Math 631 Notes

David Jin

February 17, 2020

Lectures by Bhargav Bhatt. Notes on topics which were particularly difficult for me.

1 Closed Subschemes and Ideal sheaves

Let A be a ring. The closed subschemes of $\operatorname{Spec} A$ are in bijection with the ideals $I \subseteq A$. For a scheme X, we have a similar correspondence.
Given a closed subscheme $i : Z \subseteq X$, we have a surjective map of sheaves $\mathcal{O}_X \to i_*\mathcal{O}_Z$. Let $\mathscr{I}$ be the kernel of this map. I claim that $\mathscr{I}$ is a sheaf of ideals of $\mathcal{O}_X$. Indeed, since on each affine open, $\operatorname{Spec} A \subseteq X$, we have that $i_*\mathcal{O}_Z(\operatorname{Spec} A) = \mathcal{O}_Z(i^{-1}(\operatorname{Spec} A)) = \mathcal{O}_Z(\operatorname{Spec} A/I) = \operatorname{Spec} A/I$. Thus, $\mathscr{I}(\operatorname{Spec} A) = I$.
Conversely, given a sheaf of ideals $\mathscr{I} \subseteq \mathcal{O}_X$, we can define a closed subscheme as follows. Let $Z = Supp\mathcal{O}_x/\mathscr{I}$. Then, $(Z, \mathcal{O}_X/\mathscr{I})$ is a closed subscheme of X.

2 Cartier Divisors on an Integral Scheme

Let X be an integral scheme. Let $\mathscr{K}_X$ be the sheaf of meromorphic functions on X. Since X is integral, $\mathscr{K}_X$ is the constant sheaf on $K(X)$, the rational functions of X, or equivalently, $\mathcal{O}_{X,\eta}$, the local ring at the generic point η.

Definition 2.1. A **Cartier Divisor** on X is a set of ordered pairs (U_i, f_i) with $f_i \in K(X)^*$ such that $\{U_i\}$ covers X and for each pair i, j, we have $f_i = uf_j$ for $u \in \mathcal{O}_X^*$ on the intersection $U_i \cap U_j$.
We identify the divisors $D_1 = (U_i, f_i)$ and $D_2 = (V_i, g_i)$ provided that for each pair i, j, we have that $f_i = ug_j$ for $u \in \mathcal{O}_X^*$ on the intersection $U_i \cap V_j$.
For two divisors D_1 and D_2 as above, we define $D_1 + D_2$ as the divisor $(U_i \cap V_j, f_ig_j)$. It is easy to check that this is well defined.
We write **Div(X)** for the abelian group of divisors under the above equivalence relation.

Remark 1. For $f \in K(X)^*$, the pair (X, f) is a divisor. The subgroup of divisors of this form are called the **principal divisors of** X.
We have an exact sequence

$$1 \longrightarrow \mathcal{O}_X^*(X) \longrightarrow K(X)^* \longrightarrow Div(X) \longrightarrow \text{coker} \longrightarrow 0$$

We define **Cl(X)** as the cokernel in the above diagram. We say that two divisors are **linearly equivalent** if they are equivalent in $\mathrm{Cl}(X)$.

Remark 2. Let $\mathscr{K}_X^*/\mathcal{O}_X^*$ be the sheaf associated the presheaf $(U \mapsto \mathscr{K}_X^*(U)/\mathcal{O}_X^*(U))$. An alternative definition of $\mathrm{Div}(X)$ is $\Gamma(X, \mathscr{K}_X^*/\mathcal{O}_X^*)$.

Proposition 2.2. *To a divisor $D = (U_i, f_i)$ we can associate a line bundle $\mathcal{O}_X(D)$ as follows. For $U \subseteq X$, define $\Gamma(U, \mathcal{O}_X(D)) = \{g \in K(X) : \forall i,\ f_i g \in \Gamma(U \cap U_i, \mathcal{O}_X)\}$.*
On the open set U_i, $\mathcal{O}_X(D)|_{U_i}$ is isomorphic to $\mathcal{O}_X|_{U_i}$ via the map which sends f_i to 1.
Thus, we have a map $\varphi : Div(X) \to Pic(X)$. This map descends to an isomorphism $Cl(X) \to Pic(X)$.

Proof. To show that the map $Div(X) \to Pic(X)$ descends to a map $Cl(X) \to Pic(X)$, it suffices to show that principal divisors map to $\mathcal{O}_X \in Pic(X)$. For a divisor $D = (X, f)$ with $f \in K(X)$, we have the associated sheaf $\mathcal{O}_X(D) = (U \mapsto \{g \in K(X) : fg \in \Gamma(U, \mathcal{O}_x)\})$. Thus, we have a global isomorhpism $\mathcal{O}_X \to \mathcal{O}_X(D)$ given by $1 \mapsto g$. ∎

3 Construction of the Symmetric Algebra

We begin with the construction in the case of A-modules and then globalize to the case of $\mathcal{O}_X$-modules over a scheme X.

