The No Free Lunch Theorem, Kolmogorov Complexity, and the Role of Inductive Biases in Machine Learning

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Abstract

No free lunch theorems for supervised learning state that no learner can solve all problems or that all learners achieve exactly the same accuracy on average over a uniform distribution on learning problems. Accordingly, these theorems are often referenced in support of the notion that individual problems require specially tailored inductive biases. While virtually all uniformly sampled datasets have high complexity, real-world problems disproportionately generate low-complexity data, and we argue that neural network models share this same preference, formalized using Kolmogorov complexity. Notably, we show that architectures designed for a particular domain, such as computer vision, can compress datasets on a variety of seemingly unrelated domains. Our experiments show that pre-trained and even randomly initialized language models prefer to generate low-complexity sequences. Whereas no free lunch theorems seemingly indicate that individual problems require specialized learners, we explain how tasks that often require human intervention such as picking an appropriately sized model when labeled data is scarce or plentiful can be automated into a single learning algorithm. These observations justify the trend in deep learning of unifying seemingly disparate problems with an increasingly small set of machine learning models.

1. Introduction

The problem of justifying inductive reasoning has challenged epistemologists since at least the 1700s (Hume, 1748). How can we justify our belief that patterns we observed previously are likely to continue into the future without appealing to this same inductive reasoning in a circular fashion? Nonetheless, we adopt inductive reasoning in everyday life whenever we learn from our mistakes or make decisions based on past experience. Likewise, the feasibility of machine learning is entirely dependent on in-

duction, as models extrapolate from patterns found in previously observed training data to new samples at inference time.

More recently, in the late 1990s, *no free lunch theorems* emerged from the computer science community as rigorous arguments for the impossibility of induction in contexts seemingly relevant to real machine learning problems (Wolpert, 1996; Wolpert & Macready, 1997). One such no free lunch theorem for supervised learning states that no single learner can achieve high accuracy on every problem (Shalev-Shwartz & Ben-David, 2014). Another states that, assuming a world where labeling functions of learning problems are drawn from a uniform distribution, every learner is equally good on expectation, achieving the same accuracy as a random guess (Wolpert, 1996). Such a world would be hostile to inductive reasoning. The assumption that labelings are drawn uniformly ensures that training data is uninformative about unseen samples.

In contrast to this dismal outlook on machine learning, naturally occurring inference problems involve highly structured data, and basic structures are often shared even across seemingly disparate different problems. If we can design learning algorithms with inductive biases that are aligned with this structure, then we may hope to perform inference on a wide range of problems. In this work, we explore the alignment between structure in real-world data and machine learning models through the lens of *Kolmogorov complexity*.

The Kolmogorov complexity of an output is defined as the length of the shortest program under a fixed language that produces that output. In Section 3, we explain the connection between Kolmogorov complexity and compressibility. We note that virtually all data drawn from a uniform distribution as assumed by the no free lunch theorem of Wolpert (1996) cannot be significantly compressed, yet relevant real-world datasets are highly compressible. In particular, neural networks themselves can be used to create compressions of data labelings, upper bounding their Kolmogorov complexity.

We then demonstrate in Section 4 that modern neural networks prefer low Kolmogorov complexity too, complementing the low complexity of actual data. While models implemented on a computer cannot generate data with complexity exceeding the length of their associated pro-

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gram, we find they actually prefer data that is far simpler. We formulate simple languages for generating numerical sequences, under which we can directly measure the Kolmogorov complexity of a sequence. We use these languages to inspect the simplicity bias of both pre-trained and randomly initialized language models. GPT-3 (Brown et al., 2020) reliably favors less complex sequences, and bigger and better GPT-3 variants even more so. Notably, randomly initialized GPT models share this simplicity bias.

To further emphasize the universality of this simplicity bias, we reshape tabular data from diverse domains, including click prediction and airline delay prediction, into images and feed them through convolutional computer vision architectures, showing that these vision architectures prefer correct labelings to random ones, even on data which do not remotely resemble natural images and which contain no spatial structure whatsoever. We use this property to compute cross-domain generalization bounds via Kolmogorov complexity.

A common intuition associated with no free lunch theorems dictates that since a single learner cannot solve all problems, practitioners must inspect data and manually select an appropriate learner for the specific problem at hand. For example, a practitioner might select a more constrained model to avoid overfitting on small datasets, or they may select convolutional architectures to accommodate natural image data. To the contrary, we show in Section 5 that the meta learner which selects the best learning algorithm from cross validation suffers little from overfitting even when the number of models investigated is large, and this cost is quickly overtaken by gains in validation accuracy under reasonably sized validation splits.

Moreover, a single learner, which on the one hand supports a variety of functions but on the other hand prefers simple ones, can solve a wide range of problems. We show that flexible models accompanied by a penalty which encourages simple solutions can solve problems with a variety of sample sizes. For example, a single learner which combines a small convolutional neural network and a transformer, while encouraging use of the small model insofar as it fits the training data, simultaneously achieves strong performance both on small datasets like CIFAR-10 (Krizhevsky, 2009) and also large datasets like ImageNet (Deng et al., 2009). In fact, the historic evolution of machine learning systems supports the ability of a single learner to perform diverse tasks (see Figure 1) as highly task-specific pre-neural algorithms, such as LDA (Blei et al., 2003) and HOG (Dalal & Triggs, 2005), were replaced by neural architectures such as convolutional or recurrent models, and transformers can now handily perform all tasks listed in Figure 1.

We summarize our contributions as follows:

- We demonstrate the direct connection between compressibility and learnability that is implicit in no free lunch theorems by deriving a new no free lunch theorem using Kolmogorov complexity.

- We show that the low Kolmogorov complexity of real datasets can be directly derived from the machine learning models used to fit them.

- We compute the first cross-domain PAC-Bayes generalization bounds which show that neural networks such as convolutional architectures have low complexity biases that are relevant even on diverse tabular data far from what they were designed for.

- We demonstrate GPT-3's preference for sequences generated by short expression trees, and we find that even randomly initialized language models have a simplicity bias.

- We argue through the PAC-Bayes lens why in principle model selection can be automated into a larger algorithm, and we provide explicit examples demonstrating how multiple models can be subsumed into a single powerful learner.
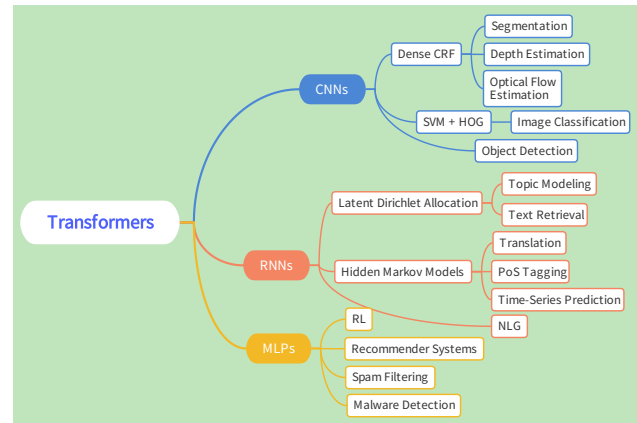

Figure 1: Over time, tasks that were performed by domain-specialized ML systems are increasingly performed by unified neural network architectures.

We also emphasize several conceptual insights and findings throughout the paper:

- While there is a valid discussion to be had about the role of inductive biases in machine learning, the no free lunch theorems have no direct bearing on that discussion. In particular, the no free lunch theorems should not be used to rationalize claims about how we can't have relatively general-purpose learners.

- This is because many real-world modeling problems are highly structured, and that structure is shared

across problems to a surprisingly large extent, in contrast to the assumptions of the no free lunch theorems. For instance, it is therefore entirely possible for one model to be better than another at solving all problems of interest. There may be a universe where Leonardo da Vinci is not a polymath, a universe where learning problems have very different structure and therefore preclude general-purpose solvers, but it is not our universe.

- Some approaches to model construction will be much better at general purpose problem solving than others. Embracing a flexible hypothesis space, combined with soft (not restrictive) inductive biases for high level structures we often see in reality (such as equivariance to certain transformations), and a low-complexity bias, is a good recipe for general problem solving.

- Using a single model for multiple different problem settings may even be optimal in some instances. For example, we may in principle want to use the same model for large and small datasets, contrary to common practice.

- Neural networks have a low-complexity bias, which helps make them general-purpose problem solvers. Convolutional neural networks even have a bias towards tabular data, despite clearly being designed without tabular data in mind.

We make our code available here. We summarize several key takeaways throughout the paper in $\boxed{\texttt{blue}}$.

2. Background

In this section, we introduce no free lunch theorems for supervised learning and Kolmogorov complexity.

No free lunch theorems. No free lunch theorems (NFL) state that without making strong assumptions, a single algorithm cannot simultaneously solve all problems well. No free lunch theorems for search and optimization indicate that all optimizers and search algorithms satisfying certain conditions perform exactly the same on average over all such search or optimization problems (Wolpert et al., 1995; Wolpert & Macready, 1997). In this work, we narrow our focus to NFL for supervised learning. Wolpert (1996), and similarly Schaffer (1994), proves an analogous theorem for supervised learning under which every learner—a function that takes in labeled data and outputs a labeling function for the associated domain—achieves exactly the same accuracy of 50% on average over all binary classification problems where accuracy is only evaluated on unseen samples.

In order to prove no free lunch theorems, one needs to place

very strong assumptions on the lack of structure in the labeling function such as a uniform distribution, so that conditioning on the training labels does not modify the probability over labelings on unseen points (Rao et al., 1995). To illustrate the severity of this condition, imagine that a person is presented a sequence of one million 1s and asked whether they predict that the next element will be 1 or 0. If labelings of sequence elements were distributed uniformly, then we should assign equal probability to both options, despite the fact that intuition and Bayesian probabilistic models tell us overwhelmingly to favor 1.

Shalev-Shwartz & Ben-David (2014) instead do not assume a particular distribution over learning problems and prove that for every learner, there exists a task on which the learner achieves poor accuracy with high probability over training splits, whereas another learner achieves perfect accuracy. Notably, the latter NFL computes accuracy over all data, not just "off-training" samples. While this statement of NFL does not explicitly require uniformly distributed data, the existence of catastrophic failure modes for our learners would not matter if our learners never encountered them in practice. After all, we do not care if our learners cannot solve problems we do not want to solve. Thus, the practical relevance of this theorem again hinges on the distribution over real-world learning problems and how well it aligns with the inductive bias of a learner. In this paper, we argue that the real-world learning problems we care about share a high degree of common structure, and the inductive biases of neural networks are well-aligned with such problems.

