MONOIDAL GROTHENDIECK CONSTRUCTION

JOE MOELLER AND CHRISTINA VASILAKOPOULOU

Abstract. We lift the standard equivalence between fibrations and indexed categories to an equivalence between monoidal fibrations and monoidal indexed categories. In doing so, we investigate the relation between this 'global' monoidal structure and a 'fibrewise' one, first hinted in [Shu08]. Finally, we give various examples where this correspondence appears.

Contents

1. Introduction

The classic Grothendieck construction [Gro61] exhibits one of the most fundamental relations in category theory, namely the equivalence between fibrations and pseudofunctors into **Cat**. This equivalence allows us to freely move between the worlds of indexed categories and fibred categories, having access to tools and results from each theory of very different flavor.

Due to its importance, it is only natural that one would be interested in what sort of extra structure these objects could have, and furthermore how they correspond to one another. One way of expressing this is to 'lift' the Grothendieck construction to specific settings of interest. The goal of this paper is to establish the appropriate correspondence in the monoidal setting.

This is accomplished via restricting the standard equivalence to the pseudomonoids of each monoidal 2-category using abstract 2-categorical machinery. Therefore we obtain a canonical correspondence between *monoidal fibrations* (fibrations which are strict monoidal functors with a cartesian condition on the domain tensor product functor) and *monoidal indexed categories* (weak monoidal pseudofunctors into **Cat**). The monoidal Grothendieck construction in this sense uses the weak monoidal structure of the pseudofunctor to equip the corresponding total category with a monoidal product, which is strictly preserved by the fibration.

In [Shu08], Shulman introduced the notion of a monoidal fibration, and gave a different but not unrelated monoidal version of the Grothendieck construction under certain assumptions. Namely, he directly constructs an equivalence between monoidal fibrations over a cartesian monoidal base and pseudofunctors into **MonCat**. This implies that the respective monoidal fibrations have monoidal fibres and strong monoidal reindexing functors between them, which is certainly not always the case for an arbitrary monoidal fibration.

This striking dissimilarity between Shulman's equivalence and the one established here motivated an investigation regarding the 'fibrewise' monoidal structure of a fibration as opposed to a 'global' monoidal structure. Neither of these imply the other. Abstractly, this is explained via the difference between pseudomonoids in different monoidal 2-categories, namely **Fib** and $\mathbf{Fib}(\mathbb{X})$. These two versions only meet when the base category has a (co)cartesian monoidal structure, clearly expressed as a bijection between ordinary pseudofunctors into **MonCat** and weak monoidal pseudofunctors into $(\mathbf{Cat}, \times, \mathbf{1})$. This interesting subtlety concerning the transfer of monoidality from the target category to the very structure of the functor and vice versa could potentially bring new perspective into future variations of Grothendieck constructions, such as an enriched one. The recent work in [BW18] seems to be more in the 'fibrewise' direction, hence future work could address the 'global' one.

Finally, but perhaps most importantly, the examples of the monoidal Grothendieck construction are those that render the clarification of this correspondence essential. As is the case for the ordinary Grothendieck construction, applications seem to pop up in diverse settings, which span from pure categorical and algebraic frameworks, to more applied contexts like network theory. We gather some of them in the last section of the paper, and we are convinced that many more exist and would benefit from such a viewpoint.

Outline of the paper. In Section 2, we review the basic theory of fibrations, indexed categories, and the Grothendieck construction, as well as that of monoidal 2-categories and pseudomonoids. Section 3 contains the eponymous construction in the form of 2-equivalences between the respective 2-categories of monoidal objects. In more detail, Sections 3.1 and 3.2 contains elementary descriptions of (braided/symmetric) monoidal variations of the categories involved in the classic Grothendieck construction, whereas Section 3.3 details the construction which gives equivalences between these various categories. In Section 4, we investigate the bijection between the 'global' and 'fibrewise' monoidal Grothendieck construction for cartesian bases. Finally, Section 5 highlights some examples of this construction as it arises in various contexts.

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2. Preliminaries

We assume familiarity with basic notions and constructions in 2-category theory, see e.g. [KS74, Lac10].

