J. Math. Anal. Appl. 421 (2015) 697–717

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The well-posedness of a nonlocal hyperbolic model for type-I superconductors

K. Van Bockstal *, M. Slodička

Research Group NaM[2], Department of Mathematical Analysis, Ghent University, Galglaan 2, 9000 Ghent, Belgium

ARTICLE INFO

Article history:
Received 19 June 2013
Available online 25 July 2014
Submitted by Goong Chen

Keywords:
Hyperbolic equation
Integro-partial differential equation
Nonlocal superconductors
Singular convolution kernel
Time discretization
Error estimates

ABSTRACT

A vectorial nonlocal linear hyperbolic problem with applications in superconductors of type-I is studied. The nonlocal term is represented by a (space) convolution with a singular kernel, which is arising in Eringen's model. The well-posedness of the problem is discussed under low regularity assumptions and the error estimates for two time-discrete schemes (based on backward Euler approximation) are established.

1. Introduction

Industrial applications require macroscopic models and their mathematical analysis for superconductivity. In their phenomenological theory of superconductivity in 1935, London and London explained that a macroscopic description of type-I superconductors involves a two-fluid model [7,2]. Namely, the current density $\boldsymbol{J}$ is supposed to be the sum of a normal ($\boldsymbol{J}_n$) and a superconducting part ($\boldsymbol{J}_s$). In this contribution, a superconductive material of type-I occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$. The symbol $\boldsymbol{\nu}$ denotes the outward unit normal vector on $\partial\Omega$. The full Maxwell's equations ($\tilde{\delta} = 1$) and quasi-static Maxwell's equations ($\tilde{\delta} = 0$) for linear materials are considered. Thus, a linear dependence of the magnetic induction $\boldsymbol{B}$ and the electric displacement field $\boldsymbol{D}$ on respectively the magnetic field $\boldsymbol{H}$ and the electric field $\boldsymbol{E}$ is assumed, namely

$$\boldsymbol{B} = \mu \boldsymbol{H} \quad \text{and} \quad \boldsymbol{D} = \epsilon \boldsymbol{E}, \tag{1}$$

* Corresponding author.
E-mail addresses: Karel.VanBockstal@UGent.be (K. Van Bockstal), Marian.Slodicka@UGent.be (M. Slodička).
URLs: http://cage.UGent.be/~kvb/ (K. Van Bockstal), http://cage.UGent.be/~ms/ (M. Slodička).

http://dx.doi.org/10.1016/j.jmaa.2014.07.039
0022-247X/

where the constant $\mu > 0$ stands for the magnetic permeability and the constant $\epsilon > 0$ for the electric permittivity. In agreement with our previous notations, the quasi-static and full Maxwell's equations can be combined as

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \tilde{\delta}\partial_t \boldsymbol{D} = \boldsymbol{J}_n + \boldsymbol{J}_s + \tilde{\delta}\epsilon\partial_t \boldsymbol{E}, \quad \text{Ampère's law} \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} = -\mu\partial_t \boldsymbol{H}. \quad \text{Faraday's law} \tag{3}$$

Applying the divergence operator to Faraday's law (3) and integrating in time gives

$$\nabla \cdot \boldsymbol{H}(t) = \nabla \cdot \boldsymbol{H}(t = 0).$$

Therefore, assuming $\nabla \cdot \boldsymbol{H}(t = 0) = 0$, it is ensured that the magnetic field remains divergence free for any time. The normal density current $\boldsymbol{J}_n$ is required to satisfy Ohm's law $\boldsymbol{J}_n = \sigma\boldsymbol{E}$, $\sigma > 0$ being the conductivity of the normal electrons. For the superconductive part of the current $\boldsymbol{J}_s$, the nonlocal representation of the superconductive current by Eringen is considered [1]. This representation identifies the state of the superconductor, at time t, with the field $\boldsymbol{H}(\cdot, t)$ and is given by the linear functional

$$\boldsymbol{J}_s(\boldsymbol{x}, t) = \int_\Omega \sigma_0\big(|\boldsymbol{x} - \boldsymbol{x}'|\big)\big(\boldsymbol{x} - \boldsymbol{x}'\big) \times \boldsymbol{H}\big(\boldsymbol{x}', t\big)\,\mathrm{d}\boldsymbol{x}' =: -(\mathcal{K}_0 \star \boldsymbol{H})(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T),$$

where the singular kernel $\sigma_0 : (0, \infty) \to \mathbb{R}$ is defined by

$$\sigma_0(s) = \begin{cases} \frac{\widetilde{C}}{2s^2}\exp(-\frac{s}{r_0}) & s < r_0; \\ 0 & s \geqslant r_0, \end{cases}$$

with $\widetilde{C} := \frac{3}{4\pi\xi_0\Lambda} > 0$. The length ξ_0 is called the coherence length of the material and $\Lambda := \frac{m_e}{n_s e^2}$, with n_s the number of superelectrons per unit volume, m_e and $-e$ the mass and the electric charge of an electron respectively. The points which contribute to the integral are separated by distances of order r_0 or less, where r_0 is defined by

$$r_0 = \frac{\xi_0 l}{\xi_0 + l},$$

with l the mean free path of the electrons in the material. Taking the curl of (2) and the time derivative of (3) results into the following parabolic ($\tilde{\delta} = 0$) and hyperbolic ($\tilde{\delta} = 1$) integro-differential equation

$$\tilde{\delta}\epsilon\mu\partial_{tt}\boldsymbol{H} + \sigma\mu\partial_t\boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} + \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{0}. \tag{4}$$

The well-posedness of the nonlocal parabolic model ($\tilde{\delta} = 0$ in (4)) is studied in detail in [9,12]. Also the error estimates for two time-discrete schemes based on backward Euler method are derived in [12]. In the first scheme, the convolution is taken implicitly (from the actual time step). In the second one, the convolution is taken explicitly (from the previous time step). This second scheme is considered, because it is easier to implement than the first scheme and it gives the same order of convergence. For both schemes, the error estimates for the time discretization have been obtained using a priori estimates, which were based on Grönwall's argument. The convergence rates are of order $\mathcal{O}(\tau) = e^{CT}\tau$ in the space $C([0, T], \mathbf{L}^2(\Omega)) \cap L_2((0, T), \mathbf{H}(\mathbf{curl}, \Omega))$ under appropriate conditions, where τ is the discretization parameter. To get rid of the exponential (in time) character of this constant, the use of Grönwall's lemma should be avoided. For this reason, a convolution kernel $\mathcal{K}$ is derived in [12, Lemma 3], more specific

$$\nabla \times \boldsymbol{J}_s(\boldsymbol{x}, t) = -\int_{\Omega} \mathcal{K}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{H}(\boldsymbol{x}', t)\,\mathrm{d}\boldsymbol{x}' =: -(\mathcal{K} \star \boldsymbol{H})(\boldsymbol{x}, t)$$

when $\boldsymbol{H}$ is divergence free and $\boldsymbol{H} \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$ (see also [2, §11.7] and [1]), where the kernel $\mathcal{K}$ is defined by

$$\mathcal{K} : \Omega \times \Omega \to \mathbb{R} : (\boldsymbol{x}, \boldsymbol{x}') \mapsto \kappa(|\boldsymbol{x} - \boldsymbol{x}'|),$$

with

$$\kappa : (0, \infty) \to \mathbb{R} : s \mapsto \begin{cases} \frac{\tilde{C}}{2s^2}(1 - \frac{s}{r_0}) \exp(-\frac{s}{r_0}) & s < r_0; \\ 0 & s \geqslant r_0. \end{cases}$$

This leads to a different model. Indeed, using the vector identity

$$-\Delta \boldsymbol{H} = \nabla \times (\nabla \times \boldsymbol{H}) - \nabla(\nabla \cdot \boldsymbol{H}),$$

Eq. (4) can be rewritten as

$$\tilde{\delta}\epsilon\mu\partial_{tt}\boldsymbol{H} + \sigma\mu\partial_t\boldsymbol{H} - \Delta\boldsymbol{H} + \mathcal{K} \star \boldsymbol{H} = \boldsymbol{0}. \tag{5}$$

One major advantage of this model is the positive definiteness of the kernel $\mathcal{K}$ [12, Lemma 5]. Using this property, it is possible to avoid the use of Grönwall's lemma in the case that $\tilde{\delta} = 0$. This leads to a convergence rate of order $\mathcal{O}(\tau) = C\tau$ for both schemes. For more details, the reader is referred to [12, §6]. Note that a fully discrete approximation scheme in the case that $\tilde{\delta} = 0$ is proposed in [14].

In this paper, the main focus is on the case $\tilde{\delta} = 1$. The analysis follows the same lines as [12]. Recent engineering applications can be found in [13,4,15,5]. Section 2 summarizes the mathematical tools. Problem (4) for $\tilde{\delta} = 1$ is presented in detail in Section 3 and the well-posedness of the problem is shown in Section 4. A time-discrete numerical scheme is developed. The existence of a weak solution for each time step is shown. Also the convergence of the method is discussed and error estimates are derived. A modified scheme is considered in Section 5. In Section 6 model (5) is shortly studied for $\tilde{\delta} = 1$. Now, the use of Grönwall's lemma with exponential in time character of the constant cannot be avoided and no better error estimates for the time discretization can be obtained.

2. Functional setting

First, some standard notations are introduced. The Euclidean norm of a vector $\boldsymbol{v}$ in $\mathbb{R}^3$ is expressed by $|\boldsymbol{v}|$. The Lebesgue spaces of vector-valued functions with componentwise p-th power integrable functions are denoted by $\mathbf{L}^p(\Omega)$ with the usual norm $\|\cdot\|_p$. For instance, in the special case $p = 2$, the $\mathbf{L}^2(\Omega)$ scalar product is denoted by $(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$ and the corresponding norm is $\|\boldsymbol{v}\| = \sqrt{(\boldsymbol{v}, \boldsymbol{v})}$. The following spaces are used in our analysis: $\mathbf{H}^1(\Omega)$, $\mathbf{H}^2(\Omega)$, $\mathbf{H}(\mathbf{curl}, \Omega)$ and the fractional Sobolev spaces $\mathbf{H}^s(\Omega)$ – see [8]. The Hilbert space $\mathbf{H}^1(\Omega)$ is endowed with the norm

$$\|\boldsymbol{\varphi}\|^2_{\mathbf{H}^1(\Omega)} = \|\boldsymbol{\varphi}\|^2 + \|\nabla\boldsymbol{\varphi}\|^2.$$

The norm in the Hilbert space $\mathbf{H}^2(\Omega)$ is $\|\boldsymbol{\varphi}\|^2_{\mathbf{H}^2(\Omega)} = \|\boldsymbol{\varphi}\|^2_{\mathbf{H}^1(\Omega)} + \|\Delta\boldsymbol{\varphi}\|^2$. The space $\mathbf{H}(\mathbf{curl}, \Omega)$ is a Banach space with respect to the graph norm

$$\|\boldsymbol{\varphi}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} = \|\boldsymbol{\varphi}\|^2 + \|\nabla \times \boldsymbol{\varphi}\|^2.$$

Further, the spaces of test functions will be $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}_0(\mathbf{curl},\Omega)$ depending on the problem under consideration. They inherit the norms $\|\boldsymbol{\varphi}\|_{\mathbf{H}^1(\Omega)}$ and $\|\boldsymbol{\varphi}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$. Their dual spaces are denoted by $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^{-1}(\mathbf{curl},\Omega)$ respectively. The following Friedrichs inequality holds true for every $\boldsymbol{\varphi}\in\mathbf{H}_0^1(\Omega)$

$$\|\boldsymbol{\varphi}\|^2_{\mathbf{H}_0^1(\Omega)} \leqslant C\|\nabla\boldsymbol{\varphi}\|. \tag{6}$$

The space of Lipschitz continuous functions $\boldsymbol{f}:[0,T]\to\mathbf{L}^2(\Omega)$ is denoted by $\mathrm{Lip}([0,T],\mathbf{L}^2(\Omega))$. Consider an abstract Banach space X with norm $\|\cdot\|_X$. The spaces $L^p((0,T),X)$ and $C([0,T],X)$ consist of functions $u:[0,T]\to X$ satisfying

$$\|u\|_{L^p((0,T),X)} = \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty,$$
$$\|u\|_{C([0,T],X)} = \max_{[0,T]}\|u(t)\|_X < \infty.$$

To reduce the number of arbitrary constants, the notation $a \lesssim b$ is used if there exists a positive constant C such that $a \leqslant Cb$. Moreover, the values C, ε and C_ε are generic and positive constants independent of the discretization parameter τ. The value ε is small and $C_\varepsilon \lesssim 1+\varepsilon^{-1}$. Finally, some useful (in)equalities are stated, which can easily be derived:

$$2\sum_{i=1}^n (a_i-a_{i-1})a_i = a_n^2 - a_0^2 + \sum_{i=1}^n (a_i-a_{i-1})^2, \quad a_i\in\mathbb{R}; \quad \text{Abel's summation rule}$$

and

$$ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon}b^2 = \varepsilon a^2 + C_\varepsilon b^2, \quad a,b\in\mathbb{R},\ \varepsilon>0. \quad \text{Young's inequality}$$

2.1. Important estimates

In this section, some useful estimates on the singular kernels and the related convolutions appearing in (4) and (5) are mentioned. They are of crucial importance for the calculations. Using spherical coordinates one can deduce that

- $\sigma_0(|\boldsymbol{x}|)\boldsymbol{x}$ belongs to $\mathbf{L}^p(\Omega)$ for $1\leqslant p<3$;
- $\mathcal{K}(\boldsymbol{x},\cdot)\in L_p(\Omega)$ if $1\leqslant p<\frac{3}{2}$, $\forall\boldsymbol{x}\in\Omega$.

