The k-Facility Location Problem Via Optimal Transport: A Bayesian Study of the Percentile Mechanisms

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1 Introduction

- **Scope of Mechanism Design**: Procedures to aggregate agents' private information to optimize a social objective, while addressing self-interested behavior and manipulation.
- **Truthfulness in Mechanism Design**: A crucial property ensuring no agent benefits from misreporting their private information, often requiring a compromise on social objective optimization.
- **Approximation Ratio**: Introduced by Nisan and Ronen to quantify efficiency loss, comparing the social objective achieved by a truthful mechanism to the optimal objective over all possible reports.
- **k-Facility Location Problem (k-FLP)**:
  - Central authority locates k facilities among n agents.
  - Agents prefer facilities to be as close as possible.
  - Applications: Disaster relief, supply chain management, healthcare, public facilities accessibility.
- **Algorithmic Mechanism Design in k-FLP**:
  - Initiated by Procaccia and Tennenholtz, focusing on locating one facility among agents on a line.
  - Various methods with fixed approximation ratios for one or two facilities on different structures (e.g., trees, circles, graphs, metric spaces) have been developed.
  - Positive outcomes for limited agents or up to 2 facilities; **negative results** for $k \geq 3$ facilities.
- **Challenges for k $\geq$ 3 Facilities**:
  - No deterministic, anonymous, and truthful mechanisms with bounded approximation ratio for $k \geq 3$ on the line.
  - Possible to define truthful mechanisms with bounded approximation ratio for specific cases (number of agents = facilities + 1) or using randomized mechanisms.
- **Percentile Mechanisms**:
  - そもそもこれを理解したほうがよさそう
  - 下のほうで例を挙げて理解しました。したつもりカモ。。。。
  - Class of truthful mechanisms for generic k-FLP introduced in [45].
  - Typically have an **unbounded approximation ratio**.
  - Study shows they can be optimal if agents' types are sampled from a probability distribution (Bayesian Mechanism Design).
  - Main contribution: Selecting a percentile mechanism that asymptotically minimizes the expected social objective.

1.1 Our Contribution

- Comprehensive investigation of k-Facility Location Problem (k-FLP) from a Bayesian Mechanism Design perspective.

- Assumption: Agents' positions on the line follow a distribution μ [18,29].
- **Focus on percentile mechanisms** [45] and explore conditions for bounded Bayesian approximation ratio.
- **Establish that percentile mechanisms exhibit different performances based on the measure** μ.
- Identify the optimal percentile mechanism tailored to distribution μ.
- **Connection between k-FLP and a projection problem in the Wasserstein space.**
  - ここを一番理解したい
  - 理解できればそこからは OT の教科書に戻るだけ
  - 最適化した距離が何を表すのか。degree of happiness? hate?
  - 表裏一体か。
- ↓からはあまり興味ない。
- Use of tools and techniques from Optimal Transport theory to approach k-FLP.
- Demonstrate convergence of the expected cost ratio to a bounded value as the number of agents increases.
- Characterize both the limit value of the ratio and the speed of convergence.
- Use of Bahadur's representation formula to relate the j-th ordered statistic of a random variable to a suitable quantile.
- Derive a bound on the performances of percentile mechanisms for any finite number of agents.
- Tackle the problem of retrieving the best percentile mechanism for distribution μ and number of facilities k.
- Show existence of a percentile vector $v_\mu \in (0,1)^k$ that induces the optimal percentile mechanism.
- Characterize the percentile vector as the solution to a system of k equations.
- Compute the optimal percentile vector for **common probability measures** (e.g., Uniform, Gaussian distributions).
  - Gumbel distribution でも成り立ってくれればめっちゃ嬉しい
- Show that the optimal percentile vector is invariant under positive affine transformations of the probability measures.
- Present a study on the stability of the optimal percentile vector.
- Demonstrate that the Bayesian approximation ratio limit deviates from 1 proportionally to the infinity Wasserstein distance between μ and an approximation $\tilde{\mu}$.
- Highlight that the more precise the approximation of μ, the better the performance of the optimal percentile mechanism.

