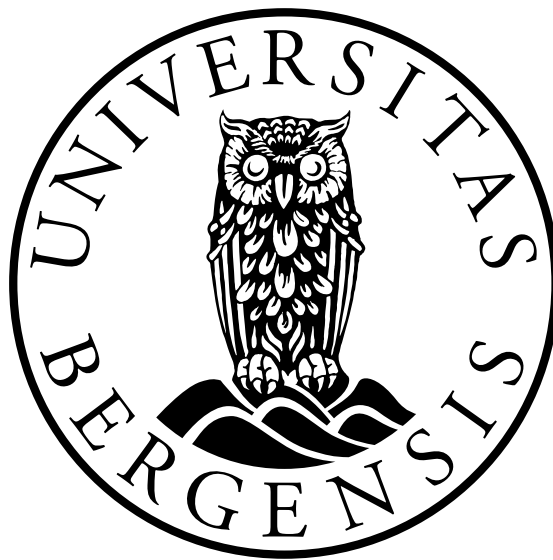


Numerical methods for nonlinear nonlocal water wave models

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Dissertation for the degree of Philosophiae Doctor (PhD)

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April 2017

Preface

This dissertation is submitted as a partial fulfillment of the requirements for the degree Doctor of Philosophy (PhD) at the Department of Mathematics, University of Bergen. The research was supported by the Research Council of Norway on grant no. 213474/F20.

The advisory committee has consisted of Henrik Kalisch (University of Bergen, Bergen, Norway), Dmitry Pelinovsky (McMaster University, Hamilton, Ontario, Canada) and Angel Durán (Universidad de Valladolid, Valladolid, Spain).

Acknowledgements

First of all I would like to thank Henrik Kalisch and Magnus Svärd who carried out the selection process of candidates for the PhD position. I am sincerely thankful to Professor Henrik Kalisch, my main supervisor, for sharing knowledge and guiding me, for his enthusiasm, support and encouragement. The thesis would not have been possible without you.

In the course of the doctorate studies, I have had fruitful collaboration with Olivier Verdier, Evgueni Dinvaï, Denys Dutykh, Ben Segal and others. I would like to thank them for productive discussions, advices on research questions and some inspiring ideas we came up with.

As a PhD Candidate I have been lucky to meet my colleagues and fellow students at the Department of Mathematics. They have always provided a friendly and supportive environment. I appreciate everything we have experienced during the time of our studies.

I would like to express my gratitude to my family for their continuous support and encouragement over the years of my studies. I am also thankful for all of my teachers and professors from schools and universities I studied at.

I am heartily grateful for my wife Assem. You have always been there for me. Without your love, support and understanding I would not have been able to go that far.

Outline

This thesis is organised in the following way. Part I contains general theoretical background on nonlinear wave models as well as description of methods used to solve the equations involved. Some properties of solutions to the equations and a summary of results are also given in the first part. Part 2 consists of the research papers that present scientific results in detail.

List of reseach papers included in Part II

Paper A:

Moldabayev, D., Kalisch, H., Dutykh, D.: *The Whitham equation as a model for surface water waves*, Phys. D **309**, 99–107 (2015),
<http://dx.doi.org/10.1016/j.physd.2015.07.010>.

Paper B:

Dinvay, E., Moldabayev, D., Dutykh, D., Kalisch, H.: *The Whitham equation with surface tension*, Nonlinear Dynamics, 1–14 (2017),
<http://dx.doi.org/10.1007/s11071-016-3299-7>.

Paper C:

Henrik Kalisch, Daulet Moldabayev, Olivier Verdier: *A numerical study of nonlinear dispersive wave models with SpecTraVVave*, **specify status of the paper**.

Paper D:

Benjamin Segal, Daulet Moldabayev, Henrik Kalisch, Bernard Deconinck: *Explicit solutions for a long-wave model with constant vorticity*, submitted to European Journal of Mechanics - B/Fluids.

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Part I

Background

Chapter 1

Introduction

This is the introduction [\[3\]](#)...

Chapter 2

Summary of results

This chapter provides an overview of the results achieved in the course of research work. Detailed

2.1 The Whitham equation as a model for surface water waves

The Whitham equation was proposed as an alternate model equation for the simplified description of unidirectional wave motion at the surface of an inviscid fluid. As the Whitham equation incorporates the full linear dispersion relation of the water wave problem, it is thought to provide a more faithful description of shorter waves of small amplitude than traditional long wave models such as the KdV equation. In this work, we identify a scaling regime in which the Whitham equation can be derived from the Hamiltonian theory of surface water waves. A Hamiltonian system of Whitham type allowing for two- way wave propagation is also derived. The Whitham equation is integrated numerically, and it is shown that the equation gives a close approximation of inviscid free surface dynamics as described by the Euler equations. The performance of the Whitham equation as a model for free surface dynamics is also compared to different free surface models: the KdV equation, the BBM equation, and the Padé (2,2) model. It is found that in a wide parameter range of amplitudes and wavelengths, the Whitham equation performs on par with or better than the three considered models.

