Assignment 8

Real Space Renormalization Group
Infinite Density Matrix Renormalization Group

December 12th 2023

REAL SPACE RENORMALIZATION GROUP

Given the quantum Ising Hamiltonian in the transverse field on a one-dimensional lattice with nearest neighbor interaction:

$$\hat{H} = \lambda \sum_{i}^{N} \sigma_i^z + \sum_{i}^{N-1} \sigma_i^x \sigma_{i+1}^x$$

where σ_x and σ_z are the Pauli matrices and λ is the transverse field.

1) Compute the ground state energy as a function of the transverse field λ using the real-space RG algorithm.

2) Optional: Compute the ground state energy as a function of λ using the INFINITE DMRG algorithm.

Compare the results between them and with the mean-field solution.

Real Space Renormalization Group is a method to approximate the ground state of a many-body quantum system (coarse-graining). The general idea is starting from an initial system size n_0 whose Hamiltonian H_{n_0} can be diagonalized exactly and iteratively increasing the system size while keeping the (approximated) Hilbert space dimension fixed.

This is achieved by performing a truncation, projecting onto a fixed number of states considered to be appropriate for a low-energy description of the system.

Sketch of the algorithm:

Let d be the local Hilbert space dimension and $n_0 = 2m$. As initial condition set $N = n_0$.

(i) Find the d_m lowest energy eigenvectors $|v\rangle$ of H_N and build the projector $P = \sum |v\rangle\langle v|$.

(ii) Truncate the system description (states and operators) using P; compute $\tilde{H} = P^\dagger H P$

(iii) Double the system $n \to 2n$ using the truncated basis for each side; in particular $H_{2N} = H_N \otimes \mathbb{I} + \mathbb{I} \otimes H_N + H_{int}$.

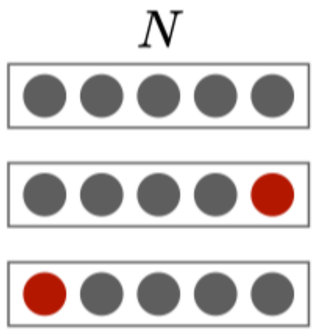
REAL SPACE RENORMALIZATION GROUP

Initial: N spin Hamiltonian can be solved exactly

$$H_N^{(0)} = \lambda \sum_i^N \sigma_i^z + \sum_i^{N-1} \sigma_i^x \sigma_{i+1}^x$$

$$A^{(0)} = \mathbb{I}_{N-1} \otimes \sigma^x$$

$$B^{(0)} = \sigma^x \otimes \mathbb{I}_{N-1}$$



Algorithm:

(1) Build the Hamiltonian and state for the system of size 2N from the previous step:

$$H_{2N}^{(n)} = H_N^{(n)} \otimes \mathbb{I}_N + \mathbb{I}_N \otimes H_N^{(n)} + A_N^{(n)} \otimes B_N^{(n)}$$

(2) Compute the d^N lowest eigenvectors of H_{2N} and construct the projector

$$P = \sum_{i=1}^{d^N} |v_i\rangle\langle v_i|$$

(3) Check the convergence of the energy density $\epsilon = E_0/N$ with a threshold criterion:

$$\Delta_n = |\epsilon^{(n)} - \epsilon^{(n-1)}| < \tau$$

(4) if it is not converged: update the Hamiltonian and the interaction for the next iteration

$$H_N^{n+1} \leftarrow \frac{1}{2} P^\dagger H_{2N}^n P$$

$$A_N^{n+1} \leftarrow \frac{1}{\sqrt{2}} P^\dagger (\mathbb{I}_N \otimes A_N^n) P$$

$$B_N^{n+1} \leftarrow \frac{1}{\sqrt{2}} P^\dagger (B_N^n \otimes \mathbb{I}_N) P$$

INFINITE DENSITY MATRIX RENORMALIZATION GROUP

Sketch of the algorithm:

Let d be the local Hilbert space dimension and $n_0 = 2m$. As initial condition set $n = n_0$.

(i) Find the d_m lowest energy eigenvectors $|k\rangle$ of H_n and build the projector $P = \sum |k\rangle\langle k|$.

