec\dfrac{l}{2}\sqrt{\left(\dfrac{P}{EI}\right)}}{Z}$$

$$e = \left(\frac{Z}{A}\right)\left(\frac{1}{\sec 10^\circ 40'}\right) = \frac{463.7 \times 10^3}{11310 \times 1.017} = 44.32 \text{ mm}$$

Hence, $e = 44.32$ mm is the value of the eccentricity in order that there may be no tension anywhere on the section.

Example 15.7

Find the necessary diameter of a mild steel strut 2 meter long hinged at its ends if it has to carry a load of 100 kN with possible deviation from the axis of 1/10th of the diameter. The greatest compressive stress is not to exceed 80 MPa and value of $E = 200$ GPa.

Solution

Given : $P = 100$ kN, $\sigma_{max} = 80$ MPa , $L = 2$ m

$E = 200$ GPa , $e = d/10$, hinged ends

$$\frac{kL}{2} = \frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)}$$

$$= \left[\frac{100 \times 10^3}{2 \times 10^5 \times \dfrac{\pi d^4}{64}}\right]^{1/2} \times 1000 = \frac{3191.54}{d^2}$$

$$M_{max} = Pe \sec\left(\frac{kL}{2}\right) = 100 \times 10^3 \left(\frac{d}{10}\right) \sec\left(\frac{3191.54}{d^2}\right)$$

where, $\frac{kL}{2}$ is in radians.

Maximum stress, $\sigma_{max} = \frac{P}{A} + \frac{M}{Z}$

On putting $A = \frac{\pi d^2}{4}$, $\sigma_{max} = 80$ MPa, $P = 100 \times 10^3$, $Z = \frac{\pi d^3}{32}$,

we get,
$$80 = \frac{100 \times 10^3}{\left(\frac{\pi d^2}{4}\right)} + \frac{100 \times 10^3 \left(\frac{d}{10}\right) \sec\left(\frac{3191.5}{d^2}\right)}{\left(\frac{\pi d^3}{32}\right)}$$

or,
$$\frac{80\pi}{10^5} d^2 = 4 + 3.2 \sec\left(\frac{3191.5}{d^2}\right)$$

On solving by trial and error method, $d = 60.03$ mm. Provide diameter as 60.1 mm.

SAQ 6

A steel tube is initially straight, has an external diameter of 38 mm and internal diameter 35 mm. It is 1.5 m long and carries a compressive load of 20 kN acting parallel to the axis of the tube but 2 mm from it. Calculate the maximum stress in the tube.

Take $E = 210$ GPa.

15.3.2 Theory of Beam Column or Laterally Loaded Compression Member

A beam that is acted upon by an axial compressive force in addition to the transverse loads is referred to as beam-column. Consider a column with a concentrated load at mid height as shown in Figure 15.7 (a). Let W be the lateral concentrated load applied and P be the axial load (Figure 15.7).

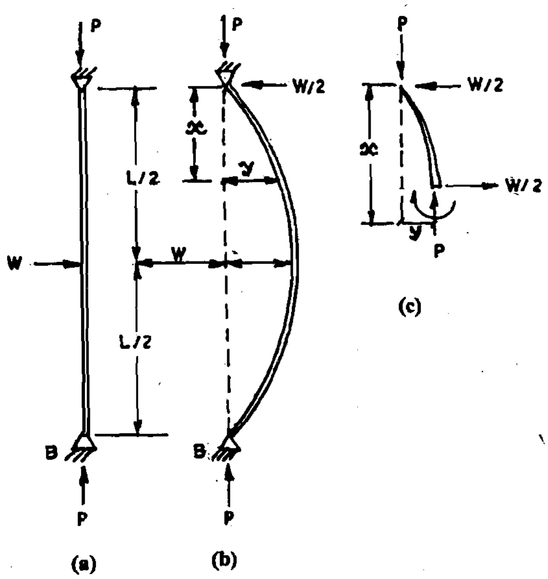


Figure 15.7

From Figure 15.7 (b), the moment at a section, distance x from origin A is as follows :

$$M = \frac{Wx}{2} + Py$$

$$EI\frac{d^2y}{dx^2} = -M = -\left(\frac{Wx}{2} + Py\right) \quad (15.15)$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = -\frac{Wx}{2\,EI}$$

