Boolos and Jeffrey - HW1

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1 A question about $\cap$

Proposition:

The intersection of a finite set $\mathbf{S}$ and an enumerable set $\mathbf{T}$ is enumerable.

Lemma 1.1. *Any finite set is enumerable.*

Proof. Let $\mathbf{S}$ be a finite set with n elements. Let $\mathbf{K} = \{1,2,\ldots,n\}$. Choose an element $\mathbf{s} \in \mathbf{S}$ and assign $f(n) = \mathbf{s}$. Set $\mathbf{S}'$ to $\mathbf{S}$ - $\{\mathbf{s}\}$. Choose an element $\mathbf{s}' \in \mathbf{S}'$ and assign $f(n-1) = \mathbf{s}'$. Repeat this procedure until $\mathbf{S}$ is exhausted. The resulting function $f : \mathbf{K} \to \mathbf{S}$ is an enumeration of $\mathbf{S}$. □

Theorem 1.1. *The intersection of two enumerable sets is enumerable.*

Proof. Let $f : \mathbb{N} \to \mathbf{A}$ represent a function that enumerates the first set. Let $g : \mathbb{N} \to \mathbf{B}$ represent a function that enumerates the second set. Let $h : \mathbb{N} \to \mathbf{A} \cap \mathbf{B}$ be a new function defined as follows:

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathbf{B} \\ undefined & \text{if } f(x) \notin \mathbf{B}. \end{cases}$$

□

Conclusion:

By **Lemma 1.1** the set $\mathbf{S}$ is enumerable. By **Theorem 1.1** the intersection of $\mathbf{S}$ and $\mathbf{T}$ is enumerable.

2 A slightly harder question about $\cap$

Proposition:

The intersection of an enumerable set of enumerable sets is itself enumerable.

Proof. Let **S** be a enumerable set of enumerable sets. Pick a set $\mathbf{A} \in \mathbf{S}$. Let **B** be $\bigcap(\mathbf{S} - \mathbf{A})$. By **Theorem 1.1** we can define a function $h : \mathbb{N} \to \mathbf{A} \cap \mathbf{B}$ that enumerates $\bigcap \mathbf{S}$. □

3 It takes two...

Proposition:

Let **F** be a set of *one to one* functions that both i) have a domain that's a subset of the positive integers, and ii) are *onto* a two element set {a,b}. **F** is enumerable.

Conclusion:

(work in progress)

4 Enumerate all the things!

Proposition:

The set of all finite sequences of positive integers is enumerable.

Lemma 4.1. *The Cartesian product of two finite sets is enumerable.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be two sets with a finite number n many members. The Cartesian product $\mathbf{A} \times \mathbf{B}$ has $n \cdot n$ members which is also a finite number. By **Lemma 1.1** this finite set is enumerable. $\square$

Theorem 4.1. *The union of an enumerable set of enumerable sets is itself enumerable.*

Proof. Let $\mathbf{A}$ be an enumerable set of enumerable sets. The members of $\mathbf{A}$ can be enumerated as $(a_1, a_2, a_3, \dots)$. The members of each a_i can be enumerated as $(a_{i1}, a_{i2}, a_{i3}, \dots)$. We can arrange them on a two-dimensional grid as follows:

$$\begin{matrix} a_1 & a_{11} & a_{21} & \cdots \\ a_2 & a_{12} & a_{22} & \cdots \\ a_3 & a_{13} & a_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

$\bigcup \mathbf{A}$ can now be enumerated by sweeping through the grid in a triangular fashion: $(a_1, a_{11}, a_2, a_{21}, a_{12}, a_3, \dots)$. $\square$

Theorem 4.2. *The set of all finite sequences of positive integers is enumerable.*

Proof. Let $\mathbf{S}$ be the set of all finite sequences of positive integers. $\mathbf{S}$ is the union of length-1 sequences, length-2 sequences, length-3 sequences, etc. Each length-n sequence is a Cartesian product of two finite sets which by **Lemma 4.1** is enumerable. Therefore by **Theorem 4.1** the $\bigcup \mathbf{S}$ is enumerable. $\square$