System Dynamics (24.509)

VI. Mathematical Modeling of Engineering Systems

Case Study F - Dynamic Model of a Permanent Magnet DC Motor

Mathematical Model

The goal in the development of the mathematical model is to relate the voltage applied to the armature to the velocity of the motor. Two balance equations can be developed by considering the electrical and mechanical characteristics of the system.

Electrical Characteristics

The equivalent electrical circuit of a dc motor is illustrated in Fig. 6F.2. It can be represented by a voltage source (V_a) across the coil of the armature. The electrical equivalent of the armature coil can be described by an inductance (L_a) in series with a resistance (R_a) in series with an induced voltage (V_c) which opposes the voltage source. The induced voltage is generated by the rotation of the electrical coil through the fixed flux lines of the permanent magnets. This voltage is often referred to as the back emf (electromotive force).

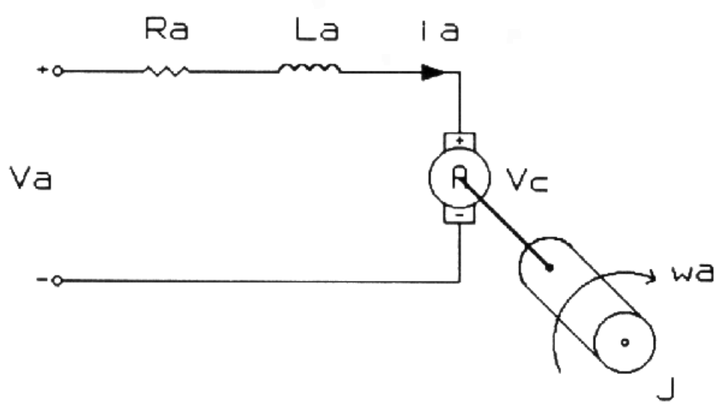

Fig. 6F.2 Electrical representation of a dc motor.

A differential equation for the equivalent circuit can be derived by using Kirchoff's voltage law around the electrical loop. Kirchoff's voltage law states that the sum of all voltages around a loop must equal zero, or

$$V_a - V_{Ra} - V_{La} - V_c = 0 \qquad (6F.1)$$

According to Ohm's law, the voltage across the resistor can be represented as

$$V_{Ra} = i_a R_a \qquad (6F.2)$$

where i_a is the armature current. The voltage across the inductor is proportional to the change of current through the coil with respect to time and can be written as

$$V_{La} = L_a \frac{d}{dt} i_a \qquad (6F.3)$$

where L_a is the inductance of the armature coil. Finally, the back emf can be written as

$$V_c = k_v \omega_a \qquad (6F.4)$$

where k_v is the velocity constant determined by the flux density of the permanent magnets, the reluctance of the iron core of the armature, and the number of turns of the armature winding. w $_a$ is the rotational velocity of the armature.

Substituting eqns. (6F.2), (6F.3), and (6F.4) into eqn. (6F.1) gives the following differential equation:

$$V_a - i_a R_a - L_a \frac{d}{dt} i_a - k_v \omega_a = 0 \qquad (6F.5)$$

Mechanical Characteristics

Performing an energy balance on the system, the sum of the torques of the motor must equal zero. Therefore,

$$T_e - T_{\omega'} - T_{\omega} - T_L = 0 \qquad (6F.6)$$

where T_e is the electromagnetic torque, Tw ' is the torque due to rotational acceleration of the rotor, Tw is the torque produced from the velocity of the rotor, and T_L is the torque of the mechanical load. The electromagnetic torque is proportional to the current through the armature winding and can be written as

$$T_e = k_t i_a \qquad (6F.7)$$

where k_t is the torque constant and like the velocity constant is dependent on the flux density of the fixed magnets, the reluctance of the iron core, and the number of turns in the armature winding. Tw ' can be written as

$$T_{\omega'} = J\frac{d}{dt}\omega_a \qquad (6F.8)$$

where J is the inertia of the rotor and the equivalent mechanical load. The torque associated with the velocity is written as

$$T_\omega = B\omega_a \qquad (6F.9)$$

where B is the damping coefficient associated with the mechanical rotational system of the machine.

