

Solution: Sonny Bayes Robs a Train

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1 A Bayesian Approach

I know of no way of solving this problem with using a Bayesian approach. It demonstrates most starkly the need to introduce a reasonable prior distribution. Let's introduce two variables that define the uncertainty in the problem. Let x , a continuous value, denote the unknown amount of gold initially loaded into each of the trains. We'll make it an integer in dollars just so we can use sums rather than integrals. Therefore one of the running trains has x amount of gold and the other has $2x$. Let us also define another parameter L , a Boolean defining their Luck. L is True if their random guess resulted in them arriving first at the train with more gold and False otherwise. Initially these are independent variables but they become perfectly correlated once some data come in as we shall see. Finally let D denote the data which in this case is just the fact that they have discovered about 10 million in gold on one of the trains.

2 The General Solution

We will proceed, as always, by writing down the probability that we desire and apply Bayes theorem to get something that we can calculate numerically. We are interested in the posterior probability of L given the new data. This is

$$P(L|D) = \sum_x P(L, x|D) \tag{1}$$

This is a common pattern in Bayesian statistics. Though we only care about the marginal distribution $P(L|D)$, we need to demarginalize it in order to be able to do the calculation. Now we can apply Bayes theorem to $P(L, x|D)$

$$P(L, x|D) = P(D|L, x)P(L, x)/P(D) \quad (2)$$

$$\propto P(D|L, x)P(L|x)P(x) \quad (3)$$

As is common, we don't care about the normalization constant $P(D)$ so we throw it away and change it from an equation to a proportionality statement. If that is confusing, you can keep it. You will see that it will drop out anyway.

We can put this into the equation above and now we have

$$P(L|D) \propto \sum_x P(D|L, x)P(L|x)P(x) \quad (4)$$

This is now in a form that we can work with. That is, we can logically assign numbers to each of these three terms in the sum. Let's start with $P(D|L, x)$. This is called the likelihood of the data. More specifically, we can call it the *complete* likelihood because we are including both variables. What is the value of this term? It's usually helpful to spell these things out in words. This is: What is the probability of observing $D=10$ (or any amount) given that we know L and also know x . For example, if $x = 5$ million and $L = \text{True}$, the probability is 1 because we were lucky to choose the best train and 2 times $x = 5$ is 10. For any other value of x it will be zero. If instead we know $L = \text{False}$, this will be zero for every value except when $x = 10$.

What about $P(L|x)$. We chose the first train at random and so it cannot depend on x . This implies that $P(L|x) = 0.5$ for both values of L . It's just another constant factor that will drop out.

Finally, there is $P(x)$. This is the prior distribution on x . It embodies our expectation of the value of x before receiving the new data that there is 10 million worth of gold on one of the trains. Often, it is the case that the prior is not that important and in fact it gets less important as you add new data. In this case, the only other piece of data would be the amount of gold on the other train. If we knew that we wouldn't care about the prior or even bother thinking about this in a Bayesian fashion. But in this problem, with only one data point, it turns out that the prior is not just important; it determines the problem entirely.

Let's get back to our equation and calculate $P(L|D)$ for each of the possible values of L which are True and False not that we know that $P(L|x)$ is a

negligible constant and $P(L, x|D)$ acts to collapse the sum down to one single term (depending on L).

We can put this into the equation above and now we have

$$P(L = True|D) \propto P(x = 5) \quad (5)$$

and

$$P(L = False|D) \propto P(x = 10) \quad (6)$$

Now we know (assuming we trust our informant) that $P(L = True|D) + P(L = False|D) = 1$ because, regardless of the data, we know that L is either True or False. This is a common trick. We wait until the end to normalize our posteriors rather than carrying around a lot of proportionality factors that will just be absorbed into this constant anyway. The properly normalized $P(L|D)$, for either L , is clearly.

$$P(L|D) = \frac{P(L|D)}{P(L = True|D) + P(L = False|D)} \quad (7)$$

So for example,

$$P(L = True|D) = \frac{1}{1 + \frac{P(L=False|D)}{P(L=True|D)}} \quad (8)$$

The last step in the formal solution is to plug in the value of this ratio. Again notice how any common constant factors drop out.

$$P(L = True|D) = \frac{1}{1 + \frac{P(x=10)}{P(x=5)}} \quad (9)$$

As mentioned before, the answer of whether $P(L = True|D)$ (we should take the gold) is greater than $P(L = False|D)$ (we should rob the other train) depends entirely on our prior distribution evaluated at our data point and half that value.

3 Uniform Prior

For example, if we use a uniform prior, i.e. one that is the same for all x , $P(L = False|D) = 0.5$ and so there is no incentive to drop the gold and rob the other train though if they did, they should expect the same amount

of gold. More precisely, the expectation value of gold is the same whether or not they switch. But if they take the gold, they know exactly how much they get. There is zero variance. If they switch, the expectation value is the same but the variance is non zero. If like most people you decide to not accept risk (uncertainly) with no expected reward, you'll take the gold and go. If you are the gambling sort who likes risk even with no expected reward you might want to try your luck with the other train. Given that guns are out, you are probably better ending the job here.

This is the general solution for any prior but in this problem we do have real prior information. There is no reason to assume a uniform prior. The train robbers have each seen three trains and counted the gold on each. Thus, they can construct a useful prior and make a more educated guess.

So if the train robbers observations of these 9 trains has yielded the following list of approximate amounts {14 14 7 13 10 15 7 8 12} you can build some kind of distribution that you can evaluate at $x = 5$ and $x = 10$. Note that the mean of these is 11.1 with a standard deviation of about 3. From that alone we can decide what to do. Both $x = 5$ and $x = 10$ are lower than the mean and $x = 5$ would be the lowest of 10 sampled points. It seems that $x = 10$ has a higher prior probability and so we should indeed drop the gold and rob the other train assuming the risk of failure in robbing that other train is assumed minimal.

If we assume some risk of failure in robbing the second train, we might want to try to calculate this more accurately. One way of doing this is to assume some distribution.

4 Normal distribution prior

Let's try the normal distribution. The sample mean 11.1 is also the maximum likelihood estimator for the normal distribution mean and the $N/(N-1)s^2$ is the ML estimator for the distribution variance with $N=9$ which is still about 3. The prior is

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-1/2((x - \mu)/\sigma)^2 \right] \quad (10)$$

Now look at Equation 11. When $L = True$ we know $x = 5 = D/2$. When $L = False$ we know $x = 10 = D$. Let's leave this in terms of D to get a formula for the general solution. Pluggin this in, we can write Equation 11

after a little algebra as

$$P(L = True|D) = \frac{1}{1 + \exp[-D(3D/4 - \mu)/(2\sigma^2)]} \quad (11)$$

From this we can see right away what we should do, assuming no risk of failing to rob the second train. The decision depends on the sign of $3/4D - \mu$ because this determines whether the exponential is greater than 1 which determines whether $P(L = True|D) > 0.5$. For $\mu = 11.1$, the crossing point is $D = 14.8$. If D were greater than this, we should keep the gold and not rob the second train.

5 The Professional Solution

If these train robbers were true professionals and considered real risks of being successful, they would perform the following calculation for the cut-off value for the gold on the first train. If the amount was clearly less than the cut-off value, they could leave immediately and rob the other train. If the opposite were true, they would just pack it up. If it was unclear, they could count it and then decide.

5.1 utility function

5.2 factoring in risk

6 Conclusion: What we have learned