Moreover, the following inequalities can be derived

$$\left|\boldsymbol{J}_s(\boldsymbol{x},t)\right| = \left|(\mathcal{K}_0\star\boldsymbol{H})(\boldsymbol{x},t)\right| \leqslant C(q)\left\|\boldsymbol{H}(t)\right\|_q, \quad q>\frac{3}{2},\ \forall\boldsymbol{x}\in\Omega; \tag{7}$$

and

$$\left|(\mathcal{K}\star\boldsymbol{H})(\boldsymbol{x},t)\right| = \left|\int_\Omega \mathcal{K}(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{H}(\boldsymbol{x}',t)\,\mathrm{d}\boldsymbol{x}'\right| \leqslant C(q)\left\|\boldsymbol{H}(t)\right\|_q, \quad \forall q>3,\ \forall\boldsymbol{x}\in\Omega. \tag{8}$$

Therefore, using Young's inequality it is true that

$$(\mathcal{K}_0\star\boldsymbol{h}_1,\nabla\times\boldsymbol{h}_2) \overset{(7)}{\leqslant} C_\varepsilon\|\boldsymbol{h}_1\|^2 + \varepsilon\|\nabla\times\boldsymbol{h}_2\|^2, \quad \forall\boldsymbol{h}_1\in\mathbf{L}^2(\Omega),\ \boldsymbol{h}_2\in\mathbf{H}_0(\mathbf{curl},\Omega). \tag{9}$$

The values ε and C_ε in the right-hand side (RHS) of this inequality can be switched. Due to the Sobolev embeddings theorem in $\mathbb{R}^3$ holds that $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$ [8, Theorem 3.6]. Employing this, together with the positive definiteness of $\mathcal{K}$ and the Friedrichs inequality (6), gives for all $\boldsymbol{h}_1 \in \mathbf{H}_0^1(\Omega)$ and $\boldsymbol{h}_2 \in \mathbf{L}^2(\Omega)$ that

$$(\mathcal{K} \star \boldsymbol{h}_1, \boldsymbol{h}_2) \overset{(8)}{\leqslant} C_\varepsilon \|\boldsymbol{h}_1\|^2_{\mathbf{H}^1(\Omega)} + \varepsilon\|\boldsymbol{h}_2\|^2 \leqslant C_\varepsilon \|\nabla \boldsymbol{h}_1\|^2 + \varepsilon\|\boldsymbol{h}_2\|^2, \tag{10}$$

and

$$(\mathcal{K} \star \boldsymbol{h}_1, \boldsymbol{h}_1) \geqslant 0.$$

3. Hyperbolic nonlocal problem for superconductivity

It is assumed without loss of generality that $\tilde{\delta} = \epsilon = \mu = \sigma = 1$ in (4). Also a possible source term $\boldsymbol{f}$ is considered in the RHS. The aim of this paper is to address the well-posedness of the following problem

$$\begin{cases} \partial_{tt}\boldsymbol{H} + \partial_t \boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} + \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}) = \boldsymbol{f} & \text{in } Q_T := \Omega \times (0,T); \\ \boldsymbol{H} \times \boldsymbol{\nu} = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T); \\ \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0 & \text{in } \Omega; \\ \partial_t \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0' & \text{in } \Omega; \end{cases} \tag{11}$$

to design a numerical scheme for computations and to derive error estimates for the time discretization. The variational formulation of (11) is

$$(\partial_{tt}\boldsymbol{H}, \boldsymbol{\varphi}) + (\partial_t \boldsymbol{H}, \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\varphi}) + (\mathcal{K}_0 \star \boldsymbol{H}, \nabla \times \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}), \tag{12}$$

for all $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$.

Theorem 1 *(Uniqueness). The problem (11) admits at most one solution $\boldsymbol{H} \in C([0,T], \mathbf{H}_0(\mathbf{curl}, \Omega))$ such that $\partial_t \boldsymbol{H} \in C([0,T], \mathbf{L}^2(\Omega))$.*

Proof. Assume that we have two different solutions $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. Then $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2$ fulfills (12) with $\boldsymbol{H}_0' = \boldsymbol{H}_0 = \boldsymbol{f} = \boldsymbol{0}$. We set $\boldsymbol{\varphi} = \partial_t \boldsymbol{H}(t)$ and integrate over the time variable $t \in (0,\eta) \subset (0,T)$ to get

$$\frac{1}{2}\|\partial_t \boldsymbol{H}(\eta)\|^2 + \int_0^\eta \|\partial_t \boldsymbol{H}\|^2 + \frac{1}{2}\|\nabla \times \boldsymbol{H}(\eta)\|^2 = -\int_0^\eta (\mathcal{K}_0 \star \boldsymbol{H}, \nabla \times \partial_t \boldsymbol{H}).$$

For the term in the RHS, we obtain, using the integration by parts formula and (9), that

$$\begin{aligned} \int_0^\eta (\mathcal{K}_0 \star \boldsymbol{H}, \nabla \times \partial_t \boldsymbol{H}) &= (\mathcal{K}_0 \star \boldsymbol{H}, \nabla \times \boldsymbol{H})|_0^\eta - \int_0^\eta (\mathcal{K}_0 \star \partial_t \boldsymbol{H}, \nabla \times \boldsymbol{H}) \\ &\overset{(9)}{\leqslant} \varepsilon\|\nabla \times \boldsymbol{H}(\eta)\|^2 + C_\varepsilon\|\boldsymbol{H}(\eta)\|^2 + C\int_0^\eta \|\partial_t \boldsymbol{H}\|^2 + C\int_0^\eta \|\nabla \times \boldsymbol{H}\|^2 \\ &\leqslant \varepsilon\|\nabla \times \boldsymbol{H}(\eta)\|^2 + C_\varepsilon \int_0^\eta \|\partial_t \boldsymbol{H}\|^2 + C\int_0^\eta \|\nabla \times \boldsymbol{H}\|^2. \end{aligned}$$

In the last step, we have used that $\boldsymbol{H}(\eta) = \int_0^\eta \partial_t \boldsymbol{H}$ because $\boldsymbol{H}_0 = \mathbf{0}$. Using the estimate, we arrive at

$$\frac{1}{2}\left\|\partial_t \boldsymbol{H}(\eta)\right\|^2 + \int_0^\eta \|\partial_t \boldsymbol{H}\|^2 + \left(\frac{1}{2} - \varepsilon\right)\left\|\nabla \times \boldsymbol{H}(\eta)\right\|^2 \leqslant C_\varepsilon \int_0^\eta \|\partial_t \boldsymbol{H}\|^2 + C \int_0^\eta \|\nabla \times \boldsymbol{H}\|^2.$$

Fixing a sufficiently small positive ε and applying Grönwall's argument, we get that $\partial_t \boldsymbol{H} = \mathbf{0}$ a.e. in Q_T. Therefore, due to $\boldsymbol{H}_0 = \mathbf{0}$, we have that $\boldsymbol{H} = \mathbf{0}$ a.e. in Q_T. Thus, $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are identical. $\square$

4. Existence of a solution

To address the existence of a solution to (11), the semidiscretization in time is employed. This discretization is based on Rothe's method [3]. The interval $[0, T]$ is divided into n equidistant subintervals $[t_{i-1}, t_i]$ with time step $\tau = \frac{T}{n} < 1$, thus $t_i = i\tau$, $i = 1, \ldots, n$. With the standard notation for the discretized fields

$$\boldsymbol{h}_i = \boldsymbol{H}(t_i), \qquad \delta \boldsymbol{h}_i = \frac{\boldsymbol{h}_i - \boldsymbol{h}_{i-1}}{\tau}, \qquad \delta^2 \boldsymbol{h}_i = \frac{\delta \boldsymbol{h}_i - \delta \boldsymbol{h}_{i-1}}{\tau} = \frac{\boldsymbol{h}_i}{\tau^2} - \frac{\boldsymbol{h}_{i-1}}{\tau^2} - \frac{\delta \boldsymbol{h}_{i-1}}{\tau},$$

the following linear recurrent scheme is proposed to approximate the original problem

$$\begin{cases} (\delta^2 \boldsymbol{h}_i, \boldsymbol{\varphi}) + (\delta \boldsymbol{h}_i, \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) + (\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) = (\boldsymbol{f}_i, \boldsymbol{\varphi}); \\ \boldsymbol{h}_0 = \boldsymbol{H}_0 \end{cases} \tag{13}$$

for all $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, which is equivalent to

$$\begin{aligned} a(\boldsymbol{h}_i, \boldsymbol{\varphi}) &:= \left(\frac{1}{\tau^2} + \frac{1}{\tau}\right)(\boldsymbol{h}_i, \boldsymbol{\varphi}) + (\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) + (\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) \\ &= (\boldsymbol{f}_i, \boldsymbol{\varphi}) + \left(\frac{1}{\tau^2} + \frac{1}{\tau}\right)(\boldsymbol{h}_{i-1}, \boldsymbol{\varphi}) + \left(\frac{\delta \boldsymbol{h}_{i-1}}{\tau}, \boldsymbol{\varphi}\right) =: f_i(\boldsymbol{\varphi}). \end{aligned}$$

Theorem 2. *Suppose that $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$. Then the variational problem (13) admits a unique solution $\boldsymbol{h}_i \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ for $i = 1, \ldots, n$ if $\tau < \tau_0$.*

Proof. The bilinear form a is elliptic for $\tau < \tau_0$:

$$\begin{aligned} a(\boldsymbol{h}, \boldsymbol{h}) &\geqslant \left(\frac{1}{\tau^2} + \frac{1}{\tau}\right)\|\boldsymbol{h}\|^2 + \|\nabla \times \boldsymbol{h}\|^2 - \left|(\mathcal{K}_0 \star \boldsymbol{h}, \nabla \times \boldsymbol{h})\right| \\ &\overset{(9)}{\geqslant} \left(\frac{1}{\tau} - C_\varepsilon\right)\|\boldsymbol{h}\|^2 + \frac{1}{\tau^2}\|\boldsymbol{h}\|^2 + (1 - \varepsilon)\|\nabla \times \boldsymbol{h}\|^2 \\ &\geqslant C(\tau)\|\nabla \times \boldsymbol{h}\|^2_{\mathbf{H}_0(\mathbf{curl}, \Omega)}, \end{aligned}$$

with $\varepsilon < 1$ fixed. Moreover, a is continuous in $\mathbf{H}_0(\mathbf{curl}, \Omega)$. The functional $f_i(\boldsymbol{\varphi})$ is linear and bounded in $\mathbf{H}_0(\mathbf{curl}, \Omega)$ if $\boldsymbol{h}_{i-1} \in \mathbf{L}^2(\Omega)$ and $\delta \boldsymbol{h}_{i-1} \in \mathbf{L}^2(\Omega)$. Therefore, if $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$, applying the Lax–Milgram lemma gives the existence of a unique solution to (13) for any $i = 1, \ldots, n$. $\square$

4.1. A priori estimates

First, basic stability results for $\boldsymbol{h}_i$ are derived. The a priori estimates in parts (i), (ii) and (iii) in the following theorem will serve as uniform bounds to prove convergence.

