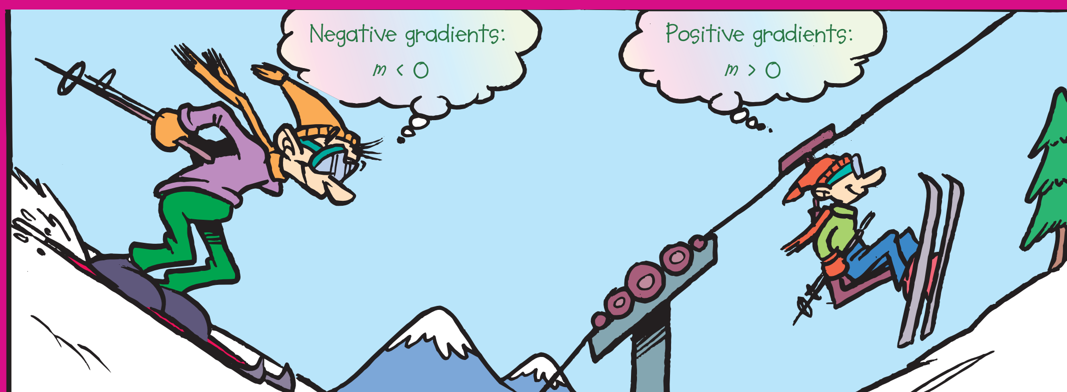
8

Coordinate Geometry



Chapter Contents

Learning Outcomes

Students will be able to:

- Find the distance between two points.
- Find the midpoint of an interval.
- Find the gradient of an interval.
- Graph straight lines on the Cartesian plane.
- Use the gradient-intercept form of a straight line.
- Find the equation of a straight line given a point and the gradient, or two points on the line.
- Identify parallel and perpendicular lines.
- Graph linear inequalities on the Cartesian plane.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Communication, Logical Thinking, IT Skills, Reflection), Human Ingenuity

The French mathematician René Descartes first introduced the number plane. He realised that using two sets of lines to form a square grid allowed the position of a point in the plane to be recorded using a pair of numbers or coordinates.

Coordinate geometry is a powerful mathematical technique that allows algebraic methods to be used in the solution of geometrical problems.

In this chapter, we will look at the basic ideas of:
- the distance between two points on the number plane
- the midpoint of an interval
- gradient (or slope)
- the relationship between a straight line and its equation.

We shall then see how these can be used to solve problems.

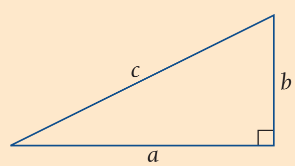
8:01 | The Distance Between Two Points

The number plane is the basis of coordinate geometry, an important branch of mathematics. In this chapter, we will look at some of the basic ideas of coordinate geometry and how they can be used to solve problems.

prep quiz 8:01

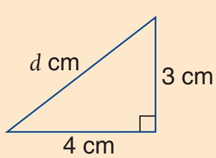
1 Which of the following is the correct statement of Pythagoras' theorem for the triangle shown?

A $a^2 = b^2 + c^2$ **B** $b^2 = a^2 + c^2$ **C** $c^2 = a^2 + b^2$



For questions 2 to 4, use Pythagoras' theorem to find the value of d.

2



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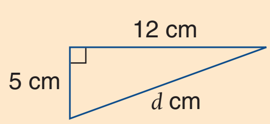

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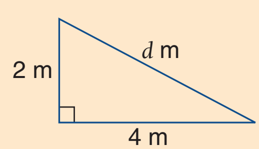
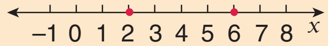
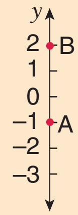


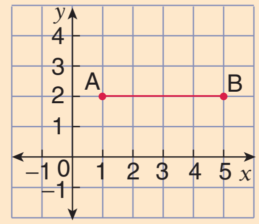
5 

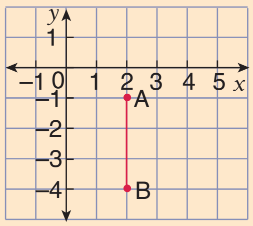
Distance AB = . . . units.

6 

Distance AB = . . . units.

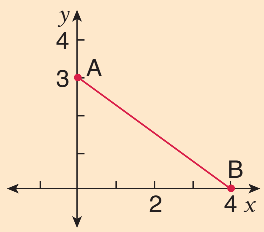
7 Distance AB = . . . units.



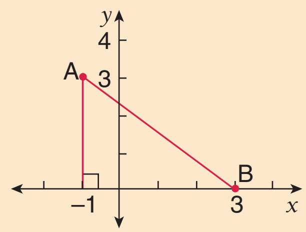
8 

Distance AB = . . . units.

9 Find the distance AB.



10 Find the distance AB.

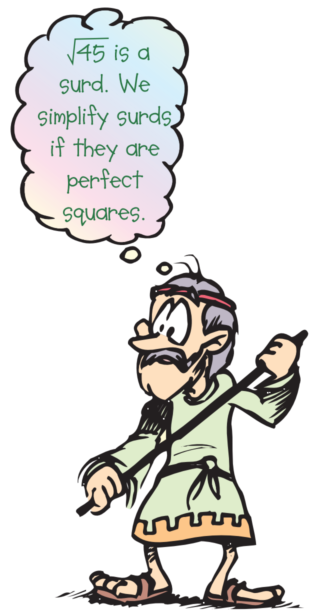


Pythagoras' theorem can be used to find the distance between two points on the number plane.

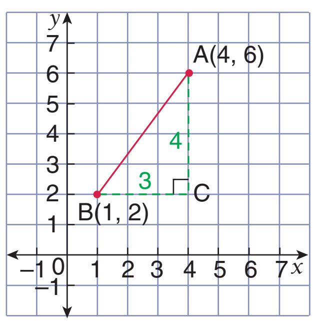
worked examples

1 Find the distance between the points (1, 2) and (4, 6).

2 If A is (–2, 2) and B is (4, 5) find the length of AB.



Solutions

1 

$c^2 = a^2 + b^2$

$AB^2 = AC^2 + BC^2$

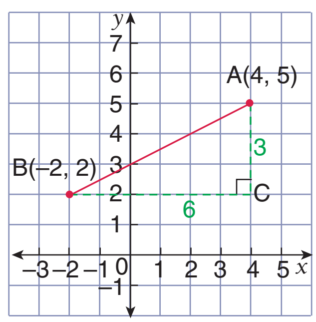
$= 4^2 + 3^2$

$= 16 + 9$

$= 25$

$\therefore$ AB = $\sqrt{25}$

$\therefore$ the length of AB is 5 units.

2 

$c^2 = a^2 + b^2$

$AB^2 = AC^2 + BC^2$

$= 3^2 + 6^2$

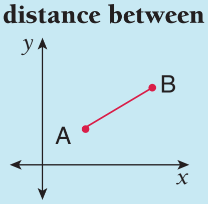
$= 9 + 36$

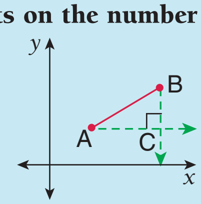
$= 45$

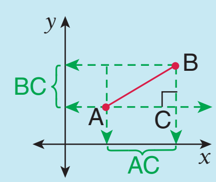
$\therefore$ AB = $\sqrt{45}$

$\therefore$ the length of AB is $\sqrt{45}$ unit.

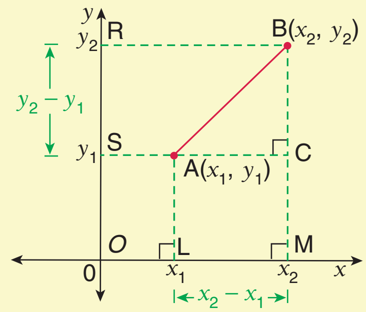
By drawing a right-angled triangle we can use Pythagoras' theorem to find the distance between any two points on the number plane.







Distance formula



A formula for finding the distance between two points, A(x_1, y_1) and B(x_2, y_2), can be found using Pythagoras' theorem. We wish to find the length of interval AB.

Now $\quad \text{LM} = x_2 - x_1$ (since LM = MO − LO)

$\therefore \quad \boxed{\text{AC} = x_2 - x_1}$ (ACML is a rectangle)

and $\quad \text{RS} = y_2 - y_1$ (since RS = RO − SO)

$\therefore \quad \boxed{\text{BC} = y_2 - y_1}$ (BCSR is a rectangle)

Now $\quad \text{AB}^2 = \text{AC}^2 + \text{BC}^2$ (Pythagoras' theorem)

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \text{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance AB between A(x_1, y_1) and B(x_2, y_2) is given by:

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

worked examples

1 Find the distance between the points (3, 8) and (5, 4).

2 Find the distance between the points (−2, 0) and (8, −5)

Solutions

1 Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$(x_1, y_1) = (3, 8)$ and $(x_2, y_2) = (5, 4)$

$$\therefore d = \sqrt{(5-3)^2 + (4-8)^2}$$
$$= \sqrt{(2)^2 + (-4)^2}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20}$$

∴ Distance ≑ 4·47 (using a calculator to answer to 2 decimal places).

2 Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (8, -5)$

$$\therefore d = \sqrt{(8 - -2)^2 + (-5 - 0)^2}$$
$$= \sqrt{(10)^2 + (-5)^2}$$
$$= \sqrt{100 + 25}$$
$$= \sqrt{125}$$

∴ Distance ≑ 11·18 (using a calculator to answer to 2 decimal places).

- You should check that the formula will still give the same answer if the coordinates are named in the reverse way. Hence, in example **1**, if we call $(x_1, y_1) = (5, 4)$ and $(x_2, y_2) = (3, 8)$, we would produce the same answer.

Exercise 8:01

Foundation Worksheet 8:01

Distance between points

1 Use Pythagoras' theorem to find the length of the hypotenuse in each of the following.

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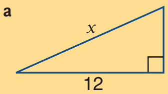

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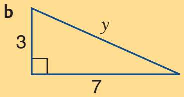
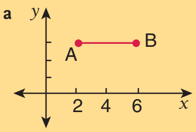


2 Find the distance AB in each of the following.

a


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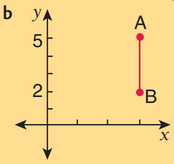

3 Find the length of AB in each of the following.

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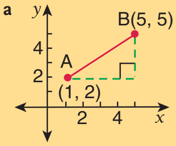

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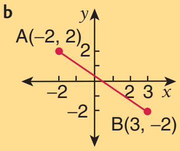

1 Use Pythagoras' theorem to find the length of each of the following. (Leave your answer as a surd, where necessary.)

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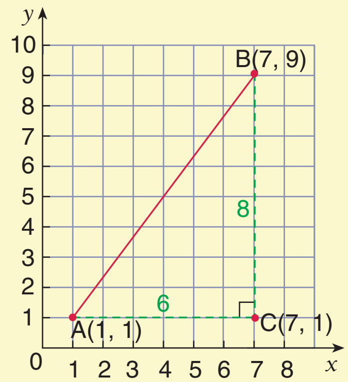

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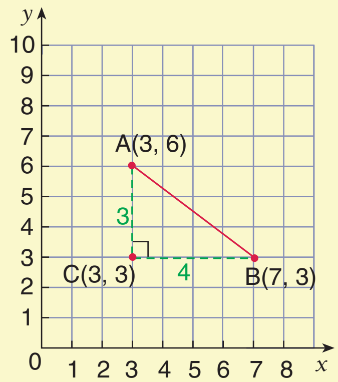
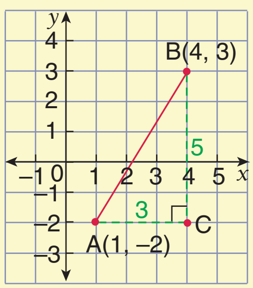
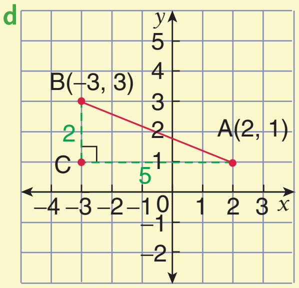
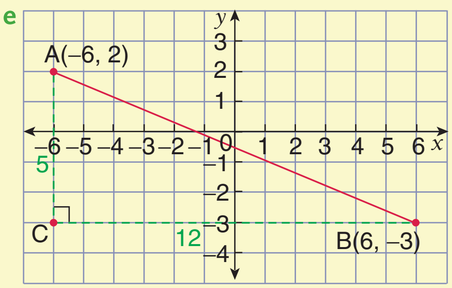


c


d


e


2 Find the lengths BC and AC and use these to find the lengths of AB. (Leave your answers in surd form.)

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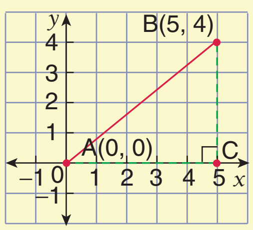

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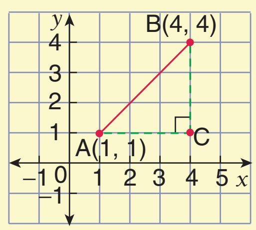

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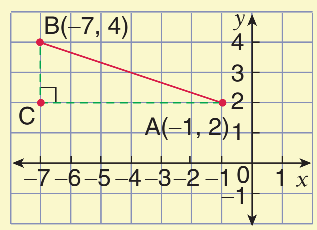
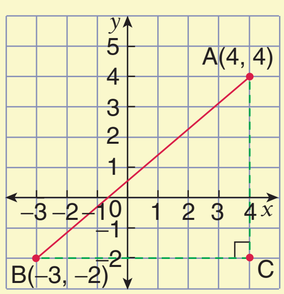


d


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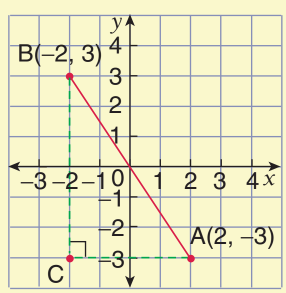

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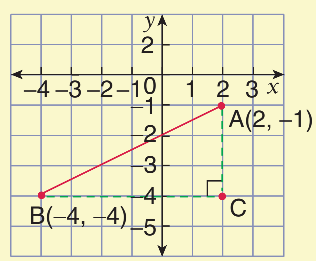
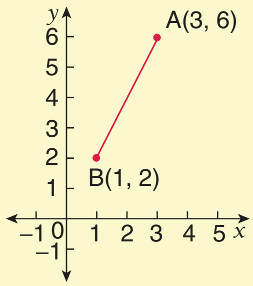


