Modeling and Numerical Simulations of the Effect of Angular Rotation on Continuous Wave Radar

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August 14, 2019

Model

This is an attempt at a derivation of a model for the effect that the angular rotation (spin) of a sphere has on an incident plane wave. The motivating application is for detecting the spin of a golf ball using continuous wave (CW) radar.

Consider a sphere of radius R traveling with velocity $\vec{\mathbf{v}}$, rotating with angular velocity $\tilde{\omega}$, with a radar transmitter/receiver antenna some distance away. We will assume that the sphere is far enough away from the radar antennas that a) the radar transmit and receive antennas can be modeled as a single point source, and b) the radio waves incident on the sphere can be modeled as plane waves. We also assume diffuse reflection from the surface of the sphere, and that any frequency-dependent effects are negligible. Finally, for the sake of this first pass, we will ignore any travelling wave effects, and instead focus on the Doppler effect.

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We will use a ball-centered frame of reference. In this geometry, the ball is centered at the origin, and we assume, without loss of generality, that the radar is at an initial distance d_0 away along the positive y axis. The plane waves, then, are travelling in the negative y direction, and the ball is travelling with a translational velocity $\vec{\mathbf{v}} = (v_x, v_y, v_z)$. In order to calculate the combined Doppler effect of the ball, we require the velocity of a surface element, $\vec{\mathbf{v}}_{SE}$ which is given by the combined effects of the translational and rotational velocities of the ball:

$$\vec{\mathbf{v}}_{SE} = \vec{\omega} \times \vec{\mathbf{r}} + \vec{\mathbf{v}} \tag{1}$$

where $\vec{\mathbf{r}}$ is the position vector. Thus, if the plane wave incident on the surface element can be modeled by

$$E(t) = E_0 \cos(2\pi f_0 t) \tag{2}$$

then the reflected wave has frequency

$$f = f_0 \left(1 + 2\frac{\vec{\mathbf{v}}_{SE} \cdot \hat{\mathbf{d}}}{c}\right) \tag{3}$$

where $\vec{\mathbf{d}} = (0, d_0, 0) - \vec{\mathbf{v}}t - \vec{\mathbf{r}}$ is the vector from the surface element to the radar antenna, $\hat{\mathbf{d}} = \vec{\mathbf{d}}/||\vec{\mathbf{d}}||$, c is the speed of light, and the factor of two is due to the fact that the surface element both receives and transmits a Doppler-shifted signal. The reflected signal, $G(t)$, viewed from the radar, then, can be written as

$$G(t) = \int_{\Omega} E_0 \cos\left[2\pi f_0 \left(1 + 2\frac{\vec{\mathbf{v}}_{SE} \cdot \hat{\mathbf{d}}}{c}\right) t\right] \left(\hat{\mathbf{d}} \cdot \hat{\mathbf{n}}\right) (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}})\, dS \tag{4}$$

where Ω is the region of the sphere visible to the radar antenna, $\hat{\mathbf{n}}$ is the surface normal vector and the additional $\hat{\mathbf{d}} \cdot \hat{\mathbf{n}}$ and $\hat{\mathbf{y}} \cdot \hat{\mathbf{n}}$ factors are due to the angle of reflection and the plane wave's angle of incidence, respectively.

Equation (4) gives the signal as viewed by the radar. However, we are ultimately interested in the Doppler only signal which is produced by the digital signal processing. For our purposes, the Doppler only signal can be modeled by removing the carrier frequency f_0 from the integral, leaving us with

$$\widetilde{G}(t) = \int_{\Omega} E_0 \cos\left[4\pi \frac{\vec{\mathbf{v}}_{SE} \cdot \hat{\mathbf{d}}}{c} f_0 t\right] \left(\hat{\mathbf{d}} \cdot \hat{\mathbf{n}}\right) (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}})\, dS \tag{5}$$

This (somewhat) general model can be simplified in a few ways to give us a more tractable integral.

