Optimal Repair of (k + 2, k, 2) MDS Array Codes *

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Abstract

Maximum distance separable (MDS) codes are widely used in distributed storage systems as they provide optimal fault tolerance for a given amount of storage overhead. The seminal work of Dimakis $et\ al.$ first established a lower bound on the repair bandwidth for a single failed node of MDS codes, known as the cut-set bound. MDS codes that achieve this bound are called minimum storage regenerating (MSR) codes. Numerous constructions and theoretical analyses of MSR codes reveal that they typically require exponentially large sub-packetization levels, leading to significant disk I/O overhead. To mitigate this issue, many studies explore the trade-offs between the sub-packetization level and repair bandwidth, achieving reduced sub-packetization at the cost of suboptimal repair bandwidth. Despite these advances, the fundamental question of determining the minimum repair bandwidth for a single failure of MDS codes with fixed sub-packetization remains open.

In this paper, we address this challenge for the case of two parity nodes (n - k = 2) and sub-packetization $\ell = 2$. We derive tight lower bounds on both the minimum repair bandwidth and the minimum I/O overhead. Furthermore, we present two explicit MDS array code constructions that achieve these bounds, respectively, offering practical code designs with provable repair efficiency.

1 Introduction

Erasure codes are widely adopted in distributed storage systems as they provide better storage efficiency for the same level of fault tolerance compared to replication [1]. Consider an (n, k) erasure-coded distributed storage system. For a file of size \mathcal{M} bits to be stored, it is first divided into k data packets, denoted by C_1, C_2, \ldots, C_k , each of size \mathcal{M}/k . Then, using the erasure code, r = n - k parity packets are generated, denoted by $C_{k+1}, C_{k+2}, \ldots, C_n$, each also of size \mathcal{M}/k . Finally, these n data and parity packets are stored on n distinct storage nodes. In distributed system literature, these n data and parity packets are called a stripe. If a packet in a stripe is lost, it can be recovered by accessing some remaining data or parity packets.

Reed-Solomon (RS) codes. Reed-Solomon (RS) codes are the most prevalent class of erasure codes. They are widely deployed in distributed storage systems (e.g., Google [2], Facebook [3]) because they support general (n, k) (where n > k) and achieve the optimal trade-off between storage overhead and fault tolerance. More precisely, a key advantage is their maximum distance separable (MDS) property, i.e., in an (n, k) RS-coded stripe, any k packets suffice to reconstruct the stripe.

Traditional RS code repair strategies incur high repair bandwidth—the total amount of data transmitted over the network to recover failed nodes. Specifically, repairing up to r failed packets in an RS-coded stripe requires transferring k packets, which is equivalent to the size of the original file.

Fig. 1 illustrates an (6,4) RS-coded stripe. In this example, if any two packets fail, the repair process requires transferring all four remaining packets, resulting in a repair bandwidth of \mathcal{M} bits.

MSR codes. To reduce the repair bandwidth of RS codes, Dimakis *et al.* introduced the concept of minimum storage regenerating (MSR) codes, which minimize the repair bandwidth for a single failed node while maintaining the MDS property [4]. The minimum repair bandwidth of MDS codes is known as the *cut-set bound*. The key idea behind MSR codes is to further partition each packet into ℓ *sub-packets*, allowing for a more granular encoding process. The number of sub-packets in each packet ℓ is referred to as the *sub-packetization*. MSR codes achieve

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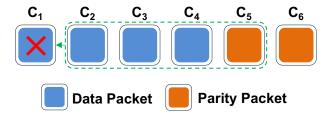


Figure 1: An (n, k) = (6, 4) RS-coded stripe.

reduced repair bandwidth by allowing each helper node to transmit only a portion of its sub-packets during the repair process.

Fig. 2 illustrates an (6,4,8) MSR-coded stripe, where each packet is divided into $\ell=8$ sub-packets. In this example, if any single packet is lost, the repair process requires transferring four sub-packets from each of the remaining five packets; the resulting repair bandwidth is twenty sub-packets. Since each packet is divided into eight sub-packets, the repair bandwidth is equivalent to 2.5 packets, which significantly saves 37.5% repair bandwidth compared to the (6,4) RS-coded stripe in Fig. 1.

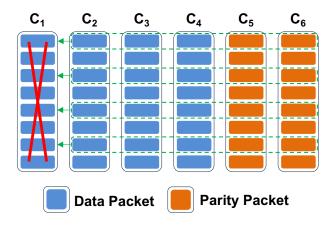


Figure 2: An $(n, k, \ell) = (6, 4, 8)$ MSR-coded stripe.

Since the introduction of MSR codes, numerous MSR code constructions have been proposed. Product-Matrix (PM) MSR codes [5] and PM-RBT [6] proposed by Rashmi et~al. are the first explicit MSR codes, but they have high storage overhead $(n \ge 2k-1)$. Functional MSR (F-MSR) codes [7] support n-k=2 and n-k=3, yet they are non-systematic, meaning original data cannot be directly read from the stripes. Butterfly codes [8] are the first binary MSR codes, but they only support n-k=2. Subsequent MSR constructions supporting general n and k emerged, such as Ye-Barg codes [9, 10], the generic transformation from MDS codes to MSR codes [11], and Clay codes [12]. Notably, Clay codes [12] have been deployed in the distributed storage system Ceph [13]. The above constructions only support n-1 helper nodes. Recently, Li et~al. proposed a universal MSR code construction with the smallest sub-packetization that supports any number of helper nodes [14]. A comprehensive review of MSR constructions can be found in the monograph [15].

Among known MSR codes, all systematic code constructions with low storage overhead require exponential sub-packetization levels $\ell = \exp(O(n))$. For example, The sub-packetization of an (n,k) Clay code is $\ell = (n-k)^{\lceil n/(n-k) \rceil}$. Concurrently, multiple theoretical lower bounds [16–18] demonstrate that exponential sub-packetization is unavoidable for high-rate systematic MSR codes.

Exponential sub-packetization severely hinders the practical adoption of MSR codes in distributed storage systems. While fine-grained partitioning of packets reduces repair bandwidth, it incurs high I/O overheads. Specifically, when the data and parity packets are divided into numerous sub-packets, the repair process of single failures requires helper nodes to transmit partial sub-packets to failed nodes, generating extensive non-contiguous disk I/O. For systems where disk I/O efficiency is the bottleneck, this significantly degrades performance.

As shown in Fig. 2, to repair the first packet in a (6,4,8) MSR-coded stripe, each of the remaining five packets transmits four sub-packets, resulting in a total of 20 non-contiguous I/O operations on the stripe. This is a significant overhead compared to the four I/O operations required for repairing a single packet in the RS-coded stripe in Fig. 1.

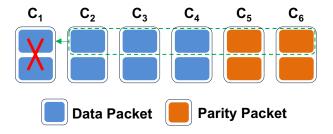


Figure 3: An $(n, k, \ell) = (6, 4, 2)$ MDS-coded stripe.

MDS array codes with small sub-packetization. To reduce disk I/O overhead while maintaining low repair bandwidth, many works consider the trade-off between the exponential sub-packetization level of MSR codes and optimal repair bandwidth, focusing on MDS array codes with small sub-packetization. Such schemes preserve the MDS property like RS codes, while introducing reduced sub-packetization to achieve moderate repair bandwidth between that of RS codes and MSR codes.

As shown in Fig. 3, a (6, 4, 2) MDS-coded stripe divides each packet into two sub-packets. In this example, if any single packet fails, the repair process requires transferring five (or six) sub-packets from the remaining five packets, resulting in a saving of 25-37.5% repair bandwidth across all the data and parity packets compared to the RS-coded stripe in Fig. 1. Compared to the MSR-coded stripe in Fig. 2, this coding scheme achieves a suboptimal repair bandwidth without incurring non-contiguous disk I/O within one stripe.

The piggybacking framework [19] pioneered the construction of MDS codes with small sub-packetization and low repair bandwidth. This framework inherently retains the MDS property of RS codes while supporting arbitrary sub-packetization. Hitchhiker codes [20], a specialized piggybacking design with $\ell=2$, have been implemented in HDFS, reducing data packet repair bandwidth by 25%-45% compared to RS codes, though parity packet repair bandwidth remains unchanged. HashTag codes [21] similarly reduce repair bandwidth for data blocks. HashTag+ codes [22] further optimize repair bandwidth for both data and parity blocks. The Elastic Transformation [23] also converts RS codes into repair-efficient codes and supports configurable sub-packetization. Additionally, ε -MSR codes [24,25] demonstrate that for sufficiently large n, logarithmic sub-packetization suffices to achieve $(1 + \varepsilon)$ -times the optimal repair bandwidth.

Despite significant progress in constructing such MDS codes, the fundamental lower bound on repair bandwidth subject to a prescribed sub-packetization level remains largely unexplored. The following open problem posed by Ramkumar *et al.* in their survey [15] addresses this fundamental question.

Open Problem 1. [15, Open Problem 9] Characterize the tradeoff between repair bandwidth, sub-packetization level, and field size for the general class of vector MDS codes.

In [26], Guruswami and Wootters first studied the repair problem of RS codes. By viewing each element of $\mathbb{F}_{q^{\ell}}$ as an ℓ -length vector over \mathbb{F}_q , they proposed repair schemes that outperform the naive approach and proved a lower bound on repair bandwidth that holds for any scalar MDS codes. Here, the degree of field extension ℓ is the sub-packetization. Dau and Milenkovic [27] subsequently tightened this bound for RS codes.

More recently, attention has turned to the repair I/O of scalar MDS codes over $\mathbb{F}_{q^{\ell}}$, defined as the total number of \mathbb{F}_{q} -elements accessed during repair. Dau *et al.* [28] provided the first nontrivial repair I/O lower bound for full-length RS codes over fields of characteristic 2 with 2 parity nodes. Subsequent works [29–31] have extended and refined these bounds for broader classes of RS codes.

1.1 Our contributions

We investigate the special case of Open Problem 1 under fixed parameters r = n - k = 2 and $\ell = 2$. Wu et al. addressed a variant of Open Problem 1 also in the special case of n - k = 2 and $\ell = 2$; their work requires the codes to be degraded-repair-friendly [32]. We impose no degraded-repair-friendly restriction. The redundancy r = 2 is widely deployed in practice (e.g., RAID-6 [33] and Tencent Ultra-Cold Storage [34].) The sub-packetization $\ell = 2$ further ensures that, within one stripe, every packet is accessed contiguously during repair, eliminating non-contiguous I/O.

On the one hand, we give the theoretical limits on the repair efficiency of $(n = k + 2, k, \ell = 2)$ MDS array code. Firstly, we consider the minimum repair bandwidth (denoted by β_i) of each packet C_i $(1 \le i \le n)$ in one $(n = k + 2, k, \ell = 2)$ MDS-coded stripe, where the repair bandwidth is the number of sub-packets transferred during the repair of C_i . Then, we consider the minimum repair I/O (denoted by γ_i) of each packet C_i $(1 \le i \le n)$ in one

 $(n = k + 2, k, \ell = 2)$ MDS-coded stripe, where the repair I/O is the number of sub-packets accessed on the helper packets during the repair of C_i . We provide the following lower bounds on the minimum repair bandwidth and the minimum repair I/O.

