Real Analysis - A Long Form Mathematics Textbook Chapter 1

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1.5 The Completness Axiom

The set R has two binary operations, addition(+) and multiplication (*), and is the unique set satisfying the following axioms:

- Axiom 1: (Commutative Law), If a, b $\in$ R, then $a + b = b + a$ and $a * b = b * a$
- Axiom 2: (Distributive Law), If a,b,c $\in$ R, then $a*(b+c) = a*b+a*c$
- Axiom 3: (Associative Law), If a,b,c $\in$ R, then $(a+b)+c = a+(b+c)$ and $(a*b)*c = a*(b*c)$
- Axiom 4: (Identity Law). There are special elements 0,1 $\in$ R, where $a + 0 = a$ and $a * 1 = a$ for all a $\in$ R
- Axiom 5: (Inverse Law). For each a $\in$ R, there is an element -$a \in R$ such that $a+(-a) = 0$. If a $\neq$ 0, then there is also an element $a^{-1} \in R$ such that $a * a^{-1} = 1$
- Axiom 6: (Order Axiom). There is nonempty subset $P \subseteq R$, called the positive elements, such that
  - If a,b $\in$ P, then a + b $\in$ P and a*b $\in$ P
  - If a $\in$ R and a $\neq$ 0, then either a $\in$ P or -a $\in$ P, but not both.

- Axiom 7: (Completeness Axiom). Given any nonempty A $\subseteq$ R where A is bounded above, A has a least upper bound. In other words, $sup(A) \in R$ for every such A.

Theorem 1.24 (Suprema analytically). Let A $\subseteq$ R. Then $sup(A) = \alpha$ if and only if

- α is an upper bound of A, and
- Given any $\epsilon > 0, \alpha - \epsilon$ is not an upper bound of A. That is, there is some $x \in A$ for which $x > \alpha - \epsilon$

Likewise, $inf(A) = \beta$ iff

- β is a lower bound of A, and
- Given any $\epsilon > 0$, $\beta + \epsilon$ is not a lower bound of A. That is there is some $x \in A$ for which $x < \beta + epsilon$

1.7 The Archimedean Principle

Lemma 1.26 (The Archimedean Property). If a and b are real numbers with a ¿ 0, then there exists a natural number n such that $na > b$. In particular, for any $\epsilon > 0$ there exists $n \in N$ such that $1/n < \epsilon$.

Principle 1.34 (Well-ordering principle). Every non-empty subset of natural numbers contains a smallest element.

Exercises

Exercise 1.1

Explain the error in the following proof that 2=1. Let x = y. Then

$$\begin{aligned} x^2 &= xy \\ x^2 - y^2 &= xy - y^2 \\ (x+y)(x-y) &= y(x-y) \\ x + y &= y \\ 2y &= y \\ 2 &= 1 \end{aligned} \tag{1}$$

answer: by dividing by (x-y), the proof is essentially dividing by zero, which is mathematicall undefined; thus this invalidates the entire proof.

Exercise 1.2

Which of the following statements are true? Give a short explanation for each of your answers.

- For ever $n \in N$, there is $m \in N$ such that $m > n$.
- For every $m \in N$, there is an $n \in N$ such that $m > n$.
- There is an $m \in N$ such that for every $n \in N$, $m \geq n$.
- There is an $n \in N$ such that for every $m \in N$, $m \geq n$.
- There is an $n \in R$ such that for every $m \in R$, $m \geq n$.
- For every pair $x < y$ of integers, there is an integer z such that $x < z < y$.
- For every pair $x < y$ of real numbers, there is a real number z such that $x < z < y$.

answer:

- True. For any natural number n, choosing $m = n + 1$ satisfies $m > n$. Natural numbers are unbounded above.
- False. If m=1, there is no $n \in N$ with $n < 1$.
- False. There is no largest natural number; N is infinite.
- True. Let n=1. For all $m \in N$, $m \geq 1$.
- False. Real numbers extend to $-\infty$; no universal lower bound $n \in R$.
- False. If x and y are consecutive integers (e.g., $x = 2, y = 3$), no integer z exists between them.
- True. For real numbers, $z = (x + y)/2$ always satisfies $x < z < y$.

Exercise 1.3

If A and B are two boxes (possibly with things inside), describe the following in terms of boxes:

- $A\ B$
- $P(A)$
- $|A|$

answers:

- Imagine looking inside box A and taking out any item that is also found in box B. What remains in A are only those things that are not in B.
- Think of every possible way you could select items from box A (including selecting none, or all). Each possible selection is itself a (possibly empty) box. The collection of all these possible boxes is $P(A)$.
- Open box A and count how many items are inside. That count is $|A|$.

