Real Analysis - A Long Form Mathematics Textbook Chapter 1

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1.5 The Completness Axiom

The set R has two binary operations, addition(+) and multiplication (*), and is the unique set satisfying the following axioms:

- Axiom 1: (Commutative Law), If a, b $\in$ R, then $a + b = b + a$ and $a * b = b * a$
- Axiom 2: (Distributive Law), If a,b,c $\in$ R, then $a*(b+c) = a*b+a*c$
- Axiom 3: (Associative Law), If a,b,c $\in$ R, then $(a+b)+c = a+(b+c)$ and $(a * b) * c = a * (b * c)$
- Axiom 4: (Identity Law). There are special elements 0,1 $\in$ R, where $a + 0 = a$ and $a * 1 = a$ for all a $\in$ R
- Axiom 5: (Inverse Law). For each a $\in$ R, there is an element -$a \in R$ such that $a + (-a) = 0$. If a $\neq$ 0, then there is also an element $a^{-1} \in R$ such that $a * a^{-1} = 1$
- Axiom 6: (Order Axiom). There is nonempty subset $P \subseteq R$, called the positive elements, such that
  - If a,b $\in$ P, then a + b $\in$ P and a*b $\in$ P
  - If a $\in$ R and a $\neq$ 0, then either a $\in$ P or -a $\in$ P, but not both.

- Axiom 7: (Completeness Axiom). Given any nonempty A $\subseteq$ R where A is bounded above, A has a least upper bound. In other words, $sup(A) \in R$ for every such A.

Theorem 1.24 (Suprema analytically). Let A $\subseteq$ R. Then $sup(A) = \alpha$ if and only if

- α is an upper bound of A, and
- Given any $\epsilon > 0, \alpha - \epsilon$ is not an upper bound of A. That is, there is some $x \in A$ for which $x > \alpha - \epsilon$

Likewise, $inf(A) = \beta$ iff

- β is a lower bound of A, and
- Given any $\epsilon > 0$, $\beta + \epsilon$ is not a lower bound of A. That is there is some $x \in A$ for which $x < \beta + epsilon$

1.7 The Archimedean Principle

Lemma 1.26 (The Archimedean Property). If a and b are real numbers with a ¿ 0, then there exists a natural number n such that $na > b$. In particular, for any $\epsilon > 0$ there exists $n \in N$ such that $1/n < \epsilon$.

Principle 1.34 (Well-ordering principle). Every non-empty subset of natural numbers contains a smallest element.

Exercises

Exercise 1.1

Explain the error in the following proof that 2=1. Let x = y. Then

$$\begin{aligned} x^2 &= xy \\ x^2 - y^2 &= xy - y^2 \\ (x+y)(x-y) &= y(x-y) \\ x + y &= y \\ 2y &= y \\ 2 &= 1 \end{aligned} \tag{1}$$

answer: by dividing by (x-y), the proof is essentially dividing by zero, which is mathematicall undefined; thus this invalidates the entire proof.

Exercise 1.2

Which of the following statements are true? Give a short explanation for each of your answers.

- For ever $n \in N$, there is $m \in N$ such that $m > n$.
- For every $m \in N$, there is an $n \in N$ such that $m > n$.
- There is an $m \in N$ such that for every $n \in N$, $m \geq n$.
- There is an $n \in N$ such that for every $m \in N$, $m \geq n$.
- There is an $n \in R$ such that for every $m \in R$, $m \geq n$.
- For every pair $x < y$ of integers, there is an integer z such that $x < z < y$.
- For every pair $x < y$ of real numbers, there is a real number z such that $x < z < y$.

answer:

- True. For any natural number n, choosing $m = n + 1$ satisfies $m > n$. Natural numbers are unbounded above.
- False. If m=1, there is no $n \in N$ with $n < 1$.
- False. There is no largest natural number; N is infinite.
- True. Let n=1. For all $m \in N$, $m \geq 1$.
- False. Real numbers extend to $-\infty$; no universal lower bound $n \in R$.
- False. If x and y are consecutive integers (e.g., $x = 2, y = 3$), no integer z exists between them.
- True. For real numbers, $z = (x + y)/2$ always satisfies $x < z < y$.

Exercise 1.3

If A and B are two boxes (possibly with things inside), describe the following in terms of boxes:

- $A \ B$
- $P(A)$
- $|A|$

answers:

- Imagine looking inside box A and taking out any item that is also found in box B. What remains in A are only those things that are not in B.
- Think of every possible way you could select items from box A (including selecting none, or all). Each possible selection is itself a (possibly empty) box. The collection of all these possible boxes is $P(A)$.
- Open box A and count how many items are inside. That count is $|A|$.