Theorem 1. Let C be a (k+2,k,2) MDS array code and β_i the minimal repair bandwidth for each packet C_i (where $1 \le i \le n$). Then the avg-min repair bandwidth $\bar{\beta}(C) := \frac{1}{n} \sum_{i=1}^{n} \beta_i$ satisfies that

$$\bar{\beta}(\mathcal{C}) \geq \frac{5k}{4}.$$

Corollary 2. Let C be a (k+2, k, 2) MDS array code and β_i the minimal repair bandwidth for each packet C_i (where $1 \le i \le n$). Then the max-min repair bandwidth $\beta(C) := \max_{i \in [n]} \{\beta_i\}$ satisfies that

$$\bar{\beta}(\mathcal{C}) \ge \left\lceil \frac{5k}{4} \right\rceil.$$

Theorem 3. Let C be a (k+2,k,2) MDS array code and γ_i the minimal repair bandwidth for each packet C_i (where $1 \leq i \leq n$). Then the avg-min repair IO $\bar{\gamma}(C) := \frac{1}{n} \sum_{i=1}^{n} \gamma_i$ satisfies that

$$\bar{\gamma}(\mathcal{C}) \ge \frac{4k+1}{3}.$$

Corollary 4. Let C be a (k+2,k,2) MDS array code and γ_i the minimal repair bandwidth for each packet C_i (where $1 \leq i \leq n$). Then the max-min repair IO $\gamma(C) := \max_{i \in [n]} \{\gamma_i\}$ satisfies that

$$\gamma(\mathcal{C}) \ge \left\lceil \frac{4k+1}{3} \right\rceil.$$

On the other hand, we construct two classes of $(n = k + 2, k, \ell = 2)$ MDS array codes that achieve the lower bound on repair bandwidth (Theorem 1) and the lower bound on repair I/O (Theorem 3), respectively, which shows that those lower bounds are all tight.

The rest of the paper proceeds as follows. Section 2 describes the linear repair process of MDS array codes. Section 3 characterizes the theoretical lower bounds on repair bandwidth and repair I/O for $(n = k + 2, k, \ell = 2)$ MDS array codes. Section 4 constructs two classes of such codes that achieve the proposed lower bounds. Section 5 concludes the paper.

2 MDS codes and their repair schemes

For convenience, we assume that each storage node contains exactly one packet, with each packet represented as a column vector of length ℓ over the finite field \mathbb{F}_q . As a result, each sub-packet can be seen as a symbol from \mathbb{F}_q . One stripe corresponds to a codeword of an MDS code. In addition, we use the notation [n] to denote the set $\{1, 2, \ldots, n\}$. **MDS codes.** An (n, k, ℓ) linear MDS array code \mathcal{C} over the finite field \mathbb{F}_q comprises n nodes (denoted by C_1, C_2, \ldots, C_n), where some k nodes are data nodes and the rest r = n - k nodes are parity nodes, each node is a column vector of length ℓ over \mathbb{F}_q . In addition, any r out of the n nodes can be recovered from the remaining k nodes.

The relationship among the n nodes of an (n, k, ℓ) MDS array code \mathcal{C} can be described by the following parity check equations:

$$\mathbf{H}_1 C_1 + \mathbf{H}_2 C_2 + \dots + \mathbf{H}_n C_n = \mathbf{0},\tag{1}$$

where each \mathbf{H}_i $(i \in [n])$ is an $r\ell \times \ell$ parity check sub-matrix over \mathbb{F}_q . Then, the r nodes $C_{i_1}, C_{i_2}, \ldots, C_{i_r}$ can be recovered from the remaining k nodes if and only if the square matrix $[\mathbf{H}_{i_1} \ \mathbf{H}_{i_2} \ \ldots \ \mathbf{H}_{i_r}]$ is invertible.

Repair scheme driven by a repair matrix. For any node C_i of an (n, k, ℓ) MDS code \mathcal{C} , any linear repair process of C_i can be described by an $\ell \times r\ell$ repair matrix \mathbf{M} over \mathbb{F}_q . Conversely, any $\ell \times r\ell$ matrix \mathbf{M} satisfying specific conditions can characterize a linear repair process for an MDS code \mathcal{C} .

Let **M** be an $\ell \times r\ell$ matrix over \mathbb{F}_q . Multiplying **M** with the parity check equations (1), we obtain the following repair equations:

$$\mathbf{MH}_1 C_1 + \mathbf{MH}_2 C_2 + \dots + \mathbf{MH}_n C_n = \mathbf{0}. \tag{2}$$

If the square matrix \mathbf{MH}_i is invertible for some i (where $1 \leq i \leq n$), then the repair equations (2) can be rewritten as

$$C_i = -(\mathbf{MH}_i)^{-1} \sum_{j \in [n] \setminus \{i\}} \mathbf{MH}_j C_j,$$

which means C_i can be computed from the n-1 vectors $\mathbf{MH}_jC_j, j \in [n] \setminus \{i\}$.

Repair bandwidth and repair I/O. Now we analyze the repair bandwidth and repair I/O of the repair process above, driven by the repair matrix M. Suppose that the $\ell \times \ell$ matrix \mathbf{MH}_i is invertible for some $i \in [n]$. Then, each helper node C_j ($j \in [n] \setminus \{i\}$) needs to transmit the vector $\mathbf{MH}_j C_j$ to the failed node C_i . For each $j \in [n]$, we denote the rank of the $\ell \times \ell$ matrix \mathbf{MH}_j as rank (\mathbf{MH}_j), and the number of nonzero columns in \mathbf{MH}_j as nz (\mathbf{MH}_j). Then, the square matrix \mathbf{MH}_j can be decomposed as

$$\mathbf{MH}_j = \mathbf{L}_j \cdot \mathbf{R}_j,$$

where \mathbf{L}_j is an $\ell \times \operatorname{rank}(\mathbf{MH}_j)$ matrix and \mathbf{R}_j is an $\operatorname{rank}(\mathbf{MH}_j) \times \ell$ matrix. We can obtain \mathbf{L}_j by selecting $\operatorname{rank}(\mathbf{MH}_j)$ independent columns of \mathbf{MH}_j , and obtain \mathbf{R}_j from the linear combination coefficients for the columns of \mathbf{L}_j to generate the columns in \mathbf{MH}_j . We have $\operatorname{rank}(\mathbf{R}_j) = \operatorname{rank}(\mathbf{MH}_j)$, $\operatorname{nz}(\mathbf{R}_j) = \operatorname{nz}(\mathbf{MH}_j)$, and the repair equations (2) can be further rewritten as

$$C_i = -(\mathbf{M}\mathbf{H}_i)^{-1} \cdot \sum_{j \in [n] \setminus \{i\}} \mathbf{L}_j(\mathbf{R}_j C_j).$$

Since \mathbf{L}_j , $j \in [n] \setminus \{i\}$, can be calculated from \mathbf{M} and the parity check sub-matrix \mathbf{H}_j , it is sufficient to transmit the n-1 vectors $\mathbf{R}_j C_j$, $j \in [n] \setminus \{i\}$ to the failed node C_i for the repair.

Note that each $\mathbf{R}_j C_j$ is a vector of length rank $(\mathbf{M}\mathbf{H}_j)$ over \mathbb{F}_q , and transmitting $\mathbf{R}_j C_j$ requires to read nz (\mathbf{R}_j) symbols from C_j . Therefore, during the repair process of C_i driven by the repair matrix \mathbf{M} , the repair bandwidth, i.e., the total number of symbols transmitted from the helper nodes to the failed node, is

$$BW(\mathbf{M}) = \sum_{j \in [n] \setminus \{i\}} \operatorname{rank}(\mathbf{M}\mathbf{H}_j) = \sum_{j \in [n]} \operatorname{rank}(\mathbf{M}\mathbf{H}_j) - \ell.$$
(3)

Similarly, the repair I/O, i.e., the total number of symbols read from the helper nodes, is

$$IO(\mathbf{M}) = \sum_{j \in [n] \setminus \{i\}} \operatorname{nz}(\mathbf{R}_j) = \sum_{j \in [n] \setminus \{i\}} \operatorname{nz}(\mathbf{M}\mathbf{H}_j)$$
$$= \sum_{j \in [n]} \operatorname{nz}(\mathbf{M}\mathbf{H}_j) - \ell.$$
(4)

Further, the repair degree, i.e., the number of helper nodes during the repair, is

$$d(\mathbf{M}) = |\{i \in [n] : \mathbf{MH}_i \neq \mathbf{O}\}| - 1.$$

Lemma 5. For an (n, k, ℓ) MDS code C and a nonzero matrix \mathbf{M} of size $\ell \times r\ell$, we have $d(\mathbf{M}) > k$.

Proof. If $d(\mathbf{M}) \leq k-1$, then there exist r distinct $a_1, \ldots, a_r \in [n]$ such that $\mathbf{M} \cdot [\mathbf{H}_{a_1}, \mathbf{H}_{a_2}, \cdots, \mathbf{H}_{a_r}] = \mathbf{O}$. By the MDS property, $[\mathbf{H}_{a_1}, \mathbf{H}_{a_2}, \cdots, \mathbf{H}_{a_r}]$ is invertible, and hence \mathbf{M} is a zero matrix, which leads to a contradiction. \square

We denote the index set of nodes which can be repaired by an $\ell \times r\ell$ matrix **M** as

$$\mathcal{R}(\mathbf{M}) := \{ i \in [n] : \operatorname{rank}(\mathbf{MH}_i) = \ell \}. \tag{5}$$

Further, for each node C_i , $i \in [n]$, we denote the matrix set which can repair C_i as

$$\mathcal{M}_i := \left\{ \mathbf{M} \in \mathbb{F}_q^{\ell \times r\ell} : i \in \mathcal{R}(\mathbf{M}) \right\}. \tag{6}$$

Based on the analysis above, we can represent the avg-min repair bandwidth $\bar{\beta}(\mathcal{C})$, the max-min repair bandwidth $\beta(\mathcal{C})$, the avg-min repair I/O $\bar{\gamma}(\mathcal{C})$ and the max-min repair I/O $\gamma(\mathcal{C})$ of an (n, k, ℓ) MDS code \mathcal{C} in terms of

$$\bar{\beta}(\mathcal{C}) = \frac{1}{n} \sum_{i \in [n]} \beta_i = \frac{1}{n} \sum_{i \in [n]} \min_{\mathbf{M} \in \mathcal{M}_i} \{ BW(\mathbf{M}) \}, \tag{7}$$

$$\beta(\mathcal{C}) = \max_{i \in [n]} \{\beta_i\} = \max_{i \in [n]} \min_{\mathbf{M} \in \mathcal{M}_i} \{BW(\mathbf{M})\},$$
(8)

$$\bar{\gamma}(\mathcal{C}) = \frac{1}{n} \sum_{i \in [n]} \gamma_i = \frac{1}{n} \sum_{i \in [n]} \min_{\mathbf{M} \in \mathcal{M}_i} \{ \text{IO}(\mathbf{M}) \},$$
(9)

$$\gamma(\mathcal{C}) = \max_{i \in [n]} \{\beta_i\} = \max_{i \in [n]} \min_{\mathbf{M} \in \mathcal{M}_i} \{ \mathrm{IO}(\mathbf{M}) \},.$$
(10)

3 Fundamental Limits

This section establishes the lower bounds of 1) the avg-min repair bandwidth (7), and 2) the avg-min repair I/O (9) of a (k+2,k,2) MDS array code \mathcal{C} over finite field \mathbb{F}_q , where $r=n-k=2,\ell=2$ and the parity check sub-matrix \mathbf{H}_i for node C_i (where $i \in [n]$) is of size 4×2 . By Lemma 5, we have $d(\mathbf{M}) \in \{k,k+1\}$ for any nonzero repair matrix \mathbf{M} of size 2×4 . The following lemma shows that we only need to consider the case of $d(\mathbf{M}) = k+1$ when we measure the lower bounds on avg-min repair bandwidth and avg-min repair I/O.