2 Preliminaries

- **The k-Facility Location Problem (k-FLP)**:
  - Given a set of **self-interested agents** $\mathcal{N} = [n] := \{1, 2, \ldots, n\}$.
  - Denote the set of their positions as $\mathcal{X} := \{x_i\}_{i\in[n]}$ over $\mathbb{R}$.
  - Assume agents are indexed such that positions x_i are in non-decreasing order.
    * agent は数直線上にいる
  - Vector $\boldsymbol{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ contains the elements of $\mathcal{X}$.
  - k facilities are located at entries of vector $\boldsymbol{y} := (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$.
  - An agent at position x_i incurs a cost $c_i(x_i, \boldsymbol{y}) = \min_{j\in[k]} |x_i - y_j|$ to access a facility.
  - Social Cost (SC) of $\boldsymbol{y}$ is the sum of all agents' costs: $SC(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i\in[n]} c_i(x_i, \boldsymbol{y})$.
  - Goal: Find locations for k facilities that minimize $SC(\boldsymbol{x}, \boldsymbol{y})$.
  - Rescaled Social Cost: $SC(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{n}\sum_{i\in[n]} c_i(x_i, \boldsymbol{y})$.
- **Mechanism Design and Worst-Case Analysis**:
  - A k-facility location **mechanism is a function** $f : \mathbb{R}^n \to \mathbb{R}^k$.
  - Takes agents' reports $\boldsymbol{x}$ and returns a set of k locations $\boldsymbol{y}$.
  - **Agents may misreport their positions to reduce their incurred cost.**

- A mechanism f is truthful (**strategyproof**) if, for every agent, reporting their true position minimizes their cost: $c_i(x_i, f(\boldsymbol{x})) \leq c_i(x_i, f(\boldsymbol{x}_{-i}, x_i'))$ for any misreport $x_i' \in \mathbb{R}$, where $\boldsymbol{x}_{-i}$ is $\boldsymbol{x}$ without its i-th component.
  - agent が嘘をつくと逆にコストが増える関数を仮定
  - すると agent は self-interested でありながらも正直に報告するよう動機づけができると解釈した
  - 要するに各 agent は正直に位置申告するのが最適
- Truthful mechanisms prevent misreporting **but result in efficiency loss.**
- To evaluate this efficiency loss, approximation ratio of a truthful mechanism f (introduced by Nisan and Ronen) is defined as:

$$\mathrm{ar}(f) := \sup_{\boldsymbol{x} \in \mathbb{R}^n} \frac{SC_f(\boldsymbol{x})}{SC_{\mathrm{opt}}(\boldsymbol{x})},$$

  where $SC_f(\boldsymbol{x})$ is the Social Cost of placing facilities at $f(\boldsymbol{x})$ and $SC_{\mathrm{opt}}(\boldsymbol{x})$ is the optimal Social Cost.
- **The worst-case approximation ratio defined above is referred to as the approximation ratio.**
- Evaluating a mechanism f from its approximation ratio is known as the worst-case analysis of f.

- **Bayesian Analysis**:
  - **Bayesian Mechanism Design**:
    - Assumes agents' types follow a probability distribution.
    - Performance of mechanisms is studied from a probabilistic viewpoint.
    - Each agent's type is described by a random variable X_i.
    - Each X_i is identically distributed according to a law μ and independent from other random variables.
  - **Truthful Mechanism**:
    - A mechanism is truthful if, for every agent i, the following holds:

$$\mathbb{E}_{\boldsymbol{X}_{-i}}\left[c_i\left(x_i, f\left(x_i, \boldsymbol{X}_{-i}\right)\right)\right] \leq \mathbb{E}_{\boldsymbol{X}_{-i}}\left[c_i\left(x_i, f\left(x_i', \boldsymbol{X}_{-i}\right)\right)\right] \quad \forall x_i \in \mathbb{R},$$