Lemma 1 *(A priori estimates). Suppose that* $\boldsymbol{f} : [0,T] \to \mathbf{L}^2(\Omega)$ *obeys* $\boldsymbol{f} \in L^2((0,T), \mathbf{L}^2(\Omega))$.

(i) *Let* $\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ *and* $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$. *Then, there exists a positive constant* C *such that*

$$\max_{1\leqslant j\leqslant n} \|\boldsymbol{h}_j\|^2 + \sum_{i=1}^n \|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^n \|\tau\nabla\times\boldsymbol{h}_i\|^2 \leqslant C$$

for all $\tau < \tau_0$;

(ii) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl},\Omega)$ *and* $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$, *then*

$$\max_{1\leqslant i\leqslant n} \|\delta\boldsymbol{h}_i\|^2 + \max_{1\leqslant i\leqslant n} \|\nabla\times\boldsymbol{h}_i\|^2 + \sum_{i=1}^n \|\delta\boldsymbol{h}_i - \delta\boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^n \left\|\nabla\times(\boldsymbol{h}_i - \boldsymbol{h}_{i-1})\right\|^2 \leqslant C$$

for all $\tau < \tau_0$;

(iii) *If* $\nabla\cdot\boldsymbol{f} = \nabla\cdot\boldsymbol{H}_0 = \nabla\cdot\boldsymbol{H}_0' = 0$ *then* $\nabla\cdot\boldsymbol{h}_i = 0$ *for all* $i = 1,\ldots,n$. *Moreover, we have that*

$$\tau\sum_{i=1}^n \left\|\delta^2\boldsymbol{h}_i\right\|^2_{\mathbf{H}_0^{-1}(\mathbf{curl},\Omega)} \leqslant C$$

for all $\tau < \tau_0$;

(iv) *If* $\partial_t\boldsymbol{f} \in L^2((0,T),\mathbf{L}^2(\Omega))$, $\nabla\times(\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl},\Omega)$, $\boldsymbol{H}_0' \in \mathbf{H}_0(\mathbf{curl},\Omega)$ *and* $\nabla\times\nabla\times\boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ *then*

$$\max_{1\leqslant i\leqslant n} \left\|\delta^2\boldsymbol{h}_i\right\|^2 + \max_{1\leqslant i\leqslant n} \|\nabla\times\delta\boldsymbol{h}_i\|^2 + \sum_{i=1}^n \left\|\delta^2\boldsymbol{h}_i - \delta^2\boldsymbol{h}_{i-1}\right\|^2 + \sum_{i=1}^n \left\|\nabla\times(\delta\boldsymbol{h}_i - \delta\boldsymbol{h}_{i-1})\right\|^2 \leqslant C$$

for all $\tau < \tau_0$.

Proof. (i) First, we multiply (13) by τ and sum up for $i = 1,\ldots,k$. We define the sequence $\boldsymbol{s}_k : \Omega \to \mathbb{R}$ by

$$\boldsymbol{s}_k = \sum_{i=1}^k \tau\nabla\times\boldsymbol{h}_i, \quad k \geqslant 1; \qquad \boldsymbol{s}_0 = 0.$$

Using this notation, we can write for all $\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl},\Omega)$ that

$$(\delta\boldsymbol{h}_k,\boldsymbol{\varphi}) + (\boldsymbol{h}_k,\boldsymbol{\varphi}) + (\boldsymbol{s}_k,\nabla\times\boldsymbol{\varphi}) + \left(\sum_{i=1}^k \tau\mathcal{K}_0\star\boldsymbol{h}_i,\nabla\times\boldsymbol{\varphi}\right) = \left(\sum_{i=1}^k \tau\boldsymbol{f}_i,\boldsymbol{\varphi}\right) + (\delta\boldsymbol{h}_0,\boldsymbol{\varphi}) + (\boldsymbol{h}_0,\boldsymbol{\varphi}).$$

Then, we put $\boldsymbol{\varphi} = \boldsymbol{h}_k$, multiply this by τ, sum up for $k = 1,\ldots,j$, and obtain

$$\begin{aligned}\sum_{k=1}^j (\delta\boldsymbol{h}_k,\boldsymbol{h}_k)\tau + \sum_{k=1}^j \|\boldsymbol{h}_k\|^2\tau + \sum_{k=1}^j (\boldsymbol{s}_k,\delta\boldsymbol{s}_k)\tau + \sum_{k=1}^j\left(\sum_{i=1}^k \tau\mathcal{K}_0\star\boldsymbol{h}_i,\nabla\times\boldsymbol{h}_k\right)\tau \\ = \sum_{k=1}^j\left(\sum_{i=1}^k \tau\boldsymbol{f}_i,\boldsymbol{h}_k\right)\tau + \sum_{k=1}^j(\delta\boldsymbol{h}_0,\boldsymbol{h}_k)\tau + \sum_{k=1}^j(\boldsymbol{h}_0,\boldsymbol{h}_k)\tau.\end{aligned}$$

For the first and third terms on the left-hand side (LHS), we use Abel's summation rule

$$2\sum_{k=1}^{j}(\delta\boldsymbol{h}_k,\boldsymbol{h}_k)\tau = \|\boldsymbol{h}_j\|^2 - \|\boldsymbol{H}_0\|^2 + \sum_{k=1}^{j}\|\boldsymbol{h}_k-\boldsymbol{h}_{k-1}\|^2,$$

$$2\sum_{k=1}^{j}(\delta\boldsymbol{s}_k,\boldsymbol{s}_k)\tau = \|\boldsymbol{s}_j\|^2 + \sum_{k=1}^{j}\|\boldsymbol{s}_k-\boldsymbol{s}_{k-1}\|^2 = \left\|\sum_{k=1}^{j}\tau\nabla\times\boldsymbol{h}_k\right\|^2 + \sum_{k=1}^{j}\|\tau\nabla\times\boldsymbol{h}_k\|^2.$$

For the last term on the LHS, we apply Cauchy's and Young's inequalities

$$\begin{aligned}\left|\sum_{k=1}^{j}\left(\sum_{i=1}^{k}\tau\mathcal{K}_0\star\boldsymbol{h}_i,\tau\nabla\times\boldsymbol{h}_k\right)\right| &\leqslant C_\varepsilon\sum_{k=1}^{j}\left\|\sum_{i=1}^{k}\tau\mathcal{K}_0\star\boldsymbol{h}_i\right\|^2 + \varepsilon\sum_{k=1}^{j}\|\tau\nabla\times\boldsymbol{h}_k\|^2\\ &\overset{(7)}{\leqslant} C_\varepsilon\sum_{k=1}^{j}\left(\sum_{i=1}^{k}\|\boldsymbol{h}_i\|^2\tau\right)\tau + \varepsilon\sum_{k=1}^{j}\|\tau\nabla\times\boldsymbol{h}_k\|^2\\ &\leqslant C_\varepsilon\sum_{i=1}^{j}\|\boldsymbol{h}_i\|^2\tau + \varepsilon\sum_{k=1}^{j}\|\tau\nabla\times\boldsymbol{h}_k\|^2.\end{aligned}$$

Analogue, for the first term on the RHS, we apply Cauchy's and Young's inequalities together with the assumption on the source function

$$\left|\sum_{k=1}^{j}\left(\sum_{i=1}^{k}\tau\boldsymbol{f}_i,\boldsymbol{h}_k\right)\tau\right| \leqslant \sum_{k=1}^{j}\tau\sum_{i=1}^{k}\left(\frac{\|\boldsymbol{f}_i\|^2+\|\boldsymbol{h}_k\|^2}{2}\right)\tau \lesssim 1 + \sum_{k=1}^{j}\|\boldsymbol{h}_k\|^2\tau.$$

Using the assumptions on the initial conditions, we can easily deduce that

$$\left|\sum_{k=1}^{j}(\delta\boldsymbol{h}_0,\boldsymbol{h}_k)\tau\right| \lesssim 1 + \sum_{k=1}^{j}\|\boldsymbol{h}_k\|^2\tau \quad \text{and} \quad \left|\sum_{k=1}^{j}(\boldsymbol{h}_0,\boldsymbol{h}_k)\tau\right| \lesssim 1 + \sum_{k=1}^{j}\|\boldsymbol{h}_k\|^2\tau.$$

Eventually, we arrive at the following inequality (after changing summation indices)

$$\begin{aligned}&\|\boldsymbol{h}_j\|^2 + \sum_{i=1}^{j}\|\boldsymbol{h}_i-\boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^{j}\|\boldsymbol{h}_i\|^2\tau + \left\|\sum_{i=1}^{j}\tau\nabla\times\boldsymbol{h}_i\right\|^2 + \sum_{i=1}^{j}\|\tau\nabla\times\boldsymbol{h}_i\|^2\\ &\quad\leqslant C + C\|\boldsymbol{H}_0\|^2 + \varepsilon\sum_{i=1}^{j}\|\tau\nabla\times\boldsymbol{h}_i\|^2 + C_\varepsilon\sum_{i=1}^{j}\|\boldsymbol{h}_k\|^2\tau.\end{aligned}$$

Fixing ε sufficiently small and applying Grönwall's argument, we conclude the proof.

(ii) Setting $\boldsymbol{\varphi} = \delta\boldsymbol{h}_i$ in (13), multiplying by τ and summing up for $i = 1,\dots,j$ we have

$$\sum_{i=1}^{j}(\delta^2\boldsymbol{h}_i,\delta\boldsymbol{h}_i)\tau + \sum_{i=1}^{j}\|\delta\boldsymbol{h}_i\|^2\tau + \sum_{i=1}^{j}(\nabla\times\boldsymbol{h}_i,\nabla\times\delta\boldsymbol{h}_i)\tau + \sum_{i=1}^{j}(\mathcal{K}_0\star\boldsymbol{h}_i,\nabla\times\delta\boldsymbol{h}_i)\tau = \sum_{i=1}^{j}(\boldsymbol{f}_i,\delta\boldsymbol{h}_i)\tau.$$

For the first and third terms on the LHS, we use Abel's summation rule

$$2\sum_{i=1}^{j}(\delta^2\boldsymbol{h}_i,\delta\boldsymbol{h}_i)\tau = \|\delta\boldsymbol{h}_j\|^2 - \left\|\boldsymbol{H}_0'\right\|^2 + \sum_{i=1}^{j}\|\delta\boldsymbol{h}_i-\delta\boldsymbol{h}_{i-1}\|^2,$$

$$2\sum_{i=1}^{j}(\nabla\times\boldsymbol{h}_i,\nabla\times\delta\boldsymbol{h}_i)\tau = \|\nabla\times\boldsymbol{h}_j\|^2 - \|\nabla\times\boldsymbol{H}_0\|^2 + \sum_{i=1}^{j}\left\|\nabla\times(\boldsymbol{h}_i-\boldsymbol{h}_{i-1})\right\|^2.$$

Also the following partial summation formula is satisfied

$$\sum_{i=1}^{j}(\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \delta \boldsymbol{h}_i)\tau = (\mathcal{K}_0 \star \boldsymbol{h}_j, \nabla \times \boldsymbol{h}_j) - (\mathcal{K}_0 \star \boldsymbol{H}_0, \nabla \times \boldsymbol{H}_0) - \sum_{i=1}^{j}(\mathcal{K}_0 \star \delta \boldsymbol{h}_i, \nabla \times \boldsymbol{h}_{i-1})\tau.$$

Hence, using (i), (7) and (9), we obtain

$$\left|\sum_{i=1}^{j}(\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \delta \boldsymbol{h}_i)\tau\right| \leqslant C_\varepsilon + \varepsilon\|\nabla \times \boldsymbol{h}_j\|^2 + C\sum_{i=1}^{j}\|\delta \boldsymbol{h}_i\|^2\tau + C\sum_{i=1}^{j}\|\nabla \times \boldsymbol{h}_i\|^2\tau.$$

The RHS can be estimated using Cauchy's and Young's inequalities as follows

$$\left|\sum_{i=1}^{j}(\boldsymbol{f}_i, \delta \boldsymbol{h}_i)\tau\right| \leqslant C\sum_{i=1}^{j}\|\boldsymbol{f}_i\|^2\tau + C\sum_{i=1}^{j}\|\delta \boldsymbol{h}_i\|^2\tau \lesssim 1 + \sum_{i=1}^{j}\|\delta \boldsymbol{h}_i\|^2\tau.$$

Combining the previous results gives the following inequality

$$\begin{aligned}
&\|\delta \boldsymbol{h}_j\|^2 + \sum_{i=1}^{j}\|\delta \boldsymbol{h}_i - \delta \boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^{j}\|\delta \boldsymbol{h}_i\|^2\tau + \|\nabla \times \boldsymbol{h}_j\|^2 + \sum_{i=1}^{j}\left\|\nabla \times (\boldsymbol{h}_i - \boldsymbol{h}_{i-1})\right\|^2 \\
&\leqslant C_\varepsilon + C\left\|\boldsymbol{H}_0'\right\|^2 + C\|\nabla \times \boldsymbol{H}_0\|^2 + \varepsilon\|\nabla \times \boldsymbol{h}_j\|^2 + C\sum_{i=1}^{j}\|\delta \boldsymbol{h}_i\|^2\tau + C\sum_{i=1}^{j}\|\nabla \times \boldsymbol{h}_i\|^2\tau.
\end{aligned}$$

Fixing a sufficiently small positive ε, an application of Grönwall's lemma concludes the proof.

