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help with abstract math

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Contents

FUNCTIONS: SPECIFICATION AND DEFINITION

This section describes precisely what is meant by “function”.

- We start by giving a specification of “function”.
- After that, we get into the technicalities of the definitions of the general concept of function.

Specification of function

See the abstractmath discussion of specification.

I will use two running examples throughout this discussion:

- F is the function defined on the set $\{1, 2, 3, 6\}$ as follows: $F(1) = 3$, $F(2) = 3$, $F(3) = 2$, $F(6) = 1$. This is the function called “Finite” in the chapter on examples of functions.
- G is the real-valued function defined by the formula $G(x) = x^2 + 2x + 5$.

Specification: functions

A function f is a mathematical object which ***determines and is completely determined by*** the following data:

- f has a **domain**, which is a set. The domain may be denoted by $\operatorname{dom} f$.
- f has a **codomain**, which is also a set and may be denoted by $\operatorname{cod} f$.
- For each element a of the domain of f, f has a **value** at a, denoted by $f(a)$.
- The value of f at a is completely determined by a and f .
- The value of f at a must be an element of the codomain of f.

When someone wants to talk about a function, they may not mention the domain or codomain. In many cases, as in calculus books, it may not *matter* what the domain or codomain is. See Functions: Notation and Terminology.

The operation of finding $f(a)$ given f and a is called **evaluation**.

Examples

• The definition above of the finite function *F* specifies that the domain is the set $\{1, 2, 3, 6\}$. The value of *F* at each element of the domain is given explicitly. The value at 3, for example, is 2, ***because the definition says that*** $F(2) = 3$. The codomain of *F* is not specified, but must include the set $\{1, 2, 3\}$.

• The definition of *G* above gives the value at each element of the domain by a formula. The value at 3, for example, is $G(3) = 3^2 + 2 \cdot 3 + 5 = 20$. The definition does not specify the domain or the codomain. The convention in the case of functions defined on the real numbers by a formula is to take the domain to be ***all real numbers at which the formula is defined***. In this case, that is ***every*** real number, so the domain is $\mathbb{R}$. The codomain must include all real numbers greater than or equal to 4. (Why?)

• In the chapter on examples of functions, the Split function is defined by something more complicated than a formula. It is given by two formulas; which one you apply depends on where the input is in the real line. The Split function is explicitly given the domain the closed interval $[0, 1]$. Its codomain is not given. It must include $[0, 1]$.

Definition of function

This section gives the strict modern definition of function.

In the nineteenth century, mathematicians realized that it was necessary for some purposes (particularly harmonic analysis) to give a mathematical definition of the concept of function. A stricter version of this definition turned out to be necessary in algebraic topology and other fields, and that is the one I give here.

Other weaker definitions are also used. See graphs and functions.

To state this definition we need a preliminary idea.

The functional property

A set R of ordered pairs has the **functional property** if two pairs in R with the same first coordinate have to have the same second coordinate (which means they are the same pair).

Examples

• The set $\{(1, 2), (2, 4), (3, 2), (5, 8)\}$ has the functional property, since no two different pairs have the same first coordinate.

• The set $\{(1, 2), (2, 4), (3, 2), (2, 8)\}$ does ***not*** have the functional property. There are two different pairs with first coordinate 2.

Note that in both sets there are two different pairs with the same ***second*** coordinate. ***This is irrelevant to the functional property.***

• The empty set $\emptyset$ has the function property vacuously.

Example: graph of a function defined by a formula

The graph of any function studied in beginning calculus has the functional property. For example, the graph of the function *G* is the set

$$\{(x, x^2 + 2x + 5) \mid x \in \mathbb{R}\}$$

This set has the functional property because if *x* is any real number, the formula $x^2 + 2x + 5$ defines a ***specific real number***.

The graph is a mathematical object: a set of ordered pairs. The *illustration* called the "graph" in a calculus book is a *picture* of the graph in this sense.

• if $x = 0$, then $x^2 + 2x + 5 = 5$, so the pair $(0, 5)$ is an element of the graph of *G*. Each time you plug in 0 in the formula you get 5.

• if $x = 1$, then $x^2 + 2x + 5 = 8$.

• if $x = -2$, then $x^2 + 2x + 5 = 5$.

No other pair whose first coordinate is -2 is in the graph of G, only $(-2, 5)$. That is because when you plug -2 into the formula $x^2 + 2x + 5$, you get 5 and nothing else. Of course, $(0, 5)$ is in the graph, but that does not contradict the functional property. $(0, 5)$ and $(-2, 5)$ have the same *second* coordinate, but that is OK.

How to think about the functional property

The point of the functional property is that for any pair in the set of ordered pairs, ***the first coordinate determines what the second one is***. That's why you can write "$G(x)$" for any x in the domain of G and not be ambiguous.