3 Use Pythagoras' theorem to find the length of interval AB in each of the following. (Leave answers in surd form.)

a



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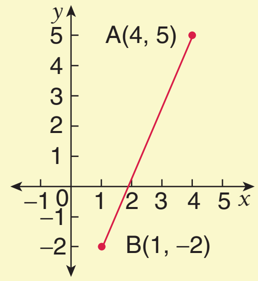

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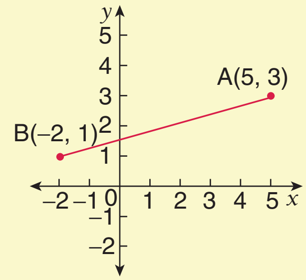

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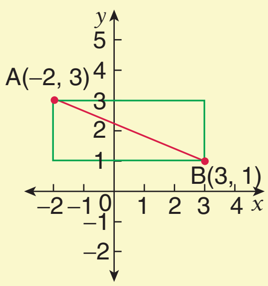

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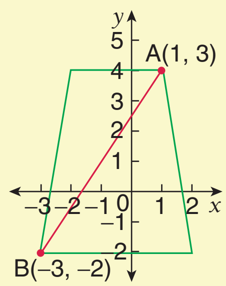

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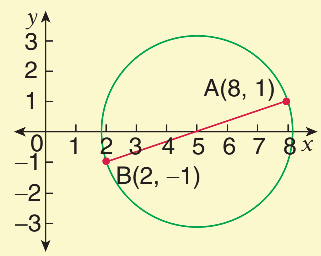


4 Use the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to find the distance between the points:

a (4, 2) and (7, 6)
b (0, 1) and (8, 7)
c (−6, 4) and (−2, 1)
d (−2, −4) and (4, 4)
e (−6, 2) and (6, 7)
f (4, 9) and (−1, −3)
g (3, 0) and (5, −4)
h (8, 2) and (7, 0)
i (6, −1) and (−2, 4)
j (−3, 2) and (−7, 3)
k (6, 2) and (1, 1)
l (4, 4) and (3, 3)

5 **a** Find the distance from the point (4, 2) to the origin.
b Which of the points (−1, 2) or (3, 5) is closer to the point (3, 0)?
c Find the distance from the point (−2, 4) to the point (3, −5).
d Which of the points (7, 2) or (−4, −4) is further from (0, 0)?

6 **a** The vertices of a triangle are A(0, 0), B(3, 4) and C(−4, 5). Find the length of each side.
b ABCD is a parallelogram where A is the point (2, 3), B is (5, 5), C is (4, 3) and D is (1, 1). Show that the opposite sides of the parallelogram are equal.
c Find the length of the two diagonals of the parallelogram in part **b**.
d EFGH is a quadrilateral, where E is the point (0, 1), F is (3, 2), G is (2, −1) and H is (−1, −2). Prove that EFGH is a rhombus. (The sides of a rhombus are equal.)
e (3, 2) is the centre of a circle. (6, 6) is a point on the circumference. What is the radius of the circle?
f Prove that the triangle ABC is isosceles if A is (−2, −1), B is (4, 1) and C is (2, −5). (Isosceles triangles have two sides equal.)
g A is the point (−13, 7) and B is (11, −3). M is halfway between A and B. How far is M from B?

8:02 | The Midpoint of an Interval

prep quiz 8:02

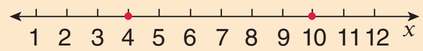
1 $\dfrac{4+10}{2}$

2 $\dfrac{-2+4}{2}$

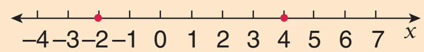
3 What is the average of 4 and 10?

4 What is the average of -2 and 4?

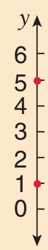
5 What number is halfway between 4 and 10?



6 What number is halfway between -2 and 4?

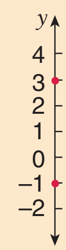


7 What number is halfway between 1 and 5?



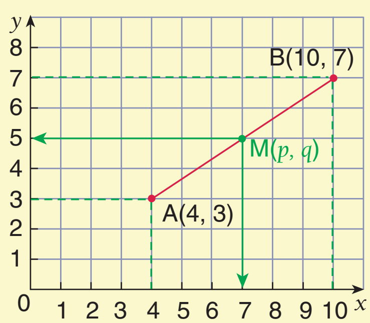
8 $\dfrac{1+5}{2} = ?$

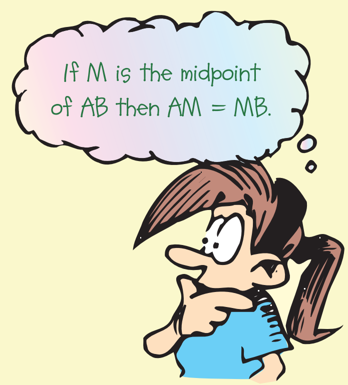
9 What number is halfway between -1 and 3?



10 $\dfrac{-1+3}{2} = ?$

- The midpoint of an interval is the halfway position. If M is the midpoint of AB then it will be halfway between A and B.





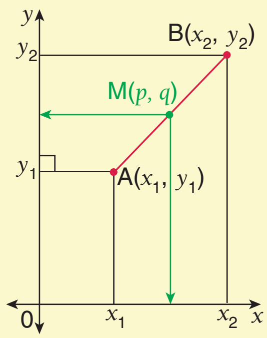
Consider the *x*-coordinates.

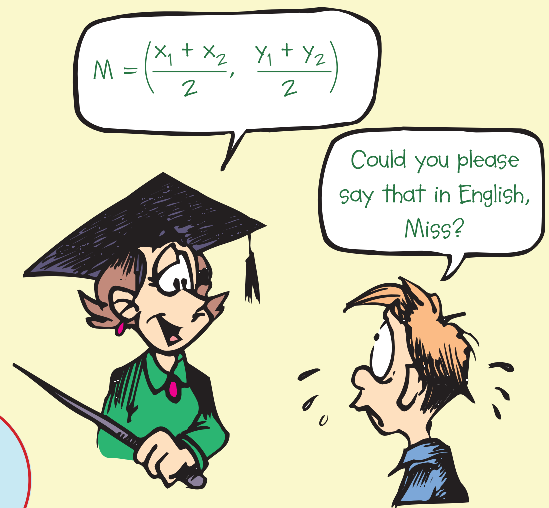
Note: 7 is halfway between 4 and 10. → The average of 4 and 10 is 7. → $\dfrac{4+10}{2} = 7$ → Formula: $p = \dfrac{x_1 + x_2}{2}$

Consider the *y*-coordinates.

Note: 5 is halfway between 3 and 7. → The average of 3 and 7 is 5. → $\dfrac{3+7}{2} = 5$ → Formula: $q = \dfrac{y_1 + y_2}{2}$

Midpoint formula





The midpoint, M, of interval AB, where A is (x_1, y_1) and B is (x_2, y_2), is given by:

$$\mathbf{M} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

worked examples

1 Find the midpoint of the interval joining (2, 6) and (8, 10).

2 Find the midpoint of interval AB, if A is the point (−3, 5) and B is (4, −2).

Solutions

1 $$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2+8}{2}, \frac{6+10}{2}\right) = (5, 8)$$

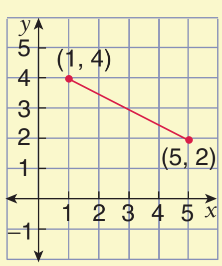
2 $$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-3+4}{2}, \frac{5+-2}{2}\right) = \left(\tfrac{1}{2}, \tfrac{3}{2}\right) \text{ or } \left(\tfrac{1}{2}, 1\tfrac{1}{2}\right)$$

Exercise 8:02

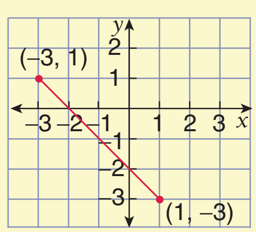
Foundation Worksheet 8:02
Midpoint
1 Read the midpoint of the interval AB from the graph
2 Find the midpoint of the interval that joins:
 a (3, 4) to (10, 8) **b** . . .
3 Find the midpoint of the interval that joins:
 a (−4, 6) to (−3, −5) **b** . . .

1 Use the graph to find the midpoint of each interval.

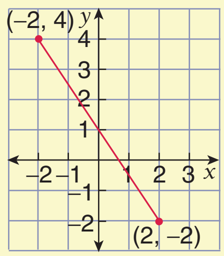
a



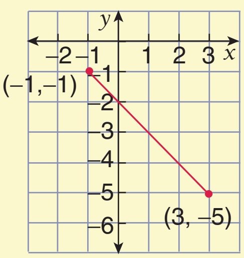
b



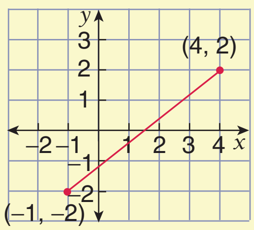
c



d

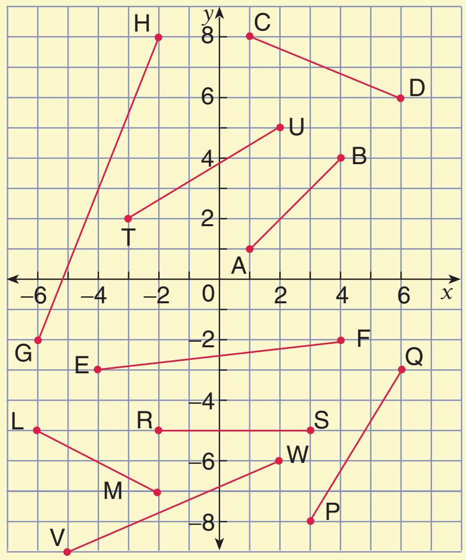


e



2 Use the graph to find the midpoints of the intervals:

a AB	**b** CD	**c** GH
d EF	**e** LM	**f** PQ
g RS	**h** TU	**i** VW



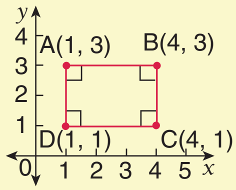
3 Find the midpoint of each interval AB if:

a A is (2, 4), B is (6, 10)	**b** A is (1, 8), B is (5, 6)	**c** A is (4, 1), B is (8, 7)
d A is (0, 0), B is (−4, 2)	**e** A is (−1, 0), B is (5, 4)	**f** A is (−2, −6), B is (4, 2)
g A is (−8, −6), B is (0, −10)	**h** A is (−2, 4), B is (−4, −6)	**i** A is (−2, −4), B is (−6, −7)

4 Find the midpoint of the interval joining:

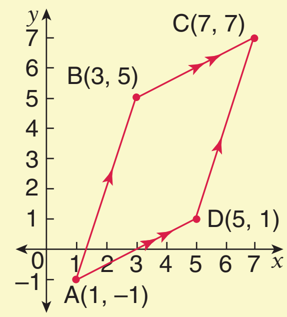
a (−3, −3) and (2, −3)	**b** (8, −1) and (7, −1)	**c** (5, 5) and (5, −5)
d (6, −7) and (−7, 6)	**e** (0, −4) and (−4, 0)	**f** (6, −6) and (5, −5)
g (111, 98) and (63, 42)	**h** (68, −23) and (72, −29)	**i** (400, 52) and (124, 100)

5 **a** **i** Find the midpoint of AC.
ii Find the midpoint of BD.
iii Are the answers for **i** and **ii** the same?
iv What property of a rectangle does this result demonstrate?

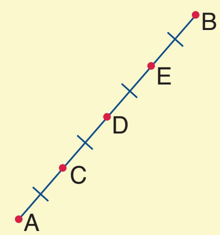


b If (4, 6) and (2, 10) are points at opposite ends of a diameter of a circle, what are the coordinates of the centre?

c **i** Find the midpoint of AC.
ii Find the midpoint of BD.
iii Are the answers for **i** and **ii** the same?
iv What property of a parallelogram does this result demonstrate?



6 **a** If the midpoint of (3, k) and (13, 6) is (8, 3), find the value of k.
b The midpoint of AB is (7, −3). Find the value of d and e if A is the point (d, 0) and B is (−1, e).
c The midpoint of AB is (−6, 2). If A is the point (4, 4), what are the coordinates of B?
d A circle with centre (3, 4) has a diameter of AB. If A is the point (−1, 6) what are the coordinates of B?



7 **a** If A is the point (1, 4) and B is the point (15, 10), what are the coordinates of the points C, D and E?
b If A is the point (1, 4) and D is the point (15, 10), what are the coordinates of the points B, C and E?

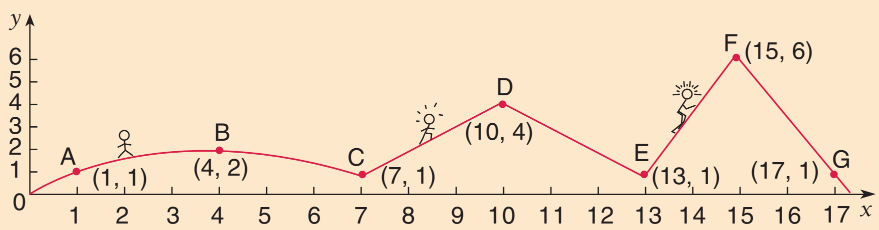
8 **a** Use coordinate geometry to show that the points A(−12, 10), B(8, 0), C(4, −6) and D(−16, 4) form a parallelogram.
b Use coordinate geometry to show that the points (−3, 2), (5, −2), (4, −4) and (−4, 0) form a rectangle.

Roger has 4 different pizza toppings. How many different pizzas could be made using:

- 1 topping?
- 2 toppings?
- any number of toppings?

8:03 | The Gradient of a Line

prep quiz
8:03



1 Which is steepest, AB or EF?

2 **3** and **4** If I travel from left to right, between which 3 pairs of letters am I travelling upwards?

5 **6** and **7** Between which 3 pairs of letters am I travelling downwards?

Say whether the hill is sloping up, down, or not at all, at the points

8 A　　**9** G　　**10** F

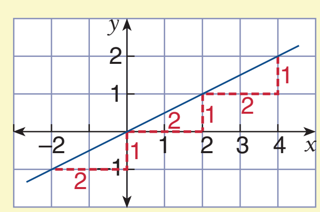
The **gradient** or **slope** of a line is a measure of *how steep* it is.

Negative gradient　　Positive gradient　　A higher positive gradient

- If we move from left to right the line going down is said to have a *negative* gradient (or slope). The line going up is said to have a *positive* gradient (or slope).
- If the line is horizontal (not going up or down) its gradient is *zero*.
- We find the gradient of a line by comparing its rise (change in y) with its run (change in x).

$$\textbf{Gradient} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } y}{\textbf{change in } x}$$

- So a gradient of $\frac{1}{2}$ means that for every run of 2 there is a rise of 1 (or for every 2 that you go across you go up 1).



