Co-ordinate Geometry Workbook

AS91256

Table of contents

1 Lesson 1: Introduction to Cartesian Coordinates

1.1 Learning Objectives

By the end of this lesson, students will be able to:

- Understand the structure of the Cartesian coordinate system
- Plot points accurately on a coordinate plane
- Identify quadrants and understand positive/negative coordinates
- Recognize points that lie on axes
- Interpret coordinates from a diagram and a table

1.2 The Cartesian Coordinate System

The Cartesian coordinate system uses two perpendicular number lines: - The **x-axis** (horizontal) - The **y-axis** (vertical)

The point where they intersect is called the **origin**, denoted as (0, 0).

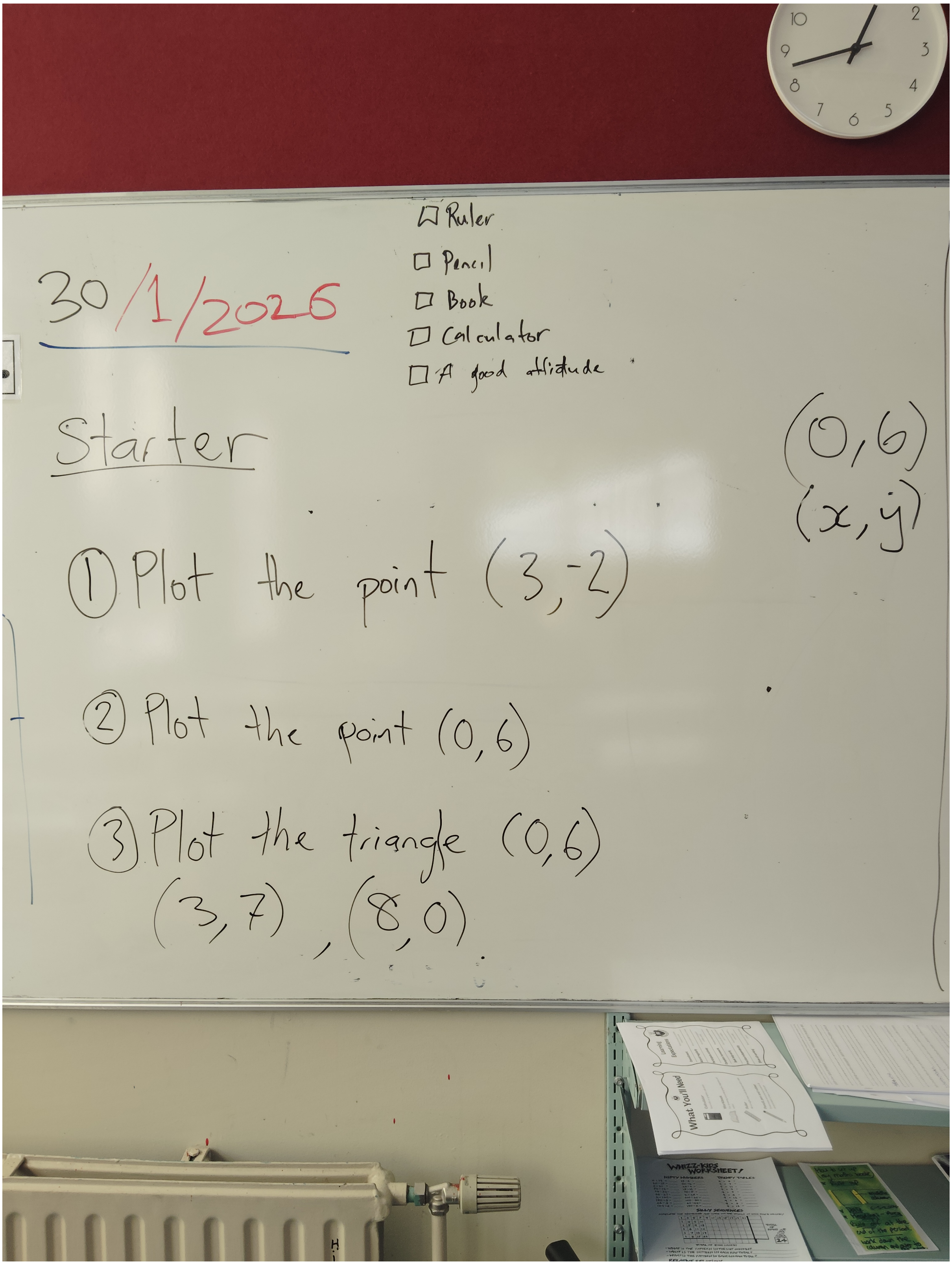

Figure 1: Simple example of points on a Cartesian plane

1.3 Plotting Points

Each point in the plane is represented by an ordered pair (x, y): - The first number (x) tells us how far to move horizontally - The second number (y) tells us how far to move vertically

1.3.1 Example 1: Plotting Points

Plot the following points: - A(2, 3) - B(-1, 4) - C(-3, -2) - D(4, -1)