Lemma 6. For each node C_i of an (k+2,k,2) MDS array code, there exists $\mathbf{M} \in \mathcal{M}_i$ of $d(\mathbf{M}) = k+1$ such that $IO(\mathbf{M}) \leq 2k$, and therefore $BW(\mathbf{M}) \leq 2k$.

Proof. Without loss of generality, we assume i = 1 and parity check matrix

$$\mathbf{H} = egin{bmatrix} \mathbf{A}_1 & \dots & \mathbf{A}_k & \mathbf{I}_2 & \mathbf{O} \\ \mathbf{B}_1 & \dots & \mathbf{B}_k & \mathbf{O} & \mathbf{I}_2 \end{bmatrix}.$$

Let $\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ & 1 & 0 \end{bmatrix}$ and $\mathbf{M}_2 = \begin{bmatrix} 1 & 0 \\ & 0 & 1 \end{bmatrix}$. As $\mathbf{A}_i, \mathbf{B}_i, i \in [k]$ are all invertible matrices, it's obvious that $d(\mathbf{M}_1) = d(\mathbf{M}_2) = k+1$ and $IO(\mathbf{M}_1), IO(\mathbf{M}_2) \leq 2k$. Now we claim that $\mathbf{M}_1 \in \mathcal{M}_1$ or $\mathbf{M}_2 \in \mathcal{M}_1$. If $\mathbf{M}_1 \notin \mathcal{M}_1$ and $\mathbf{M}_2 \notin \mathcal{M}_1$, then

$$\operatorname{rank}\left(\mathbf{M}_{1}\begin{bmatrix}\mathbf{A}_{1}\\\mathbf{B}_{1}\end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix}\mathbf{e}_{1}\mathbf{A}_{1}\\\mathbf{e}_{1}\mathbf{B}_{1}\end{bmatrix}\right) = 1$$

and

$$\operatorname{rank}\left(\mathbf{M}_{2}\begin{bmatrix}\mathbf{A}_{1}\\\mathbf{B}_{1}\end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix}\mathbf{e}_{1}\mathbf{A}_{1}\\\mathbf{e}_{2}\mathbf{B}_{1}\end{bmatrix}\right) = 1$$

where $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Therefore, we have $\langle \mathbf{e}_1 \mathbf{B}_1 \rangle = \langle \mathbf{e}_1 \mathbf{A}_1 \rangle = \langle \mathbf{e}_2 \mathbf{B}_1 \rangle$. This contradicts the fact that \mathbf{B}_1 is invertible.

In conclusion, there exists $\mathbf{M} \in \mathcal{M}_i$ of $d(\mathbf{M}) = k + 1$ such that $IO(\mathbf{M}) \leq 2k$.

Definition 1 describes a total order on $2^{[n]}$. It first compares subsets by their cardinality. Then, for subsets of the same cardinality, it compares them by the dictionary order.

Definition 1 (Total order \prec on the power set $2^{[n]}$). For any two subsets $S, T \subseteq [n]$, we define their order as follows:

- If |S| < |T|, then $S \prec T$.
- If $|\mathcal{S}| = |\mathcal{T}|$, we suppose that $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$ and $T = \{t_1, t_2, \ldots, t_m\}$ where $s_1 < s_2 < \cdots < s_m$ and $t_1 < t_2 < \cdots < t_m$. Then, we say $\mathcal{S} \prec \mathcal{T}$ if and only if there exists an index $i \in [m]$ such that $s_i < t_i$, and for all j < i, $s_j = t_j$.

Now we consider the linear repair process for node C_i , $i \in [n]$, and measure the corresponding lower bounds on repair bandwidth and repair I/O.

3.1 The lower bound on avg-min repair bandwidth

Recall the definitions of set $\mathcal{R}(\mathbf{M})$ in (5) and set \mathcal{M}_i in (6). According to $d(\mathbf{M}) = k + 1$, we have

$$\mathrm{BW}(\mathbf{M}) = \sum_{j \in [n]} \mathrm{rank} \left(\mathbf{M} \mathbf{H}_j \right) - 2 = k + |\mathcal{R}(\mathbf{M})|,$$

which means that for a node C_i , $i \in [n]$, and a repair matrix $\mathbf{M} \in \mathcal{M}_i$, the smaller $|\mathcal{R}(\mathbf{M})|$ is, the smaller the repair bandwidth of C_i is. We denote the repair matrix in \mathcal{M}_i that minimizes $\mathcal{R}(\mathbf{M})$ under the order \leq as \mathbf{F}_i , i.e.,

$$\mathbf{F}_i = \operatorname{argmin}_{\mathbf{M} \in \mathcal{M}_i} \{ \mathcal{R}(\mathbf{M}) \}, \text{ under the order } \prec.$$
 (11)

Thus, the avg-min repair bandwidth of code \mathcal{C} is

$$\bar{\beta}(\mathcal{C}) = \frac{1}{n} \sum_{i \in [n]} \{BW(\mathbf{F}_i)\}.$$

Remark 1. Note that, for every node C_i , $i \in [n]$, we always have $i \in \mathcal{R}(\mathbf{F}_i)$.

Lemma 7. For any two distinct nodes C_i and C_j , $i, j \in [n]$, if $\mathcal{R}(\mathbf{F}_i) \prec \mathcal{R}(\mathbf{F}_j)$, then $j \notin \mathcal{R}(\mathbf{F}_i)$.

Proof. We prove this by contradiction. Suppose that $j \in \mathcal{R}(\mathbf{F}_i)$. When $\mathcal{R}(\mathbf{F}_i) = \mathcal{R}(\mathbf{F}_i)$ (or $\mathcal{R}(\mathbf{F}_i) = \mathcal{N}(\mathbf{F}_i')$), we have $\mathbf{F}_i \in \mathcal{M}_j$ (or $\mathbf{F}_i' \in \mathcal{M}_j'$). In either case, this implies that $\mathcal{R}(\mathbf{F}_j) \leq \mathcal{R}(\mathbf{F}_i)$ since $\mathcal{R}(\mathbf{F}_j)$ (or $\mathcal{N}(\mathbf{F}_j')$) is minimized under \prec . This contradicts the assumption that $\mathcal{R}(\mathbf{F}_i) \prec \mathcal{R}(\mathbf{F}_i)$. Therefore, we conclude that $j \notin \mathcal{R}(\mathbf{F}_i)$.

Without loss of generality, we assume the code \mathcal{C} satisfies $\mathcal{R}(\mathbf{F}_1) \preceq \mathcal{R}(\mathbf{F}_2) \preceq \cdots \preceq \mathcal{R}(\mathbf{F}_n)$. Therefore, there must exist an integer $i \in [k]$ such that $\mathcal{R}(\mathbf{F}_i) \prec \mathcal{R}(\mathbf{F}_{k+1})$. Recall that $i \in \mathcal{R}(\mathbf{F}_i)$ for any $i \in [n]$. If $\mathcal{R}(\mathbf{F}_1) = \mathcal{R}(\mathbf{F}_2) = \cdots = \mathcal{R}(\mathbf{F}_{k+1})$, then $[k+1] \subseteq \mathcal{R}(\mathbf{F}_1)$, and $|\mathcal{R}(\mathbf{F}_i)| \geq |\mathcal{R}(\mathbf{F}_1)| \geq k+1$ for any $i \in [n]$. It follows that the repair bandwidth and repair I/O for any failed node are larger than 2k, the trivial upper bound. Then we define t as follows:

$$t = \max\{i \in [k] : \mathcal{R}(\mathbf{F}_i) \prec \mathcal{R}(\mathbf{F}_{k+1})\}. \tag{12}$$

Lemma 8. The integer t satisfies $n - |\mathcal{R}(\mathbf{F}_{k+1})| - 1 \le t$ and $\bigcup_{j \in [t]} \mathcal{R}(\mathbf{F}_j) = [t]$.

Proof. By the definition of t, we have

$$\mathcal{R}(\mathbf{F}_t) \prec \mathcal{R}(\mathbf{F}_{t+1}) = \cdots = \mathcal{R}(\mathbf{F}_{k+1}).$$

In particular, for any $i \in [t]$ and $j \ge t + 1$, we have $\mathcal{R}(\mathbf{F}_i) \prec \mathcal{R}(\mathbf{F}_j)$. Combining this with Lemma 7, we get

$$j \notin \bigcup_{i \in [t]} \mathcal{R}(\mathbf{F}_i), \ \forall \ j \ge t + 1.$$
 (13)

Recall that $i \in \mathcal{R}(\mathbf{F}_i)$ for any $i \in [n]$. It follows that

$$\{t+1,\dots,k+1\} \subseteq \mathcal{R}(\mathbf{F}_{k+1}),\tag{14}$$

and

$$[t] \subseteq \bigcup_{i \in [t]} \mathcal{R}(\mathbf{F}_i). \tag{15}$$

By (14), we have

$$n-1-t = |\{t+1,\ldots,k+1\}| \le |\mathcal{R}(\mathbf{F}_{k+1})|.$$

By (13) and (15), we can get

$$\bigcup_{j \in [t]} \mathcal{R}(\mathbf{F}_j) = [t].$$

Now, we can fix the parity-check matrix of the code \mathcal{C} in a systematic form as follows:

$$H = \begin{bmatrix} \mathbf{A}_1 & \dots & \mathbf{A}_k & \mathbf{I}_2 \\ \mathbf{B}_1 & \dots & \mathbf{B}_k & & \mathbf{I}_2 \end{bmatrix},$$

where all the block $\mathbf{A}_i, \mathbf{B}_i$ are 2×2 matrices. Further, by the MDS property, all \mathbf{A}_i and $\mathbf{B}_i, i \in [k]$ are invertible matrices.

By (13), we have $k+1 \notin \bigcup_{i \in [t]} \mathcal{R}(\mathbf{F}_i)$ and $k+2 \notin \bigcup_{i \in [t]} \mathcal{R}(\mathbf{F}_i)$. For any $i \in [t]$ and $j \in \{k+1, k+2\}$, rank $(\mathbf{M}_i \mathbf{H}_j) \leq 1$ is guaranteed by the definition of $\mathcal{R}(\mathbf{F}_i)$. Combining with $d(\mathbf{F}_i) = k+1$, it implies that when we repair the node $C_i, i \in [t]$, each of the two nodes C_{k+1} and C_{k+2} only needs to transmit one symbol.

From the above discussion, during the repairing of node $C_i, i \in [t]$, we assume that the nodes C_{n-1} and C_n transmit $\mathbf{u}_i C_{n-1}$ and $\mathbf{v}_i C_n$, respectively, for some nonzero vectors $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^2$.

Lemma 9. For each $i \in [t]$, the repair matrix \mathbf{F}_i can be written as

$$\mathbf{F}_i = \Lambda egin{bmatrix} \mathbf{u}_i & \ & \mathbf{v}_i \end{bmatrix},$$

where Λ is a 2 × 2 invertible matrix.