Kolmogorov complexity and compression. Kolmogorov complexity quantifies the structure in a bitstring, measuring the extent to which it can be compressed (an algorithmic definition of information content). For a fixed programming language L, the Kolmogorov complexity of data x, $K(x)$, is the length of the shortest program in that language that outputs x (Kolmogorov, 1963). Analogous to conditional entropy, $K(y|x)$ is defined as the length of the shortest program which inputs x and outputs y. Kolmogorov complexity provides a mathematical formalization of the notion of simplicity and Occam's razor, which encompasses many related concepts like Shannon information, compression, and minimum description length (MDL) (Li et al., 2008).

While objects with large Kolmogorov complexity are impossible to verify (Chaitin, 1974), they are abundant over all possible bitstrings. All but exponentially few sequences of a given length have near maximal Kolmogorov complexity and are thus incompressible. Taken over the uniform distribution over bitstrings x, $P(K(x) \leq n - k) \leq 2^{1-k}$. However as we will discuss, these high complexity objects are extremely uncommon in practice.

Universal induction. Inspired by Kolmogorov complexity, a line of work introduces *universal induction* methods, which prefer low complexity answers (Solomonoff, 1964; Hutter, 2000; Nakkiran, 2021). Notably, Solomonoff induction (Solomonoff, 1964; Rathmanner & Hutter, 2011) makes predictions by applying Bayes rule to the universal prior which favors low complexity, and provides learning guarantees. The existence of universal learners calls into question the broader message of no free lunch theorems, showing that Occam's razor or a preference for low-complexity data labelings is sufficient for learning on low-complexity data (Lattimore & Hutter, 2013).

Rather than formalizing theoretical learners that rely on Kolmogorov complexity, which is in general uncomputable, Fernández-Delgado et al. (2014) and Gómez & Rojas (2016) test out popular machine learning algorithms on a diverse array of datasets to see if any existing algorithms are plausibly universal. These works, which directly respond to no free lunch theorems, indicate that Wolpert's no free lunch theorem (Wolpert, 1996) may not restrict machine learning in practice. Another line of work similarly shows that a single transformer model can perform well on a vast test bed of problems by mimicking Bayesian inference with in-context learning, and in fact achieves state-of-the-art on small tabular datasets (Müller et al.; Hollmann et al., 2022).

Whereas Fernández-Delgado et al. (2014) and Gómez & Rojas (2016) demonstrate empirically that single learners can perform well on average across real-world problems, our own work explains why this phenomenon is possible, relating the success of machine learning to the low Kolmogorov complexity of real-world data along with the preference of our models for low complexity (see Section 3.2). In this work, we clarifying the role that incompressibility plays in preventing learning on random data by deriving a no free lunch theorem directly from Kolmogorov complexity.

PAC-Bayes and the marginal likelihood. The PAC-Bayes framework is a convenient paradigm for proving generalization bounds on parametric models, while avoiding the pitfalls of uniform convergence. Rather than considering all elements of the hypothesis class on equal footing, we choose prior and posterior distributions over the parameters, and the generalization gap for elements of the posterior depends merely on the discrepancy between the two as measured by the KL divergence. This framework can explain many favorable properties of neural networks like flat minima (Hochreiter & Schmidhuber, 1997), noise resilience (Arora et al., 2018), compressibility (Zhou et al., 2018), and it can also provide nonvacuous generalization bounds with most recent bounds drawing directly from Kolmogorov complexity and the universal prior (Lotfi

et al., 2022a).

The marginal likelihood, or the probability that a random sample from the prior generates the data, is a natural statement of Occam's razor or the idea that simple models should be preferred (MacKay, 2003; Lotfi et al., 2022b). A prior which achieves high marginal likelihood will prefer the observed training set to random data, indicating that it provides a compression of the observations. The marginal likelihood is also directly related to PAC-Bayes generalization theory since the popular Catoni generalization bound (Catoni, 2007) can be re-written as a function of the marginal likelihood when the PAC-Bayes posterior is assumed to be the true Bayesian posterior (Germain et al., 2016; Lotfi et al., 2023).

Existing works in the generalization bound literature show that a model generalizes with high probability on a particular data distribution whenever it is compressible (i.e. has low complexity) with respect to the prior, but the prior is chosen specifically for the dataset at hand (e.g. CNNs for image classification) and furthermore the prior is often tuned directly on a fraction of the training set (Dziugaite & Roy, 2017; 2018; Pérez-Ortiz et al., 2021). In fact, there exists a widely held belief in the generalization community that problem-specific priors, notably ones which are tuned on the training set, are necessary for strong generalization bounds (Dziugaite et al., 2021), and this belief manifests in data-dependent prior bounds across the literature.

In this paper, we argue against the necessity of problem-specific priors. Our generalization bounds and compression experiments show that a single low-complexity biased prior can suffice on a wide variety of problems due to the low Kolmogorov complexity of data. Whereas previous generalization theory literature is in line with the notion supported by no free lunch theorems that problems require specially tailored solutions, our work fights back against this widely held belief.

Complexity in deep learning. Several works have related Kolmogorov complexity to neural networks. One line of study proves that multi-layer perceptrons with Boolean input features are biased towards low-entropy functions, namely ones which classify disproportionately many or few points into the same class (Mingard et al., 2019) or are insensitive to flips in Boolean input features (De Palma et al., 2019). Pearlmutter & Rosenfeld (1990) argue that random initialization and noise in data increase the complexity of neural networks but that ensembling such models reduces complexity in expectation. Schmidhuber (1997) proposes another method for reducing Kolmogorov complexity by instead explicitly searching for simple neural networks and finds improvements in generalization on very small problems where such a search is computationally feasible.

The compressibility of neural networks has also been studied for purposes other than PAC-Bayes generalization bounds. For instance, Blier & Ollivier (2018) explore the compressibility of datasets using neural networks, including their parameters, and show that variational methods yield poor compression whereas prequential coding can be used to obtain shorter code lengths. Hinton & Van Camp (1993) find that adding noise to neural network weights during training reduces the information contained in the weights, getting bits back and making the parameter vector compressible. Our work shows that the compressibility of datasets using neural networks is universal and does not require domain-specific models.

In contrast to existing works, we show that neural networks naturally encode a preference for simple solutions, and this structure is shared with real-world data. Compressibility emerges naturally and does not require special interventions.

On the relationship between our contributions and existing literature. We push back against a pervasive narrative that one needs to specially select machine learning setups using domain knowledge on a per-problem basis, often inspired by no free lunch theorems. In particular, we summarize the relationship between our contributions and existing literature below:

- In contrast to previous works which counter the no free lunch theorem by observing that a single model can achieve better-than-average empirical accuracy across diverse datasets (Fernández-Delgado et al., 2014; Gómez & Rojas, 2016), we explain and formalize the structures which are universal across such data distributions using Kolmogorov complexity. Relating this formalism to learning, we then show why low complexity is fundamental to such successes of machine learning models by proving a novel no free lunch theorem directly using Kolmogorov complexity.

- The preference we demonstrate for low complexity emerges naturally in a variety of models, from transformer language models to convolutional neural networks, and requires no special interventions as proposed in Schmidhuber (1997) or Hinton & Van Camp (1993).

- Existing generalization bound literature tunes priors on specific data distributions (Dziugaite & Roy, 2017; 2018; Pérez-Ortiz et al., 2021; Dziugaite et al., 2021) in line with the idea often drawn from no free lunch theorems that each domain requires a specially tailored model. In contrast, we demonstrate that neural networks can compress a wide range of datasets in domains they were not even designed for, and that this

compressibility can explain generalization via PAC-Bayes generalization bounds.

- Common wisdom dictates that neural network architectures must be carefully chosen for specific problems or training set sizes (Grinsztajn et al., 2022; Brigato & Iocchi, 2021; Lee et al., 2021), but we instead show through the formalism of complexity and also through empirical experiments that specialized models can in principle be combined into a single learner which can perform well on a wide variety of problems and sample sizes.

3. Unpacking the No Free Lunch Theorem with Kolmogorov Complexity

The often cited no free lunch theorem of Wolpert (1996) states that all learners perform the same when averaged over a uniform distribution on all possible datasets. However, since most possible datasets are incompressible, the assumption of uniform samples subtly selects high complexity incompressible data, a set of problems on which learning is fundamentally impossible. Through hypothesis testing, we conclusively rule out the possibility that real datasets are as high complexity as randomly drawn ones. We bring to the surface the centrality of complexity in NFL theorems by deriving a new NFL theorem which uses the incompressibility of random data to show why on this data learning is impossible.

3.1. An Exercise in Bounding Dataset Complexity

We first consider the hypothesis that unlabeled machine learning datasets are drawn uniformly at random and use a bound on the Kolmogorov complexity as a test statistic. One can produce upper bounds on $K(x)$ by compressing x, but then it is necessary to include both the size of the compressed file and the size of the program required to decompress it. Alternatively, if one can construct a short program which directly outputs x, this program also forms a compression of x. Using bzip2, and including the size of the decompression program, we compress text dataset Amazon Review Full (McAuley & Leskovec, 2013) and audio dataset LibriSpeech (Panayotov et al., 2015) to 393.2 MB and 8.36 GB respectively, providing upper bounds on the Kolmogorov complexity with respect to the Python programming language. Computing the number of possible text and audio datasets of these sizes and supposing such datasets were in fact uniformly sampled at random, the probability of observing complexities this size or smaller is less than $10^{-1292913987}$ and $10^{-47632711550}$, astronomically low p-values, conclusively ruling out the possibility that they were sampled uniformly in this way. If we randomly shuffle the datasets, we instead obtain bounds of only 836.7 MB and 9.69 GB, considerably larger, showing

that the compressibility results not just from an inefficient encoding, but from structure in the dataset.

Other works have also examined Kolmogorov complexity in data, for example EEG patterns (Petrosian, 1995) or animal behavior (Zenil et al., 2015), and confirm that such data is simple. Our experiments above show that low Kolmogorov complexity is not specific to EEG patterns or animal behavior and is in fact a generic characteristic of common datasets we use in machine learning.

3.2. NNs as Compressors of the Labeling Function

Particularly relevant to supervised learning, we show that not only are the unlabeled datasets compressible—the labeling functions are too. Further, we can demonstrate their compressibility concretely using trained models as compressors. Given a labeled dataset $\mathcal{D} = (X, Y) = \{(x_i, y_i)\}_{i=1}^n$, any likelihood model $p(y|x)$—regardless of whether the top predictions are correct—can be used to generate a lossless compression scheme to encode the dataset labels Y given the inputs X. Using a stream code such as arithmetic coding (Witten et al., 1987), in combination with the probability model $p(y|x)$, the labels can be encoded in $K(Y|X, p) \leq -\sum_{i=1}^n \log_2 p(y_i|x_i) + 3$ bits. Models which maximize the log likelihood of the data also implicitly minimize the length of this encoding.