(iii) Take the divergence of the strong formulation

$$\delta^2 \boldsymbol{h}_i + \delta \boldsymbol{h}_i + \nabla \times \nabla \times \boldsymbol{h}_i + \nabla \times (\mathcal{K}_0 \star \boldsymbol{h}_i) = \boldsymbol{f}_i.$$

Multiply the result by τ and sum up for $i = 1, \ldots, j$ to arrive at

$$\nabla \cdot \delta \boldsymbol{h}_j + \nabla \cdot \boldsymbol{h}_j = 0 \quad \text{or} \quad (1+\tau)\nabla \cdot \boldsymbol{h}_j = \nabla \cdot \boldsymbol{h}_{j-1},$$

where $1 \leqslant j \leqslant n$. Therefore, $\nabla \cdot \boldsymbol{h}_i = 0$ for $i = 1, \ldots, n$ if $\nabla \cdot \boldsymbol{H}_0 = 0$. It holds

$$\left(\delta^2 \boldsymbol{h}_i, \boldsymbol{\varphi}\right) = (\boldsymbol{f}_i, \boldsymbol{\varphi}) - (\delta \boldsymbol{h}_i, \boldsymbol{\varphi}) - (\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}) - (\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi}), \quad \boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega).$$

Further, we may write

$$\left|(\boldsymbol{f}_i, \boldsymbol{\varphi})\right| \leqslant \|\boldsymbol{f}_i\|\|\boldsymbol{\varphi}\|, \qquad \left|(\delta \boldsymbol{h}_i, \boldsymbol{\varphi})\right| \leqslant \|\delta \boldsymbol{h}_i\|\|\boldsymbol{\varphi}\|, \qquad \left|(\nabla \times \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi})\right| \leqslant \|\nabla \times \boldsymbol{h}_i\|\|\nabla \times \boldsymbol{\varphi}\|$$

and

$$\left|(\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi})\right| \overset{(7)}{\lesssim} \|\boldsymbol{h}_i\|\|\nabla \times \boldsymbol{\varphi}\|.$$

Thus using

$$\left\|\delta^2 \boldsymbol{h}_i\right\|_{\mathbf{H}_0^{-1}(\mathbf{curl}, \Omega)} = \sup_{\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl}, \Omega)} \frac{(\delta^2 \boldsymbol{h}_i, \boldsymbol{\varphi})}{\|\boldsymbol{\varphi}\|_{\mathbf{H}_0(\mathbf{curl}, \Omega)}},$$

(i) and (ii), we deduce that

$$\tau \sum_{i=1}^{n} \left\| \delta^2 \boldsymbol{h}_i \right\|^2_{\mathbf{H}_0^{-1}(\mathbf{curl},\Omega)} \leqslant C.$$

(iv) First, we set

$$\delta^2 \boldsymbol{h}_0 := \boldsymbol{f}(0) - \boldsymbol{H}_0' - \nabla \times \nabla \times \boldsymbol{H}_0 - \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega).$$

We subtract (13) for $i = i-1$ from (13) for $i = i$, then we set $\boldsymbol{\varphi} = \delta^2 \boldsymbol{h}_i$ and we sum the result for $i = 1, \ldots, j$ with $1 \leqslant j \leqslant n$ to get

$$\sum_{i=1}^{j} \left(\delta^3 \boldsymbol{h}_i, \delta^2 \boldsymbol{h}_i\right)\tau + \sum_{i=1}^{j} \left\|\delta^2 \boldsymbol{h}_i\right\|^2 \tau + \sum_{i=1}^{j} \left(\nabla \times \delta \boldsymbol{h}_i, \nabla \times \delta^2 \boldsymbol{h}_i\right)\tau + \sum_{i=1}^{j} \left(\mathcal{K}_0 \star \delta \boldsymbol{h}_i, \nabla \times \delta^2 \boldsymbol{h}_i\right)\tau$$
$$= \sum_{i=1}^{j} \left(\delta \boldsymbol{f}_i, \delta^2 \boldsymbol{h}_i\right)\tau.$$

Further, we follow the same way as in (ii) when considering $\delta^2 \boldsymbol{h}_i$ instead of $\delta \boldsymbol{h}_i$. $\square$

4.2. Convergence

The existence of a weak solution is proved using Rothe's method. The following piecewise linear in time vector fields $\boldsymbol{h}_n$ and $\boldsymbol{v}_n$

$$\begin{aligned} &\boldsymbol{h}_n(0) = \boldsymbol{H}_0, \qquad \boldsymbol{h}_n(t) = \boldsymbol{h}_{i-1} + (t - t_{i-1})\delta \boldsymbol{h}_i \quad \text{for } t \in (t_{i-1}, t_i],\ i = 1, \ldots, n; \\ &\boldsymbol{v}_n(0) = \boldsymbol{H}_0', \qquad \boldsymbol{v}_n(t) = \delta \boldsymbol{h}_{i-1} + (t - t_{i-1})\delta^2 \boldsymbol{h}_i \quad \text{for } t \in (t_{i-1}, t_i],\ i = 1, \ldots, n \end{aligned}$$

and the piecewise constant in time fields $\overline{\boldsymbol{h}}_n$ and $\overline{\boldsymbol{v}}_n$ are introduced

$$\begin{aligned} &\overline{\boldsymbol{h}}_n(0) = \boldsymbol{H}_0, \qquad \overline{\boldsymbol{h}}_n(t) = \boldsymbol{h}_i, \quad \text{for } t \in (t_{i-1}, t_i],\ i = 1, \ldots, n; \\ &\overline{\boldsymbol{v}}_n(0) = \boldsymbol{H}_0', \qquad \overline{\boldsymbol{v}}_n(t) = \delta \boldsymbol{h}_i, \quad \text{for } t \in (t_{i-1}, t_i],\ i = 1, \ldots, n. \end{aligned}$$

Similarly, the vector field $\overline{\boldsymbol{f}}_n$ is defined. Note that $\overline{\boldsymbol{v}}_n = \partial_t \boldsymbol{h}_n$. The variational formulation (13) can be rewritten as

$$\left(\partial_t \boldsymbol{v}_n(t), \boldsymbol{\varphi}\right) + \left(\partial_t \boldsymbol{h}_n(t), \boldsymbol{\varphi}\right) + \left(\nabla \times \overline{\boldsymbol{h}}_n(t), \nabla \times \boldsymbol{\varphi}\right) + \left(\mathcal{K}_0 \star \overline{\boldsymbol{h}}_n(t), \nabla \times \boldsymbol{\varphi}\right) = \left(\overline{\boldsymbol{f}}_n(t), \boldsymbol{\varphi}\right). \tag{14}$$

Now, the convergence of the sequences $\boldsymbol{h}_n$ and $\overline{\boldsymbol{h}}_n$ to the unique weak solution of (11) is proved if $\tau \to 0$ or $n \to \infty$.

Theorem 3 *(Existence). Let $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$, $\boldsymbol{f} : [0, T] \to \mathbf{L}^2(\Omega)$ and $\boldsymbol{f} \in L^2((0, T), \mathbf{L}^2(\Omega))$. Assume that $\nabla \cdot \boldsymbol{H}_0 = \nabla \cdot \boldsymbol{H}_0' = 0 = \nabla \cdot \boldsymbol{f}(t)$ for any time $t \in [0, T]$. Then there exists a vector field $\boldsymbol{H}$ such that*

(i) $\overline{\boldsymbol{h}}_n \rightharpoonup \boldsymbol{H}$ in $L^2((0, T), \mathbf{H}_0(\mathbf{curl}, \Omega))$, $\boldsymbol{h}_n \rightharpoonup \boldsymbol{H}$ in $L^2((0, T), \mathbf{H}_0(\mathbf{curl}, \Omega))$;
(ii) $\boldsymbol{h}_n(t) \rightharpoonup \boldsymbol{H}(t)$ in $\mathbf{L}^2(\Omega)$ for any $t \in [0, T]$;
(iii) $\boldsymbol{v}_n \rightharpoonup \partial_t \boldsymbol{H}$ in $L^2((0, T), \mathbf{H}^{\frac{1}{2}}(\Omega))$;

(*iv*) $\boldsymbol{H}$ *is a weak solution of (12);*
(*v*) $\boldsymbol{H} \in C([0,T],\mathbf{H}^{\frac{1}{2}}(\Omega))$, $\partial_t \boldsymbol{H} \in L^2((0,T),\mathbf{H}^{\frac{1}{2}}(\Omega)) \cap C([0,T],\mathbf{L}^2(\Omega))$ *and* $\partial_{tt}\boldsymbol{H} \in L^2((0,T),\mathbf{H}_0^{-1}(\mathbf{curl},\Omega))$.

Proof. (*i*) Thanks to Lemma 1(*i*) and (*ii*), the sequences $\bar{\boldsymbol{h}}_n$ and $\boldsymbol{h}_n$ are bounded in $L^2((0,T),\mathbf{H}_0(\mathbf{curl},\Omega))$. Therefore, due to the reflexivity of $L^2((0,T),\mathbf{H}_0(\mathbf{curl},\Omega))$, the sequence $\bar{\boldsymbol{h}}_n$ contains a weakly convergence subsequence (denoted by the same symbol again) such that $\bar{\boldsymbol{h}}_n \rightharpoonup \boldsymbol{H}$ in $L^2((0,T),\mathbf{H}_0(\mathbf{curl},\Omega))$. The sequences $\bar{\boldsymbol{h}}_n$ and $\boldsymbol{h}_n$ have the same limit in the space $L^2((0,T),\mathbf{H}_0(\mathbf{curl},\Omega))$. Employing Lemma 1(*i*) gives

$$\begin{aligned}
\lim_{n\to\infty}\|\boldsymbol{h}_n - \bar{\boldsymbol{h}}_n\|^2_{L^2((0,T),\mathbf{H}_0(\mathbf{curl},\Omega))} &= \lim_{n\to\infty}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}\left\|\boldsymbol{h}_{i-1} + \frac{t-t_{i-1}}{\tau}(\boldsymbol{h}_i - \boldsymbol{h}_{i-1}) - \boldsymbol{h}_i\right\|^2_{\mathbf{H}_0(\mathbf{curl},\Omega)}\mathrm{d}t \\
&\leqslant 4\lim_{n\to\infty}\tau\sum_{i=1}^n\|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\|^2_{\mathbf{H}_0(\mathbf{curl},\Omega)} \\
&\leqslant \lim_{n\to\infty}\frac{C}{n} = 0.
\end{aligned}$$

Hence, $\boldsymbol{h}_n \rightharpoonup \boldsymbol{H}$ in $L^2((0,T),\mathbf{H}_0(\mathbf{curl},\Omega))$.