Proof. Recall that node C_{n-1} transmits $\mathbf{u}_i C_{n-1}$ and node C_n transmits $\mathbf{v}_i C_n$ during the repair of node C_i . It follows that

$$\mathbf{F}_{i}\mathbf{H}_{n-1} = \begin{bmatrix} a\mathbf{u}_{i} \\ b\mathbf{u}_{i} \end{bmatrix} \text{ and } \mathbf{F}_{i}\mathbf{H}_{n} = \begin{bmatrix} c\mathbf{v}_{i} \\ d\mathbf{v}_{i} \end{bmatrix}$$

for some $a, b, c, d \in \mathbb{F}_q$. Since $\mathbf{H}_{n-1} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{O} \end{bmatrix}$ and $\mathbf{H}_n = \begin{bmatrix} \mathbf{O} \\ \mathbf{I}_2 \end{bmatrix}$, we have

$$\begin{split} \mathbf{F}_i &= \mathbf{F}_i \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_2 \end{bmatrix} = \mathbf{F}_i \begin{bmatrix} \mathbf{H}_{n-1} & \mathbf{H}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{F}_i \mathbf{H}_{n-1} & \mathbf{F}_i \mathbf{H}_n \end{bmatrix} = \begin{bmatrix} a \mathbf{u}_i & c \mathbf{v}_i \\ b \mathbf{u}_i & d \mathbf{v}_i \end{bmatrix} = \Lambda \begin{bmatrix} \mathbf{u}_i & \\ & \mathbf{v}_i \end{bmatrix}, \end{split}$$

where $\Lambda = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Because the matrix \mathbf{F}_i is of full rank, Λ is invertible.

Since the two repair matrices \mathbf{F}_i and $\Lambda^{-1}\mathbf{F}_i$ behave the same during the repair process, we always assume that

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{u}_i & \\ & \mathbf{v}_i \end{bmatrix} \text{ for all } i \in [t].$$

Now we denote the complement set of $\mathcal{R}(\mathbf{F}_i)$ as $\overline{\mathcal{R}(\mathbf{F}_i)} = [n] \setminus \mathcal{R}(\mathbf{F}_i)$ for $i \in [t]$, and it is evident that $i \notin \overline{\mathcal{R}(\mathbf{F}_i)}$. By definition, we have

$$\operatorname{rank} \begin{bmatrix} \mathbf{u}_i \mathbf{A}_j \\ \mathbf{v}_i \mathbf{B}_j \end{bmatrix} = 1 \tag{16}$$

for any $j \in \overline{\mathcal{R}(\mathbf{F}_i)} \cap [k]$.

Let V be a row vector of length 2 or a 2×2 matrix over \mathbb{F}_q . We use $\langle V \rangle$ to denote the subspace spanned by V, i.e., $\langle V \rangle = \{\theta \cdot V : \theta \in \mathbb{F}_q\}$.

Lemma 10. For any $i, j \in [t]$,

1. if
$$\langle \mathbf{u}_i \rangle = \langle \mathbf{u}_j \rangle$$
 and $\langle \mathbf{v}_i \rangle = \langle \mathbf{v}_j \rangle$, then $\overline{\mathcal{R}(\mathbf{F}_i)} = \overline{\mathcal{R}(\mathbf{F}_j)}$;

2. if
$$\langle \mathbf{u}_i \rangle = \langle \mathbf{u}_j \rangle$$
, $\langle \mathbf{v}_i \rangle \neq \langle \mathbf{v}_j \rangle$ or $\langle \mathbf{u}_i \rangle \neq \langle \mathbf{u}_j \rangle$, $\langle \mathbf{v}_i \rangle = \langle \mathbf{v}_j \rangle$, then $\overline{\mathcal{R}(\mathbf{F}_i)} \cap \overline{\mathcal{R}(\mathbf{F}_j)} \cap [k] = \emptyset$.

Proof. We only prove the second part here. Assume $a \in \overline{\mathcal{R}(\mathbf{F}_i)} \cap \overline{\mathcal{R}(\mathbf{F}_j)} \cap [k]$. By (16),

$$\operatorname{rank} \begin{bmatrix} \mathbf{u}_i \mathbf{A}_a \\ \mathbf{v}_i \mathbf{B}_a \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathbf{u}_j \mathbf{A}_a \\ \mathbf{v}_j \mathbf{B}_a \end{bmatrix} = 1.$$

This means $\langle \mathbf{u}_i \mathbf{A}_a \rangle = \langle \mathbf{v}_i \mathbf{B}_a \rangle$ and $\langle \mathbf{u}_j \mathbf{A}_a \rangle = \langle \mathbf{v}_j \mathbf{B}_a \rangle$. So if $\langle \mathbf{u}_i \rangle = \langle \mathbf{u}_j \rangle$ or $\langle \mathbf{v}_i \rangle = \langle \mathbf{v}_j \rangle$, we can get

$$\langle \mathbf{v}_i \rangle = \langle \mathbf{u}_i \mathbf{A}_a \mathbf{B}_a^{-1} \rangle = \langle \mathbf{u}_j \mathbf{A}_a \mathbf{B}_a^{-1} \rangle = \langle \mathbf{v}_j \rangle$$

or

$$\langle \mathbf{u}_i \rangle = \langle \mathbf{v}_i \mathbf{B}_a \mathbf{A}_a^{-1} \rangle = \langle \mathbf{v}_j \mathbf{B}_a \mathbf{A}_a^{-1} \rangle = \langle \mathbf{u}_j \rangle,$$

respectively. It presents a contradiction.

Lemma 11. If a, b, c are three elements in [t] such that $\mathcal{R}(\mathbf{F}_a) \prec \mathcal{R}(\mathbf{F}_b) \prec \mathcal{R}(\mathbf{F}_c)$ and $\overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k] \neq \emptyset$, then

$$\forall \ i, j \in \overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k], \ \langle \mathbf{B}_j \mathbf{A}_j^{-1} \rangle = \langle \mathbf{B}_i \mathbf{A}_i^{-1} \rangle.$$

Proof. By Lemma 10, the conditions $\mathcal{R}(\mathbf{F}_a) \prec \mathcal{R}(\mathbf{F}_b) \prec \mathcal{R}(\mathbf{F}_c)$ and $\overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k] \neq \emptyset$ imply that

$$\langle \mathbf{u}_a \rangle \neq \langle \mathbf{u}_b \rangle, \langle \mathbf{u}_b \rangle \neq \langle \mathbf{u}_c \rangle, \langle \mathbf{u}_a \rangle \neq \langle \mathbf{u}_c \rangle$$

and

$$\langle \mathbf{v}_a \rangle \neq \langle \mathbf{v}_b \rangle, \langle \mathbf{v}_b \rangle \neq \langle \mathbf{v}_c \rangle, \langle \mathbf{v}_a \rangle \neq \langle \mathbf{v}_c \rangle.$$

For simplicity, we write $\mathbf{W}_t = \mathbf{B}_t \mathbf{A}_t^{-1}$ for all $t \in [k]$. By (16), if there exists $i, j \in \overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k]$,

$$\operatorname{rank} \begin{bmatrix} \mathbf{u}_{a} \\ \mathbf{v}_{a} \mathbf{W}_{i} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathbf{u}_{a} \\ \mathbf{v}_{a} \mathbf{W}_{j} \end{bmatrix} = 1,$$

$$\operatorname{rank} \begin{bmatrix} \mathbf{u}_{b} \\ \mathbf{v}_{b} \mathbf{W}_{i} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathbf{u}_{b} \\ \mathbf{v}_{b} \mathbf{W}_{j} \end{bmatrix} = 1,$$

$$\operatorname{rank} \begin{bmatrix} \mathbf{u}_{c} \\ \mathbf{v}_{c} \mathbf{W}_{i} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathbf{u}_{c} \\ \mathbf{v}_{c} \mathbf{W}_{j} \end{bmatrix} = 1.$$

Therefore we have

$$\langle \mathbf{v}_a \mathbf{W}_i \rangle = \langle \mathbf{v}_a \rangle \mathbf{W}_i = \langle \mathbf{u}_a \rangle = \langle \mathbf{v}_a \mathbf{W}_j \rangle = \langle \mathbf{v}_a \rangle \mathbf{W}_j,$$

$$\langle \mathbf{v}_b \mathbf{W}_i \rangle = \langle \mathbf{v}_b \rangle \mathbf{W}_i = \langle \mathbf{u}_b \rangle = \langle \mathbf{v}_b \mathbf{W}_j \rangle = \langle \mathbf{v}_b \rangle \mathbf{W}_j,$$

$$\langle \mathbf{v}_c \mathbf{W}_i \rangle = \langle \mathbf{v}_c \rangle \mathbf{W}_i = \langle \mathbf{u}_c \rangle = \langle \mathbf{v}_c \mathbf{W}_j \rangle = \langle \mathbf{v}_c \rangle \mathbf{W}_j.$$

Multiplying \mathbf{W}_{i}^{-1} on the right, we have

$$\langle \mathbf{v}_a \rangle = \langle \mathbf{v}_a \rangle \mathbf{W}_j \mathbf{W}_i^{-1},$$

 $\langle \mathbf{v}_b \rangle = \langle \mathbf{v}_b \rangle \mathbf{W}_j \mathbf{W}_i^{-1},$
 $\langle \mathbf{v}_c \rangle = \langle \mathbf{v}_c \rangle \mathbf{W}_j \mathbf{W}_i^{-1}.$

It implies that $\langle \mathbf{v}_a \rangle$, $\langle \mathbf{v}_b \rangle$ and $\langle \mathbf{v}_c \rangle$ are three distinct $\mathbf{W}_j \mathbf{W}_i^{-1}$ -invariant subspaces of dimension 1. So we have $\mathbf{W}_j \mathbf{W}_i^{-1} = \mathbf{B}_j \mathbf{A}_j^{-1} \mathbf{A}_i \mathbf{B}_i^{-1} \in \langle \mathbf{I}_2 \rangle$ and therefore $\langle \mathbf{B}_j \mathbf{A}_j^{-1} \rangle = \langle \mathbf{B}_i \mathbf{A}_i^{-1} \rangle$.

Using the above lemmas, we establish the following key lemma on the repair bandwidth.

Lemma 12. The size of set $\{\mathcal{R}(\mathbf{F}_i) : i \in [t]\}$ is not larger than 4. Particularly, if $|\{\mathcal{R}(\mathbf{F}_i) : i \in [t]\}| = 4$ then t = k.

Proof. We prove this lemma by contradiction. Assume that there exist 5 different sets in $\mathcal{R}(\mathbf{F}_i)$ for $i \in [t]$. Let $a, b, c, b, e \in [t]$ such that $\mathcal{R}(\mathbf{F}_a) \prec \mathcal{R}(\mathbf{F}_b) \prec \mathcal{R}(\mathbf{F}_c) \prec \mathcal{R}(\mathbf{F}_d) \prec \mathcal{R}(\mathbf{F}_e)$ be the first five small sets under \prec .