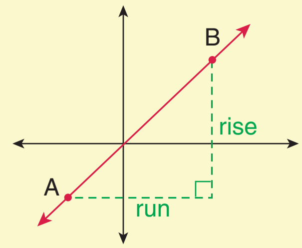
Finding the gradient of a line

1 Select any two points on the line.

2 Join the points and form a right-angled triangle by drawing a vertical line from the higher point and a horizontal side from the lower point.

3 Find the change in the y-coordinates (rise) and the change in the x-coordinates (run).

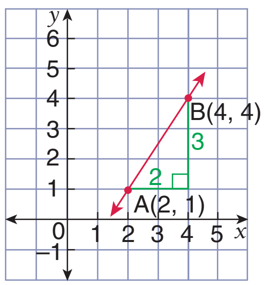
4 Use the formula above to find the gradient.



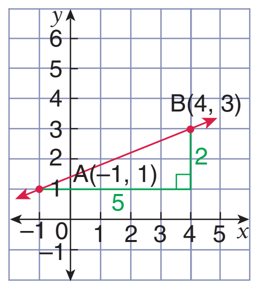
worked examples

Use the points A and B to find the gradient of the line AB in each case.

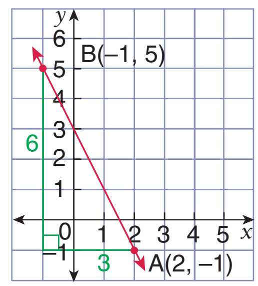
1



2



3



Solutions

1 Gradient

$$= \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{\text{up } 3}{\text{across } 2}$$

$$= \frac{3}{2}$$

2 $$m = \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{\text{up } 2}{\text{across } 5}$$

$$= \frac{2}{5}$$

3 $$m = \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{\text{down } 6}{\text{across } 3}$$

$$= -\frac{6}{3}$$

$$= -2$$

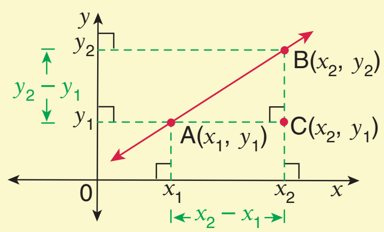
■ m is used for 'gradient'

■ Gradient is generally left as an improper fraction instead of a mixed numeral. So we write $\frac{3}{2}$ instead of $1\frac{1}{2}$.

• Architectural design often requires an understanding of gradients (slopes).

Gradient formula

We wish to find a formula for the gradient of a line AB where A is (x_1, y_1) and B is (x_2, y_2).



$$\text{Gradient of AB} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\text{change in } y}{\text{change in } x}$$
$$= \frac{BC}{AC}$$
$$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{(opposite sides of a rectangle are equal)}$$

m is used for 'gradient'

The gradient of the line that passes through the points A(x_1, y_1) and B(x_2, y_2) is given by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

worked examples

Find the gradient of the straight line passing through the following points.

1 (1, 3) and (4, 7)

2 (6, −2) and (2, −1)

Solutions

1 Let (x_1, y_1) be (1, 3) and (x_2, y_2) be (4, 7).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-3}{4-1} = \frac{4}{3}$$

$\therefore$ The gradient is $1\frac{1}{3}$.

2 Let (x_1, y_1) be (6, −2) and (x_2, y_2) be (2, −1).

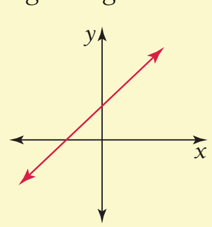
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-2)}{2-6} = \frac{1}{-4}$$

$\therefore$ The gradient is $-\frac{1}{4}$.

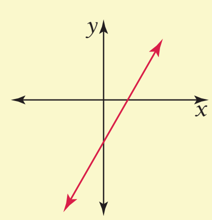
Exercise 8:03

1 For each of the following, state if the line has a positive or negative gradient.

a



b

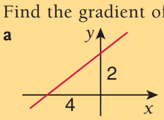


Foundation Worksheet 8:03

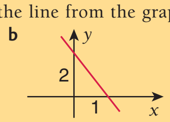
Gradients

1 Find the gradient of the line from the graph.

a

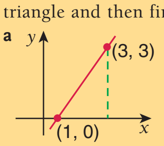


b

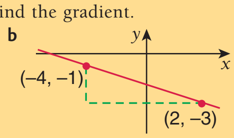


2 Find the lengths of the missing sides on each triangle and then find the gradient.

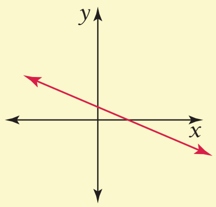
a



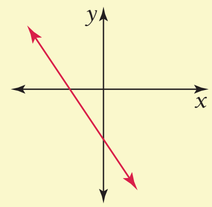
b



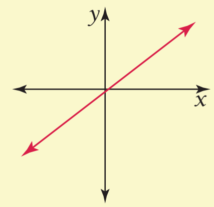
c



d

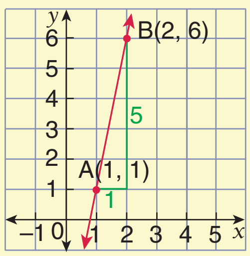


e

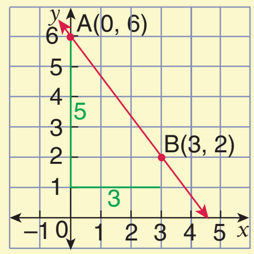


2 Using Gradient = $\frac{}{\text{Run}}$ find the gradient of AB in each of the following:

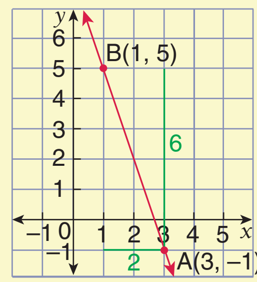
a



b

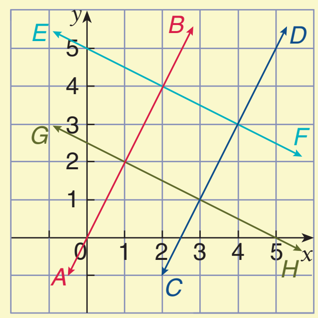


c



d A(2, 9), B(–1, 0) **e** A(0, 5), B(5, 0) **f** A(–3, –8), B(1, 8)

3
a Calculate the gradients of the four lines.
b Which lines have the same gradients?
c Which lines are parallel?

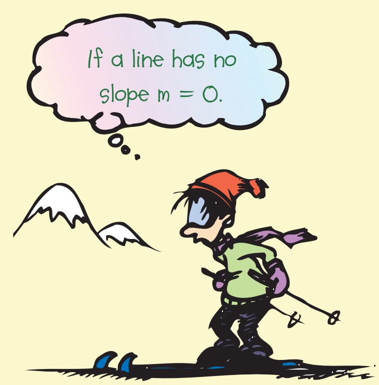


4 On the same number plane, draw:
a a line through (0, 0) with a gradient of –2
b a line through (1, 1) which is parallel to the line in **a**.
c Do the lines in **a** and **b** have the same gradient?

If two lines have the same gradient they are *parallel*.

5 Use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of the straight line passing through the points:

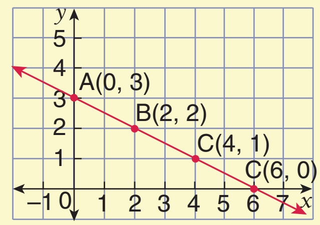
a (2, 6) and (5, 7) **b** (4, 2) and (5, 6)
c (3, 1) and (7, 3) **d** (0, 0) and (5, 2)
e (0, 5) and (6, 6) **f** (3, 0) and (5, 6)
g (6, 2) and (2, 1) **h** (7, 7) and (5, 6)
i (9, 12) and (3, 7) **j** (–4, 3) and (1, 4)
k (–3, –2) and (0, 6) **l** (4, –1) and (3, 3)
m (2, 3) and (–4, 9) **n** (–4, 1) and (–2, –4)
o (5, 2) and (7, –6) **p** (–3, –1) and (–6, –7)
q (4, –2) and (–4, –2) **r** (–6, 3) and (1, 3)



6 **a** Find the gradient of the line that passes through A(3, 1) and B(5, 11).

b Find the slope of the line that passes through O(0, 0) and B(−1, −2).

c On the graph shown, all of the points A, B, C and D lie on the same straight line, $x + 2y = 6$.



Find the gradient of the line using the points:

i A and B **ii** C and D

iii A and D **iv** B and C

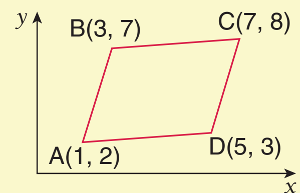
Conclusion: Any two points on a straight line can be used to find the gradient of that line. A straight line has only one gradient.

d Use the gradient of an interval to show that the points (−2, 5), (2, 13) and (6, 21) are collinear (ie, lie on the same straight line).

7 **a** **i** Find the gradient of BC and of AD.

ii Find the gradient of AB and of DC.

iii What kind of quadrilateral is ABCD? Give a reason for your answer.



b Prove that a quadrilateral that has vertices A(2, 3), B(9, 5), C(4, 0) and D(−3, −2) is a parallelogram. (It will be necessary to prove that opposite sides are parallel.)

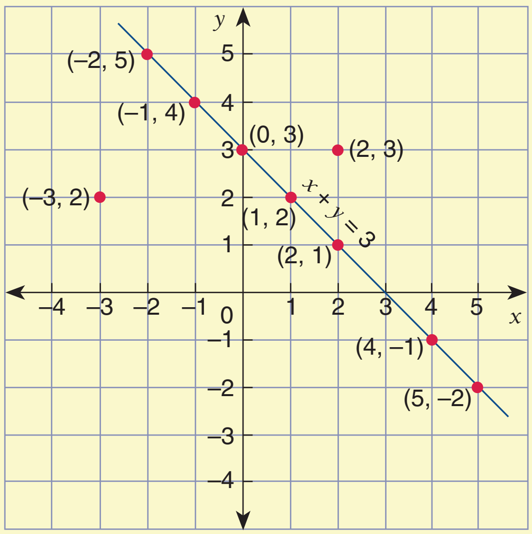
8 Use the fact that *a rhombus is a parallelogram with a pair of adjacent sides equal* to prove that the points A(−1, 1), B(11, 4), C(8, −8) and D(−4, −11) form the vertices of a rhombus.

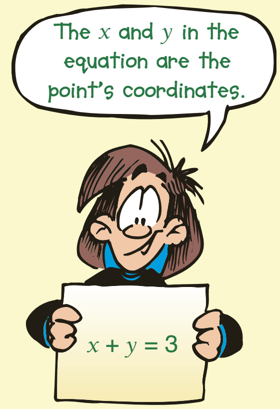
8:04 | Graphing Straight Lines

A straight line is made up of a set of points, each with its own pair of coordinates.

- Coordinate geometry uses an equation to describe the relationship between the x- and y-coordinates of any point on the line.

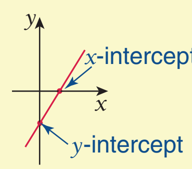
  In the diagram, the equation of the line is $x + y = 3$. From the points shown, it is clear that the relationship is that the sum of each point's coordinates is 3.
- A point can only lie on a line if its coordinates satisfy the equation of the line. For the points $(-3, 2)$ and $(2, 3)$, it is clear that the sum of the coordinates is not equal to 3. So they do not lie on the line.





To graph a straight line we need:
- an equation to allow us to calculate the x- and y-coordinates for each point on the line
- a table to store at least two sets of coordinates
- a number plane on which to plot the points.

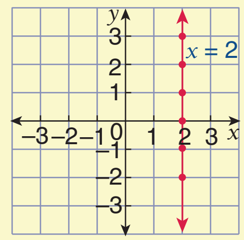
Two important points on a line are:
- the x-intercept (where the line crosses the x-axis) This is found by substituting $y = 0$ into the line's equation and then solving for x.
- the y-intercept (where the line crosses the y-axis) This is found by substituting $x = 0$ into the line's equation and then solving for y.



Horizontal and vertical lines

The line shown on the graph on the right is vertical.

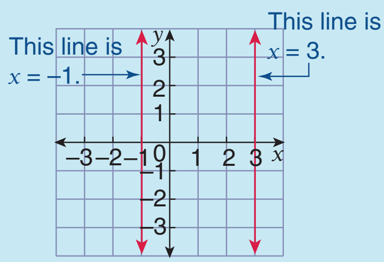
- Below, we have put the points on the line into a table.

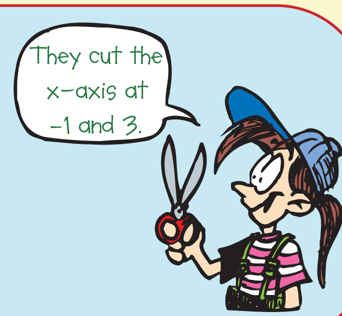


x	2	2	2	2	2	2
y	−2	−1	0	1	2	3

- There seems to be no connection between x and y. However, x is always 2. So the equation is $x = 2$.

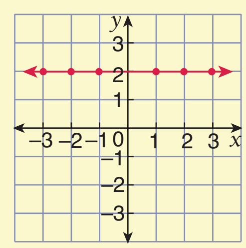
Vertical lines have equations of the form $x = a$ where a is where the line cuts the x-axis.





The line on the right is horizontal.

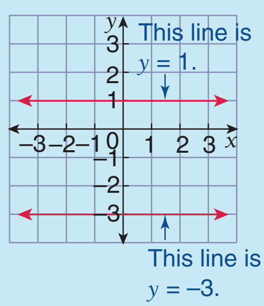
- Below, we have put the points on the line, into a table.



x	−2	−1	0	1	2	3
y	2	2	2	2	2	2

- There seems to be no connection between x and y. However, y is always 2. So the equation is $y = 2$.

Horizontal lines have equations of the form $y = b$ where b is where the line cuts the y-axis.



worked examples

Draw the graph of each straight line.

