Real Analysis - A Long Form Mathematics Textbook Chapter 2: Cardinality

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2.1 Bijections and Cardinality

Principle 2.1 (The bijection principle). Two sets have the same size if and only if there is a bijection between them.

2.2 Counting Infinities

Theorem 2.8 ($|Z| = |Q|$). There are the same number of integers as rational numbers.

Theorem 2.9 ($|R| > |N|$). There are more real numbers than natural numbers.

Theorem 2.11 (Sizes of infinity). There are different sizes of infinity, with countable infinity being the smallest. Moreover, N, Z, and Q are countable while R is uncountable.

Theorem 2.13 ($|A| < |P(A)|$). If A is a set and P(A) is the power set of A, then

$$|A| < |P(A)| \tag{1}$$

Corollary 2.14 (There exist infinitely many infinities). There exist infinitely many distinct infinite cardinalities.

Exercises

Exercise 2.1

- List all the elements of $P(\{a, b, c\})$

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \tag{2}$$

- Determine a formula for the number of elements in the power set of an n-element set.

$$|P(A)| = 2^{|A|} \tag{3}$$

  for finite sets.

Exercise 2.2

Prove that $|\{e^n : n \in N\}| = |N|$.

- Injectivity

  Suppose $f(n_1) = f(n_2)$. Then $e^{n_1} = e^{n_2}$. Since e^x is strictly increasing, $n_1 = n_2$. Thus f is injective

- Surjectivity

  For every $e^n \in \{e^n : n \in N\}$, there exists $n \in N$ such that $f(n) = e^n$. Thus, f is surjective.

- Conclusion

  Since f is both injective and surjective, it is a bijection. By the bijection principle, the cardinalities are equivalent.

Exercise 2.3

The following pairs of sets have the same size, and so there exists a bijection between them. Write down an explicit bijection in each case. You do not need to prove your answers.

- $(0, \infty)$ and $(1, \infty)$

  f(x) = x + 1
  Maps each element in $(0, \infty)$ to $(1, \infty)$ by shifting right by 1.

- $(0, \infty)$ and $(-\infty.3)$

  f(x) = 3-x
  Reflects $(0, \infty)$ over x = 1.5 covering all real numbers less than 3.

- $(0, \infty)$ and $(0, 1)$ f(x) = $\frac{1}{x+1}$
  Compresses $(0, \infty)$ into $(0, 1)$ via reciprocal transformation.

- R and $(0, \infty)$ f(x) = e^x
  Exponential function maps all reals to positive reals bijectively.

- R and $(0, 1)$ f(x) = $\frac{1}{1+e^{-x}}$
  Logistic function maps R to (0, 1) with an S-shaped curve.

- Z and $\{..., 1/8, 1/4, 1/2, 1, 2, 4, 8, ...\}$ f(k) = 2^k
  Maps integers to powers of 2 (negative integers map to reciprocals)

- $\{0, 1\} \times N$ and N

$$f(b, n) = \begin{cases} 2n \text{ if } b = 0, \\ 2n - 1 \text{ if } b = 1 \end{cases} \tag{4}$$

  interleaves pairs: (0,n) maps to even numbers, (1,n) to odds.

- $[0, 1]$ and $(0, 1)$

$$f(x) = \begin{cases} \frac{1}{2} \text{ if } x = 0, \\ \frac{1}{n+2} \text{ if } x = 1/n \text{ for some } n \in N, \\ x, \text{ otherwise.} \end{cases} \tag{5}$$

Exercise 2.4

This problem shows that "equinumerosity is an equivalence relation." (This justifies the notation $|A| = |B|$.) Let A, B, and C be sets. For this problem only, we'll write A $\sim$ B to mean that A and B are equinumerous, meaning that there is a bijection $A \to B$.

- Show that $A \sim A$. The identity function $id_A : A \to A$ defined by $id_A(x) = x$ for all $x \in A$ is a bijection. Therefore every set is equinumerous with itself.

- Show that if $A \sim B$ then $B \sim A$ If there is a bijection $f : A \to B$, then the inverse function $f^{-1} : B \to A$ is also a bijection. Thus, if $A \sim B$, then $B \sim A$.
- Show that if $A \sim B$ and $B \sim C$, then $A \sim C$ If $f : A \to B$ and $g : B \to C$ are bijections, then the composition $g \circ f : A \to C$ is also a bijection. Therefore $A \sim C$.

Exercise 2.5

- Prove that if A and B are countable sets, then $A \cup B$ is also a countable set.

  Let A and B be countable sets.

  - Both A and B are finite. Their union $A \cup B$ is finite, hence countable.
  - At least one set is infinite.
    * Assume A and B are disjoint (if not, replace B with $B \setminus A$, which is countable.)
    * Let $f : A \to N$ and $g : B \to N$ be bijections.
    * Definie h: $A \cup B \to N$ as:

$$h(x) = \begin{cases} 2f(x) \text{ if } x \in A, \\ 2g(x) + 1 \text{ if } x \in B \end{cases} \tag{6}$$

    * h is injective because even and odd numbers in N are disjoint. Thus, $A \cup B$ is countable.