(*ii*) The sequence $\boldsymbol{h}_n : [0,T] \to \mathbf{L}^2(\Omega)$, $n \in \mathbb{N}$, is equibounded and uniform equicontinuous. For every $n \in \mathbb{N}$ and $\forall t, t_1, t_2 \in [0,T]$, we have using Lemma 1(*i*) and (*ii*) that

$$\|\boldsymbol{h}_n(t)\| \leqslant \|\boldsymbol{h}_{i-1}\| + \|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\| \leqslant C$$

and

$$\begin{aligned}
\|\boldsymbol{h}_n(t_2) - \boldsymbol{h}_n(t_1)\| &= \left\|\int_{t_1}^{t_2}\partial_t\boldsymbol{h}_n(t)\mathrm{d}t\right\| \\
&\leqslant \int_{t_1}^{t_2}\|\partial_t\boldsymbol{h}_n(t)\|\mathrm{d}t \\
&\leqslant \sqrt{\int_{t_1}^{t_2}1^2\mathrm{d}t}\sqrt{\int_{t_1}^{t_2}\|\partial_t\boldsymbol{h}_n(t)\|^2\mathrm{d}t} \\
&\leqslant \sqrt{|t_2 - t_1|}\sqrt{\sum_{i=1}^n\|\delta\boldsymbol{h}_i\|^2\tau} \\
&\lesssim \sqrt{|t_2 - t_1|}.
\end{aligned}$$

An application of [3, Lemma 1.3.10] gives $\boldsymbol{h}_n(t) \rightharpoonup \boldsymbol{H}(t)$ in $\mathbf{L}^2(\Omega)$ for any $t \in [0,T]$.

(*iii*) The sequence $\partial_t\boldsymbol{h}_n$ is bounded in the reflexive space $L^2((0,T),\mathbf{L}^2(\Omega))$ by Lemma 1(*ii*). Hence, $\partial_t\boldsymbol{h}_n = \bar{\boldsymbol{v}}_n \rightharpoonup \partial_t\boldsymbol{H}$ in $L^2((0,T),\mathbf{L}^2(\Omega))$. Lemma 1(*ii*) implies

$$\begin{aligned}
\lim_{n\to\infty}\|\boldsymbol{v}_n - \bar{\boldsymbol{v}}_n\|^2_{L^2((0,T),\mathbf{L}^2(\Omega))} &\leqslant 4\lim_{n\to\infty}\tau\sum_{i=1}^n\|\delta\boldsymbol{h}_i - \delta\boldsymbol{h}_{i-1}\|^2_{\mathbf{L}^2(\Omega)} \\
&\leqslant \lim_{n\to\infty}\frac{C}{n} = 0.
\end{aligned}$$

Thus $\boldsymbol{v}_n \rightharpoonup \partial_t \boldsymbol{H}$ in $L^2((0,T),\mathbf{L}^2(\Omega))$. Lemma 1($i$), ($ii$) and ($iii$) give

$$\boldsymbol{v}_n \in L^2\big((0,T),\mathbf{H}_0(\mathbf{curl},\Omega)\big), \qquad \max_{t\in[0,T]}\big\|\boldsymbol{v}_n(t)\big\| \leqslant C, \qquad \nabla\cdot\boldsymbol{v}_n(t) = 0 \quad \forall t\in[0,T].$$

Consequently, reviewing [8, Theorem 3.47] we see that $\boldsymbol{v}_n \in L^2((0,T),\mathbf{H}^{\frac{1}{2}}(\Omega))$. Using [10, Lemma 10] we obtain that

$$\mathbf{H}^{\frac{1}{2}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega) \cong \mathbf{L}^2(\Omega)^* \hookrightarrow \mathbf{H}_0^{-1}(\mathbf{curl},\Omega).$$

Taking into account the fact that $\partial_t\boldsymbol{v}_n \in L^2((0,T),\mathbf{H}_0^{-1}(\mathbf{curl},\Omega))$, see Lemma 1($iii$), and using the generalized Aubin–Lions lemma [11, Lemma 7.7] we get that $\{\boldsymbol{v}_n\}$ is compact in the space $L_2((0,T),\mathbf{L}^2(\Omega))$ and $\boldsymbol{v}_n \rightharpoonup \partial_t\boldsymbol{H}$ in $L^2((0,T),\mathbf{H}^{\frac{1}{2}}(\Omega))$. Therefore, there exists a subsequence of $\boldsymbol{v}_n$ (denoted by the same symbol again) for which we have [6, p. 88]

$$\boldsymbol{v}_n(\boldsymbol{x},t) \to \partial_t\boldsymbol{H}(\boldsymbol{x},t) \quad \text{a.e. in } \Omega\times(0,T).$$

(iv) Let us integrate (14) in time to get (for any $t\in(0,T)$)

$$\big(\boldsymbol{v}_n(t)-\boldsymbol{H}_0',\boldsymbol{\varphi}\big) + \big(\boldsymbol{h}_n(t)-\boldsymbol{H}_0,\boldsymbol{\varphi}\big) + \int_0^t (\nabla\times\bar{\boldsymbol{h}}_n,\nabla\times\boldsymbol{\varphi}) + \int_0^t (\mathcal{K}_0\star\bar{\boldsymbol{h}}_n,\nabla\times\boldsymbol{\varphi}) = \int_0^t (\bar{\boldsymbol{f}}_n,\boldsymbol{\varphi}). \tag{15}$$

Clearly $\bar{\boldsymbol{f}}_n \rightharpoonup \boldsymbol{f}$ in $L^2([0,T],\mathbf{L}^2(\Omega))$. Both terms $\int_0^t(\nabla\times\bar{\boldsymbol{h}}_n,\nabla\times\boldsymbol{\varphi})$ and $\int_0^t(\mathcal{K}_0\star\bar{\boldsymbol{h}}_n,\nabla\times\boldsymbol{\varphi})$ are linear bounded functionals in the space $L^2((0,T),\mathbf{H}_0(\mathbf{curl},\Omega))$. Thanks to ($ii$) and ($iii$), we get

$$\begin{array}{ccccc}
\big(\boldsymbol{h}_n(t)-\boldsymbol{H}_0,\boldsymbol{\varphi}\big) & = & \displaystyle\int_0^t(\partial_t\boldsymbol{h}_n,\boldsymbol{\varphi}) & = & \displaystyle\int_0^t(\bar{\boldsymbol{v}}_n,\boldsymbol{\varphi}) \\
\downarrow & & & & \downarrow \\
\big(\boldsymbol{H}(t)-\boldsymbol{H}_0,\boldsymbol{\varphi}\big) & & & = & \displaystyle\int_0^t(\partial_t\boldsymbol{H},\boldsymbol{\varphi}),
\end{array}$$

which is valid for any $t\in[0,T]$. Using the stability result $\partial_t\boldsymbol{v}_n \in L^2((0,T),\mathbf{H}_0^{-1}(\mathbf{curl},\Omega))$ and (iii), we have for any $t\in[0,T]$ that

$$\begin{array}{ccc}
\big(\boldsymbol{v}_n(t)-\boldsymbol{H}_0',\boldsymbol{\varphi}\big) & = & \displaystyle\int_0^t(\partial_t\boldsymbol{v}_n,\boldsymbol{\varphi}) \\
\downarrow & & \downarrow \\
\big(\partial_t\boldsymbol{H}(t)-\boldsymbol{H}_0',\boldsymbol{\varphi}\big) & = & \displaystyle\int_0^t(\partial_{tt}\boldsymbol{H},\boldsymbol{\varphi}),
\end{array}$$

with $\partial_{tt}\boldsymbol{H} \in L^2((0,T),\mathbf{H}_0^{-1}(\mathbf{curl},\Omega))$. Now, we can pass to the limit for $n\to\infty$ in (15) to arrive at

$$\int_0^t(\partial_{tt}\boldsymbol{H},\boldsymbol{\varphi}) + \int_0^t(\partial_t\boldsymbol{H},\boldsymbol{\varphi}) + \int_0^t(\nabla\times\boldsymbol{H},\nabla\times\boldsymbol{\varphi}) + \int_0^t(\mathcal{K}_0\star\boldsymbol{H},\nabla\times\boldsymbol{\varphi}) = \int_0^t(\boldsymbol{f},\boldsymbol{\varphi}).$$

Differentiating this equality with respect to the time variable gives the existence of a weak solution to (12).

(v) We recall that $\mathbf{H}^{\frac{1}{2}}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$. The sequence $\boldsymbol{h}_n : [0,T] \to \mathbf{H}^{\frac{1}{2}}(\Omega)$, $n \in \mathbb{N}$, is equibounded. Using [8, Theorem 3.47], Theorem 2 and Lemma 1(i)–(iii) give for all $t \in [0,T]$ that

$$\begin{aligned}\left\|\boldsymbol{h}_n(t)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} &\leqslant \|\boldsymbol{h}_{i-1}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} + \|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \\ &\lesssim \|\boldsymbol{h}_{i-1}\| + \|\nabla \times \boldsymbol{h}_{i-1}\| + \|\boldsymbol{h}_i - \boldsymbol{h}_{i-1}\| + \left\|\nabla \times (\boldsymbol{h}_i - \boldsymbol{h}_{i-1})\right\| \\ &\leqslant C.\end{aligned}$$

In part (ii) of the proof, we have shown that the sequence $\boldsymbol{h}_n : [0,T] \to \mathbf{L}^2(\Omega)$, $n \in \mathbb{N}$, is uniform equicontinuous. Now [3, Lemma 1.3.10] implies that $\boldsymbol{H} \in C((0,T), \mathbf{H}^{\frac{1}{2}}(\Omega))$. Consider the following evolution triple (or sometimes called Gelfand's triple) of spaces

$$\mathbf{H}_0(\mathbf{curl}, \Omega) \hookrightarrow \mathbf{L}^2(\Omega) \cong \mathbf{L}^2(\Omega)^* \hookrightarrow \mathbf{H}_0^{-1}(\mathbf{curl}, \Omega).$$

We know that

$$\partial_t \boldsymbol{H} \in L^2\big((0,T), \mathbf{H}_0(\mathbf{curl}, \Omega)\big) \quad \text{and} \quad \partial_{tt} \boldsymbol{H} \in L^2\big((0,T), \mathbf{H}_0^{-1}(\mathbf{curl}, \Omega)\big).$$

Applying [11, Lemma 7.3] gives $\partial_t \boldsymbol{H} \in C([0,T], \mathbf{L}^2(\Omega))$, which concludes the proof. □

4.3. Error estimates

The following theorem addresses the error estimates for the time discretization.

Theorem 4 *(Error). Suppose that $\boldsymbol{f} \in \operatorname{Lip}([0,T], \mathbf{L}^2(\Omega))$.*

(i) *If $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$ then*

$$\max_{t\in[0,T]} \left\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\right\|^2 + \max_{t\in[0,T]} \left\| \nabla \times \int_0^t [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 \leqslant C\tau.$$

(ii) *If $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, $\boldsymbol{H}_0' \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ then*

$$\max_{t\in[0,T]} \left\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\right\|^2 + \max_{t\in[0,T]} \left\| \nabla \times \int_0^t [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 \leqslant C\tau^2.$$

Please note that the positive constant C in these estimates is of the form Ce^{CT}.