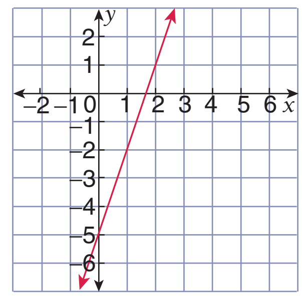
1 $y = 3x - 5$ **2** $3x + y = 5$ **3** $2x + 3y = 6$ **4** $3x - 4y + 12 = 0$

Solutions

1 $y = 3x - 5$

x	0	1	2
y	-5	-2	1

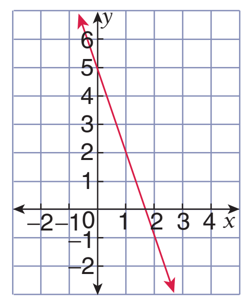
when $x = 0$, $y = 3 \times 0 - 5$
$y = -5$
when $x = 1$, $y = 3 \times 1 - 5$
$y = -2$
when $x = 2$, $y = 3 \times 2 - 5$
$y = 1$



2 $3x + y = 5$

x	0	1	2
y	5	2	-1

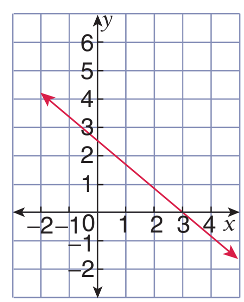
when $x = 0$, $3 \times 0 + y = 5$
$0 + y = 5$
$y = 5$
when $x = 1$, $3 \times 1 + y = 5$
$3 + y = 5$
$y = 2$
when $x = 2$, $3 \times 2 + y = 5$
$6 + y = 5$
$y = -1$



3 $2x + 3y = 6$

x	0	1	2
y	2	$1\frac{1}{3}$	$\frac{2}{3}$

when $x = 0$, $2 \times 0 + 3y = 6$
$0 + 3y = 6$
$y = 2$
when $x = 1$, $2 \times 1 + 3y = 6$
$2 + 3y = 6$
$3y = 4$
$y = 1\frac{1}{3}$
when $x = 2$, $2 \times 2 + 3y = 6$
$4 + 3y = 6$
$3y = 2$
$y = \frac{2}{3}$



Since it is difficult to plot fractions, an intercept method would be better here:
At the x axis, $y = 0$
$\therefore\ 2x + 3 \times 0 = 6$
$2x = 6$
$x = 3$
At the y axis, $x = 0$
$\therefore\ 2 \times 0 + 3y = 6$
$3y = 6$
$y = 2$

continued →→→

4 $3x - 4y + 12 = 0$

It can be seen that if a table of values is used here fractions will result. The intercept method is better again.

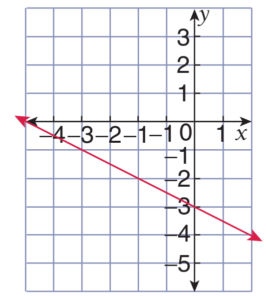
Rewrite the equation as $3x - 4y = -12$

At the x axis, $y = 0$

$$\begin{aligned} \therefore\ 2x - 4 \times 0 &= -12 \\ 2x &= -12 \\ x &= -6 \end{aligned}$$

At the y axis, $x = 0$

$$\begin{aligned} \therefore\ 2 \times 0 + 4y &= -12 \\ 4y &= -12 \\ y &= -3 \end{aligned}$$



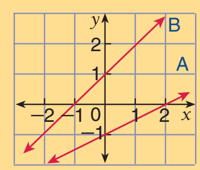
Exercise 8:04

Foundation Worksheet 8:04

Graphing lines

1 Complete the tables for the equations:
a $y = x$ **b** $y = x + 2$ **c** $x + y = 2$

2 Read off the x- and y-intercept of line:
a A **b** B



1 Using separate axes labelled from -4 to 6, draw the graph of the following lines using any of the above methods.

a $y = 2x - 3$ **b** $y = 4 - 3x$
c $2x + y = 5$ **d** $y + 3x = 3$

2 Graph the lines represented by these equations using an appropriate method.

a $x + 2y = 4$ **b** $y = 5 - x$ **c** $y = 3x + 2$ **d** $2y - 5x = 10$
e $x + 3y + 9 = 0$ **f** $3 - 2x = y$ **g** $3y = 2x + 6$ **h** $y - 2 = 4x$

3 On which of the following lines does the point $(-2, 3)$ lie?

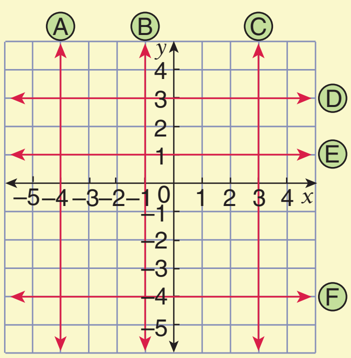
a $y = 2x + 1$ **b** $2x + y + 1 = 0$ **c** $x = 2y - 8$ **d** $y - 2x = -1$
e $3x - 2y = 0$ **f** $2y - 3x = 0$

4 The line $2x - 5y + 6 = 0$ passes through which of the following points?

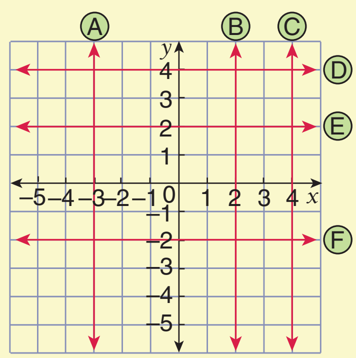
a $(2, 2)$ **b** $(7, 4)$ **c** $(-2, 2)$ **d** $(-3, 0)$ **e** $(8, 2)$ **f** $(3, 0)$

5 For each number plane, write down the equations of the lines (A) to (F).

a



b



6 Using values from −5 to 5 on each axis, draw the graphs of the following straight lines. Use a new diagram for each part.

a $y = 4, \quad x = 5, \quad y = -1, \quad x = 0$ **b** $x = 1, \quad y = 0, \quad x = 2, \quad y = 3$

c $y = 4, \quad x = 2, \quad y = -2, \quad x = -4$ **d** $x = 5, \quad y = -5, \quad x = 2, \quad y = 2$

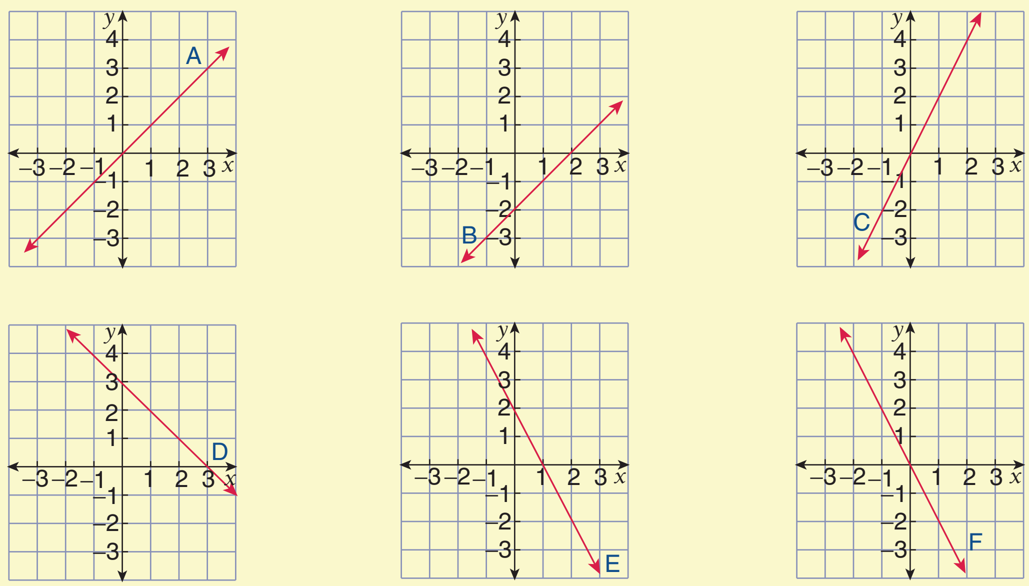
e $y = -2, \quad y = 0, \quad x = 0, \quad x = 3$

Which of these encloses a square region?

7 Match each of the graphs A to F with one of the following equations:

$y = 2x$ $\quad y = x - 2$ $\quad 2x + y = 0$

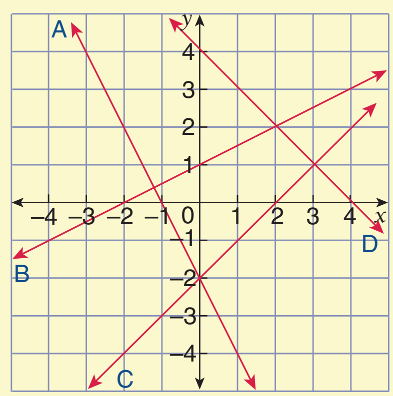
$y = x$ $\quad x + y = 3$ $\quad 2x + y = 2$



8 Which of the lines A, B, C or D could be described by the following equation.

a $x - y = 2$ **b** $x + y = 4$

c $2x + y + 2 = 0$ **d** $x - 2y + 2 = 0$



9 Use the intercept method to graph the following lines.

a $2x + y = 2$ **b** $3x + y = 6$ **c** $2x + y = 4$

d $2x - y = 4$ **e** $3x - y = 3$ **f** $4x - y = 2$

10 Draw the graph of each equation.

a $y = \dfrac{3x}{2}$ **b** $y = \dfrac{x+1}{2}$ **c** $y = \dfrac{x-1}{2}$

d $3x + 2y = 7$ **e** $5x - 2y - 6 = 0$ **f** $2x - 3y - 5 = 0$

8:05 | The Gradient–Intercept Form of a Straight Line: $y = mx + c$

prep quiz 8:05

If $x = 0$, what is the value of:

1 $2x$ **2** $mx + c$

If $x = 0$, what is the value of y when:

3 $y = 3x + 2$ **4** $y = 4x - 1$

What is the gradient of:

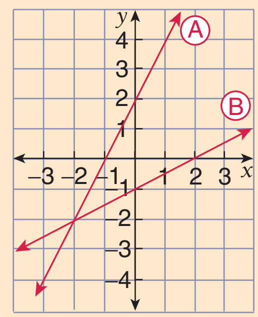
5 line A **6** line B

What are the coordinates of the y-intercept of:

7 line A **8** line B

9 Does every point on the y-axis have an x-coordinate of 0?

10 Can the y-intercept of a line be found by putting $x = 0$.



As you learnt in Book 3:

- The equation of a line can be written in several ways. For instance,
  $x - y - 4 = 0$, $y = x - 4$ and $x - y = 4$
  are different ways of writing the same equation.
- When the equation is written in the form $x - y - 4 = 0$, it is said to be in general form.
- The form $y = x - 4$ is a particularly useful way of writing the equation of a line. It allows us to get information about the line directly from the equation.

investigation 8:05

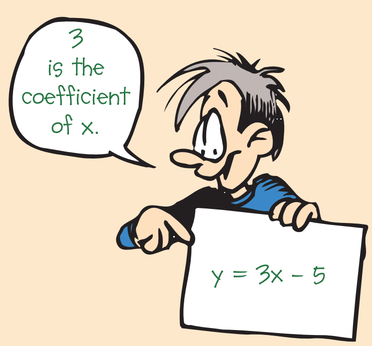
Investigation 8:05 | What does $y = mx + c$ tell us?

Please use the Assessment Grid on the page 222 to help you understand what is required for this Investigation.

For each of the following equations complete the table by:

1. graphing each line on a Cartesian grid
2. calculating the gradient of each line (use any two points that the line passes through), and
3. noting where the graph crosses the y axis (y-intercept).

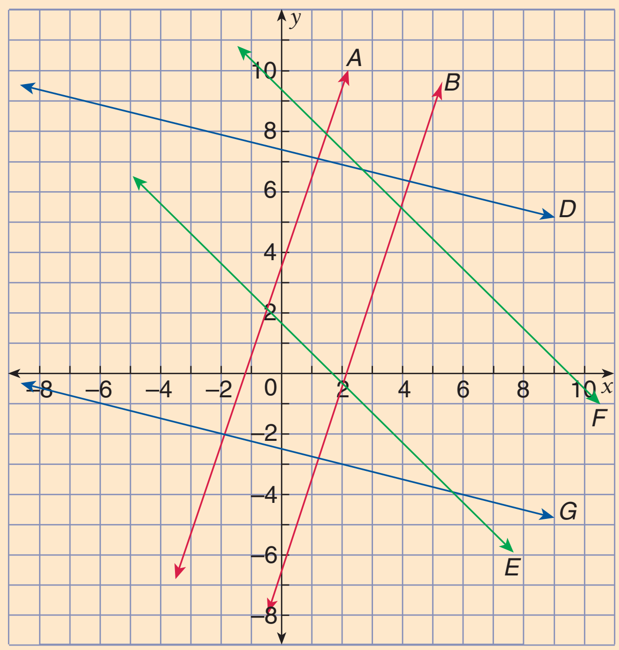
Line	Equation	Gradient	y-intercept
1	$y = 3x - 4$		
2	$y = 5 - 2x$		
3	$y = \frac{1}{2}x + 1$		
4	$y = x$		
5	$y = 3 - 2x$		



Are there any patterns you can find connecting the gradients and the y-intercepts and the numbers in the equations of the lines? Explain these patterns or connections.

Try to write a rule that will help you write down the gradient and y-intercept for a line without drawing the line.

4 Using your rule, write down the equations of the lines shown in the Cartesian grid below.



5 By making a table of values for your equations, check that your equations represent the lines given.

6 Do you notice anything special about the lines that have the same colour?

7 Can you discuss some possible real life applications of straight lines and their equations?

- **When an equation of a line is written in the form $y = mx + c$,**
  m gives the gradient of the line and
  c gives the y-intercept of the line.
- **Clearly, lines with the same gradient are parallel.**
- **When an equation of a line is written in the form $ax + by + c = 0$, where a, b and c are integers and $a > 0$, it is said to be in general form.**

Assessment Grid for Investigation 8:05 | What does $y = mx + c$ tell us?

