Boolos and Jeffrey - HW1

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1 A question about $\cap$

Lemma 1.1. *Any finite set is enumerable.*

Proof. Let $\mathbf{S}$ be a finite set with n elements. Let $\mathbf{K} = \{1,2,\ldots,n\}$. Choose an element $\mathbf{s} \in \mathbf{S}$ and assign $f(n) = \mathbf{s}$. Set $\mathbf{S}'$ to $\mathbf{S}$ - $\{\mathbf{s}\}$. Choose an element $\mathbf{s}' \in \mathbf{S}'$ and assign $f(n-1) = \mathbf{s}'$. Repeat this procedure until $\mathbf{S}$ is exhausted. The resulting function $f : \mathbf{K} \to \mathbf{S}$ is an enumeration of $\mathbf{S}$. □

Lemma 1.2. *The intersection of two enumerable sets is enumerable.*

Proof. Let $f : \mathbb{N} \to \mathbf{A}$ represent a function that enumerates the first set. Let $g : \mathbb{N} \to \mathbf{B}$ represent a function that enumerates the second set. Let $h : \mathbb{N} \to \mathbf{A} \cap \mathbf{B}$ be a new function defined as follows:

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathbf{B} \\ undefined & \text{if } f(x) \notin \mathbf{B}. \end{cases}$$

□

Theorem 1.1. *The intersection of a finite set* $\boldsymbol{S}$ *and an enumerable set* $\boldsymbol{T}$ *is enumerable.*

Proof. By **Lemma 1.1** the set $\mathbf{S}$ is enumerable. By **Lemma 1.2** the intersection of $\mathbf{S}$ and $\mathbf{T}$ is enumerable. □

2 A slightly harder question about $\cap$

Theorem 2.1. *The intersection of an enumerable set of enumerable sets is itself enumerable.*

Proof. Let $\mathbf{S}$ be a enumerable set of enumerable sets. Pick a set $\mathbf{A} \in \mathbf{S}$. Let $\mathbf{B}$ be $\bigcap(\mathbf{S} - \mathbf{A})$. By **Lemma 1.2** we can define a function $h : \mathbb{N} \to \mathbf{A} \cap \mathbf{B}$ that enumerates $\bigcap \mathbf{S}$. $\square$

3 It takes two...

Theorem 3.1. *Let **F** be the set of all one to one functions that both i) have a domain that's a subset of the positive integers, and ii) are onto a two element set* $\{a,b\}$. ***F** is enumerable.*

Proof. We can arrange each function $f \in \mathbf{F}$ in a two dimensional grid as follows:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & \dots \\ \hline 1 & f(1,1) & f(1,2) & f(1,3) & \dots \\ 2 & f(2,1) & f(2,2) & f(2,3) & \dots \\ 3 & f(3,1) & f(3,2) & f(3,3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

$\mathbf{F}$ can now be enumerated by sweeping through the grid in a triangular fashion: $(f(1,1), f(1,2), f(2,1), f(1,3), f(2,2), f(3,1), \dots)$. $\square$

4 Enumerate all the things!

Lemma 4.1. *For any n, the set of n-member sequences is enumerable.*

Proof. We proceed by induction. Let $\mathbf{A}_n$ be the set of all n-member sequences of $\mathbb{N}$. The base case of $\mathbf{A}_0$ is trivially enumerable. $\mathbf{A}_0 = \emptyset$, so a sequence of length 0 is a function $f : \emptyset \to \mathbb{N}$ which is enumerable by convention. For the inductive step suppose $\mathbf{A}_n$ is enumerable by the list $(a_1, a_2, a_3, \ldots, a_n)$ where a_i is an sequence of length n. We can construct $\mathbf{A}_{n+1}$ by appending a number $n \in \mathbb{N}$ to each $a_i \in \mathbf{A}_{n+1}$. $\mathbf{A}_{n+1}$ is then enumerable by a list $(a_1, a_2, a_3, \ldots, a_n)$ using the preceding procedure to construct each a_i. Therefore by induction for any n the set of n-member sequences is enumerable. $\square$

Lemma 4.2. *The union of an enumerable set of enumerable sets is itself enumerable.*

Proof. Let $\mathbf{A}$ be an enumerable set of enumerable sets. The members $a \in \mathbf{A}$ can be enumerated as $(a_1, a_2, a_3, \ldots)$. The members of each a_i can be enumerated as $(a_{i1}, a_{i2}, a_{i3}, \ldots)$. We can arrange them on a two-dimensional grid as follows:

$$\begin{array}{cccc} a_1 & a_{11} & a_{21} & \cdots \\ a_2 & a_{12} & a_{22} & \cdots \\ a_3 & a_{13} & a_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

$\bigcup \mathbf{A}$ can now be enumerated by sweeping through the grid in a triangular fashion: $(a_1, a_{11}, a_2, a_{21}, a_{12}, a_3, \ldots)$. $\square$

Theorem 4.1. *The set of all finite sequences of positive integers is enumerable.*

Proof. Let $\mathbf{A}$ be a set of sets where each member is a set containing all the n-member sequences of a particular n. Each member of $\mathbf{A}$ is enumerable by **Lemma 4.1**. The $\bigcup \mathbf{A}$ is enumerable by **Lemma 4.2**. Let $\mathbf{S}$ be the set of all finite sequences of positive integers. By definition $\mathbf{S} = \bigcup \mathbf{A}$, therefore $\mathbf{S}$ is enumerable. $\square$