Using Lemma 7, we have $d, e \in \mathcal{R}(\mathbf{F}_a) \cap \mathcal{R}(\mathbf{F}_b) \cap \mathcal{R}(\mathbf{F}_c) \cap [t]$. Using Lemma 12, for any $d^* \in \mathcal{R}(\mathbf{F}_a) \cap \mathcal{R}(\mathbf{F}_b) \cap \mathcal{R}(\mathbf{F}_b) \cap [t]$. Noting that $d \in \mathcal{R}(\mathbf{F}_d)$, we can conclude that

$$\operatorname{rank}\left(\mathbf{M}_{d}\mathbf{H}_{d^{*}}\right) = \operatorname{rank}\begin{bmatrix}\mathbf{u}_{d}\mathbf{A}_{d^{*}}\\\mathbf{v}_{d}\mathbf{B}_{d^{*}}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{u}_{d}\\\mathbf{v}_{d}\mathbf{B}_{d^{*}}\mathbf{A}_{d^{*}}^{-1}\end{bmatrix}$$
$$= \operatorname{rank}\begin{bmatrix}\mathbf{u}_{d}\\\mathbf{v}_{d}\mathbf{B}_{d}\mathbf{A}_{d}^{-1}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{u}_{d}\mathbf{A}_{d}\\\mathbf{v}_{d}\mathbf{B}_{d}\end{bmatrix}$$
$$= \operatorname{rank}\left(\mathbf{M}_{d}\mathbf{H}_{d}\right) = 2.$$

Therefore, by definition, we have $d^* \in \mathcal{R}(\mathbf{F}_d)$. It follows that $\overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k] \subseteq \mathcal{R}(\mathbf{F}_d)$. Since $e \in \overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k]$, we have $e \in \mathcal{R}(\mathbf{F}_d)$, contradicting Lemma 7. So we always have $|\{\mathcal{R}(\mathbf{F}_i) : i \in [t]\}| \leq 4$. Assume $|\{\mathcal{R}(\mathbf{F}_i) : i \in [t]\}| = 4$. Let $a, b, c, b \in [t]$ such that $\overline{\mathcal{R}(\mathbf{F}_a)} \vee \mathcal{R}(\mathbf{F}_b) \vee \mathcal{R}(\mathbf{F}_c) \vee \mathcal{R}(\mathbf{F}_d)$. Similarly to the previous discussion, we can conclude that $\overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k] \subseteq \mathcal{R}(\mathbf{F}_d)$. According to Lemma 8, we have $\{i : t+1 \leq i \leq k\} \subseteq \overline{\mathcal{R}(\mathbf{F}_a)} \cap \overline{\mathcal{R}(\mathbf{F}_b)} \cap \overline{\mathcal{R}(\mathbf{F}_c)} \cap [k]$. This implies that $\{i : t+1 \leq i \leq k\} \subseteq \mathcal{R}(\mathbf{F}_d)$. Recall that, in Lemma 8, $\bigcup_{j \in [t]} \mathcal{R}(\mathbf{F}_j) = \{1, \ldots, t\}$. Therefore, we can conclude that $\{i : t+1 \leq i \leq k\} \subseteq \{1, \ldots, t\}$. This leads us to deduce that $\{i : t+1 \leq i \leq k\}$ must be the empty set, which means that t = k.

Now, we are ready to prove the lower bound on the avg-min repair bandwidth in Theorem 1.

Proof of Theorem 1. Recall that

$$\bar{\beta}(\mathcal{C}) = \frac{1}{n} \sum_{i \in [n]} \mathrm{BW}(\mathbf{F}_i) = \frac{1}{n} \sum_{i \in [n]} |\mathcal{R}(\mathbf{F}_i)| + k,$$

where $\mathcal{R}(\mathbf{F}_i)$ is equal to $\mathcal{R}(\mathbf{F}_i)$. Since $|\mathcal{R}(\mathbf{F}_{k+1})| \leq |\mathcal{R}(\mathbf{F}_n)|$ and $\mathcal{R}(\mathbf{F}_{t+1}) = \cdots = \mathcal{R}(\mathbf{F}_{k+1})$, we have

$$\sum_{i \in [n]} |\mathcal{R}(\mathbf{F}_i)| \ge \sum_{i \in [t]} |\mathcal{R}(\mathbf{F}_i)| + (k+2-t)|\mathcal{R}(\mathbf{F}_{k+1})|.$$

If t < k, according to Lemma 12, we have $|\{\mathcal{R}(\mathbf{F}_i) : i \in [t]\}| \le 3$. Assume that $\{\mathcal{R}(\mathbf{F}_i) : i \in [t]\} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$. Let $s_1 = |\{i \in [t] : \mathcal{R}(\mathbf{F}_i) = \mathcal{S}_1\}|$, $s_2 = |\{i \in [t] : \mathcal{R}(\mathbf{F}_i) = \mathcal{S}_2\}|$, and $s_3 = |\{i \in [t] : \mathcal{R}(\mathbf{F}_i) = \mathcal{S}_3\}|$. It follows that $s_1 + s_2 + s_3 = t$ and $|\mathcal{S}_1| \ge s_1$, $|\mathcal{S}_2| \ge s_2$, and $|\mathcal{S}_3| \ge s_3$. Therefore, we can state that

$$\sum_{i \in [t]} |\mathcal{R}(\mathbf{F}_i)| = s_1 |\mathcal{S}_1| + s_2 |\mathcal{S}_2| + s_3 |\mathcal{S}_3|$$

$$\geq s_1^2 + s_2^2 + s_3^2 \geq (s_1 + s_2 + s_3)^2 / 3$$

$$= t^2 / 3.$$

The last inequality is derived from the Cauchy-Schwarz inequality. Since $|\mathcal{R}(\mathbf{F}_{k+1})| \ge \max\{|\mathcal{S}_1|, |\mathcal{S}_2|, |\mathcal{S}_3|\}$ and $|\mathcal{R}(\mathbf{F}_{k+1})| \ge k+1-t$, it follows that $|\mathcal{R}(\mathbf{F}_{k+1})| \ge \max\{\frac{t}{3}, k+1-t\}$. Thus, we can compute that

$$\sum_{i \in [n]} |\mathcal{R}(\mathbf{F}_i)| \ge \sum_{i \in [t]} |\mathcal{R}(\mathbf{F}_i)| + (k+2-t)|\mathcal{R}(\mathbf{F}_{k+1})|$$

$$\ge \frac{t^2}{3} + (k+2-t) \max\{\frac{t}{3}, k+1-t\}$$

$$\ge \frac{(k+1)(k+2)}{4}.$$

If t=k, by Lemma 12, we have $|\{\mathcal{R}(\mathbf{F}_i): i\in [k]\}| \leq 4$. Similar to the situation where t< k, we assume that $\{\mathcal{R}(\mathbf{F}_i): i\in [k]\} = \{\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4\}$. Let $s_1=|\{i\in [k]:\mathcal{R}(\mathbf{F}_i)=\mathcal{S}_1\}|,\ s_2=|\{i\in [k]:\mathcal{R}(\mathbf{F}_i)=\mathcal{S}_2\}|,\ s_3=|\{i\in [k]:\mathcal{R}(\mathbf{F}_i)=\mathcal{S}_3\}|\ \text{and}\ s_4=|\{i\in [k]:\mathcal{R}(\mathbf{F}_i)=\mathcal{S}_4\}|.$ It holds that $s_1+s_2+s_3+s_4=k$ and $|\mathcal{S}_1|\geq s_1$, $|\mathcal{S}_2|\geq s_2$, $|\mathcal{S}_3|\geq s_3$, $|\mathcal{S}_4|\geq s_4$. Therefore, we have

$$\sum_{i \in [k]} |\mathcal{R}(\mathbf{F}_i)| = s_1 |\mathcal{S}_1| + s_2 |\mathcal{S}_2| + s_3 |\mathcal{S}_3| + s_4 |\mathcal{S}_4|$$

$$\geq s_1^2 + s_2^2 + s_3^2 + s_4^2$$

$$\geq \frac{(s_1 + s_2 + s_3 + s_4)^2}{4}$$

$$= \frac{k^2}{4}.$$

Since $|\mathcal{R}(\mathbf{F}_{k+1})| \geq \lceil \frac{k}{4} \rceil$, we find that

$$\sum_{i \in [n]} |\mathcal{R}(\mathbf{F}_i)| \ge \sum_{i \in [k]} |\mathcal{R}(\mathbf{F}_i)| + 2|\mathcal{R}(\mathbf{F}_{k+1})|$$
$$\ge \frac{k^2}{4} + \frac{k}{2} = \frac{k(k+2)}{4}.$$

In conclusion,

$$\bar{\beta}(\mathcal{C}) = \frac{\sum_{i \in [n]} |\mathcal{R}(\mathbf{F}_i)|}{n} + k \ge \frac{5k}{4}.$$

3.2 The lower bound on avg-min repair I/O

Similar to $\mathcal{R}(\mathbf{M})$ and \mathcal{M}_i , for an 2×4 repair matrix \mathbf{M} , we define the following set

$$\mathcal{N}(\mathbf{M}) := \{ i \in [n] : \operatorname{nz}(\mathbf{M}\mathbf{H}_i) = 2 \}.$$

According to $d(\mathbf{M}) = k + 1$, the repair I/O corresponding to the repair matrix \mathbf{M} is

$$IO(\mathbf{M}) = \sum_{j \in [n]} nz (\mathbf{M}\mathbf{H}_j) - 2 = k + |\mathcal{N}(\mathbf{M})|.$$

Further, we define

$$\mathcal{M}_i' := \{ \mathbf{M} \in \mathbb{F}_q^{\ell \times r\ell} : \operatorname{rank}(\mathbf{M}) = \ell, \ i \in \mathcal{N}(\mathbf{M}) \}.$$

Similar as the definition of \mathbf{F}_i in (11), we define

$$\mathbf{F}_{i}' = \operatorname{argmin}_{\mathbf{M} \in \mathcal{M}_{i}'} \{ \mathcal{N}(\mathbf{M}) \} \text{ under the order } \prec . \tag{17}$$

Since $\mathcal{R}(\mathbf{M}) \subseteq \mathcal{N}(\mathbf{M})$, we have $\mathcal{M}_i \subseteq \mathcal{M}_i'$. Therefore,

$$\begin{split} \bar{\gamma}(\mathcal{C}) &= \sum_{i \in [n]} \min_{\mathbf{M} \in \mathcal{M}_i} \{ \mathrm{IO}(\mathbf{M}) \} \\ &\geq \sum_{i \in [n]} \min_{\mathbf{M} \in \mathcal{M}_i'} \{ \mathrm{IO}(\mathbf{M}) \} = \sum_{i \in [n]} \{ \mathrm{IO}(\mathbf{F}_i') \}. \end{split}$$

Remark 2. Note that, for every node C_i , $i \in [n]$, we always have $i \in \mathcal{N}(\mathbf{F}'_i)$.

Lemma 13. For any two distinct nodes C_i and C_j , $i, j \in [n]$, if $\mathcal{N}(\mathbf{F}'_i) \prec \mathcal{N}(\mathbf{F}'_j)$, then $j \notin \mathcal{N}(\mathbf{F}'_i)$.

Proof. We prove this by contradiction. Suppose that $j \in \mathcal{N}(\mathbf{F}_i')$, we have $\mathbf{F}_i' \in \mathcal{M}_j'$. In either case, this implies that $\mathcal{N}(\mathbf{F}_j') \preceq \mathcal{N}(\mathbf{F}_i')$. Since $\mathcal{N}(\mathbf{F}_j')$ is minimized under \prec . This contradicts the assumption that $\mathcal{N}(\mathbf{F}_i') \prec \mathcal{N}(\mathbf{F}_j')$. Therefore, we conclude that $j \notin \mathcal{N}(\mathbf{F}_i')$.