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

Assessment Criteria (B, C, D) for this investigation				Achieved ✓
Criterion B Investigating Patterns	a	None of the following descriptors has been achieved.	0	
	b	Some help was needed to complete the table and identify the simple patterns in questions 2 and 3.	1	
			2	
	c	Mathematical problem-solving techniques have been selected and applied to accurately graph the lines required and complete the table, with some suggestion of emerging patterns.	3	
			4	
	d	The student has graphed the required lines and used the patterns evident in the table to find a connecting rule to give the equations of the lines in question 4.	5	
			6	
	e	The patterns evident between the equations and their graphs have been explained and summarised as a mathematical rule. The patterns for the lines in part 6 have been explained.	7	
			8	
Criterion C Communication in Mathematics	a	None of the following descriptors has been achieved.	0	
	b	There is a basic use of mathematical language and representation. Lines of reasoning are insufficient.	1	
			2	
	c	There is satisfactory use of mathematical language and representation. Graphs, tables and explanations are clear but not always logical or complete.	3	
			4	
	d	A good use of mathematical language and representation. Graphs are accurate, to scale and fully labelled. Explanations and answers are complete and concise.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the following descriptors has been achieved.	0	
	b	An attempt has been made to explain whether the results make sense, with connection to possible real-life applications.	1	
			2	
	c	There is a correct but brief explanation of whether results make sense and how they were found. A description of the important aspects of the graphs is given and the relation to their equations.	3	
			4	
	d	There is a critical explanation of the graphs obtained and their related equations. All results are fully explained and justified, and specific reference to real-life applications of lines is given.	5	
			6	

worked examples

1 Write down the gradient and y-intercepts of these lines.

a $y = 4x - 2$

$m = 4$ $\quad b = -2$

Gradient $= 4$

y-intercept $= -2$

b $y = 3 - 5x$

$b = 3$ $\quad m = -5$

Gradient $= -5$

y-intercept $= 3$

c $2x + 3y = 12$

$3y = -2x + 12$

$y = -\frac{2}{3}x + 4$

$m = -\frac{2}{3}$ $\quad b = 4$

Gradient $= -\frac{2}{3}$

y-intercept $= 4$

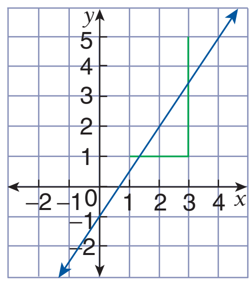
2 From the graph of the line shown, write down the gradient and y-intercept of the line and hence its equation.

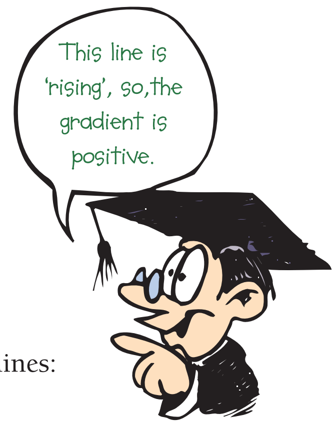
From the graph:

The y-intercept $(c) = -1$

The gradient $(m) = \frac{4}{2} = 2$

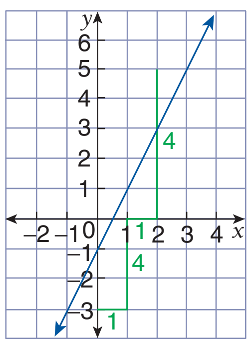
$\therefore$ the equation $y = mx + c$ becomes $y = 2x - 1$





3 Use the y-intercept and the gradient to graph the following lines:

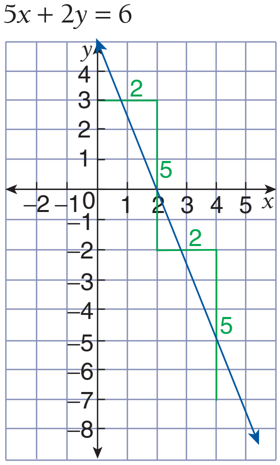
a $y = 4x - 3$



Start at the y-intercept of $(c) = -3$.

Gradient $= \frac{\text{Rise}}{\text{Run}} = 4 = \frac{4}{1}$ and is positive so for every 1 unit across there is a rise of 4 units.

b $5x + 2y = 6$



First, rearrange the equation in the form $y = mx + c$.

$$5x + 2y = 6$$
$$2y = -5x + 6$$
$$y = -\frac{5}{2}x + 3.$$

Start at the y-intercept $(c) = 3$

Gradient $= \frac{\text{Rise}}{\text{Run}} = -\frac{5}{2}$

and is negative so for every 2 units across there is a fall of 5 units.

Exercise 8:05

Foundation Worksheet 8:05

Gradient–intercept form

1 For each line find from the graph its:
a y-intercept **b** gradient
2 What is $y = mx + b$ when:
a $m = 4$ and $b = 3$ **b** $m = -3$ and $b = -1$
3 Find the equation of a line with:
a a gradient of 2 and y-intercept of 2

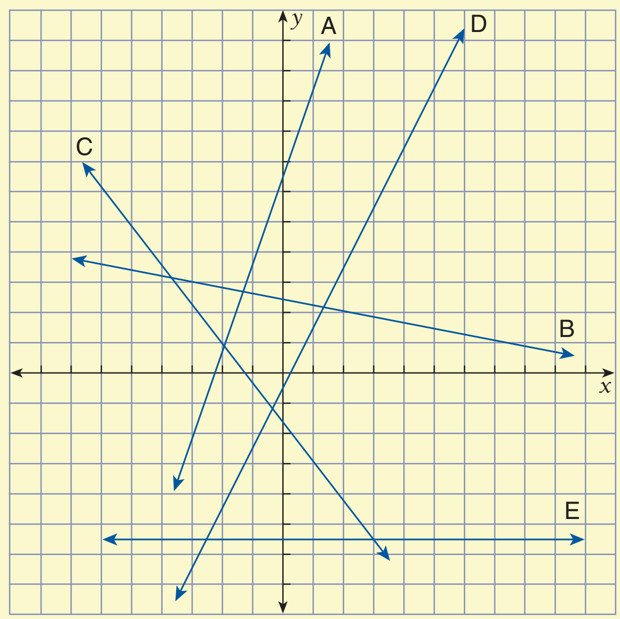
1 Write the gradient and y-intercept of the following lines:

a $y = 4x - 5$ **b** $y = 3x + 2$ **c** $y = 9 - x$
d $y = 3 + \frac{5}{7}x$ **e** $2y = 3x - 5$

2 By first rearranging the equation in the form $y = mx + c$ write the gradient and y-intercept of the following lines:

a $2y = 6x + 8$ **b** $y - 5 = 3x$ **c** $3x + 4y = 10$ **d** $2x = 6 - 3y$
e $3 - 2y = x$ **f** $2y + 4x - 10 = 0$ **g** $5x - 3y = -1$ **h** $3y - 4 = 6x$
i $x - y - 1 = 0$ **j** $6 - 5x = y$

3 Use the graph to find the gradient and y-intercept of each line and hence write the equation of each line in the general form. (The general form of a line is $ax + by + c = 0$, where a, b and c are whole numbers and a is positive.)



4 Draw the graphs of the following using only the gradient and y-intercept (follow example 3).

a $y = x - 6$
b $y = 3 - 2x$
c $2y = 3x - 6$
d $3x - 4y = 8$
e $x + 2y - 4 = 0$
f $2x + y + 4 = 0$
g $y + 6 = 0$
h $3x - 2y = -12$

Technology Applications Activities
8:05 Equation grapher

8:06 | The Equation of a Straight Line, Given Point and Gradient

prep quiz 8:06

A line has the equation $y = 3x + 5$. Find:

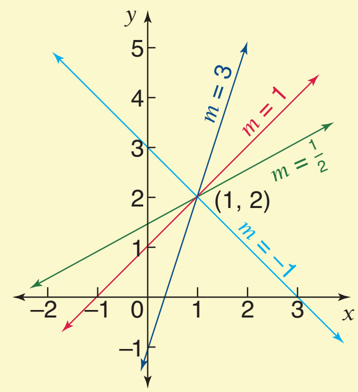
1 its gradient
2 its y-intercept
3 the value of y when $x = 2$
4 the value of x when $y = 5$.

If $y = mx + c$, find the value of c if:

5 $m = 2, x = 0, y = 4$
6 $m = -2, x = 1, y = 3$
7 $m = 3, x = -1, y = -2$
8 $m = -2, x = -4, y = 10$
9 $m = 0, x = 2, y = 3$
10 $m = 1, x = 2, y = 2$

Through a given point, any number of lines can be drawn. Each of these lines has a different gradient, so the equation of a straight line can be found if we know its gradient and a point through which it passes.

The equation of a line through (1, 2) having a gradient of 3 can be found by beginning with the formula $y = mx + b$. We know that $m = 3$, but we must also find the value of b. To do this, we substitute the coordinates of (1, 2) into the equation, as a point on the line must satisfy its equation.



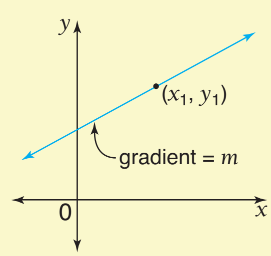
Working:
$$\begin{aligned} y &= mx + c && \text{(formula)} \\ \therefore\ y &= 3x + c && (m = 3 \text{ is given}) \\ 2 &= 3(1) + c && [(1, 2) \text{ lies on the line}] \\ 2 &= 3 + c \\ \therefore\ c &= -1 \end{aligned}$$

$\therefore$ The equation of the line is $y = 3x - 1$.

To find the equation of a straight line that has a gradient of 2 and passes through (7, 5):
(1) Substitute $m = 2$, $x = 7$ and $y = 5$ into the formula $y = mx + c$ to find the value of c.
(2) Rewrite $y = mx + c$ replacing m and c with their numerical values.

We can use the method above to discover a formula that could be used instead.

Question: What is the equation of a line that has a gradient m and passes through the point (x_1, y_1)?



Working:
$$\begin{aligned} y &= mx + c && \text{(formula; gradient is } m) \\ \therefore\ y_1 &= mx_1 + c && [(x_1, y_1) \text{ lies on the line}] \\ \therefore\ c &= y_1 - mx_1 \end{aligned}$$

$\therefore$ The equation of the line is

$$y = mx + (y_1 - mx_1)$$

ie $y - y_1 = mx - mx_1$

$\therefore\ y - y_1 = m(x - x_1)$

This last form of the answer is the easiest to remember as it could be written as $\dfrac{y - y_1}{x - x_1} = m$.

The equation of a line with gradient m, that passes through the point (x_1, y_1) is given by:

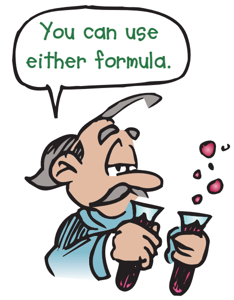
$$y - y_1 = m(x - x_1) \text{ or } \frac{y - y_1}{x - x_1} = m.$$

worked examples

1 Find the equation of the line that passes through (1, 4) and has gradient 2.

2 A straight line has gradient $-\frac{1}{2}$ and passes through the point (1, 3). Find the equation of this line.

$y - y_1 = m(x - x_1)$
or
$y = mx + c$



Solutions

1 Let the equation of the line be:

$y = mx + c$

$\therefore y = 2x + c$ ($m = 2$ is given)

$4 = 2(1) + c$ [(1, 4) lies on the line]

$4 = 2 + c$

$\therefore c = 2$

$\therefore$ The equation is $y = 2x + 2$.

or 1 $y - y_1 = m(x - x_1)$

(x_1, y_1) is (1, 4), $m = 2$

$\therefore y - 4 = 2(x - 1)$

$y - 4 = 2x - 2$

$\therefore y = 2x + 2$ is the equation of the line.

2 Let the equation be:

$y = mx + c$

$\therefore y = -\frac{1}{2}x + c$ ($m = -\frac{1}{2}$ is given)

$3 = -\frac{1}{2}(1) + c$ [(1, 3) is on the line]

$3 = -\frac{1}{2} + c$

$\therefore c = 3\frac{1}{2}$

$\therefore$ The equation is $y = -\frac{1}{2}x + 3\frac{1}{2}$.

or 2 $y - y_1 = m(x - x_1)$

(x_1, y_1) is (1, 3), $m = -\frac{1}{2}$

$\therefore y - 3 = -\frac{1}{2}(x - 1)$

$y - 3 = -\frac{1}{2}x + \frac{1}{2}$

$\therefore y = -\frac{1}{2}x + 3\frac{1}{2}$ is the equation of the line.

Exercise 8:06

Foundation Worksheet 8:06
Point–gradient form
Use $y - y_1 = m(x - x_1)$ to find the equation of a line when:
1 $m = 2$, $(x_1, y_1) = (1, 4)$
2 $m = -2$, $(x_1, y_1) = (-1, 3)$

1 For each part, find c if the given point lies on the given line.

a (1, 3), $y = 2x + c$ c (2, 10), $y = 4x + c$ c (−1, 3), $y = 2x + c$
d (5, 5), $y = 2x + c$ e (3, 1), $y = x + c$ f (−1, −9), $y = -2x + c$

2 Find the equation of the straight line (giving answers in the form $y = mx + c$) if it has:

a gradient 2 and passes through the point (1, 3)
b gradient 5 and passes through the point (0, 0)
c gradient 3 and passes through the point (2, 2)
d slope 4 and passes through the point (−1, 6)

e gradient −1 and passes through the point (−2, 8)

f gradient −2 and passes through the point (0, 7)

g slope −5 and passes through the point (1, 0)

h gradient $\frac{1}{2}$ and passes through the point (4, 5)

i gradient $\frac{1}{4}$ and passes through the point $(6, 3\frac{1}{2})$

j slope $-\frac{1}{2}$ and passes through the point (−4, −1)

3 a A straight line has a gradient of 2 and passes through the point (3, 2). Find the equation of the line.

b A straight line has a gradient of −1. If the line passes through the point (2, 1), find the equation of the line.

c What is the equation of a straight line that passes through the point (−2, 0) and has a gradient of 3?

d A straight line that passes through the point (1, −2) has a gradient of −3. What is the equation of this line?

e A straight line that has a gradient of 3 passes through the origin. What is the equation of this line?

f Find the equation of the straight line that has a gradient of 4 and passes through the point (−1, −2).

g (2, 8) is on a line that has a gradient of 4. Find the equation of this line.

h The point (−6, 4) lies on a straight line that has a gradient of −2. What is the equation of this line?

i Find the equation of the straight line that has a gradient of 2 and passes through the midpoint of the interval joining (1, 3) and (5, 5).

j A straight line passes through the midpoint of the interval joining (0, 0) and (−6, 4). Find the equation of the line if its gradient is $\frac{1}{2}$.

8:07 | The Equation of a Straight Line, Given Two Points

Only one straight line can be drawn through two points. Given two points on a straight line, we can always find the equation of that line.