Let $$\frac{P}{EI} = k^2$$

Then, we get $$\frac{d^2y}{dx^2} + k^2y = -\frac{Wx}{2\,EI} \quad (15.16)$$

The solution of the above equation is as follows :

$$y = A\sin kx + B\cos kx - \frac{Wx}{2k^2EI} \quad (15.17)$$

$$y = A\sin kx + B\cos kx - \frac{Wx}{2P}$$

At $x = 0$, $y = 0$, i.e. $B = 0$

On subsituting the values, $$y = A\sin kx - \frac{Wx}{2P} \quad (15.18)$$

i.e., $$\frac{dy}{dx} = Ak\sin kx - \frac{W}{2P}$$

At $x = \frac{l}{2}, \frac{dy}{dx} = 0$

On subsituting the values, $$0 = Ak\cos\frac{kl}{2} - \frac{W}{2P}$$

i.e., $$A = \left(\frac{W}{2Pk}\right)\sec\frac{kl}{2}$$

Substituting value of A in Eq. (15.17)

$$y = \left(\frac{W}{2\,Pk}\right)\sec\frac{kl}{2}\sin kx - \frac{Wx}{2\,P} \quad (15.19)$$

At $x = \frac{l}{2}$; we get,

$$y_{max} = \left(\frac{W}{2Pk}\right)\sec\frac{kl}{2}\sin\frac{kl}{2} - \frac{Wl}{4P}$$

$$= \left(\frac{W}{2Pk}\right)\tan\frac{kl}{2} - \frac{Wl}{4P}$$

$$= \left(\frac{W}{2Pk}\right)\left[\tan\frac{kl}{2} - \frac{kl}{2}\right] \quad (15.20)$$

The maximum moment at $x = \frac{l}{2}$

$$M_{max} = \left(\frac{W}{2}\times\frac{l}{2}\right) + Py_{max}$$

$$= \frac{Wl}{4} + \left(\frac{W}{2k}\right)\left[\tan\frac{kl}{2} - \frac{kl}{2}\right]$$

$$= \frac{W}{2k}\left(\tan\frac{kl}{2}\right) \quad (15.21)$$

The maximum stress can be evaluated from

$$\sigma_{max} = \frac{P}{A} + \frac{M}{Z} \tag{15.22}$$

15.4 EMPIRICAL FORMULAE FOR DESIGN OF COLUMNS

Numerous column experiments indicate that the Euler formula is reliable for designing axially loaded columns, provided the slenderness ratio is within the range in which the eccentricity has relatively little effect. This range is called the *slender range* which ranges from 120 to 140. At lower slenderness the failure stress would be the compressive strength of the material. The extent of this range is 0 – 40. In the *intermediate range* (i.e. slenderness ratio 40 to 120), the only equation of rational nature that applies to the real columns is the secant formula. Since the application of the secant formulae to centric loading requires an estimate of accidental eccentricity ratio, the equation acquires an empirical nature. Furthermore, the application is so involved that simpler empirical formulae have been developed which give results in reasonable agreement with the experimental results within the intermediate range.

15.4.1 Rankine's Formula

Rankine proposed an empirical formula for columns which cover all cases ranging from very short to very long struts. He proposed the relation

$$\frac{1}{P} = \left(\frac{1}{P_c}\right) + \left(\frac{1}{P_e}\right) \tag{15.23}$$

where, P_c = ultimate load for a short column,

$P_e = \pi^2 \frac{EI}{L^2}$ = Eulerian crippling load for the standard case, and

$\frac{1}{P_c}$ = constant for a material.

For short columns, P_e is very large and hence $1/P_e$ is small in comparison to $1/P_c$, thus making the crippling load P approximately equal to P_c.

For long columns, P_e is extremely small and hence $1/P_e$ is large as compared to $1/P_c$, thus making the crippling load P approximately equal to P_e. Thus, the value of P obtained from the above relation covers all cases ranging from short to long columns or struts.

Eq. (15.23) can be rearranged as

$$P = \frac{P_c\,P_e}{P_c + P_e} = \frac{P_c}{1 + \left(\dfrac{P_c}{P_e}\right)} = \frac{\sigma_c A}{1 + \left(\dfrac{\sigma_c A L^2}{\pi^2 EI}\right)}$$

$$P = \frac{\sigma_c A}{1 + \left(\dfrac{L}{r}\right)^2 \left(\dfrac{\sigma_c}{\pi^2 E}\right)} = \frac{\sigma_c A}{1 + a\left(\dfrac{L}{r}\right)^2} \tag{15.24}$$

where, σ_c = crushing stress for the material, and

a = Rankine's constant for the material which is determined experimentally, and should not be calculated values of σ_c and E.