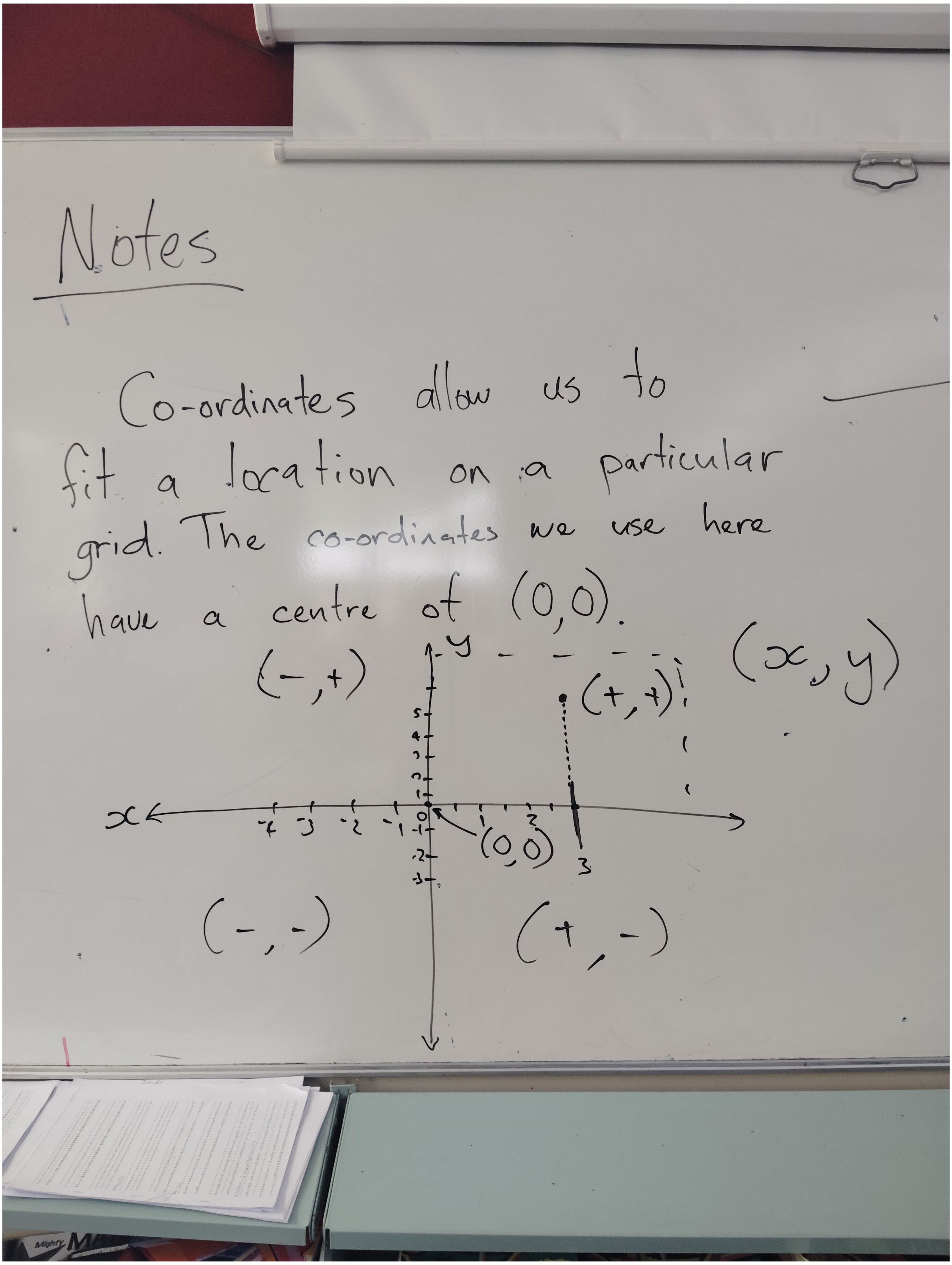

Figure 2: Coordinate basics on a grid

1.3.2 Example 2: Mixed Point Types

Plot each point and label it clearly:

- E(0, 4)
- F(5, 0)
- G(-6, 0)
- H(0, -3)
- I(2, -5)
- J(-4, 3)

Note: Points with $x = 0$ lie on the **y-axis**. Points with $y = 0$ lie on the **x-axis**.

1.4 Quadrants

The coordinate plane is divided into four quadrants: - **Quadrant I:** x > 0, y > 0 (top right) - **Quadrant II:** x < 0, y > 0 (top left) - **Quadrant III:** x < 0, y < 0 (bottom left) - **Quadrant IV:** x > 0, y < 0 (bottom right)

1.4.1 Example 2: Identifying Quadrants

In which quadrant does each point lie? - P(3, 5) -> Quadrant I (both positive) - Q(-2, 7) -> Quadrant II (x negative, y positive) - R(-4, -3) -> Quadrant III (both negative) - S(6, -2) -> Quadrant IV (x positive, y negative)

Points on the axes are not in any quadrant.

1.4.2 Example 3: Quadrant and Axis Check

For each point, state the quadrant or axis:

- U(0, 7)
- V(-8, 0)
- W(-2, 5)
- X(6, -1)
- Y(3, 0)
- Z(0, -6)

1.5 Practice Exercises

1. Plot the following points on a coordinate plane:

   - P(3, -2)
   - Q(-4, 1)
   - R(0, 5)
   - S(-2, -3)
   - T(4, 0)
   - U(-6, 2)
   - V(5, -4)

2. Identify the quadrant or axis for each point:

   - A(5, 7)
   - B(-3, 2)
   - C(-1, -6)

- D(4, -3)
- E(0, -8)
- F(9, 0)

3. Complete the table:

Point	x value	y value	Quadrant/Axis
G(2, 6)			
H(-5, 4)			
I(0, -3)			
J(-7, 0)			
K(-2, -5)			

4. Name three points that lie:
   - In Quadrant II
   - On the x-axis
   - On the y-axis

5. Create your own set of four points, one in each quadrant, and swap with a classmate to plot and check.

1.6 Exit Ticket

1. Plot and label: A(4, -2), B(-3, -1), C(0, 6).
2. State the quadrant or axis for each point above.
3. Write one coordinate that lies in Quadrant III and one that lies on the y-axis.

2 Lesson 2: Map It - Scale, Coordinates, and Bearings

2.1 Achievement Objectives

GM4-7: Communicate and interpret locations and directions, using compass directions, distances, and grid references.

2.2 Learning Objectives

By the end of this lesson, students will be able to:

- Understand and apply map scales to convert between map distances and real distances
- Use Cartesian coordinates to locate specific points on a map
- Apply polar coordinates using bearings and distances to navigate
- Interpret latitude and longitude coordinates

2.3 Description of Mathematics

This lesson investigates three mathematical concepts in the context of maps:

2.3.1 1. Scale

A map is a reduced version of a real landscape. Scale compares the size of lengths on a map to those in the real landscape.

Key Concept: If a map has a scale of 1:50,000, this means that 1 cm on the map represents 50,000 cm (or 500 m) in real life.

2.3.1.1 Example 1: Working with Scale

A map has a scale of 1:25,000.

a) If two towns are 3 cm apart on the map, how far apart are they in real life?

Solution: - Map distance = 3 cm - Real distance = 3 x 25,000 = 75,000 cm = 750 m = 0.75 km

b) If two landmarks are 2 km apart in real life, how far apart would they be on the map?

Solution: - Real distance = 2 km = 200,000 cm - Map distance = 200,000 / 25,000 = 8 cm

2.3.2 2. Cartesian (Rectangular) Coordinates

Cartesian coordinates refer to the location of a specific point using a combination of horizontal distance and vertical distance.

Example: The star below is located at (7, 4).

[Imagine a coordinate grid with x-axis from 0 to 9 and y-axis from 0 to 6. A star is plotted at the point where x=7 and y=4.]

2.3.2.1 With Real Maps: Latitude and Longitude

- **Longitude** is the number of degrees 'around the world' from the Prime Meridian
- **Latitude** is measured in degrees north or south from the equator

2.3.2.2 Example 2: Using Grid References

On a map with a grid system: - Location A is at grid reference (345, 678) - Location B is at grid reference (348, 682)

If each grid unit represents 100 meters, how far apart are A and B?

