Babenko's equation for steady gravity waves on water of finite depth

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Abstract

The two-dimensional problem describing periodic steady waves on water of finite depth is considered.

MSC: 76B15, 35Q35

Keywords: Steady water waves; Babenko's equation; Froude number

1 Introduction

The present paper is concerned with a classical nonlinear problem in the mathematical theory of water waves, namely, the two-dimensional problem of steady waves. It was Stokes [19], who had initiated studies in this field as early as 1847. On the basis of approximations developed for periodic waves with a single crest per wavelength (now, they are referred to as *Stokes waves*), he made some conjectures about the behaviour of such waves on deep water. From the paper [17] by Plotnikov and Toland (see also references cited therein), one gets an idea how these conjectures were proved. In particular, the so-called Nekrasov's equation was essential for this purpose. The first version of this nonlinear integral equation for waves on deep water was derived by Nekrasov in 1921 (see [13] and also [15], Part 1); soon after that, he generalized his equation to the case of finite depth (see [14] and [15], Part 2). Much later, Amick and Toland [1] proposed and investigated a more sophisticated version of the latter equation.

In the 1920s, Levi-Civita [11] and Struik [20] considered (independently of Nekrasov) the problem of periodic waves on deep water and on water of finite depth respectively. The hodograph transform allowed them to reduce the question of existence of waves to that of finding a pair of conjugate harmonic functions satisfying nonlinear Neumann boundary conditions. The existence proofs given in [11] and [20] are based on a majorant method for demonstrating the convergence of power series that provide formal solutions. However, Zeidler [23], ch. 71, writes that these proofs are 'very complicated' because they involve 'voluminous computations'. By now, both techniques (Nekrasov's equations and that of Levi-Civita and Struik) are investigated in detail by means of analytic bifurcation theory. An account of this theory can be found in the books by Buffoni and Toland [6], and Zeidler

[22], whereas its application to equations describing steady water waves are given by many authors and summarised by Toland [21] (deep water) and Zeidler [23], ch. 71 (water of finite depth), who also provide detailed historical remarks.

Another method was developed by Buffoni, Dancer and Toland [4, 5] for periodic waves on deep water (with and without surface tension). In the absence of surface tension, it is based on the so-called Babenko's equation:

$$\mu\, \mathcal{C} w' = w + w\, \mathcal{C} w' + \mathcal{C}(w' w). \tag{1}$$

bid

Here μ is a dimensionless bifurcation parameter related to the speed of wave propagation and $w(t)$, where $t \in [-\pi, \pi]$, describes the unknown steady wave profile which is given parametrically in certain dimensionless coordinates; $'$ denotes differentiation with the respect to t and $\mathcal{C}$ is the periodic Hilbert transform (also known as the conjugation operator in the theory of Fourier series [24]). It is defined on $L^2(-\pi, \pi)$ by linearity from the following relations:

$$\mathcal{C}(\cos nt) = \sin nt \quad \text{for } n \geq 0, \qquad \mathcal{C}(\sin nt) = -\cos nt \quad \text{for } n \geq 1. \tag{2}$$

?HT?

The original form of equation (1) (it was discovered by Babenko in 1987 and announced in [2]) is outlined in the book [16] by Okamoto and Shōji. A generalization of Babenko's equation was later studied by Shargorodsky and Toland [18].

Interesting results concerning the secondary or sub-harmonic bifurcations from branches describing Stokes-wave solutions of (1) are proved in the articles [4, 5]. In the first of them, it is shown that such bifurcations do not occur in a neighbourhood of those points, where Stokes waves bifurcate from a trivial solution. On the other hand, significant numerical evidence about the existence of steady periodic waves that distinguish from Stokes waves had appeared after 1980. These new waves have more than one crest per period and bifurcate from Stokes waves. Branches of sub-harmonic bifurcations for deep water were first computed by Chen and Saffman [7]. Craig and Nicholls [8] proposed a different method for obtaining similar numerical results for water of finite depth. References to other works containing numerical results on sub-harmonic bifurcations can be found in [8] and in the paper [3] by Baesens and MacKay, who also give some theoretical insights concerning these bifurcations. The above mentioned numerical observations were confirmed rigorously by Buffoni, Dancer, and Toland [5], who 'concluded that the sub-harmonic bifurcations [...] are *an inevitable consequence* of the formation of Stokes highest wave'. A characteristic property of the latter wave is the angle equal to $2\pi/3$ formed at the crest by two smooth, symmetric curves. Concerning this wave see the article [17] by Plotnikov and Toland and references cited therein.

The aim of this paper is to derive Babenko's equation describing periodic waves on water of finite depth and to investigate its properties similar to those obtained in [4] in the case of deep water.

1.1 Statement of the problem

In its simplest form, the problem of steady surface waves concerns the two-dimensional, irrotational motion of an inviscid, incompressible, heavy fluid, say water, bounded above by a free surface and below by a rigid horizontal bottom. (For example, this kind of motion occurs in water occupying an infinitely long channel of rectangular cross-section.) In an appropriate frame of reference the velocity field of steady motion is time-independent as well as the free-surface profile, and we choose Cartesian coordinates (X, Y) so that the latter is given by the graph of an unknown positive function ξ, that is, $Y = \xi(X)$, $X \in \mathbb{R}$, whereas the bottom coincides with the X-axis and gravity acts in the negative Y-direction. We suppose that ξ is continuously differentiable, 2ℓ-periodic and even, and has the *given mean value* $H > 0$ over the period.