One can show by delimiting, as is done in Fortnow (2000), that $K(Y|X) \leq K(Y|X, p) + K(p) + 2\log_2 K(p) + c$, where c is some small constant depending on the language. Writing the negative log likelihood in terms of the empirical cross entropy, combining our two inequalities, and dividing by the size of the dataset n yields

$$\frac{1}{n} K(Y|X) \leq \frac{CE}{\ln 2} + n^{-1}(K(p) + 2\log_2 K(p) + c), \quad (1)$$

where CE is the cross entropy of the classifier p averaged over dataset $\mathcal{D}$. This inequality implies that, regardless of how large the model is, it provides a non-trivial compression of the dataset as the size n of the dataset grows sufficiently large, as long as CE is better than random guess. To demonstrate this fact, we employ the compression scheme from Lotfi et al. (2022a) in order to find a compressed representation of MLPs on several class balanced tabular classification datasets[1]. As shown in Figure 2 (left), we are able to compress the labels on most of the datasets by well over the naive $n\log_2 C$ encoding length where C is the number of classes. We also apply the method with convolutional architectures to compress labels on CIFAR-10 and CIFAR-100 in Figure 2 (middle), allowing us to reject the hypothesis that the labeling functions are drawn uniformly at random with extremely high confidence.

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[1] https://www.openml.org/

3.3. A Kolmogorov-Style No Free Lunch Theorem

A corollary of Equation 1 is that if the dataset is incompressible, then no model can do better than random chance in the limit of a large dataset, as we show in Theorem 1. Since compressible datasets in uniformly sampled data are exponentially unlikely, we can prove our own version of the no free lunch theorem directly from this constraint on complexity. With very high probability, on any given uniformly sampled dataset, learning is impossible.

Theorem 1. *Let (X, Y) be a labeled dataset with n data points and uniformly sampled random labels from C classes. Then, with probability at least $1 - \delta$, for every classifier $p(y|x)$,*

$$CE(p) \geq \ln C - \frac{\ln 2}{n}\left(K(p) + 2\log_2\frac{K(p)}{\delta} + c\right), \quad (2)$$

where $CE(p)$ is the empirical cross entropy of the classifier $p(y|x)$ on the data. Thus for any model of bounded size, if the size of the dataset is large enough, the model cannot represent any classifier with cross entropy appreciably smaller than that attained from random guess.

Proof: *See Appendix A*

Like any of the no free lunch theorems, the necessary existence of unsolvable problems initially seems limiting, but here we see that the incompressiblility of the datasets from the uniform distribution is the fundamental issue that would impair learning. On compressible datasets (ones with less than maximal complexity), learning is possible.

> Real datasets are highly unlike the high complexity samples from the uniform distribution, associated with no free lunch theorems, where learning is impossible. The common structure shared by real datasets nullifies the limitations imposed by no free lunch theorems.

4. Low-Complexity Bias in Machine Learning Models

In the previous section, we saw that real-world data distributions across domains share a common bias towards low Kolmogorov complexity. If we can construct models which also prefer low-complexity data, then we can hope to perform inference with a single model across many such domains. While early machine learning systems incorporated highly domain-specific design elements, such as hand-crafted image features (Dalal & Triggs, 2005) or graphical models for language modeling (Mnih & Hinton, 2007), modern neural network architectures across domains are converging on transformers (Vaswani et al., 2017; Dosovitskiy et al., 2020; Gulati et al., 2020; Somepalli et al.,
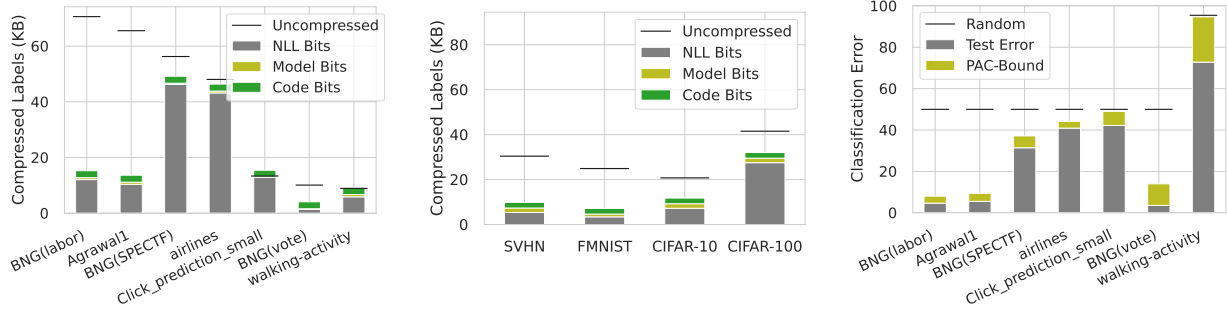
Figure 2: (Left): Compressed sizes of tabular labels where compression is performed via a trained MLP model (as in Section 3.2) vs. direct encoding of labels ($n \log_2 C$). (Middle): Compression of image classification datasets using CNNs. Note the breakdown of the total compressed size of the labels into model fit (NLL Bits), compressed parameters (Model Bits), and architecture and decompressor (Code Bits). In both cases, models can greatly compress a diverse suite of datasets, highlighting a common structure shared by models and real-world data. (Right): Compression based generalization bounds (Lotfi et al., 2022a) for CNNs on tabular data, fed in with each pixel representing a tabular feature. The bounds are able to explain the majority of the model performance as shown by the test error, indicating that even CNNs designed for computer vision have a generic inductive bias appropriate for a wide range of datasets containing no spatial structure at all.

2021), some of which can simultaneously achieve impressive performance on a variety of data types with a single architecture (Jaegle et al., 2021).

In this section, we finally argue that neural networks have a generic simplicity bias that extends well beyond the datasets for which they are designed. To this end, we: (1) feed tabular datasets from diverse domains such as click prediction and airline delay prediction into convolutional networks designed specifically for computer vision and find that they provably generalize well due to their simplicity bias, (2) formulate a language with respect to which we can measure the Kolmogorov complexity of numerical sequences and observe that GPT-3 generates low-complexity sequences with exponentially higher probability than complex ones, (3) predict the next term in a sequence with randomly initialized language models. Whereas the no free lunch theorem of Wolpert (1996) implies that such an inference procedure cannot outperform a random guess on average, we find that randomly initialized neural networks prefer sequence completions which generate low-complexity completed sequences, demonstrating that they can make accurate guesses as long as the ground truth sequence distribution also favors low complexity.

4.1. Bounding Generalization by Complexity

Generalization bounds limit how the expected risk $R(h)$ for a model h will differ from its train risk $\hat{R}(h)$. One simple such generalization bound is the finite hypothesis bound under a prior $P(h)$ (Langford & Seeger, 2001): with probability $1 - \delta$: $R(h) \leq \hat{R}(h) + \sqrt{\frac{\log 1/P(h) + \log 1/\delta}{2n}}$. Relating to Occam's razor and Solomonoff induction, consider the universal prior that assigns higher likelihood to compressible hypotheses: $P(h) = 2^{-K_p(h)}/Z$ where $K_p(h) \leq K(h) + 2\log_2 K(h)$ is the *prefix* Kolmogorov complexity

and $Z \leq 1$. Combining the two, we have with probability $1 - \delta$,

$$R(h) \leq \hat{R}(h) + \sqrt{\frac{K_p(h)\log 2 + \log 1/\delta}{2n}}. \qquad (3)$$

Despite the simplicity of the finite hypothesis bound, when combined with the universal prior, it provides strong nontrivial statements about the generalization of machine learning models even for neural networks which have many more parameters than data points (Lotfi et al., 2022a). Solutions found by many machine learning models on real datasets are highly compressible, and this reflects their bias for simple low Kolmogorov complexity functions. Even under an arbitrarily large or even infinite hypothesis space, generalization is possible if we assign prior mass disproportionately to the highly structured data that typically occurs.

4.2. Neural Networks Prefer Naturally Occurring Labelings Across Domains

The inductive biases of even specialized architectures like convolutional neural networks facilitate broad learning abilities. To demonstrate this fact, we take tabular classification datasets and encode the tabular features as an image by simply forming images where each pixel corresponds to a different feature, zero padding as necessary. We train a small convolutional network using this input data to predict the classification labels. Since the data has no local or translation equivariant structure, learning with the convolutional network requires overcoming its strong inductive bias which was hand tailored for settings with such structure. Even in spite of this mismatch, the convolutional networks perform well. Using the compression and PAC-Bayes bound methodology from Lotfi et al. (2022a) (a variant of Equation 3), we show the generalization bounds on these models along with their test error in Figure 2 (right).

The strong generalization of convolutional networks on these tabular datasets is almost entirely explainable through simplicity bias as shown by the fact that a finite hypothesis compression bound nearly matches the test error.

> Though CNNs were designed for vision, they generalize on unrelated tabular domains, a phenomenon which is almost entirely explained by their preference for low-complexity solutions.

4.3. GPT-3 Assigns Exponentially Higher Probability to Simpler Sequences

We now study the preference of GPT-3—a line of high-performance autoregressive text generators—for simpler sequences. The ability of language models to solve reasoning problems has recently been studied by Zelikman et al. (2022), who develop an effective prompting framework, and d'Ascoli et al. (2022), who develop transformers for predicting symbolic expressions directly from the sequence. To perform our own study, we need a well-defined, computable notion of complexity. We thus define a simple, non-Turing-complete language and measure Kolmogorov complexity with respect to this simple language. Namely, we generate integer sequences with binary expression trees. We then define the complexity of a sequence as the size of the smallest expression tree, measured as the number of internal nodes—or equivalently the number of operators in the expression represented by the tree—that generates that sequence. By using a small set of terms for the leaves and binary operators for the nodes, we can enumerate over all possible expression trees for small sizes at most L and compute all sequences with complexities 0 through L.

In our experiments, we use operations $+$, $\times$, and $//$, where $//$ denotes integer division. For leaves, we use 2 and i, where i is the index within the sequence. For example, $(2 + i) \times i$ could be implemented with a tree of size 2 and would generate the sequence $a_i = 0, 3, 8, 15, ...$ Using this setup, we can generate sequences of varying complexity, according to a well-defined metric, and quantify the preference of GPT-3 models for simpler sequences over more complex ones. We provide details on how we tokenize sequences and extract their probabilities in Appendix C.

In Figure 3, we measure the average log-probability GPT-3 models assign to sequences of a given Kolmogorov complexity, where we fix the number of numerical tokens input into the model to be 30, and we observe that the probabilities assigned by these language models decrease exponentially with sequence complexity, similar to the Solomonoff prior discussed in Section 2. In contrast, a uniform prior would be described by a flat line. Since autoregressive language models like GPT-3 output

per-token softmax probabilities conditional on all previous tokens within their context, we can compute the log-probability of a sequence of tokens as $\log P(\text{Sequence}) = \sum_i \log P(\text{Token}_i | \{\text{Token}_{<i}\})$. Note that for very complex sequences, we cannot easily measure their minimum description length, so we limit our experiments to expression trees with at most 7 operators. In this low-complexity regime, we observe that big GPT-3 models which excel at language modeling, e.g. Davinci which contains 175 billion parameters, assign higher probability to these simple sequences than much smaller GPT-3 models such as Ada. The legend lists GPT-3 variants in order of increasing size.