Consider the line passing through (1, 1) and (2, 4). Let the equation of the line be:

$$y = mx + c \text{ (formula)}$$

First find the gradient using the two points.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$(x_1, y_1) = (1, 1)$
$(x_2, y_2) = (2, 4)$

$$= \frac{4 - 1}{2 - 1}$$

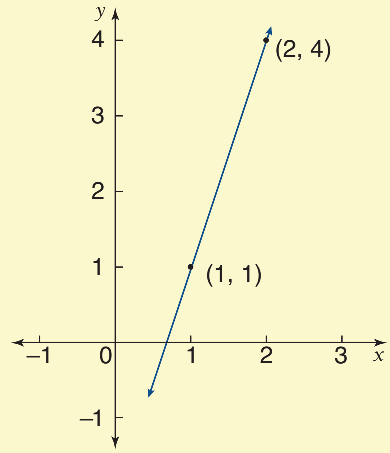
$$= 3$$

$\therefore\ y = 3x + c$ (since $m = 3$)

$4 = 3(2 + c)$ [(2, 4) lies on the line]

$\therefore\ c = -2$

$\therefore$ The equation of the line is $y = 3x - 2$.



To find the equation of a straight line that passes through the two points (1, 2) and (3, 6):

1. **Find the value of the gradient m, using the given points.**
2. **For $y = mx + c$, find the value of c by substituting the value of m and the coordinates of one of the given points.**
3. **Rewrite $y = mx + c$ replacing m and c with their numerical values.**

Another method is to use the formula:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

where (x_1, y_1) and (x_2, y_2) are points on the line.

worked example

Find the equation of the line that passes through the points (–1, 2) and (2, 8).

Solution

Let the equation of the line be:

$$y = mx + c$$

Now $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (-1, 2)$
$(x_2, y_2) = (2, 8)$

$$= \frac{8 - 2}{2 - (-1)}$$

$$= \frac{6}{3}$$

$\therefore\ m = 2$

$\therefore\ y = 2x + c$ (since $m = 2$)

(2, 8) lies on the line.

$\therefore\ 8 = 2(2) + c$

$\therefore\ c = 4$

$\therefore$ The equation is $y = 2x + 4$.

or

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

(x_1, y_1) is $(-1, 2)$, (x_2, y_2) is $(2, 8)$

$$\therefore\ y - 2 = \frac{8 - 2}{2 - (-1)}[x - (-1)]$$

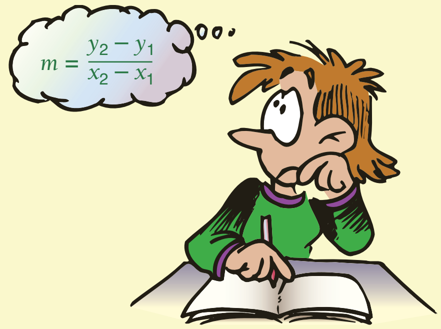
$$y - 2 = \frac{6}{3}(x + 1)$$

$$y - 2 = 2(x + 1)$$

$$y - 2 = 2x + 2$$

$\therefore\ y = 2x + 4$ is the equation of the line.

Exercise 8:07



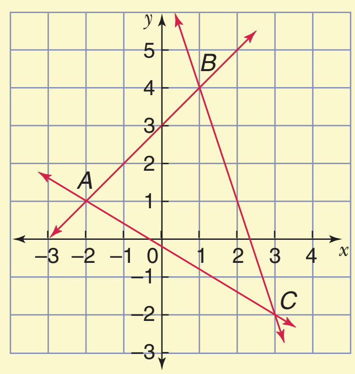
1 Find the gradient of the line that passes through the points:

a (2, 0) and (3, 4) **b** (−1, 3) and (2, 6)
c (3, 1) and (1, 5) **d** (−2, −1) and (0, 9)
e (−2, 1) and (2, 2) **f** (5, 2) and (4, 3)
g (0, 0) and (1, 3) **h** (1, 1) and (4, 4)
i (−1, 8) and (1, −2) **j** (0, 0) and (1, −3)

2 Use your answers for question **1** to find the equations of the lines passing through the pairs of points in question **1**.

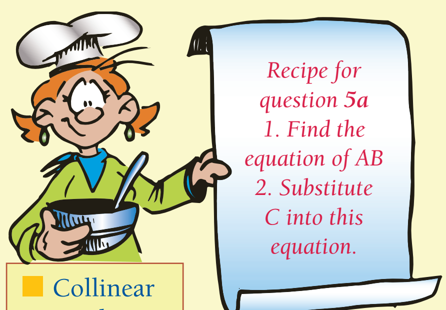
3 **a** Find the equation of the line that passes through the points (−2, −2) and (1, 4).
b The points A(4, 3) and B(5, 0) lie on the line AB. What is the equation of AB?
c What is the equation of the line AB if A is the point (−2, −4) and B is (2, 12)?
d Find the equation of the line that passes through the points (1, 6) and (2, 8). By substitution in this equation, show that (3, 10) also lies on this line.
e What is the equation of the line CD if C is the point (2, 3) and D is the point (4, 5)?

4 A is the point (−2, 1), B is the point (1, 4) and C is the point (3, −2).



a Find the gradient of each side of ΔABC.
b Find the equation of each of the lines AB, BC and AC.
c Find the *y*-intercept of each of the lines AB, BC and AC.
d Find the equation of the line passing through point A and the midpoint of interval BC.
e Find the gradient and *y*-intercept of the line passing through point A and the midpoint of interval BC.

5 **a** Find the equation of the line joining A(1, 2) and B(5, −6). Hence show that C(3, −2) also lies on this line.
b A(−2, 2), B(1, −4) and C(3, −8) are points on the number plane. Show that they are collinear.
c Show that the points (−2, −11), (3, 4) and (4, 7) are collinear.



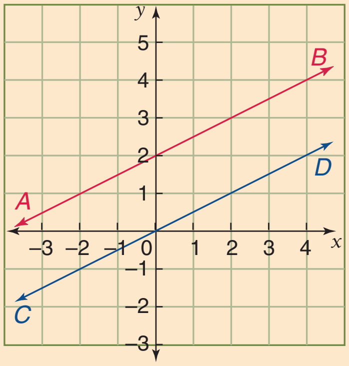
Collinear points lie on the same straight line.

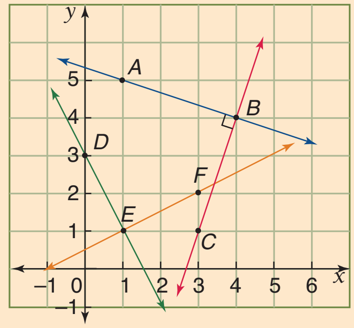
6 Find the equation of the lines in general form that pass through the points:

a (3, −2) and (−4, 1)
b (−2, −4) and (3, 2)
c (1·3, −2·6) and (4, −7·3)
d $(1\frac{1}{2}, -\frac{2}{3})$ and $(-2\frac{1}{3}, \frac{1}{2})$

8:08 | Parallel and Perpendicular Lines

prep quiz 8:08





In the diagram, AB is perpendicular to BC, and DE is perpendicular to EF.

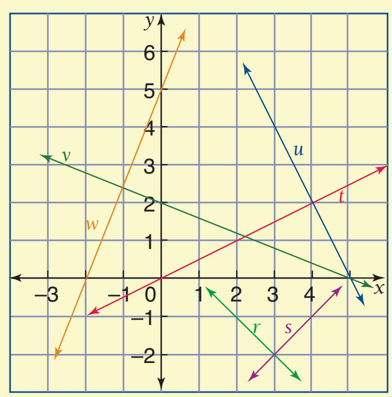
1 What is the gradient of each line?
2 Are the lines parallel?
3 If EF was drawn parallel to AB, what would its gradient be?
4 Is it possible for two lines with with different gradients to be parallel?

5 Find the gradient of AB. Call this m_1.
6 Find the gradient of BC. Call this m_2.
7 Using your answers to **5** and **6**, find the product of the gradients, $m_1 m_2$.
8 Find the gradient of DE. Call this m_3.
9 Find the gradient of EF. Call this m_4.
10 Using your answers to **8** and **9**, find the product of the gradients, $m_3 m_4$.

Questions **1** to **4** in the Prep Quiz remind us that:
- two straight lines are parallel if their gradients are equal
- the gradients of two lines are equal if the lines are parallel.

Questions **5** to **10** of the Prep Quiz suggest that a condition for two lines to be perpendicular might be that the product of their gradients is equal to −1.

We do not intend to prove this here, but let us look at several pairs of lines where the product of the gradient is −1 to see if the angle between the lines is 90°.



A Line r has gradient −1. $\therefore\ m_1 = -1$
Line s has gradient 1. $\therefore\ m_2 = 1$
Note that $m_1 m_2 = -1$.

B Line t has gradient $\frac{1}{2}$. $\therefore\ m_1 = \frac{1}{2}$
Line u has gradient −2. $\therefore\ m_2 = -2$
Note that $m_1 m_2 = -1$.

C Line v has gradient $-\frac{2}{5}$. $\therefore\ m_1 = -\frac{2}{5}$
Line w has gradient $\frac{5}{2}$. $\therefore\ m_2 = \frac{5}{2}$
Note that $m_1 m_2 = -1$.

- By measurement, or use of Pythagoras' theorem, we can show that the angle between each pair of lines is 90°.
- If two lines are perpendicular, then the product of their gradients is −1.
- $m_1 m_2 = -1$ (where neither gradient is zero).

- If the product of the gradients of two lines is -1, then the lines are perpendicular.

Two lines with gradients of m_1 and m_2 are:
- **parallel if $m_1 = m_2$**
- **perpendicular if $m_1 m_2 = -1$**

(or $m_1 = \dfrac{-1}{m_2}$) where neither m_1 nor m_2 can equal zero.

worked examples

1 Which of the lines $y = 4x$, $y = 3x + 2$ and $y = x$ is perpendicular to $x + 4y + 2 = 0$?

2 Find the equation of the line that passes through the point (2, 4) and is perpendicular to $y = 3x - 2$.

3 Find the equation of the line that passes through the point (1, 4) and is parallel to $y = 3x - 2$.

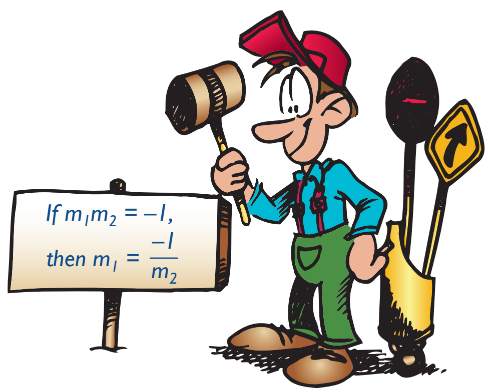
Solutions

1 Step 1: Find the gradient of $x + 4y + 2 = 0$.
Writing this in gradient form gives:
$y = -\frac{1}{4}x - 2$
$\therefore$ The gradient of this line is $-\frac{1}{4}$.

Step 2: Find the gradients of the other lines.
The gradient of $y = 4x$ is 4.
The gradient of $y = 3x + 2$ is 3.
The gradient of $y = x$ is 1.

Step 3: Find which gradient in step **2** will multiply $-\frac{1}{4}$ to give -1.

Conclusion: $-\frac{1}{4} \times 4 = -1$
$\therefore$ $x + 4y + 2 = 0$ is perpendicular to $y = 4x$.



2 Let the equation of the line be $y = mx + b$.
Now the gradient of $y = 3x - 2$ is 3.
$\therefore$ $m = -\frac{1}{3}$ (since $-\frac{1}{3} \times 3 = -1$)
$\therefore$ $y = -\frac{1}{3}x + c$
$4 = -\frac{1}{3}(2) + c$ [since (2, 4) lies on line]
$4 = -\frac{2}{3} + c$
$\therefore$ $c = 4\frac{2}{3}$
$\therefore$ The equation of the line is $y = -\frac{1}{3}x + 4\frac{2}{3}$.

3 Let the equation of the line be $y = mx + c$.
$y = 3x - 2$ has gradient 3
$\therefore$ $m = 3$ (Parallel lines have equal gradients.)
$\therefore$ $y = 3x + c$
$4 = 3(1) + c$, [(1, 4) lies on the line]
$\therefore$ $c = 1$
$\therefore$ The equation of the line is $y = 3x + 1$.

Exercise 8:08

Foundation Worksheet 8:08
Parallel and perpendicular lines
1 Use $y = mx + c$ to find the slope of the lines in columns A and B.
2 Which lines in columns A and B are:
a parallel? **b** perpendicular?

1 Are the following pairs of lines parallel or not?

a $y = 3x + 2$ and $y = 3x - 1$
b $y = 5x - 2$ and $y = 2x - 5$
c $y = x + 7$ and $y = x + 1$
d $y = x - 3$ and $y = 1x + 2$
e $y = 3x + 2$ and $2y = 6x - 3$
f $y = 2x + 1$ and $2x - y + 3 = 0$
g $3x + y - 5 = 0$ and $3x + y + 1 = 0$
h $x + y = 6$ and $x + y = 8$

2 Are the following pairs of lines perpendicular or not?

a $y = \frac{1}{5}x + 3, y = -5x + 1$
b $y = 3x - 2, y = -\frac{1}{3}x + 7$
c $y = 2x - 1, y = -\frac{1}{2}x + 3$
d $y = \frac{2}{3}x + 4, y = -\frac{3}{2}x - 5$
e $y = 4x, y = \frac{1}{4}x - 3$
f $y = \frac{3}{4}x - 1, y = -\frac{4}{3}x$
g $y = 3x - 1, x + 3y + 4 = 0$
h $x + y = 6, x - y - 3 = 0$

3 a Which of the following lines are parallel to $y = 2x + 3$?
$y = 3x + 2$ $\quad 2x - y + 6 = 0$ $\quad 2y = x + 3$ $\quad y = 2x - 3$

b Two of the following lines are parallel. Which are they?
$y = x - 3$ $\quad x + y = 3$ $\quad y = 3x$ $\quad 3y = x$ $\quad y = -x + 8$

c A is the point (1, 3), B is (3, 4), C is (6, 7) and D is (8, 8).
Which of the lines AB, BC, CD and DA are parallel?

4 a Which of the following lines are perpendicular to $y = 2x$?
$y = 3x$ $\quad y = 2x - 3$ $\quad x + 2y = 4$ $\quad y = -0{\cdot}5x + 5$

b Two of the following lines are perpendicular. Which are they?
$y = -1\frac{1}{2}x + 2$ $\quad y = \frac{1}{2}x - 1$ $\quad y = \frac{2}{3}x$

c A is the point (2, −1), B is the point (3, −2) and C is (4, −1). Prove that AB ⊥ BC.

5 a Find the equation of the line that has y-intercept 3 and is parallel to $y = 5x - 1$.
b Line AB is parallel to $y = 3x - 4$. Find the equation of AB if its y-intercept is −1.
c Line EF is parallel to $y = x + 5$. Its y-intercept is 3. What is the equation of EF?
d A line has a y-intercept of 10 and is parallel to the line $x + y = 4$. What is the equation of this line?