Material	σ_c **(MPa)**	***a* for Hinged Ends**
Wrought iron	255	1/9000
Cast iron	550	1/1600
Mild steel	330	1/7500
Strong timber	50	1/750

Eq. (15.24) is the Rankine's formula for the standard case of two end-hinged column. It is sometimes known as Rankine-Gordon formula. For columns with other end conditions, the value of the constant will be changed accordingly. However, since 'a' is a constant for a particular material used as a hinged column, it is better to modify the Rankine's formula as

$$P = \frac{\sigma_c A}{1 + a\left(\frac{L_e}{r}\right)^2} \tag{15.25}$$

where,

a = Rankine's constant for a particular material, and

L_e = Effective length of the column.

Eq. (15.25) can be re-arranged in terms of an average axial stress and is given as

$$\frac{P}{A} = \frac{\sigma_c}{\left[1 + a\left(\frac{L_e}{r}\right)^2\right]} \tag{15.26}$$

where,

σ_c = allowable stress, and $a = \frac{\sigma_c}{\pi^2 E}$.

The only difference is that this formula includes factor of safety.

15.4.2 Straight Line Formula

This is proposed by T. H. Johnson and written in the form

$$\frac{P}{A} = \sigma_0 - c_1\left(\frac{L}{r}\right) \tag{15.27}$$

where, σ_0 and c_1 are experimentally determined and these constants are depending on the material. (P/A) may be expressed either in terms of critical stress or safe working stress. Following table gives one of the empirical values.

(P/A) (MPa)	**Material**	**σ_0 (MPa)**	**c_1 (MPa)**
Critical stress	Structural steel	367.5	2
Critical stress	Cast iron	23.8	0.6
Safe working stress	Mild steel	150	0.57

15.4.3 Johnson's Parabolic Formula

Prof. J. B. Johnson modified the straight line formula as under

$$\frac{P}{A} = \sigma_0 - c_2\left(\frac{L}{r}\right)^2 \tag{15.28}$$

where,

σ_0 = compressive yield stress, and $c_2 = \frac{\sigma_0^2}{42E}$.

Example 15.8

Using Rankine's formula find the crippling load for a mild steel strut of 500 mm long with a rectangular cross-section 50 mm × 12.5 mm having

(a) hinged ends, and

(b) both ends fixed.

Take σ_c = 330 MPa and a for hinged ends = 1/7500.

Solution

$$A = 50 \times 12.5 = 625 \text{ mm}^2$$

$$I_{min} = \frac{50 \times 12.5^3}{12} = 8140 \text{ mm}^4$$

$$r^2 = \frac{I}{A} = \frac{8140}{625} = 13.02 \text{ mm}^2$$

(a) For hinged ends

$$P = \frac{\sigma_c A}{\left[1 + a\left(\frac{l_e}{r}\right)^2\right]}$$

Taking, $l_e = l$, we get,

$$P = \frac{330 \times 625}{\left[\left(1 + \frac{1}{7500}\right)\left(\frac{500^2}{13.02}\right)\right]} = 57933 \text{ N} = 57.93 \text{ kN}.$$

(b) For fixed ends

$$P = \frac{\sigma_c A}{\left[1 + a\left(\frac{l_e}{r}\right)^2\right]}$$

Taking, $l_e = \frac{l}{2}$, we get,

$$= \frac{330 \times 625}{\left[\left(1 + \frac{1}{7500}\right)\left(\frac{1}{4}\right)\left(\frac{500^2}{13.02}\right)\right]} = 125759 \text{ N}$$

$$= 125.76 \text{ kN}.$$

SAQ 7

A hollow cylindrical cast iron column 150 mm external diameter and 20 mm thick is 6 meter in length having both ends hinged. Find the load using Rankine's formula. Compare this load with that given by Euler's formula.

Take $\sigma_c = 550 \text{ N/mm}^2$ and $a = 1/160$. For what length of strut of this cross-section does the Euler formula ceases to apply.

15.4.4 Indian Standard Code Formulae

IS : 800-1962

In old version of IS : 800, secant formula was adopted with $ec/r^2 = 0.2$, as it was found to be average value for a large number of columns experimentally tested. But factor of safety of 1.68 was adopted additionally; the Eq. (15.13) was modified to

$$\frac{P}{A} = \frac{\left(\frac{f_y}{1.68}\right)}{1 + 0.20 \sec\left(\frac{L}{2r}\right)\sqrt{\left(\frac{1.68\,P}{AE}\right)}} \qquad (15.29)$$

where, f_y = yield stress which was taken as 250 MPa.

