Delayed Constraint Generation

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2024 年 10 月 10 日

1 Notation

- 以下、NTT データより引用
- 車両スケジューリングや訪問スケジューリング問題には、以下の二つの決定が含まれている:
  - エージェント（車両や人）を訪問先に割り当てる。
  - 各エージェントの訪問時刻を決める。
- 「割り当てる」問題が上位層 (非線形)、「訪問時刻を決める」問題が下位層 (線形) としてモデル化できる。
- Benders 分解手法は、階層構造の問題を効率的に解くための手法であり、原問題を上位層と下位層を分けて解くことが可能になる。
- しかし、上位層は非線形計画問題。。。こちら側のお気持ちとしてはできるだけ線形計画問題を解いていきたい
- そこで非線形項に対し変数を固定してしまえばそれは定数項になる！
- 固定した変数に対して上位問題の解は一応求まるが、しかしそれが原問題の最適解かは不明
- うまい具合に変数を固定しなければならないが、それはどのようにして行うのか？ (これが今回のテーマ)

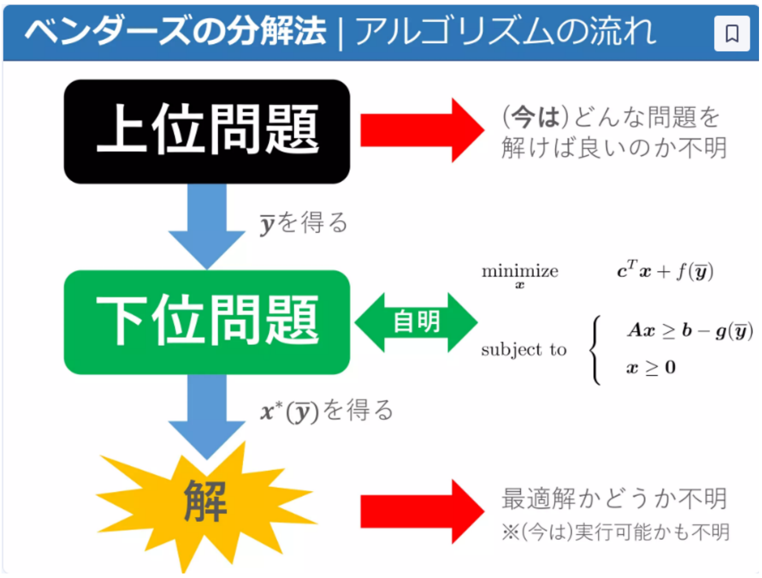


図 1　Benders decomposition の大雑把な説明

2 what is DCG

- We have an optimization problem given by

$$
\begin{aligned}
\text{minimize} \quad & c^\top x + f(y) \\
\text{subject to} \quad & Ax + F(y) \geq b \\
& x \geq 0,\ y \in Y
\end{aligned}
\tag{P}
$$

- Here, A is an $m \times n$ matrix, $f : \mathbb{R}^q \mapsto \mathbb{R}$, $F : \mathbb{R}^q \mapsto \mathbb{R}^m$, and $Y \subseteq \mathbb{R}^q$.
- x については線形となっているが、y は線形ではない (二次関数とか)
- The difficulty in this problem comes from the y variables, **as the problem becomes a linear program for a fixed y.**
- In a typical application, the set Y can also be complicated, such as in the case of integrality constraints.
- We can rewrite the optimization problem in the following form:

$$\begin{aligned} \text{minimize} \quad & f(y) + g(y) \\ \text{subject to} \quad & y \in Y \end{aligned}$$

