

Boolos and Jeffrey - HW1

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September 7, 2014

1 A question about \cap

Proposition:

The intersection of a finite set \mathbf{S} and an enumerable set \mathbf{T} is enumerable.

Lemma 1.1. *Any finite set is enumerable.*

Proof. Let \mathbf{S} be a finite set with n elements. Let $\mathbf{K} = \{1, 2, \dots, n\}$. Choose an element $\mathbf{s} \in \mathbf{S}$ and assign $f(n) = \mathbf{s}$. Set \mathbf{S}' to $\mathbf{S} - \{\mathbf{s}\}$. Choose an element $\mathbf{s}' \in \mathbf{S}'$ and assign $f(n-1) = \mathbf{s}'$. Repeat this procedure until \mathbf{S} is exhausted. The resulting function $f : \mathbf{K} \rightarrow \mathbf{S}$ is an enumeration of \mathbf{S} . \square

Lemma 1.2. *The intersection of two enumerable sets is enumerable.*

Proof. Let $f : \mathbb{N} \rightarrow \mathbf{A}$ represent a function that enumerates the first set. Let $g : \mathbb{N} \rightarrow \mathbf{B}$ represent a function that enumerates the second set. Let $h : \mathbb{N} \rightarrow \mathbf{A} \cap \mathbf{B}$ be a new function defined as follows:

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathbf{B} \\ \text{undefined} & \text{if } f(x) \notin \mathbf{B}. \end{cases}$$

\square

Conclusion:

Proof. By **Lemma 1.1** the set \mathbf{S} is enumerable. By **Lemma 1.2** the intersection of \mathbf{S} and \mathbf{T} is enumerable. \square

2 A slightly harder question about \cap

Proposition:

The intersection of an enumerable set of enumerable sets is itself enumerable.

Conclusion:

Proof. Let \mathbf{S} be a enumerable set of enumerable sets. Pick a set $\mathbf{A} \in \mathbf{S}$. Let \mathbf{B} be $\bigcap(\mathbf{S} - \mathbf{A})$. By **Lemma 1.2** we can define a function $h : \mathbb{N} \rightarrow \mathbf{A} \cap \mathbf{B}$ that enumerates $\bigcap \mathbf{S}$. \square

3 It takes two...

Proposition:

Let \mathbf{F} be the set of all *one to one* functions that both i) have a domain that's a subset of the positive integers, and ii) are *onto* a two element set $\{a,b\}$. \mathbf{F} is enumerable.

Conclusion:

Proof. We can arrange each function $f \in \mathbf{F}$ in a two dimensional grid as follows:

	1	2	3	...
1	$f(1,1)$	$f(1,2)$	$f(1,3)$...
2	$f(2,1)$	$f(2,2)$	$f(2,3)$...
3	$f(3,1)$	$f(3,2)$	$f(3,3)$...
\vdots	\vdots	\vdots	\vdots	\ddots

\mathbf{F} can now be enumerated by sweeping through the grid in a triangular fashion: $(f(1,1), f(1,2), f(2,1), f(1,3), f(2,2), f(3,1), \dots)$. \square

4 Enumerate all the things!

Proposition:

The set of all finite sequences of positive integers is enumerable.

Lemma 4.1. *For any n , the set of n -member sequences is enumerable.*

Proof. We proceed by induction. Let \mathbf{A}_n be the set of all n -member sequences of \mathbb{N} . The base case of \mathbf{A}_0 is trivially enumerable. $\mathbf{A}_0 = \emptyset$, so a sequence of length 0 is a function $f : \emptyset \rightarrow \mathbb{N}$ which is enumerable by convention. For the inductive step suppose \mathbf{A}_n is enumerable by the list $(a_1, a_2, a_3, \dots, a_n)$ where a_i is an sequence of length n . We can construct \mathbf{A}_{n+1} by appending a number $n \in \mathbb{N}$ to each $a_i \in \mathbf{A}_{n+1}$. \mathbf{A}_{n+1} is then enumerable by a list $(a_1, a_2, a_3, \dots, a_n)$ using the preceding procedure to construct each a_i . Therefore by induction for any n the set of n -member sequences is enumerable. \square

Lemma 4.2. *The union of an enumerable set of enumerable sets is itself enumerable.*

Proof. Let \mathbf{A} be an enumerable set of enumerable sets. The members $a \in \mathbf{A}$ can be enumerated as (a_1, a_2, a_3, \dots) . The members of each a_i can be enumerated as $(a_{i1}, a_{i2}, a_{i3}, \dots)$. We can arrange them on a two-dimensional grid as follows:

$$\begin{array}{cccc} a_1 & a_{11} & a_{21} & \dots \\ a_2 & a_{12} & a_{22} & \dots \\ a_3 & a_{13} & a_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

$\bigcup \mathbf{A}$ can now be enumerated by sweeping through the grid in a triangular fashion: $(a_1, a_{11}, a_2, a_{21}, a_{12}, a_3, \dots)$. \square

Conclusion:

Proof. Let \mathbf{A} be a set of sets where each member is a set containing all the n -member sequences of a particular n . Each member of \mathbf{A} is enumerable by **Lemma 4.1**. The $\bigcup \mathbf{A}$ is enumerable by **Lemma 4.2**. Let \mathbf{S} be the set of all finite sequences of positive integers. By definition $\mathbf{S} = \bigcup \mathbf{A}$, therefore \mathbf{S} is enumerable. \square