Exercise 1.4

If $A_1, A_2, A_3, ..., A_n$ are all boxes (possibly with things inside), describe the following terms of boxes:

- $\bigcup_{i=1}^{n} A_i$
- $\bigcap_{i=1}^{n} A_i$

answers:

- If an item exists in at least one box, it appears once in the union box.
- If even one box lacks an item, it is excluded from the intersection.

Exercise 1.5

Prove that each of the following holds for any sets A and B.

- $A \cup B = A$ iff $B \subseteq A$
- $A \cap B$ iff $A \subseteq B$
- $A\ B = A$ iff $A \cap B = \emptyset$
- $A\ B = \emptyset$ iff $A \subseteq B$

answers:

- (forward)If $A \cup B = A$, every element of B must already be in A. Thus $B \subseteq A$. (reverse) If $B \subseteq A$, combining A and B adds no new elements, so $A \cup B = A$
- (forward) if $A \cap B = A$, all elements of A are in B, so $A \subseteq B$. (reverse) if $A \subseteq B$, the intersection $A \cap B$ contains exactly A.
- (forward) if $A \setminus B = A$, no elements of A are in B, so $A \cap B = \emptyset$. (reverse) if $A \cap B = \emptyset$, removing B from A leaves A unchanged.
- (forward) if $A \setminus B = \emptyset$, all elements of A are in B, so $A \subseteq B$. (reverse) if $A \subseteq B$, removing B from A removes all elements, leaving $\emptyset$

Exercise 1.6

Suppose f:$X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

- Prove that $f(f^{-1}(B)) \subseteq B$
- Give an example where $f(f^{-1}(B)) \neq B$
- Prove that $A \subseteq f^{-1}(f(A)))$
- Give an example where $A \neq f^{-1}(f(A)))$

answers:

- Let $y \in f(f^{-1}(B))$. By definition, there exists $x \in f^{-1}(B)$ such that $f(x) = y$. Since $x \in f^{-1}(B)$, $f(x) \in B$ by the definition of preimage. Thus $y = f(x) \in B$. Therefore, $f(f^{-1}(B)) \subseteq B$

- Let f: $\{1,2\} \to \{a,b,c\}$ with $f(1) = a$ and $f(2) = b$. Take $B = \{a,b,c\}.f^{-1}(B) = \{1,2\}.f(f^{-1}(B)) = f(\{1,2\}) = \{a,b\} \neq B$

- Let $x \in A$. Then $f(x) \in f(X)$. By definition of preimage, $x \in f^{-1}(f(A))$. Thus, $A \subseteq f^{-1}(f(A))$

- Let f: $\{1,2\} \to \{a\}$ with $f(1) = f(2) = a$. Take $A = \{1\}.f(A) = \{a\}$. $f^{-1}(f(A)) = f^{-1}(\{a\}) = \{1,2\} \neq A$

Exercise 1.7

Suppose that f:$X \to Y$ and g: $Y \to X$ are functions and that the composite g of f is the identity function id" $X \to X$. (The identity function sends every element to itself: $id(x) = x$) Show that f must be a one-to-one function and that g must be an onto function.

answer:

- f is injective

  Assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Applying g to both sides: $g(f(x_1)) = g(f(x_2))$ since g of f = id_X, this simplifies to: $x_1 = x_2$. Thus, $f(x_1) = f(x_2) \to x_1 = x_2$, proving f is injective.

- g is surjective

  For any $x \in X$, let $y = f(x) \in Y$. Applying g to y: $g(y) = g(f(x)) = x$. Thus, every $x \in X$ has preimage $y = f(x) \in Y$ under g, proving g is surjective.

Exercises 1.8

The following are special cases of De Morgan's laws

- Prove that $(A \cap B)^c = A^c \cup B^c$

  (a) $(A \cap B)^c \subseteq A^c \cup B^c$

  Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$, so $x \notin A$ or $x \notin B$. Thus, $x \in A^c$ or $x \in B^c$. Therefore $x \in A^c \cup B^c$.

  (b) $A^c \cup B^c \subseteq (A \cap B)^c$

  Let $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, so $x \notin A$ or $x \notin B$. Thus, $x \notin A \cap B$, so $x \in (A \cap B)^c$

- Prove that $(A \cup B)^c = A^c \cap B^c$

  (a) $(A \cup B)^c \subseteq A^c \cap B^c$

  Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.