Proof. We subtract (12) from (14) and integrate (12) in time over $t \in (0,\xi)$ to get

$$\begin{aligned}&\big(\boldsymbol{v}_n(\xi) - \partial_t \boldsymbol{H}(\xi), \boldsymbol{\varphi}\big) + \big(\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi), \boldsymbol{\varphi}\big) + \left(\nabla \times \int_0^\xi \big[\overline{\boldsymbol{h}}_n(t) - \boldsymbol{H}(t)\big], \nabla \times \boldsymbol{\varphi}\right) \\ &+ \left(\mathcal{K}_0 \star \int_0^\xi \big[\overline{\boldsymbol{h}}_n(t) - \boldsymbol{H}(t)\big], \nabla \times \boldsymbol{\varphi}\right) = \left(\int_0^\xi \big[\overline{\boldsymbol{f}}_n(t) - \boldsymbol{f}(t)\big], \boldsymbol{\varphi}\right).\end{aligned}$$

Now, putting $\boldsymbol{\varphi} = \boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)$, using $\overline{\boldsymbol{v}}_n = \partial_t \boldsymbol{h}_n$ and integrating in time over the variable $\xi \in (0, \eta) \subset (0, T)$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\big\|\boldsymbol{h}_n(\eta) - \boldsymbol{H}(\eta)\big\|^2 + \int_0^\eta \|\boldsymbol{h}_n - \boldsymbol{H}\|^2 \\
&\quad + \frac{1}{2}\left\|\nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 + \int_0^\eta \left(\mathcal{K}_0 \star \int_0^\xi \big[\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\big], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right) \\
&= \int_0^\eta \left(\int_0^\xi \big[\overline{\boldsymbol{f}}_n(t) - \boldsymbol{f}(t)\big], \boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\right) + \int_0^\eta (\overline{\boldsymbol{v}}_n - \boldsymbol{v}_n, \boldsymbol{h}_n - \boldsymbol{H}) \\
&\quad + \int_0^\eta \left(\nabla \times \int_0^\xi [\boldsymbol{h}_n - \overline{\boldsymbol{h}}_n], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right) \\
&\quad + \int_0^\eta \left(\mathcal{K}_0 \star \int_0^\xi [\boldsymbol{h}_n - \overline{\boldsymbol{h}}_n], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right). \qquad (16)
\end{aligned}$$

We may write due to the Lipschitz continuity of $\boldsymbol{f}$ that

$$\begin{aligned}
\left|\int_0^\eta \left(\int_0^\xi \big[\overline{\boldsymbol{f}}_n(t) - \boldsymbol{f}(t)\big], \boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\right)\right| &\lesssim \int_0^\eta \int_0^\eta \|\overline{\boldsymbol{f}}_n - \boldsymbol{f}\|^2 + \int_0^\eta \|\boldsymbol{h}_n - \boldsymbol{H}\|^2 \\
&\lesssim \tau^2 + \int_0^\eta \|\boldsymbol{h}_n - \boldsymbol{H}\|^2.
\end{aligned}$$

The integration by parts formula gives the following estimate

$$\begin{aligned}
&\int_0^\eta \left(\mathcal{K}_0 \star \int_0^\xi \big[\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\big], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right) \\
&= \left(\mathcal{K}_0 \star \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}], \nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}]\right) - \int_0^\eta \left(\mathcal{K}_0 \star [\boldsymbol{h}_n - \boldsymbol{H}], \nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}]\right) \\
&\overset{(9)}{\leqslant} \varepsilon \left\|\nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 + C_\varepsilon \int_0^\eta \|\boldsymbol{h}_n - \boldsymbol{H}\|^2 + C \int_0^\eta \left\|\nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2.
\end{aligned}$$

It holds

$$\big\|\boldsymbol{h}_n(t) - \overline{\boldsymbol{h}}_n(t)\big\| \lesssim \tau \big\|\partial_t \boldsymbol{h}_n(t)\big\| \quad \text{for } t \in [0, T]$$

and

$$\big\|\boldsymbol{v}_n(t) - \overline{\boldsymbol{v}}_n(t)\big\| \lesssim \tau \big\|\partial_t \boldsymbol{v}_n(t)\big\| \quad \text{for } t \in [0, T].$$

Analogue as in the previous estimate, we get using Lemma 1(ii) that

$$\begin{aligned}
&\int_0^\eta \left(\mathcal{K}_0 \star \int_0^\xi [\boldsymbol{h}_n - \bar{\boldsymbol{h}}_n], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right) \\
&\quad \leqslant \varepsilon \left\| \nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 + C_\varepsilon \int_0^\eta \|\boldsymbol{h}_n - \bar{\boldsymbol{h}}_n\|^2 + C \int_0^\eta \left\| \nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 \\
&\quad \leqslant \varepsilon \left\| \nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 + C_\varepsilon \tau^2 + C \int_0^\eta \left\| \nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2.
\end{aligned}$$

It remains to estimate the second and third terms on the RHS in (16). We have to distinguish between two cases depending on the assumptions on the initial conditions. If $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$, we get

$$\begin{aligned}
\left| \int_0^\eta (\bar{\boldsymbol{v}}_n - \boldsymbol{v}_n, \boldsymbol{h}_n - \boldsymbol{H}) \right| &\leqslant \varepsilon \int_0^\eta \|\boldsymbol{h}_n - \boldsymbol{H}\|^2 + C_\varepsilon \int_0^\eta \|\bar{\boldsymbol{v}}_n - \boldsymbol{v}_n\|^2 \\
&\leqslant \varepsilon \int_0^\eta \|\boldsymbol{h}_n - \boldsymbol{H}\|^2 + C_\varepsilon \tau
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\eta \left(\nabla \times \int_0^\xi [\boldsymbol{h}_n - \bar{\boldsymbol{h}}_n], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right) \\
&\quad \leqslant \varepsilon \left\| \nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 + C_\varepsilon \int_0^\eta \|\nabla \times [\boldsymbol{h}_n - \bar{\boldsymbol{h}}_n]\|^2 + C \int_0^\eta \left\| \nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 \\
&\quad \leqslant \varepsilon \left\| \nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 + C_\varepsilon \tau + C \int_0^\eta \left\| \nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2.
\end{aligned}$$

If $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, $\boldsymbol{H}_0' \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ then

$$\left| \int_0^\eta (\bar{\boldsymbol{v}}_n - \boldsymbol{v}_n, \boldsymbol{h}_n - \boldsymbol{H}) \right| \leqslant \varepsilon \int_0^\eta \|\boldsymbol{h}_n - \boldsymbol{H}\|^2 + C_\varepsilon \tau^2$$

and

$$\int_0^\eta \left(\nabla \times \int_0^\xi [\boldsymbol{h}_n - \bar{\boldsymbol{h}}_n], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right) \leqslant \varepsilon \left\| \nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2 + C_\varepsilon \tau^2 + C \int_0^\eta \left\| \nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}] \right\|^2.$$

Combining the previous results, choosing a sufficiently small positive ε and applying Grönwall's argument, we conclude the proof. □

From this estimate, the uniqueness of the solution can also be proved. If $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ satisfy (12), then (if $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$)

$$\max_{\eta\in[0,T]}\left\|\boldsymbol{H}_1(\eta)-\boldsymbol{H}_2(\eta)\right\| \leqslant \max_{\eta\in[0,T]}\left\|\boldsymbol{h}_n(\eta)-\boldsymbol{H}_1(\eta)\right\| + \max_{\eta\in[0,T]}\left\|\boldsymbol{h}_n(\eta)-\boldsymbol{H}_2(\eta)\right\| \lesssim \sqrt{\tau},$$

which is arbitrarily small.

5. Modified scheme

In this section, the following time-discrete scheme is considered, which represents a slight modification of (13)

$$\begin{cases} (\delta^2 \boldsymbol{h}_i, \boldsymbol{\varphi}) + (\delta \boldsymbol{h}_i, \boldsymbol{\varphi}) + (\nabla\times\boldsymbol{h}_i, \nabla\times\boldsymbol{\varphi}) = (\boldsymbol{f}_i, \boldsymbol{\varphi}) - (\mathcal{K}_0 \star \boldsymbol{h}_{i-1}, \nabla\times\boldsymbol{\varphi}); \\ \boldsymbol{h}_0 = \boldsymbol{H}_0 \end{cases} \tag{17}$$

for all $\boldsymbol{\varphi}\in\mathbf{H}_0(\mathbf{curl},\Omega)$, which is equivalent to

$$\begin{aligned} a(\boldsymbol{h}_i,\boldsymbol{\varphi}) &:= \left(\frac{1}{\tau^2}+\frac{1}{\tau}\right)(\boldsymbol{h}_i,\boldsymbol{\varphi}) + (\nabla\times\boldsymbol{h}_i, \nabla\times\boldsymbol{\varphi}) \\ &= (\boldsymbol{f}_i,\boldsymbol{\varphi}) - (\mathcal{K}_0\star\boldsymbol{h}_{i-1},\nabla\times\boldsymbol{\varphi}) + \left(\frac{1}{\tau^2}+\frac{1}{\tau}\right)(\boldsymbol{h}_{i-1},\boldsymbol{\varphi}) + \left(\frac{\delta\boldsymbol{h}_{i-1}}{\tau},\boldsymbol{\varphi}\right) \\ &=: f_i(\boldsymbol{\varphi}). \end{aligned}$$

In this scheme, the convolution term is taken explicitly (from the last time step), while in the scheme (13) an implicit form (from the actual time step) is considered. The main advantage is that this scheme is easier to implement than scheme (13).

An application of the Lax–Milgram lemma gives the existence of a unique solution to (17) in $\mathbf{H}_0(\mathbf{curl},\Omega)$ for any $i=1,\dots,n$ and any $\tau>0$. Indeed, the bilinear form $a(\boldsymbol{h},\boldsymbol{\varphi})$ is elliptic and continuous in $\mathbf{H}_0(\mathbf{curl},\Omega)$. Moreover, according to (7), the functional $f_i(\boldsymbol{\varphi})$ is linear and bounded in $\mathbf{H}_0(\mathbf{curl},\Omega)$ if $\boldsymbol{H}_0\in\mathbf{L}^2(\Omega)$ and $\boldsymbol{H}_0'\in\mathbf{L}^2(\Omega)$ for $i=1,\dots,n$.

Handling this scheme is very similar to the way used for (13). For short, only the differences between both algorithms are pointed out.

Lemma 2 *(A priori estimates). Suppose that $\boldsymbol{f}:[0,T]\to\mathbf{L}^2(\Omega)$ obeys $\boldsymbol{f}\in L^2((0,T),\mathbf{L}^2(\Omega))$.*

(i) Let $\boldsymbol{H}_0\in\mathbf{L}^2(\Omega)$ and $\boldsymbol{H}_0'\in\mathbf{L}^2(\Omega)$. Then, there exists a positive constant C such that

$$\max_{1\leqslant j\leqslant n}\|\boldsymbol{h}_j\|^2 + \sum_{i=1}^n \|\boldsymbol{h}_i-\boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^n\|\tau\nabla\times\boldsymbol{h}_i\|^2 \leqslant C$$

for all $\tau<\tau_0$;

(ii) If $\boldsymbol{H}_0\in\mathbf{H}_0(\mathbf{curl},\Omega)$ and $\boldsymbol{H}_0'\in\mathbf{L}^2(\Omega)$, then

$$\max_{1\leqslant i\leqslant n}\|\delta\boldsymbol{h}_i\|^2 + \max_{1\leqslant i\leqslant n}\|\nabla\times\boldsymbol{h}_i\|^2 + \sum_{i=1}^n\|\delta\boldsymbol{h}_i-\delta\boldsymbol{h}_{i-1}\|^2 + \sum_{i=1}^n\left\|\nabla\times(\boldsymbol{h}_i-\boldsymbol{h}_{i-1})\right\|^2 \leqslant C$$

for all $\tau<\tau_0$;

(iii) If $\nabla\cdot\boldsymbol{f}=\nabla\cdot\boldsymbol{H}_0=\nabla\cdot\boldsymbol{H}_0'=0$ then $\nabla\cdot\boldsymbol{h}_i=0$ for all $i=1,\dots,n$. Moreover, we have that

$$\tau\sum_{i=1}^n\left\|\delta^2\boldsymbol{h}_i\right\|^2_{\mathbf{H}_0^{-1}(\mathbf{curl},\Omega)} \leqslant C$$

for all $\tau<\tau_0$;

(*iv*) *If* $\partial_t \boldsymbol{f} \in L^2((0,T),\mathbf{L}^2(\Omega))$, $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl},\Omega)$, $\boldsymbol{H}_0' \in \mathbf{H}_0(\mathbf{curl},\Omega)$ *and* $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ *then*

$$\max_{1\leqslant i\leqslant n}\left\|\delta^2 \boldsymbol{h}_i\right\|^2 + \max_{1\leqslant i\leqslant n}\|\nabla \times \delta \boldsymbol{h}_i\|^2 + \sum_{i=1}^n \left\|\delta^2 \boldsymbol{h}_i - \delta^2 \boldsymbol{h}_{i-1}\right\|^2 + \sum_{i=1}^n \left\|\nabla \times (\delta \boldsymbol{h}_i - \delta \boldsymbol{h}_{i-1})\right\|^2 \leqslant C$$

for all $\tau < \tau_0$.