Similarly, we suppose that the code C satisfies $\mathcal{N}(\mathbf{F}'_1) \leq \mathcal{N}(\mathbf{F}'_2) \leq \cdots \leq \mathcal{N}(\mathbf{F}'_n)$. Therefore, there must exist an integer $i \in [k]$ such that $\mathcal{N}(\mathbf{F}'_i) \prec \mathcal{N}(\mathbf{F}_{k+1})$. Then we define t as follows:

$$t = \max\{i \in [k] : \mathcal{N}(\mathbf{F}_i') \prec \mathcal{N}(\mathbf{F}_{k+1}')\}. \tag{18}$$

Lemma 14. The integer t satisfies $n - |\mathcal{N}(\mathbf{F}'_{k+1})| - 1 \le t$ and $\bigcup_{i \in [t]} \mathcal{N}(\mathbf{F}'_i) = [t]$.

Now, we fix the parity-check matrix of the code C in a systematic form as follows:

$$H = \begin{bmatrix} \mathbf{A}_1 & \dots & \mathbf{A}_k & \mathbf{I}_2 \\ \mathbf{B}_1 & \dots & \mathbf{B}_k & & \mathbf{I}_2 \end{bmatrix},$$

where all the block $\mathbf{A}_i, \mathbf{B}_i$ are 2×2 matrices. Further, by the MDS property, all \mathbf{A}_i and $\mathbf{B}_i, i \in [k]$ are invertible matrices.

Recall that $k+1 \notin \bigcup_{i \in [t]} \mathcal{N}(\mathbf{F}'_i)$ and $k+2 \notin \bigcup_{i \in [t]} \mathcal{N}(\mathbf{F}'_i)$. For any $i \in [t]$ and $j \in \{k+1, k+2\}$, $\operatorname{nz}(\mathbf{F}'_i\mathbf{H}_j) \leq 1$ is guaranteed by the definition of $\mathcal{N}(\mathbf{F}'_i)$. Combining with $d(\mathbf{F}'_i) = k+1$, it implies that when we repair the node C_i , $i \in [t]$, each of the two nodes C_{k+1} and C_{k+2} only needs to transmit one symbol.

From the above discussion, during the repairing of node $C_i, i \in [t]$, we assume that the nodes C_{n-1} and C_n transmit $\mathbf{u}_i C_{n-1}$ and $\mathbf{v}_i C_n$, respectively, for some nonzero vectors $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^2$.

Since

$$\operatorname{nz}\left(\mathbf{F}_{i}'\mathbf{H}_{k+1}\right) = \operatorname{nz}\begin{bmatrix}\mathbf{u}_{i}\\\mathbf{0}\end{bmatrix} \le 1$$

and

$$\operatorname{nz}\left(\mathbf{F}_{i}'\mathbf{H}_{k+2}\right) = \operatorname{nz}\begin{bmatrix}\mathbf{0}\\\mathbf{v}_{i}\end{bmatrix} \leq 1,$$

combining with the fact that vectors \mathbf{u}_i and \mathbf{v}_i are nonzero, we have

$$\{\langle \mathbf{u}_i \rangle, \langle \mathbf{v}_i \rangle : i \in [t]\} \subseteq \{\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle\},\$$

where $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. So there are only four choices of $(\langle \mathbf{u}_i \rangle, \langle \mathbf{v}_i \rangle)$ for all $i \in [t]$: $(\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_1 \rangle), (\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle)$, $(\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_1 \rangle), (\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_2 \rangle)$.

Lemma 15. For each $i \in [t]$, the repair matrix \mathbf{F}'_i can be written as

$$\mathbf{F}_i' = \Lambda \begin{bmatrix} \mathbf{u}_i & \\ & \mathbf{v}_i \end{bmatrix},$$

where Λ is a 2×2 invertible matrix.

Since the two repair matrices \mathbf{F}'_i and $\Lambda^{-1}\mathbf{F}'_i$ behave the same during the repair process, we always assume that

$$\mathbf{F}_i' = \begin{bmatrix} \mathbf{u}_i & \\ & \mathbf{v}_i \end{bmatrix} \text{ for all } i \in [t].$$

Let $\overline{\mathcal{N}(\mathbf{F}_i')} = [n] \setminus \mathcal{N}(\mathbf{F}_i')$ be the complement set of $\mathcal{N}(\mathbf{F}_i')$. It is evident that $i \notin \overline{\mathcal{N}(\mathbf{F}_i')}$. By definition, we have

$$1 \le \operatorname{rank} \begin{bmatrix} \mathbf{u}_i \mathbf{A}_j \\ \mathbf{v}_i \mathbf{B}_j \end{bmatrix} \le \operatorname{nz} \begin{bmatrix} \mathbf{u}_i \mathbf{A}_j \\ \mathbf{v}_i \mathbf{B}_j \end{bmatrix} = 1 \tag{19}$$

for any $j \in \overline{\mathcal{N}(\mathbf{F}_i')} \cap [k]$.

Lemma 16. For any $i, j \in [t]$,

- 1. if $\langle \mathbf{u}_i \rangle = \langle \mathbf{u}_j \rangle$ and $\langle \mathbf{v}_i \rangle = \langle \mathbf{v}_j \rangle$, then $\overline{\mathcal{N}(\mathbf{F}_i')} = \overline{\mathcal{N}(\mathbf{F}_i')}$;
- 2. if $\langle \mathbf{u}_i \rangle = \langle \mathbf{u}_j \rangle$, $\langle \mathbf{v}_i \rangle \neq \langle \mathbf{v}_j \rangle$ or $\langle \mathbf{u}_i \rangle \neq \langle \mathbf{u}_j \rangle$, $\langle \mathbf{v}_i \rangle = \langle \mathbf{v}_j \rangle$, then $\overline{\mathcal{N}(\mathbf{F}_i')} \cap \overline{\mathcal{N}(\mathbf{F}_j')} \cap [k] = \emptyset$.

Using the above lemmas, we establish the following properties of $\mathcal{N}(\mathbf{F}_i')$ where $i \in [t]$.

Lemma 17. 1. Let $a, b \in [t]$ and assume $a \leq b$. If $\mathcal{N}(\mathbf{F}'_a) \cup \mathcal{N}(\mathbf{F}'_b) = [t]$, $\mathcal{N}(\mathbf{F}'_b) = \mathcal{N}(\mathbf{F}'_b)$.

- 2. For any a in [t], there exists $b \in [t]$ such that $\mathcal{N}(\mathbf{F}'_a) \cup \mathcal{N}(\mathbf{F}'_b) = [t]$.
- 3. $|\{\mathcal{N}(\mathbf{F}_i'): i \in [t]\}| \le 3.$
- Proof. 1. Recall that $\mathcal{N}(\mathbf{F}'_a) \leq \mathcal{N}(\mathbf{F}'_b) \leq \mathcal{N}(\mathbf{F}'_t)$. If we assume $\mathcal{N}(\mathbf{F}'_b) \neq \mathcal{N}(\mathbf{F}'_t)$, it implies that $\mathcal{N}(\mathbf{F}'_b) \prec \mathcal{N}(\mathbf{F}'_t)$. By Lemma 13 we have $t \notin \mathcal{N}(\mathbf{F}'_a) \cup \mathcal{N}(\mathbf{F}'_b) = [t]$. It presents a contradiction. Therefore, we have $\mathcal{N}(\mathbf{F}'_b) = \mathcal{N}(\mathbf{F}'_t)$.
 - 2. Let a be an element in [t]. According to Lemma 16 (2), if there exists $b \in [t]$ such that $\langle \mathbf{u}_b \rangle = \langle \mathbf{u}_a \rangle$, $\langle \mathbf{v}_b \rangle \neq \langle \mathbf{v}_a \rangle$ or $\langle \mathbf{u}_b \rangle \neq \langle \mathbf{u}_a \rangle$, $\langle \mathbf{v}_b \rangle = \langle \mathbf{v}_a \rangle$, then $\mathcal{N}(\mathbf{F}_a) \cap \mathcal{N}(\mathbf{F}_b) \cap [t] = \emptyset$. It follows that $\mathcal{N}(\mathbf{F}_a) \cup \mathcal{N}(\mathbf{F}_b) = [t]$, which completes the proof. Otherwise, for any $b \in [t]$, either $\langle \mathbf{u}_b \rangle = \langle \mathbf{u}_a \rangle$, $\langle \mathbf{v}_b \rangle = \langle \mathbf{v}_a \rangle$, or $\langle \mathbf{u}_b \rangle \neq \langle \mathbf{u}_a \rangle$, $\langle \mathbf{v}_b \rangle \neq \langle \mathbf{v}_a \rangle$. This restricts $(\langle \mathbf{u}_i \rangle, \langle \mathbf{v}_i \rangle)$ to choosing only two of the four situations. For example, if $(\langle \mathbf{u}_a \rangle, \langle \mathbf{v}_a \rangle) = (\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_1 \rangle)$, then

$$\{(\langle \mathbf{u}_i \rangle, \langle \mathbf{v}_i \rangle) : i \in [t]\} \subseteq \{(\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_1 \rangle), (\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_2 \rangle)\}.$$

According to Lemma 16 (1), we have $|\{\mathcal{N}(\mathbf{F}'_i): i \in [t]\}| \le 2$. Recall that in Lemma 14, we have $\bigcup_{i \in [t]} \mathcal{N}(\mathbf{F}'_i) = [t]$. There exist $b \in [t]$ such that $\mathcal{N}(\mathbf{F}'_a) \cup \mathcal{N}(\mathbf{F}'_b) = \bigcup_{i \in [t]} \mathcal{N}(\mathbf{F}'_i) = [t]$.

3. Recall that the choices of $(\langle \mathbf{u}_i \rangle, \langle \mathbf{v}_i \rangle)$ for all $i \in [t]$ are $(\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_1 \rangle)$, $(\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle)$, $(\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_1 \rangle)$, $(\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_1 \rangle)$, $(\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_2 \rangle)$. By Lemma 16 (1), we have $|\{\mathcal{N}(\mathbf{F}'_i) : i \in [t]\}| \le 4$. If $|\{\mathcal{N}(\mathbf{F}'_i) : i \in [t]\}| = 4$, it implies that

$$\{(\langle \mathbf{u}_i \rangle, \langle \mathbf{v}_i \rangle) : i \in [t]\}$$

$$=\{(\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_1 \rangle), (\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle), (\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_1 \rangle), (\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_2 \rangle)\}.$$

Assume $\{\mathcal{N}(\mathbf{F}_i'): i \in [t]\} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\}$ and $\mathcal{S}_1 \prec \mathcal{S}_2 \prec \mathcal{S}_3 \prec \mathcal{S}_4$. For any 3 elements of $\{(\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_1 \rangle), (\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_1 \rangle), (\langle \mathbf{e}_2 \rangle, \langle \mathbf{e}_2 \rangle)\}$, we can find 2 elements $(\langle \mathbf{e}_a \rangle, \langle \mathbf{e}_b \rangle), (\langle \mathbf{e}_c \rangle, \langle \mathbf{e}_d \rangle)$ such that either $\langle \mathbf{e}_a \rangle = \langle \mathbf{e}_c \rangle, \langle \mathbf{e}_b \rangle \neq \langle \mathbf{e}_d \rangle$ or $\langle \mathbf{e}_a \rangle \neq \langle \mathbf{e}_c \rangle, \langle \mathbf{e}_b \rangle = \langle \mathbf{e}_d \rangle$, where $a, b, c, d \in [2]$. According to Lemma 16 (2), for any 3 elements of $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\}$, there must exist a pair of elements whose cup is [t]. Thus, we can find two elements of $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$ whose cup is [t]. This contradicts Lemma 17 (2). In conclusion, we have $|\{\mathcal{N}(\mathbf{F}_i'): i \in [t]\}| \leq 3$.

