Boolos and Jeffrey - HW5

David Maldonado, *david.m.maldonado@gmail.com*

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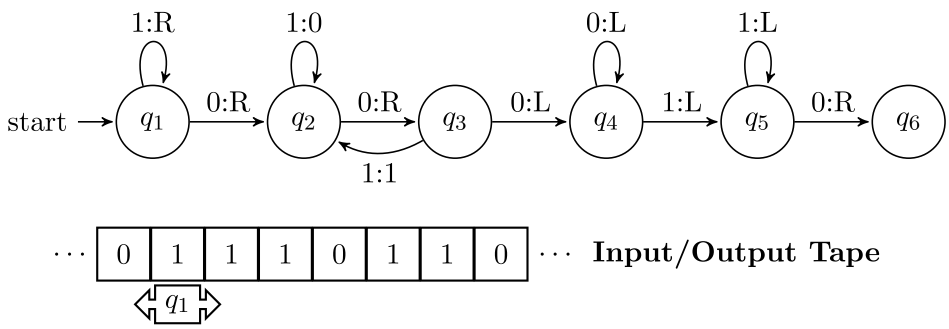
f(x,y) = x, when x=3 and y=2

Graphical representation of turing machine:



Translation into first order logic:

Our representation will take the form $\Delta \models H$, where Δ is a set of statements in first order logic and H is a single statement logically entailed from Δ. Our language has the domain $\mathbb{Z}$ of negative integers, 0 and positive integers. Special features of the language referred to in the following logical statements include the successor function ($'$), the constant $\mathbf{0}$ (which denotes the integer 0), the two-place ordering predicate $<$, the two-place state predicate $\mathbf{Q}$, and the two-place "scanning" predicate $\mathbf{S}$.

We first have to encode some mathematical facts about $'$ and $<$:

$$\forall z \exists x\ z = x' \wedge \forall z \forall x \forall y([z = x' \wedge z = y'] \rightarrow x = y) \tag{1}$$

and

$$\forall x \forall y \forall z[(x < y \wedge y < z) \rightarrow x < z] \ \wedge \ \forall x \forall y(x' = y \rightarrow x < y) \\ \wedge \ \forall x \forall y(x < y \rightarrow x \neq y). \quad (2)$$

Next we need to encode the initial configuration of the tape:

$$\mathbf{0} Q_1 \mathbf{0} \wedge \mathbf{0} S_1 \mathbf{0} \wedge \mathbf{0} S_1 \mathbf{0}' \wedge \mathbf{0} S_1 \mathbf{0}'' \wedge \mathbf{0} S_1 \mathbf{0}'''' \wedge \mathbf{0} S_1 \mathbf{0}''''' \wedge \\ \forall y[(y \neq \mathbf{0} \wedge y \neq \mathbf{0}' \wedge y \neq \mathbf{0}'' \wedge y \neq \mathbf{0}'''' \wedge y \neq \mathbf{0}''''') \rightarrow \mathbf{0} S_0 y] \quad (3)$$

To finish Δ we encode each of the state transitions (arrows in the flow graph) as logical statements:

$$\forall t \forall x[(tQ_1 x \wedge tS_1 x) \rightarrow (t' Q_1 x' \wedge \forall y[(tS_0 y \rightarrow t' S_0 y) \wedge (tS_1 y \rightarrow t' S_1 y)])] \quad (4)$$

$$\forall t \forall x[(tQ_1 x \wedge tS_0 x) \rightarrow (t' Q_2 x' \wedge \forall y[(tS_0 y \rightarrow t' S_0 y) \wedge (tS_1 y \rightarrow t' S_1 y)])] \quad (5)$$

$$\forall t \forall x[(tQ_2 x \wedge tS_1 x) \rightarrow \\ ([t' Q_2 x \wedge t' S_0 x] \wedge \forall y[(y \neq x) \rightarrow ([tS_0 y \rightarrow t' S_0 y] \wedge [tS_1 y \rightarrow t' S_1 y])])] \quad (6)$$

$$\forall t \forall x[(tQ_2 x \wedge tS_0 x) \rightarrow (t' Q_3 x' \wedge \forall y[(tS_0 y \rightarrow t' S_0 y) \wedge (tS_1 y \rightarrow t' S_1 y)])] \quad (7)$$

$$\forall t \forall x[(tQ_3 x' \wedge tS_0 x') \rightarrow (t' Q_4 x \wedge \forall y[(tS_0 y \rightarrow t' S_0 y) \wedge (tS_1 y \rightarrow t' S_1 y)])] \quad (8)$$

$$\forall t \forall x[(tQ_3 x \wedge tS_1 x) \rightarrow \\ ([t' Q_2 x \wedge t' S_1 x] \wedge \forall y[(y \neq x) \rightarrow ([tS_0 y \rightarrow t' S_0 y] \wedge [tS_1 y \rightarrow t' S_1 y])])] \quad (9)$$

$$\forall t \forall x[(tQ_4x' \wedge tS_0x') \rightarrow (t'Q_4x \wedge \forall y[(tS_0y \rightarrow t'S_0y) \wedge (tS_1y \rightarrow t'S_1y)])] \quad (10)$$

$$\forall t \forall x[(tQ_4x' \wedge tS_1x') \rightarrow (t'Q_5x \wedge \forall y[(tS_0y \rightarrow t'S_0y) \wedge (tS_1y \rightarrow t'S_1y)])] \quad (11)$$

$$\forall t \forall x[(tQ_5x' \wedge tS_1x') \rightarrow (t'Q_5x \wedge \forall y[(tS_0y \rightarrow t'S_0y) \wedge (tS_1y \rightarrow t'S_1y)])] \quad (12)$$

$$\forall t \forall x[(tQ_5x \wedge tS_0x) \rightarrow (t'Q_6x' \wedge \forall y[(tS_0y \rightarrow t'S_0y) \wedge (tS_1y \rightarrow t'S_1y)])] \quad (13)$$

The last statement to encode is H, which is the disjunction of all sentences stating the machine is in a potential halting state:

$$\begin{gathered}
\exists t \exists x(tQ_1x \wedge tS_0x) \vee \exists t \exists x(tQ_1x \wedge tS_1x) \vee \\
\exists t \exists x(tQ_2x \wedge tS_0x) \vee \exists t \exists x(tQ_2x \wedge tS_1x) \vee \\
\exists t \exists x(tQ_3x \wedge tS_0x) \vee \exists t \exists x(tQ_3x \wedge tS_1x) \vee \\
\exists t \exists x(tQ_4x \wedge tS_0x) \vee \exists t \exists x(tQ_4x \wedge tS_1x) \vee \\
\exists t \exists x(tQ_5x \wedge tS_0x) \vee \exists t \exists x(tQ_5x \wedge tS_1x) \vee \\
\exists t \exists x(tQ_6x \wedge tS_0x) \vee \exists t \exists x(tQ_6x \wedge tS_1x) \vee
\end{gathered} \quad (14)$$