      where x_i is agent i's true type, $\boldsymbol{X}_{-i}$ is the $(n-1)$-dimensional random vector describing other agents' types, and $\mathbb{E}_{\boldsymbol{X}_{-i}}$ is the expectation with respect to the joint distribution of $\boldsymbol{X}_{-i}$.
      - $\mathbb{E}_{\boldsymbol{X}_{-i}}$ is the expectation with respect to the joint distribution of $\boldsymbol{X}_{-i}$
      - これ何？？
      - strategyproof との違いがわからない
  - **β-Approximation**:
    - A mechanism f is a β-approximation if:

$$\mathbb{E}\left[SC_f\left(\boldsymbol{X}_n\right)\right] \leq \beta \mathbb{E}\left[SC_{\mathrm{opt}}\ \left(\boldsymbol{X}_n\right)\right],$$

    - The lower the β, the better the mechanism.
    - 不等式右辺が optimal なので $\beta \geq 1$
  - **Bayesian Approximation Ratio**:
    - Defined as the ratio between the expected Social Cost of a mechanism and the expected Social Cost of the optimal solution.
    - For a mechanism f, the Bayesian approximation ratio is:

$$B_{\mathrm{ar}}^{(n)}(f) := \frac{\mathbb{E}\left[SC_f\left(\boldsymbol{X}_n\right)\right]}{\mathbb{E}\left[SC_{\mathrm{opt}}\ \left(\boldsymbol{X}_n\right)\right]},$$

    - The expected value is taken over the joint distribution of the vector $\boldsymbol{X}_n := (X_1, \ldots, X_n)$.
    - If $B_{\mathrm{ar}}^{(n)}(f) < +\infty$, then f is a $B_{\mathrm{ar}}^{(n)}(f)$ approximation.
  - **Notation**:
    - Use $\boldsymbol{x}$ to denote the vector containing the agents' reports and the agents' real positions interchangeably.
    - Use the capital letter $\boldsymbol{X}_n$ to denote the random vector describing the agents' types.
- **The Percentile Mechanisms**:
  - **Definition**:

    * Given a vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_k)$, such that $0 \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq 1$.
    * The percentile mechanism induced by $\boldsymbol{v}$, denoted as $\mathcal{PM}_{\boldsymbol{v}}$, proceeds as follows:
      · **Step (i)**: The mechanism designer collects all the reports of the agents, namely $\{x_1, \ldots, x_n\}$ and reorders them non-decreasingly. Assume the reports are already ordered, i.e., $x_i \leq x_{i+1}$.
      · **Step (ii)**: The designer places the k facilities at the positions $y_j = x_{i_j}$, where $i_j = \lfloor (n-1)v_j \rfloor + 1$.
  - **Properties**:
    * If the values x_i are sampled from a distribution, the output of any percentile mechanism is composed by the $(\lfloor (n-1)v_j \rfloor + 1)$-th order statistics of the sample.
    * Percentile mechanisms are truthful whenever the cost of an agent placed at x_i is $c_i = \min_{j \in [k]} |x_i - y_j|$, where y_j are the positions of the facilities.
    * **エージェントの位置**: $\{2, 8, 4, 6, 1\}$
    * **reorder non–decreasingly**: $\{1, 2, 4, 6, 8\}$
    * **ベクトル $\boldsymbol{v}$**: $\{0.2, 0.5, 0.8\}$ (3 つの施設を配置する)
      · $v_1 = 0.2$ **の場合**:

$$i_1 = \lfloor (5-1) \cdot 0.2 \rfloor + 1 = \lfloor 4 \cdot 0.2 \rfloor + 1 = \lfloor 0.8 \rfloor + 1 = 1$$