Substituting eqns. (6F.7), (6F.8), and (6F.9) into eqn. (6F.6) gives the following differential equation:

$$k_t i_a - J\frac{d}{dt}\omega_a - B\omega_a - T_L = 0 \qquad (6F.10)$$

State Space Representation

The differential equations given in eqns. (6F.5) and (6F.10) for the armature current and the angular velocity can be written as

$$\frac{d}{dt}i_a = -\frac{R_a}{L_a}i_a - \frac{k_v}{L_a}\omega_a + \frac{V_a}{L_a} \qquad (6F.11)$$

$$\frac{d}{dt}\omega_a = \frac{k_t}{J}i_a - \frac{B}{J}\omega_a - \frac{T_L}{J} \qquad (6F.12)$$

which describe the dc motor system. Putting the differential equations into state space form gives

$$\frac{d}{dt}\begin{bmatrix} i_a \\ \omega_a \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{k_v}{L_a} \\ \frac{k_t}{J} & -\frac{B}{J} \end{bmatrix}\begin{bmatrix} i_a \\ \omega_a \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{J} \end{bmatrix}\begin{bmatrix} V_a \\ T_L \end{bmatrix} \qquad (6F.13)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i_a \\ \omega_a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_a \\ T_L \end{bmatrix} \qquad (6F14)$$

which is expressed symbolically as

$$\frac{d}{dt}\underline{x} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\underline{u} \qquad (6F.15)$$

$$\underline{y} = \underline{\underline{C}}\underline{x} + \underline{\underline{D}}\underline{u} \qquad (6F16)$$

where $\underline{x}$ is the state vector, $\underline{u}$ is the input vector, and $\underline{y}$ is the output vector.

Transfer Function Block Diagram

A block diagram for the system can be developed from the differential equations given in eqns. (6F.11) and (6F.12). Taking the Laplace transform of each equation gives

$$sI_a(s) - i_a(0) = -\frac{R_a}{L_a}I_a(s) - \frac{k_v}{L_a}\Omega_a(s) + \frac{1}{L_a}V_a(s) \qquad (6F.17)$$

$$s\Omega_a(s) - \omega_a(0) = \frac{k_t}{J}I_a(s) - \frac{B}{J}\Omega_a(s) - \frac{1}{J}T_L(s) \qquad (6F.18)$$

If perturbations around some steady state value are considered, the initial conditions go to zero and all the variables become some change around a reference state, and the equations can be expressed as follows:

$$I_a(s) = \frac{-k_v\Omega_a(s) + V_a(s)}{L_a s + R_a} \qquad (6F19)$$

$$\Omega_a(s) = \frac{-k_t I_a(s) - T_L(s)}{Js + B} \qquad (6F.20)$$

The above equations can then easily be put into block diagram form. The block diagram obtained from these equations for a permanent magnet dc motor is shown in Fig. 6F.3.

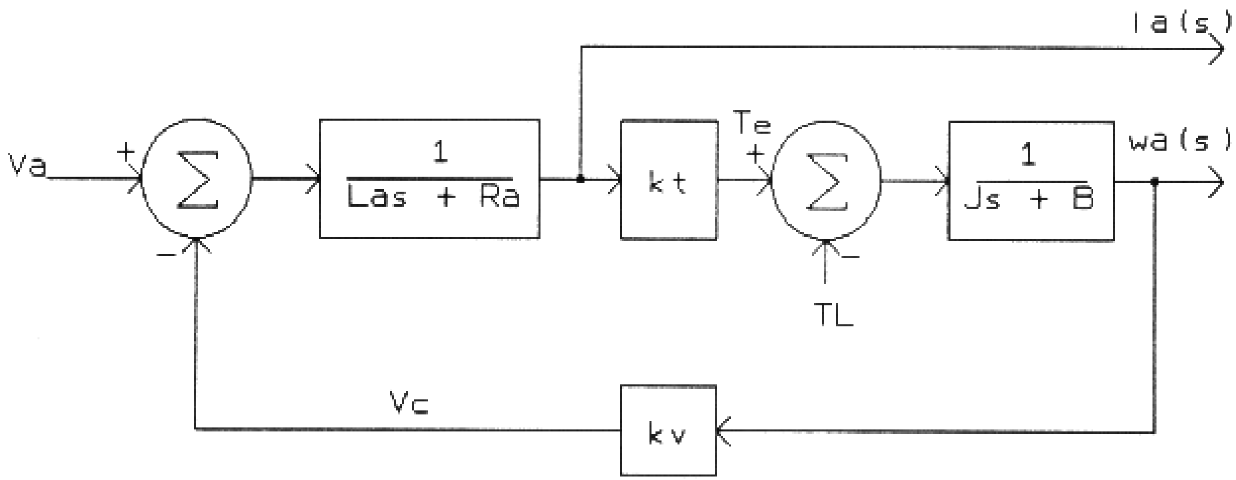

Fig. 6F.3 Block diagram representation of eqns. (6F.19) and (6F.20).

The block diagram in Fig. 6F.3 can be simplified by making the assumption that the load torque is constant. In the case of a sun tracking servo system, the only load torque to be concerned with is the friction in the system, which is relatively constant while the motor is moving. Since the change in T_L is zero, it does not need to appear in the block diagram. Also, if one only focuses on the angular velocity as the response of interest, the block diagram becomes as shown in Fig. 6F.4.

This block diagram is then easily reduced by block diagram algebra to an overall transfer function. Several steps in this process are shown in Fig. 6F.5, with the overall transfer function between the output angular velocity and input applied voltage given within the last block in Fig. 6F.5.