Exercise 1.4

If $A_1, A_2, A_3, ..., A_n$ are all boxes (possibly with things inside), describe the following terms of boxes:

- $\bigcup_{i=1}^{n} A_i$
- $\bigcap_{i=1}^{n} A_i$

answers:

- If an item exists in at least one box, it appears once in the union box.
- If even one box lacks an item, it is excluded from the intersection.

Exercise 1.5

Prove that each of the following holds for any sets A and B.

- $A \cup B = A$ iff $B \subseteq A$
- $A \cap B$ iff $A \subseteq B$
- $A \ B = A$ iff $A \cap B = \emptyset$
- $A \ B = \emptyset$ iff $A \subseteq B$

answers:

- (forward)If $A \cup B = A$, every element of B must already be in A. Thus $B \subseteq A$. (reverse) If $B \subseteq A$, combining A and B adds no new elements, so $A \cup B = A$
- (forward) if $A \cap B = A$, all elements of A are in B, so $A \subseteq B$. (reverse) if $A \subseteq B$, the intersection $A \cap B$ contains exactly A.
- (forward) if $A \setminus B = A$, no elements of A are in B, so $A \cap B = \emptyset$. (reverse) if $A \cap B = \emptyset$, removing B from A leaves A unchanged.
- (forward) if $A \setminus B = \emptyset$, all elements of A are in B, so $A \subseteq B$. (reverse) if $A \subseteq B$, removing B from A removes all elements, leaving $\emptyset$

Exercise 1.6

Suppose f:$X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

- Prove that $f(f^{-1}(B)) \subseteq B$
- Give an example where $f(f^{-1}(B)) \neq B$
- Prove that $A \subseteq f^{-1}(f(A)))$
- Give an example where $A \neq f^{-1}(f(A)))$

answers:

- Let $y \in f(f^{-1}(B))$. By definition, there exists $x \in f^{-1}(B)$ such that $f(x) = y$. Since $x \in f^{-1}(B)$, $f(x) \in B$ by the definition of preimage. Thus $y = f(x) \in B$. Therefore, $f(f^{-1}(B)) \subseteq B$

- Let f: $\{1,2\} \to \{a,b,c\}$ with $f(1) = a$ and $f(2) = b$. Take $B = \{a,b,c\}.f^{-1}(B) = \{1,2\}.f(f^{-1}(B)) = f(\{1,2\}) = \{a,b\} \neq B$
- Let $x \in A$. Then $f(x) \in f(X)$. By definition of preimage, $x \in f^{-1}(f(A))$. Thus, $A \subseteq f^{-1}(f(A))$
- Let f: $\{1,2\} \to \{a\}$ with $f(1) = f(2) = a$. Take $A = \{1\}.f(A) = \{a\}$. $f^{-1}(f(A)) = f^{-1}(\{a\}) = \{1,2\} \neq A$

Exercise 1.7

Suppose that f:$X \to Y$ and g: $Y \to X$ are functions and that the composite g of f is the identity function id" $X \to X$. (The identity function sends every element to itself: $id(x) = x$) Show that f must be a one-to-one function and that g must be an onto function.

answer:

- f is injective

  Assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Applying g to both sides: $g(f(x_1)) = g(f(x_2))$ since g of f = id_X, this simplifies to: $x_1 = x_2$. Thus, $f(x_1) = f(x_2) \to x_1 = x_2$, proving f is injective.

- g is surjective

  For any $x \in X$, let $y = f(x) \in Y$. Applying g to y: $g(y) = g(f(x)) = x$. Thus, every $x \in X$ has preimage $y = f(x) \in Y$ under g, proving g is surjective.

Exercises 1.8

The following are special cases of De Morgan's laws

- Prove that $(A \cap B)^c = A^c \cup B^c$

  (a) $(A \cap B)^c \subseteq A^c \cup B^c$

  Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$. Thus, $x \in A^c$ or $x \in B^c$. Therefore $x \in A^c \cup B^c$.

  (b) $A^c \cup B^c \subseteq (A \cap B)^c$

  Let $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, so $x \notin A$ or $x \notin B$. Thus, $x \notin A \cap B$, so $x \in (A \cap B)^c$

- Prove that $(A \cup B)^c = A^c \cap B^c$

  (a) $(A \cup B)^c \subseteq A^c \cap B^c$

  Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.