We can also examine the decay of such log-probabilities as we feed more tokens, corresponding to digits of sequence elements, into the model. As the sequences get longer, we see in Figure 3 that the probabilities assigned to sequences decay sub-exponentially, indicating that these models, especially bigger variants such as Davinci, become more and more confident about later sequence elements.

4.4. Even Randomly Initialized Language Models Prefer Low Complexity

The previous section solely examined pre-trained language models, but these models were trained on massive corpora of data. Do they prefer low complexity at initialization before they have even seen any data at all? While the initialization of neural network parameters is highly diffuse, these random parameters can induce a highly structured distribution over functions.

Trained language models are known to repeat themselves when generating text (Holtzman et al., 2020; Fu et al., 2021). One might think that this behavior is learned from training data which often contains repeated text, but we show in this section that randomly initialized GPT models repeat themselves too. Interestingly, we can formalize the preference for repetition as a preference for low Kolmogorov complexity. In order to disentangle the impact of initialization from training, we adopt a simple language for generating binary sequences under which we can quickly measure Kolmogorov complexity. We consider a program to be a bitstring, and then the program upon execution simply repeats the bitstring until output reaches length 10. Under this language, the sequence $0, 0, 0, ...$ has Kolmogorov complexity 1, and $0, 1, 0, 1, ...$ has complexity 2, yet randomly generated sequences are exponentially more likely to have high complexity. We conduct our evaluations exhaustively on all such sequences of length 10.

We now generate sequences of length 10 with randomly initialized GPT-2 language models (Radford et al., 2019), using each initialization to generate one sequence, and we measure the frequency with which each sequence is generated. We provide estimated generation probabilities
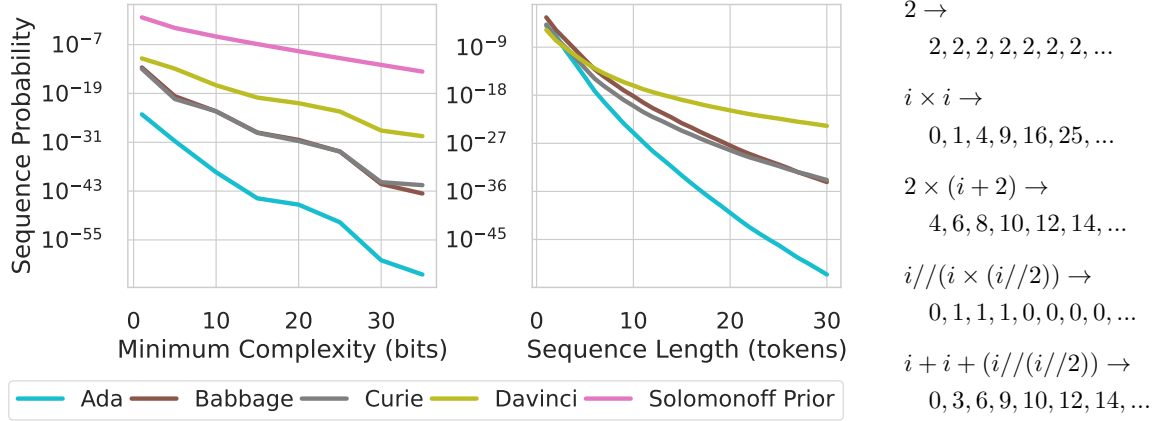
$$2 \rightarrow$$
$$2, 2, 2, 2, 2, 2, 2, ...$$

$$i \times i \rightarrow$$
$$0, 1, 4, 9, 16, 25, ...$$

$$2 \times (i + 2) \rightarrow$$
$$4, 6, 8, 10, 12, 14, ...$$

$$i // (i \times (i//2)) \rightarrow$$
$$0, 1, 1, 1, 0, 0, 0, 0, ...$$

$$i + i + (i // (i//2)) \rightarrow$$
$$0, 3, 6, 9, 10, 12, 14, ...$$

Figure 3: GPT-3 prefers low-complexity sequences generated by expression trees. **Left:** Average log-probability of sequences by complexity. **Middle:** Average log-probability by sequence length, restricted to decimal digit tokens. **Right:** Example sequences of complexity 0 through 4 with minimal generating expressions. GPT-3 variants ordered by increasing size. Observe that GPT-3 variants assign exponentially lower probabilities to higher complexity sequences (left), as in the Solomonoff prior, and bigger more powerful models especially exhibit this behavior. Moreover, the models become more confident as they see more tokens, and the more powerful GPT-3 variants such as `Davinci` learn faster (middle).

by Kolmogorov complexity in Appendix D where we see again that low-complexity sequences are assigned exponentially higher probabilities. Here, we compare (1) the uniform distribution over sequences, (2) randomly initialized GPT-2, as well as (3) pre-trained GPT-2 models. We see that whereas the uniform distribution lacks any preference for low-complexity sequences, randomly initialized parameters induce a structured distribution over sequences, and pre-trained checkpoints exhibit an even stronger preference for low complexity as they are trained on structured text. We can also use the randomly initialized language models to perform next element prediction by estimating the probabilities they assign to the next element in a sequence given having correctly generated the previous terms. While Wolpert's no free lunch theorem (Wolpert, 1996) ensures that the average completion accuracy over all possible length 10 bitstrings is exactly 0.5, we verify in Appendix D that randomly initialized networks can be used for sequence completion when the sequence has low complexity.

We can further generate very long length-100 sequences with randomly initialized and also pre-trained GPT-2 models and run a simple hypothesis test, demonstrating both randomly initialized and pre-trained models generate lower Kolmogorov complexity sequences on average than a uniform distribution. We generate 100,000 samples from each of these three generative distributions and perform a one-tailed t-test on the null hypothesis that $\mu(K(S_{\text{GPT}})) \geq \mu(K(S_{\mathcal{U}}))$, where S_{GPT} and $S_{\mathcal{U}}$ respectively denote random sequences generated by the language model or a uniform distribution. Performing this hypothesis test, we reject this null hypothesis in both randomly initialized and pre-trained models with an extremely low p-value, indicat-

ing that language models are indeed more likely to generate simple sequences, pre-trained models even more so. The details of these hypothesis tests are found in Appendix D. We conclude that neural networks for language generation, both trained and randomly initialized, express a bias towards low Kolmogorov complexity which mirrors that of data as demonstrated in Section 3.

> Language models, both pre-trained and randomly initialized, prefer to generate low-complexity sequences. As a result, we can use even such randomly initialized models to predict the next element in a sequence, as long as the sequence is low-complexity.

5. Model Selection with a Simplicity Bias

In typical industrial workflows, practitioners examine their data and select an appropriate learner. For example, if data are linearly separable, then a practitioner might use logistic regression, whereas if the data is more complex, the practitioner may instead choose a neural network. We can then consider the human model selector and the model they select as a single meta-learner. Whereas the no free lunch theorems seemingly preclude automated meta-learners which select performant models on any task, empirical works show that model selection can in fact be automated in practice (Vilalta & Drissi, 2002). Giraud-Carrier & Provost (2005) show that with minimal assumptions, the defeating conclusion of Wolpert's no free lunch theorem is escaped by meta-learners as long as datasets share structure so that the model selector generalizes to new datasets. Sub-

sequent works prove the existence of meta-induction algorithms, for selecting amongst all induction methods, which are near-optimal among learners (Schurz, 2008; Schurz & Thorn, 2017).

In this section, we argue why in principle, model selection can be automated from the view of Kolmogorov complexity.

5.1. Model Selection and Generalization Bounds

When developing a machine learning approach for a given application, it is often helpful to leverage domain knowledge in constructing or choosing the right model for the task. One might start by choosing from families like MLPs, CNNs, GNNs, PointNets, or Transformers and then decide on the appropriate way of featurizing inputs, possibly incorporating knowledge of data symmetries via hard-coded equivariances or data augmentations. Even just enumerating the possible models a practitioner would consider and choosing based on their expertise, the number is not tremendously large. Even if we are extremely generous and suppose that the practitioner is choosing from 100 million models, we can consider the impractical algorithm of selecting between them via cross validation. While one might expect that such a procedure would overfit, in fact even finite hypothesis bounds show this is not the case. Using cross validation on a validation set of size $n = 20000$ for a classification problem, plugging in a uniform prior $P(h) = 1/|\mathcal{H}| = 10^{-8}$ to the finite hypothesis bound derived in Equation 3, we get that the gap between *validation* and test error will be less than 3.4% with probability greater than 99%. Ultimately, we avoid overfitting because we only need a number of data points proportional to the log of the size of the hypothesis space for finite hypothesis bounds.

For an even more general class of models, one may consider the number of bits of prior knowledge needed to specify model architectures like MLPs, CNNs, or GNNs, as well as symmetry groups and any other required information. In each of these cases, the model architectures can in fact be expressed in few bits. A near state-of-the-art computer vision model can be expressed in only 280 characters (Trockman & Kolter, 2022) of PyTorch. Similarly, important symmetry groups like translations, rotations, reflections, and other matrix groups can be expressed in only a few lines of code (Finzi et al., 2021b) and can be used to encode equivariances or for augmentation. Therefore, even in selecting from all possible models that can be expressed in that short amount of code, we can expect to generalize with only tens of thousands of data points.

> In principle, automating model selection directly via cross validation provably generalizes well across millions of models with only thousands of data points.

5.2. One Model for Big and Small Training Sets

It is commonly believed that small training datasets demand compact neural architectures, whereas large training sets can accommodate flexible ones. Accordingly, practitioners hand select appropriate models for their datasets. In this section, we argue that a single learner can be effective for all data sizes as long as we encode our a priori preference for simplicity. Our prior should prefer simple functions we believe are more likely yet support a wide variety of functions in case training data rules them out. We begin with a simple illustration on polynomial regression.

Polynomial regression. Common intuition dictates that high degree polynomials easily overfit small training sets. In contrast, low degree polynomials cannot fit complicated functions so they should be avoided when training data is plentiful. However, we find that a single high degree polynomial can be effective across a wide variety of sample sizes and regression problems as long as we encode a preference for low-complexity solutions, namely ones which rely on low degree coefficients.