  (b) $A^c \cap B^c \subseteq (A \cup B)^c$

  Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A and x \notin B$. Thus $x \notin A \cup B$, so $x \in (A \cup B)^c$. Therefore, $(A \cup B)^c = A^c \cap B^c$

0.1 Exercise 1.9

- Prove that $\sqrt{3}$ is irrational

  Assume $\sqrt{3}$ is rational, so $\sqrt{3} = a/b$ where $a, b \in Z$ are coprime. Squaring both sides: $3 = \frac{a^2}{b^2} \implies 3b^2 = a^2$. This implies a^2 is divisible by 3, so a must also be divisable by 3 (by the fundamental theorem of arithmetic). Let a = 3k. Substituting: $3b^2 = (3k)^2 \implies 3b^2 = 9k^2 \implies b^2 = 3k^2$. Thus, b^2 is divisible by 3, so b is also divisible by 3. This contradicts a and b being coprime. Hence, $\sqrt{3}$ is irrational.

- What goes wrong when you try to adapt your argument from part (a) to show that $\sqrt{4}$ is irrational?

- In part (a) you proved that $\sqrt{3}$ to be irrational, and essentially the same proof shows that $\sqrt{5}$ is irrational. By considering their product or otherwise, prove that $\sqrt{3} - \sqrt{5}$ and $\sqrt{3} + \sqrt{5}$ are either both rational or both irrational. Deduce that they must both be irrational.

Exercise 1.10

Prove that the multiplicative identity in a field is unique.

Let F be a field. Suppose there exist two multiplicative identities 1 and e in F. By definition of a multiplicative identity, for all $a \in F$:

$1 \times a = a$ and $e \times a = a$

Consider the case where a = e. Applying the identity property of 1:

$1 \times e = e$

Now consider the case where a = 1. Applying the identity property of e:

$e \times 1 = 1$

In a field, multiplication is commutative ($a \times b = b \times a$), so:

$1 \times e = e \times 1$

From (1),(2) and (3), we conclude:

$e = 1$

Thus, there cannot be two distinct multiplicative identities in F. The multiplicative identity is unique.

Exercise 1.11

Given an ordered field F, recall that we defined the positive elements to be a nonempty subset $P \subseteq F$ that satisfies both the following conditions:

(i) If $a, b \in P$, then $a + b \in P$ and $a \times b \in P$

(ii) If $a \in F$ and $a \neq 0$, then either $a \in P$ or $-a \in P$, but not both.

- Give an example of some $P_1 \subseteq R$ that satisfies (i) but not (ii)

  $P_1 =$ R(the entire set of real numbers)

  Satisfies (i): Closed under addition and multiplication

  Fails (ii): For any $a \neq 0$, both a and -a belong to P_1, violating the "exactly one" requirement.

- Give an example of some $P_2 \subseteq R$ that satisfies (ii) but not (i)

  $P_2 = \{x \in R | (x > 0 and x \notin Z) or (x < 0 and x \in Z)\}$

  Satisfies (ii): For every $a \neq 0$: if a is positive non-integer, $a \in P_2$. If a is a positive integer, -$a \in P_2$. If a is negative integer, $a \in P_2$. If a is negative non-integer, $-a \in P_2$.

  Fails (i): $0.5 \in P_2$ and $0.5 \in P_2$ but $0.5 + 0.5 = 1 \notin P_2$. $-1 \in P_2$ and $0.5 \in P_2$, but $-1 + 0.5 = -0.5 \notin P_2$.

Exercises 1.12

Assume that F is an ordered field and $a, b, c, d \in F$ with $a < b$ and $c < d$.

- Show that $a+c < b+d$ Step 1: use the additive property of inequalities in ordered fields: If $a < b$, then $a+c < b+c$. If $c < d$, then $b+c < b+d$.

  Step 2: By transitivity of <: $a + c < b + c < b + d \implies a + c < b + d$

- Prove that it is not necessarily true that $ac < bd$.

  Counter example:

  Let F = R, and choose: a = -2, b = -1, c = -3, d = -2.

  - a ¡ b(since -2 ¡ -1) and c ¡ d (since -3 ¡ -2)
  - Compute ac = (-2)(-3) = 6 and bd = (-1)(-2) = 2
  - 6 $\not< 2$, so $ac < bd$ fails.

Exercise 1.13

Let a,b and ϵ be elements of an ordered field.

- Show that if $a < b + \epsilon$ for every $\epsilon > 0$, then $a \leq b$.

  Suppose, for contradiction, that $a > b$. Then $a - b > 0$. Let $\epsilon = a - b$, which is a positive number. Plug this into the assumption $a < b+\epsilon = a$. This says $a < a$, which is impossible. Therefore, our assumption that $a > b$ must be false. So it must be that $a \leq b$.

