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Decision Making II

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The solutions for these exercises (comprising source code, discussion and interpretation as an IPython Notebook) should be handed in before **9 June at 8am** through the Moodle interface (in emergency cases send them to `owen.mackwood@bccn-berlin.de`). The solutions will be discussed in the computer course on Monday, 13 June.

Exercise 1: Chapman-Kolmogorov

In this exercise we will use the Chapman-Kolmogorov equation to find the time-dependent distribution of the decision variable in a drift-diffusion model for decision making.

1. Since we are interested in the time evolution of the distribution, you should first generate an $N \times M$ matrix where the first column contains an initial condition ensuring the particle rests in the center bin. Use $N = 2 \cdot 50 + 1$ dimensionless positions (each a distance of 1 apart, so $\Delta x = 1$) and $M = 1000/\Delta t$, where $\Delta t = 1$.

2. Use the transition probability
$$P(x|x'; \Delta t) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left\{-\frac{(x - x' - \mu\Delta t)^2}{2\sigma^2\Delta t}\right\} \tag{1}$$
to compute the matrix of probabilities of transitioning to any x from any x' (take $\mu = 0.1$, $\sigma = 1.0$). This should result in an $N \times N$ matrix where each row contains the probabilities of transition from a particular x' (the columns) into the corresponding x (the row) in the next time step.

3. Now use your transition matrix to compute the distribution at each successive time step for all t. Plot your result using *imshow* (process your data in some fashion to ensure that the distribution is well visualized past the first 100 steps), and at several different time points that illustrate the development over time.

4. If you look at the results, which boundary condition do you think you have implemented? Which part of the code is responsible for that? Does the implemented boundary condition make sense for what we want to simulate?

5. Plot the sum of the "probabilities" as a function of time (that is, sum along the position axis). Why are the probabilities not normalized? How can you interpret this result?

6. Plot the negative derivative (*diff*) of the summed probability computed in the previous step. Compare this plot to the histogram of response times obtained in exercise 1.4 of the **Decision Making I** sheet. What can you infer about their relationship?

7. Could you determine from these results, how the probability of making each of the two decisions evolves over time? If not, how could you modify your solution to make this possible?

Exercise 2: Fokker-Planck

In this exercise we will integrate the Fokker-Planck equation to accomplish the same as in Exercise 1.

1. Once again create a matrix for the distribution over time as per 1.1.
2. Instead of using the Chapman-Kolmogorov equation to determine the transition probabilities, this time we will integrate the Fokker-Planck equation $\partial P(x,t|x_0)/\partial t = -\mu \partial P/\partial x + \frac{1}{2}\sigma^2 \partial^2 P/\partial x$ that results from (1). If one performs a discrete integration using Euler, the distribution evolves according to:

$$P(x,t+\Delta t) = \left(1 - r\sigma^2\right) P(x,t) + \frac{1}{2}\left(\frac{r}{\Delta x}\mu + r\sigma^2\right) P(x+1,t) + \frac{1}{2}\left(-\frac{r}{\Delta x}\mu + r\sigma^2\right) P(x-1,t),$$

   where $r = \Delta t/\Delta x^2$. Create an $N \times N$ matrix that implements this discretized Fokker-Planck differential operator (hint: the rows should be zero everywhere aside from the three values centered on the diagonal).
3. Use the transition matrix from the previous step to compute the time-dependent distribution as in Exercise 1. Plot the result. Look closely at the first several time steps, and comment on whether the system behaves as you expect.
4. Recompute the $N \times N$ transition matrix but with $\Delta t = 0.5$ (you'll also need a new $N \times M$ matrix for the distribution), and see if that improves on the previous result. Can you guess why this might be?
5. Plot $-1/\log(P(t) + \epsilon)$ for a few early time steps. Make the same plots for the Chapman-Kolmogorov solution. How are they different? Can you explain why? (Hint: Look closely at the two transformation matrices).
6. Now let's try a different initial condition. Create a new $N \times M$ matrix for the distribution over time (using the new Δt), but this time use an initial distribution $P(x,0)$ that is Gaussian, centered at the midpoint with $\sigma = 2$.
7. Use the Fokker-Planck differential operator to compute the time-dependent distribution. Again plot $-1/\log(P(t) + \epsilon)$ at several time points.
8. Why does the initial condition have this effect on the computed distributions? Look at the derivation of the Fokker-Planck equation from the lecture and pay close attention to which step could fail for the first initial condition.