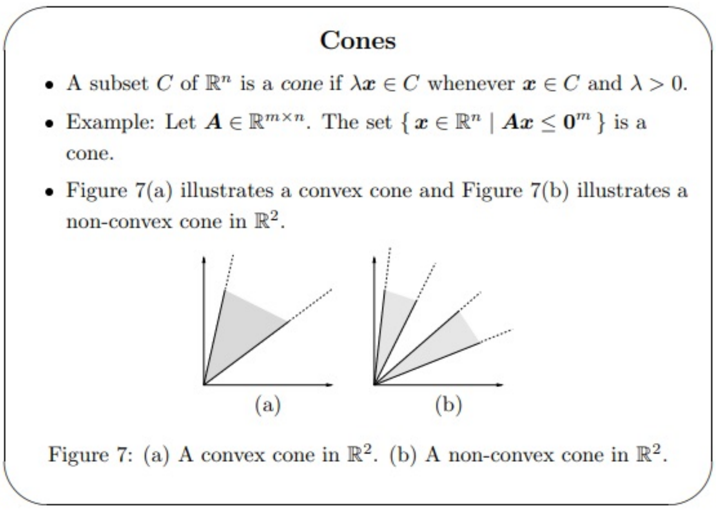
where,

$$\begin{aligned} g(y) = \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b - F(y), \\ & x \geq 0. \end{aligned} \tag{BSP}$$

- As we have observed before, **if we fix** y, then finding $g(y)$ is equivalent to solving the linear program
- We refer to $g(y)$ as the **Benders subproblem**.
- To derive the dual of this Benders subproblem, let $u \in \mathbb{R}^m$ be the dual variables associated with the constraints. The dual Benders subproblem then becomes:

$$\begin{aligned} g(y) = \text{maximize} \quad & (b - F(y))^\top u \\ \text{subject to} \quad & A^\top u \leq c, \\ & u \geq 0. \end{aligned} \tag{DBSP}$$

- Here is a **key observation**: The feasible region of (DBSP) is independent of the y variables.
- This implies that if the feasible region of (DBSP) is empty, then either the feasible region of (BSP) is empty or its optimal objective function value is unbounded.
- In the former case, the original problem (P) becomes infeasible, whereas in the latter case, the optimal objective function value of problem (P) is also unbounded.
  - 強双対性より、もし主問題が最適解を持つならば、双対問題も最適解を持ち、その最適値は一致
  - またもし双対問題が非有界であれば、主問題は実行不可能
- Therefore, we can just assume that the feasible region of (DBSP) is not empty and hence, it can be written in terms of its extreme points and extreme directions by using the Minkowski's Representation theorem.
- extreme directions は錐のこと
- the Minkowski's Representation theorem は以下。ここを参照した



Cones

- A subset C of $\mathbb{R}^n$ is a *cone* if $\lambda \boldsymbol{x} \in C$ whenever $\boldsymbol{x} \in C$ and $\lambda > 0$.
- Example: Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. The set $\{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{0}^m \}$ is a cone.
- Figure 7(a) illustrates a convex cone and Figure 7(b) illustrates a non-convex cone in $\mathbb{R}^2$.

(a) (b)

Figure 7: (a) A convex cone in $\mathbb{R}^2$. (b) A non-convex cone in $\mathbb{R}^2$.