- Use part (a) to show that if $|a - b| < \epsilon$ for all $\epsilon > 0$, then $a = b$

  Suppose, for contradiction, that $a \neq b$. Then $|a - b| > 0$. Let $\epsilon = |a - b|/2$, which is still positive. By assumption $|a - b| < \epsilon = |a - b|/2$. But this says $|a - b| < |a - b|/2$, which is impossible unless $|a - b| = 0$. Therefore, our assumption that $a \neq b$ must be false. So $a = b$.

Exercise 1.14

Prove that the equality —ab— = —a——b— holds for all real numbers a and b.

Definition of absolute value: For any real number

$$x : |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- case 1: both $a \leq 0$ and $b \leq 0$

  - $ab \geq 0$
  - $|ab| = ab$

- $|a| = a, |b| = b$
- $|a||b| = ab$
- So, $|ab| = |a||b|$

- case 2: $a \geq 0, b < 0$
  - $ab \leq 0$
  - $|ab| = -(ab) = -ab$
  - $|a| = a, |b| = -b$
  - $|a||b| = a(-b) = -ab$
  - So, $|ab| = |a||b|$
- case 3: $a < 0, b \geq 0$ similar to case 2
- case 4: $a < 0, b < 0$
  - $ab \geq 0$
  - $|ab| = (-a)(-b) = ab$
  - $|a| = -a, |b| = -b$
  - $|a||b| = (-a)(-b) = ab$
  - So, $|ab| = |a||b|$

Exercise 1.15

For each of the following, find all numbers x which satisfy the expression.

- $|x - 4| = 7$

  $\{-3, 11\}$
- $|x - 4| < 7$

  $(-3, 11)$
- $|x + 2| < 1$

  $(-3, -1)$
- $|x - 1| + |x - 2| > 1$

  $(-\infty, 1) \cup (2, \infty)$

- $|x-1|+|x+1|>1$
  R
- $|x-1||x+1|=0$
  $\{-1,1\}$
- $|x-1||x+2|=3$
  $\{\frac{-1+\sqrt{21}}{2},\frac{-1-\sqrt{21}}{2}\}$

Exercise 1.16

Let $max\{x,y\}$ denote the maximum of the real numbers x and y, and let $min\{x,y\}$ denote the minimum. For example, $min\{-1,4\}=-1$, and also min{-1,-1} = -1. Prove that

$max\{x,y\}=\frac{x+y+|y-x|}{2}$ and $min\{x,y\}=\frac{x+y-|y-x|}{2}$

Then find a formula for $max\{x,y,z\}$ and $min\{x,y,z\}$.

- Consider two cases to prove for maximum, minimum
  - $y \geq x$
    maximum: $\frac{x+y+|y-x|}{2}=\frac{x+y+y-x}{2}=y$
    minimum: $\frac{x+y-|y-x|}{2}=\frac{x+y-y+x}{2}=x$
  - $y < x$
    maximum: $\frac{x+y+|y-x|}{2}=\frac{x+y+x-y}{2}=x$
    minimum: $\frac{x+y-|y-x|}{2}=\frac{x+y-x+y}{2}=y$

  Thus the formula holds for all $x,y \in R$
- extension to three variables
  - maximum of three variables $max\{x,y,z\}=max(max(x,y),z)=\frac{\frac{x+y+|y-x|}{2}+z+|z-\frac{x+y+|y-x|}{2}|}{2}$
  - minimum of three variables $min\{x,y,z\}=min(min(x,y),z)=\frac{\frac{x+y-|y-x|}{2}+z-|z-\frac{x+y-|y-x|}{2}|}{2}$

Exercise 1.17

Prove that if $a, b \in R$ and $0 < a < b$, then $a^n < b^n$ for any positive integer n.

- base case

  given $0 < a < b$, the inequality holds by assumption.

- inductive step:

  Assume $a^k < b^k$ for some integer $k \geq 1$. $a^{k+1} < ab^k$. Since a ¡ b, then $ab^k < bb^k = b^k + 1$. Thus, $a^{k+1} < b^{k+1}$.

- By induction, $a^n < b^n$ holds for all positive integers n.

Exercise 1.18

Prove that if $a_1, a_2, ..., a_n$ are real numbers, then: $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$.

Use induction based on triangle inequality of two numbers to prove.

Exercise 1.19

Prove that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for every natural number n.

With partial fractions, we know $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = (1/1-1/2)+(1/2-1/3)+...+(1/n-1/(n+1)) = \frac{n}{n+1} \tag{2}$$