Proof. (*i*) We follow Lemma 1(*i*). Using (9) we have

$$\left|\sum_{k=1}^j \left(\sum_{i=1}^k \tau \mathcal{K}_0 \star \boldsymbol{h}_{i-1}, \tau \nabla \times \boldsymbol{h}_k\right)\right| \leqslant C_\varepsilon \sum_{i=0}^j \|\boldsymbol{h}_i\|^2 \tau + \varepsilon \sum_{k=1}^j \|\tau \nabla \times \boldsymbol{h}_k\|^2.$$

After fixing a sufficiently small positive ε, an application of Grönwall's lemma completes the proof.

(*ii*) Note that

$$\sum_{i=1}^j (\mathcal{K}_0 \star \boldsymbol{h}_{i-1}, \nabla \times \delta \boldsymbol{h}_i)\tau = (\mathcal{K}_0 \star \boldsymbol{h}_j, \nabla \times \boldsymbol{h}_j) - (\mathcal{K}_0 \star \boldsymbol{H}_0, \nabla \times \boldsymbol{H}_0) - \sum_{i=1}^j (\mathcal{K}_0 \star \delta \boldsymbol{h}_i, \nabla \times \boldsymbol{h}_i)\tau.$$

The rest of the proof runs as before.

(*iii*) The proof is the same as in Lemma 1(*iii*) replacing $(\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi})$ by $(\mathcal{K}_0 \star \boldsymbol{h}_{i-1}, \nabla \times \boldsymbol{\varphi})$.

(*iv*) We set

$$\delta^2 \boldsymbol{h}_0 := \boldsymbol{f}(0) - \boldsymbol{H}_0' - \nabla \times \nabla \times \boldsymbol{H}_0 - \nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0), \qquad \boldsymbol{h}_{-1} := \boldsymbol{h}_0 - \delta \boldsymbol{h}_0 \tau.$$

Note that $\delta \boldsymbol{h}_0, \boldsymbol{h}_{-1} \in \mathbf{L}^2(\Omega)$. The proof follows very closely the proof of Lemma 1(*iv*), except for the appearance of the term $(\mathcal{K}_0 \star \boldsymbol{h}_{i-1}, \nabla \times \boldsymbol{\varphi})$ instead of $(\mathcal{K}_0 \star \boldsymbol{h}_i, \nabla \times \boldsymbol{\varphi})$. $\square$

The variational formulation (17) can be rewritten as ($\boldsymbol{\varphi} \in \mathbf{H}_0(\mathbf{curl},\Omega)$)

$$\left(\partial_t \boldsymbol{v}_n(t), \boldsymbol{\varphi}\right) + \left(\partial_t \boldsymbol{h}_n(t), \boldsymbol{\varphi}\right) + \left(\nabla \times \overline{\boldsymbol{h}}_n(t), \nabla \times \boldsymbol{\varphi}\right) = \left(\overline{\boldsymbol{f}}_n(t), \boldsymbol{\varphi}\right) - \left(\mathcal{K}_0 \star \overline{\boldsymbol{h}}_n(t-\tau), \nabla \times \boldsymbol{\varphi}\right).$$

Next theorem derives the error estimates for the scheme (17). The same convergence rate is obtained as in the error estimates in Theorem 4.

Theorem 5 *(Error). Suppose that* $\boldsymbol{f} \in \mathrm{Lip}([0,T],\mathbf{L}^2(\Omega))$.

(*i*) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl},\Omega)$ *and* $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$ *then*

$$\max_{t\in[0,T]}\left\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\right\|^2 + \max_{t\in[0,T]}\left\|\nabla \times \int_0^t [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 \leqslant C\tau.$$

(*ii*) *If* $\nabla \times (\mathcal{K}_0 \star \boldsymbol{H}_0) \in \mathbf{L}^2(\Omega)$, $\boldsymbol{H}_0 \in \mathbf{H}_0(\mathbf{curl},\Omega)$, $\boldsymbol{H}_0' \in \mathbf{H}_0(\mathbf{curl},\Omega)$ *and* $\nabla \times \nabla \times \boldsymbol{H}_0 \in \mathbf{L}^2(\Omega)$ *then*

$$\max_{t\in[0,T]}\left\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\right\|^2 + \max_{t\in[0,T]}\left\|\nabla \times \int_0^t [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 \leqslant C\tau^2.$$

Please note that the positive constant C *in these estimates is of the form* Ce^{CT}.

Proof. The proof follows the same lines as Theorem 4. The term $\int_0^\eta (\mathcal{K}_0 \star \int_0^\xi [\boldsymbol{h}_n - \bar{\boldsymbol{h}}_n], \nabla \times [\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)])$ in (16) is now replaced by $\int_0^\eta (\mathcal{K}_0 \star \int_0^\xi [\boldsymbol{h}_n(t) - \bar{\boldsymbol{h}}_n(t-\tau)], \nabla \times [\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)])$. This can be handled using integration by parts, (9) and Lemma 2(ii) as follows

$$\begin{aligned}
&\left|\int_0^\eta \left(\mathcal{K}_0 \star \int_0^\xi \big[\boldsymbol{h}_n(t) - \bar{\boldsymbol{h}}_n(t-\tau)\big], \nabla \times \big[\boldsymbol{h}_n(\xi) - \boldsymbol{H}(\xi)\big]\right) \mathrm{d}\xi\right| \\
&= \left|\left(\mathcal{K}_0 \star \int_0^\eta \big[\boldsymbol{h}_n(t) - \bar{\boldsymbol{h}}_n(t-\tau)\big], \nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}]\right)\right. \\
&\quad \left. - \int_0^\eta \left(\mathcal{K}_0 \star \big[\boldsymbol{h}_n(\xi) - \bar{\boldsymbol{h}}_n(\xi-\tau)\big], \nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}]\right) \mathrm{d}\xi\right| \\
&\leqslant \varepsilon \left\|\nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 + C_\varepsilon \int_0^\eta \big\|\boldsymbol{h}_n(t) - \bar{\boldsymbol{h}}_n(t-\tau)\big\|^2 + C \int_0^\eta \left\|\nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 \\
&\leqslant \varepsilon \left\|\nabla \times \int_0^\eta [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 + C_\varepsilon \tau^2 + C \int_0^\eta \left\|\nabla \times \int_0^\xi [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2.
\end{aligned}$$

The rest is the same as in Theorem 4. □

6. Higher regularity

In this section, problem (5) for $\tilde{\delta} = \varepsilon = \mu = \sigma = 1$ is considered

$$\begin{cases}
\partial_{tt}\boldsymbol{H} + \partial_t \boldsymbol{H} - \Delta \boldsymbol{H} + \mathcal{K} \star \boldsymbol{H} = \boldsymbol{f} & \text{in } Q_T; \\
\boldsymbol{H} = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T); \\
\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0 & \text{in } \Omega; \\
\partial_t \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0' & \text{in } \Omega; \\
\nabla \cdot \boldsymbol{H}_0 = \nabla \cdot \boldsymbol{H}_0' = \boldsymbol{0} & \text{in } \Omega.
\end{cases} \tag{18}$$

In this hyperbolic problem, the use of Grönwall's lemma with exponential in time character of the constant cannot be avoided despite of the positive definiteness of $\mathcal{K}$. Therefore, the analysis follows closely the lines of the analysis of problem (11). Only the proof of the uniqueness of the solution of problem (18) is carried out because it uses the main ingredients who are necessary to prove the well-posedness of the problem. Therefore, the other proofs are skipped in this section. The same results are obtained as in the previous section, where the curl-spaces are replaced by analogous $\mathbf{H}^s(\Omega)$-spaces.

The variational formulation of (18) reads as

$$(\partial_{tt}\boldsymbol{H}, \boldsymbol{\varphi}) + (\partial_t \boldsymbol{H}, \boldsymbol{\varphi}) + (\nabla \boldsymbol{H}, \nabla \boldsymbol{\varphi}) + (\mathcal{K} \star \boldsymbol{H}, \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega). \tag{19}$$

Theorem 6 *(Uniqueness). The problem (18) admits at most one solution $\boldsymbol{H} \in C([0,T], \mathbf{H}_0^1(\Omega))$ such that $\partial_t \boldsymbol{H} \in C([0,T], \mathbf{L}^2(\Omega))$.*

Proof. Assume that we have two solutions $\boldsymbol{H}_1$, $\boldsymbol{H}_2$. Then $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2$ fulfills (18) with $\boldsymbol{H}_0 = \boldsymbol{H}_0' = \boldsymbol{0} = \boldsymbol{f}$. Setting $\boldsymbol{\varphi} = \partial_t \boldsymbol{H}$ in (19) and integrating in time for $t \in (0,T)$ we find that

$$\frac{1}{2}\left\|\partial_t \boldsymbol{H}(\eta)\right\|^2+\int_0^\eta\|\partial_t \boldsymbol{H}\|^2+\frac{1}{2}\left\|\nabla \boldsymbol{H}(\eta)\right\|^2=-\int_0^\eta(\mathcal{K}\star \boldsymbol{H},\partial_t \boldsymbol{H}).$$

According to (10), we deduce that

$$\left|\int_0^\eta(\mathcal{K}\star \boldsymbol{H},\partial_t \boldsymbol{H})\right|\leqslant\varepsilon\int_0^\eta\|\partial_t \boldsymbol{H}\|^2+C_\varepsilon\int_0^\eta\|\nabla \boldsymbol{H}\|^2.$$

Hence, fixing a sufficiently small ε and applying Grönwall's argument, we obtain that $\partial_t \boldsymbol{H}=\mathbf{0}$ a.e. in Q_T. Then $\boldsymbol{H}=\mathbf{0}$ a.e. in Q_T because $\boldsymbol{H}_0=\mathbf{0}$. $\square$

As before, the following linear recurrent scheme (convolution implicitly) is proposed

$$\begin{cases}\left(\delta^2 \boldsymbol{h}_i,\boldsymbol{\varphi}\right)+(\delta \boldsymbol{h}_i,\boldsymbol{\varphi})+(\nabla \boldsymbol{h}_i,\nabla\boldsymbol{\varphi})+(\mathcal{K}\star \boldsymbol{h}_i,\boldsymbol{\varphi})=(\boldsymbol{f}_i,\boldsymbol{\varphi}), \quad \boldsymbol{\varphi}\in\mathbf{H}_0^1(\Omega);\\ \boldsymbol{h}_0=\boldsymbol{H}_0\end{cases}\tag{20}$$

which is equivalent to

$$\begin{aligned}a(\boldsymbol{h}_i,\boldsymbol{\varphi}) &:= \left(\frac{1}{\tau^2}+\frac{1}{\tau}\right)(\boldsymbol{h}_i,\boldsymbol{\varphi})+(\nabla \boldsymbol{h}_i,\nabla\boldsymbol{\varphi})+(\mathcal{K}\star \boldsymbol{h}_i,\boldsymbol{\varphi})\\ &= (\boldsymbol{f}_i,\boldsymbol{\varphi})+\left(\frac{1}{\tau^2}+\frac{1}{\tau}\right)(\boldsymbol{h}_{i-1},\boldsymbol{\varphi})+\left(\frac{\delta \boldsymbol{h}_{i-1}}{\tau},\boldsymbol{\varphi}\right)\\ &=: f_i(\boldsymbol{\varphi}).\end{aligned}$$

The bilinear form $a(\boldsymbol{h},\boldsymbol{\varphi})$ is elliptic and continuous in $\mathbf{H}_0^1(\Omega)$ due to the positive definiteness of $\mathcal{K}$, inequality (10) and $\mathbf{H}_0^1(\Omega)\hookrightarrow\mathbf{L}^6(\Omega)$. If $\boldsymbol{H}_0\in\mathbf{L}^2(\Omega)$ and $\boldsymbol{H}_0'\in\mathbf{L}^2(\Omega)$, then the functional $f_i(\boldsymbol{\varphi})$ is linear and bounded in $\mathbf{H}_0^1(\Omega)$, $i=1,\dots,n$. An application of the Lax–Milgram lemma gives the well-posedness of (20) for any $i=1,\dots,n$ and any $\tau>0$.