  (b) $A^c \cap B^c \subseteq (A \cup B)^c$

  Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A and x \notin B$. Thus $x \notin A \cup B$, so $x \in (A \cup B)^c$. Therefore, $(A \cup B)^c = A^c \cap B^c$

Exercise 1.9

- Prove that $\sqrt{3}$ is irrational

  Assume $\sqrt{3}$ is rational, so $\sqrt{3} = a/b$ where $a, b \in Z$ are coprime. Squaring both sides: $3 = \frac{a^2}{b^2} \implies 3b^2 = a^2$. This implies a^2 is divisible by 3, so a must also be divisable by 3 (by the fundamental theorem of arithmetic). Let a = 3k. Substituting: $3b^2 = (3k)^2 \implies 3b^2 = 9k^2 \implies b^2 = 3k^2$. Thus, b^2 is divisible by 3, so b is also divisible by 3. This contradicts a and b being coprime. Hence, $\sqrt{3}$ is irrational.

- What goes wrong when you try to adapt your argument from part (a) to show that $\sqrt{4}$ is irrational?

- In part (a) you proved that $\sqrt{3}$ to be irrational, and essentially the same proof shows that $\sqrt{5}$ is irrational. By considering their product or otherwise, prove that $\sqrt{3} - \sqrt{5}$ and $\sqrt{3} + \sqrt{5}$ are either both rational or both irrational. Deduce that they must both be irrational.

Exercise 1.10

Prove that the multiplicative identity in a field is unique.

Let F be a field. Suppose there exist two multiplicative identities 1 and e in F. By definition of a multiplicative identity, for all $a \in F$:

$1 \times a = a$ and $e \times a = a$

Consider the case where a = e. Applying the identity property of 1:

$1 \times e = e$

Now consider the case where a = 1. Applying the identity property of e:

$e \times 1 = 1$

In a field, multiplication is commutative ($a \times b = b \times a$), so:

$1 \times e = e \times 1$

From (1),(2) and (3), we conclude:

$e = 1$

Thus, there cannot be two distinct multiplicative identities in F. The multiplicative identity is unique.

Exercise 1.11

Given an ordered field F, recall that we defined the positive elements to be a nonempty subset $P \subseteq F$ that satisfies both the following conditions:

(i) If $a, b \in P$, then $a + b \in P$ and $a \times b \in P$

(ii) If $a \in F$ and $a \neq 0$, then either $a \in P$ or $-a \in P$, but not both.

- Give an example of some $P_1 \subseteq R$ that satisfies (i) but not (ii)

  $P_1 =$ R(the entire set of real numbers)

  Satisfies (i): Closed under addition and multiplication

  Fails (ii): For any $a \neq 0$, both a and -a belong to P_1, violating the "exactly one" requirement.

- Give an example of some $P_2 \subseteq R$ that satisfies (ii) but not (i)

  $P_2 = \{x \in R | (x > 0 and x \notin Z) or (x < 0 and x \in Z)\}$

  Satisfies (ii): For every $a \neq 0$: if a is positive non-integer, $a \in P_2$. If a is a positive integer, -$a \in P_2$. If a is negative integer, $a \in P_2$. If a is negative non-integer, $-a \in P_2$.

  Fails (i): $0.5 \in P_2$ and $0.5 \in P_2$ but $0.5 + 0.5 = 1 \notin P_2$. $-1 \in P_2$ and $0.5 \in P_2$, but $-1 + 0.5 = -0.5 \notin P_2$.

Exercises 1.12

Assume that F is an ordered field and $a, b, c, d \in F$ with $a < b$ and $c < d$.

- Show that $a+c < b+d$ Step 1: use the additive property of inequalities in ordered fields: If $a < b$, then $a+c < b+c$. If $c < d$, then $b+c < b+d$.

  Step 2: By transitivity of $<$: $a + c < b + c < b + d \implies a + c < b + d$

- Prove that it is not necessarily true that $ac < bd$.

  Counter example:

  Let F = R, and choose: a = -2, b = -1, c = -3, d = -2.

  - a ¡ b(since -2 ¡ -1) and c ¡ d (since -3 ¡ -2)
  - Compute ac = (-2)(-3) = 6 and bd = (-1)(-2) = 2
  - 6 $\not<$ 2, so $ac < bd$ fails.

Exercise 1.13

Let a,b and ϵ be elements of an ordered field.

- Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

  Suppose, for contradiction, that $a > b$. Then $a - b > 0$. Let $\epsilon = a - b$, which is a positive number. Plug this into the assumption $a < b+\epsilon = a$. This says $a < a$, which is impossible. Therefore, our assumption that $a > b$ must be false. So it must be that $a \leq b$.