図 2　cone の説明

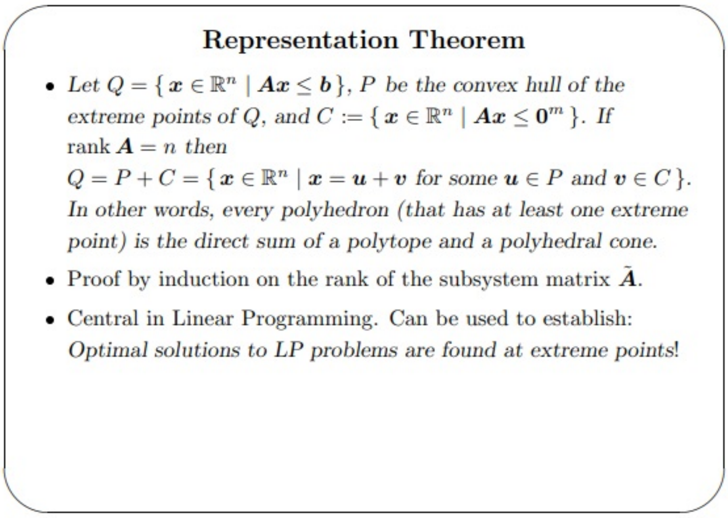


図 3　Minkowski's Representation theorem の説明

- Let $\mathcal{U}$ be the feasible region of (DBSP) given by

$$\mathcal{U} = \left\{ u \in \mathbb{R}^m : A^\top u \le c,\, u \ge 0 \right\}$$

- Representation theorem より凸多面体 $\mathcal{U}$ を以下のように点と錐に分割することが可能
- Denote its sets of extreme points and extreme rays by

$$\mathcal{P}_{\mathcal{U}} = \{u^p,\, p \in \mathcal{I}_{\mathcal{P}}\} \quad \text{and} \quad \mathcal{R}_{\mathcal{U}} = \{u^r,\, r \in \mathcal{I}_{\mathcal{R}}\}$$

  respectively.
- Given y, suppose for an extreme ray u^r that $(b - F(y))^\top u^r > 0$. This would imply that (DBSP) is unbounded, and hence, (BSP) is infeasible, which in turn makes (P) infeasible.
- 一般に n 変数の線形計画問題の場合凸多面体の頂点の中に，必ず最適解が存在
- u^p は端点なのでこの中に最適解が存在する
- So we can rewrite (DBSP) as follows:

$$\begin{aligned} g(y) = \quad \text{minimize} \quad & v \\ \text{subject to} \quad & (b - F(y))^\top u^p \le v, \quad p \in \mathcal{I}_{\mathcal{P}}, \\ & (b - F(y))^\top u^r \le 0, \quad r \in \mathcal{I}_{\mathcal{R}}. \end{aligned}$$

- Now we have a reformulation of (DBSP) with only one decision variable v **but too many constraints**. However, this model gives us a chance to rewrite the original problem (P) as:

$$\begin{aligned} \text{minimize} \quad & f(y) + v \\ \text{subject to} \quad & (b - F(y))^\top u^p \le v, \quad p \in \mathcal{I}_{\mathcal{P}} \\ & (b - F(y))^\top u^r \le 0, \quad r \in \mathcal{I}_{\mathcal{R}} \\ & y \in Y \end{aligned} \tag{BMP}$$

  This problem is known as the **Benders master problem** (BMP).
- It is impractical to generate all extreme points and extreme rays. (CFLP でもここが問題になっている)
- Thus, Benders decomposition works with only a subset of those exponentially many constraints, and depending on the solution of the subproblem, adds more constraints iteratively until the optimal solution of (BMP) is attained. This procedure is known as **delayed constraint generation**.
- なので制約を小分けにして考えていこうぜって感じ
- 制約を小分けにして都度考えることを delayed constraint generation というのだと思う
- これ以下の考え方は benders 分解の一例であることに注意

- Let $\overline{\mathcal{I}}_\mathcal{P} \subseteq \mathcal{I}_\mathcal{P}$ and $\overline{\mathcal{I}}_\mathcal{R} \subseteq \mathcal{I}_\mathcal{R}$ designate the subsets of the considered constraints. Then, we can write the **restricted Benders master problem** (RBMP) as:

$$\begin{aligned} \text{minimize} \quad & f(y) + v \\ \text{subject to} \quad & (b - F(y))^\top u^p \leq v, \quad p \in \overline{\mathcal{I}}_\mathcal{P} \\ & (b - F(y))^\top u^r \leq 0, \quad r \in \overline{\mathcal{I}}_\mathcal{R} \\ & y \in Y \end{aligned} \tag{RBMP}$$