The following lemma is analogous to Lemma 1.

Lemma 3 *(A priori estimates). Suppose that $\boldsymbol{f}:[0,T]\to\mathbf{L}^2(\Omega)$ obeys $\boldsymbol{f}\in L^2((0,T),\mathbf{L}^2(\Omega))$.*

(i) *Let $\boldsymbol{H}_0\in\mathbf{L}^2(\Omega)$ and $\boldsymbol{H}_0'\in\mathbf{L}^2(\Omega)$. Then, there exists a positive constant C such that*

$$\max_{1\leqslant j\leqslant n}\|\boldsymbol{h}_j\|^2+\sum_{i=1}^n\|\boldsymbol{h}_i-\boldsymbol{h}_{i-1}\|^2+\sum_{i=1}^n\|\tau\nabla \boldsymbol{h}_i\|^2\leqslant C$$

for all $\tau<\tau_0$;

(ii) *If $\boldsymbol{H}_0\in\mathbf{H}_0^1(\Omega)$ and $\boldsymbol{H}_0'\in\mathbf{L}^2(\Omega)$, then*

$$\max_{1\leqslant i\leqslant n}\|\delta \boldsymbol{h}_i\|^2+\max_{1\leqslant i\leqslant n}\|\nabla \boldsymbol{h}_i\|^2+\sum_{i=1}^n\|\delta \boldsymbol{h}_i-\delta \boldsymbol{h}_{i-1}\|^2+\sum_{i=1}^n\|\nabla \boldsymbol{h}_i-\nabla \boldsymbol{h}_{i-1}\|^2\leqslant C$$

for all $\tau<\tau_0$;

(iii) *Moreover, we have that*

$$\tau\sum_{i=1}^n\left\|\delta^2 \boldsymbol{h}_i\right\|^2_{\mathbf{H}^{-1}(\Omega)}\leqslant C;$$

(*iv*) *If* $\partial_t \boldsymbol{f} \in L^2((0,T),\mathbf{L}^2(\Omega))$, $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ *and* $\boldsymbol{H}_0' \in \mathbf{H}_0^1(\Omega)$ *then*

$$\max_{1\leqslant i\leqslant n} \left\|\delta^2 \boldsymbol{h}_i\right\|^2 + \max_{1\leqslant i\leqslant n} \|\nabla\delta\boldsymbol{h}_i\|^2 + \sum_{i=1}^n \left\|\delta^2\boldsymbol{h}_i - \delta^2\boldsymbol{h}_{i-1}\right\|^2 + \sum_{i=1}^n \|\nabla\delta\boldsymbol{h}_i - \nabla\delta\boldsymbol{h}_{i-1}\|^2 \leqslant C$$

for all $\tau < \tau_0$.

The variational formulation (20) can be rewritten as

$$\left(\partial_t \boldsymbol{v}_n(t),\boldsymbol{\varphi}\right) + \left(\partial_t \boldsymbol{h}_n(t),\boldsymbol{\varphi}\right) + \left(\nabla\overline{\boldsymbol{h}}_n(t),\nabla\boldsymbol{\varphi}\right) + \left(\mathcal{K}\star\overline{\boldsymbol{h}}_n(t),\boldsymbol{\varphi}\right) = \left(\overline{\boldsymbol{f}}_n(t),\boldsymbol{\varphi}\right), \quad \boldsymbol{\varphi}\in\mathbf{H}_0^1(\Omega). \tag{21}$$

The main point of the existence theorem is the embedding

$$\mathbf{H}_0^1(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega) \cong \mathbf{L}^2(\Omega)^* \hookrightarrow \mathbf{H}^{-1}(\Omega).$$

Theorem 7 *(Existence). Let* $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$, $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$, $\boldsymbol{f} : [0,T] \to \mathbf{L}^2(\Omega)$ *and* $\boldsymbol{f} \in L^2((0,T),\mathbf{L}^2(\Omega))$. *Assume that* $\nabla\cdot\boldsymbol{H}_0 = \nabla\cdot\boldsymbol{H}_0' = 0 = \nabla\cdot\boldsymbol{f}(t)$ *for any time* $t\in[0,T]$. *Then there exists a solution* $\boldsymbol{H} \in C([0,T],\mathbf{H}_0^1(\Omega))$ *with* $\partial_t\boldsymbol{H} \in L^2((0,T),\mathbf{H}_0^1(\Omega)) \cap C([0,T],\mathbf{L}^2(\Omega))$ *and* $\partial_{tt}\boldsymbol{H} \in L^2((0,T),\mathbf{H}^{-1}(\Omega))$, *which solves (19).*

Now, the following error estimates can be derived. There may be no smaller constants C in comparison with the constants appearing in Theorem 4 because Grönwall's argument with exponential in time character of the constant cannot be avoided.

Theorem 8 *(Error). Assume that* $\boldsymbol{f} \in \mathrm{Lip}([0,T],\mathbf{L}^2(\Omega))$.

(*i*) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega)$ *and* $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$ *then*

$$\max_{t\in[0,T]} \left\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\right\|^2 + \max_{t\in[0,T]} \left\|\nabla\int_0^t [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 \leqslant C\tau.$$

(*ii*) *If* $\boldsymbol{H}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ *and* $\boldsymbol{H}_0' \in \mathbf{H}_0^1(\Omega)$ *then*

$$\max_{t\in[0,T]} \left\|\boldsymbol{h}_n(t) - \boldsymbol{H}(t)\right\|^2 + \max_{t\in[0,T]} \left\|\nabla\int_0^t [\boldsymbol{h}_n - \boldsymbol{H}]\right\|^2 \leqslant C\tau^2.$$

Please note that the positive constant C *in these estimates is of the form* Ce^{CT}.

6.1. Modified scheme in $\mathbf{H}^1(\Omega)$

Last, the following time-discrete scheme is considered, where the convolution term is taken explicitly (from the last time step)

$$\begin{cases} \left(\delta^2\boldsymbol{h}_i,\boldsymbol{\varphi}\right) + (\delta\boldsymbol{h}_i,\boldsymbol{\varphi}) + (\nabla\boldsymbol{h}_i,\nabla\boldsymbol{\varphi}) = (\boldsymbol{f}_i,\boldsymbol{\varphi}) - (\mathcal{K}\star\boldsymbol{h}_{i-1},\boldsymbol{\varphi}), \quad \boldsymbol{\varphi}\in\mathbf{H}_0^1(\Omega); \\ \boldsymbol{h}_0 = \boldsymbol{H}_0 \end{cases} \tag{22}$$

which is equivalent to

$$a(\boldsymbol{h}_i, \boldsymbol{\varphi}) := \left(\frac{1}{\tau^2} + \frac{1}{\tau}\right)(\boldsymbol{h}_i, \boldsymbol{\varphi}) + (\nabla \boldsymbol{h}_i, \nabla \boldsymbol{\varphi})$$
$$= (\boldsymbol{f}_i, \boldsymbol{\varphi}) - (\mathcal{K} \star \boldsymbol{h}_{i-1}, \boldsymbol{\varphi}) + \left(\frac{1}{\tau^2} + \frac{1}{\tau}\right)(\boldsymbol{h}_{i-1}, \boldsymbol{\varphi}) + \left(\frac{\delta \boldsymbol{h}_{i-1}}{\tau}, \boldsymbol{\varphi}\right) =: f_i(\boldsymbol{\varphi}).$$

Via the Lax–Milgram lemma, the existence of a unique solution in $\mathbf{H}_0^1(\Omega)$ is obtained to (22) for any $i = 1, \ldots, n$ and any $\tau > 0$ if $\boldsymbol{H}_0 \in \mathbf{L}^q(\Omega)$, $q > 3$, and $\boldsymbol{H}_0' \in \mathbf{L}^2(\Omega)$.

This scheme (22) can be analyzed in the same way as (20). Remark that the error estimates from Theorem 8 are also valid for (22).

7. Conclusion

The well-posedness of a vectorial nonlocal linear hyperbolic problem (11) with applications in superconductors of type-I is studied. This model is derived from the full Maxwell's equations, the two-fluid model of London and London, and the nonlocal representation (by a space convolution with a singular kernel) of the superconductive current by Eringen. Two time-discrete schemes (based on an explicit and implicit handling of the convolution term) are established. The error estimates are derived for this schemes. Also a symmetrification of this problem is considered in problem (18). The convolution kernel in that problem is positive definite, but this doesn't lead to better error estimates for the time discretization.

Acknowledgments

The authors are supported by the IAP P7/02-project of the Belgian Science Policy. The authors thank the reviewers for closely reading the manuscript and their useful suggestions.

References

[1] A.C. Eringen, Electrodynamics of memory-dependent nonlocal elastic continua, J. Math. Phys. 25 (1984) 3235–3249.
[2] M. Fabrizio, A. Morro, Electromagnetism of Continuous Media, Oxford University Press Inc., New York, 2003.
[3] J. Kačur, Method of Rothe in Evolution Equations, Teubner-Texte Math., vol. 80, Teubner, Leipzig, 1985.
[4] V.F. Kozhevnikov, C.V. Giuraniuc, M.J. Van Bael, K. Temst, C. Van Haesendonck, T.M. Mishonov, T. Charlton, R.M. Dalgliesh, Yu.N. Khaidukov, Yu.V. Nikitenko, V.L. Aksenov, V.N. Gladilin, V.M. Fomin, J.T. Devreese, J.O. Indekeu, Evidence for nonmonotonic magnetic field penetration in a type-I superconductor, Phys. Rev. B 78 (July 2008) 012502.
[5] V. Kozhevnikov, A. Suter, H. Fritzsche, V. Gladilin, A. Volodin, T. Moorkens, M. Trekels, J. Cuppens, B.M. Wojek, T. Prokscha, E. Morenzoni, G.J. Nieuwenhuys, M.J. Van Bael, K. Temst, C. Van Haesendonck, J.O. Indekeu, Nonlocal effect and dimensions of cooper pairs measured by low-energy muons and polarized neutrons in type-I superconductors, Phys. Rev. B 87 (March 2013) 104508.
[6] A. Kufner, O. John, S. Fučík, Function Spaces, Monogr. Textb. Mech. Solids Fluids, Noordhoff International Publishing, Leyden, 1977.
[7] F. London, H. London, The electromagnetic equations of the supraconductor, Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. 149 (866) (1935) 71–88.
[8] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press Inc., New York, 2003.
[9] F.M. Odeh, On the integrodifferential equations of the nonlocal theory of superconductivity, J. Math. Phys. 5 (8) (1964) 1168–1182.
[10] G. Palatucci, O. Savin, E. Valdinoci, Local and global minimizers for a variational energy involving a fractional norm, Ann. Mat. Pura Appl. 192 (4) (2013) 673–718.
[11] T. Roubíček, Nonlinear Partial Differential Equations with Applications, Internat. Ser. Numer. Math., vol. 153, Birkhäuser Verlag, Basel, Boston, Berlin, 2005.
[12] M. Slodička, K. Van Bockstal, A nonlocal parabolic model for type-I superconductors, Numer. Methods Partial Differential Equations (2014), http://dx.doi.org/10.1002/num.21880, in press.
[13] A. Suter, E. Morenzoni, N. Garifianov, R. Khasanov, E. Kirk, H. Luetkens, T. Prokscha, M. Horisberger, Observation of nonexponential magnetic penetration profiles in the Meissner state: a manifestation of nonlocal effects in superconductors, Phys. Rev. B 72 (July 2005) 024506.
[14] K. Van Bockstal, M. Slodička, Error estimates for the full discretization of a nonlocal parabolic model for type-I superconductors, J. Comput. Appl. Math. (2014), http://dx.doi.org/10.1016/j.cam.2014.01.022, in press.
[15] T. Van Cauteren, J.O. Indekeu, Non-local electrodynamics effect on surface superconductivity, Europhys. Lett. 94 (4) (2011) 46001.