- Use part (a) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$

  Suppose, for contradiction, that $a \neq b$. Then $|a - b| > 0$. Let $\epsilon = |a - b|/2$, which is still positive. By assumption $|a - b| < \epsilon = |a - b|/2$. But this says $|a - b| < |a - b|/2$, which is impossible unless $|a - b| = 0$. Therefore, our assumption that $a \neq b$ must be false. So $a = b$.

Exercise 1.14

Prove that the equality —ab— = —a——b— holds for all real numbers a and b.

Definition of absolute value: For any real number

$$x : |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- case 1: both $a \leq 0$ and $b \leq 0$
  - $ab \geq 0$
  - $|ab| = ab$

  - $|a| = a, |b| = b$
  - $|a||b| = ab$
  - So, $|ab| = |a||b|$

- case 2: $a \geq 0, b < 0$
  - $ab \leq 0$
  - $|ab| = -(ab) = -ab$
  - $|a| = a, |b| = -b$
  - $|a||b| = a(-b) = -ab$
  - So, $|ab| = |a||b|$

- case 3: $a < 0, b \geq 0$ similar to case 2

- case 4: $a < 0, b < 0$
  - $ab \geq 0$
  - $|ab| = (-a)(-b) = ab$
  - $|a| = -a, |b| = -b$
  - $|a||b| = (-a)(-b) = ab$
  - So, $|ab| = |a||b|$

Exercise 1.15

For each of the following, find all numbers x which satisfy the expression.

- $|x - 4| = 7$

  $\{-3, 11\}$

- $|x - 4| < 7$

  $(-3, 11)$

- $|x + 2| < 1$

  $(-3, -1)$

- $|x - 1| + |x - 2| > 1$

  $(-\infty, 1) \cup (2, \infty)$

- $|x-1| + |x+1| > 1$

  R

- $|x-1||x+1| = 0$

  $\{-1, 1\}$

- $|x-1||x+2| = 3$

  $\{\frac{-1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}\}$

Exercise 1.16

Let $max\{x, y\}$ denote the maximum of the real numbers x and y, and let $min\{x, y\}$ denote the minimum. For example, $min\{-1, 4\} = -1$, and also min{-1,-1} = -1. Prove that

$max\{x, y\} = \frac{x+y+|y-x|}{2}$ and $min\{x, y\} = \frac{x+y-|y-x|}{2}$

Then find a formula for $max\{x, y, z\}$ and $min\{x, y, z\}$.

- Consider two cases to prove for maximum, minimum

  - $y \geq x$

    maximum: $\frac{x+y+|y-x|}{2} = \frac{x+y+y-x}{2} = y$

    minimum: $\frac{x+y-|y-x|}{2} = \frac{x+y-y+x}{2} = x$

  - $y < x$

    maximum: $\frac{x+y+|y-x|}{2} = \frac{x+y+x-y}{2} = x$

    minimum: $\frac{x+y-|y-x|}{2} = \frac{x+y-x+y}{2} = y$

  Thus the formula holds for all $x, y \in R$

- extension to three variables

  - maximum of three variables $max\{x, y, z\} = max(max(x, y), z) = \frac{\frac{x+y+|y-x|}{2} + z + |z - \frac{x+y+|y-x|}{2}|}{2}$

  - minimum of three variables $min\{x, y, z\} = min(min(x, y), z) = \frac{\frac{x+y-|y-x|}{2} + z - |z - \frac{x+y-|y-x|}{2}|}{2}$

Exercise 1.17

Prove that if $a, b \in R$ and $0 < a < b$, then $a^n < b^n$ for any positive integer n.

- base case

  given $0 < a < b$, the inequality holds by assumption.

- inductive step:

  Assume $a^k < b^k$ for some integer $k \geq 1$. $a^{k+1} < ab^k$. Since a ¡ b, then $ab^k < bb^k = b^k + 1$. Thus, $a^{k+1} < b^{k+1}$.

- By induction, $a^n < b^n$ holds for all positive integers n.

Exercise 1.18

Prove that if $a_1, a_2, ..., a_n$ are real numbers, then: $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$.

Use induction based on triangle inequality of two numbers to prove.

Exercise 1.19

Prove that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for every natural number n.

With partial fractions, we know $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = (1/1-1/2)+(1/2-1/3)+...+(1/n-1/(n+1)) = \frac{n}{n+1} \quad (2)$$

Exercise 1.20

Determine which natural numbers, n, have the property that $\sqrt{n}$ is irrational.

A natural number n has an irrational square root if and only if it's not a perfect square.

Exercise 1.21

Let $f : X \to Y$, and assume $A_1, A_2 \subseteq X$. Show that: $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Recall that if A is a set, then $f(A) = \{f(x) : x \in A\}$.