Solution: Using the distance formula:

$$d = \sqrt{(348 - 345)^2 + (682 - 678)^2}$$
$$d = \sqrt{3^2 + 4^2}$$
$$d = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ grid units}$$

Real distance = 5 x 100 = 500 meters

2.3.3 3. Polar Coordinates (Bearings)

Polar coordinates specify a location using the angle or bearing from a given point and the distance from that point.

Bearings are measured clockwise from North and are written as three-digit numbers: - North = 000 degrees (or 360 degrees) - East = 090 degrees - South = 180 degrees - West = 270 degrees

2.3.3.1 Example 3: Using Bearings

From point A, a landmark is located on a bearing of 045 degrees at a distance of 3 km.

a) In which general direction is the landmark?

Solution: 045 degrees is halfway between North (000) and East (090), so it's Northeast.

b) If you walk on a bearing of 045 degrees for 3 km, how far East and North have you traveled?

Solution: Using trigonometry: - East component = 3 x sin(45 degrees) = 3 x 0.707 approx 2.12 km - North component = 3 x cos(45 degrees) = 3 x 0.707 approx 2.12 km

2.4 Practical Activities

2.4.1 Activity 1: Local Area Mapping

Using a map of your local area:

1. Identify your school on the map
2. Measure the distance from your school to a local landmark
3. Use the map scale to calculate the real distance
4. Record the grid reference or coordinates of both locations

2.4.2 Activity 2: Treasure Hunt Coordinates

Create a treasure map with: - A coordinate grid system - At least 5 landmarks marked with coordinates - Write instructions using coordinates to move from one landmark to another

2.4.3 Activity 3: Navigation Challenge

Using a compass or compass app:

1. Stand at a starting point
2. Follow these instructions:

   - Walk 20 paces on bearing 090 degrees
   - Walk 15 paces on bearing 180 degrees
   - Walk 20 paces on bearing 270 degrees
   - Walk 15 paces on bearing 000 degrees

3. Where should you end up? Why?

2.5 Te Reo Maori Vocabulary

English	Te Reo Maori
Latitude	Ahopae
Longitude	Ahopou
Map	Mahere
Scale drawing	Tuhinga awhata
Scale	Awhata
Cartesian coordinates	Taunga tukutuku
North	Raki
South	Runga
East	Rawhiti
West	Rato

2.6 Cultural Connections

2.6.1 Traditional Polynesian Navigation

Tipuna (Ancestors) had clever ways to navigate in the absence of sophisticated GPS systems. Polynesian navigators used:

- Ocean currents
- The sun and stars
- The migration of whales and birds
- Wave patterns and swells

These traditional methods allowed Pacific peoples to travel vast distances across the ocean with remarkable accuracy.

2.7 Practice Problems

2.7.1 Problem Set 1: Scale

1. A map has a scale of 1:50,000. What real distance does 4.5 cm on the map represent?
2. Two cities are 150 km apart. How far apart would they be on a map with a scale of 1:1,000,000?
3. On a treasure map, the scale is 1 cm = 5 m. If the treasure is 7.5 cm from the starting point on the map, how far is it in real life?

2.7.2 Problem Set 2: Coordinates

4. Plot the following locations on a coordinate grid:
   - School: (2, 5)
   - Library: (7, 5)
   - Park: (2, 1)
   - Shop: (7, 1)
5. Using your plotted points from question 4, calculate the distance from:
   a) School to Library
   b) School to Park
   c) Library to Shop

2.7.3 Problem Set 3: Bearings

6. From your starting position:
   - Point A is on a bearing of 045 degrees at 5 km
   - Point B is on a bearing of 135 degrees at 5 km
   - Point C is on a bearing of 225 degrees at 5 km
   - Point D is on a bearing of 315 degrees at 5 km

   Sketch these points. What shape do they form?
7. You walk 3 km on a bearing of 060 degrees. How far North and how far East have you traveled?
8. To get from home to school, you walk 400 m East then 300 m North.
   a) What is the direct distance from home to school?
   b) What bearing would take you directly from home to school?

2.8 Extension Activities

2.8.1 For Advanced Learners

1. **Magic Carpet Journey:** Plan a journey from your location to a Pacific Island nation. Provide:

   - Starting and ending coordinates (latitude/longitude)
   - Bearing from start to finish
   - Distance of the journey

2. **Multiple Navigation:** Starting from point A, follow these instructions:

   - 5 km on bearing 030 degrees
   - 3 km on bearing 120 degrees
   - 4 km on bearing 210 degrees

   Calculate your final position relative to point A using coordinates.

3. **Map Making Project:** Create a detailed scale map of your school grounds including:

   - A clear scale
   - Grid reference system
   - All major buildings and landmarks
   - Compass rose showing North

2.9 Digital Tools

Consider using: - **Google Maps** for exploring scale and real-world coordinates - **Compass apps** on mobile phones for practicing bearings - **GPS tracking apps** to see how movement in real life corresponds to movement on a map - **Online mapping tools** to create custom maps with coordinate grids

2.10 Assessment Ideas

Students could demonstrate their understanding by:

1. Creating a detailed treasure map with scale, coordinates, and bearing-based instructions
2. Planning and executing a navigation course using bearings and distances
3. Converting between map distances and real distances using various scales
4. Using latitude and longitude to identify locations around New Zealand and the Pacific

2.11 Home Learning Connection

Letter to Parents and Whanau:

This week in maths, we are learning about maps. We're exploring how maps use mathematics including: - Scale (to show big places on small maps) - Coordinates (to locate specific places) - Bearings (to give directions)

You can help by: - Looking at maps together at home (road maps, street directories, online maps) - Asking your child to explain how the scale works - Having them show you how to describe where things are using coordinates - Discussing how GPS and digital maps work

If you have any paper maps at home, ask your child to show you the important parts and demonstrate how to describe locations and give directions using the map.

2.12 Required Materials

- Compasses (for finding north, not for drawing circles) or phones with compass app
- Protractors
- Rulers
- Pencils and paper
- Maps of local area or access to online maps
- Calculator (for trigonometry in bearing calculations)

2.13 Learning Outcomes

By completing this lesson, students will have developed skills in: - **Mathematical thinking:** Understanding scale relationships and coordinate systems - **Problem solving:** Converting between map and real distances - **Communication:** Using mathematical language to describe locations and directions - **Cultural awareness:** Appreciating traditional navigation methods of Pacific peoples - **Practical application:** Using mathematical concepts in real-world contexts

2.14 Reflection Questions

1. Why is it important to understand map scales?
2. How do coordinates help us communicate locations precisely?
3. What are the advantages of using bearings instead of just saying "go northeast"?
4. How do digital maps like Google Maps use these mathematical concepts?
5. What can we learn from traditional Polynesian navigation methods?