      1 番目の順序統計量は 1
      · $v_2 = 0.5$ **の場合**:

$$i_2 = \lfloor (5-1) \cdot 0.5 \rfloor + 1 = \lfloor 4 \cdot 0.5 \rfloor + 1 = \lfloor 2 \rfloor + 1 = 3$$

      3 番目の順序統計量は 4
      · $v_3 = 0.8$ **の場合**:

$$i_3 = \lfloor (5-1) \cdot 0.8 \rfloor + 1 = \lfloor 4 \cdot 0.8 \rfloor + 1 = \lfloor 3.2 \rfloor + 1 = 4$$

      4 番目の順序統計量は 6
      · 施設の配置: メカニズムは、1 番目の順序統計量（1)、3 番目の順序統計量（4)、および 4 番目の順序統計量（6）の位置に施設を配置する
      · したがって、施設は位置 1、4、6 に配置される
  - **Approximation Ratio**:
    * For $k > 2$, the approximation ratio of any percentile mechanism becomes unbounded.
    * Percentile mechanisms are anonymous and deterministic, hence $\mathrm{ar}(\mathcal{PM}_{\boldsymbol{v}}) = +\infty$ for every percentile vector $\boldsymbol{v}$.
  - **Truthfulness**:
    * Since percentile mechanisms are truthful in the classic setting, they also retain their truthfulness within the Bayesian framework [29].
- **Basic Notions on Optimal Transport**:
  - **Probability Measures**:
    * $\mathcal{P}(\mathbb{R})$: Set of probability measures over $\mathbb{R}$
    * For $\gamma \in \mathcal{P}(\mathbb{R})$, $\mathrm{spt}(\gamma) \subset \mathbb{R}$ denotes the support of γ
    * The support is the smallest closed set $C \subset \mathbb{R}$ such that $\gamma(C) = 1$
  - **Probability Measures with Finite Support**:
    * $\mathcal{P}_k(\mathbb{R})$: Set of probability measures over $\mathbb{R}$ whose support consists of k points
    * $\nu \in \mathcal{P}_k(\mathbb{R})$ if and only if $\nu = \sum_{j=1}^{k} \nu_j \delta_{x_j}$, where $x_j \in \mathbb{R}$, $\nu_j \geq 0$, $\sum_{j=1}^{k} \nu_j = 1$, and δ_{x_j} is Dirac's delta centered at x_j[*1]
  - **Wasserstein Distance**:

[*1] 可測空間 $(X, \mathcal{F})$ に対し，$\{x\} \in \mathcal{F}$ とする。このとき，$A \in \mathcal{F}$ に対し

$$\delta_x(A) = \begin{cases} 1 & x \in A, \\ 0 & x \in X \backslash A \end{cases}$$

となる測度をディラック測度という。

- $*$ Given two measures $\alpha, \beta \in \mathcal{P}(\mathbb{R})$, the Wasserstein distance $W_1(\alpha, \beta)$ is defined as:

$$W_1(\alpha,\beta) = \min_{\pi\in\Pi(\alpha,\beta)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|\, d\pi$$

- $*$ $\Pi(\alpha, \beta)$ is the set of probability measures over $\mathbb{R}\times\mathbb{R}$ whose first marginal is equal to α and the second marginal is equal to β
- $*$ The infinity Wasserstein distance $W_\infty(\alpha,\beta)$ is defined as:

$$W_\infty(\alpha,\beta) = \min_{\pi\in\Pi(\alpha,\beta)} \max_{(x,y)\in\mathrm{spt}(\pi)} |x-y|$$

- $*$ Both W_1 and W_∞ are metrics over $\mathcal{P}(\mathbb{R})$

– **References**:

- $*$ For a complete introduction to Optimal Transport theory, refer to [47]

