Counting permutations under constraints

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Setting & Notations

Let $n \in \mathbb{N}$ and $\mathcal{F}$ be a **set**[1] of *forbidden edges*, where an edge is defined as an (directed) pair of distinct natural numbers smaller than n, e.g. $(1, 3)$. If a permutation $\sigma \in \mathfrak{S}_n$ of n includes at least one forbidden edge from $\mathcal{F}$, we say σ is forbidden by $\mathcal{F}$. For instance, assuming $n = 8$ and $\mathcal{F} = \{(1, 3), (4, 6)\}$, $\sigma = (8 \to 1 \to 3 \to 7 \to 4 \to 6 \to 3 \to 2)$ is forbidden because it includes $(4, 6)$ and $(1, 3)$.

Our goal is to compute the number of permutations not forbidden by $\mathcal{F}$:

$$f(\mathcal{F}) = n! - |\{\sigma \in \mathfrak{S}_n; \sigma \text{ is forbidden by } \mathcal{F}\}|$$

As a special case, let us denote $|S_0| = n!$, $|S_1| = f(\mathcal{F})$ where $|\mathcal{F}| = n - 1$ and more generally $|S_i| = f(\mathcal{F})$ where $|\mathcal{F}| = i(n - 1)$. The question is then how many times can we be told than $n - 1$ edges do not exist before $|S_i| \leq 1$.

Fun with Combinatorics!

First, assume $\mathcal{F} = \{(u, v)\}$. Let's build a forbidden permutation. Because v should always follows u, we can pretend they form a single node and collapse them. Left with $n-1$ nodes, we can freely order them in one of the $(n-1)!$ possible way and be guaranteed this will include $u \to v$.

Now if $\mathcal{F} = \{(u, v), (w, x)\}$, how many forbidden permutations are there? Well $(n - 1)!$ permutations include (u, v), $(n - 1)!$ include (w, x) but we are double counting those with both (u, v) and (w, x). By the same collapsing argument applied to the two pairs of nodes, there are $(n - 2)!$ permutations include the two edges. Note furthermore that this still holds if the edges are not node disjoint (e.g. $\mathcal{F} = \{(u, v), (v, w)\}$).

Armed with the fact that there are $(n - i)$ permutations which include i forbidden

[1] thus not containing any repetition

edges[2], we can use the inclusion-exclusion principle to compute $|S_1|$

$$\begin{aligned}|S_1| &= n! - \sum_{i=1}^{n-1}(-1)^{i+1}\binom{n-1}{i}(n-i)! \\ &= \sum_{i=0}^{n-1}(-1)^i \frac{(n-1)!}{i!(n-1-i)!}(n-i)! \\ &= (n-1)!\sum_{i=0}^{n-1}(-1)^i\frac{(n-i)}{i!} \sim \frac{n!}{e}\end{aligned}$$

In S_1, each node has only one forbidden successor. In S_2 it has two. However, some combinations counted by the binomial coefficients in the formula cannot appear. For instance, while we are still double counting $[(0,1),(1,2)]$, it makes not sense to subtract $[(0,1),(0,2)]$ as it is not feasible in any permutation anyway. Therefore we need to replace the binomial coefficients $\binom{|\mathcal{F}|}{i}$ by D_i, which is the number of sequences of i edges from $\mathcal{F}$ such that each node appear at most once at the head of an edge.

For instance, if $n = 4$ and $\mathcal{F} = \{(0,1),(0,2),(1,2),(2,3)\}$

$D_1 = 4$	$D_2 = 4$	$D_3 = 1$
$[(0,1)]$	$[(0,1),(1,2)], [(0,1),(2,3)]$	$[(0,1),(1,2),(2,3)]$
$[(0,2)]$	$[(0,2),(1,2)]$	
$[(1,2)]$	$[(1,2),(2,3)]$	
$[(2,3)]$		

Beginning of the handwavy part ;)

More generally for S_2, denoting $|\mathcal{F}| = 2(n-1) = m$, $D_0 = 1$ (by convention) $D_1 = m$ (as any single edge is admissible), $D_2 = \frac{(m-1)(m-2)}{2}$, $D_3 = \frac{(m-2)(m-3)(m-4)}{6}$ and so on (although I'm not sure exactly why). Empirically, $D_i = \binom{m+1-i}{i}$, even though it gets a bit off as i gets closer to $n-1$. Especially, $D_{n-1} = \lceil \frac{n}{2} \rceil$. Still

$$\begin{aligned}|S_2| &\approx \sum_{i=0}^{n-1}(-1)^i\binom{2n-1-i}{i}(n-i)! \\ &= \sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\underbrace{\frac{(2n-1-i)!(n-i)!}{(2(n-1)-i)!}}_{a_i}\end{aligned}$$

[2]when each node can only appear once at the head of an edge

Removing the -1 of a_i for clarity, we see that $a_0 = n!$ and $a_n = n!$[3], while the maximum is reached at $i = \lceil \frac{2n}{3} \rceil$[4]: let $n = 3p$,

$$a_i = \frac{(2 \cdot 3p - 2p)!(3p - 2p)!}{2(3p - 2p)!} = 4p \cdots \underbrace{3p}_{n} \cdot 1 \cdots 1 \cdot p \cdots 1$$

[3]even though it doesn't appear in the sum
[4]Again no matter how convinced I am it's true, that's a leap of faith