3 Lesson 3: The Distance Formula

3.1 Learning Objectives

By the end of this lesson, students will be able to:

- Understand the connection between the Pythagorean theorem and the distance formula
- Apply the distance formula to calculate the distance between any two points
- Simplify radical expressions in distance calculations
- Solve real-world problems involving distance on coordinate planes
- Find unknown coordinates when given distance information

3.2 Introduction

One of the most important applications of coordinate geometry is finding the distance between two points. Whether you're calculating how far apart two cities are on a map, finding the straight-line distance a delivery drone needs to travel, or determining if two locations are within wireless signal range, the distance formula is essential.

3.3 The Pythagorean Theorem Review

Before we explore the distance formula, let's recall the Pythagorean theorem:

For a right triangle with legs of length a and b, and hypotenuse of length c:

$$a^2 + b^2 = c^2$$

Example: If a right triangle has legs of 3 and 4 units, find the hypotenuse.

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25$$
$$c = \sqrt{25} = 5$$

3.4 Deriving the Distance Formula

When we plot two points on a coordinate plane, we can create a right triangle:

Consider points A(x_1, y_1) and B(x_2, y_2): - The horizontal leg has length $|x_2 - x_1|$ - The vertical leg has length $|y_2 - y_1|$ - The distance between A and B is the hypotenuse

Using the Pythagorean theorem:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Taking the square root of both sides:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is the **distance formula**.

3.5 Using the Distance Formula

3.5.1 Example 1: Basic Distance Calculation

Find the distance between points A(1, 2) and B(4, 6).

Solution: Identify the coordinates: $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (4, 6)$

$$d = \sqrt{(4-1)^2 + (6-2)^2}$$
$$d = \sqrt{3^2 + 4^2}$$
$$d = \sqrt{9 + 16}$$
$$d = \sqrt{25} = 5$$

The distance is exactly 5 units.

3.5.2 Example 2: Distance with Negative Coordinates

Find the distance between C(-2, 3) and D(4, -1).

Solution:

$$d = \sqrt{(4-(-2))^2 + (-1-3)^2}$$
$$d = \sqrt{(4+2)^2 + (-4)^2}$$
$$d = \sqrt{6^2 + 16}$$
$$d = \sqrt{36 + 16}$$
$$d = \sqrt{52}$$

To simplify: $\sqrt{52} = \sqrt{4 \times 13} = 2\sqrt{13}$

Approximately: $d \approx 7.21$ units

3.5.3 Example 3: Points on the Same Horizontal or Vertical Line

Find the distance between E(2, 5) and F(2, 9).

Solution:

$$d = \sqrt{(2-2)^2 + (9-5)^2}$$
$$d = \sqrt{0 + 4^2}$$
$$d = \sqrt{16} = 4$$

Note: When points share the same x-coordinate (vertical line) or y-coordinate (horizontal line), the distance is simply the difference in the other coordinate.

3.5.4 Example 4: Distance from the Origin

Find the distance from the origin O(0, 0) to point P(6, 8).

Solution:

$$d = \sqrt{(6-0)^2 + (8-0)^2}$$
$$d = \sqrt{36 + 64}$$
$$d = \sqrt{100} = 10$$

The distance from the origin to any point (x, y) is $\sqrt{x^2 + y^2}$.

3.6 Simplifying Radicals

When the distance is not a perfect square, we should simplify:

3.6.1 Example 5: Simplifying Radical Answers

Find the distance between G(-1, 2) and H(5, 4), leaving your answer in simplest radical form.

Solution:

$$d = \sqrt{(5-(-1))^2 + (4-2)^2}$$
$$d = \sqrt{6^2 + 2^2}$$
$$d = \sqrt{36 + 4}$$
$$d = \sqrt{40}$$

Factor out perfect squares: $\sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}$

Decimal approximation: $d \approx 6.32$ units

3.7 Real-World Applications

3.7.1 Example 6: Map Distances

A park ranger at point P(2, 5) needs to reach a hiker in distress at point H(8, 13). Each coordinate unit represents 100 meters. How far must the ranger travel in a straight line?

Solution:

$$d = \sqrt{(8-2)^2 + (13-5)^2}$$
$$d = \sqrt{6^2 + 8^2}$$
$$d = \sqrt{36 + 64}$$
$$d = \sqrt{100} = 10 \text{ units}$$

Real distance = 10 x 100 = **1000 meters = 1 km**

3.7.2 Example 7: Cell Tower Coverage

A cell tower is located at coordinates (3, 4) and has a signal range of 10 km. A house is located at (11, 10). Is the house within range? (Units are in kilometers)

Solution: Distance from tower to house:

$$d = \sqrt{(11-3)^2 + (10-4)^2}$$
$$d = \sqrt{8^2 + 6^2}$$
$$d = \sqrt{64 + 36}$$
$$d = \sqrt{100} = 10 \text{ km}$$

The house is **exactly** at the edge of the signal range (10 km).

3.7.3 Example 8: Comparing Distances

A delivery company needs to determine which warehouse is closer to a customer. - Warehouse A is at (2, 3) - Warehouse B is at (-1, 7) - Customer is at (5, 6)

Which warehouse should make the delivery?

Solution:

Distance from A to Customer:

$$d_A = \sqrt{(5-2)^2 + (6-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2} \approx 4.24$$

Distance from B to Customer:

$$d_B = \sqrt{(5-(-1))^2 + (6-7)^2} = \sqrt{36+1} = \sqrt{37} \approx 6.08$$

Warehouse A is closer (4.24 units vs 6.08 units).

3.8 Working Backwards: Finding Unknown Coordinates

3.8.1 Example 9: Finding an Unknown x-coordinate

Point A is at (3, 4) and point B is at (x, 10). The distance between them is 10 units. Find all possible values of x.

Solution:

$$10 = \sqrt{(x-3)^2 + (10-4)^2}$$
$$10 = \sqrt{(x-3)^2 + 36}$$

Square both sides:

$$100 = (x-3)^2 + 36$$
$$(x-3)^2 = 64$$
$$x - 3 = \pm 8$$

Therefore: $x = 3 + 8 = 11$ or $x = 3 - 8 = -5$

Answer: B could be at (11, 10) or (-5, 10).

3.8.2 Example 10: Finding an Unknown y-coordinate

Points P(2, y) and Q(6, 3) are 5 units apart. Find y.