Recall that the paradigmatic example of a 3-category [Gur13], **2-Cat**, consists of of 2-categories, 2-functors, 2-natural transformations and modifications between them. If we consider pseudofunctors (or homomorphisms) $\mathscr{F}\colon\mathcal{K}\to\mathcal{L}$ between 2-categories, i.e. equipped with natural isomorphisms $\delta_{g,f}\colon(\mathscr{F}g)\circ(\mathscr{F}f)\xrightarrow{\sim}\mathscr{F}(g\circ f)$ and $\gamma_A\colon 1_{\mathscr{F}A}\xrightarrow{\sim}\mathscr{F}(1_A)$, and pseudonatural transformations $\tau\colon\mathscr{F}\Rightarrow\mathscr{G}$, i.e.

equipped with natural isomorphisms $\tau_f\colon (\mathscr{G}f)\circ\tau_A \xrightarrow{\sim} \tau_B\circ(\mathscr{F}f)$ for any $f\colon A\to B$, we obtain the tricategory $\mathbf{2\text{-}Cat}_{ps}$.

Also recall that a 2-equivalence $\mathcal{K}\simeq\mathcal{L}$ consists of two 2-functors $S\colon\mathcal{K}\to\mathcal{L}$, $T\colon\mathcal{L}\to\mathcal{K}$ and two 2-natural isomorphisms $1_{\mathcal{K}}\cong TS$, $ST\cong 1_{\mathcal{L}}$; this is of course obtained from the definition of an equivalence internal in the 2-category $\mathbf{2\text{-}Cat}$. Equivalently, it is given by a 2-functor S for which $S_{A,B}\colon\mathcal{K}(A,B)\xrightarrow{\sim}\mathcal{L}(SA,SB)$ are isomorphisms of categories, and every $C\in\mathcal{L}$ is isomorphic to some SA.

2.1. **Fibrations.** We recall some basic facts and constructions from the theory of fibrations and opfibrations. A few indicative references for the general theory are [Gra66, Her94, Bor94].

Consider a functor $P:\mathcal{A}\to\mathbb{X}$. A morphism $\phi:A\to B$ in $\mathcal{A}$ over a morphism $f=P(\phi):X\to Y$ in $\mathbb{X}$ is called *cartesian* if and only if, for all $g:X'\to X$ in $\mathbb{X}$ and $\theta:A'\to B$ in $\mathcal{A}$ with $P\theta=f\circ g$, there exists a unique arrow $\psi:A'\to A$ such that $P\psi=g$ and $\theta=\phi\circ\psi$:

$$\begin{tikzcd}
A' \arrow[rrd, "\theta"] \arrow[rd, dashed, "\exists!\psi"'] \arrow[dd, dotted] & & & \\
& A \arrow[r, "\phi"'] \arrow[dd, dotted] & B \arrow[dd, dotted] & \text{in } \mathcal{A} \\
X' \arrow[rd, "g"'] \arrow[rrd, "f\circ g=P\theta"] & & & \\
& X \arrow[r, "f=P\phi"'] & Y & \text{in } \mathbb{X}
\end{tikzcd} \tag{1}$$

For $X\in\mathrm{ob}\mathbb{X}$, the *fibre of* P *over* X written $\mathcal{A}_X$, is the subcategory of $\mathcal{A}$ which consists of objects A such that $P(A)=X$ and morphisms ϕ with $P(\phi)=1_X$, called *vertical* morphisms. The functor $P:\mathcal{A}\to\mathbb{X}$ is called a *fibration* if and only if, for all $f:X\to Y$ in $\mathbb{X}$ and $B\in\mathcal{A}_Y$, there is a cartesian morphism ϕ with codomain B above f; it is called a *cartesian lifting* of B along f. The category $\mathbb{X}$ is then called the *base* of the fibration, and $\mathcal{A}$ its *total* category.

Dually, the functor $U:\mathcal{C}\to\mathbb{X}$ is an *opfibration* if U^{op} is a fibration, *i.e.* for every $C\in\mathcal{C}_X$ and $g:X\to Y$ in $\mathbb{X}$, there is a cocartesian morphism with domain C above g, the *cocartesian lifting* of C along g.

If $P:\mathcal{A}\to\mathbb{X}$ is a fibration, assuming the axiom of choice we may select a cartesian arrow over each $f:X\to Y$ in $\mathbb{X}$ and $B\in\mathcal{A}_Y$, denoted by $\mathrm{Cart}(f,B):f^*(B)\to B$. Such a choice of cartesian liftings is called a *cleavage* for P, which is then called a *cloven* fibration; any fibration is henceforth assumed to be cloven. Dually, if U is an opfibration, for any $C\in\mathcal{C}_X$ and $g:X\to Y$ in $\mathbb{X}$ we can choose a cocartesian lifting of C along g, $\mathrm{Cocart}(g,C):C\longrightarrow g_!(C)$. The choice of (co)cartesian liftings in an (op)fibration induces a so-called *reindexing functor* between the fibre categories

$$f^*:\mathcal{A}_Y\to\mathcal{A}_X \quad\text{ and }\quad g_!\colon\mathcal{C}_X\to\mathcal{C}_Y \tag{2}$$

respectively, for each morphism $f:X\to Y$ and $g\colon X\to Y$ in the base category. It can be verified that $1_{\mathcal{A}_X}\cong(1_X)^*$ and that for composable morphism in the base category, $(g\circ f)^*\cong g^*\circ f^*$. If these isomorphisms are equalities, we have the notion of a *split* fibration.