Let $y \in f(A_1 \cap A_2)$, then there exists $x \in A_1 \cap A_2$ that results in $y = f(x)$. Since $x \in A_1 \cap A_2$, $x \in A_1$ and $x \in A_2$. Therefore, $y = f(x) \in f(A_1)$ and

$y = f(x) \in f(A_1)$; this gives $y \in f(A_1) \cap f(A_2)$. Thus proves $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Exercise 1.22

Give an example of a function f, and a pair of sets A and B, for which

$$f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2) \tag{3}$$

Recall that if A is a set, then $f(A) = \{f(x) : x \in A\}$.

Let $f : R \to R$ be $f(x) = x^2$ $A_1 = \{1\}, A_2 = \{-1\}$.

$$\begin{aligned} f(A_1 \cap A_2) = f(\emptyset) = \emptyset \\ f(A_1) \cap f(A_2) = \{1\} \cap \{1\} = \{1\} \end{aligned} \tag{4}$$

Exercise 1.23

Assume that $A \subseteq B$ and both are bounded above. Prove that $sup(A) \leq sup(B)$.

Suppose for contradiction, that $sup(A) > sup(B)$. Then, by the definition of supremum, there exists some $a \in A$ such that $a > sup(B)$. But since $a \in B$, this contradicts the fact that $sup(B)$ is an upper bound for B.

Exercise 1.24

Suppose $A \subseteq R$ has a maximal element - that is, there is an element $M \in A$ such that $x \leq M$ for all $x \in A$. Likewise, assume $B \subseteq R$ has a minimal element m.

- Prove that $sup(A) = M$.
  - Since M is the maximal element of A, by definition $\forall x \in A, x \leq M$. Thus M is an upper bound of A.
  - To show that M is the least upper bound, suppose there exists another upper bound S of A such that $S < M$. But since $M \in A$, we must have $M \leq S$. This contradicts $S < M$. Therefore, no such S exists, and M is the smallest upper bound.
- Prove that $inf(B) = m$.

- Since m is the minimal element of B, by definition $\forall x \in B, x \geq m$. Thus, m is a lower bound of B.
- To show m is the greatest lower bound, suppose there exists another lower bound t of B such that t ¿ m. But since $m \in B$, we must have $t \leq m$. This contradicts $t > m$. Therefore, no such t exists, and m is the largest lower bound.

Exercise 1.25

Suppose that A is a nonempty set containing finitely many integers. Prove by induction that A has a maximal element, and that $max(A) \in A$.

- base case: if A contains exactly one integer a, then is trivially the maximal element, and $max(A) = a \in A$.

- inductive step: Assume every nonempty finite set of integers with k elements has a maximal element that belongs to the set. Let $A = \{a_1, a_2, ..., a_{k+1}\}$. $A' = A \setminus \{x\}$ with k elements. By the induction hypothesis, A' has a maximal element $m \in A'$. Compare x with m, if x ¿ m, then x is the maximal element of A; if $x \leq m$, then m remains the maximal element of A. This proves that the maximal element of A is an element of A. By mathematical induction, every nonempty finite set of integers has a maximal element that belongs to the set.

Exercise 1.26

Prove that N is complete

To prove that N is complete, we first clarify the definition of completeness. Completeness means that every nonempty subset of N that is bounded above has a least upper bound in N.

- Let $S \subseteq$ N be a nonempty subset bounded above.

  By definition, there exsists some $M \in N$ such that $s \leq M$ for all $s \in S$

- S is finite.

  Since $S \subseteq \{1, 2, ..., M\}$ and this set is finite, S must also be finite.

- Every finite nonempty subset of N has a maximum.

  By the result proven in the previous induction problem, every finite nonempty set of integers contains a maximal element. Let max(S) denote this maximum.

- max(S) is the least upper bound of S

  max(S) is an upper bound because $s \leq max(S)$ for all $s \in S$. It is the least upper bound because no smaller natural number than max(S) can be an upper bound for S.

- Since every nonempty bounded-above subset $S \subseteq N$ has a least upper bound $max(S) \in N$. N is complete.

Exercise 1.27

For each item, compute the requested supremum or infimum or carefully explain why it does not exist. Either way, prove that your answer is correct.

- Determine sup(A) for A = $\{\frac{(-1)^n}{n} : n \in N\}$

  The set A alternates between positive and negative terms: A = {-1, 1/2, -1/3, 1/4, -1/5 ...}. The positive terms are 1/2, 1/4, 1/6 ... approaching 0; the negative terms are -1, -1/3, -1/5 ... approaching 0. The supremum is 1/2.