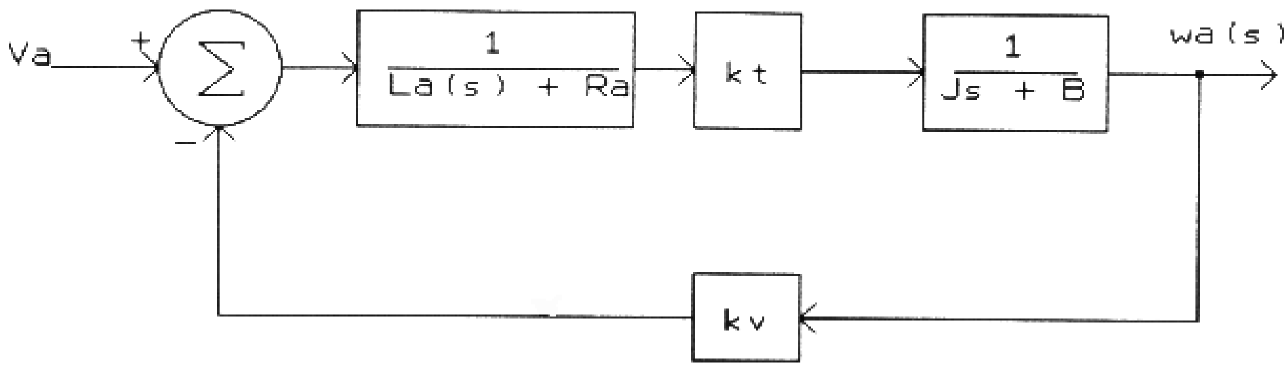

Fig. 6F.4 Block diagram of the dc motor as modeled in this study.

$$G(s) = \frac{G1(s)}{1 + G1(s)H(s)}$$

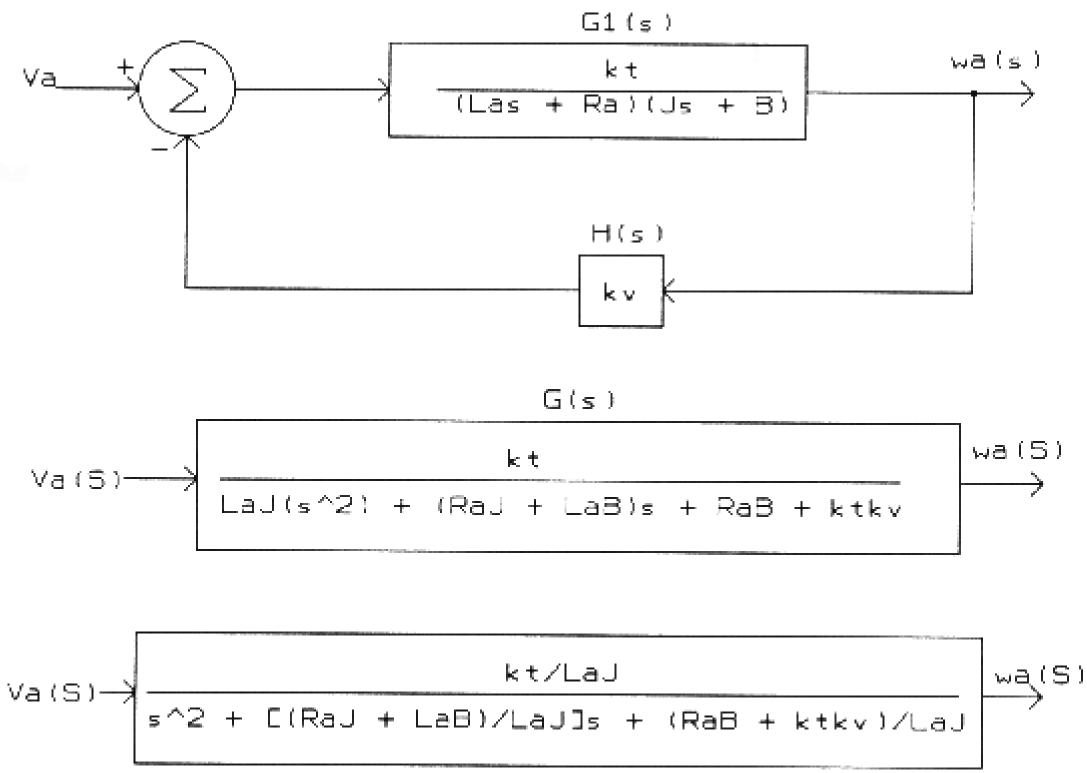

Fig. 6F.5 Overall transfer function for the dc motor.

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24.509 Lecture Notes by Dr. J. R. White, UMass-Lowell (Spring 1997).

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