Solution:

$$5 = \sqrt{(6-2)^2 + (3-y)^2}$$
$$5 = \sqrt{16 + (3-y)^2}$$
$$25 = 16 + (3-y)^2$$
$$(3-y)^2 = 9$$
$$3 - y = \pm 3$$

Therefore: $y = 3 - 3 = 0$ or $y = 3 + 3 = 6$

Answer: P could be at (2, 0) or (2, 6).

3.9 Connection to Lesson 2: Maps and Scale

Remember from Lesson 2 that maps use scales. When using the distance formula with map coordinates, don't forget to convert to real distances:

Example 11: On a map with scale 1:50,000, two landmarks are at grid coordinates (120, 80) and (180, 140). Each grid unit represents 1 meter. What is the real distance between the landmarks?

Solution: Map distance (in grid units):

$$d = \sqrt{(180-120)^2 + (140-80)^2}$$
$$d = \sqrt{60^2 + 60^2}$$
$$d = \sqrt{3600 + 3600}$$
$$d = \sqrt{7200} = 60\sqrt{2} \approx 84.85 \text{ grid units}$$

Since each unit = 1 meter, map distance = 84.85 m

Real distance = 84.85 m x scale = 84.85 x 1 = 84.85 m (the scale tells us about map:real ratio, but here we're working with grid units that are already scaled)

3.10 Practice Exercises

3.10.1 Set 1: Basic Calculations

1. Find the distance between:
   a) A(0, 0) and B(5, 12)
   b) C(2, 3) and D(8, 11)
   c) E(-4, 1) and F(2, -7)
   d) G(-3, -2) and H(-7, -5)
2. Simplify your answers in simplest radical form:
   a) Distance between (0, 0) and (3, 7)
   b) Distance between (1, 2) and (6, 5)
   c) Distance between (-2, 3) and (4, 8)

3.10.2 Set 2: Special Cases

3. Find the distance between:

   a) Two points on a horizontal line: (2, 5) and (9, 5)
   b) Two points on a vertical line: (3, 1) and (3, 8)
   c) Two points where one is the origin: (0, 0) and (-6, -8)

3.10.3 Set 3: Applications

4. A drone flies from point A(10, 20) to point B(40, 50). Each unit represents 10 meters.

   a) How far does the drone travel?
   b) Express your answer in kilometers.

5. Which point is closer to (5, 5)?

   - Point A at (1, 2)
   - Point B at (9, 9)

6. A circular running track has its center at (0, 0) and a radius of 50 meters. A water station is at point (30, 40). Is the water station inside, on, or outside the track?

3.10.4 Set 4: Unknown Coordinates

7. Point M is at (x, 6) and N is at (4, 2). If MN = 5, find all possible values of x.

8. Points P(3, y) and Q(-1, 5) are $2\sqrt{5}$ units apart. Find y.

9. A point on the x-axis is 13 units from point (5, 12). Find the coordinates of the point.

3.10.5 Set 5: Challenge Problems

10. Three vertices of a square are at A(1, 1), B(4, 1), and C(4, 4). Find the coordinates of the fourth vertex D and verify your answer by showing all four sides have equal length.

11. Prove that the triangle with vertices A(0, 0), B(4, 0), and C(2, $2\sqrt{3}$) is equilateral by showing all three sides have the same length.

12. A lighthouse at position (0, 0) has a light that can be seen up to 20 km away. A ship is traveling in a straight line from point (-15, 20) to point (25, 15). Will the ship be visible from the lighthouse at any point during its journey? (Hint: Find the closest point on the path to the lighthouse)

3.11 Additional Resources

3.11.1 Online Tools

- Use graphing calculators or Desmos to visualize distances
- GeoGebra has excellent tools for exploring coordinate geometry
- Google Earth for real-world distance applications

3.11.2 Video Resources

- Search for "distance formula derivation" to see animated proofs
- "Pythagorean theorem applications" shows real-world uses

3.11.3 Extension Topics

- **3D Distance Formula:** For three dimensions: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Taxicab Distance:** In cities with grid layouts, you can't travel "as the crow flies"
- **GPS and Great Circle Distance:** How GPS calculates distance on the curved Earth

3.12 Homework

Complete Practice Exercises Sets 1 and 2 completely, and at least 3 problems from Set 3.

4 Lesson 4: The Midpoint Formula

4.1 Learning Objectives

By the end of this lesson, students will be able to:

- Apply the midpoint formula to find the center point of a line segment
- Understand midpoint as the average of coordinates
- Use midpoints to solve geometric problems
- Find unknown endpoints when given a midpoint
- Apply midpoint concepts to real-world situations

4.2 Introduction

The midpoint of a line segment is the point that divides it into two equal parts - exactly halfway between the two endpoints. This concept is crucial in many applications: finding the center of a bridge, determining meeting points, locating optimal facility placements, and solving geometric problems.

4.3 The Midpoint Formula

For a line segment with endpoints (x_1, y_1) and (x_2, y_2), the midpoint M is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

4.3.1 Understanding the Formula

The midpoint formula simply takes the **average** of the x-coordinates and the **average** of the y-coordinates: - Midpoint x-coordinate: $\frac{x_1+x_2}{2}$ - Midpoint y-coordinate: $\frac{y_1+y_2}{2}$

Think of it as finding the "balance point" between two locations.

4.4 Basic Examples

4.4.1 Example 1: Simple Midpoint Calculation

Find the midpoint of the line segment joining A(2, 3) and B(8, 7).

Solution:

$$M = \left(\frac{2+8}{2}, \frac{3+7}{2}\right)$$

$$M = \left(\frac{10}{2}, \frac{10}{2}\right)$$

$$M = (5, 5)$$

The midpoint is at (5, 5).

4.4.2 Example 2: Midpoint with Negative Coordinates

Find the midpoint of the line segment joining P(-4, 6) and Q(2, -2).

Solution:

$$M = \left(\frac{-4+2}{2}, \frac{6+(-2)}{2}\right)$$
$$M = \left(\frac{-2}{2}, \frac{4}{2}\right)$$
$$M = (-1, 2)$$

4.4.3 Example 3: Midpoint with the Origin

Find the midpoint of the line segment joining O(0, 0) and R(10, 6).

Solution:

$$M = \left(\frac{0+10}{2}, \frac{0+6}{2}\right)$$
$$M = (5, 3)$$

Note: The midpoint between the origin and any point (a, b) is $\left(\frac{a}{2}, \frac{b}{2}\right)$.

4.4.4 Example 4: Midpoint with Fractional Coordinates

Find the midpoint of S(1.5, 2.5) and T(4.5, 6.5).