  Proof

  - For all n, $\frac{(-1)^n}{n} \leq 1/2$
  - if $n \geq 2$, positive terms $1/n \leq 1/2$ and negative terms are ¡ 1/2
  - Thus, $sup(A) = 1/2$ and $1/2 \in A$

- Fix $a \in (0, 1)$. Determine inf(B) for $B = \{a^n : n \in N\}$.

  For $a \in (0, 1)$, a^n is a strictly decreasing sequence bounded below by 0. For example, with a = 0.5: $B = \{0.5, 0.25, 0.125\}$

  To prove inf(B) = 0, firstly, $a^n > 0$ for all n, so 0 is a lower bound. For any $\epsilon > 0$, $chooseN > \frac{ln\epsilon}{ln\alpha}$. Then $a^N \leq \epsilon$, proving 0 is the greatest lower bound.

- Fix $a \in (1, \infty)$. Determine sup(C) for $C = \{a^n : n \in N\}$

  For $a > 1, a^n$ is strictly increasing sequence unbounded above. For example a = 2. Thus the supremum doesn't exist in the real numbers. Since for any M ¿ 0, choose N ¿ $\frac{lnM}{lna}$. Then $a^N > M$, proving the sequence grows without bound.

Exercise 1.28

Prove the infimum case of Theorem 1.24

To prove the infimum case of theorem 1.24, we demonstrate that for a set $A \subseteq R$, $inf(A) = \beta$ if and only if 1. β is a lower bound of A 2. For every $\epsilon > 0$, there exists $x \in A$ such that $x < \beta + \epsilon$

- forward direction: assume $\beta = inf(A)$
  - Lower bound property: by definition of infimum, β is a lower bound of A, so $x \leq \beta$ for all $x \in A$.
  - ϵ condition: Let $\epsilon > 0$. Since β is the greatest lower bound, $\beta + \epsilon$ is not a lower bound of A. Thus, there exists $x \in A$ such that $x < \beta + \epsilon$
- reverse direction: assume β is a lower bound of A and for every $\epsilon > 0$, there exists $x \in A$ with $x < \beta + \epsilon$. We prove $\beta = inf(A)$ by showing it is the greatest lower bound.
  - β is already a lower bound by assumption
  - Suppose there exists a greater lower bound γ such that $\gamma > \beta$. Let $\epsilon = \gamma - \beta > 0$. By assumption, there exists $x \in A$ such that $x < \beta + \epsilon = \gamma$. This contradicts γ being a lower bound. Thus β is the greatest lower bound.

Exercise 1.29

Prove that $sup(\{\frac{n}{n+1} : n \in N\}) = 1$ and $inf(\{\frac{n}{n+1} : n \in N\}) = 1/2$.

Sup(S) = 1

- show that 1 is an upper bound

For all $n \in N$,

$$\frac{n}{n+1} = 1 - \frac{1}{n+1} < 1 \tag{5}$$

So every element of S is less than 1, hence 1 is an upper bound.

- Show that 1 is the least upper bound

  Let $\epsilon > 0$. We want to show there exists $x \in S$ such that $x > 1 - \epsilon$. Set $\frac{n}{n+1} > 1 - \epsilon$

  $$1 - \frac{1}{n+1} > 1 - \epsilon \rightarrow \frac{1}{n+1} < \epsilon \rightarrow n > 1/\epsilon - 1 \tag{6}$$

  For any $\epsilon > 0$, pick n large enough so that $n > 1/\epsilon - 1$. Then $\frac{n}{n+1} > 1 - \epsilon$. Thus, for any $\epsilon > 0$, there is an element of S within ϵ of 1 from below. Therefore, 1 is the supremum of S.

- inf(S) = 1/2

  - Show that 1/2 is a lower bound. The smallest value occurs at $n = 1 : 1/2$. For $n > 1$, $\frac{n}{n+1} \geq 1/2$
  - Show that 1/2 is the greatest lower bound For any $\epsilon > 0$, $1/2 + \epsilon$ is greater than 1/2. For n=1, $1/2 < 1/2 + \epsilon$. For all n ¿ 1, $\frac{n}{n+1} > 1/2$. Thus 1/2 is the smallest element and hence the infimum.

Exercise 1.30

Let $A, B \subseteq R$, and assume that $sup(A) < sup(B)$

- Show that there exists an element $b \in B$ that is an upper bound for A.

  Since sup(A) ¡ sup(B), sup(A) is not an upper bound for B. There exists an element $b \in B$ such that $b > sup(A)$. Since sup(A) is an upper bound for A, any $b > sup(A)$ is also an upper bound for A. Thus, $b \in B$ serves as an upper bound for A.

- Give an example to show that this is not necessarily the case if we instead only assume that $sup(A) \leq sup(B)$. You do not need to prove your answer.