- Suppose the optimal solution of (RBMP) is $(\bar{y}, \bar{v})$. Clearly, $f(\bar{y}) + \bar{v}$ is a lower bound on the optimal objective function value of (P). We next solve (DBSP) with fixed $\bar{y}$. There are three possible cases:
  1. If (DBSP) is unbounded, then we have an extreme ray $\boldsymbol{u}^r$ so that we can add the following constraint to the (RBMP):

  $$(b - F(y))^\top u^r \leq 0.$$

  This type of constraint is known as a Benders feasibility cut as it implies the feasibility of problem (BSP), and consequently, of problem (P).
  2. If $\bar{v} < g(\bar{y})$ and (DBSP) is optimal, then we have an extreme point u^p so that we can add the following constraint to the (RBMP):

  $$(b - F(y))^\top u^p \leq v.$$

  This type of constraint is known as a Benders optimality cut. Moreover, $f(\bar{y}) + g(\bar{y})$ is an upper bound on the optimal objective function value of (P).
  3. When $\bar{v} = g(\bar{y})$, that is the lower bound is equal to the upper bound, the algorithm stops.
- アルゴリズムを図示したものが以下

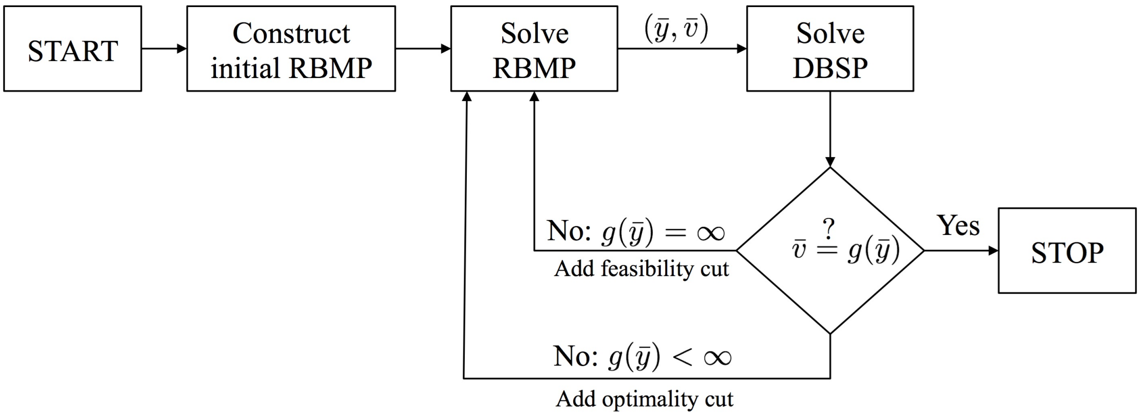


図 4　Benders decomposition のイメージ

3　例題 (liner case)

- 実例をもとに考えてみよう
- 主問題を以下とする

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{j=1}^5 x_j + \sum_{j=1}^5 7y_j \\ \text{subject to} \quad & x_1 + x_4 + x_5 = 8, \\ & x_2 + x_5 = 3, \\ & x_3 + x_4 = 5, \\ & x_1 \leq 8y_1, x_2 \leq 3y_2, x_3 \leq 5y_3, x_4 \leq 5y_4, x_5 \leq 3y_5, \\ & x_j \geq 0, y_j \in \{0,1\}, j = 1, \ldots, 5. \end{aligned} \tag{P}$$

- For a fixed $\bar{y}$、(BSP) は以下となる。

$$
\begin{aligned}
g(\bar{y}) = \text{minimize} \quad & \sum_{j=1}^{5} x_j \\
\text{subject to} \quad & x_1 + x_4 + x_5 = 8, \\
& x_2 + x_5 = 3, \\
& x_3 + x_4 = 5, \\
& x_1 \leq 8\bar{y}_1,\ x_2 \leq 3\bar{y}_2,\ x_3 \leq 5\bar{y}_3,\ x_4 \leq 5\bar{y}_4,\ x_5 \leq 3\bar{y}_5, \\
& x_j \geq 0 \quad \text{for } j = 1, \ldots, 5.
\end{aligned}
\tag{BSP}
$$