Solution:

$$M = \left(\frac{1.5+4.5}{2}, \frac{2.5+6.5}{2}\right)$$
$$M = \left(\frac{6}{2}, \frac{9}{2}\right)$$
$$M = (3, 4.5)$$

4.5 Real-World Applications

4.5.1 Example 5: Meeting Point

Two friends live at coordinates A(2, 8) and B(10, 4). They want to meet exactly halfway between their houses. Where should they meet?

Solution:

$$M = \left(\frac{2+10}{2}, \frac{8+4}{2}\right) = (6, 6)$$

They should meet at location (6, 6).

4.5.2 Example 6: Facility Placement

A straight road connects town A at coordinates (10, 20) to town B at coordinates (50, 60). The council wants to place a rest stop at the midpoint. What are the coordinates?

Solution:

$$M = \left(\frac{10+50}{2}, \frac{20+60}{2}\right) = (30, 40)$$

The rest stop should be built at (30, 40).

4.5.3 Example 7: Bridge Center Point

A bridge spans across a river from point P(-12, 5) to point Q(8, 15). Engineers need to place a support at the center of the bridge. What are its coordinates?

Solution:

$$M = \left(\frac{-12+8}{2}, \frac{5+15}{2}\right)$$

$$M = \left(\frac{-4}{2}, \frac{20}{2}\right)$$

$$M = (-2, 10)$$

The support should be at (-2, 10).

4.5.4 Example 8: Emergency Services

An ambulance station needs to be equidistant from two hospitals: - Hospital A: (5, 12) - Hospital B: (15, 8)

Where should the station be located to be exactly halfway between them?

Solution:

$$M = \left(\frac{5+15}{2}, \frac{12+8}{2}\right) = (10, 10)$$

The station should be at (10, 10).

4.6 Working Backwards: Finding Unknown Endpoints

Sometimes we know the midpoint and one endpoint, and need to find the other endpoint.

4.6.1 Example 9: Finding the Other Endpoint

The midpoint of segment AB is M(5, 7). If point A is at (2, 3), find the coordinates of point B.

Solution:

Let B = (x, y)

Using the midpoint formula:

$$\frac{2+x}{2} = 5 \text{ and } \frac{3+y}{2} = 7$$

Solve for x:

$$2 + x = 10$$
$$x = 8$$

Solve for y:

$$3 + y = 14$$
$$y = 11$$

Therefore, **B = (8, 11)**

Check: Midpoint of (2, 3) and (8, 11) = $\left(\frac{2+8}{2}, \frac{3+11}{2}\right) = (5, 7)$

4.6.2 Example 10: Finding the Starting Point

A delivery driver ends their route at point E(20, 30). The midpoint of their route was at M(12, 18). Where did they start?

Solution:

Let the starting point be S(x, y)

$$\frac{x+20}{2} = 12 \text{ and } \frac{y+30}{2} = 18$$

Solve for x:

$$x + 20 = 24$$
$$x = 4$$

Solve for y:

$$y + 30 = 36$$
$$y = 6$$

The driver started at **(4, 6)**.

4.6.3 Example 11: Symmetric Point

Point P(3, 5) is given. Find the point Q that is symmetric to P with respect to the origin (0, 0).

Solution:

If the origin is the midpoint between P and Q, then:

Let Q = (x, y)

$$\frac{3+x}{2} = 0 \text{ and } \frac{5+y}{2} = 0$$

$$x = -3 \text{ and } y = -5$$

Point Q is at **(-3, -5)**.

This is reflection through the origin.

4.7 Multiple Midpoints

4.7.1 Example 12: Trisecting a Line Segment

Points A(0, 0) and B(9, 6) are the endpoints of a segment. Find two points that divide the segment into three equal parts.

Solution:

First, find the midpoint M of AB:

$$M = \left(\frac{0+9}{2}, \frac{0+6}{2}\right) = (4.5, 3)$$

Now find the midpoint of AM:

$$P = \left(\frac{0+4.5}{2}, \frac{0+3}{2}\right) = (2.25, 1.5)$$

And the midpoint of MB:

$$Q = \left(\frac{4.5+9}{2}, \frac{3+6}{2}\right) = (6.75, 4.5)$$

However, this doesn't trisect. To trisect, we need: - First point at 1/3 of the way: $(x_1 + \frac{1}{3}(x_2 - x_1), y_1 + \frac{1}{3}(y_2 - y_1)) = (3, 2)$ - Second point at 2/3 of the way: $(x_1 + \frac{2}{3}(x_2 - x_1), y_1 + \frac{2}{3}(y_2 - y_1)) = (6, 4)$

The two trisection points are **(3, 2) and (6, 4)**.

4.7.2 Example 13: Median of a Triangle

A triangle has vertices at A(0, 0), B(6, 0), and C(3, 6). Find the median from vertex A to side BC.

Solution:

Step 1: Find the midpoint M of BC:

$$M = \left(\frac{6+3}{2}, \frac{0+6}{2}\right) = (4.5, 3)$$

Step 2: The median is the line segment from A(0, 0) to M(4.5, 3).

The median from A has endpoint at **(4.5, 3)**.

4.8 Geometric Properties Using Midpoints

4.8.1 Example 14: Proving a Parallelogram

Show that quadrilateral ABCD with vertices A(1, 2), B(5, 3), C(6, 6), and D(2, 5) is a parallelogram by showing that the diagonals bisect each other.

Solution:

Find the midpoint of diagonal AC:

$$M_{AC} = \left(\frac{1+6}{2}, \frac{2+6}{2}\right) = (3.5, 4)$$

Find the midpoint of diagonal BD:

$$M_{BD} = \left(\frac{5+2}{2}, \frac{3+5}{2}\right) = (3.5, 4)$$

Since both diagonals have the same midpoint, they bisect each other, proving ABCD is a **parallelogram**.

4.8.2 Example 15: Finding the Center of a Circle

Three points on a circle are A(0, 4), B(4, 0), and C(0, -4). The center lies on the perpendicular bisector of any chord. Find an approximation of the center.

Solution:

Find midpoint of AB:

$$M_{AB} = \left(\frac{0+4}{2}, \frac{4+0}{2}\right) = (2, 2)$$

The center is somewhere on the perpendicular bisector through (2, 2).

Find midpoint of AC:

$$M_{AC} = \left(\frac{0+0}{2}, \frac{4+(-4)}{2}\right) = (0, 0)$$

This suggests the center might be near the origin. By symmetry with points A and C, the center is at **(0, 0)** and the radius is 4.