Definition 3.1. Let A be a commutative ring, M an A-module. We define the **tensor algebra of** M as

$$T(M) = \bigoplus_{n \geq 0} T_n(M)$$

where $T_n(M) = M \otimes \cdots \otimes M$. We let $T_0(M) = A$.
We define the **symmetric algebra of** M, denoted $Sym(M)$ is the tensor algebra $T(M)$ modulo the submodule generated by the elements of the form $m \otimes m' - m' \otimes m$ where $m, m' \in M$.

Proposition 3.2. *Sym is a functor from the category of A-modules to A-algebras which is left adjoint to the forgetful functor. Thus, for any A-module M and A-algebra B, we have an isomorphism*

$$\mathrm{Hom}_{A-alg}(Sym(M), B) \cong \mathrm{Hom}_{A-mod}(M, B)$$

This adjunction gives rise to the following isomorphisms:

$$Sym_A(M) \otimes_A B \cong Sym_B(M \otimes_A B)$$

$$Sym(M \oplus N) \cong Sym(M) \otimes Sym(N)$$

If M is a free module of rank n with basis $(m_1, \ldots, m_n)$, there is a unique isomorphism $Sym(M) \cong R[x_1, \ldots, x_n]$ such that $m_i \mapsto x_i$.

Now we address the case where X is a scheme and $\mathscr{F}$ is a quasi-coherent sheaf over X.

Definition 3.3. We define **symmetric algebra of** $\mathscr{F}$ as the sheaf associated to the presheaf $(U \mapsto Sym(\Gamma(U, \mathscr{F}))$

Proposition 3.4. *For X an affine scheme and $\mathscr{F} = \tilde{M}$, $Sym(\mathscr{F}) = \widetilde{Sym(M)}$, since $\Gamma(D(f), Sym(\tilde{M})) = Sym(M_f) \cong Sym(M)_f$. Thus, for general X and $\mathscr{F}$, $Sym(\mathscr{F})$ is a quasi-coherent algebra.*
Further, Sym is a functor from the category of quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_X$-algebras which is left adjoint to the forgetful functor. Thus, for any QC $\mathcal{O}_X$-module $\mathscr{F}$ and QC $\mathcal{O}_X$-algebra $\mathscr{B}$, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X-alg}(Sym(\mathscr{F}), \mathscr{B}) \cong \mathrm{Hom}_{\mathcal{O}_X-mod}(\mathscr{F}, \mathscr{B})$$

Analogously to the affine case (extension of scalars in above prop), for a morphism of schemes $f : X \to Y$ and $\mathscr{F}$ a quasi-coherent sheaf on Y,

$$f^* Sym(\mathscr{F}) \cong Sym(f^* \mathscr{F})$$

4 Spec of quasi-coherent $\mathcal{O}_X$-algebras

Let X and $\operatorname{Spec} B$ be schemes over $\operatorname{Spec} A$. Observe that $\operatorname{Spec} B$ is the scheme which represents the functor $(X \mapsto \mathrm{Hom}_{A-alg}(B, \Gamma(X, \mathcal{O}_X))$. That is, we have an isomorphism

$$\mathrm{Hom}_{\operatorname{Spec} A}(X, \operatorname{Spec} B) \cong \mathrm{Hom}_{A-alg}(B, \Gamma(X, \mathcal{O}_X))$$

This motivates the general case where $\mathscr{B}$ is a quasi-coherent $\mathcal{O}_X$-algebra.

Proposition 4.1. *Let X be a scheme and $\mathscr{B}$ a quasi-coherent $\mathcal{O}_X$-algebra. Let $f : T \to X$ be a scheme over X. The functor $(T \mapsto \mathrm{Hom}_{\mathcal{O}_X-alg}(\mathscr{B}, f_*\mathcal{O}_T))$ is representable and we call the representing object* $\operatorname{Spec} \mathscr{B}$*. So, we have an isomorphism*

$$\mathrm{Hom}(T, \operatorname{Spec} \mathscr{B}) \cong \mathrm{Hom}_{\mathcal{O}_X-alg}(\mathscr{B}, f_*\mathcal{O}_T)$$

Proof. Observe that the functor above is a sheaf, since pushforward and Hom are left-exact. Then, since it is locally representable on affine schemes by above, it is representable. □

