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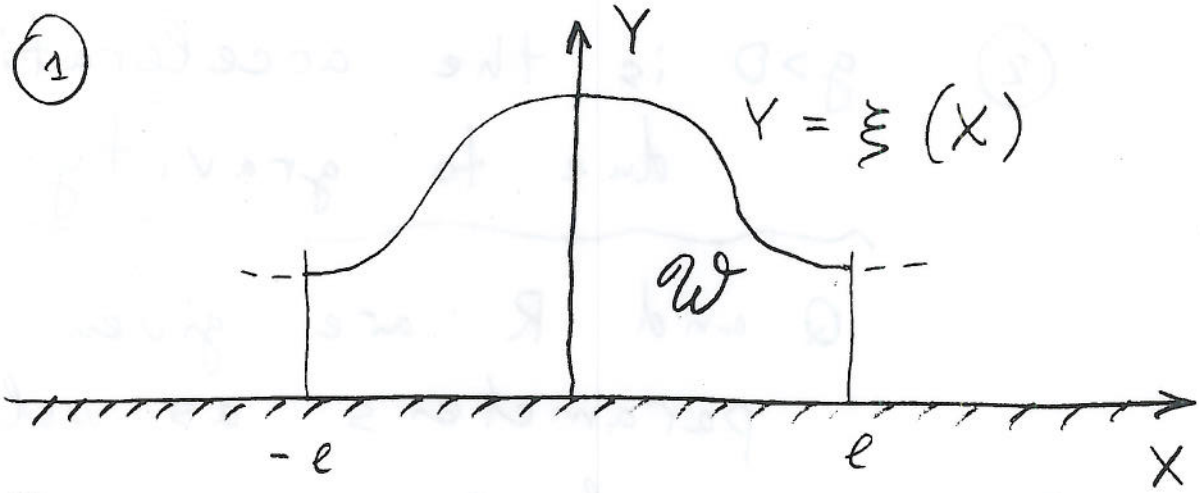


- ξ is continuous, positive,
- 2ℓ-periodic

$\Psi \in C^1(\overline{\mathcal{W}}) \cap C^2(\mathcal{W})$

Ψ is 2ℓ-periodic in X

- $\Rightarrow$ Ψ is bounded

- (1) $\Psi_{XX} + \Psi_{YY} = 0$ in $\mathcal{W}$;

(2) $\Psi = -Q$, $Y = 0$, $X \in \mathbb{R}$;

(3) $\Psi = 0$, $Y = \xi(X)$, $X \in \mathbb{R}$;

(4) $\frac{1}{2}|\nabla \Psi|^2 + g\xi = R$.

(2) $g>0$ is the acceleration due to gravity

Q and R are given parameters as well as $\ell > 0$.

Moreover, $Q \neq 0$ and $R > R_c$.

Let $H = \frac{1}{2\ell} \int_{-\ell}^{\ell} \xi(X)\, dX > 0$,

then (4) $\Longrightarrow$

$$c^2 := \frac{1}{2\ell} \int_{-\ell}^{\ell} |\nabla \Psi(X, Y(X))|^2 dX$$

$$= 2(R - gH)$$

c is the mean velocity along the free surface. in some sense, ③ (5) Let $c^2 = \frac{1}{2\ell} \int_{-\ell}^{\ell} |\nabla \Psi(X, Y(X))|^2 dX$

be the unknown value of the mean (per the unit length of the bottom) velocity squared on the free surface.

(6) Then (4) $\Longrightarrow c^2 = 2(R - gH)$.

