Boolos and Jeffrey - HW1

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1 A question about $\cap$

Proposition:

The intersection of a finite set **S** and an enumerable set **T** is enumerable.

Lemma 1.1. *Any finite set is enumerable.*

Proof. Let **S** be a finite set with n elements. Let $\mathbf{K} = \{1,2,\ldots,n\}$. Choose an element $\mathbf{s} \in \mathbf{S}$ and assign $f(n) = \mathbf{s}$. Set $\mathbf{S}'$ to **S** - $\{\mathbf{s}\}$. Choose an element $\mathbf{s}' \in \mathbf{S}'$ and assign $f(n-1) = \mathbf{s}'$. Repeat this procedure until **S** is exhausted. The resulting function $f : \mathbf{K} \to \mathbf{S}$ is an enumeration of **S**. □

Theorem 1.1. *The intersection of two enumerable sets is enumerable.*

Proof. Let $f : \mathbb{N} \to \mathbf{A}$ represent a function that enumerates the first set. Let $g : \mathbb{N} \to \mathbf{B}$ represent a function that enumerates the second set. Let $h : \mathbb{N} \to \mathbf{A} \cap \mathbf{B}$ be a new function defined as follows:

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathbf{B} \\ undefined & \text{if } f(x) \notin \mathbf{B}. \end{cases}$$

□

Conclusion:

Proof. By **Lemma 1.1** the set **S** is enumerable. By **Theorem 1.1** the intersection of **S** and **T** is enumerable. □

2 A slightly harder question about $\cap$

Proposition:

The intersection of an enumerable set of enumerable sets is itself enumerable.

Conclusion:

Proof. Let **S** be a enumerable set of enumerable sets. Pick a set $\mathbf{A} \in \mathbf{S}$. Let **B** be $\bigcap(\mathbf{S} - \mathbf{A})$. By **Theorem 1.1** we can define a function $h : \mathbb{N} \to \mathbf{A} \cap \mathbf{B}$ that enumerates $\bigcap \mathbf{S}$. □

3 It takes two...

Proposition:

Let **F** be the set of all *one to one* functions that both i) have a domain that's a subset of the positive integers, and ii) are *onto* a two element set {a,b}. **F** is enumerable.

Lemma 3.1. *The Cartesian product of two finite sets is enumerable.*

Proof. Let **A** and **B** be two sets with a finite number n many members. The Cartesian product $\mathbf{A} \times \mathbf{B}$ has $n \cdot n$ members which is also a finite number. By **Lemma 1.1** this finite set is enumerable. □

Theorem 3.1. *Any set of enumerable sets is enumerable*

I'm probably going the wrong direction with this proof as I can't figure out how to prove this. Or even if this is correct (I have a gut feeling it's not).

Conclusion: IN PROGRESS - NOT YET PROVED

Proof. each $\mathbf{f} \in \mathbf{F}$ can be represented as a Cartesian product of finite sets: $(\{k_1, k_2\} \subset \mathbb{N}) \times \{a, b\}$. By **Lemma 3.1** each $\mathbf{f} \in \mathbf{F}$ is enumerable. By **Theorem 3.1** the set F is enumerable. □

4 Enumerate all the things!

Proposition:

The set of all finite sequences of positive integers is enumerable.

Theorem 4.1. *The union of an enumerable set of enumerable sets is itself enumerable.*

Proof. Let **A** be an enumerable set of enumerable sets. The members of **A** can be enumerated as $(a_1, a_2, a_3, \ldots)$. The members of each a_i can be enumerated as $(a_{i1}, a_{i2}, a_{i3}, \ldots)$. We can arrange them on a two-dimensional grid as follows:

$$\begin{matrix} a_1 & a_{11} & a_{21} & \cdots \\ a_2 & a_{12} & a_{22} & \cdots \\ a_3 & a_{13} & a_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

$\bigcup \mathbf{A}$ can now be enumerated by sweeping through the grid in a triangular fashion: $(a_1, a_{11}, a_2, a_{21}, a_{12}, a_3, \ldots)$. □

Conclusion:

Proof. Let **S** be the set of all finite sequences of positive integers. **S** is the union of one member sequences, two member sequences, three member sequences, etc. Each n-member sequence is a Cartesian product of two finite sets which by **Lemma 3.1** is enumerable. Therefore by **Theorem 4.1** the $\bigcup \mathbf{S}$ is enumerable. □