  Let $A = (0, 1)$ and $B = (0, 1).sup(A) = sup(B) = 1$. However, every element $b \in B$ satisfies $b < 1$, so no $b \in B$ is an upper bound for A.

Exercise 1.31

Suppose that $A, B \subseteq R$ are nonempty and bounded above. Find a formula for $sup(A \cup B)$ and prove that it is correct.

The supremum of the union of two nonempty, bounded above sets $A, B \subseteq R$ is given by:

$$sup(A \cup B) = max\{sup(A), sup(B)\} \tag{7}$$

- Show $max\{sup(A), sup(B)\}$ is an upper bound for $A \cup B$

  Let $\alpha = sup(A)$ and $\beta = sup(B)$. For any $x \in A \cup B$:

  - If $x \in A$, then $x \leq \alpha \leq max\{\alpha, \beta\}$
  - If $x \in B$, then $x \leq \beta \leq max\{\alpha, \beta\}$

  Thus, $max\{\alpha, \beta\}$ is an upper bound for $A \cup B$

- Prove $max\{sup(A), sup(B)\}$ is the least upper bound

  Assume M is another upper bound for $A \cup B$ with $M < max\{\alpha, \beta\}$. Without loss of generality, suppose $\alpha \geq \beta$, so $max\{\alpha, \beta\} = \alpha$. Then:

  - $M < \alpha$
  - Since $\alpha = sup(A)$, there exists $a \in A$ such that $M < a \leq \alpha$. But $a \in A \cup B$, which contradicts M being an upper bound for $A \cup B$. Similar logic applies for $\beta > \alpha$. Therefore, no such M can exists, and $max\{\alpha, \beta\}$ is the least upper bound.

Exercise 1.32

Suppose $A \subseteq R$ is bounded above and $c \in R$. Define $c + A = \{c + a : a \in A\}$ and $cA = \{ca : a \in A\}$

- Prove that $sup(c + A) = c + sup(A)$

  - Let $s = sup(A)$. For any $x \in c + A$, we have $x = c + a$ for some $a \in A$. Since $a \leq s$, it follows that $c + a \leq c + s$. Thus $c + s$ is an upper bound for $c + A$.
  - Suppose $M < c + s$ is another upper bound for $c + A$. Then $M - c < s$. By definition of supremum, there exists $a \in A$ such that $M - c < a \leq s$. This implies $c + a > M$, which contradicts M being an upper bound. Hence c+s is the least upper bound.

- Determine neccessary and sufficient conditions on c and A for $sup(cA) = csup(A)$. Give an example of a set A and number c where $sup(cA) \neq csup(A)$.

  The equality holds if and only if $c \geq 0$

  - $c \geq 0$

    For any $x \in cA$, we have $x = ca$ where $a \leq sup(A)$. Since $c \geq 0$, multiplying preserves the inequality: $ca \leq csup(A)$ if $M < csupA$ were an upper bound, then $M/c < sup(A)$, implying there exists $a \in A$ such that $M/c < a$. This leads to $ca > M$, a contradiction. Thus $sup(cA) = csup(A)$.

  - $c < 0$

    Multiplying by a negative scalar reverse inequalities. Here $sup(cA) = cinf(A)$, which generally differs from $csup(A)$. For example, let $A = \{2, 1\}$ and $c = -1$. Then

    * $sup(A) = 2$
    * $csup(A) = -2$
    * $cA = \{-2, -1\}$
    * $sup(cA) = -1 \neq -2$

Exercise 1.33

For $A \subseteq R$, we denote -A to be the set obtained by taking the opposite of everything in A. That is

$$-A = \{-x : x \in A\} \tag{8}$$

Suppose that$A \neq \emptyset$ and that A is bounded below. Prove that $-A \neq \emptyset, -A$ is bounded above, and $sup(-A) = -inf(A)$

- Since $A \neq \emptyset$, there exists at least one $x \in A$. Then $-x \in A$ by definition. Thus -A is also nonempty.