IS : 800-1984 (Merchant-Rankine's Formula)

This approach has been changed in IS : 800-1984. Close fit with test data on axially loaded columns is obtained by expressing the axial compresive stress, σ_c, in terms of following form which is called Merchant-Rankine's Formula

$$\sigma_c = \frac{1}{\left[\left(\frac{1}{(f_{cc})}\right)^n + \left(\frac{1}{f_y}\right)^n\right]^{\frac{1}{n}}} \qquad (15.30)$$

where,

f_y = yield stress of steel in MPa,

(The table has been given in the code for various values of $f_y = 230$ to 540 MPa),

$f_{cc} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$ = Euler's critical buckling stress,

$\frac{L}{r}$ = slenderness ratio,

L = effective length of the compression member,

r = minimum radius of gyration,

$E = 2.0 \times 10^5$ MPa or 200 GPa, and

n = an imperfection index assumed as 1.4

When slenderness ratio tends to zero $\sigma_c = f_y$.

Using factor of safety of 1.67, the Eq. (15.30) will give the allowable stress in axial compression in form

$$f_{ac} = \frac{0.6\ (f_{cc} \cdot f_y)}{\left[(f_{cc})^{1.4} + (f_y)^{1.4}\right]^{0.71}} \qquad (15.31)$$

Table below shows the various values of allowable stress in axial compression for IS : 800-1984 and IS : 800-1962 for $f_y = 250$ MPa.

L/r	**Allowable Stress in Axial Compression (MPa) as per**	
	IS : 800-1984	**IS : 800-1962**
10	150	122
50	132	115
100	80	82
150	45	46
200	28	27
250	18	16

15.4.5 Prof. Perry's Formula

From the secant formula, it can be seen that if column section, length and end conditions are given, it is easy to workout the extreme stress intensities due to given load and eccentricity or to calculate permissible eccentricity for a given load and permissible stresses. However, if the safe load for a given section, stresses and limit of eccentricity has to be determined, the necessary formula will have to be thrown into workable form

$$\sigma_{max} = \sigma_o + \sigma_b = \frac{P}{A} + \frac{\left[Pe \sec\left(\frac{l}{2}\right)\sqrt{\left(\frac{P}{EI}\right)}\right]}{Z}$$

$$\sigma_{max} = \frac{P}{A} + Pe\frac{\gamma_c}{Ar^2}\sec\left(\frac{l}{2}\right)\sqrt{\left(\frac{P}{EI}\right)} \qquad \left(\text{since } Z = \frac{I}{\gamma_c} = \frac{Ar^2}{\gamma_c}\right)$$

$$= \sigma_o\left[1 + e\frac{\gamma_c}{r^2}\left(\sec\left(\frac{l}{2}\right)\sqrt{\left(\frac{P}{EI}\right)}\right)\right]$$

$$= \sigma_o\left[1 + \frac{e\,\gamma_c}{r^2}\left(\sec\left(\frac{\pi}{2}\right)\sqrt{\left(\frac{P}{P_e}\right)}\right)\right]$$

where, $P_e = \dfrac{\pi^2 EI}{l^2}$

Prof. Perry found out that the expression, $\sec\left(\frac{l}{2}\right)\sqrt{\left(\frac{P}{EI}\right)}$ or $\sec\left(\frac{\pi}{2}\right)\sqrt{\left(\frac{P}{P_e}\right)}$

approximated very closely to $\dfrac{1.2\,P_e}{(P_e - P)}$ or $\dfrac{1.2\,\sigma_e}{(\sigma_e - \sigma_o)}$.

On substituting, we get

$$\sigma_{max} = \sigma_o\left[1 + \left(\frac{e\,\gamma_c}{r^2}\right)\left(\frac{1.2\,\sigma_e}{(\sigma_e - \sigma_o)}\right)\right]$$

$$\sigma_{max} - \sigma_o = \left(\frac{e\,\gamma_c}{r^2}\right)\left(\frac{1.2\,\sigma_e\,\sigma_o}{(\sigma_e - \sigma_o)}\right)$$

$$\frac{(\sigma_{max} - \sigma_o)(\sigma_e - \sigma_o)}{\sigma_e\,\sigma_o} = 1.2\left(\frac{e\,\gamma_c}{r^2}\right)$$

On simplifying, we have,

$$\left(\frac{\sigma_{max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = 1.2\left(\frac{e\,\gamma_c}{r^2}\right)$$

from which it is easy to workout σ_o from given σ_{max} and e.

