Boolos and Jeffrey - HW2

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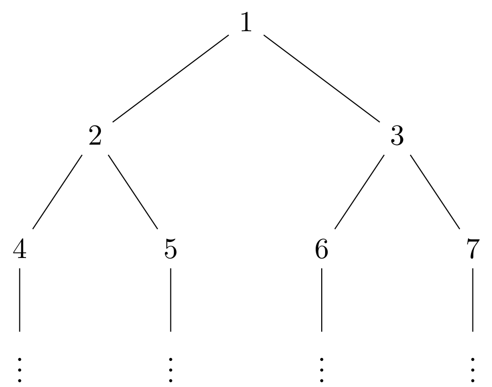
1 All nodes lead to Rome.

Theorem 1.1. *The set of nodes of an infinite binary tree is enumerable.*

Proof. Starting from the single origin node at the first level $d = 1$ the amount of nodes on each level is 2^d. The nodes can simply be counted by starting at the origin and moving left to right at each level:



$\square$

2 What a long, strange trip it's been.

Theorem 2.1. *The set of infinite paths beginning at the origin down an infinite binary tree is not enumerable.*

Proof. Let each pair of paths from a particular node be represented by 0 and 1. With this encoding each path p_n, beginning from the origin, can be represented as a binary string of 0's and 1's. We can arrange the paths in a two dimensional grid:

$$\begin{array}{c|ccccc} p_1 & 0 & 1 & 0 & 0 & \dots \\ p_2 & 1 & 0 & 0 & 0 & \dots \\ p_3 & 1 & 1 & 0 & 1 & \dots \\ p_3 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We can create a new path not contained in our representation by constructing the anti-diagonal $(1, 1, 1, 0, \dots)$. Therefore by diagonalization we have shown the paths are *not* enumerable. $\square$

3 $\mathbb{N}$ *into* $\mathbb{N}$

Theorem 3.1. *Where $\mathbb{N}$ is the set of positive integers, the set of all one-to-one, total functions from $\mathbb{N}$ into $\mathbb{N}$ is not enumerable.*

Proof. We can represent the set of all total, injective, non-surjective functions on $\mathbb{N}$ as a two dimensional grid of sequences s_n of ordered pairs:

$$\begin{array}{c|ccccc} s_1 & (1,2) & (2,3) & (3,4) & (4,5) & \dots \\ s_2 & (1,2) & (2,4) & (3,6) & (4,8) & \dots \\ s_3 & (1,1) & (2,3) & (3,5) & (4,7) & \dots \\ s_3 & (1,3) & (2,6) & (3,9) & (4,12) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We can create a sequence s not contained in our representation by constructing the anti-diagonal $< (1, 1), (2, 3), (3, 6), (5, 11), \cdots >$. Each item d_n of the anti-diagonal can be generated by the function:

$$d(n) = (n, \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even.} \end{cases})$$

Therefore by diagonalization we have shown the set of all total, injective, non-surjective functions on $\mathbb{N}$ is not enumerable. $\square$

4 $\mathbb{N}$ *onto* $\mathbb{N}$

Theorem 4.1. *Where $\mathbb{N}$ is the set of positive integers, the set of all one-to-one, total functions from $\mathbb{N}$ onto $\mathbb{N}$ is not enumerable.*

Proof. We can represent the set of all total, bijective functions on $\mathbb{N}$ as a two dimensional grid of permutations p_n :

$$\begin{array}{c|ccccc} p_1 & 1 & 2 & 3 & 4 & \dots \\ p_2 & 3 & 2 & 6 & 7 & \dots \\ p_3 & 1 & 4 & 3 & 8 & \dots \\ p_3 & 8 & 4 & 2 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We can create a permutation p not contained in our representation by constructing the anti-diagonal d_n by the formula:

$$d(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

Therefore by diagonalization we have shown the set of all total, bijective functions on $\mathbb{N}$ is not enumerable. □