- **Basic Assumptions**:
  - **Underlying Distribution** μ:
    - $*$ The measure μ is absolutely continuous with density ρ_μ
    - $*$ The support of μ is an interval (bounded or unbounded) and ρ_μ is strictly positive on the interior of the support
    - $*$ The density function ρ_μ is differentiable on the support of μ
  - **Cumulative Distribution Function (c.d.f.)**:
    - $*$ The c.d.f. F_μ of a probability measure μ satisfying these properties is locally bijective
    - $*$ The pseudo-inverse function of F_μ, denoted $F_\mu^{[-1]}$, is well-defined on $(0,1)$

3 The Bayesian Analysis of the Percentile Mechanism

- Study the percentile mechanisms within the Bayesian Mechanism Design framework
- Consider a scenario where agents' reports are drawn from a **shared** distribution μ
- The distribution μ satisfies the basic assumptions outlined in Section 2
- Establish a connection between the k-Facility Location Problem (k-FLP) and the Wasserstein distance
- Use the connection to investigate the convergence behaviour of the Bayesian approximation ratio
- Focus on the scenario as the number of agents tends to infinity

3.1 The k-FLP as a Wasserstein Projection problem

- Given a vector $\boldsymbol{x} := (x_1, x_2, \ldots, x_n)$ containing the reports of n agents
- Define the measure $\mu_{\boldsymbol{x}} := \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$
- Using the map $\boldsymbol{x} \to \mu_{\boldsymbol{x}}$, associate any agents' profile to a probability measure in $\mathcal{P}_n(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$
- Consider the following minimization problem:

$$\min_{\lambda\in\mathcal{P}_k(\mathbb{R})} W_1(\mu_{\boldsymbol{x}}, \lambda) \tag{5}$$

  - 今回の問題は外部性を持たない (agent 同士のいざこざがない) ため W_1 でイケる
  - エージェントの報告位置: $\{2, 5, 8, 10\}$
  - これに基づく分布 $\mu_{\boldsymbol{x}}$ は次のように定義される：

$$\mu_{\boldsymbol{x}} = \frac{1}{4}\delta_2 + \frac{1}{4}\delta_5 + \frac{1}{4}\delta_8 + \frac{1}{4}\delta_{10}$$

  - 最小化問題: サポートが $k=2$ の分布 λ を見つける。つまり、次の形の分布を見つける：

$$\lambda = \nu_1\delta_{y_1} + \nu_2\delta_{y_2}$$

ここで、$\nu_1+\nu_2 = 1$ かつ $\nu_1, \nu_2 \geq 0$

  - この問題の目的は、$\mu_{\boldsymbol{x}}$ と λ の間の Wasserstein 距離を最小化するような λ を見つけること
  - これにより、エージェントの報告位置を基にして、2 つの位置に施設を配置することになる
- Due to the metric properties of W_1, this problem is also known as the Wasserstein projection problem on $\mathcal{P}_k(\mathbb{R})$
- Since $\mathcal{P}_k(\mathbb{R})$ is closed with respect to any W_1 metric, any Wasserstein projection problem admits at least one solution [3]
- When $\mu_{\boldsymbol{x}}$ is clear from the context, denote the solution to this problem as $\nu^{(k,n)}$
- Given a measure ζ, say that ν is the projection of ζ over $\mathcal{S} \subset \mathcal{P}(\mathbb{R})$ with respect to W_1 if $\nu \in \mathcal{S}$ and $W_1(\zeta, \nu) \leq W_1(\zeta, \rho)$ for every $\rho \in \mathcal{S}$
  - 射影とは、ある対象を別の対象に最も近づける操作のこと
  - ζ と一番距離が近いのが ν だからそれはそう
- In particular, $\nu^{(k,n)}$ is the projection of $\mu_{\boldsymbol{x}}$ over $\mathcal{P}_k(\mathbb{R})$ with respect to W_1
- The starting point of the Bayesian analysis of percentile mechanisms connects the k-FLP to a Wasserstein projection problem
- **The objective value of the Wasserstein projection problem is the same as the objective value of the k-FLP**