Proposition 4.2. *Let $h : \operatorname{Spec} \mathscr{B} \to X$ be the structure morphism. For each affine open subset $U \subseteq X$, $h^{-1}(U) \cong \operatorname{Spec} \Gamma(U, \mathscr{B})$ and $h_*\mathcal{O}_{\operatorname{Spec} \mathscr{B}} \cong \mathscr{B}$ by construction.*
Further, given $\mathscr{B}, \mathscr{C}$ quasi-coherent $\mathcal{O}_X$-algebras, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X-alg}(\mathscr{B}, \mathscr{C}) \cong \mathrm{Hom}_X(\operatorname{Spec} \mathscr{C}, \operatorname{Spec} \mathscr{B})$$

In particular, Spec *is a fully faithful functor from quasi-coherent $\mathcal{O}_X$-algebras to schemes over X. The essential image of this functor are the schemes for which the structure morphism is affine.*

5 Proj of quasi-coherent graded $\mathcal{O}_X$-algebras

Similarly to above, we present the globalized construction of Proj to a quasi-coherent sheaf of graded algebras.

We review the case of projective space first. Recall that $\mathbb{P}^n$ is the scheme which represents the functor which sends a scheme T to isomorphism classes of surjections $\mathcal{O}_Y^{n+1} \to \mathscr{L}$ where $\mathscr{L}$ is a line bundle on T.
More generally, for A a graded ring generated in degree 1, $\operatorname{Proj} A$ is the scheme which represents the functor which, for a scheme Y, associates a pair $(\psi, \mathscr{L})$, such that $\mathscr{L}$ is a line bundle of Y and $\psi : A \to \Gamma_*(Y, \mathscr{L})$ is a graded ring homomorphism such that $\mathscr{L}$ is generated by the global sections $\psi(f)$ for $f \in A_1$.

Proposition 5.1. *Let X be a scheme and $\mathscr{A}$ be a quasi-coherent sheaf of graded algebras over X. There exists a unique scheme $\operatorname{Proj}_X \mathscr{A}$ over X such that for each affine open $U \subseteq X$, we have an isomorphism $\pi^{-1}(U) \cong \operatorname{Proj} \Gamma(U, \mathscr{A})$, where $\pi : \operatorname{Proj}_X \mathscr{A} \to X$ is the structure morphism.*
$\operatorname{Proj} \mathscr{A}$ has the following properties:

1. *The structure morphism π is separated. (Check locally on the base)*
2. *There is a functor $(M \mapsto \tilde{M})$ which assigns to a quasi-coherent graded $\mathscr{A}$-module M a quasi-coherent $\operatorname{Proj} \mathscr{A}$-module. This functor is exact and has the property that on each affine open $U \subseteq X$, we have that $\Gamma(U, \tilde{M}) \cong \widetilde{\Gamma(U, M)}$ in the category of quasi-coherent modules over $\operatorname{Proj} \Gamma(U, \mathscr{A})$.*
3. *Let $\mathscr{A}(n)$ be the graded $\mathscr{A}$-module such that on each affine open U, we have $\Gamma(U, \mathscr{A}(n)) \cong \Gamma(U, \mathscr{A})(n)$. We define $\mathcal{O}_{\operatorname{Proj} \mathscr{A}}(n) = \widetilde{\mathscr{A}(n)}$.*
4. *Proj is compatible with base change. That is, for a morphism $f : Y \to X$ of schemes and $\mathscr{A}$ a quasi-coherent sheaf of algebras on X, we have*

$$\operatorname{Proj}(\mathscr{A}) \times_X Y \cong \operatorname{Proj}(g^* \mathscr{A})$$

Proposition 5.2. *Let X be a scheme, $\mathscr{A}$ a quasi-coherent sheaf of algebras over X such that for each affine open $U \subseteq X$, $\Gamma(U, \mathscr{A})$ is generated in degree 1. In this case, we say that $\mathscr{A}$ is generated by $\mathscr{A}_1$. Then we have the following results:*

1. *$\mathcal{O}_{\operatorname{Proj} \mathscr{A}}(n)$ is invertible for all n.*
2. *We have a natural map $\mathscr{A}_n \to \pi_* \mathcal{O}_{\operatorname{Proj} \mathscr{A}}(n)$ of $\mathscr{A}_0$-modules which together give a map*

   $$\mathscr{A} \to \bigoplus \pi^* \mathcal{O}_{\operatorname{Proj} \mathscr{A}}(n)$$

   of graded $\mathscr{A}$-modules.
3. *We have a functor Γ_* which takes a quasi-coherent $\operatorname{Proj} \mathscr{A}$-module $\mathscr{F}$, to a graded $\mathscr{A}$-module, defined by*

   $$\Gamma_*(\mathscr{F}) = \bigoplus \pi^* (\mathscr{F} \otimes \mathcal{O}_{\operatorname{Proj} \mathscr{A}}(n))$$

   There is a functorial isomorphism

   $$\Gamma_*(\mathscr{F}) \tilde{\to} \mathscr{F}$$

Proof. (1) and (3) can both be checked affine-locally, which reduces the problem to the Proj of a graded ring. □