To this end, we adopt Tikhonov regularization with Tikhonov matrix given by $\mathrm{diag}(\{\alpha k^2\}_{k=0}^d)$; in particular, we impose an ℓ_2 penalty that increases quadratically with the degree of the corresponding monomial. In Appendix E, we see that this model, which is flexible yet has a strong simplicity bias, performs at least on par with a low degree polynomial when training data is scarce and at least on par with a high degree polynomial when training data is abundant.
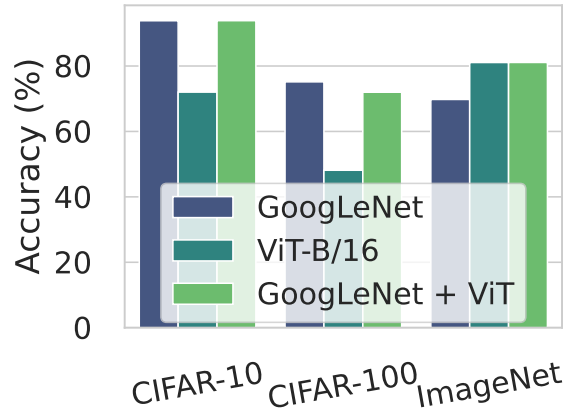

Figure 4: A single learner, which is more expressive than a ViT but also prefers simple solutions representable by a GoogLeNet, can simultaneously solve small and large scale problems.

Neural networks. We illustrate a similar concept with neural networks. We consider a small network, GoogLeNet (Szegedy et al., 2015), which performs well on small datasets such as CIFAR-10 and CIFAR-100 (Krizhevsky, 2009) but poorly on larger datasets like ImageNet (Deng et al., 2009). We also consider a large network, ViT-B/16 (Dosovitskiy et al., 2020), which performs significantly worse on CIFAR variants but much better on ImageNet. As in the polynomial regression example above, we can combine these two architectures, specifying our preference for the simpler GoogLeNet to the extent that it fits the training data. To learn on a dataset, we train both models and then take a convex combination of their logits, $c*\text{logits}_{\text{ViT}} + (1-c)*\text{logits}_{\text{G}}$, controlled by a single parameter c with ℓ_2 regularization in favor of GoogLeNet (i.e., by adding λc^2 to the loss function). In Figure 4, we observe that while GoogLeNet and ViT each have strengths and weaknesses, combining them with a preference for simplicity achieves the best of both worlds. While aggressively restricting our architectures can decrease computational cost, it is unnecessary for generalization. Experimental details and additional experiments with Swin Transformer (Liu et al., 2021) are found in Appendix E.

> A single model can work well with both small and large training sets, so long as we embrace flexibility combined with a soft simplicity bias.

6. Discussion

While large ML models are highly flexible, we saw that they reliably prefer low Kolmogorov complexity solutions—aligning well with relevant learning problems—despite not being designed with complexity in mind. This observation raises the question: why exactly do neural networks encode such a strong preference for low complexity and how can we tune this preference? Complementing the above observation, we also saw that a single expressive model which simultaneously supports a diversity of solutions but prefers simple ones can solve both simple and hard problems across sample sizes. Such learners present clear advantages over the current paradigm in deep learning in which we manually select small constrained architectures or large ones with mild inductive biases, depending on the problem. Keeping this possibility in mind, can we design expressive yet simplicity-biased models with affordable computational costs?

We finish with a discussion of several fundamental themes that surface throughout the paper.

Are the no free lunch theorems relevant to real-world model construction? Not directly. They should not be invoked in discussions about the "need for strong inductive biases", or used as an argument that we cannot significantly automate machine learning or science, as they so often are. The assumptions of these theorems — such as datasets drawn from a uniform distribution over all datasets — are completely misaligned with the real world, where data are often highly structured, unlike uniformly sampled datasets (Section 3). There is a valid discussion to be had about the role of inductive biases in model construction, but the no free lunch theorem should play no part in this discussion. Our paper provides evidence, with PAC-Bayes bounds (Section 4.2), and generative likelihoods (Section 4.3 and Section 4.4), that the structure across many real-world datasets is shared to a surprising extent. These findings are aligned with the current movement towards similar transformer based architectures for many tasks, spanning vision, NLP, and tabular data, and away from more specialized models for each task (Figure 1).

How do we build models that are broadly applicable? Discovering the principles for building general purpose models is a fascinating research ambition. An emerging principle is that *we should embrace a flexible hypothesis space, while providing soft encouragement to learn salient structures in many real-world applications* (Wilson & Izmailov, 2020). In other words, we should leave behind restriction biases, often represented by hard architectural constraints, such as strict locality, parameter sharing, and translation equivariance, in favour of softer inductive biases. This principle is clearly exemplified by *residual pathway priors* (RPPs), which turn hard architectural constraints into soft inductive biases for equivariances (e.g., rotation or translation symmetries), efficiently guiding models towards structured solutions where that structure exists (Finzi et al., 2021a). These approaches have particularly strong performance where there are approximate equivariances, without suffering poor performance where there is not. Indeed, more generally, hard constraints are often unrealistic, and soft constraints in practice are sufficient for efficiently finding structure, while protecting against the misspecification and narrow applicability associated with the hard constraints.

We see this principle surfacing in many contexts. Transformers, for instance, lack hard constraints but have recently been found to discover more equivariant solutions even than models with hard constraints (Gruver et al., 2023). Outside deep learning, there is also work showing how soft inductive biases can be used to automate kernel selection on a variety of tasks which previously called for carefully hand-constructed kernels (Benton et al., 2019; Wilson & Adams, 2013; Lloyd et al., 2014). In this paper, we provide evidence that models can be made data efficient, while providing strong performance in larger data

regimes, by embracing flexibility combined with soft inductive biases (Section 5).

Should the model (i.e. learner) we use depend on how much training data are available? Should we use a different model on $n = 5$ points than $n = 10^6$ points? The conventional wisdom is yes. We believe in principle, the answer is no. In practice, it depends.

- The *conventional wisdom* is that a model with a flexible hypothesis space cannot be well-determined from a small number of data points, and therefore will likely overfit and provide poor generalization. This conventional wisdom partly arises from early generalization theory, regarding Rademacher complexity (Mohri & Rostamizadeh, 2009), and VC dimension (Vapnik, 1998), as well as empirical overfitting of large models to small datasets. But there are now many empirical counterexamples to this thinking; indeed, we often use flexible models with many more parameters than datapoints, even without explicit regularization, and achieve good generalization (Zhang et al., 2016). It is not uncommon to train a neural network with millions of parameters on problems with tens of thousands of points, such as a ResNet-18 (He et al., 2016) on CIFAR-10 (Krizhevsky, 2009).

  But such counterexamples in fact long predate deep learning. For instance, Gaussian processes, which with standard kernels represent models with an infinite number of parameters, can perfectly fit most datasets. However, GPs rarely overfit and in fact generalize particularly well on small datasets (Rasmussen & Williams, 2006). Indeed, the examples of *benign overfitting* in *Understanding Deep Learning Requires Rethinking Generalization* (Zhang et al., 2016), where a convolutional net perfectly fits images with random labels, can be reproduced by several other model classes (Smith & Le, 2018; Wilson & Izmailov, 2020). Moreover, there are generalization theories, such as PAC-Bayes, which do not penalize large hypothesis spaces, but instead focus on which solutions are *likely* under the model, rather than which solutions are expressible (McAllester, 1998; Alquier, 2021).

- *In principle*, our beliefs about the generative process for data would not typically depend on how many points we happen to observe (Neal, 1996). Moreover, honestly representing our beliefs should not lead to poor generalization in an ideal modeling paradigm. Typically we would believe that there are many solutions that are a priori *possible*, even if most of them are not a priori *likely*. We should therefore embrace a large hypothesis space, regardless of how much data we have access to (Wilson & Izmailov, 2020). While a small model may be quickly constrained by the data,

it will be erroneously constrained, which can be particularly problematic in representing epistemic uncertainty (which solutions are possible given limited information) (MacKay, 1995). We have shown several examples of how we can construct models that are competitive in both small and large data regimes, by embracing the principle of a flexible hypothesis space combined with soft inductive biases towards simplicity.

- *In practice*, it can sometimes be computationally cumbersome to work with large models (or learners incorporating multiple architectures as in Section 5.1) on small data, such that it is not worth the computational trade-offs. It can also be more convenient to take a "brute force" approach, manually scaling models with more data points. But if one is willing to put the necessary care into representing our honest assumptions, then a single model approach is likely to provide better performance across different data regimes. We have seen here how this single model approach can be possible, by combining flexibility with soft simplicity biases (Section 5.2).

How crucial are the implicit biases of the optimization procedure in finding simple generalizable solutions? The ability for deep models to generalize is often attributed to the implicit biases of stochastic optimizers like SGD or Adam (Amir et al., 2021; Wu et al., 2020). However, good generalization can be achieved with full-batch training procedures, even in the absence of explicit regularization (Geiping et al., 2021; Izmailov et al., 2021) or without gradient-based optimization at all (Chiang et al., 2023). Indeed, while stochastic optimizers can sometimes confer a small gain in performance, it is largely the design of the architecture that makes generalizable solutions more easily accessible (occupy a greater volume in the loss landscape), rather than the optimizer selecting for particularly generalizable low-loss solutions. For example, as we increase the dimensionality of the parameter space, flat solutions occupy an exponentially increasing volume, and these solutions often provide more compressed representations of data, typically leading to better generalization (Hochreiter & Schmidhuber, 1997; Kawaguchi et al., 2017; Huang et al., 2020; Maddox et al., 2020; Foret et al., 2020). In Bayesian terms, the posterior volume of low-loss solutions that provide good generalization is much larger than the volume of low-loss solutions that provide poor generalization, as a consequence of the architectural prior (Wilson & Izmailov, 2020).

Are recent demonstrations of GPT and large language models surprising? Large language models have recently shown versatility, capable of problem solving in a wide variety of contexts, ranging from passing the bar exam (Katz

et al., 2023), university-level physics or chemistry problems (Lewkowycz et al., 2022), or even general-purpose reasoning in science and math (Taylor et al., 2022). Part of the reason these demonstrations are perceived as surprising is due to the influence of the no free lunch theorems, which imply one model can't effectively solve many tasks. We have shown how this intuition is misguided. Real-world modeling problems are often highly structured, and that structure is shared across problems to a large extent. We have also demonstrated that GPT-3 and GPT-2 models, even randomly initialized, have a simplicity bias. This simplicity bias, in combination with their flexibility, makes large language models good candidates as general-purpose problem solvers, though of course they will not be a magic bullet for every problem, or without limitations.

References

Alquier, P. User-friendly introduction to PAC-Bayes bounds. *arXiv preprint arXiv:2110.11216*, 2021.

Amir, I., Koren, T., and Livni, R. Sgd generalizes better than gd (and regularization doesn't help). In *Conference on Learning Theory*, pp. 63–92. PMLR, 2021.

Arora, S., Ge, R., Neyshabur, B., and Zhang, Y. Stronger generalization bounds for deep nets via a compression approach. In *International Conference on Machine Learning*, pp. 254–263. PMLR, 2018.

Benton, G., Maddox, W. J., Salkey, J., Albinati, J., and Wilson, A. G. Function-space distributions over kernels. *Advances in Neural Information Processing Systems*, 32, 2019.

Blei, D. M., Ng, A. Y., and Jordan, M. I. Latent dirichlet allocation. *Journal of Machine Learning Research*, 3 (Jan):993–1022, 2003.