4.9 Connection to Other Lessons

4.9.1 Link to Distance Formula (Lesson 3)

The two endpoints are equidistant from the midpoint:

Example 16: Verify that M(4, 5) is equidistant from A(1, 2) and B(7, 8).

Distance from M to A:

$$d_{MA} = \sqrt{(4-1)^2 + (5-2)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Distance from M to B:

$$d_{MB} = \sqrt{(7-4)^2 + (8-5)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Both distances equal $3\sqrt{2}$, confirming M is the midpoint.

4.9.2 Link to Maps (Lesson 2)

Example 17: On a map with scale 1:10,000, two landmarks are at (240, 180) and (360, 260). Each unit represents 10 meters. Where should a information sign be placed at the midpoint?

Solution:

$$M = \left(\frac{240+360}{2}, \frac{180+260}{2}\right) = (300, 220)$$

The sign should be at grid position **(300, 220)**.

4.10 Practice Exercises

4.10.1 Set 1: Basic Midpoint Calculations

1. Find the midpoint of the line segment joining:
   a) A(4, 6) and B(10, 12)
   b) C(-2, 5) and D(6, -3)
   c) E(0, 8) and F(8, 0)
   d) G(-5, -7) and H(-1, -3)
2. Find the midpoint between:
   a) The origin and (12, 16)
   b) (3.5, 7.2) and (8.5, 4.8)
   c) $(-\frac{1}{2}, 3)$ and $(\frac{5}{2}, 7)$

4.10.2 Set 2: Finding Unknown Endpoints

3. The midpoint of segment PQ is M(6, 4). If P is at (2, 1), find Q.
4. The midpoint of segment RS is M(-3, 5). If S is at (1, 9), find R.
5. Point A(5, 8) is one endpoint. The midpoint is M(3, 4). Find the other endpoint B.
6. A segment has midpoint (0, 0) and one endpoint at (7, -3). Find the other endpoint.

4.10.3 Set 3: Applications

7. Two warehouses are located at A(12, 18) and B(28, 34). A distribution center will be built exactly halfway between them. What are its coordinates?
8. A park bench should be placed at the midpoint between two trees at positions T1(-8, 6) and T2(4, 14). Where should the bench go?
9. On a map, a school is at (15, 20) and a library is at (35, 40). Where should a crosswalk be placed on a straight path connecting them, exactly at the midpoint?

4.10.4 Set 4: Geometric Applications

10. A rectangle has opposite vertices at (2, 3) and (10, 11). Find the coordinates of the center of the rectangle.
11. Triangle ABC has vertices A(0, 0), B(8, 0), and C(4, 6). Find:
    a) The midpoint of side AB
    b) The midpoint of side BC
    c) The midpoint of side AC
12. Show that the quadrilateral with vertices P(1, 1), Q(5, 2), R(6, 6), and S(2, 5) is a parallelogram by proving the diagonals bisect each other.

4.10.5 Set 5: Challenge Problems

13. Points A, B, and C are collinear (on the same line) with B between A and C. If A is at (2, 3), B is at (5, 7), and B is the midpoint of AC, find C.
14. The midpoint of XY is M(4, 6) and the midpoint of YZ is N(7, 9). If X is at (1, 3), find Z.
15. Three vertices of a parallelogram are A(1, 2), B(5, 4), and C(7, 7). Find all possible locations for the fourth vertex D. (Hint: There are three possible parallelograms)
16. A segment AB is divided into four equal parts by points P, Q, and R (in that order). If A is at (0, 0) and B is at (12, 8), find the coordinates of P, Q, and R.

4.11 Real-World Applications

4.11.1 Urban Planning

City planners use midpoints to: - Determine optimal locations for public facilities - Place street lights evenly along roads - Design bus stops at convenient intervals

4.11.2 Sports

- Finding the center circle of a sports field
- Placing mid-court lines in basketball
- Determining fair meeting points for tournaments

4.11.3 Engineering

- Locating support beams at the center of bridges
- Finding balance points in structural design
- Calculating center of mass for symmetric objects

4.11.4 Navigation

- Waypoints in flight paths
- Meeting coordinates for rescue operations
- Optimal refueling locations

4.12 Additional Resources

4.12.1 Interactive Tools

- Desmos: Plot segments and automatically show midpoints
- GeoGebra: Explore midpoint properties dynamically
- Online midpoint calculators for checking work

4.12.2 Extension Topics

- **Weighted Midpoints:** When one endpoint is "heavier" than the other
- **Centroids:** The "average" point of a triangle (uses midpoints)
- **Perpendicular Bisectors:** Lines through midpoints perpendicular to segments

4.13 Key Formulas Summary

Midpoint Formula:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Finding Other Endpoint: If M(m_x, m_y) is the midpoint of AB and A(x_1, y_1) is known:

$$B = (2m_x - x_1, 2m_y - y_1)$$

Verification: Distance from A to M equals distance from M to B.

4.14 Homework

Complete Practice Exercises Sets 1 and 2, and at least 3 problems from Set 3.

5 Lesson 5: Gradients and Linear Equations

5.1 Learning Objectives

By the end of this lesson, students will be able to:

- Calculate the gradient (slope) of a line between two points
- Understand the meaning of positive, negative, zero, and undefined gradients
- Write equations of lines in various forms
- Convert between different forms of linear equations

5.2 Gradient (Slope) of a Line

The gradient m of a line passing through points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

5.2.1 Interpreting Gradient

- **Positive gradient**: Line slopes upward from left to right
- **Negative gradient**: Line slopes downward from left to right
- **Zero gradient**: Horizontal line
- **Undefined gradient**: Vertical line

5.2.2 Example 1: Calculating Gradient

Find the gradient of the line passing through A(1, 2) and B(5, 10).

Solution:

$$m = \frac{10 - 2}{5 - 1} = \frac{8}{4} = 2$$

5.3 Equations of Lines

5.3.1 Slope-Intercept Form

$$y = mx + c$$

where m is the gradient and c is the y-intercept.

5.3.2 Point-Slope Form

$$y - y_1 = m(x - x_1)$$

where m is the gradient and (x_1, y_1) is a point on the line.

5.3.3 General Form

$$ax + by + c = 0$$

5.3.4 Example 2: Writing Line Equations

Write the equation of the line with gradient 3 passing through the point (2, 5).

Solution using point-slope form:

$$y - 5 = 3(x - 2)$$

$$y - 5 = 3x - 6$$

$$y = 3x - 1$$

5.3.5 Example 3: Finding Equation from Two Points

Find the equation of the line passing through P(1, 3) and Q(4, 9).

