Fairness Criteria Definitions

1. **Demographic Parity (DP):**

$$P(\hat{Y} = 1 \mid A = 0) = P(\hat{Y} = 1 \mid A = 1)$$

2. **Equal Opportunity (EO):**

$$P(\hat{Y} = 1 \mid Y = 1, A = 0) = P(\hat{Y} = 1 \mid Y = 1, A = 1)$$

3. **Predictive Parity (PP):**

$$P(Y = 1 \mid \hat{Y} = 1, A = 0) = P(Y = 1 \mid \hat{Y} = 1, A = 1)$$

Definitions of Metrics

- **True Positive Rate ($\mathbf{TPR}_a$)** for group $A = a$:

$$\mathrm{TPR}_a = P(\hat{Y} = 1 \mid Y = 1, A = a)$$

- **False Positive Rate ($\mathbf{FPR}_a$)** for group $A = a$:

$$\mathrm{FPR}_a = P(\hat{Y} = 1 \mid Y = 0, A = a)$$

- **Prevalence (p_a)** for group $A = a$:

$$p_a = P(Y = 1 \mid A = a)$$

- **Positive Predictive Value ($\mathbf{PPV}_a$)** for group $A = a$:

$$\mathrm{PPV}_a = P(Y = 1 \mid \hat{Y} = 1, A = a)$$

Proof of the PPV Identity

$$\begin{aligned}
\mathrm{PPV}_a &= \frac{P(\hat{Y} = 1 \mid Y = 1, A = a) \cdot P(Y = 1 \mid A = a)}{P(\hat{Y} = 1 \mid A = a)} \\
&= \frac{\mathrm{TPR}_a \cdot p_a}{P(\hat{Y} = 1 \mid A = a)} \\
&= \frac{\mathrm{TPR}_a \cdot p_a}{\mathrm{TPR}_a \cdot p_a + \mathrm{FPR}_a \cdot (1 - p_a)}
\end{aligned}$$

Derivation Under DP and EO

Assuming **Demographic Parity (DP)** and **Equal Opportunity (EO)** hold:
From **EO**:

$$\mathrm{TPR}_0 = \mathrm{TPR}_1 = \mathrm{TPR}$$

The true positive rates are equal across groups. From **DP**:

$$P(\hat{Y} = 1 \mid A = 0) = P(\hat{Y} = 1 \mid A = 1)$$

The overall positive prediction rates are equal across groups. Compute $P(\hat{Y} = 1 \mid A = a)$ for each group:

$$P(\hat{Y} = 1 \mid A = a) = \mathrm{TPR} \cdot p_a + \mathrm{FPR}_a \cdot (1 - p_a)$$

From **DP**, we have:

$$\mathrm{TPR} \cdot p_0 + \mathrm{FPR}_0 \cdot (1 - p_0) = \mathrm{TPR} \cdot p_1 + \mathrm{FPR}_1 \cdot (1 - p_1)$$

Left and right sides are denominators of PPV_a. Therefore:

$$\frac{\mathrm{PPV}_0}{\mathrm{PPV}_1} = \frac{p_0}{p_1}$$

The ratio $\frac{\mathrm{PPV}_0}{\mathrm{PPV}_1} = \frac{p_0}{p_1}$ which is possible only in trivial cases, such as when the decision is group-independent. Similar steps can be used to show that other pairs of fairness criteria cannot be satisfied simultaneously in non-trivial situations.