6 a Find the equation of the line that has y-intercept 5 and is perpendicular to $y = -\frac{1}{3}x + 1$.
b The line AB is perpendicular to $y = -x + 4$. Its y-intercept is 3. What is the equation of AB?
c Find the equation of CD if CD is perpendicular to the line $y = -\frac{1}{2}x$ and has a y-intercept of 0.
d A line has a y-intercept of 1·5 and is perpendicular to the line $y = -2x + 1$. Find the equation of the line.

7 **a** AB is a line which passes through the point (2, 3). What is the equation of AB if it is parallel to $y = 5x + 2$?

b Find the equation of the line that passes through (1, 0) and is parallel to $y = -3x - 1$.

c A is the point (0, 0) and B is the point (1, 3). Find the equation of the line that has y-intercept 5 and is parallel to AB.

d Find the equation of the line that has y-intercept -3 and is parallel to the x-axis.

e What is the equation of the line that is parallel to the x-axis and passes through the point $(-2, -3)$?

8 **a** If AB passes through the point (2, 3) and is perpendicular to $y = 2x - 7$, find the equation of AB in general form.

b Find the equation of the line that passes through (1, 0) and is perpendicular to $y = -3x - 1$. Write your answer in general form.

c A is the point (0, 0) and B is the point (1, 3). Find the equation of the line that has y-intercept 5 and is perpendicular to AB. Give the answer in general form.

d Find the equation of the line that has y-intercept -3 and is perpendicular to the y-axis.

e What is the equation of a line that is perpendicular to the x-axis and passes through $(3, -2)$?

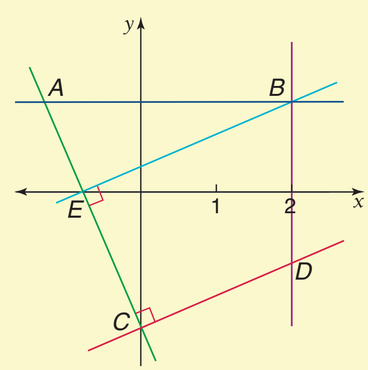
9 **a** Find the equation of the line that is parallel to the line $2x - 3y + 6 = 0$ and passes through the point $(3, -4)$. Give the answer in general form.

b A line is drawn through $(-1, 2)$, perpendicular to the line $4x + 3y - 6 = 0$. Find its equation in general form.

c A line is drawn through the point $(-1, -1)$, parallel to the line $2x - 3y + 9 = 0$. Where will it cross the x-axis?

d A line is drawn parallel to $4x - 3y + 1 = 0$, through the points (1, 3) and $(6, a)$. What is the value of a?

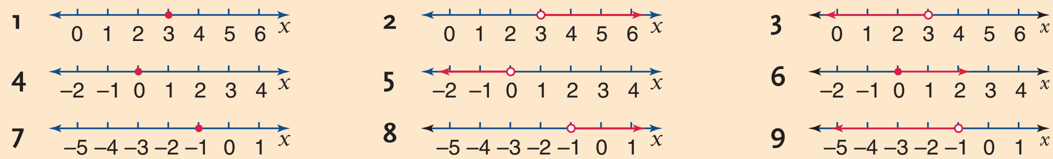
10 In the diagram, the line $5x + 2y + 5 = 0$ cuts the x-axis and y-axis at E and C respectively. BD is the line $x = 2$, AB is parallel to the x-axis and BE and CD are perpendicular to AC. Find the coordinates of the points A, B, C, D and E.



8:09 | Graphing Inequalities on the Number Plane

prep quiz 8:09

For each number line graph, write down the appropriate equation or inequation.



10 On a number line, draw the graph of $x < -2$ where x is a real number.

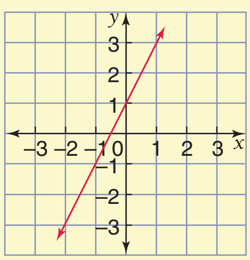
In Prep Quiz 8:09 questions **1**, **2** and **3**, we see that, once $x = 3$ is graphed on the number line, all points satisfying the inequation $x > 3$ lie on one side of the point and all points satisfying the inequation $x < 3$ lie on the other side.

On the number plane, all points satisfying the equation $y = 2x + 1$ lie on one straight line.

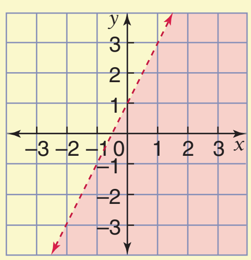
All points satisfying the inequation $y < 2x + 1$ will lie on one side of the line.

All points satisfying the inequation $y > 2x + 1$ will lie on the other side of the line.

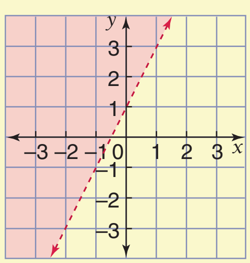
A $y = 2x + 1$



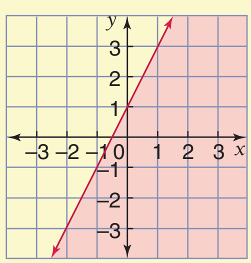
B $y < 2x + 1$



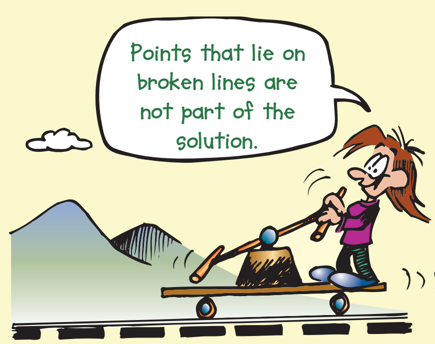
C $y > 2x + 1$



D $y \leqslant 2x + 1$



Note:
- Inequations **B**, **C** and **D** are often called 'half planes'.
- In **D**, the line is part of the solution set. In **B** and **C**, the line acts as a boundary only, and so is shown as a broken line.
- Choose points at random in each of the half planes in **B**, **C** and **D** to confirm that all points in each half plane satisfy the appropriate inequation.



worked examples

1 Graph the region $3x + 2y > 6$ on the number plane.

2 Graph **a** the union and **b** the intersection of the half planes representing the solutions of $x + 2y \geqslant 2$ and $y < 3x - 1$.

continued →→→

Solutions

1 Graph the boundary line $3x + 2y = 6$ as a broken line since it is not part of $3x + 2y > 6$.

$3x + 2y = 6$

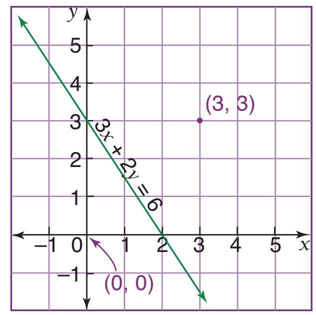
x	0	1	2
y	3	1·5	0

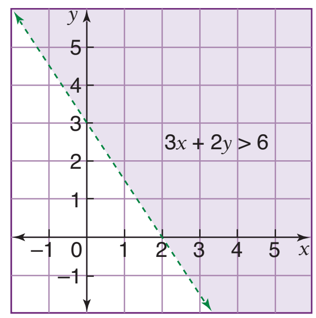
Discover which half plane satisfies the inequation $3x + 2y > 6$ by substituting a point from each side of the boundary into $3x + 2y > 6$.

(0, 0) is obviously to the left of $3x + 2y = 6$.
$\therefore$ substitute (0, 0) into $3x + 2y > 6$.
$3(0) + 2(0) > 6$, which is false.
$\therefore$ (0, 0) does not lie in the half plane.

(3, 3) is obviously to the right of $3x + 2y = 6$.
$\therefore$ substitute (3, 3) into $3x + 2y > 6$.
$3(3) + 2(3) > 6$, which is true.
$\therefore$ (3, 3) lies in the half plane $3x + 2y > 6$.
Shade in the half plane on the (3, 3) side.

Points that lie on broken lines are not part of the solution.



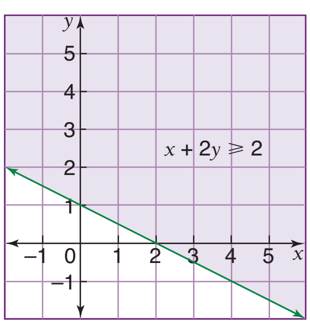


2 Graph the two half planes using the method above.

$x + 2y = 6$

x	0	1	2
y	1	0·5	0

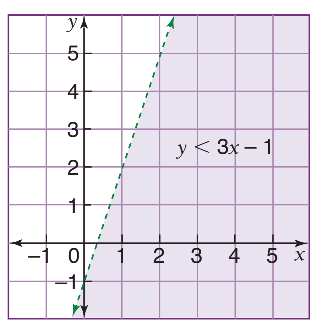
Points above the boundary line satisfy $x + 2y \geqslant 2$.



$y = 3x - 1$

x	0	1	2
y	−1	2	5

Points to the right of the boundary satisfy $y < 3x - 1$.



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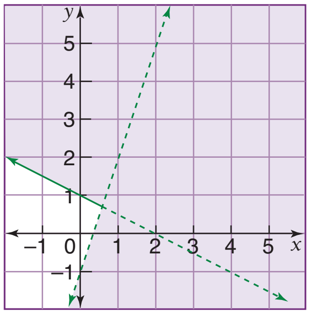
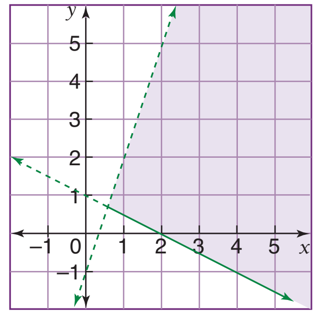
a The union of the two half planes is the region that is part of one or the other or both graphs.



The union is written:
$\{(x, y): x + 2y \geqslant 2 \cup y < 3x - 1\}$

b The intersection is the region that belongs to both half planes. It is the part that the graphs have in common.



The intersection is written:
$\{(x, y): x + 2y \geqslant 2 \cap y < 3x - 1\}$

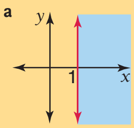
Note:
- Initially draw the boundary lines as broken lines.
- Part of each region has a part of the boundary broken and a part unbroken.

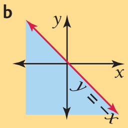
Exercise 8:09

Foundation Worksheet 8:09

Graphing inequalities

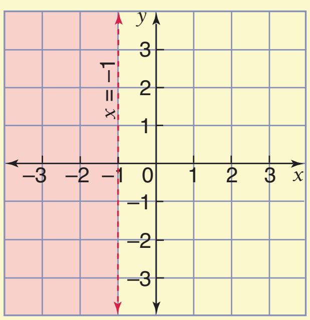
1 Write down the inequality that describes each region.

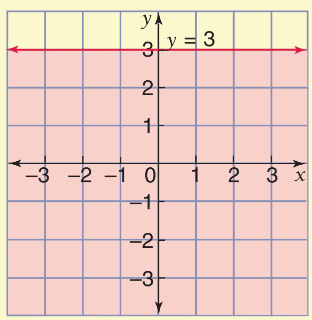
a


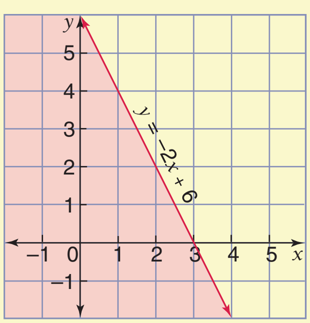
b


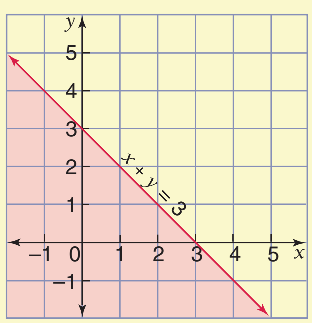
2 Graph the inequalities:
a $y < 2$ **b** $y \geqslant x - 2$

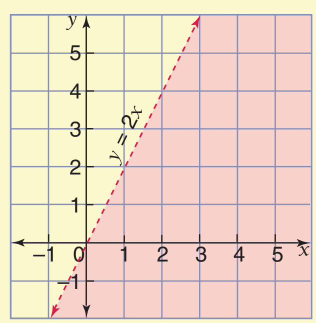
1 By testing a point from each side of the line, write down the inequation for each solution set graphed below.

a


b


c


d


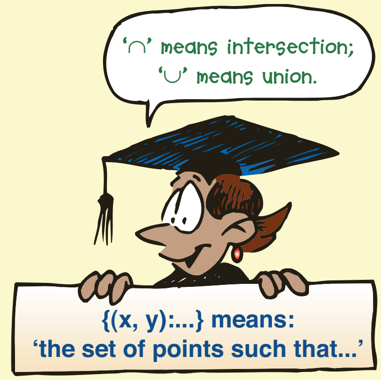
e


2 On separate number plane diagrams, draw the graph of:

a $x \geqslant 1$ **b** $y > x$ **c** $y < 2$ **d** $x + y \leqslant 4$
e $y < -x + 5$ **f** $y \geqslant 2x - 2$ **g** $2x + 3y - 6 \leqslant 0$ **h** $y > 0$

3 Draw a separate diagram to show the part of the number plane where:

a $x \geqslant 0$ and $y \geqslant 0$, ie $\{(x, y): x \geqslant 0 \cap y \geqslant 0\}$
b $x \geqslant 0$ and $y \leqslant 0$, ie $\{(x, y): x \geqslant 0 \cap y \leqslant 0\}$
c $x \leqslant 0$ and/or $y \geqslant 0$, ie $\{(x, y): x \leqslant 0 \cup y \geqslant 0\}$
d $x \leqslant 0$ and/or $y \leqslant 0$, ie $\{(x, y): x \leqslant 0 \cup y \leqslant 0\}$



4 Use question **1** of this exercise to sketch the region described by the intersection of:

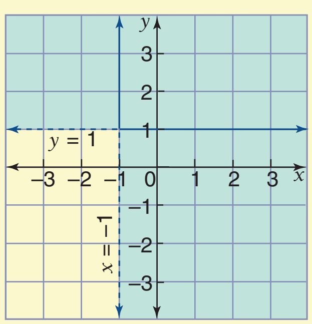
a $x < -1$ and $y \leqslant 3$ **b** $y \leqslant 3$ and $y < 2x$
c $y \leqslant -2x + 6$ and $y \leqslant 3$ **d** $x + y \leqslant 3$ and $y < 2x$
e $y \leqslant -2x + 6$ and $x + y \leqslant 3$ **f** $y \leqslant -2x + 6$ and $y < 2x$

5 Use question **1** of this exercise to sketch the region described by the union of:

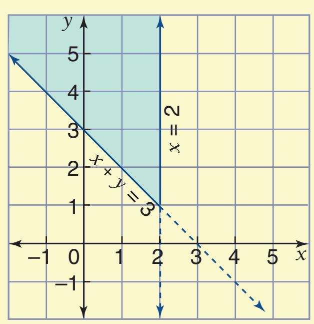
a $x < -1$ and $y \leqslant 3$ **b** $x + y \leqslant 3$ and $y < 2x$
c $x + y \leqslant 3$ and $y \leqslant 3$ **d** $y \leqslant -2x + 6$ and $y < 2x$

6 Describe, in terms of the union or intersection of inequations, the regions drawn below.

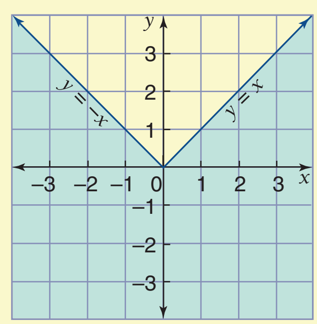
a



b



c



7 Sketch the regions described below.

a the intersection of $y \geqslant 2x$ and $x + y \leqslant 3$
b the union of $y < 1$ and $y < x - 2$
c the intersection of $y < 2x + 1$ and $5x + 4y < 20$
d the union of $y \geqslant 2$ and $y < x$

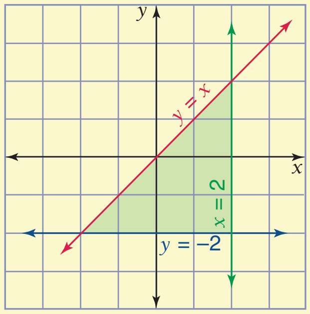
Fill in only that part of the boundary which is part of the answer.