No velocity, no spin

Here, we limit the ball to have $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ and $\omega = 0$. With these simplifications, equation (4) can be written as

$$G(t) = \int_0^\pi d\phi \int_0^\pi d\theta R^2 \sin\phi E_0 \cos(2\pi f_0 t) \left(\hat{\mathbf{d}} \cdot \hat{\mathbf{n}}\right) (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}}) \tag{6}$$

where

$$\vec{\mathbf{d}} = (0, d_0, 0) - \vec{\mathbf{r}} \tag{7}$$

and I have defined ϕ as the polar coordinate and θ as the azimuthal coordinate, with the surface parameterization

$$\begin{aligned} x &= R\sin\phi\cos\theta \\ y &= R\sin\phi\sin\theta \\ z &= R\cos\phi \end{aligned} \tag{8}$$

In this case, equation (5) evaluates to a constant since $\vec{\mathbf{v}}_{SE} = \vec{\mathbf{0}}$, so the cosine term is equal to 1, removing the time dependence.

Velocity in y direction only

Here, we limit the ball to travel only in the $-y$ direction, with velocity $\vec{\mathbf{v}} = (0, -v, 0)$. Equation (5) can then be rewritten as

$$\widetilde{G}(t) = \int_0^\pi d\phi \int_0^\pi d\theta R^2 \sin\phi E_0 \cos\left[4\pi \frac{\vec{\mathbf{v}}_{SE} \cdot \hat{\mathbf{d}}}{c} f_0 t\right] \left(\hat{\mathbf{d}} \cdot \hat{\mathbf{n}}\right) (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}}) \tag{9}$$

where

$$\vec{\mathbf{v}}_{SE} = -v\hat{\mathbf{y}} \tag{10}$$

$$\vec{\mathbf{d}} = (0, d_0 - vt, 0) - \vec{\mathbf{r}} \tag{11}$$

Velocity in y direction, spin along z axis

This is the model closest to the situation that we are interested in. Here, we limit the sphere to a rotation along the z axis, with an angular velocity of magnitude ω. Second, we limit the ball to travel only in the $-y$ direction, with velocity $\vec{\mathbf{v}} = (0, -v, 0)$. With these simplifications, equation (5) can be rewritten as

$$\widetilde{G}(t) = \int_0^\pi d\phi \int_0^\pi d\theta R^2 \sin\phi E_0 \cos\left[4\pi \frac{\vec{\mathbf{v}}_{SE} \cdot \hat{\mathbf{d}}}{c} f_0 t\right] \left(\hat{\mathbf{d}} \cdot \hat{\mathbf{n}}\right) (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}}) \tag{12}$$

where

$$\vec{\mathbf{v}}_{SE} = \left(\omega R \sin\phi\, \hat{\theta} - v\hat{\mathbf{y}}\right) \tag{13}$$

$$\vec{\mathbf{d}} = (0, d_0 - vt, 0) - \vec{\mathbf{r}} \tag{14}$$

Some Preliminary Numerical Results

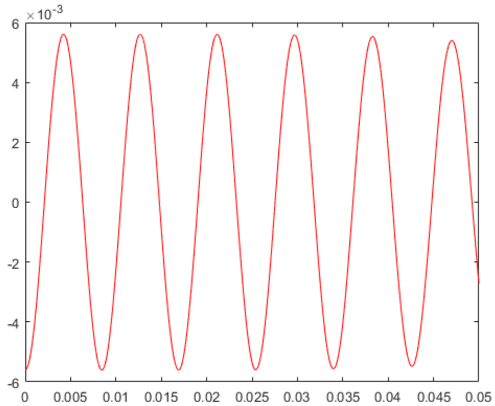
Next, we will attempt to test our model by computing the result of equations (9) and (12). We begin with equation (9) by writing the full forms of the vectors and unit vectors:

$$\begin{aligned} \vec{\mathbf{v}}_{SE} = (0, -v, 0) \quad &, \quad \hat{\mathbf{d}} = \frac{(-x, d_0 - vt - y, -z)}{\sqrt{x^2 + (d_0 - vt - y)^2 + z}} \\ \hat{\mathbf{y}} = (0, 1, 0) \quad &, \quad \hat{\mathbf{n}} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \end{aligned} \tag{15}$$

