RD+Mechanics Formulation

Reaction Diffusion

$$\frac{\partial \phi_T}{\partial t} = \nabla \cdot D_T \nabla \phi_T + \alpha_T \phi_T (1 - \phi_T)$$

Where:

$$\phi_T = \text{Tumor volume fraction or number of tumor cells}$$

And parameters D_T and α_T can be functions of a deformation measure:

$$D_T = \overline{D}_T \exp(-\gamma_T^{Pa} J_T)$$
$$\alpha_T = \overline{\alpha}_T \exp(-\gamma_T^{mob} J_T)$$

Elastic Deformation

Let $\boldsymbol{x} = \chi(\boldsymbol{X}, t)$ be the current (deformed) configuration, a function of the reference (undeformed) configuration $\boldsymbol{X}$ and time. The deformation gradient is:

$$\boldsymbol{F} = \frac{\partial \chi}{\partial \boldsymbol{X}}$$

And it can be decomposed into:

$$\boldsymbol{F} = \boldsymbol{F}^S \boldsymbol{F}^G$$

Where $\boldsymbol{F}^G$ is the right stretch tensor,

$$\boldsymbol{F}^G = \lambda^G \boldsymbol{I} = (\beta \phi_T + 1)\boldsymbol{I}$$

Small deformations

When $||\nabla \boldsymbol{u}|| << 1$, the formulation can be rewritten in terms of infinitesimal strain,

$$\boldsymbol{E}_T = \frac{1}{2}(\nabla \boldsymbol{u}_T + \nabla \boldsymbol{u}_T^T)$$

This can be decomposed:

$$\boldsymbol{E}_T = \boldsymbol{E}_T^S + \boldsymbol{E}_T^G$$

$\boldsymbol{F}^G = \lambda^G \boldsymbol{I}$ is equivalent to the following growth strain:

$$\begin{aligned} \boldsymbol{E}_T^G &= \frac{1}{2}((\boldsymbol{F}_T^G)^T + \boldsymbol{F}_T^G) - \boldsymbol{I} \\ &= \frac{1}{2}(\lambda^G \boldsymbol{I} + \lambda^G \boldsymbol{I}) - \boldsymbol{I} \\ &= (\lambda^G - 1)\boldsymbol{I} \\ \boldsymbol{E}_T^G &= \beta \phi_T \boldsymbol{I} \end{aligned}$$

The Cauchy stress tensor is:

$$\boldsymbol{T}_T = \boldsymbol{C}_T(\boldsymbol{E}_T - \boldsymbol{E}_T^G) = \boldsymbol{C}_T\boldsymbol{E}_T^S$$

Where $\boldsymbol{C}_T$ is the 4th order elasticity tensor. Under linear and isotropic assumptions:

$$\begin{aligned}\boldsymbol{T}_T &= 2G\boldsymbol{E}^S + \frac{2G}{1-2\nu}(\mathrm{tr}\boldsymbol{E}^S)\boldsymbol{I} \\ &= 2G(\boldsymbol{E} - \beta\phi_T\boldsymbol{I}) + \frac{2G}{1-2\nu}\boldsymbol{I}(\mathrm{tr}\boldsymbol{E} - \mathrm{tr}\boldsymbol{E}^G)\end{aligned}$$

Where $\mathrm{tr}\boldsymbol{E}^G = 3\beta\phi_T$. Thus, for linear elasticity:

$$\begin{aligned}\nabla \cdot \boldsymbol{T}_T &= 0 \\ \boldsymbol{T}_T &= 2G(\boldsymbol{E} - \beta_T\phi_T\boldsymbol{I}) + \frac{2G}{1-2\nu}(\mathrm{tr}\boldsymbol{E} - 3\beta_T\phi_T)\boldsymbol{I}\end{aligned}$$

Large Deformations: Hyperelasticity

Compressible Neo-Hookean strain energy:

$$\begin{aligned}W &= \frac{G}{2}(I_{C_1}^S - 3) + \frac{K}{2}(J^S - 1)^2 \\ I_{C_1}^S &= \mathrm{tr}(\boldsymbol{C}^S) \\ J^S &= \sqrt{\det(\boldsymbol{C}^S)} = \det(\boldsymbol{F}^S)\end{aligned}$$

1st PK stress:

$$\begin{aligned}\boldsymbol{T} &= \frac{\partial W}{\partial \boldsymbol{F}} = \frac{\partial W}{\partial \boldsymbol{F}^S}\frac{\partial \boldsymbol{F}^S}{\partial \boldsymbol{F}} \\ \frac{\partial W}{\partial \boldsymbol{F}^S} &= \frac{G}{(J^S)^{5/3}}(\boldsymbol{B}^S - \frac{1}{3}\mathrm{tr}(\boldsymbol{B}^S)\boldsymbol{I}) + K(J^S - 1)\boldsymbol{I} \\ \frac{\partial \boldsymbol{F}^S}{\partial \boldsymbol{F}} &= \boldsymbol{I} \otimes (\boldsymbol{F}^G)^{-1} = \frac{1}{\lambda^G}\end{aligned}$$

For nonlinear (Neo-Hookean Hyperelasticity):

$$\begin{aligned}\nabla \cdot \boldsymbol{T}_T &= 0 \\ \boldsymbol{T}_T &= \frac{1}{1+\beta\phi_T}\left[\frac{G}{(J^S)^{5/3}}(\boldsymbol{B}^S - \frac{1}{3}\mathrm{tr}(\boldsymbol{B}^S\boldsymbol{I}) + K(J^S - 1)\boldsymbol{I}\right]\end{aligned}$$