8 Graph the regions which satisfy all of the following inequalities:

a $x \geqslant 1, x \leqslant 4, y \geqslant 0, y \leqslant 4$ **b** $y \geqslant 0, y \leqslant 2x, x + y \leqslant 6$

9 Write down the inequalities that describe each region.

a



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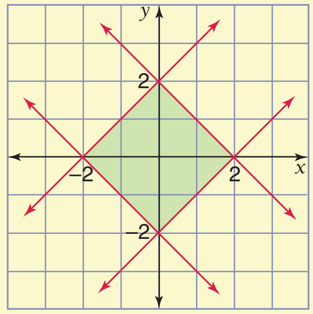

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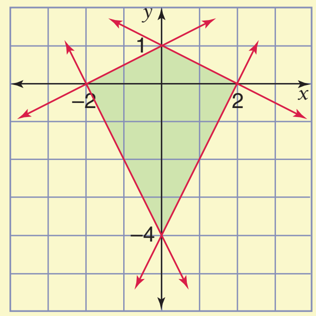


Fun Spot 8:09 | Why did the banana go out with a fig?

Answer each question and put the letter for that question in the box above the correct answer.

For the points A(2, 5), B(−1, 3), C(4, 2), D(0, 6), *find:*

E distance AC

E distance CD

E distance BD

A slope of AB

A slope of AD

T slope of BC

T midpoint of DC

N midpoint of AC

I midpoint of AB

What is the gradient of the following lines:

C $y = 2x - 1$

U $2y = x - 5$

G $2x + y + 1 = 0$

What is the y-intercept of the following lines.

D $y = 2x - 1$

T $2y = x - 5$

L $2x + y - 1 = 0$

Find the equation of the line with:

C gradient of 2 and a y-intercept of -1

O slope of 3 and a y-intercept of 5

T y-intercept of 4 and a slope of -2

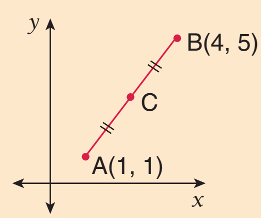
S y-intercept of -2 and a slope of 4

Write in the form $y = mx + b$:

B $2x - y - 5 = 0$ **E** $2x - y + 5 = 0$ **A** $x = \frac{1}{2}y - 2$

In the diagram find:

D slope of AB **U** coordinates of C



$y = 2x - 5$	$y = 2x + 5$	2	$-\frac{1}{2}$	$\frac{1}{2}$	$y = 4x - 2$	$\sqrt{13}$	$(\frac{1}{2}, 4)$	$(2, 4)$	$y = 2x - 1$	$y = 3x + 5$	$(2\frac{1}{2}, 3)$	1	-1	$(3, 3\frac{1}{2})$	$-\frac{1}{5}$	-2	$\sqrt{10}$	$-2\frac{1}{2}$	$\frac{2}{3}$	$\frac{4}{3}$	$y = 2x + 4$	$y = 4 - 2x$	$\sqrt{32}$

Mathematical Terms 8

coordinates
- A pair of numbers that gives the position of a point in a number plane relative to the origin.
- The first of the coordinates is the x-coordinate. It tells how far right (or left) the point is from the origin.
- The second of the coordinates is called the y-coordinate. It tells how far the point is above (or below) the origin.

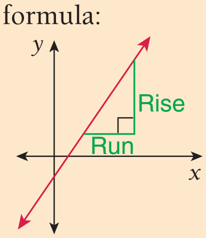
distance formula
- Gives the distance between the points (x_1, y_1) and (x_2, y_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

general form
- A way of writing the equation of a line.
- The equation is written in the form $ax + by + c = 0$. where a, b, c are integers and $a > 0$.

gradient
- The slope of a line or interval. It can be measured using the formula:

$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$



gradient formula
- Gives the gradient of the interval joining (x_1, y_1) to (x_2, y_2).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

gradient–intercept form
- A way of writing the equation of a line. eg $y = 2x - 5$, $y = \frac{1}{2}x + 2$
  When an equation is rearranged and written in the form $y = mx + c$ then m is the gradient and c is the y-intercept.

graph (a line)
- All the points on a line.
- To plot the points that lie on a line.

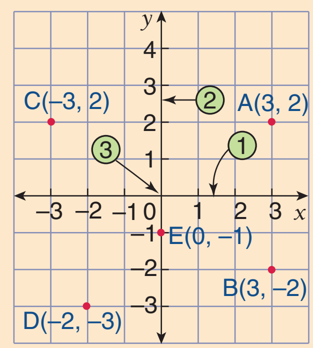
interval
- The part of a line between two points.

midpoint
- Point marking the middle of an interval.

midpoint formula
- Gives the midpoint of the interval joining (x_1, y_1) to (x_2, y_2).

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

number plane
- A rectangular grid that allows the position of points in a plane to be identified by an ordered pair of numbers.



origin
- The point where the x-axis and y-axis intersect, (0, 0). See 3 under **number plane**.

plot
- To mark the position of a point on the number plane.

***x*-axis**
- The horizontal number line in a number plane. See 1 under **number plane**.

***x*-intercept**
- The point where a line crosses the x-axis.

***y*-axis**
- The vertical number line in a number plane. See 2 under **number plane**.

***y*-intercept**
- The point where a line crosses the y-axis.

Mathematical terms 8

diagnostic test 8

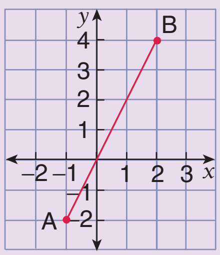
Diagnostic Test 8: | Coordinate Geometry

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

Section

1 Find the length of the interval AB in each of the following. (Leave answers in surd form.) — 8:01

a



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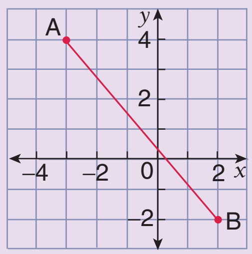

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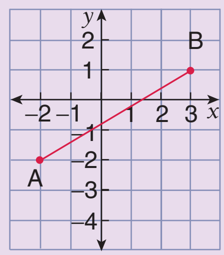


2 Use the distance formula to find the distance between the points: — 8:01

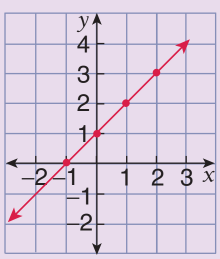
a (1, 2) and (7, 10) **b** (3, 0) and (5, 3) **c** (−3, −2) and (1, −3)

3 Find the midpoint of the interval joining: — 8:02

a (1, 2) and (7, 10) **b** (3, 0) and (5, 3) **c** (−3, −2) and (1, −3)

4 What is the gradient of each line? — 8:03

a



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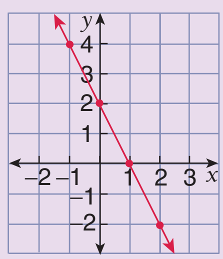
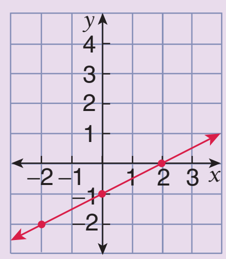


c



5 Find the gradient of the line that passes through: — 8:03

a (1, 3), (2, 7) **b** (−2, 8), (4, 5) **c** (0, 3), (3, 5)

6 — 8:04

a Does the point (3, 2) lie on the line $x + y = 5$?

b Does the point (−1, 3) lie on the line $y = x + 2$?

c Does the point (2, −2) lie on the line $y = x - 4$?

7 Graph the lines: — 8:04

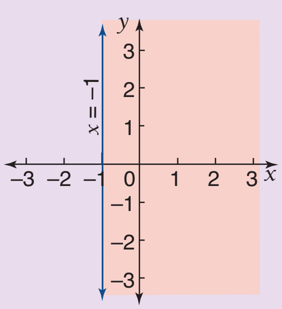
a $y = 2x + 1$ **b** $2x - y = 3$ **c** $3x + 2y = 6$

8 State the x- and y-intercepts of the lines: — 8:04

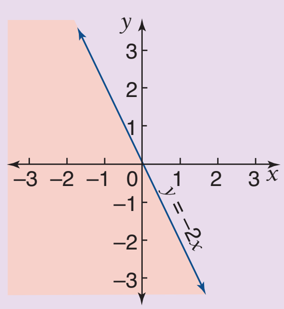
a $2x - y = 3$ **b** $x + 3y = 6$ **c** $x + 2y = 4$

9 Graph the lines: **a** $x = 2$ **b** $y = -1$ **c** $x = -2$	8:04
10 Write down the equation of the line which has: **a** a gradient of 3 and a y-intercept of 2 **b** a gradient of $\frac{1}{2}$ and a y-intercept of -3 **c** a y-intercept of 3 and a gradient of -1	8:05
11 Write each of the answers to question **10** in general form.	8:05
12 What is the gradient and y-intercept of the lines: **a** $y = 2x + 3$? **b** $y = 3 - 2x$? **c** $y = -x + 4$?	8:05
13 Rearrange these equations into gradient–intercept form. **a** $4x - y + 6 = 0$ **b** $2x + 3y - 3 = 0$ **c** $5x + 2y + 1 = 0$	8:05
14 Find the equation of the line that: **a** passes through (1, 4) and has a gradient of 2 **b** has a gradient of -3 and passes through (1, 3) **c** has a gradient of $\frac{1}{2}$ and passes through $(-2, 0)$	8:06
15 Find the equation of the line that: **a** passes through the points (1, 1) and (2, 3) **b** passes through the points $(-1, 2)$ and $(1, -4)$ **c** passes through the origin and (3, 4)	8:07
16 Find the equation of the line that: **a** has a y-intercept of 2 and is parallel to $y = 4x - 1$ **b** passes through (1, 7) and is parallel to $y = -3x + 4$ **c** is perpendicular to $y = \frac{2}{3}x + 1$ and passes through $(-1, 4)$ **d** is perpendicular to $y = 1 - 2x$ and passes through $(-1, 4)$	8:08
17 Write down the inequation for each region.	8:09
18 Graph **a** the union and **b** the intersection of the half planes representing the solutions of $x + 2y \geqslant 2$ and $y < 3x - 1$.	8:09

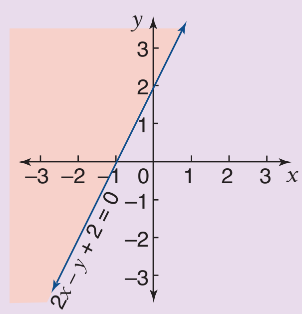
17 a



17 b



17 c

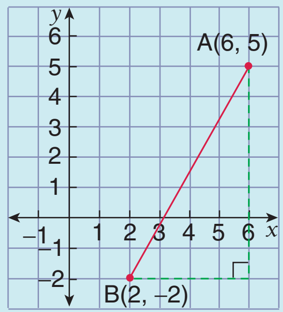


assignment

8A

Chapter 8 | Revision Assignment

1 Find:
- **a** the length AB as a surd
- **b** the slope of AB
- **c** the midpoint of AB.



2 A is the point (2, 5) and B is the point (7, 17).
- **a** What is the length AB (as a surd)?
- **b** What is the slope of AB?
- **c** What is the midpoint of AB?

3 A is the point (6, 5) and B is the point (2, −2).
- **a** What is the equation of the line AB?
- **b** The line AB passes through the point (100, b). What is the value of b?
- **c** AC is perpendicular to AB. Find its equation in general form.

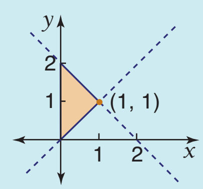
4
- **a** A line has an x-intercept of 3 and a gradient of 1. Find where the line crosses the y-axis and hence write down its equation.
- **b** A line has a slope of $-\frac{1}{2}$ and a y-intercept of 6. What is its equation? What is its x-intercept?
- **c** A line has an x-intercept of 3 and a y-intercept of 6. What is its equation?

5 The points X(2, 2), Y(−2, 4) and Z(−4, 0) form a triangle. Show that the triangle is both isosceles and right-angled.

6 A line is drawn perpendicular to the line $2x - 3y + 4 = 0$ through its y-intercept. What is the equation of the line? Give the answer in general form.

7 A median of a triangle is a line drawn from a vertex to the midpoint of the opposite side. Find the equation of the median through A of the triangle formed by the points A(3, 4), B(−2, −4) and C(−6, 8).

8 What inequalities describe the region shown?



Technology Applications
Drag and Drops

1 x and y intercept and graphs
2 Using $y = mx + c$ to find the gradient
3 General form of a line
4 Parallel and perpendicular lines
5 Inequalities and regions

Technology Applications
Animations

Linear graphs and equations

Chapter Review
Questions

The coordinate system for locating points on the earth is based on circles.