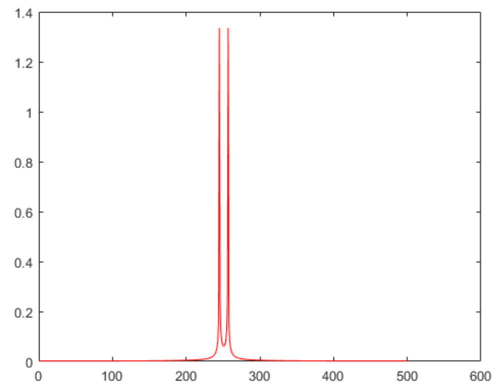
Plugging in equations (15) and (8) into equation (9) gives us

$$\begin{aligned} &\widetilde{G}(t) = \int_0^\pi d\phi \int_0^\pi d\theta\ R^2 \sin^2\phi \sin\theta E_0\ \times \\ &\cos\left[\frac{4\pi v f_0 t \left(R \sin\phi \sin\theta + vt - d_0\right)}{c\sqrt{R^2 + d_0^2 - 2d_0 vt + v^2 t^2 - 2R\left(vt - d_0\right) \sin\phi \sin\theta}}\right] \times \\ &\frac{(vt - d_0) \sin\phi \sin\theta - R}{\sqrt{R^2 + d_0^2 - 2d_0 vt + v^2 t^2 - 2R\left(vt - d_0\right) \sin\phi \sin\theta}} \end{aligned} \tag{16}$$

Figure 1 shows a numerically integrated solution of equation (16), and its corresponding discrete Fourier transform. Numerical integration was performed using Matlab's `integral2` function.



(a)



(b)

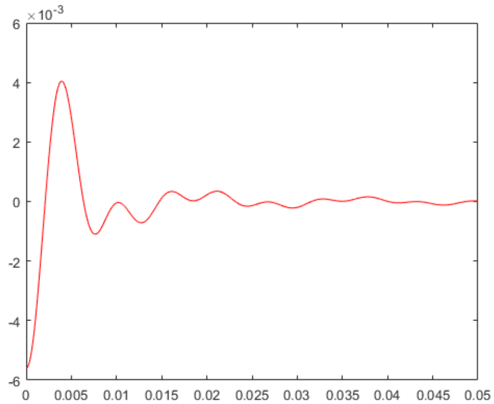
Figure 1: Left: numerical integration of equation (16) for $t \in [0, 0.05]$, $N = 500$, $R = 0.042$, $E_0 = 1$, $c = 3 \times 10^8$, $f_0 = 300 \times 10^6$, $v = 60$ and $d_0 = 5$. Right: Fast Fourier transform of the function in left figure.

Performing a similar calculation for equation (12), and noting that $\hat{\theta} = (-\sin\theta, \cos\theta, 0)$ we have

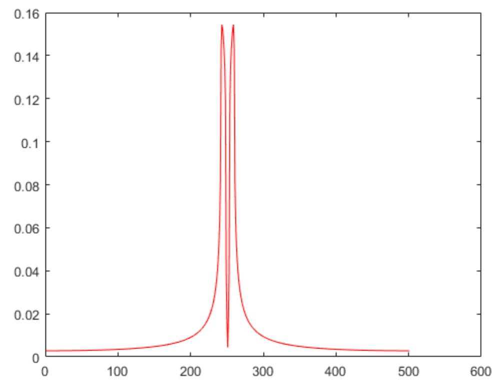
$$\begin{aligned} \vec{\mathbf{v}}_{SE} &= (\omega R \sin\theta \sin\phi, \omega R \cos\theta \sin\phi - v, 0) \\ \hat{\mathbf{d}} &= \frac{(-x, d_0 - vt - y, -z)}{\sqrt{x^2 + (d_0 - vt - y)^2 + z}} \\ \hat{\mathbf{y}} = (0,1,0) \quad &, \quad \hat{\mathbf{n}} = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \end{aligned} \tag{17}$$