Theorem 1. *Let $\boldsymbol{x}$ be the reports of n agents. Let $\boldsymbol{y}$ be the solution to the k-FLP, i.e., the facility locations that minimize the Social Cost. Then the set $\{y_j\}_{j\in[k]}$ is the support of a measure $\nu^{(k,n)}$ that solves problem (5). Moreover, we have that*

$$SC_{opt}(\boldsymbol{x}) = W_1\left(\mu_{\boldsymbol{x}}, \nu^{(k,n)}\right) = \min_{\lambda \in \mathcal{P}_k(\mathbb{R})} W_1\left(\mu_{\boldsymbol{x}}, \lambda\right).$$

Vice-versa, if $\nu \in \mathcal{P}_k(\mathbb{R})$ is a solution to problem (5), then its support $\{y_j\}_{j\in[k]}$ is a solution to the k-FLP.

Proof.

- Let $\boldsymbol{x}$ be the vector containing the reports of n agents, and let $\boldsymbol{y}$ be the vector containing the optimal location for k facilities when the agents are located according to $\boldsymbol{x}$.
- Assume that the closest facility to each agent x_i is **unique** so that the sets A_j, defined as $A_j := \left\{x_i : \min_{l\in[k]} |x_i - y_l| = |x_i - y_j|\right\}$, are well-defined and disjoint.
  - 各 agent に対してどの集合に属するかが明確に決まる、かつ、一つの施設にのみ割り当てられる
  - ??
  - こいつは x_i の集合？ y_j の集合？ x_i の集合っぽいけどなぜ A の添え字が j 何だろうか
  - y_j に一番近い agent の集合ですね。
- Show that, given an optimal facility location $\boldsymbol{y}$, it is possible to retrieve a measure $\nu \in \mathcal{P}_k(\mathbb{R})$ that solves the projection problem (5) and whose support is $\{y_j\}_{j\in[k]}$.
- For every y_j, set $\nu_j = \frac{\ell_j}{n}$, where $\ell_j := |A_j|$ is the number of agents whose closest facility is located at y_j. Then set $\nu = \sum_{j\in[k]} \nu_j \delta_{y_j}$. **Since A_j are disjoint sets**, we have $\nu \in \mathcal{P}_k(\mathbb{R})$.
- Consider the transportation plan, namely π, between $\mu_{\boldsymbol{x}}$ and ν defined as

$$\pi_{i,j} := \pi_{x_i, y_j} = \begin{cases} \frac{1}{n} & \text{if } x_i \in A_j \\ 0 & \text{otherwise} \end{cases}$$

- Since according to π every agent goes to its closest facility, π is optimal, thus we have

$$W_1\left(\mu_{\boldsymbol{x}}, \nu\right) = \sum_{i\in[n], j\in[k]} |x_i - y_j| \, \pi_{i,j} = \frac{1}{n} \sum_{j\in[k]} \sum_{x_i \in A_j} |x_i - y_j|$$

- Show that ν solves problem (5). Toward a **contradiction**(背理法), **assume** that $\tilde{\nu} = \sum_{j=1}^{k} \tilde{\nu}_j \delta_{\tilde{y}_j} \in \mathcal{P}_k(\mathbb{R})$ is such that $W_1\left(\mu_{\boldsymbol{x}}, \tilde{\nu}\right) < W_1\left(\mu_{\boldsymbol{x}}, \nu\right)$.
- Define the partition of agents A_j' related to the set of points $\left\{y_j'\right\}_{j\in[k]}$. Then we have

$$\frac{1}{n} \sum_{j\in[k]} \sum_{x_i \in A_j'} \left|x_i - y_j'\right| = W_1\left(\mu_x, \tilde{\nu}_j\right) < W_1\left(\mu_x, \nu\right) = \frac{1}{n} \sum_{j\in[k]} \sum_{x_i \in A_j} |x_i - y_j|,$$

which contradicts the optimality of $\boldsymbol{y}$, proving the first part of the Theorem.

