Numerical methods for nonlinear nonlocal water wave models

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Dissertation for the degree of Philosophiae Doctor (PhD)

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April 2017

Preface

This dissertation is submitted as a partial fulfillment of the requirements for the degree Doctor of Philosophy (PhD) at the Department of Mathematics, University of Bergen. The research was supported by the Research Council of Norway on grant no. 213474/F20.

The advisory committee has consisted of Henrik Kalisch (University of Bergen, Bergen, Norway), Dmitry Pelinovsky (McMaster University, Hamilton, Ontario, Canada) and Angel Durán (Universidad de Valladolid, Valladolid, Spain).

Acknowledgements

First of all I would like to thank Henrik Kalisch and Magnus Svärd who carried out the selection process of candidates for the PhD position. I am sincerely thankful to Professor Henrik Kalisch, my main supervisor, for sharing knowledge and guiding me, for his enthusiasm, support and encouragement. The thesis would not have been possible without you.

In the course of the doctorate studies, I have had fruitful collaboration with Olivier Verdier, Evgueni Dinvay, Denys Dutykh, Ben Segal and others. I would like to thank them for productive discussions, advices on research questions and some inspiring ideas we came up with.

As a PhD Candidate I have been lucky to meet my colleagues and fellow students at the Department of Mathematics. They have always provided a friendly and supportive environment. I appreciate everything we have experienced during the time of our studies.

I would like to express my gratitude to my family for their continuous support and encouragement over the years of my studies. I am also thankful for all of my teachers and professors from schools and universities I studied at.

I am heartily grateful for my wife Assem. You have always been there for me. Without your love, support and understanding I would not have been able to go that far.

Outline

This thesis is organised in the following way. Part I contains general theoretical background on nonlinear wave models as well as description of methods used to solve the equations involved. Some properties of solutions to the equations and a summary of results are also given in the first part. Part 2 consists of the research papers that present scientific results in detail.

List of reseach papers included in Part II

Paper A:
Moldabayev, D., Kalisch, H., Dutykh, D.: *The Whitham equation as a model for surface water waves*, Phys. D **309**, 99–107 (2015),
http://dx.doi.org/10.1016/j.physd.2015.07.010.

Paper B:
Dinvay, E., Moldabayev, D., Dutykh, D., Kalisch, H.: *The Whitham equation with surface tension*, Nonlinear Dynamics, 1–14 (2017),
http://dx.doi.org/10.1007/s11071-016-3299-7.

Paper C:
Henrik Kalisch, Daulet Moldabayev, Olivier Verdier: *A numerical study of nonlinear dispersive wave models with SpecTraVVave*, specify status of the paper.

Paper D:
Benjamin Segal, Daulet Moldabayev, Henrik Kalisch, Bernard Deconinck: *Explicit solutions for a long-wave model with constant vorticity*, submitted to European Journal of Mechanics - B/Fluids.

Contents

Part I

Background

Chapter 1

Introduction

This is the introduction [6]. . .

Chapter 2

Summary of results

This chapter provides an overview of the results achieved in the course of research work. Detailed

2.1 The Whitham equation as a model for surface water waves

2.1.1 Introduction.

The Whitham equation was proposed as an alternate model equation for the simplified description of unidirectional wave motion at the surface of an inviscid fluid. As the Whitham equation incorporates the full linear dispersion relation of the water wave problem, it is thought to provide a more faithful description of shorter waves of small amplitude than traditional long wave models such as the KdV equation. In this work, we identify a scaling regime in which the Whitham equation can be derived from the Hamiltonian theory of surface water waves. A Hamiltonian system of Whitham type allowing for two- way wave propagation is also derived. The Whitham equation is integrated numerically, and it is shown that the equation gives a close approximation of inviscid free surface dynamics as described by the Euler equations. The performance of the Whitham equation as a model for free surface dynamics is also compared to different free surface models: the KdV equation, the BBM equation, and the Padé (2,2) model. It is found that in a wide parameter range of amplitudes and wavelengths, the Whitham equation performs on par with or better than the three considered models.