$$\begin{aligned} &\widetilde{G}(t) = \int_0^{\pi} d\phi \int_0^{\pi} d\theta \; R^2 \sin^2\phi \sin\theta E_0 \times \\ &\cos\left[\frac{4\pi f_0 t \left[\omega R \sin\phi \cos\theta (d - vt) + v^2 t - dv + vR \sin\phi \sin\theta\right]}{c\sqrt{R^2 + d_0^2 - 2d_0 vt + v^2t^2 - 2R\,(vt - d_0) \sin\phi \sin\theta}}\right] \times \\ &\frac{(vt - d_0) \sin\phi \sin\theta - R}{\sqrt{R^2 + d_0^2 - 2d_0 vt + v^2t^2 - 2R\,(vt - d_0) \sin\phi \sin\theta}} \end{aligned} \tag{18}$$

Figure 2 shows the numerically integrated solution and discrete Fourier transform of equation (18).



(a)



(b)

Figure 2: Left: numerical integration of equation (18) for $t \in [0, 0.05]$, $N = 500$, $R = 0.042$, $E_0 = 1$, $c = 3 \times 10^8$, $f_0 = 300 \times 10^6$, $v = 60$, $d_0 = 5$ and $\omega = 733$ rad$/s$. Right: Fast Fourier transform of the function in left figure.

Finally, Figure 3 shows a comparison between equation (16) and (18) for a higher value of v to separate the two DFT peaks.

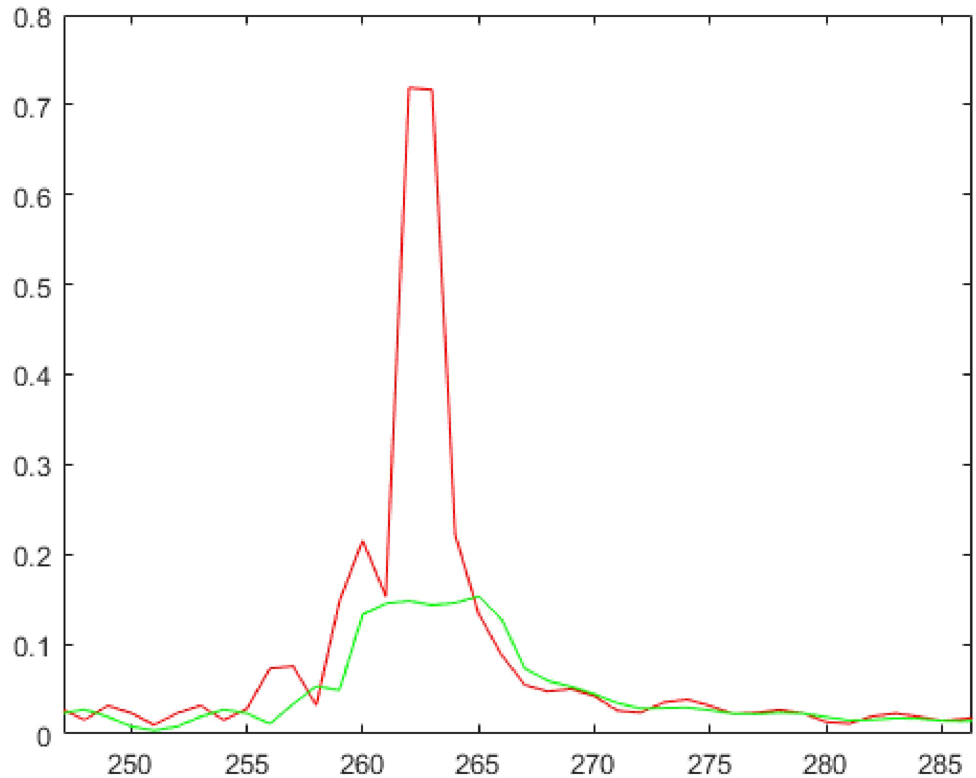
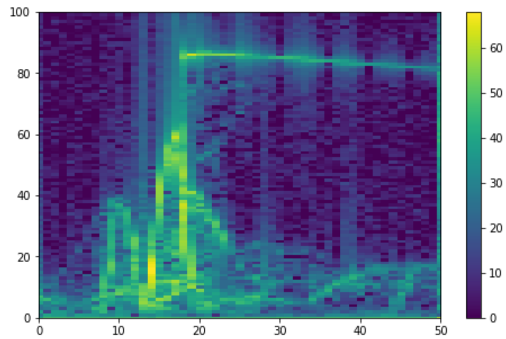