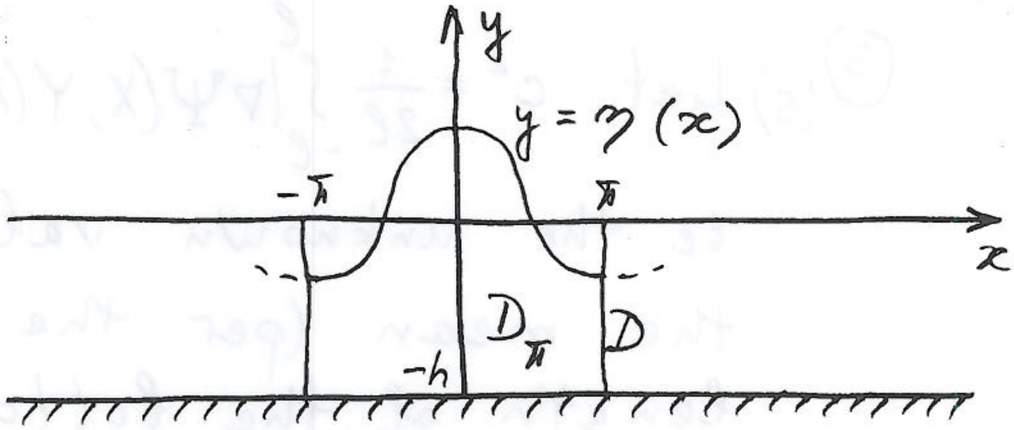
Non-dimensional problem

(7) $h = \pi H/\ell$ is the non-dimensional mean depth

(8)
$$x = \frac{\pi}{\ell} X, \quad y = \frac{\pi}{\ell} Y - h,$$
$$\eta(x) = \frac{\pi}{\ell} \xi(X) - h$$

$\Longrightarrow \eta$ is 2π-periodic and

(9) $$\int_{-\pi}^{\pi} \eta(x)\, dx = 0$$



$$\psi(x,y) = \left[g\left(\frac{\ell}{\pi}\right)^3 \right]^{-1/2} \Psi(X,Y), \tag{10}$$

$$Q_0 = \left[g\left(\frac{\ell}{\pi}\right)^3 \right]^{-1/2} Q$$

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(11) $\psi_{xx} + \psi_{yy} = 0 \quad \text{in } D;$

(12) $\psi = -Q_0, \quad y = -h, \quad x \in \mathbb{R};$

(13) $\psi = 0, \quad y = \eta(x), \quad x \in \mathbb{R};$

(14) $|\nabla \psi|^2 + 2\eta = \mu,$.

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(15) $$\mu = \frac{\pi c^2}{g\ell} = \frac{2\pi}{g\ell}(R - gH)$$

is the Froude number squared, which is part of solution to

be found along with
η and ψ.

Let ϕ be the harmonic conjugate to $-\psi$ in D with the additive constant chosen so that

(16) $\phi(0,y) = 0 \quad , \quad y \in [-h, \eta(0)]$

Then

(17) $\zeta(z) = \phi(x,y) + i\psi(x,y)$

is the complex potential, that is, an analytic function of $z = x + iy \in D$; moreover,

$$\left(\mathrm{Re}\,\overline{\zeta_z}\ ,\ \mathrm{Im}\,\overline{\zeta_z}\right)$$

gives the velocity field in D. Finally, ζ maps

⑥ conformally the one-period water domain $\overline{D_\pi}$ onto a rectangle $[-b, b] \times [-Q_0, 0]$.

Now

(18) $w = e^{-id\zeta}$ ($d = \pi/b$ is a scaling parameter)

maps conformally $[-b, b] \times [-Q_0, 0]$ onto the annulus

(19) $\{w \in \mathbb{C} : r \le |w| \le 1\} = A_r$,

where $r = e^{-dQ_0}$.

Thus we have the following conformal mapping:

(20) $z \longmapsto w(\zeta(z))$.

It maps $\overline{D_\pi}$ onto the

(7) annulus (19) so that

$C_1 = \{|w| = 1\}$ and $\{|w| = r\} = C_r$

are the images of the free surface and the bottom, respectively; the right and left vertical segments (the rest of ∂D_π) are mapped onto the lower and upper sides of the cut

(21) $\{\operatorname{Re} w \in [-1, -r], \operatorname{Im} w = 0\}$.

The inverse mapping to (20) has the following form:

(22) $z(w) =$

$$\frac{i}{d}\left[\log w + a_0 + \sum_{k=1}^{\infty} a_k \left(w^k - \frac{r^{2k}}{w^k}\right)\right]$$

where $\{a_k\}_{k=0}^{\infty} \subset \mathbb{R}$.