In its simplest form, the water-wave problem concerns the flow of an incompressible inviscid fluid with a free surface over a horizontal impenetrable bed. In this situation, the fluid flow is described by the Euler equations with appropriate boundary conditions, and the dynamics of the free surface are of particular interest in the solution of this problem. There are a number of model equations which allow the approximate description of the evolution of the free surface without having to provide a complete solution of the fluid flow below the surface. In the present contribution, interest is focused

on the derivation and evaluation of a nonlocal water-wave model known as the Whitham equation. The equation is written as

$$\eta_t + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + K_{h_0} * \eta_x = 0, \tag{2.1}$$

where the convolution kernel K_{h_0} is given in terms of the Fourier transform by

$$\mathcal{F}K_{h_0}(\xi) = \sqrt{\frac{g\tanh(h_0\xi)}{\xi}}. \tag{2.2}$$

g is the gravitational acceleration, h_0 is the undisturbed depth of the fluid, and $c_0 = \sqrt{gh_0}$ is the corresponding long-wave speed. The convolution can be thought of as a Fourier multiplier operator, and (2.2) represents the Fourier symbol of the operator. The Whitham equation was proposed by Whitham [18] as an alternative to the well known Korteweg–de Vries (KdV) equation

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{1}{6}c_0h_0^2\eta_{xxx} = 0. \tag{2.3}$$

The validity of the KdV equation as a model for surface water waves can be described as follows. Suppose a wave field with a prominent amplitude a and characteristic wavelength l is to be studied. The KdV equation is known to produce a good approximation of the evolution of the waves if the amplitude of the waves is small and the wavelength is large when compared to the undisturbed depth, and if in addition, the two non-dimensional quantities a/h_0 and h_0^2/l^2 are of similar size. The latter requirement can be written in terms of the Stokes number as

$$\mathcal{S} = \frac{al^2}{h_0^3} \sim 1. \tag{2.4}$$

While the KdV equation is a good model for surface waves if $\mathcal{S} \sim 1$, one notorious problem with the KdV equation is that it does not model accurately the dynamics of shorter waves. Recognizing this shortcoming of the KdV equation, Whitham proposed to use the same nonlinearity as the KdV equation, but couple it with a linear term which mimics the linear dispersion relation of the full water-wave problem. Thus, at least in theory, the Whitham equation can be expected to yield a description of the dynamics of shorter waves which is closer to the solutions of the more fundamental Euler equations which govern the flow.

The Whitham equation has been studied from a number of vantage points during recent years. In particular, the existence of traveling and solitary waves has been established in [8, 9]. Well posedness of a similar equation was investigated in [13-12], and a model with variable depth has been studied numerically in [2]. Moreover, it has been shown in [11, 17] that periodic solutions of the Whitham equation feature modulational instability for short enough waves in a similar way as small-amplitude periodic wave solutions of the water-wave problem. However, even though the equation is routinely mentioned in texts on nonlinear waves [7, 19], it appears that the performance of the Whitham equation in the description of surface water waves has not been investigated so far. The purpose of the present article is to give an asymptotic derivation of the Whitham equation as a model for surface water waves, and to confirm Whitham's expectation that the equation is a fair model for the description of time-dependent surface water waves. For the purpose of the derivation, we introduce an expo- nential scaling regime in which the Whitham equation can be de- rived asymptotically from an approximate Hamiltonian principle for surface water waves. In order to motivate the use of this scaling, note that the KdV equation has the property that wide classes of initial data decompose into a number of solitary waves and small- amplitude dispersive residue [1]. For the KdV equations, solitary- wave solutions are known in closed form, and are given by

$$\eta = \frac{a}{h_0}\mathrm{sech}^2\left(\sqrt{\frac{3a}{4h_0^3}}(x-ct)\right) \tag{2.5}$$

for a certain wave celerity c. These waves clearly comply with the amplitude–wavelength relation $a/h_0 \sim h_0^2/l^2$ which was mentioned above. It appears that the Whitham equation – as indeed do many other nonlinear dispersive equations – also has the property that broad classes of initial data rapidly decompose into ordered trains of solitary waves (see Fig. 1). Quantifying the amplitude–wavelength relation for these solitary waves yields an asymptotic regime which is expected to be relevant to the validity of the Whitham equation as a water wave model.