The proof of the lower bound on the avg-min repair I/O in Theorem 3 is as follows.

Proof of Theorem 3. Recall that

$$\bar{\gamma}(\mathcal{C}) = \frac{1}{n} \sum_{i \in [n]} \mathrm{IO}(\mathbf{F}'_i) = \frac{1}{n} \sum_{i \in [n]} |\mathcal{N}(\mathbf{F}'_i)| + k,$$

where $\mathcal{N}(\mathbf{F}_i')$ is equal to $\mathcal{N}(\mathbf{F}_i')$. Since $|\mathcal{N}(\mathbf{F}_{k+1}')| \le |\mathcal{N}(\mathbf{F}_n')|$ and $\mathcal{N}(\mathbf{F}_{t+1}') = \cdots = \mathcal{N}(\mathbf{F}_{k+1}')$, we have

$$\sum_{i \in [n]} |\mathcal{N}(\mathbf{F}_i')| \geq \sum_{i \in [t]} |\mathcal{N}(\mathbf{F}_i')| + (k+2-t)|\mathcal{N}(\mathbf{F}_{k+1}')|.$$

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Lemma 17 (3) tell us that $|\{\mathcal{N}(\mathbf{F}_i'): i \in [t]\}| \leq 3$. In the following, we will discuss the value of $\sum_{i \in [t]} |\mathcal{N}(\mathbf{F}_i')|$ case by case.

Case 1: $|\{\mathcal{N}(\mathbf{F}'_i) : i \in [t]\}| = 1.$

It implies that $\mathcal{N}(\mathbf{F}'_1) = \cdots = \mathcal{N}(\mathbf{F}'_t)$. Recall that $i \in \mathcal{N}(\mathbf{F}'_i)$, we have $\sum_{i \in [t]} |\mathcal{N}(\mathbf{F}'_i)| \ge t^2$.

Case 2: $|\{\mathcal{N}(\mathbf{F}'_i) : i \in [t]\}| = 2.$

Assume that $\{\mathcal{N}(\mathbf{F}_i'): i \in [t]\} = \{\mathcal{S}_1, \mathcal{S}_2\}$ and $\mathcal{S}_1 \prec \mathcal{S}_2$. Let $\mathcal{T}_1 = \{i \in [t]: \mathcal{N}(\mathbf{F}_i') = \mathcal{S}_1\}$ and $\mathcal{T}_2 = \{i \in [t]: \mathcal{N}(\mathbf{F}_i') = \mathcal{S}_2\}$. Recall that $i \in \mathcal{N}(\mathbf{F}_i')$, we have $\mathcal{T}_1 \subseteq \mathcal{S}_1$, $\mathcal{T}_2 \subseteq \mathcal{S}_2$ and $|\mathcal{T}_1| + |\mathcal{T}_2| = t$. We can compute that

$$\sum_{i \in [t]} |\mathcal{N}(\mathbf{F}_i')| = |\mathcal{T}_1||\mathcal{S}_1| + |\mathcal{T}_2||\mathcal{S}_2| \ge |\mathcal{T}_1|^2 + |\mathcal{T}_2|^2$$

$$\ge (|\mathcal{T}_1| + |\mathcal{T}_2|)^2/2 \quad \text{(Cauchy-Schwarz inequality)}$$

$$\ge t^2/2.$$

Case 3: $|\{\mathcal{N}(\mathbf{F}'_i) : i \in [t]\}| = 3.$

Assume that $\{\mathcal{N}(\mathbf{F}_i'): i \in [t]\} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$ and $\mathcal{S}_1 \prec \mathcal{S}_2 \prec \mathcal{S}_3$. Let $\mathcal{T}_1 = \{i \in [t]: \mathcal{N}(\mathbf{F}_i') = \mathcal{S}_1\}$, $\mathcal{T}_2 = \{i \in [t]: \mathcal{N}(\mathbf{F}_i') = \mathcal{S}_2\}$ and $\mathcal{T}_3 = \{i \in [t]: \mathcal{N}(\mathbf{F}_i') = \mathcal{S}_3\}$. It implies that $\mathcal{T}_1 \subseteq \mathcal{S}_1, \mathcal{T}_2 \subseteq \mathcal{S}_2, \mathcal{T}_3 \subseteq \mathcal{S}_3$ and $|\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = t$. By Lemma 13, we have $(\mathcal{T}_2 \cup \mathcal{T}_3) \cap \mathcal{S}_1 = \emptyset$. And so $\mathcal{S}_1 = \mathcal{T}_1$. According to Lemma 17 (1) and (2), we can get $\mathcal{S}_1 \cup \mathcal{S}_3 = [t]$ and $\mathcal{S}_2 \cup \mathcal{S}_3 = [t]$. It implies that $[t] \setminus \mathcal{S}_3 \subseteq \mathcal{S}_1 \cap \mathcal{S}_2$. Recall that $\mathcal{T}_2 \subseteq \mathcal{S}_2$, we can get

$$\begin{aligned} |\mathcal{S}_{2}| &\geq |([t] \setminus \mathcal{S}_{3}) \cup \mathcal{T}_{2}| = |[t] \setminus \mathcal{S}_{3}| + |\mathcal{T}_{2}| - |([t] \setminus \mathcal{S}_{3}) \cap \mathcal{T}_{2}| \\ &\geq |[t] \setminus \mathcal{S}_{3}| + |\mathcal{T}_{2}| - |\mathcal{S}_{1} \cap \mathcal{T}_{2}| \\ &= |[t] \setminus \mathcal{S}_{3}| + |\mathcal{T}_{2}| - |\mathcal{T}_{1} \cap \mathcal{T}_{2}| \\ &= |[t] \setminus \mathcal{S}_{3}| + |\mathcal{T}_{2}|. \end{aligned}$$

Let $c = |[t] \setminus S_3|$. Since $[t] \setminus S_3 \subseteq S_1 \cap S_2$, we have $0 \le c \le |S_1| = |T_1|$. Note that $|S_3| = t - |[t] \setminus S_3| = t - c$, we can compute that

$$\begin{split} & \sum_{i \in [t]} |\mathcal{N}(\mathbf{F}_i')| = |\mathcal{T}_1||\mathcal{S}_1| + |\mathcal{T}_2||\mathcal{S}_2| + |\mathcal{T}_3||\mathcal{S}_3| \\ & \geq |\mathcal{T}_1|^2 + |\mathcal{T}_2|(|\mathcal{T}_2| + c) + |\mathcal{T}_3|(t - c) \\ & = |\mathcal{T}_1|^2 + |\mathcal{T}_2|^2 + |\mathcal{T}_3|t + (|\mathcal{T}_2| - |\mathcal{T}_3|)c \\ & \geq \min\{|\mathcal{T}_1|^2 + |\mathcal{T}_2|^2 + |\mathcal{T}_3|t, |\mathcal{T}_1|^2 + |\mathcal{T}_3|^2 + |\mathcal{T}_2|t\} \\ & \geq \min\{\frac{(|\mathcal{T}_1| + |\mathcal{T}_2|)^2}{2} + |\mathcal{T}_3|t, \frac{(|\mathcal{T}_1| + |\mathcal{T}_3|)^2}{2} + |\mathcal{T}_2|t\} \\ & = \min\{\frac{(t - |\mathcal{T}_3|)^2}{2} + |\mathcal{T}_3|t, \frac{(t - |\mathcal{T}_2|)^2}{2} + |\mathcal{T}_2|t\} \\ & = \min\{\frac{t^2 + |\mathcal{T}_3|^2}{2}, \frac{t^2 + |\mathcal{T}_2|^2}{2}\} \\ & \geq \frac{t^2 + 1}{2}. \end{split}$$

From the discussion of Cases 1, 2, and 3, we can know that $\sum_{i \in [t]} |\mathcal{N}(\mathbf{F}_i')| \ge t^2/2$. Since $|\mathcal{N}(\mathbf{F}_{k+1}')| \ge |\mathcal{N}(\mathbf{F}_t')| \ge t/2$ and $|\mathcal{N}(\mathbf{F}_{k+1}')| \ge k+1-t$, we can compute that

$$\sum_{i \in [n]} |\mathcal{N}(\mathbf{F}_i')| \ge \sum_{i \in [t]} |\mathcal{N}(\mathbf{F}_i')| + (k+2-t)|\mathcal{N}(\mathbf{F}_{k+1}')|$$

$$\ge t^2/2 + (k+2-t)\max\{t/2, k+1-t\}$$

$$\ge \frac{(k+1)(k+2)}{3}$$

Hence, $\bar{\gamma}(\mathcal{C}) = \frac{\sum_{i \in [n]} |\mathcal{N}(\mathbf{F}'_i)|}{n} + k \ge \frac{4k+1}{3}$.

4 (k+2,k,2) MDS array codes with optimal repair

We construct two classes of explicit $(n = k+2, k, \ell = 2)$ MDS array codes. The first class of MDS array codes achieves the lower bounds on the avg-min repair bandwidth in Theorem 1 and max-min repair bandwidth in Corollary 2

asymptotically. The second class of MDS array codes achieves the lower bounds on the avg-min repair I/O in Theorem 3 and the max-min repair I/O in Corollary 4. These two classes of codes show that all the proposed lower bounds are tight.

Recall that an $(n = k + 2, k, \ell = 2)$ MDS array code comprises n nodes, denoted by C_1, C_2, \ldots, C_n , where each node is a column vector of length $\ell = 2$. An MDS array code can be defined by the parity check sub-matrices \mathbf{H}_i , $i \in [n]$, and the parity-check equations (1) as described in Section 2. The defined code is MDS if and only if any two parity check sub-matrices $\mathbf{H}_i, \mathbf{H}_j$ (where $1 \le i < j \le n$) can form an invertible matrix. The linear repair process for each node C_i can be described by a 2×4 repair matrix \mathbf{M}_i (where $i \in [n]$). The repair bandwidth and repair I/O for each node C_i can be calculated using the formulas (3) and (4), respectively.

4.1 (k+2,k,2) MDS array codes with optimal bandwidth

Code construction (C_1). Let \mathbb{F}_q be the finite field of size $q \geq n+3$, and α a primitive element of \mathbb{F}_q . Then, for each $i, 0 \leq i \leq n+2$, we define an element λ_i and the corresponding Vandermonde column vector as

$$\lambda_i = \alpha^i$$
 and $\lambda_i = [1, \lambda_i]^T$.

Any two distinct λ_i and λ_j (where $0 \le i, j \le n+2$) are linearly independent.

We evenly partition the n nodes C_1, C_2, \ldots, C_n into four node groups with index sets are denoted by $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$. Each of the first $n \mod 4$ groups contains $\lceil n/4 \rceil$ nodes, while each of the remaining groups contains $\lfloor n/4 \rfloor$ nodes. For example, if n = 6, then $\mathcal{G}_1 = \{1, 2\}$, $\mathcal{G}_2 = \{3, 4\}$, $\mathcal{G}_3 = \{5\}$, and $\mathcal{G}_4 = \{6\}$.