In its simplest form, the water-wave problem concerns the flow of an incompressible inviscid fluid with a free surface over a horizontal impenetrable bed. In this situation, the fluid flow is described by the Euler equations with appropriate boundary conditions, and the dynamics of the free surface are of particular interest in the solution of this problem. There are a number of model equations which allow the approximate description of the evolution of the free surface without having to provide a complete solution of the fluid flow below the surface. In the present contribution, interest is focused

on the derivation and evaluation of a nonlocal water-wave model known as the Whitham equation. The equation is written as

$$\eta_t + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x + K_{h_0} * \eta_x = 0, \quad (2.1)$$

where the convolution kernel K_{h_0} is given in terms of the Fourier transform by

$$\mathcal{F}K_{h_0}(\xi) = \sqrt{\frac{g \tanh(h_0 \xi)}{\xi}}. \quad (2.2)$$

g is the gravitational acceleration, h_0 is the undisturbed depth of the fluid, and $c_0 = \sqrt{gh_0}$ is the corresponding long-wave speed. The convolution can be thought of as a Fourier multiplier operator, and (2.2) represents the Fourier symbol of the operator. The Whitham equation was proposed by Whitham [11] as an alternative to the well known Korteweg–de Vries (KdV) equation

$$\eta_t + c_0 \eta_x + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x + \frac{1}{6} c_0 h_0^2 \eta_{xxx} = 0. \quad (2.3)$$

The validity of the KdV equation as a model for surface water waves can be described as follows. Suppose a wave field with a prominent amplitude a and characteristic wavelength l is to be studied. The KdV equation is known to produce a good approximation of the evolution of the waves if the amplitude of the waves is small and the wavelength is large when compared to the undisturbed depth, and if in addition, the two non-dimensional quantities a/h_0 and h_0^2/l^2 are of similar size. The latter requirement can be written in terms of the Stokes number as

$$\mathcal{S} = \frac{al^2}{h_0^3} \sim 1. \quad (2.4)$$

While the KdV equation is a good model for surface waves if $\mathcal{S} \sim 1$, one notorious problem with the KdV equation is that it does not model accurately the dynamics of shorter waves. Recognizing this shortcoming of the KdV equation, Whitham proposed to use the same nonlinearity as the KdV equation, but couple it with a linear term which mimics the linear dispersion relation of the full water-wave problem. Thus, at least in theory, the Whitham equation can be expected to yield a description of the dynamics of shorter waves which is closer to the solutions of the more fundamental Euler equations which govern the flow.

The Whitham equation has been studied from a number of vantage points during recent years. In particular, the existence of traveling and solitary waves has been established in [5, 6]. Well posedness of a similar equation was investigated in [9–8], and a model with variable depth has been studied numerically in [2]. Moreover, it has been shown in [7, 10] that periodic solutions of the Whitham equation feature modulational instability for short enough waves in a similar way as small-amplitude periodic wave solutions of the water-wave problem. However, even though the equation is routinely mentioned in texts on nonlinear waves [4, 12], it appears that the performance of the Whitham equation in the description of surface water waves has not been investigated so far. The purpose of the present article is to give an asymptotic derivation of the Whitham equation as a model for surface water waves, and to confirm Whitham's expectation that the equation is a fair model for the description of time-dependent surface water waves. For the purpose of the derivation, we introduce an exponential scaling regime in which the Whitham equation can be derived asymptotically from an approximate Hamiltonian principle for surface water waves. In order to motivate the use of this scaling, note that the KdV equation has the property that wide classes of initial data decompose into a number of solitary waves and small-amplitude dispersive

residue [1]. For the KdV equations, solitary- wave solutions are known in closed form, and are given by

$$\eta = \frac{a}{h_0} \operatorname{sech}^2 \left(\sqrt{\frac{3a}{4h_0^3}} (x - ct) \right) \quad (2.5)$$

for a certain wave celerity c . These waves clearly comply with the amplitude–wavelength relation $a/h_0 \sim h_0^2/l^2$ which was mentioned above. It appears that the Whitham equation – as indeed do many other nonlinear dispersive equations – also has the property that broad classes of initial data rapidly decompose into ordered trains of solitary waves (see Fig. 1). Quantifying the amplitude–wavelength relation for these solitary waves yields an asymptotic regime which is expected to be relevant to the validity of the Whitham equation as a water wave model.

2.2 The Whitham equation with surface tension

2.3 A numerical study of nonlinear dispersive wave models with SpecTraVWave

2.4 Explicit solutions for a long-wave model with constant vorticity

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Part II

Scientific results