- For the inverse implication, repeat the same argument backwards. Let ν' be a solution to the W_1 Projection problem. Toward a contradiction, assume that the support of ν' is not a solution to the k-FLP. Given a solution to the k-FLP problem, use the argument used in the first part of the proof to build a new measure that has a lower cost than ν', which would contradict the optimality of the initial solution.
- こんなの言わなくても分かるくないか？

□

- By restricting the set on which the projection problem is defined, we retrieve a similar characterization for the cost of any k-facility location mechanism.

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a k-facility location mechanism. Then, the following identity holds*

$$SC_f(\boldsymbol{x}) = \min_{\{\lambda_j\}_{j\in[k]}\subset\mathbb{R}} W_1(\mu_{\boldsymbol{x}}, \lambda) \tag{7}$$

where $\lambda = \sum_{j\in[k]} \lambda_j \delta_{y_j}$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_k) = f(\boldsymbol{x})$.

- 証明の前に少し復習
- Rescaled Social Cost: $SC(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{n}\sum_{i\in[n]} c_i(x_i, \boldsymbol{y})$.
- $SC_f(\boldsymbol{x})$ is the Social Cost of placing facilities at $f(\boldsymbol{x})$
- つまり $SC_f(\boldsymbol{x}) = SC(\boldsymbol{x}, f(\boldsymbol{x}))$

Proof.

- Let f be a mechanism, $\boldsymbol{x}$ the vector containing the reports of n agents, and let $\boldsymbol{y}$ be the vector containing the positions returned by the mechanism f, so that $\boldsymbol{y} = f(\boldsymbol{x})$.
  - theorem 1 では $\boldsymbol{y}$ は最適解であったが、今回はそうでない。
- For every $j \in [k]$, denote A_j as the set of agents that are closer to the facility placed at y_j.
- Assume without loss of generality that every A_j is disjoint from the others, so that $A_j \cap A_r = \emptyset$ for every $j \neq r$.
- Define $\nu^{(n)}$ as

$$\nu^{(n)} = \sum_{j\in[k]} \nu_j^{(n)} \delta_{y_j}$$

  where $\nu_j^{(n)} = \frac{\ell_j}{n}$ and $\ell_j = |A_j|$.(facility j に近いエージェントの数)
- Show that $\nu^{(n)}$ is a solution to problem (7). The discrete probability measure π is defined as

$$\pi_{i,j} := \pi_{x_i,y_j} = \begin{cases} \frac{1}{n} & \text{if } x_i \in A_j \\ 0 & \text{otherwise} \end{cases}$$

  which is a transportation plan between $\mu_{\boldsymbol{x}}$ and $\nu^{(n)}$.
- Since according to π every agent goes to its closest facility, we have

$$\sum_{i\in[n]} |x_i - y_j|\, \pi_{i,j} = W_1\left(\mu_{\boldsymbol{x}}, \nu^{(n)}\right).$$

- If $\tilde{\nu}$ is such that $\operatorname{spt}(\tilde{\nu}) = \operatorname{spt}(\nu) = \{y_j\}_{j\in[k]}$ and $W_1(\mu_{\boldsymbol{x}}, \tilde{\nu}) < W_1\left(\mu_{\boldsymbol{x}}, \nu^{(n)}\right)$, we infer that there exists at least one agent that can be reallocated to a closer facility, which would contradict the definition of A_j.

□

- Notice that the projection problem (7) is a further restricted version of the projection problem (5).
- Indeed, in (5), the support of the solution can be any subset of $\mathbb{R}$ containing k elements.
- While in (7), the support of the solution is fixed by the mechanism f.
- 言ってる意味は分かるが、これが今後どのように影響してくるかまでは想像できない