Boolos and Jeffrey - HW4

David Maldonado, *david.m.maldonado@gmail.com*

October 21, 2014

In the following theorems Q stands for *either* quantifier and Q' for its counterpart. v stands for a quantified variable, and F and G for first-order logical formulas with no free variables.

1 some equivalence proofs...

Theorem 1.1. $\neg QvF \cong Q'v\neg F$

Proof. We'll begin with the first case:

$$\neg\forall vF \cong \exists v\neg F \tag{1}$$

The implication $\neg\forall vF \implies \exists v\neg F$ is proven as follows. Let $\mathcal{M}$ be a model that satisfies $\neg\forall vF$. That means there exists at least one $a \in \mathcal{M}$ that makes $\neg F$ **true**, which is precisely the statement on the right-hand side.

The converse implication $\neg\forall vF \impliedby \exists v\neg F$ is proven in the same way by assuming $\exists v\neg F$ to be satisfied by $\mathcal{M}$. It follows directly that because there is at least one $a \in \mathcal{M}$ that makes $\neg F$ **true** *not all* $a \in \mathcal{M}$ make F **true** which is the statement on the left-hand side.

For the second case:

$$\neg\exists vF \cong \forall v\neg F \tag{2}$$

The implication $\neg\exists vF \implies \forall v\neg F$ is proven by first assuming $\neg\exists vF$ is satisfied by $\mathcal{M}$. With this assumption we can say that there *does not* exist $a \in \mathcal{M}$ such that F is **true**, this leads to the right-hand statement that for all $a \in \mathcal{M}$ $\neg F$ is **true**.

The converse implication $\neg\exists vF \impliedby \forall v\neg F$ is proven by assuming $\forall v\neg F$ is satisfied by $\mathcal{M}$. Now we can see that for all $a \in \mathcal{M}$ $\neg F$ is **true**, therefore there *does not* exist $a \in \mathcal{M}$ that makes F **true**, which is the left-hand statement. $\square$

Theorem 1.2. $QvF \wedge G \cong Qv(F \wedge G)$

Proof. For the first case:

$$\forall vF \wedge G \cong \forall v(F \wedge G) \tag{1}$$

The implication $\forall vF \wedge G \implies \forall v(F \wedge G)$ is proven as follows. Let $\mathcal{M}$ represent a model that satisfies $\forall vF \wedge G$, this means $\mathcal{M}$ satisfies $\forall vF$ and $\mathcal{M}$ satisfies G. For $\forall vF$ since all $a \in \mathcal{M}$ make F **true** we can say more generally that $\mathcal{M}$ satisfies F. We can now combine F and G and say $\mathcal{M}$ satisfies $(F \wedge G)$. Finally because v does not appear free in G and all $a \in \mathcal{M}$ make F **true** $\mathcal{M}$ satisfies $\forall v(F \wedge G)$.

The converse implication $\forall vF \wedge G \impliedby \forall v(F \wedge G)$ is proven using the same argument in reverse. If we assume a model $\mathcal{M}$ that satisfies $\forall v(F \wedge G)$ we can say that all $a \in \mathcal{M}$ make $F \wedge G$ **true**. We can then say that $\mathcal{M}$ satisfies F and $\mathcal{M}$ satisfies G. (kind of got stuck here, so we have F and G seperately satisfied by $\mathcal{M}$, how do I recombine them on the other side to make $\forall vFv \wedge G$????)

$$\exists vF \wedge G \cong \exists v(F \wedge G) \tag{2}$$

In the second case we'll start with the implication $\exists vF \wedge G \implies \exists v(F \wedge G)$. We'll assume a model $\mathcal{M}$ that satisfies $\exists vF \wedge G$. This means that $\mathcal{M}$ satisfies $\exists vF$ and $\mathcal{M}$ satisfies G. (I'm assuming I'm going to use the same argument as $\forall$ but if I can only say there exists an $a \in \mathcal{M}$ that makes F **true** can I generalize and say $\mathcal{M}$ satisfies F? or do I have to think about another strategy?)

□

2 proof of prenex normal form

Theorem 2.1. *Where* ***prenex normal form*** *is a logical formula where all the quantifiers are written as a string at the front and range over the quantifier-free matrix, every formula in first-order logic has an equivalent prenex normal form.*

Proof. We will proceed by induction on the complexity of the formula. Let us first agree on the following equivalences:

$$\neg QvF \cong Q'v\neg F \tag{1}$$

$$QvF \wedge G \cong Qv(F \wedge G) \tag{2}$$

$$G \wedge QvF \cong Qv(G \wedge F) \tag{3}$$

$$QvF \vee G \cong Qv(F \vee G) \tag{4}$$

$$G \vee QvF \cong Qv(G \vee F) \tag{5}$$

$$QvF \to G \cong Q'v(F \to G) \tag{6}$$

$$G \to QvF \cong Qv(G \to F) \tag{7}$$

and the **principle of substitution of equivalents**:

$$QvF \cong QwF_v w. \tag{8}$$

Our *base case* is the logical formula with no quantified variables. This formula is automatically prenex normal form. The *inductive step* assumes that every formula F with n or fewer logical symbols is equivalent to a formula F' in prenex normal form. Now we consider a formula with $n+1$ logical symbols. This well-formed formula is one of the following:

$$\neg\phi$$

$$\phi \wedge \psi$$

$$\phi \vee \psi$$

$$\phi \to \psi$$

By repeated applications of (1) - (7) and the usage of (8) to change bound variables it is possible to convert any of these formulas into prenex normal form as we'll show below.

Case 1. For $\neg\phi$ if ϕ has no quantifiers it is in prenex normal form. If ϕ has one or more quantifiers by the induction hypothesis it is in prenex normal form $(Q_1 v \ldots Q_n vF)$. The negation can be moved to the matrix by n applications of (1) using (8) to change bound variables when needed.

Case 2. For $\phi \wedge \psi$ if ϕ and ψ both have no quantifiers $\phi \wedge \psi$ is in prenex normal form. If either (or both) have quantifiers mixed applications of (2) and (3), depending on which side the quantifier is on, and use of (8) to change bound variables when needed will yield a formula in prenex normal form.

Case 3. For $\phi \vee \psi$ the proof is the same as *Case 2* except using (4) and (5) as the main operations.

Case 4. For $\phi \to \psi$ the proof is the same as *Case 2* and *Case 3* except using (6) and (7) as the main operations.

$\square$