Mathematical definition of function

A **function** f is a mathematical structure consisting of the following objects:

- A set called the **domain of** f, denoted by $\operatorname{dom} f$.
- A set called the **codomain of** f, denoted by $\operatorname{cod} f$.
- A set of ordered pairs called the **graph of** f, with the following properties:
  - $\operatorname{dom} f$ is the set of all first coordinates of pairs in the graph of f.
  - Every second coordinate of a pair in the graph of f is in $\operatorname{cod} f$ (but $\operatorname{cod} f$ may contain other elements).
  - The graph of f has the functional property. Using arrow notation, this implies that $f : A \to B$.

Examples

- Let F have graph $\{(1, 2), (2, 4), (3, 2), (5, 8)\}$ and define $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 8\}$. Then $F : A \to B$ is a function.

In speaking, we would usually say, "F is a function from A to B."

- Let G have graph $\{(1, 2), (2, 4), (3, 2), (5, 8)\}$ (same as above), and define $A = \{1, 2, 3, 5\}$ and $C = \{2, 4, 8, 9, 11, \pi, 3/2\}$. Then $G : A \to C$ is a (admittedly ridiculous) function. Note that all the second coordinates of the graph are in C, along with a bunch of miscellaneous suspicious characters that are ***not*** second coordinates of pairs in the graph.
- Let H have graph $\{(1, 2), (2, 4), (3, 2), (5, 8)\}$. Then $H : A \to \mathbb{R}$ is a function.

According to the definition
F, G and H are three different functions.

- Using the same definition of the graph, let $D = \{1, 2, 5\}$ and $E = \{1, 2, 3, 4, 5\}$. ***Then there is no function $K : D \to A$ and no function $L : E \to A$.*** Neither D nor E ***has exactly the same*** elements as the first coordinates of the graph.

Identity and inclusion

Suppose we have two sets A and B with $A \subseteq B$.

- The identity function on A is the function $\operatorname{id}_A : A \to A$ defined by $\operatorname{id}_A(x) = x$ for all $x \in A$. (Many authors call it 1_A).
- The inclusion function from A to B is the function $i : A \to B$ defined by $i(x) = x$ for all $x \in A$. Note that there is a different function for each pair of sets A and B for which $A \subseteq B$. Some authors call it $i_{A,B}$ or $\operatorname{inc}_{A,B}$.

Remark

The identity function and an inclusion function for the same set A have exactly the same graph, namely $\{(a, a) | a \in A\}$.

Graphs and functions

• If f is a function, the **domain** of f is the set of first coordinates of all the pairs in f.

• If $x \in \operatorname{dom} f$, then $f(x)$ ***is the second coordinate of the only ordered pair in*** f ***whose first coordinate is*** x***.***

Example

The set $\{(1,2),(2,4),(3,2),(5,8)\}$ has the functional property, so it is a function. Call it H. Then its domain is $\{1,2,3,5\}$ and $H(1) = 2$ and $H(2) = 4$. $H(4)$ is not defined because there is no ordered pair in H beginning with 4 (hence 4 is not in dom H.)

I showed above that the graph of the function G, ordinarily described as "the function $G(x) = x^2 + 2x + 5$", has the functional property. The specification of function requires that we say what the domain is and what the value is at each point. These two facts are ***determined*** by the graph.

Because of this, many authors ***define*** a function as a graph with the functional property. Now, the graph of a function G may be denoted by $\Gamma(G)$. For those authors, $G = \Gamma(G)$, which causes a clash of our mental images of "graph" and "function". ***In every important way except the definition, they are different!***

Γ is the Greek letter uppercase "gamma".

A definition is a device for making the meaning of math technical terms precise. When a mathematician think of "function" they think of ***many aspects of functions***, such as a map of one shape in another, a computational process, a renaming, and so on. These ways of thinking are spelled out in the chapter Functions: Images and Metaphors. ***One*** of those ways of thinking is function-as-graph, and that happens to be the best way to define the concept of function, but that doesn't make thinking of a function-as-a-graph-with-the-functional-property the most important way of thinking about it.

In my opinion, it is not even in the top three.

Functions in math, logic and computing science

Until late in the nineteenth century, functions were usually thought of as defined by formulas. Problems arose in the theory of harmonic analysis which made mathematicians require a more general notion of function. The definitions of function given here is the modern version of that more general concept. It replaces the algorithmic and dynamic idea of a function as a way of computing an output value given an input value by the static, abstract concept of a function as having a domain, a codomain and a value lying in the codomain for each element of the domain. Of course, often a definition by formula will give a function in this modern sense. However, there is no requirement that a function be given by a formula. The modern concept of function has been obtained from the formula-based idea by abstracting basic properties the old concept had (specifically properties of the graph) and using them as the basis of the new definition. The concept of function as a formula never disappeared entirely, but was studied mostly by logicians who generalized it to the study of function-as-algorithm. (This is an oversimplification of history.) Of course, the study of algorithms is one of the central topics of modern computing science, so the notion of function-as-formula (more generally, function-as-algorithm) has achieved a new importance in recent years. Nevertheless, computer science needs the abstract definition of function given here. Functions such as sine may be (and quite often are) programmed to look up their values in a table instead of calculating them by a formula, an arrangement which gains speed at the expense of using more memory. The example of a finite function given above is a baby example of a table look-up function.