Boolos and Jeffrey - HW4

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In the following theorems Q stands for *either* quantifier and Q' for its counterpart. v stands for a quantified variable, and F and G for first-order logical formulas with no free variables.

1 some equivalence proofs...

Theorem 1.1. $\neg QvF \cong Q'v\neg F$

Proof. We'll begin with the first case:

$$\neg\forall vF \cong \exists v\neg F \tag{1}$$

The implication $\neg\forall vF \implies \exists v\neg F$ is proven simply by noting that if we assume $\neg\forall vF$ to be **true** that means there exists at least one term in a model that makes $\neg F$ **true**, which is precisely the statement on the right-hand side.

The converse implication $\neg\forall vF \impliedby \exists v\neg F$ is proven in the same way by assuming $\exists v\neg F$ to be **true**. It follows directly that because there is at least one term in a model that makes $\neg F$ **true** not all terms make F **true** which is the statement on the left-hand side.

For the second case:

$$\neg\exists vF \cong \forall v\neg F \tag{2}$$

The implication $\neg\exists vF \implies \forall v\neg F$ is proven by first assuming $\neg\exists vF$ is **true**. With this assumption we can say that there does not exist a term v such that F is **true**, this leads to the right-hand statement that for all terms $\neg F$ is **true**.

The converse implication $\neg\exists vF \impliedby \forall v\neg F$ is proven by assuming $\forall v\neg F$ is **true**. Now we can see that for all terms $\neg F$ is **true**, therefore there *does not* exist a term that makes F **true**, which is the left-hand statement. $\square$

2 proof of prenex normal form

Theorem 2.1. *Where* ***prenex normal form*** *is a formula where all the quantifiers are written as a string at the front and range over the quantifier-free portion, every formula in first-order logic has an equivalent prenex normal form.*

Proof. We will proceed by induction. Let us first agree on the following equivalences:

$$\neg QvF \cong Q'v\neg F \tag{1}$$

$$QvF \wedge G \cong Qv(F \wedge G) \tag{2}$$

$$G \wedge QvF \cong Qv(G \wedge F) \tag{3}$$

$$QvF \vee G \cong Qv(F \vee G) \tag{4}$$

$$G \vee QvF \cong Qv(G \vee F) \tag{5}$$

$$QvF \to G \cong Q'v(F \to G) \tag{6}$$

$$G \to QvF \cong Qv(G \to F) \tag{7}$$

and the **principle of substitution of equivalents**:

$$QvF \cong QwF_v w \tag{8}$$

wow

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