- Prove that if A_n is a countable set for each $n \in N$, then the set $\bigcup_{n=1}^{\infty} A_n$ is also countable.

  Let $\{A_n\}_{n \in N}$ be countable sets.

  - Enumerate elements of each A_n
  - Arrange elements in a grid and traverse diagonally. Use the pairing function $\pi(i,j) = (i+j-1)(i+j-2)/2 + j$ to map $(i,j) \to N$
  - Define a surjection $\phi : N \to \bigcup_{n=1}^{\infty} A_n$ via $\phi(\pi(i,j)) = a_{ij}$. By the axiom of countable choice, such an enumeration exists.

Exercise 2.6

Show that $|N| = |Z|$ by finding an explicit bijection from N to Z. You do not need to prove your bijection works.

An explicit bijection $f : N \to Z$ is given by:

$$f(n) = \begin{cases} \dfrac{n}{2} \text{ if n is even} \\ -\dfrac{n+1}{2} \text{ if n is odd} \end{cases} \tag{7}$$

This maps the natural numbers 0, 1, 2, 3, 4, 5,... to the integers 0, -1, 1, -2, 2, -3, ... in order.

Exercise 2.7

Let $A, B \subseteq R$, we define

$$A \cdot B = \{a \cdot b : a \in A \text{ and } b \in B\} \tag{8}$$

- Give an example of sets A_1 and B_1 where $|A_1 \cdot B_1| < max\{|A_1|, |B_1|\}$
  Let $A_1 = \{0\}$ and $B_1 = \{1, 2\}$
  - Product set $A_1 \cdot B_1 = \{0\}$
  - Cardinalities $|A_1| = 1, |B_1| = 2, |A_1 \cdot B_1| = 1$
  - result: $1 < max\{1, 2\} = 2$
- Give an example of sets A_2 and B_2 where $|A_2 \cdot B_2| > max\{|A_2|, |B_2|\}$
  Let $A_2 = \{1, 2\}$ and $B_2 = \{3, 4\}$
  - Product set $A_2 \cdot B_2 = \{3, 4, 6, 8\}$
  - Cardinalities $|A_2| = 2, |B_2| = 2, |A_2 \cdot B_2| = 4$
  - result: $4 > max\{2, 2\} = 2$
- Give an example of sets A_3 and B_3 where $|A_3 \cdot B_3| = max\{|A_3|, |B_3|\}$
  Let $A_3 = \{2\}$ and $B_3 = \{1, 3, 5\}$
  - Product set $A_3 \cdot B_3 = \{2, 6, 10\}$
  - Cardinalities $|A_3| = 1, |B_3| = 3, |A_3 \cdot B_3| = 3$
  - result: $3 = max\{1, 3\} = 3$

Exercise 2.8

- Describe a way to partition the set N into 6 subsets, each containing infinitely many elements.

  One simple way to partition N into 6 subsets, each containing infinitely many elements, is to use modular arithmetic. For each k = 0,1,2,3,4,5, define the subset:

  $$A_k = \{n \in N : n \equiv k (\bmod 6)\} \tag{9}$$

  Each A_k contains all natural numbers congruent to k modulo 6. Since there are infinitely many natural numbers in each residue class modulo 6, each A_k is infinite, and together, the six sets are disjoint and cover all of N.

- Describe a way to partition the set N into infinitely many subsets, each containing infinitely many elements.

  A classic construction is to use the following approach: For each $k \in N$ (where $k \geq 1$), define the subset:

  $$B_k = \{n \in N : \text{n is divisible by k but not by any } j < k\} \tag{10}$$

Exercise 2.9

Is $|Z \times N|$ countable or uncountable?

- key reasoning: a cartesian product of two countable sets is also countable.
- arrange $Z \times N$ in an infinite grid and traverse diagonally to list all pairs, ensuring every element is included exactly once.
- Thus, $|Z \times N| = \aleph_0$, confirming its countability.

Exercise 2.10

Let S be the set of sequences (a_n) where, for each n, $a_n \in \{0, 1\}$. Is S countable or uncountable?

The set S is uncountable.

- Assume for contradiction that S is countable. Then there exists a bijection $f : N \to S$, listing all sequences $f(1), f(2), f(3), ...$
- Construct a new sequence $A = (a_n)$ such that

$$a_n = \begin{cases} 1 \text{ if the nth digit of f(n) is 0} \\ 0 \text{ if the nth digit of f(n) is 1} \end{cases} \tag{11}$$

  This sequence A differs from every f(n) at the nth position.

- contradiction: Since A is not in the list f(1), f(2), f(3),..., f cannot be a bijection. Thus, S is uncountable.

Exercise 2.11

Suppose that X is a nonempty set. Prove that the following three assertions are equivalent.