We give the parity check sub-matrix \mathbf{H}_i and repair matrix \mathbf{M}_i for each node C_i (where $1 \leq i \leq n$) in Table 1. The following lemma shows that the resulting code C_1 is MDS.

i		M	· i		\mathbf{H}_i
$i\in\mathcal{G}_1$	$\left[\begin{array}{c}1\\0\end{array}\right]$	0 1	0	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$egin{bmatrix} egin{bmatrix} oldsymbol{\lambda}_{i-1} & -oldsymbol{\lambda}_i \ oldsymbol{0} & oldsymbol{\lambda}_i \end{bmatrix}$
$i\in\mathcal{G}_2$	$\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$	0 0	1 0	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$egin{bmatrix} oldsymbol{\lambda}_i & 0 \ -oldsymbol{\lambda}_i & oldsymbol{\lambda}_{i+1} \end{bmatrix}$
$i\in\mathcal{G}_3$		0 1	1 0	0 1	$\left[egin{array}{ccc} oldsymbol{\lambda}_i & oldsymbol{0} \ oldsymbol{0} & oldsymbol{\lambda}_{i+2} \end{array} ight]$
$i\in\mathcal{G}_4$	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$0 \\ \alpha$	$\begin{array}{c} \alpha \\ 0 \end{array}$	$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$	$egin{bmatrix} oldsymbol{\lambda}_{i+2} & oldsymbol{0} \ oldsymbol{0} & oldsymbol{\lambda}_{i+2} \end{bmatrix}$

Table 1: Repair matrix \mathbf{M}_i and parity check sub-matrix \mathbf{H}_i for each node C_i (where $1 \leq i \leq n$) of code C_1 .

$\operatorname{rank}\left(\mathbf{M}_{i}\mathbf{H}_{j}\right)$	$j \in \mathcal{G}_1$	$j \in \mathcal{G}_2$	$j \in \mathcal{G}_3$	$j \in \mathcal{G}_4$
$i \in \mathcal{G}_1$	2	1	1	1
$i \in \mathcal{G}_2$	1	2	1	1
$i \in \mathcal{G}_3$	1	1	2	1
$i \in \mathcal{G}_4$	1	1	1	2

Table 2: The value of rank $(\mathbf{M}_i \mathbf{H}_j)$ for each $i, j \in [n]$ of code \mathcal{C}_1 .

Lemma 18. The code C_1 defined by the parity check sub-matrices in Table 1 is a (k+2,k,2) MDS array code.

Proof. We only need to show that each square matrix $[\mathbf{H}_i, \mathbf{H}_j]$ is invertible for all distinct indices i, j (where $1 \le i < j \le n$).

One key observation for the structure of the parity check sub-matrices in Table 1 is that any square matrix $[\mathbf{H}_i, \mathbf{H}_j]$ can be transformed into a block triangular matrix by an invertible transformation, and the blocks on the main block diagonal are invertible, which implies that the square matrix $[\mathbf{H}_i, \mathbf{H}_j]$ is also invertible.

If $i \in \mathcal{G}_1$ and $j \in \mathcal{G}_2$, we can add the second block row of $[\mathbf{H}_i, \mathbf{H}_j]$ to the first block row, and then permute the columns of the matrix to obtain a block triangular matrix. The procedure is as follows:

$$\begin{bmatrix} \lambda_{i-1} & -\lambda_i & \lambda_j & \mathbf{0} \\ \mathbf{0} & \lambda_i & -\lambda_j & \lambda_{j+1} \end{bmatrix}$$

$$\xrightarrow{\text{row transformation}} \begin{bmatrix} \lambda_{i-1} & \mathbf{0} & \mathbf{0} & \lambda_{j+1} \\ \mathbf{0} & \lambda_i & -\lambda_j & \lambda_{j+1} \end{bmatrix}$$

$$\xrightarrow{\text{column permutation}} \begin{bmatrix} \lambda_{i-1} & \lambda_{j+1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \lambda_{j+1} & \lambda_i & -\lambda_j \end{bmatrix}.$$

Since the square matrix on the main block diagonal is invertible, the original matrix $[\mathbf{H}_i, \mathbf{H}_j]$ is also invertible. For other cases, we can directly permute the columns of the matrix to obtain a block triangular matrix.

Repair bandwidth. Recall the linear repair process of each C_i driven by the repair matrix \mathbf{M}_i (where $i \in [n]$) described in Section 2. We can calculate the repair bandwidth for C_i by the formulas (3).

Firstly, one can directly check from Table 1 that for each $i \in [n]$, rank $(\mathbf{M}_i \mathbf{H}_i) = 2$. Thus, \mathbf{M}_i can define a linear repair process for node C_i . Then we summarize the values of rank $(\mathbf{M}_i \mathbf{H}_j)$ for all $i, j \in [n]$ in Table 2. Now we can calculate the repair bandwidth for each node C_i (where $i \in [n]$) by the formula (3):

$$BW(\mathbf{M}_i) = \sum_{j=1}^{n} rank (\mathbf{M}_i \mathbf{H}_j) - 2$$
$$= 2|\mathcal{G}_z| + (n - |\mathcal{G}_z|) - 2$$
$$= k + |\mathcal{G}_z|,$$

where $i \in \mathcal{G}_z$. According to the group partition, we know that $|\mathcal{G}_z| = \lceil n/4 \rceil$ or $|\mathcal{G}_z| = \lfloor n/4 \rfloor$. The max-min repair bandwidth for code \mathcal{C}_1 is

$$\beta(C_1) = k + \lceil n/4 \rceil = \left\lceil \frac{5k+2}{4} \right\rceil,$$

and the avg-min repair bandwidth for code C_1 is

$$\bar{\beta}(\mathcal{C}_1) = k + \lfloor n/4 \rfloor + \frac{1}{n} (n \mod 4) \lceil n/4 \rceil.$$

We can conclude that the max-min repair bandwidth of code C_1 achieves the lower bound in Corollary 2 when $k \mod 4 = 1$ or $k \mod 4 = 2$. In addition, the avg-min repair bandwidth of code C_1 achieves the lower bound in Theorem 1 when k goes to infinity.

4.2 (k+2,k,2) MDS array codes with optimal repair I/O

Code construction (C_2). Let \mathbb{F}_q be a finite field of size $q \geq n + 1$, and $\lambda_0, \lambda_1, \ldots, \lambda_n$ be n + 1 distinct elements in \mathbb{F}_q . We also denote a Vandermonde column vector $\boldsymbol{\lambda}_i = [1, \lambda_i]^T$ for each λ_i (where $0 \leq i \leq n$).

We evenly partition the n nodes C_1, C_2, \ldots, C_n into three node groups with index sets denoted by $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$. Each of the first $n \mod 3$ groups contains $\lceil n/3 \rceil$ nodes, while each of the remaining groups contains $\lfloor n/3 \rfloor$ nodes. For example, if n = 8, then $\mathcal{G}_1 = \{1, 2, 3\}$, $\mathcal{G}_2 = \{4, 5, 6\}$, $\mathcal{G}_3 = \{7, 8\}$.

We give the parity check sub-matrix \mathbf{H}_i and repair matrix \mathbf{M}_i for each node C_i (where $1 \leq i \leq n$) in Table 3. Lemma 19 shows that the resulting code C_2 is MDS. We omit the proof of this lemma since it is similar to the proof of Lemma 18.

Lemma 19. The code C_2 defined by the parity check sub-matrices in Table 3 is a (k+2,k,2) MDS array code.

Repair I/O. Recall the linear repair process of each C_i driven by the repair matrix \mathbf{M}_i (where $i \in [n]$) described in Section 2. We can calculate the repair I/O for C_i using the formula (4).

i		\mathbf{M}	$\cdot i$		\mathbf{H}_i
$i\in\mathcal{G}_1$	$\left[\begin{array}{c}1\\0\end{array}\right]$	0 1	0 0	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$egin{bmatrix} m{\lambda}_{i-1} & -m{\lambda}_i \ m{0} & m{\lambda}_i \end{bmatrix}$
$i\in\mathcal{G}_2$	$\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$	0	1 0	0 1	$egin{bmatrix} oldsymbol{\lambda}_i & 0 \ -oldsymbol{\lambda}_i & oldsymbol{\lambda}_{i-1} \end{bmatrix}$
$i\in\mathcal{G}_3$	$\left[\begin{array}{c}1\\0\end{array}\right]$	0	1 0	0 1	$\left[egin{array}{ccc} oldsymbol{\lambda}_i & 0 \ 0 & oldsymbol{\lambda}_{i+1} \end{array} ight]$

Table 3: Repair matrix \mathbf{M}_i and parity check sub-matrix \mathbf{H}_i for each node \mathcal{C}_i (where $1 \leq i \leq n$) of code \mathcal{C}_2 .

$\operatorname{nz}\left(\mathbf{M}_{i}\mathbf{H}_{j}\right)$	$j \in \mathcal{G}_1$	$j\in\mathcal{G}_2$	$j \in \mathcal{G}_3$
$i\in\mathcal{G}_1$	2	1	1
$i\in\mathcal{G}_2$	1	2	1
$i \in \mathcal{G}_3$	1	1	2

Table 4: The value of $\operatorname{nz}(\mathbf{M}_i\mathbf{H}_i)$ for each $i,j\in[n]$ of code \mathcal{C}_2 .

Firstly, from Table 3, one can directly check that for each $i \in [n]$, rank $(\mathbf{M}_i \mathbf{H}_i) = 2$. Thus, \mathbf{M}_i can define a linear repair process for node C_i . Then we summarize the values of $\operatorname{nz}(\mathbf{M}_i \mathbf{H}_j)$ for all $i, j \in [n]$ in Table 4. Now we can calculate the repair I/O for each node C_i (where $i \in [n]$) using the formula (4):

$$IO(\mathbf{M}_i) = \sum_{j=1}^n \operatorname{nz} (\mathbf{M}_i \mathbf{H}_j) - 2$$
$$= 2|\mathcal{G}_z| + (n - |\mathcal{G}_z|) - 2$$
$$= k + |\mathcal{G}_z|,$$

where $i \in \mathcal{G}_z$. According to the group partition of code \mathcal{C}_2 , we know that $|\mathcal{G}_z| = \lceil n/3 \rceil$ or $|\mathcal{G}_z| = \lfloor n/3 \rfloor$. The max-min repair I/O for code \mathcal{C}_1 is

$$\gamma(\mathcal{C}_1) = k + \lceil n/3 \rceil = \left\lceil \frac{4k+2}{3} \right\rceil,$$

and the avg-min repair I/O for code C_2 is

$$\bar{\gamma}(\mathcal{C}_2) = k + \lfloor n/3 \rfloor + \frac{1}{n} (n \operatorname{mod} 3) \lceil n/3 \rceil.$$

We can conclude that the max-min repair I/O of code C_2 achieves the lower bound in Corollary 4 when $k \mod 3 = 0$ or $k \mod 3 = 1$. In addition, the avg-min repair I/O of code C_2 achieves the lower bound in Theorem 3 when k goes to infinity.

5 Conclusion

We investigate lower bounds on the repair bandwidth and repair I/O for single-node failures in MDS array codes that have fixed sub-packetization. Using a linear-algebraic framework, we prove these bounds for codes with redundancy n - k = 2 and sub-packetization $\ell = 2$. We further present two explicit code constructions that meet the bounds, thereby establishing their tightness.

An interesting future research direction is to establish tight lower bounds for the repair bandwidth and repair I/O of the MDS array codes with general parameters (n, k, ℓ) . Deriving such bounds for specific small sub-packetization levels (e.g., $\ell = 2, 4, 8$) is also valuable, as these values align with practical implementations in distributed storage systems.

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