As the curve fit in the right panel of Fig. 1 shows, the relationship between wavelength and amplitude of the Whitham solitary waves can be approximately described by the relation $a/h_0 \sim e^{\kappa(l/h_0)^\nu}$ for certain values of κ and ν. Since the Whitham solitary waves are not known in exact form, the values of κ and ν have to be found numerically. Then one may define a Whitham scaling regime

$$\mathcal{W}(\kappa,\nu) = \frac{a}{h_0}e^{\kappa(l/h_0)^\nu} \sim 1, \tag{2.6}$$

and it will be shown in Sections 2 and 3 that this scaling can be used advantageously in the derivation of the Whitham equation. The derivation proceeds by examining the Hamiltonian formulation of the water-wave problem due to Zakharov, Craig and Sulem [5, 21], and by restricting to wave motion which is predominantly in the direction of increasing values of x. The approach is similar to the method of [4], but relies on the new relation (5). First, in Section 2, a Whitham system is derived which allows for two-way propagation of waves. The Whitham equation is found in Section 3. Finally, in Section 4, a comparison of modeling properties of the KdV and Whitham equations is given. The comparison also includes the regularized long-wave equation

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x - \frac{1}{6}h_0^2\eta_{xxt} = 0, \tag{2.7}$$

which was put forward in [15] and studied in depth in [3], and which is also known as the BBM or PBBM equation. The linearized dispersion relation of this equation is not an exact match to the dispersion relation of the full water-wave problem, but it is much closer than the KdV equation, and it might also be expected that this equation may be able to model shorter waves more success- fully than the KdV equation. In order to obtain an even better match of the linear dispersion relation, one may make use of Padé expansions. In the context of simplified evolution equations, this approach was pioneered in [20]. For uni-directional models, this approach was advocated in [10], and in particular, the equation based on the Padé (2,2) approximation was studied in depth. In dimensional variables, this model takes the form

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x - \frac{3}{20}c_0h_0^2\eta_{xxx} - \frac{19}{60}h_0^2\eta_{xxt} = 0. \tag{2.8}$$

The dispersion relations for the KdV, BBM and Padé (2,2) models are respectively

$$c(k) = c_0 - \frac{1}{6} c_0 h_0^2 k^2 \qquad \text{(KdV)},$$
$$c(k) = c_0 \frac{1}{1 + \frac{1}{6} h_0^2 k^2} \qquad \text{(BBM)},$$
$$c(k) = c_0 \frac{1 + \frac{3}{20} h_0^2 k^2}{1 + \frac{19}{60} h_0^2 k^2} \qquad \text{(Padé (2,2))}.$$

These approximate dispersion relations are compared to the full dispersion relation in Fig. 2. It appears clearly that the Padé (2,2) approximation remains much closer to the full dispersion relation than the dispersion relations based on either the linear KdV or linear BBM equations. As will be seen in Section 4, solutions of both the Whitham and Padé (2,2) equations give closer approximations to solutions of the full Euler equations than either the KdV or BBM equations in most cases investigated. However, the Whitham equation still keeps a slight edge over the Padé model.

2.1.2 Derivation of evolution systems of Whitham type.

The surface water-wave problem is generally described by the Euler equations with slip conditions at the bottom, and kinematic and dynamic boundary conditions at the free surface. Assuming weak transverse effects, the unknowns are the surface elevation $\eta(x,t)$, the horizontal and vertical fluid velocities $u_1(x,z,t)$ and $u_2(x,z,t)$, respectively, and the pressure $P(x,z,t)$. If the assumption of irrotational flow is made, then a velocity potential $\phi(x,z,t)$ can be used. In order to nondimensionalize the problem, the undisturbed depth h_0 is taken as a unit of distance, and the parameter $\sqrt{h_0/g}$ as a unit of time. For the remainder of this article, all variables appearing in the water-wave problem are considered as being non-dimensional. The problem is posed on a domain $\{(x,z)^T \in R^2 \mid 1 < z < \eta(x,t)\}$ which extends to infinity in the positive and negative x-direction. Due to the incompressibility of the fluid, the potential then satisfies Laplace's equation in this domain. The fact that the fluid cannot penetrate the bottom is expressed by a homogeneous Neumann boundary condition at the flat bottom. Thus we have

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 \text{ in } -1 < z < \eta(x,t) \\ \phi_{zz} &= 0 \text{ on } z = -1. \end{aligned}$$

The pressure is eliminated with the help of the Bernoulli equation, and the free-surface boundary conditions are formulated in terms of the potential and the surface excursion by

$$\left.\begin{aligned} \eta_t + \phi_x \eta_x - \phi_z &= 0, \\ \phi_t + \tfrac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + \eta &= 0, \end{aligned}\right\} \text{on } z = \eta(x,t).$$