- X is finite or countably infinite.
- There is one-to-one function $f : X \to N$.
- There is an onto function $g : N \to X$.
- (1) $\implies$ (2)

  If X is finite, say with n elements, we can enumerate its elements and define an injective function $f : X \to N$ by assigning each element a distinct natural number between 1 and n. If X is countably infinite, then by definition, there exists a bijection $h : X \to N$, which is certainly injective. Thus, in both cases, there is a one-to-one function $f : X \to N$

- (2) $\implies$ (3)

  Suppose there is an injective function $f : X \to N$. Let $T = f(X) \subseteq N$.

  - If X is finite, then T is finite, and we can define $g : N \to X$ by mapping the first —X— natural numbers to all elements of X, and the rest to any fixed element of X. This function is onto.
  - If X is infinite, then T is an infinite subset of N, and by standard results, T is countably infinite and there exists a bijection $h : N \to T$. Composing h with $f^{-1} : T \to X$ yields surjection $g : N \to X$.

- (3) $\implies$ (1)

  Suppose there is a surjective function $g : N \to X$. Then X is either finite or countably infinite:

  - If X is finite, the image of g is finite.
  - If X is infinite, then X is the image of N under g, so X is countable, and since it is infinite, it is countably infinite.

Exercise 2.12

- Give an example of a collection of countably many disjoint open intervals, or prove that this does not exist.

  The collection $\{(n, n+1)|n \in Z\}$ consists of infinitely many disjoint open intervals.

  - Disjointness: each interval $(n, n+1)$ does not overlap with others.
  - Countability: The set of integers Z is countable, so the collection is countable.

- Give an example of a collection of uncountably many disjoint open intervals, or prove that this does not exist.

  Assume, for contradiction, that there exists an uncountable collection $\{I_\alpha\}_{\alpha \in A}$ of disjoint open intervals in R.

  - Density of Rationals: each open interval I_α contains at least one rational number $q_\alpha \in Q$
  - Injection in Q: Map each interval I_α to $q_\alpha \in Q$. Since intervals are disjoint, $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$, forming an injection $f : A \to Q$.
  - contradiction: Q is countable, but A is uncountable. Thus, no such collection exists.

Exercise 2.13

Show that there are uncountably many irrational numbers.

- Assume for contradiction that the set of irrational numbers $R \setminus Q$ is countable

- Known results
  - The rational numbers Q are countable
  - the real numbers R are uncountable
- Union of sets
  - $R = Q \cup (R \setminus Q)$
  - If both Q and $R \setminus Q$ were countable, their union R would also be countable (since the union of two countable sets is countable)
- Contradiction: This directly contradicts the fact that R is uncountable.
- Therefore, the set of irrational numbers is uncountable.

Exercise 2.14

Prove that $N \times N$ is countably infinite by showing that the function $f : N \times N \to N$ defined by $f(m, n) = 2^{n-1}(2m - 1)$ is a bijection.

- Prove Injectivity

  Assume $f(m_1, n_1) = f(m_2, n_2)$. Then: $2^{n_1-1}(2m_1-1) = 2^{n_2-1}(2m_2-1)$.

  - Suppose $n_1 \neq n_2$. Without loss of generality, let $n_1 > n_2$. Dividing both sides by 2^{n_2-1} gives $2^{n_1-n_2}(2m_1 - 1) = 2m_2 - 1$. The left side is even while the right side is odd. This contradiction implies $n_1 = n_2$
  - $n_1 = n_2 \implies m_1 = m_2$

  Thus, f is injective.

- Prove surjectivity

  For any $k \in N$, we can write k as $k = 2^{n-1}\cdot$ (odd natural number).

  - Factorization: every natural number k has a unique prime factorization. Let 2^{n-1} be the highest power of 2 dividing k, Then $k = 2^{n-1} \cdot q$, where q is odd.
  - Define m: Since q is odd, write $q = 2m - 1$ for some $m \in N$.

Thus, $k = 2^{n-1}(2m-1) = f(m,n)$, proving surjectivity.

- Since f is both injectivve and surjective, it is a bijection. Therefore $N \times N$ is countably infinite.

Exercise 2.15

Let F be the collection of all functions $f : R \to R$. Prove that F is uncountable.

To prove that the collection F of all functions $f : R \to R$ is uncountable, we use a cardinality argument based on the power set of R:

- Subset of Functions:

  Consider the subset $G \subseteq F$ consisting of all characteristic functions $X_A : R \to \{0, 1\}$, where $A \subseteq R$. each X_A maps elements of A to 1 and all others to 0.

- Bijection with Power Set:

  There is a bijection between G and P(R): every subset $A \subseteq R$ corresponds to a unique characteristic function X_A. By Cantor's theorem, $|P(R)| = 2^c$, where $c = |R|$

- Uncountability of G:

  Since P(R) is uncountable (its cardinality exceeds c), the subset $G \subseteq F$ is also uncountable

- Counclusion for F: If G is uncountable, then F, which contains G, must also be uncountable.