In the longitudinal section of the water domain $\mathcal{D} = \{X \in \mathbb{R},\ 0 < Y < \xi(X)\}$, the velocity field is described by the stream function $\Psi(X, Y)$, that is, the projections of the velocity vector at (X, Y) on the X- and Y-axis are $-\Psi_Y$ and Ψ_X respectively. This unknown function is assumed to belong to $C^2(\mathcal{D}) \cap C^1(\bar{\mathcal{D}})$ and to be 2ℓ-periodic, odd function of X (hence Ψ is bounded on $\bar{\mathcal{D}}$).

Since the surface tension is neglected, Ψ and ξ must satisfy the following free-boundary problem:

$$\Psi_{XX} + \Psi_{YY} = 0, \quad (X, Y) \in \mathcal{D}; \qquad (3)\ \boxed{\texttt{1}}$$

$$\Psi(X, 0) = 0, \quad X \in \mathbb{R}; \qquad (4)\ \texttt{?2?}$$

$$\Psi(X, \xi(X)) = Q, \quad X \in \mathbb{R}; \qquad (5)\ \boxed{\texttt{3}}$$

$$\frac{1}{2}|\nabla\Psi(X, \xi(X))|^2 + g\xi(X) = R, \quad X \in \mathbb{R}. \qquad (6)\ \boxed{\texttt{4}}$$

In the boundary condition (5), Q — the constant rate of flow per the channel's unit span — is prescribed along with the parameter H characterising ξ. In the last relation known as Bernoulli's equation, g is the acceleration due to gravity, whereas R is a positive constant (its meaning is the total head and it is also referred to as the Bernoulli constant). It is known that non-trivial solutions of problem (3)–(6) exist only when $Q \neq 0$ (this fact is proved in [9], Proposition 1.1) and $R > R_c = \frac{3}{2}(Qg)^{2/3}$ (see [10] for the proof of this inequality under weaker assumptions than listed above). In what follows, we suppose that the restriction on Q is fulfilled (without loss of generality we assume that $Q > 0$). On the other hand, the parameter R, or equivalently the Froude number squared (see equation ?? below), is part of the solution to be found.

1.2 A non-dimensional statement of the problem

In order to write (3)–(6) in a non-dimensional form we average Bernoulli's equation over $(-\ell, \ell)$. In view of condition (5) that Ψ is constant on the free surface and recollecting that $H = \int_{-\ell}^{\ell} \xi(X)\,\mathrm{d}X/(2\ell)$, we obtain

$$c^2 = 2(R - gH), \quad \text{where} \quad \text{and} \quad c^2 = \frac{1}{2\ell}\int_{-\ell}^{\ell}\left|\frac{\partial\Psi}{\partial n}(X, \xi(X))\right|^2 \mathrm{d}X. \qquad (7)\ \texttt{?abe?}$$

Here n is the unit normal to $Y = \xi(X)$ directed out of $\mathcal{D}$. It is natural to call c the root-mean-square value of the velocity along the free surface. It is clear that this quantity is unknown.

Let $h = \pi H/\ell$ be the non-dimensional mean depth of flow and $Q_0 = Q/\sqrt{g(\ell/\pi)^3}$ be the non-dimensional rate of flow. Now we scale the dimensional variables and shift the vertical variables downwards as follows:

$$x = \frac{\pi}{\ell}X,\ y = \frac{\pi}{\ell}Y - h; \quad \eta(x) = \frac{\pi}{\ell}\xi(X) - h; \quad \psi(x, y) = \frac{Q_0}{Q}\Psi(X, Y). \qquad (8)\ \boxed{\texttt{dlv}}$$

This is advantageous because the new unknown η is a 2π-periodic and even function of x, and it must satisfy the following condition:

$$\int_{-\pi}^{\pi} \eta(x)\,\mathrm{d}x = 0. \tag{9}$$ eta

Furthermore, the function ψ has the same properties on the closure of the strip

$$D = \{x \in \mathbb{R}, -h < y < \eta(x)\}$$

as Ψ has on $\bar{\mathcal{D}}$, that is, $\psi \in C^1(\bar{D}) \cap C^2(D)$ and is a 2π-periodic and odd function of x. Moreover, the change of variables (8) reduces relations (3)–(6) to the following

$$\psi_{xx} + \psi_{yy} = 0, \quad (x, y) \in D; \tag{10}$$ lapp
$$\psi(x, -h) = 0, \quad x \in \mathbb{R}; \tag{11}$$?bcp?
$$\psi(x, \eta(x)) = Q_0, \quad x \in \mathbb{R}; \tag{12}$$?kcp?
$$|\nabla\psi(x, \eta(x))|^2 + 2\eta(x) = \mu, \quad x \in \mathbb{R}. \tag{13}$$ bep

In the non-dimensional Bernoulli equation, the parameter $\mu = \pi c^2/(g\ell)$ is the Froude number squared which must be found along with η and ψ. Thus, the non-dimensional statement of the problem is as follows.

Definition 1 Let Q_0 and h be given positive numbers, find (μ, η, ψ) from relations (10)–(13) so that $\mu > 0$ and η satisfies condition (9), whereas other properties of η and ψ (smoothness, periodicity and symmetry) are as described above.

2 Reduction of the non-dimensional problem to Babenko's equation

The aim of this section is to show that steady periodic waves on water of finite depth described in Definition 1 are given by solutions of a single nonlinear pseudo-differential operator equation. The latter is the same as equation (1) with the operator $\mathcal{C}$ changed to $\mathcal{B}_r = \mathcal{C} + \mathcal{K}_r$, where $r \in (0, 1)$ depends on h and

$$(\mathcal{K}_r f)(t) = \frac{2}{\pi} \int_{-\pi}^{\pi} f(\tau) K_r(t - \tau)\,\mathrm{d}\tau \quad \text{with} \quad K_r(t - \tau) = \sum_{k=1}^{\infty} \frac{r^{2k}}{1 - r^{2k}} \sin k(t - \tau). \tag{14}$$?K_r?

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