Blier, L. and Ollivier, Y. The description length of deep learning models. *Advances in Neural Information Processing Systems*, 31, 2018.

Brigato, L. and Iocchi, L. A close look at deep learning with small data. In *2020 25th International Conference on Pattern Recognition (ICPR)*, pp. 2490–2497. IEEE, 2021.

Brown, T., Mann, B., Ryder, N., Subbiah, M., Kaplan, J. D., Dhariwal, P., Neelakantan, A., Shyam, P., Sastry, G., Askell, A., et al. Language models are few-shot learners. *Advances in neural information processing systems*, 33:1877–1901, 2020.

Catoni, O. PAC-Bayesian Supervised Classification: the Thermodynamics of Statistical Learning. *Institute of Mathematical Statistics Lecture Notes*, 2007.

Chaitin, G. J. Information-theoretic limitations of formal systems. *Journal of the ACM (JACM)*, 21(3):403–424, 1974.

Chiang, P.-y., Ni, R., Miller, D. Y., Bansal, A., Geiping, J., Goldblum, M., and Goldstein, T. Loss landscapes are all you need: Neural network generalization can be explained without the implicit bias of gradient descent. In *The Eleventh International Conference on Learning Representations*, 2023.

Dalal, N. and Triggs, B. Histograms of oriented gradients for human detection. In *2005 IEEE computer society conference on computer vision and pattern recognition (CVPR'05)*, volume 1, pp. 886–893. Ieee, 2005.

d'Ascoli, S., Kamienny, P.-A., Lample, G., and Charton, F. Deep symbolic regression for recurrent sequences. *arXiv preprint arXiv:2201.04600*, 2022.

De Palma, G., Kiani, B., and Lloyd, S. Random deep neural networks are biased towards simple functions. *Advances in Neural Information Processing Systems*, 32, 2019.

Deng, J., Dong, W., Socher, R., Li, L.-J., Li, K., and Fei-Fei, L. Imagenet: A large-scale hierarchical image database. In *2009 IEEE conference on computer vision and pattern recognition*, pp. 248–255. Ieee, 2009.

Dosovitskiy, A., Beyer, L., Kolesnikov, A., Weissenborn, D., Zhai, X., Unterthiner, T., Dehghani, M., Minderer, M., Heigold, G., Gelly, S., et al. An image is worth 16x16 words: Transformers for image recognition at scale. In *International Conference on Learning Representations*, 2020.

Dziugaite, G. K. and Roy, D. M. Computing nonvacuous generalization bounds for deep (stochastic) neural networks with many more parameters than training data. *arXiv preprint arXiv:1703.11008*, 2017.

Dziugaite, G. K. and Roy, D. M. Data-dependent pac-bayes priors via differential privacy. *Advances in neural information processing systems*, 31, 2018.

Dziugaite, G. K., Hsu, K., Gharbieh, W., Arpino, G., and Roy, D. On the role of data in pac-bayes bounds. In *International Conference on Artificial Intelligence and Statistics*, pp. 604–612. PMLR, 2021.

Fernández-Delgado, M., Cernadas, E., Barro, S., and Amorim, D. Do we need hundreds of classifiers to solve real world classification problems? *The journal of machine learning research*, 15(1):3133–3181, 2014.

Finzi, M., Benton, G., and Wilson, A. G. Residual pathway priors for soft equivariance constraints. In *Advances in Neural Information Processing Systems*, volume 34, pp. 30037–30049, 2021a.

Finzi, M., Welling, M., and Wilson, A. G. A practical method for constructing equivariant multilayer perceptrons for arbitrary matrix groups. In *International Conference on Machine Learning*, pp. 3318–3328. PMLR, 2021b.

Foret, P., Kleiner, A., Mobahi, H., and Neyshabur, B. Sharpness-aware minimization for efficiently improving generalization. *arXiv preprint arXiv:2010.01412*, 2020.

Fortnow, L. Kolmogorov complexity. In *Aspects of Complexity (Short Courses in Complexity from the New Zealand Mathematical Research Institute Summer 2000 Meeting, Kaikoura)*, volume 4, pp. 73–86, 2000.

Fu, Z., Lam, W., So, A. M.-C., and Shi, B. A theoretical analysis of the repetition problem in text generation. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pp. 12848–12856, 2021.

Geiping, J., Goldblum, M., Pope, P. E., Moeller, M., and Goldstein, T. Stochastic training is not necessary for generalization. *arXiv preprint arXiv:2109.14119*, 2021.

Germain, P., Bach, F., Lacoste, A., and Lacoste-Julien, S. PAC-Bayesian theory meets Bayesian inference. *Advances in Neural Information Processing Systems*, 29, 2016.

Giraud-Carrier, C. and Provost, F. Toward a justification of meta-learning: Is the no free lunch theorem a showstopper. In *Proceedings of the ICML-2005 Workshop on Meta-learning*, pp. 12–19, 2005.

Gómez, D. and Rojas, A. An empirical overview of the no free lunch theorem and its effect on real-world machine learning classification. *Neural computation*, 28(1):216–228, 2016.

Grinsztajn, L., Oyallon, E., and Varoquaux, G. Why do tree-based models still outperform deep learning on tabular data? *arXiv preprint arXiv:2207.08815*, 2022.

Gruver, N., Finzi, M., Goldblum, M., and Wilson, A. G. The lie derivative for measuring learned equivariance. In *International Conference on Learning Representations*, 2023.

Gulati, A., Qin, J., Chiu, C.-C., Parmar, N., Zhang, Y., Yu, J., Han, W., Wang, S., Zhang, Z., Wu, Y., et al. Conformer: Convolution-augmented transformer for speech recognition. *Proc. Interspeech 2020*, pp. 5036–5040, 2020.

He, K., Zhang, X., Ren, S., and Sun, J. Deep residual learning for image recognition. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 770–778, 2016.

Hinton, G. E. and Van Camp, D. Keeping the neural networks simple by minimizing the description length of the weights. In *Proceedings of the sixth annual conference on Computational learning theory*, pp. 5–13, 1993.

Hochreiter, S. and Schmidhuber, J. Flat minima. *Neural computation*, 9(1):1–42, 1997.

Hollmann, N., Müller, S., Eggensperger, K., and Hutter, F. Tabpfn: A transformer that solves small tabular classification problems in a second. *arXiv preprint arXiv:2207.01848*, 2022.

Holtzman, A., Buys, J., Du, L., Forbes, M., and Choi, Y. The curious case of neural text degeneration. In *International Conference on Learning Representations*, 2020.

Huang, W. R., Emam, Z. A. S., Goldblum, M., Fowl, L. H., Terry, J., Huang, F., and Goldstein, T. Understanding generalization through visualizations. In *"I Can't Believe It's Not Better!"NeurIPS 2020 workshop*, 2020.

Hume, D. *Philosophical Essays Concerning Human Understanding*. A. Millar, 1748.

Hutter, M. A theory of universal artificial intelligence based on algorithmic complexity. *arXiv preprint cs/0004001*, 2000.

Izmailov, P., Vikram, S., Hoffman, M. D., and Wilson, A. G. G. What are bayesian neural network posteriors really like? In *International conference on machine learning*, pp. 4629–4640. PMLR, 2021.

Jaegle, A., Borgeaud, S., Alayrac, J.-B., Doersch, C., Ionescu, C., Ding, D., Koppula, S., Zoran, D., Brock, A., Shelhamer, E., et al. Perceiver io: A general architecture for structured inputs & outputs. In *International Conference on Learning Representations*, 2021.

Katz, D. M., Bommarito, M. J., Gao, S., and Arredondo, P. Gpt-4 passes the bar exam. *Available at SSRN 4389233*, 2023.

Kawaguchi, K., Kaelbling, L. P., and Bengio, Y. Generalization in deep learning. *arXiv preprint arXiv:1710.05468*, 2017.

Kolmogorov, A. N. On tables of random numbers. *Sankhyā: The Indian Journal of Statistics, Series A*, pp. 369–376, 1963.

Krizhevsky, A. Learning Multiple Layers of Features from Tiny Images. 2009. URL https://www.cs.toronto.edu/~kriz/learning-features-2009-TR.pdf.

Langford, J. and Seeger, M. Bounds for averaging classifiers. 2001.

Lattimore, T. and Hutter, M. No free lunch versus occam's razor in supervised learning. In *Algorithmic Probability and Friends. Bayesian Prediction and Artificial Intelligence*, pp. 223–235. Springer, 2013.

Lee, S. H., Lee, S., and Song, B. C. Vision transformer for small-size datasets. *arXiv preprint arXiv:2112.13492*, 2021.

Lewkowycz, A., Andreassen, A., Dohan, D., Dyer, E., Michalewski, H., Ramasesh, V., Slone, A., Anil, C., Schlag, I., Gutman-Solo, T., et al. Solving quantitative reasoning problems with language models. *arXiv preprint arXiv:2206.14858*, 2022.

Li, M., Vitányi, P., et al. *An introduction to Kolmogorov complexity and its applications*, volume 3. Springer, 2008.

Liu, Z., Lin, Y., Cao, Y., Hu, H., Wei, Y., Zhang, Z., Lin, S., and Guo, B. Swin transformer: Hierarchical vision transformer using shifted windows. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pp. 10012–10022, 2021.

Lloyd, J., Duvenaud, D., Grosse, R., Tenenbaum, J., and Ghahramani, Z. Automatic construction and natural-language description of nonparametric regression models. In *Proceedings of the AAAI Conference on Artificial Intelligence*, 2014.

Lotfi, S., Finzi, M., Kapoor, S., Potapczynski, A., Goldblum, M., and Wilson, A. G. PAC-Bayes compression bounds so tight that they can explain generalization. In *NeurIPS*, 2022a.

Lotfi, S., Izmailov, P., Benton, G., Goldblum, M., and Wilson, A. G. Bayesian model selection, the marginal likelihood, and generalization. *Proceedings of the International Conference on Machine Learning*, 2022b.

Lotfi, S., Izmailov, P., Benton, G., Goldblum, M., and Wilson, A. G. Bayesian model selection, the marginal likelihood, and generalization. *Journal of Machine Learning Research*, 2023.

MacKay, D. J. Probable networks and plausible predictions?a review of practical Bayesian methods for supervised neural networks. *Network: computation in neural systems*, 6(3):469–505, 1995.

MacKay, D. J. *Information theory, inference and learning algorithms*. Cambridge university press, 2003.

Maddox, W. J., Benton, G., and Wilson, A. G. Rethinking parameter counting in deep models: Effective dimensionality revisited. *arXiv preprint arXiv:2003.02139*, 2020.

McAllester, D. A. Some PAC-Bayesian theorems. In *Proceedings of the eleventh annual conference on Computational learning theory*, pp. 230–234, 1998.