- Since A is bounded below, there exists some $m \in R$ such that $m \leq x$ for all $x \in A$. Take any $y \in -A$. Then $y = -x$ for some $x \in A$. So:

  $$y = -x \leq -m \tag{9}$$

  because $x \geq m$. Therefore, -m is an upper bound for -A

- Let s = inf(A). We want to show that sup(-A) = -s
  - -s is an upper bound for -A
    For any $y \in -A, y = -x$ for some $x \in A$. Since $s \leq x$, we have $-x \leq -s$, so $y \leq -s$. Thus -s is an upper bound for -A.
  - Suppose $M < -s$ is another upper bound for -A. Then $-M > s$. By the definition of infimum, for any $\epsilon > 0$, there exists $x_0 \in A$ such that $x_0 < s + \epsilon$. Take $\epsilon = (-M) - s > 0$, so $x_0 < -M$. Then $-x_0 > M$, and $-x_0 \in A$, which contradicts M being an upper bound for -A. Therefore -s is the least upper bound, i.e., $sup(-A) = -inf(A)$

Exercise 1.34

For each $n \in N$, assume we are given a closed interval $I_n = [a_n, b_n]$. Also, assume that each I_{n+1} is contained inside of I_n. This gives a sequence of increasingly smaller intervals

$$...I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 \tag{10}$$

Prove that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. That is, prove that there is some real number x such that $x \in I_n$ for every $n \in N$.

To prove that the intersedction of nested closed intervals is non-empty, we proceed as follows:

- Let $I_n = [a_n, b_n]$, where $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all n.
  - The sequence $\{a_n\}$ is monotonically increasing and bounded above by b_1
  - The sequence $\{b_n\}$ is monotonically decreasing and bounded below by a_1

  By the monotone convergence theorem, both sequences converge: $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = y$
- Show $x \leq y$

  For all n, $a_n \leq b_n$. Taking limits preserves inequalities:

$$x = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = y \tag{11}$$

- Prove $x \in I_n$ for all n
  - Since $\{a_n\}$ is increasing, $a_n \leq x$ for all n.
  - Since $\{b_n\}$ is decreasing and $x \leq y \leq b_n$, $x \leq b_n$ for all n.

  Thus $a_n \leq x \leq b_n$, so $x \leq [a_n, b_n] = I_n$ for every n.

Exercise 1.35

Give an example showing that the conclusion of Exercise 1.34 need not hold if each I_n is allowed to be an open interval.

Consider the sequence of open intervals:

$$I_n = (0, 1/n) \text{for } n \in N \tag{12}$$

Suppose there exists an $x \in \bigcap_{n=1}^{\infty} I_n$. Then $0 < x < 1/n$ for all $n \in N$. But by the archimedean property, there exists $N \in N$ such that $1/N < x$. This contradicts $x < 1/N$. Thus no such x exists and the intersection is empty.

Exercise 1.36

- Determine {1,3,5} + {-3,0,1}

$$\{-2, 0, 1, 2, 3, 4, 5, 6\} \tag{13}$$

- Assume that $A, B \in R$ and sup(A) and sup(B) exist. Prove that

$$sup(A + B) = sup(A) + sup(B) \tag{14}$$

  - $sup(A + B) \leq sup(A) + sup(B)$
    Let $a \in A, b \in B$. Then $a \leq sup(A)$, $b \leq sup(B)$, so

$$a + b \leq sup(A) + sup(B) \tag{15}$$

    Thus sup(A) + sup(B) is an upper bound for A + B. Therefore,

$$sup(A + B) \leq sup(A) + sup(B) \tag{16}$$

– $sup(A+B) \geq sup(A) + sup(B)$

Let $\epsilon > 0$. Since sup(A) is the least upper bound, there exists $a_\epsilon \in A$ such that

$$a_\epsilon > sup(A) - \epsilon/2 \tag{17}$$

Similarly, there exists $b_\epsilon \in B$ such that

$$b_\epsilon > sup(B) - \epsilon/2 \tag{18}$$

Then,

$$a_\epsilon + b_\epsilon > sup(A) + sup(B) - \epsilon \tag{19}$$

So for every $\epsilon > 0$, there is an element in $A + B$ greater than $sup(A) + sup(B) - \epsilon$. Thus,

$$sup(A+B) \geq sup(A) + sup(B) \tag{20}$$

Thus:

$$sup(A+B) = sup(A) + sup(B) \tag{21}$$

Exercise 1.37

- Determine $\{1,3,5\} \cdot \{-3,0,1\}$

$$\{-15, -9, -3, 0, 1, 3, 5\} \tag{22}$$

- Give an example of sets A and B where $sup(A \cdot B) \neq sup(A) \cdot sup(B)$

  Let $A = (-1,0), B = (-1,0)$. For a,b both negative and less than $0, ab > 0$, and the maximum value is when a, b approach 0 from below:

$$\lim_{a \to 0} \lim_{b \to 0} ab = 0 \tag{23}$$

  But the minimum value is when a,b are close to -1:

$$(-1+\epsilon)(-1+\delta) \simeq 1 \tag{24}$$

  So $A \cdot B = (0,1)$ and $sup(A \cdot B) = 1$. But $sup(A) \cdot sup(B) = 0$.