The total energy of the system is given by the sum of kinetic energy and potential energy, and normalized such that the potential energy is zero when no wave motion is present at the surface. Accordingly the Hamiltonian function for this problem is

$$H = \int_{\mathbb{R}} \int_0^{\eta} z\,dz dx + \int_{\mathbb{R}} \int_{-1}^{\eta} \frac{1}{2} |\nabla \phi|^2 \,dz dx. \tag{2.9}$$

Defining the trace of the potential at the free surface as $\Phi(x,t) = \phi(x,\eta(x,t),t)$, one may integrate in z in the first integral and use the divergence theorem on the second integral in order to arrive at the formulation

$$H = \frac{1}{2}\int_{\mathbb{R}} \left[\eta^2 + \Phi G(\eta)\Phi\right] dx. \tag{2.10}$$

This is the Hamiltonian formulation of the water wave problem as found in [5, 16, 21], and written in terms of the Dirichlet–Neumann operator $G(\eta)$. As shown in [14], the Dirichlet–Neumann operator can be expanded in a series of the form

$$G(\eta)\Phi = \sum_{j=0}^{\infty} G_j(\eta)\Phi.$$

In order to proceed, we need to understand the first few terms in this series. As shown in [4, 5], the first two terms in this series can be written with the help of the operator $D = -i\partial_x$ as

$$G_0(\eta) = D\tanh(D), \qquad G_1(\eta) = D\eta D - D\tanh(D)\eta D\tanh(D).$$

Note that it can be shown that the terms $G_j(\eta)$ for $j \geq 2$ are of quadratic or higher-order in η, and will therefore not be needed in the current analysis.

It will be convenient for the present purpose to formulate the Hamiltonian in terms of the dependent variable $u = \Phi_x$. To this end, we define the operator $\mathcal{K}(\eta)$ by

$$G(\eta) = D\mathcal{K}(\eta)D.$$

As was the case with $G(\eta)$, the operator $\mathcal{K}(\eta)$ can be expanded in a Taylor series around zero as

$$\mathcal{K}(\eta) = \sum_{j=0}^{\infty} \mathcal{K}_j(\eta), \qquad \mathcal{K}_j(\eta) = D^{-1}G_j(\eta)D^{-1}.$$

In particular, note that $\mathcal{K}_0 = \frac{\tanh D}{D}$. In non-dimensional variables, we write the operator with the integral kernel K_{h_0} as $K = \sqrt{\frac{\tanh D}{D}}$, so that we have $\mathcal{K}_0 = K^2$. The Hamiltonian is then expressed as

$$H = \frac{1}{2}\int_{\mathbb{R}} \left[\eta^2 + u\mathcal{K}(\eta)u\right] dx. \tag{2.11}$$

The water-wave problem can then be written as a Hamiltonian system using the variational derivatives of H and posing the Hamiltonian equations

$$\eta_t = -\partial_x \frac{\delta H}{\delta u}, \qquad u_t = -\partial_x \frac{\delta H}{\delta \eta}. \tag{2.12}$$

This system is not in canonical form as the associated structure map $J_{\eta,u}$ is symmetric:

$$J_{\eta,u} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}. \tag{2.13}$$

We now proceed to derive a system of equations which is similar to the Whitham equation (2.1), but allows bi-directional wave propagation. This system will be a stepping stone on the way to a derivation of (2.1), but may also be of independent interest. Consider a wave- field having a characteristic wavelength l and a characteristic amplitude a. Taking into account the nondimensionalization, the two scalar parameters $\lambda = l/h_0$ and $\alpha = a/h_0$ appear. In order to introduce the long-wave and small amplitude approximation into the non-dimensional problem, we use the scaling $\tilde{x} = \frac{1}{\lambda}x$, and $\eta = \alpha\tilde{\eta}$. This induces the transformation $\tilde{D} = \lambda D = -\lambda i\partial_x$. If the energy is nondimensionalized in accord with the nondimensionalization mentioned earlier, then the natural scaling for the Hamiltonian is $\tilde{H} = \alpha^2 H$. In addition, the unknown u is scaled as $u = \alpha\tilde{u}$. The scaled Hamiltonian (2.11) is then written as

2.2 The Whitham equation with surface tension

2.3 A numerical study of nonlinear dispersive wave models with SpecTraVVave

2.4 Explicit solutions for a long-wave model with constant vorticity

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Part II

Scientific results