(ii) Truncate the system description (states and operators) using P; compute $\tilde{H} = P^\dagger H P$

(iii) Double the system $n \to 2n$ using the truncated basis for each side; in particular $H_{2n} = H_n \otimes \mathbb{I} + \mathbb{I} \otimes H_n + H_{int}$.

INFINITE DENSITY MATRIX RENORMALIZATION GROUP

Initial: Consider a global system composed of 4 parts: Left (L) and right (R) blocks containing $m+1$ sites each.

$$H_L^{(0)} = H_R^{(0)} = \lambda \sum_i^m \sigma_i^z + \sum_i^{m-1} \sigma_i^x \sigma_{i+1}^x$$

$$H_2^{(0)} = \lambda\sigma^z \quad H_3^{(0)} = \lambda\sigma^z$$

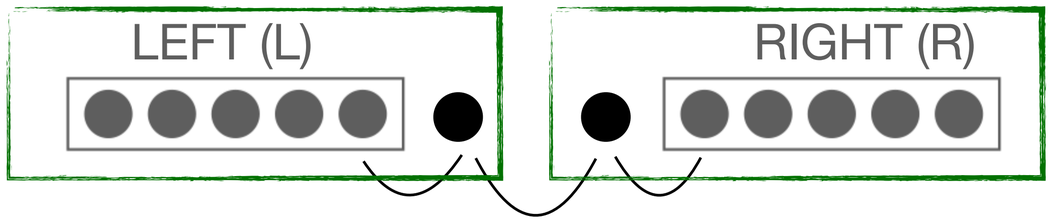
$$H_{L,R}^{(0)} = \sigma^x \otimes \sigma^x$$

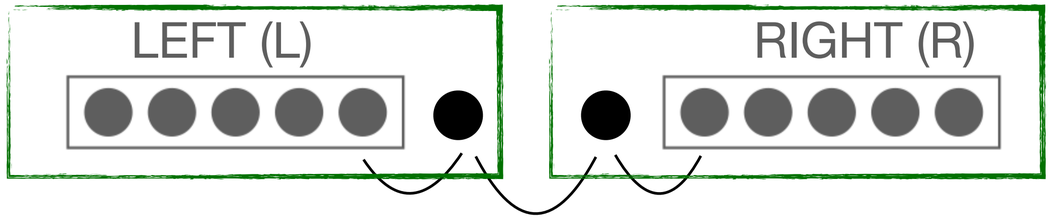
We start by an initial system that can be solved exactly. The Hamiltonian is now composed of two enlarged interacting blocks. The left block takes the form (the right is mirrored)

$$H_{L+1}^{(n)} = H_L^{(n)} \otimes \mathbb{I}_1 + A_L^{(n)} \otimes H_2 + B_L^{(n)} \otimes \sigma^x$$

For n=0 and m=1 we have:

$$H_L^{(0)} = \lambda\sigma^z \quad A_L^{(0)} = \mathbb{I}_{m=1} \quad B_L^{(0)} = \sigma^x$$





Algorithm:

(1) Enlarge the left (right) block by adding one site

$$H_{L+1}^{(n)} = H_L^{(n)} \otimes \mathbb{I}_1 + A_L^{(n)} \otimes H_2 + B_L^{(n)} \otimes \sigma^x$$

And updating $A_{L+1}^{(n)} = \mathbb{I}_m \quad B_{L+1}^{(n)} = \mathbb{I}_{m-1} \otimes \sigma^x$

(2) Build the Hamiltonian and state for the system of size $2M + 2$ from the previous step and diagonalize it.

$$H_{2m+2}^{(n)} = H_{L+1}^{(n)} \otimes \mathbb{I}_{R+1} + \mathbb{I}_{L+1} \otimes H_{R+1}^{(n)} + H_{L,R}^{(n)}$$

(3) Compute the density matrix and trace out the right block, getting the reduced density matrix ρ_{L+1}.

(4) Diagonalize ρ_{L+1} and build a projector P considering k = $min(2^{m_{max}}, 2^m)$ eigenvectors associated to the k largest eigenvalues.

(5) Project the building blocks:

$$H_L^{n+1} \leftarrow P^\dagger H_L^n P \qquad A_L^{n+1} \leftarrow P^\dagger A_L^n P \qquad B_L^{n+1} \leftarrow P^\dagger B_L^n P$$

(6) Energy is computed at each iteration and $\epsilon = \dfrac{E_0}{2(m+1)}$.