Example 15.9

A compound stanchion 3 m long is made up of two channels ISJC 200 and two 250 mm × 10 mm plates riveted one on each flange. Find the maximum load that can be applied at an eccentricity of 20 mm from the axis $y-y$. The permissible compressive stress is 80 N/mm^2. Take $E = 2 \times 10^5$ N/mm^2.

Assume hinged ends M.I. of the compound section about $y-y$ axis may be taken as 4.5×10^7 mm^4.

Solution

Firstly, we find r^2 as computed below

$$r^2 = \frac{I}{A} = \frac{4.5 \times 10^7}{8554} = 5260 \text{ mm}^2$$

Eulerian load $\quad P_e = \dfrac{\pi^2 EI}{(l^2)}$

$$= \frac{\pi^2 \times 2 \times 10^5 \times 4.5 \times 10^7}{3000^2}$$

$$= 9869600 \text{ N.}$$

Thus, $\quad \sigma_e = \dfrac{P_e}{A} = \dfrac{9869600}{8554} = 1153 \text{ N/mm}^2$

Given, $\quad \sigma_{max} = 80 \text{ N/mm}^2$

Applying Perry's formula,

$$\left(\frac{\sigma_{max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_c}\right) = \frac{1.2\ e\gamma_c}{r^2}$$

$$\left(\frac{80}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{1153}\right) = \frac{1.2 \times 20 \times 125}{5260} = 0.57$$

On solving, we get

$$\sigma_o = 50.1 \text{ N/mm}^2$$

Thus, Safe load, $P = \sigma_o \times A = 50.1 \times 8554$

$$= 428555 \text{ N} = 428.555 \text{ kN.}$$

15.5 SUMMARY

- The crippling load or buckling load of a column with different end conditions is as follows :

Case	End Condition	Euler's Buckling Load or Crippling Load	Effective Length, L_e
1.	Both hinged	$\frac{\pi^2 EI}{L^2}$	L
2.	Both fixed	$\frac{4\pi^2 EI}{L^2}$	$\frac{L}{2}$
3.	One fixed and other free	$\frac{\pi^2 EI}{4L^2}$	$2L$
4.	One fixed and other hinged	$\frac{2\pi^2 EI}{L^2}$	$\frac{L}{\sqrt{2}}$

L = actual length of column.

- The crippling stress in terms of radius of gyration for both ends hinged condition

$$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 EA}{(L/r)^2}$$

$$\sigma_{cr} = \frac{P}{A} = \frac{\pi^2 E}{(L/r)^2}$$

where, $r = \sqrt{\left(\frac{I}{A}\right)}$ = radius of gyration, and

$\frac{L}{r} = \rho$ which is ratio of effective length to least radius of gyration is called slenderness ratio of column.

- For eccentrically loaded member,

$$\sigma_{max} = \frac{P}{A} + M_{max}\frac{c}{I} = \frac{P}{A} + P\,\gamma_{max}\frac{c}{Ar^2}$$

which can be reduced to secant formula as below :

$$\sigma_{max} = \frac{P}{A}\left[1 + \left(\frac{ec}{r^2}\right)\sec\left(\frac{l}{r}\right)\sqrt{\left(\frac{P}{4AE}\right)}\right]$$

Rankine's formula for columns having wide range of slenderness ratio

$$P = \frac{\sigma_c A}{\left[1 + a\left(\frac{L}{r}\right)^2\right]}$$

where, a = Rankine's constant, and σ_c = crushing stress for the material.

Material	σ_c (MPa)	a (for Hinged Ends)
Wrought iron	255	1/9000
Cast iron	550	1/1600
Mild steel	330	1/7500
Strong timber	50	1/750

Indian Standard Code Formula

$$\sigma_c = \frac{1}{\left[\left(\frac{1}{(f_{cc})^n}\right) + \left(\frac{1}{(f_y)^n}\right)\right]^{1/n}}$$

where,

f_y = yield stress of steel (MPa) (The table has been given in the code for various values of f_y = 230 to 540 MPa,).

$f_{cc} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$ = Euler's critical buckling stress,

$\frac{L}{r}$ = slenderness ratio (ρ),

L = effective length of the compression member,

r = minimum radius of gyration,

$E = 2.0 \times 10^5$ MPa or 200 GPa (for steel),

n = an imperfection index assumed as 1.4.

Prof. Perry's formula

$$\left(\frac{\sigma_{max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = 1.2\left(\frac{e\,\gamma_c}{r^2}\right)$$

15.6 KEY WORDS

Column : A vertical member of a structure subjected to loads is known as column.