Solution:

Step 1: Find the gradient

$$m = \frac{9 - 3}{4 - 1} = \frac{6}{3} = 2$$

Step 2: Use point-slope form with point P(1, 3)

$$y - 3 = 2(x - 1)$$

$$y - 3 = 2x - 2$$

$$y = 2x + 1$$

5.4 Practice Exercises

1. Find the gradient of the line through:
   - A(2, 3) and B(6, 11)
   - C(-1, 5) and D(3, -3)
2. Write the equation of the line:
   - With gradient 4 and y-intercept -2
   - Through point (3, 7) with gradient -1
   - Through points (0, 2) and (5, 12)
3. Convert the equation $2x + 3y - 6 = 0$ to slope-intercept form.

5.5 Real-World Application

The gradient represents a rate of change. For example: - In a distance-time graph, gradient represents speed - In a cost-quantity graph, gradient represents unit price

5.6 Homework

Complete exercises 2.1 to 2.5 in the workbook.

6 Lesson 6: Parallel and Perpendicular Lines

6.1 Learning Objectives

By the end of this lesson, students will be able to:

- Identify parallel and perpendicular lines from their equations
- Determine if lines are parallel or perpendicular using gradients
- Find equations of parallel and perpendicular lines
- Solve geometric problems involving parallel and perpendicular lines

6.2 Parallel Lines

Two lines are **parallel** if they have the same gradient and never intersect.

> **!** Key Property of Parallel Lines
>
> If two lines are parallel, then $m_1 = m_2$

6.2.1 Example 1: Identifying Parallel Lines

Are the lines $y = 2x + 3$ and $y = 2x - 5$ parallel?

Solution: Both lines have gradient $m = 2$, so they are parallel.

6.2.2 Example 2: Finding Parallel Line Equation

Find the equation of the line parallel to $y = 3x + 1$ that passes through the point (2, 7).

Solution: - Parallel lines have the same gradient: $m = 3$ - Using point-slope form with (2, 7):

$$y - 7 = 3(x - 2)$$
$$y - 7 = 3x - 6$$
$$y = 3x + 1$$

Verification: When $x = 2$, $y = 3(2) + 1 = 7$

6.3 Perpendicular Lines

Two lines are **perpendicular** if they meet at a right angle (90 degrees).

> **!** Key Property of Perpendicular Lines
>
> If two lines are perpendicular, then $m_1 \times m_2 = -1$
> Or equivalently: $m_2 = -\frac{1}{m_1}$

6.3.1 Example 3: Identifying Perpendicular Lines

Are the lines $y = 2x + 1$ and $y = -\frac{1}{2}x + 3$ perpendicular?

Solution:

$$m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$$

Yes, the lines are perpendicular.

6.3.2 Example 4: Finding Perpendicular Line Equation

Find the equation of the line perpendicular to $y = 4x - 2$ that passes through the point (8, 3).

Solution: - Original gradient: $m_1 = 4$ - Perpendicular gradient: $m_2 = -\frac{1}{4}$ - Using point-slope form with (8, 3):

$$y - 3 = -\frac{1}{4}(x - 8)$$

$$y - 3 = -\frac{1}{4}x + 2$$

$$y = -\frac{1}{4}x + 5$$

6.4 Practice Exercises

1. Determine if these pairs of lines are parallel, perpendicular, or neither:
   - $y = 3x + 2$ and $y = 3x - 7$
   - $y = 2x + 1$ and $y = -\frac{1}{2}x + 4$
   - $y = 5x$ and $y = -5x + 3$
2. Find the equation of the line:
   - Parallel to $y = -2x + 3$ through point (1, 5)
   - Perpendicular to $y = \frac{1}{3}x - 1$ through point (6, 2)
3. The line L_1 passes through points A(1, 2) and B(5, 10).
   - Find the equation of L_1
   - Find the equation of the line parallel to L_1 through C(0, 0)
   - Find the equation of the line perpendicular to L_1 through D(3, 4)

6.5 Problem-Solving Strategy

When working with parallel and perpendicular lines:

1. Identify or calculate the gradient of the given line
2. Determine the gradient of the required line (same for parallel, negative reciprocal for perpendicular)
3. Use the point-slope form with the given point
4. Simplify to the required form

6.6 Real-World Applications

- Architecture: Ensuring walls are perpendicular to the floor
- Navigation: Plotting perpendicular courses
- Engineering: Designing parallel roads or railway tracks

6.7 Homework

Complete the parallel and perpendicular lines worksheet.

7 Worksheets

7.1 Worksheet 1: Coordinate Basics

Topic: Plotting points, distance, and midpoint

Exercises:

1. Plot the following points on a coordinate plane:
   - A(4, 3), B(-2, 5), C(-3, -4), D(1, -2), E(0, 6)
2. Calculate the distance between:
   - A(1, 1) and B(4, 5)
   - C(-2, 3) and D(3, -9)
   - E(0, 0) and F(-5, 12)
3. Find the midpoint of:
   - Line segment from (2, 8) to (10, 4)
   - Line segment from (-3, 7) to (5, -1)

7.2 Worksheet 2: Gradient and Equations

Topic: Finding gradients and writing equations of lines

Exercises:

1. Find the gradient of the line through:
   - P(1, 3) and Q(5, 11)
   - R(-2, 4) and S(6, -4)
   - T(0, -3) and U(4, 5)
2. Write the equation of the line:
   - With gradient 3 and y-intercept 5
   - With gradient -2 passing through (4, 1)
   - Through points (2, 3) and (6, 11)
3. Convert to slope-intercept form ($y = mx + c$):
   - $3x + y - 6 = 0$
   - $2x - 4y + 8 = 0$
   - $x + 2y - 10 = 0$

7.3 Worksheet 3: Parallel and Perpendicular Lines

Topic: Working with parallel and perpendicular lines

Exercises:

1. Determine whether each pair of lines is parallel, perpendicular, or neither:
   - $y = 4x + 1$ and $y = 4x - 3$
   - $y = 2x + 5$ and $y = -\frac{1}{2}x + 2$

- $y = 3x - 1$ and $y = 2x + 4$

2. Find the equation of:

   - The line parallel to $y = 5x + 2$ through point (2, 3)
   - The line perpendicular to $y = -3x + 1$ through point (6, 4)
   - The line parallel to $2x + y - 4 = 0$ through point (-1, 5)

3. Challenge problems:

   - Line L passes through A(1, 2) and B(7, 8). Find the equation of the line perpendicular to L that passes through the midpoint of AB.
   - Points P(2, 3), Q(6, 5), and R(x, y) form a right angle at Q. If the line PQ has gradient $\frac{1}{2}$, find the gradient of QR.