- ラグランジュ使って双対問題 (DBSP) を作る

$$
\begin{aligned}
g(\bar{y}) = \text{maximize} \quad & 8u_1 + 3u_2 + 5u_3 + 8\bar{y}_1 u_4 + 3\bar{y}_2 u_5 + 5\bar{y}_3 u_6 + 5\bar{y}_4 u_7 + 3\bar{y}_5 u_8 \\
\text{subject to} \quad & u_1 + u_4 \leq 1, \\
& u_2 + u_5 \leq 1, \\
& u_3 + u_6 \leq 1, \\
& u_1 + u_3 + u_7 \leq 1, \\
& u_1 + u_2 + u_8 \leq 1, \\
& u_j \leq 0, \quad j = 4, \ldots, 8.
\end{aligned}
\tag{DBSP}
$$

- 制約多面体は

$$
\mathcal{U} = \left\{ u \in \mathbb{R}^m : \mathbf{I}^\top \begin{pmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \leq \mathbf{1}_5,\ u_i \leq 0, i = 4, \cdots, 8 \right\}
$$

- Denote its sets of extreme points and extreme rays by

$$
\mathcal{P}_\mathcal{U} = \{u^p \mid p \in \mathcal{I}_\mathcal{P}\} \quad \text{and} \quad \mathcal{R}_\mathcal{U} = \left\{ u^r \mid -\mathbf{I}^\top \begin{pmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \leq \mathbf{0}^5, r \in \mathcal{I}_\mathcal{R} \right\}
$$

  respectively.
- Representative Theorem を用いて (BMP) を作成

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{b} 7y_j + v \\
\text{subject to} & 8u_1^p + 3u_2^p + 5u_3^p + 8y_1 u_4^p + 3y_2 u_5^p + 5y_3 u_6^p + 5y_4 u_7^p + 3y_5 u_8^p \leq v, \quad p \in \mathcal{I}_\mathcal{P} \\
& 8u_1^r + 3u_2^r + 5u_3^r + 8y_1 u_4^r + 3y_2 u_5^r + 5y_3 u_6^r + 5y_4 u_7^r + 3y_5 u_8^r \leq 0, \quad r \in \mathcal{I}_\mathcal{R} \\
& v \geq 0, y_j \in \{0,1\}, j = 1, \ldots, 5
\end{array}
\tag{BMP}
$$

- Note that we add the trivial lower bound on v.
- Now we are ready to solve this problem with Benders decomposition.
- Initially, we start **without any Benders cut**. Our first (RBMP) is given as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{5} 7y_j + v \\
\text{subject to} \quad & v \geq 0, \\
& y_j \in \{0,1\}, \quad j = 1, \ldots, 5.
\end{aligned}
$$

- Clearly, at the optimal $\bar{y} = 0$ and $\bar{v} = 0$.
- Then, we solve (DBSP) with fixed $\bar{y}$
- すると (DBSP) の目的関数は ∞ に発散する。(u_1 が ∞ に飛べるので)
- extreme ray の制約を追加する
- 制約多面体から実行可能な $u^r = (2, -1, -1, -2, 0, 0, -1, -1)^\top$ がゲットできるので、これを基にした feasibility cut を制約に追加

– 別に $u^r = \mathbf{0}$ とか $u^r = (0,0,0,-1,-1,-1,-1,-1)^\top$ でも良い。

– しかし、例えば $u^r = (0,0,0,-1,-1,-1,-1,-1)^\top$ で feasibility cut を生成してみても $-8y_1 - 3y_2 - 5y_3 - 5y_4 - 3y_5 \le 0$ と意味のない cut が生成される。(また $\bar{y} = 0$ and $\bar{v} = 0$ が選択されるため)

