Numerical methods for nonlinear nonlocal water wave models

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Preface

This dissertation is submitted as a partial fulfillment of the requirements for the degree Doctor of Philosophy (PhD) at the Department of Mathematics, University of Bergen. The research was supported by the Research Council of Norway on grant no. 213474/F20.

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Outline

This thesis is organised in the following way. Part I contains general theoretical background on nonlinear wave models as well as description of methods used to solve the equations involved. Some properties of solutions to the equations and a summary of results are also given in the first part. Part 2 consists of the research papers that present scientific results in detail.

List of reseach papers included in Part II

Paper A:
Moldabayev, D., Kalisch, H., Dutykh, D.: *The Whitham equation as a model for surface water waves*, Phys. D **309**, 99–107 (2015),
http://dx.doi.org/10.1016/j.physd.2015.07.010.

Paper B:
Dinvay, E., Moldabayev, D., Dutykh, D., Kalisch, H.: *The Whitham equation with surface tension*, Nonlinear Dynamics, 1–14 (2017),
http://dx.doi.org/10.1007/s11071-016-3299-7.

Paper C:
Henrik Kalisch, Daulet Moldabayev, Olivier Verdier: *A numerical study of nonlinear dispersive wave models with SpecTraVVave*, specify status of the paper.

Paper D:
Benjamin Segal, Daulet Moldabayev, Henrik Kalisch, Bernard Deconinck: *Explicit solutions for a long-wave model with constant vorticity*, submitted to European Journal of Mechanics - B/Fluids.

Contents

Part I

Background

Chapter 1

Introduction

This is the introduction [1–3]. . .

Chapter 2

Summary of results

This chapter provides an overview of the results achieved in the course of research work. Detailed

2.1 The Whitham equation as a model for surface water waves

The Whitham equation was proposed as an alternate model equation for the simplified description of unidirectional wave motion at the surface of an inviscid fluid. As the Whitham equation incorporates the full linear dispersion relation of the water wave problem, it is thought to provide a more faithful description of shorter waves of small amplitude than traditional long wave models such as the KdV equation. In this work, we identify a scaling regime in which the Whitham equation can be derived from the Hamiltonian theory of surface water waves. A Hamiltonian system of Whitham type allowing for two- way wave propagation is also derived. The Whitham equation is integrated numerically, and it is shown that the equation gives a close approximation of inviscid free surface dynamics as described by the Euler equations. The performance of the Whitham equation as a model for free surface dynamics is also compared to different free surface models: the KdV equation, the BBM equation, and the Padé (2,2) model. It is found that in a wide parameter range of amplitudes and wavelengths, the Whitham equation performs on par with or better than the three considered models.

In its simplest form, the water-wave problem concerns the flow of an incompressible inviscid fluid with a free surface over a horizontal impenetrable bed. In this situation, the fluid flow is described by the Euler equations with appropriate boundary conditions, and the dynamics of the free surface are of particular interest in the solution of this problem. There are a number of model equations which allow the approximate description of the evolution of the free surface without having to provide a complete solution of the fluid flow below the surface. In the present contribution, interest is focused

on the derivation and evaluation of a nonlocal water-wave model known as the Whitham equation. The equation is written as

$$\eta_t + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + K_{h_0} * \eta_x = 0, \tag{2.1}$$

where the convolution kernel K_{h_0} is given in terms of the Fourier transform by

$$\mathcal{F}K_{h_0}(\xi) = \sqrt{\frac{g\tanh(h_0\xi)}{\xi}}. \tag{2.2}$$

g is the gravitational acceleration, h_0 is the undisturbed depth of the fluid, and $c_0 = \sqrt{gh_0}$ is the corresponding long-wave speed. The convolution can be thought of as a Fourier multiplier operator, and (2.2) represents the Fourier symbol of the operator. The Whitham equation was proposed by Whitham [4] as an alternative to the well known Korteweg–de Vries (KdV) equation

2.2 The Whitham equation with surface tension

2.3 A numerical study of nonlinear dispersive wave models with SpecTraVVave

2.4 Explicit solutions for a long-wave model with constant vorticity

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Part II

Scientific results