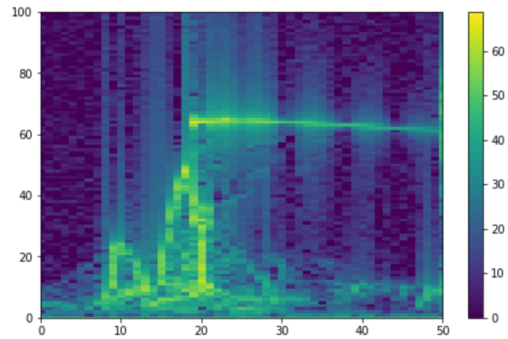
Figure 3: Comparison of FFT peaks between equation (16) (red line) and (18) (green line) for $t \in [0, 0.05]$, $N = 500$, $R = 0.042$, $E_0 = 1$, $c = 3 \times 10^8$, $f_0 = 300 \times 10^6$, $v = 120$, $d_0 = 5$ and $\omega = 733$ rad$/s$.

Investigation of "beats" in experimental FFT

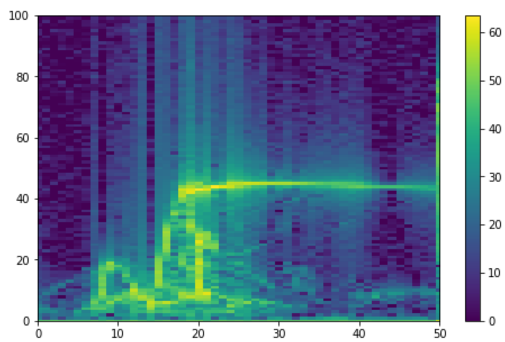
Next, we will use our model to investigate the occurance of "beats" in the FFT spectrum, which occur in our experimental radar when measuring the flight of a ball struck by various golf clubs. Three examples are shown in Figure 4 for different clubs, where the beats can be seen in the "smears" around the ball track.



(a)



(b)



(c)

Figure 4: Observed "beats" in fast Fourier transformed spectra for golf balls struck by a) a driver, b) an iron and c) a wedge. Vertical axis: FFT frequency bin. Horizontal axis: time unit. FFT performed on signal with 24.125×10^9 Hz carrier frequency, 195312.5 samples per second, and 51.2 ms sampling period. "Beats" can be observed in the "smears" around the ball track.

Our test radar has the following characteristics:

- 24.125×10^9 Hz carrier frequency
- 195312.5 samples per second
- 51.2 ms sampling period

and some typical velocity and spin values for the three clubs are listed in Table 1.

	Speed (MPH)	Spin (RPM)
Driver	177	2300
Iron	133	6100
Wedge	98	10000

Table 1: foo

We will proceed by integrating equation (18) in windows, and replicating the type of frequency bin vs. time frame plots in Figure 4.