– そのためこのサイトでは恣意的に $u^r = (2,-1,-1,-2,0,0,-1,-1)^\top$ を選び、多面体を削るように仕向けたのだと思う。

– 削るためには定数項が必要、y についての項も必要であるため、それらを満たすものを適当に選んだと思われる ($\bar{y} = 0$ なので本当に適当でいい)

– しかしこれをコンピュータにやらせると意味のない制約を大量に出力してしまう可能性がある

– そこに branch-and-cut アルゴリズムという効率よく制約を見つけるものが台頭した

– なのでこれ以降の u^r についてはこんなのを選んだらこうなったよ、という例を提示しているだけなので (*￣-￣) ふ〜んといった感じで聞いてください

- This leads to a feasibility cut that we can use to form our new (RBMP):

$$\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{5} 7y_j + v \\
\text{subject to} \quad & 8 - 16y_1 - 5y_4 - 3y_5 \le 0, \\
& v \ge 0, \\
& y_j \in \{0,1\}, \quad j = 1, \dots, 5.
\end{aligned}$$

- The optimal solution of this restricted problem is $\bar{y} = (1,0,0,0,0)^\top$ and $\bar{v} = 0$. We next solve the (DBSP) with $\bar{y}$ and observe that its optimal objective function value, $g(\bar{y})$ is unbounded. Then, an extreme ray is $u^r = (0,0,1,0,0,-1,-1,0)^\top$. We add another feasibility cut and form the next (RBMP):

$$\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{5} 7y_j + v \\
\text{subject to} \quad & 8 - 16y_1 - 5y_4 - 3y_5 \le 0, \\
& 5 - 5y_3 - 5y_4 \le 0, \\
& v \ge 0, \\
& y_j \in \{0,1\}, \quad j = 1, \dots, 5.
\end{aligned}$$

- After solving the new (RBMP), we obtain its optimal solution as $\bar{y} = (0,0,0,1,1)^\top$ and $\bar{v} = 0$. Solving now (DBSP) with the new $\bar{y}$ yields an optimal objective function value of $g(\bar{y}) = 8 > \bar{v}$.

  – 解く問題は以下

$$\begin{aligned}
g(\bar{y}) = \text{maximize} \quad & 8u_1 + 3u_2 + 5u_3 + 5u_7 + 3u_8 \\
\text{subject to} \quad & u_1 + u_4 \le 1, \\
& u_2 + u_5 \le 1, \\
& u_3 + u_6 \le 1, \\
& u_1 + u_3 + u_7 \le 1, \\
& u_1 + u_2 + u_8 \le 1, \\
& u_j \le 0, \quad j = 4, \dots, 8.
\end{aligned} \tag{3.1}$$

  – この時 u_7、u_8 は負になることから目的関数の最大化からこいつらは 0 にしたい

  – その中で最大化するには u_1 を 1 とするしかない

  – したがって $g(\bar{y}) = 8 > \bar{v}$

  – その時の実行解は $u^p = (1,1,1,0,0,0,-1,-1), (1,0,0,0,0,0,0,0)^\top$ であり、サイトでは前者を採用している

- Thus, we have a Benders optimality cut that can be added to the recent (RBMP):

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{5} 7y_j + v \\
\text{subject to} \quad & 8 - 16y_1 - 5y_4 - 3y_5 \le 0, \\
& 5 - 5y_3 - 5y_4 \le 0, \\
& 16 - 5y_4 - 3y_5 \le v, \\
& v \ge 0, \\
& y_j \in \{0,1\}, \quad j = 1, \ldots, 5.
\end{aligned}
$$

- The optimal solution of this (RBMP) is $\bar{y} = (0,0,0,1,1)^\top$ and $\bar{v} = 8$. As $\bar{y}$ has not changed, the optimal objective function value is still $g(\bar{y}) = \bar{v} = 8$. We have obtained the optimal solution.
- ちなみに $u^p = (1,0,0,0,0,0,0,0)^\top$ でも同じ結果となる。