McAuley, J. and Leskovec, J. Hidden factors and hidden topics: understanding rating dimensions with review text. In *Proceedings of the 7th ACM conference on Recommender systems*, pp. 165–172, 2013.

Mingard, C., Skalse, J., Valle-Pérez, G., Martínez-Rubio, D., Mikulik, V., and Louis, A. A. Neural networks are a priori biased towards boolean functions with low entropy. *arXiv preprint arXiv:1909.11522*, 2019.

Mnih, A. and Hinton, G. Three new graphical models for statistical language modelling. In *Proceedings of the 24th international conference on Machine learning*, pp. 641–648, 2007.

Mohri, M. and Rostamizadeh, A. Rademacher complexity bounds for non-iid processes. In *Advances in Neural Information Processing Systems*, pp. 1097–1104, 2009.

Müller, S., Hollmann, N., Arango, S. P., Grabocka, J., and Hutter, F. Transformers can do bayesian inference. In *International Conference on Learning Representations*.

Nakkiran, P. Turing-universal learners with optimal scaling laws. *arXiv preprint arXiv:2111.05321*, 2021.

Neal, R. M. *Bayesian learning for neural networks*. Springer Science & Business Media, 1996.

Panayotov, V., Chen, G., Povey, D., and Khudanpur, S. Librispeech: an asr corpus based on public domain audio books. In *2015 IEEE international conference on acoustics, speech and signal processing (ICASSP)*, pp. 5206–5210. IEEE, 2015.

Pearlmutter, B. and Rosenfeld, R. Chaitin-kolmogorov complexity and generalization in neural networks. *Advances in neural information processing systems*, 3, 1990.

Pérez-Ortiz, M., Rivasplata, O., Shawe-Taylor, J., and Szepesvári, C. Tighter risk certificates for neural networks. *The Journal of Machine Learning Research*, 22 (1):10326–10365, 2021.

Petrosian, A. Kolmogorov complexity of finite sequences and recognition of different preictal eeg patterns. In *Proceedings eighth IEEE symposium on computer-based medical systems*, pp. 212–217. IEEE, 1995.

Radford, A., Wu, J., Child, R., Luan, D., Amodei, D., Sutskever, I., et al. Language models are unsupervised multitask learners. *OpenAI blog*, 2019.

Rao, R. B., Gordon, D., and Spears, W. For every generalization action, is there really an equal and opposite reaction? analysis of the conservation law for generalization performance. In *Machine Learning Proceedings 1995*, pp. 471–479. Elsevier, 1995.

Rasmussen, C. E. and Williams, C. K. I. *Gaussian processes for Machine Learning*. The MIT Press, 2006.

Rathmanner, S. and Hutter, M. A philosophical treatise of universal induction. *Entropy*, 13(6):1076–1136, 2011.

Schaffer, C. A conservation law for generalization performance. In *Machine Learning Proceedings 1994*, pp. 259–265. Elsevier, 1994.

Schmidhuber, J. Discovering neural nets with low kolmogorov complexity and high generalization capability. *Neural networks: the official journal of the International Neural Network Society*, 10(5):857–873, 1997.

Schurz, G. The meta-inductivist's winning strategy in the prediction game: A new approach to hume's problem. *Philosophy of Science*, 75(3):278–305, 2008.

Schurz, G. and Thorn, P. A priori advantages of meta-induction and the no free lunch theorem: A contradiction? In *Joint German/Austrian Conference on Artificial Intelligence (Künstliche Intelligenz)*, pp. 236–248. Springer, 2017.

Shalev-Shwartz, S. and Ben-David, S. *Understanding machine learning: From theory to algorithms*. Cambridge university press, 2014.

Smith, S. L. and Le, Q. V. A Bayesian perspective on generalization and stochastic gradient descent. In *International Conference on Learning Representations*, 2018.

Solomonoff, R. J. A formal theory of inductive inference. part i. *Information and control*, 7(1):1–22, 1964.

Somepalli, G., Goldblum, M., Schwarzschild, A., Bruss, C. B., and Goldstein, T. Saint: Improved neural networks for tabular data via row attention and contrastive pre-training. *arXiv preprint arXiv:2106.01342*, 2021.

Szegedy, C., Liu, W., Jia, Y., Sermanet, P., Reed, S., Anguelov, D., Erhan, D., Vanhoucke, V., and Rabinovich, A. Going deeper with convolutions. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 1–9, 2015.

Taylor, R., Kardas, M., Cucurull, G., Scialom, T., Hartshorn, A., Saravia, E., Poulton, A., Kerkez, V., and Stojnic, R. Galactica: A large language model for science. *arXiv preprint arXiv:2211.09085*, 2022.

Trockman, A. and Kolter, J. Z. Patches are all you need? *arXiv preprint arXiv:2201.09792*, 2022.

Vapnik, V. N. Adaptive and learning systems for signal processing communications, and control. *Statistical learning theory*, 1998.

Vaswani, A., Shazeer, N., Parmar, N., Uszkoreit, J., Jones, L., Gomez, A. N., Kaiser, Ł., and Polosukhin, I. Attention is all you need. *Advances in neural information processing systems*, 30, 2017.

Vilalta, R. and Drissi, Y. A perspective view and survey of meta-learning. *Artificial intelligence review*, 18(2): 77–95, 2002.

Wilson, A. and Adams, R. Gaussian process kernels for pattern discovery and extrapolation. In *International conference on machine learning*, pp. 1067–1075, 2013.

Wilson, A. G. and Izmailov, P. Bayesian deep learning and a probabilistic perspective of generalization. *Advances in neural information processing systems*, 33: 4697–4708, 2020.

Witten, I. H., Neal, R. M., and Cleary, J. G. Arithmetic coding for data compression. *Communications of the ACM*, 30(6):520–540, 1987.

Wolpert, D. H. The lack of a priori distinctions between learning algorithms. *Neural computation*, 8(7):1341–1390, 1996.

Wolpert, D. H. and Macready, W. G. No free lunch theorems for optimization. *IEEE transactions on evolutionary computation*, 1(1):67–82, 1997.

Wolpert, D. H., Macready, W. G., et al. No free lunch theorems for search. Technical report, Technical Report SFI-TR-95-02-010, Santa Fe Institute, 1995.

Wu, J., Hu, W., Xiong, H., Huan, J., Braverman, V., and Zhu, Z. On the noisy gradient descent that generalizes as sgd. In *International Conference on Machine Learning*, pp. 10367–10376. PMLR, 2020.

Zelikman, E., Wu, Y., and Goodman, N. D. Star: Boot-strapping reasoning with reasoning. *arXiv preprint arXiv:2203.14465*, 2022.

Zenil, H., Marshall, J. A., and Tegnér, J. Approximations of algorithmic and structural complexity validate cognitive-behavioural experimental results. *arXiv preprint arXiv:1509.06338*, 2015.

Zhang, C., Bengio, S., Hardt, M., Recht, B., and Vinyals, O. Understanding deep learning requires rethinking generalization. *arXiv preprint arXiv:1611.03530*, 2016.

Zhou, W., Veitch, V., Austern, M., Adams, R. P., and Orbanz, P. Non-vacuous generalization bounds at the imagenet scale: a PAC-Bayesian compression approach. In *International Conference on Learning Representations*, 2018.

A. A Kolmogorov No Free Lunch Theorem Proof

Rewriting Equation (1), we have

$$\text{CE}(p) \geq \frac{\ln 2}{n} \left(K(Y|X) - K(p) - 2\log_2 K(p) - c \right).$$

Note that by simply counting all possible programs taking input X, there are less than 2^{k+1} labelings Y with $K(Y|X) \leq k$. Note that there are C^n distinct labelings, from which we are drawing uniformly. So that

$$\begin{aligned}
\mathbb{P}(K(Y|X) > n\log_2 C - m) &= 1 - \mathbb{P}(K(Y|X) \leq n\log_2 C - m) \\
&\geq 1 - \mathbb{P}(K(Y|X) \leq \lceil n\log_2 C \rceil - m) \\
&\geq 1 - \frac{2^{\lceil n\log_2 C \rceil - m + 1}}{C^n} \\
&\geq 1 - 2^{2-m}.
\end{aligned}$$

Alternatively, with probability at least $1 - \delta$,

$$K(Y|X) > n\log_2 C - \log_2 \frac{1}{\delta} - 3.$$

Thus, with probability at least $1 - \delta$, we have for every classifier p,

$$\text{CE}(p) \geq \ln C - \frac{\ln 2}{n} \left(K(p) + 2\log_2 K(p) + \log_2 \delta + c \right).$$

B. PAC-Bayes Compression Experimental Details

For the OpenML tabular classification datasets, we preprocess them first by balancing the classes, subsampling all classes down to the number of examples of the least likely class. This way, when compressing the datasets, any result achieved is nontrivial in contrast with a very class imbalanced dataset. We heavily follow the compression method of (Lotfi et al., 2022a), including the small 9 convolutional architecture which they use to generate their bounds. When cramming the tabular data into this convnet, we combine numerical features with one hot encoded categorical features and then pack these into the pixels of a 1 channel image, using however large an image as necessary to fit each of the different features inside.

With respect to the sizes of the random subspaces that we train the compressed models in, we consider 250,500,1000, and 2000. For tabular label compression, we employ a 2 hidden layer MLP with hidden dimension $k = 192$, and we consider the same 250,500,1000, and 2000 values for subspace dimension. We train for 80 epochs with 20 epochs of quantization at a batch size of 512 using Adam at lr$= 3 \times 10^{-4}$. For image classification label compression, we use the 9-layer convnet with subspace dimensions 2000, 3000, 5000, and we train for 80 epochs using SGD at learning rate 0.1 and quantize for the remaining 20 epochs, at a batch size of 50. For calculating the length of the code for model architecture and decompressor, we need only the implementation of the model, the arithmetic decoder, and the loading of the quantized values. Removing wasted bits, we minified the python file, leading to a size of approximately 2.5KB.

C. GPT-3 Experimental Details

To feed sequences into a model, we split up sequence elements into individual byte-pair encoding tokens corresponding to their decimal digits, and we place comma tokens between sequence elements as delimiters, also beginning every input with an `<|endoftext|>` token. We choose to use the byte-pair encoding of decimal digits with a space inserted before the digit, e.g. ' 0' as this is known to enhance the ability of language models to perform arithmetic (Zelikman et al., 2022). For example, the sequence $10, 11$ will be split up into ['`<|endoftext|>`', ' 1', ' 0', ',', ' 1', ' 1'], and each element of the list is tokenized individually. Then, the log-probability of a sequence is given by the sum of the log-probabilities corresponding to the correct decimal digits in their respective slots of the model's output. Note that various sequences will contain different numbers of decimal digits, and the sequence's log-probability will decrease with every token. Therefore, in order for fair comparison, we limit all sequences to 30 decimal digit tokens and truncate there.