(8) Moreover, we have that

$$\log r + a_0 = -h. \tag{23}$$

Putting

$$w = e^{it}, \quad t \in \mathbb{R},$$

we obtain a parametric representation of the free surface:

$$\begin{aligned} x(t) &= -t - \sum_{k=1}^{\infty} a_k (1 + r^{2k}) \sin kt, \\ \eta(t) &= a_0 + \sum_{k=1}^{\infty} a_k (1 - r^{2k}) \cos kt. \end{aligned} \tag{24}$$

Let $\mathcal{C}$ be the 2π-periodic Hilbert transform:

$$(\mathcal{C} f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau) \cot \frac{t-\tau}{2} \, d\tau \tag{25}$$

or

$$\mathcal{C}(\cos kt) = \sin kt, \quad k = 0, 1, \dots ;$$
$$(26) \qquad \mathcal{C}(\sin kt) = -\cos kt, \quad k = 1, 2, \dots$$

Furthermore, let

$$(27) \quad (\mathcal{K}_r f)(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) K_r(t-\tau)\, d\tau,$$

where

$$(28) \quad K_r(t-\tau) = \sum_{k=1}^{\infty} \frac{2r^{2k}}{1-r^{2k}} \sin k(t-\tau),$$

and we put

$$(29) \qquad \mathcal{B}_r = \mathcal{C} + \mathcal{K}_r .$$

It follows from (24) that

$$(30) \quad x_t = -1 - \mathcal{B}_r \eta_t$$

for the free surface parametrised by $t \in \mathbb{R}$.

(10) $\mathcal{B}_r$ is the conjugate operator in the following sense. Let $F(w)$ be analytic in A_r and let

$$(31)\quad \operatorname{Im} F(w)\Big|_{w\in C_r} \equiv 0 .$$

Then

$$(32)\quad \operatorname{Re} F(e^{it}) + \left[\mathcal{B}_r (\operatorname{Im} F)\right](t) = 0.$$

Considering

$$z_\phi = z_w \, w_\phi = z_w (-id) e^{-id\zeta} \zeta_\phi = -id w \, z_w ,$$

we see that (22) yields

$$(33)\quad z_\phi = 1 + \sum_{k=1}^{\infty} k a_k \left(w^k + \frac{r^{2k}}{w^k}\right),$$

11 which is analytic in A_r.

Since z_ϕ does not vanish in A_r, $1/z_\phi$ is also analytic there.

If $y = \eta(x)$, then (14) implies that

$$\frac{1}{z_\phi} = |\nabla\phi|^2 \overline{z_\phi} \tag{34}$$
$$= (\mu - 2\eta)(x_\phi - i y_\phi),$$

and so

$$\frac{1}{z_\phi} = (\mu - 2\eta)(i\eta_t - x_t) \tag{35}$$

when $w = e^{it}$.

Then using (30), we get

$$\frac{1}{z_\phi} = (\mu - 2\eta)(1 + \mathcal{B}_r \eta_t + i\eta_t), \tag{36}$$
$$w = e^{it}$$

(12) From (33) and (36) we see that the coefficient at w^0 in the Laurent expansion of $\frac{1}{Z_\phi} - \mu$ is equal to zero.

Moreover,

$$\operatorname{Im}\left(\frac{1}{Z_\phi} - \mu\right)\Big|_{w \in C_r} \equiv 0,$$

and so formula (32) is applicable to

$$F(e^{it}) = \left[\frac{1}{Z_\phi} - \mu\right]_{w = e^{it}}.$$

Then we obtain from (36) that

$$-2\eta + (\mu - 2\eta)\,\mathcal{B}_r \eta_t + \mathcal{B}_r\left[(\mu - 2\eta)\eta_t\right] = 0,$$

which simplifies to

$$\mu \mathcal{B}_r \eta_t = \eta + \eta \mathcal{B}_r \eta_t + \mathcal{B}_r (\eta \eta_t) . \tag{37}$$

This equation has the same form as (1.4) in BDT1; however, we have $\mathcal{B}_r$ in (30) instead of $\mathcal{C}$ in (1.4), BDT1 (see formulae (27) - (29) for the definition of $\mathcal{B}_r$).