Strut : Member of structure which may not be vertical and subjected to compressive load.

Buckling Load : Load at which column just buckles is known as buckling load or critical load or crippling load. This is generally applicable to long columns.

Crushing Load : Load at which column fails by crushing generally applicable to short column.

15.7 ANSWERS TO SAQs

SAQ 1

$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$ For both end fixed [Figure 15.4 (b)].

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$ For one end fixed and other free [Figure 15.4 (c)].

$$P_{cr} = \frac{2\pi^2 EI}{L^2}$$ For one end fixed and other end hinged [Figure 15.4 (d)].

SAQ 2

$$L_e = 6.364 \text{ m}$$

$$I_{min} = \frac{\pi(120^4 - 90^4)}{64} = 6.96 \times 10^6 \text{ mm}^4$$

$$r_{min} = 37.5 \text{ mm} ; \frac{L_e}{r_{min}} = 169.7 \text{ mm}$$

$$P_{cr} = 161.13 \text{ kN};$$

$$P_{safe} = 53.71 \text{ kN};$$

$$\sigma_{cr} = 32.56 \text{ MPa}$$

SAQ 3

$$W = wL = 160 \text{ kN}$$

$$\text{For beam} = \left(\frac{5}{384}\right)\left(\frac{WL^3}{EI}\right) = \frac{L}{100}$$

$$\frac{EI}{L^2} = 125 \text{ kN}; \quad P_{cr} = 1233.7 \text{ kN}$$

SAQ 4

$$\text{The ratio of Euler's load} = \frac{P_{E_1}}{P_{E_2}} = \frac{P_{E(\text{Rectangular})}}{P_{E(\text{Square})}}$$

$$P_{E_1} = \frac{\pi^2 E_1 I_1}{L_1^2}; \quad P_{E_2} = \frac{\pi^2 E_2 I_2}{L_2^2}$$

$$\frac{P_{E_1}}{P_{E_2}} = \frac{\pi^2 E_1 I_1}{L_1^2} \times \frac{L_2^2}{\pi^2 E_2 I_2} = \frac{I_1}{I_2}$$

$$\frac{P_{E(\text{Rectangular})}}{P_{E(\text{Square})}} = \frac{\left(\frac{3 \times 12^3}{12}\right)}{\frac{6 \times 6^3}{12}} = \frac{3 \times 12 \times 12 \times 12}{3 \times 12 \times 6 \times 6} = 4$$

$$P_{E(\text{Rectangular})} = 4\, P_{E(\text{Square})}$$

Thus, it can be seen that the buckling load for square column is four times more than the rectangular column.

SAQ 5

Here, we know,

For Case (i) $l_e = l$, and

For Case (ii) $l_e = \frac{l}{2}$.

It is also known that

$$P_E = \frac{\pi^2 EI}{l_e^2} \quad \text{which can also be written as } P_E = \frac{\pi^2 EAr^2}{l_e^2}$$

(i) For both ends hinged,

$$P_E = \frac{\pi^2 EA}{\left(\frac{l_e}{r}\right)^2} \Rightarrow \frac{l_e}{r} = \sqrt{\left[\frac{\pi^2 E}{\left(\frac{P_E}{A}\right)}\right]} = \sqrt{\left(\frac{\pi^2 E}{\sigma_E}\right)}$$

$$\frac{l_e}{r} = \left[\frac{\pi^2 \times 2.15 \times 10^5}{310}\right] = 82.73$$

$$\frac{l_e}{r} = \frac{l}{r} = 82.73 \text{ nearly } 80.$$

(ii) For both ends fixed,

$$P_E = \frac{\pi^2 EI}{\left(\frac{l}{2}\right)^2} = \frac{4\pi^2 EAr^2}{l^2} = \frac{4\pi^2 EA}{\left(\frac{l^2}{r^2}\right)}$$

$$\frac{l}{r} = 2\pi \sqrt{\left(\frac{E}{\left(\frac{P_E}{A}\right)}\right)} = 2\pi \sqrt{\left(\frac{E}{\sigma_E}\right)} = 2\sqrt{\left(\frac{215 \times 10^3 \times \pi^2}{310}\right)}$$

$$\frac{l}{r} = 2 \times 82.73 = 165.46 \text{ say } 165.5$$

SAQ 6

$\sigma_{max} = 245.8 \text{ N/mm}^2$

SAQ 7

P (Rankine) = 393.0 kN, P (Euler) = 387.0 kN and L = 1.76 m.