D. Sequence Generation and Completion with Randomly Initialized Language Models

For these experiments, we use Huggingface[2] GPT-2 architectures and pre-trained checkpoints. In order to estimate the probabilities assigned by randomly initialized language models to each bitstring, we generate one million random sequences, ensuring that many instances of each bitstring are generated as there are only $2^{10} = 1024$ bistrings of length 10.

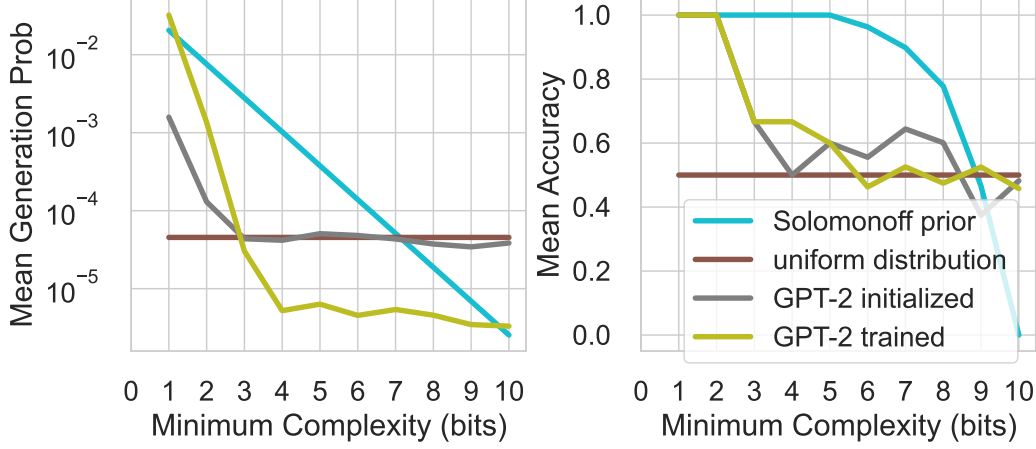We include here plots with the various sizes of GPT-2 architectures in Figure 5, Figure 6, and Figure 7.
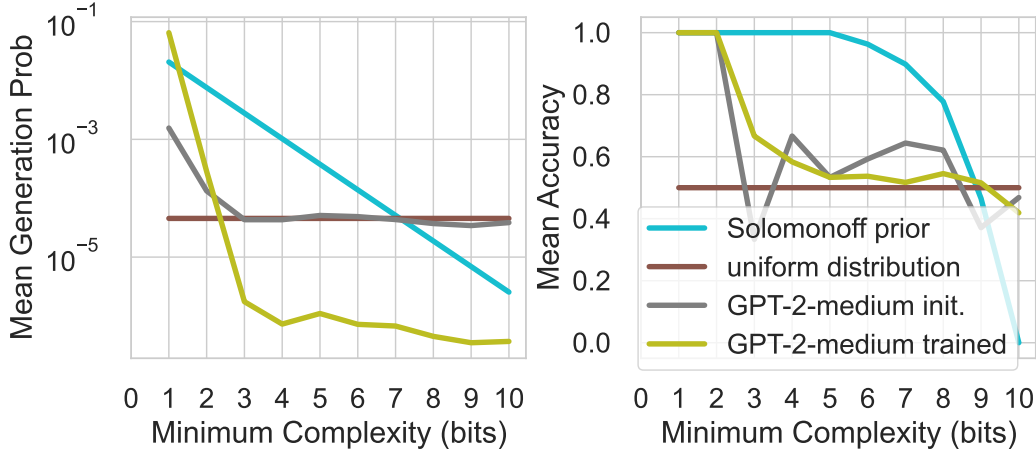

Figure 5: Randomly initialized GPT-2 Base prefers low-complexity sequences generated by bitstring repetition. **Left:** Average log-probability of sequences by complexity. **Right:** Average accuracy by complexity.

Figure 6: Randomly initialized GPT-2 Medium prefers low-complexity sequences generated by bitstring repetition. **Left:** Average log-probability of sequences by complexity. **Right:** Average accuracy.

We additionally include the hypothesis test referenced in Section 4.4. For this experiment, we generate 100,000 length-100 sequences from randomly intialized GPT-2 variants, pre-trained GPT-2 variants, and a uniform distribution. We then perform a one-sided t-test on the null hypothesis that $\mu(K(S_{\text{GPT}})) \geq \mu(K(S_{\mathcal{U}}))$, for both initialized and pre-trained models. Table 1 contains the resulting sample means, t-statistics and p-values. In all cases, we reject the null hypothesis with very low p-values, indicating that language models do prefer to generate low-complexity sequences. Notably, pre-trained language models exhibit an increased simplicity bias, and bigger and better language models even more so.
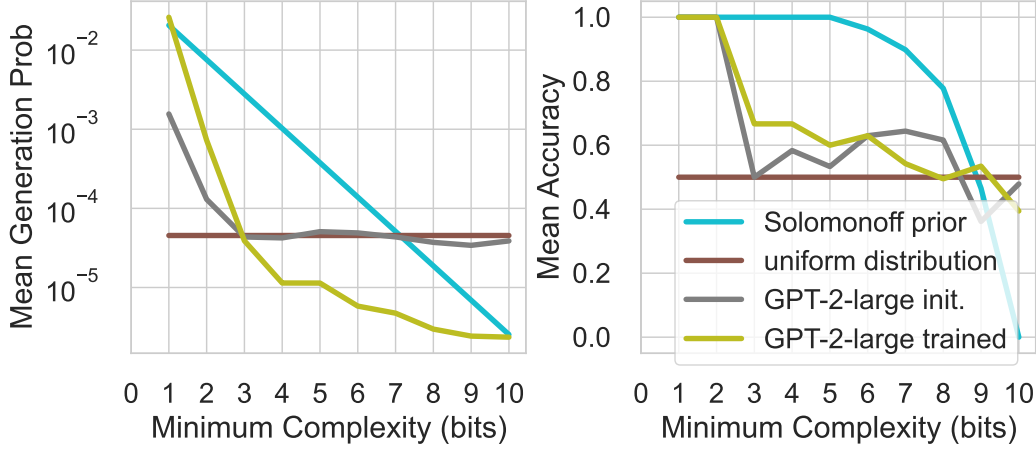
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[2]https://huggingface.co/

Figure 7: Randomly initialized GPT-2 Large prefers low-complexity sequences generated by bitstring repetition. Left: Average log-probability of sequences by complexity. **Right:** Average accuracy.

Table 1: Hypothesis test for language model simplicity bias. t-tests are one-sided, and p-values are rounded to 4 digits. We also report the mean Kolomogorov complexity of sequences generated by each language model and a uniform distribution.

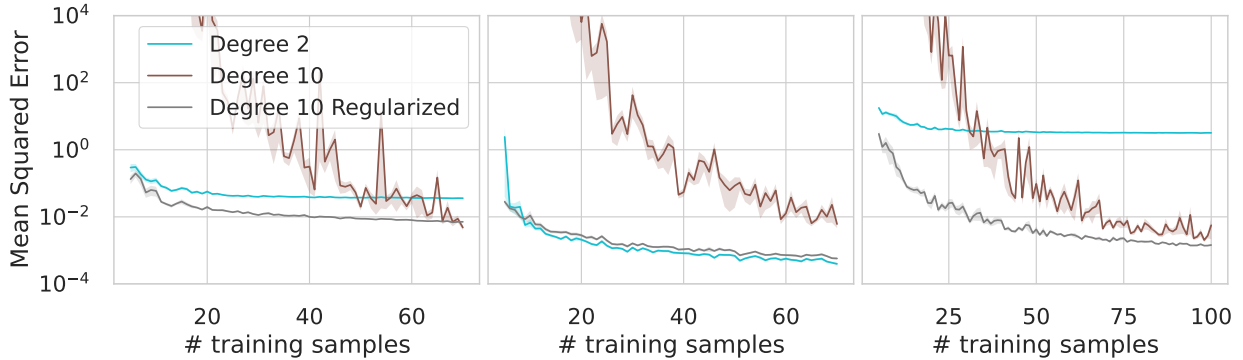| Model	$\overline{K(S_{\text{GPT}})}$	t-statistic	p-value
Uniform Distribution	98.36	-	-
GPT-2 Base Initialized	98.00	-39.95	0.0000
GPT-2 Medium Initialized	97.99	-40.91	0.0000
GPT-2 Large Initialized	98.00	-40.11	0.0000
GPT-2 Base Trained	60.81	-255.17	0.0000
GPT-2 Medium Trained	48.41	-325.16	0.0000
GPT-2 Large Trained	46.34	-342.80	0.0000

Figure 8: High-order polynomials with a complexity penalty can solve problems at a variety of sample sizes. Left: Cosine target function. **Middle:** Degree 2 polynomial target function. **Right:** Degree 10 polynomial target function.

E. Big and Small Training Sets

Polynomial regression. We choose three example target functions on which to perform regression: $\cos(\frac{3\pi}{2}x)$, x^2, and $-36x + 49x^5 - 14x^7 + x^{10}$. Training data is randomly drawn from a uniform distribution over the unit interval, and we add noise to training labels from $\mathcal{N}(0, 0.1)$. In each case, for each dataset size, we average the mean squared error over 100 randomly sampled training sets. For Tikhonov regularized polynomial regression on the cosine and degree 2 polynomial target functions, we use $\alpha = 0.01$, and we use $\alpha = 0.001$ for regression on the degree 10 polynomial target function.

Image classification with neural networks. For ImageNet trained models, we employ publicly available checkpoints

from `torchvision`[3]. We train models on CIFAR-10 and CIFAR-100 for 200 epochs with initial learning rate 0.1 and cosine annealing along with horizontal flip and random crop augmentations. We use SGD with momentum 0.9 and batches of size 128. All CIFAR images are rescaled to 224×224 so that we can use an identical model for ImageNet and CIFAR data. In order to learn the parameter c controlling the convex combination of models, we perform 10 epochs of training, where the models' parameters are frozen, and we apply weight decay with coefficient 10^{-5}. We learn the parameter c using SGD with momentum 0.9 and batch size 128, initial learning rate 0.1, and cosine annealing.

Table 2: Combinations of large and small architectures form single models that achieve high test accuracy on all dataset sizes. "GoogLeNet + ViT" denotes a model formed as a convex combination of the logits of the two constituent models with weight decay on the parameter c controlling the convex combination which multiplies the logits of the larger model, ensuring that the small model is preferred as long as it fits the data.

Model	CIFAR-10	CIFAR-100	ImageNet
GoogLeNet	93.840 %	75.160 %	69.778 %
ViT-B/16	72.020 %	48.140 %	81.072 %
Swin-B	74.710 %	64.200 %	83.582 %
GoogLeNet + ViT	93.860 %	71.990 %	81.090 %
GoogLeNet + Swin	93.760 %	75.360 %	83.150 %

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[3]https://pytorch.org/vision/stable/index.html