A morphism of fibrations $(H, F) : P \to Q$ between $P : \mathcal{A} \to \mathbb{X}$ and $Q : \mathcal{B} \to \mathbb{Y}$ is given by a commutative square of functors and categories

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\ \mathbb{X} \arrow[r, "F"'] & \mathbb{Y} \end{tikzcd} \tag{3}$$

where H preserves cartesian arrows, meaning that if ϕ is P-cartesian, then $H\phi$ is Q-cartesian. The pair (H, F) is called a *fibred 1-cell.* In particular, when P and Q are fibrations over the same base category $\mathbb{X}$, we may consider fibred 1-cells of the form $(H, 1_{\mathbb{X}})$ displayed by a commutative triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "H"] \arrow[dr, "P"'] & & \mathcal{B} \arrow[dl, "Q"] \\ & \mathbb{X} & \end{tikzcd} \tag{4}$$

when H is called a *fibred functor*. Dually, we have the notion of an *opfibred 1-cell* and *opfibred functor*. Notice that any such (op)fibred 1-cell induces functors between the fibres, by commutativity of (3):

$$H_X \colon \mathcal{A}_X \longrightarrow \mathcal{B}_{FX} \tag{5}$$

A *fibred 2-cell* between fibred 1-cells (H, F) and (K, G) is a pair of natural transformations $(\alpha : H \Rightarrow K, \beta : F \Rightarrow G)$ with α above β, *i.e.* $Q(\alpha_A) = \beta_{PA}$ for all $A \in \mathcal{A}$, displayed as

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"] \\ \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbb{Y}. \end{tikzcd} \tag{6}$$

A *fibred natural transformation* is of the form $(\alpha, 1_{1_{\mathbb{X}}}) : (H, 1_{\mathbb{X}}) \Rightarrow (K, 1_{\mathbb{X}})$ as in

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "H"] \arrow[rr, bend right, "K"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "P"'] & & \mathcal{B} \arrow[dl, "Q"] \\ & \mathbb{X} & \end{tikzcd} \tag{7}$$

which has vertical components, $Q(\alpha_A) = 1_{PA}$. Dually, we have the notion of an *opfibred 2-cell* and *opfibred natural transformation* between opfibred 1-cells and functors respectively.

We thus obtain a 2-category **Fib** of fibrations over arbitrary base categories, fibred 1-cells and fibred 2-cells. In particular, there is a 2-category $\mathbf{Fib}(\mathbb{X})$ of fibrations over a fixed base category $\mathbb{X}$, fibred functors and fibred natural transformations. Dually, we have the 2-categories **OpFib** and $\mathbf{OpFib}(\mathbb{X})$. Moreover, we also have 2-categories $\mathbf{Fib}_{\mathrm{sp}}$ and $\mathbf{OpFib}_{\mathrm{sp}}$ of split (op)fibrations, and (op)fibred 1-cells that preserve the cartesian liftings 'on the nose'. These are all in fact sub-2-categories of $\mathbf{Cat}^{\mathbf{2}} = [\mathbf{2}, \mathbf{Cat}]$ and $\mathbf{Cat}/\mathbb{X}$ respectively.

Remark 2.1. These categories are in fact fibred themselves. Explicitly, the functor $\mathrm{cod}\colon \mathbf{Fib} \to \mathbf{Cat}$ which maps a fibration to its base is a fibration, with fibres $\mathbf{Fib}(\mathbb{X})$ and cartesian liftings pullbacks along fibrations. Similarly, $\mathrm{cod}\colon \mathbf{OpFib} \to \mathbf{Cat}$ is also a fibration.

2.2. **Indexed Categories.** We review the basic theory of indexed categories; more details can be found in [Joh02].

Given a category $\mathbb{X}$, an $\mathbb{X}$-*indexed category* is a pseudofunctor $\mathscr{F}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$, where $\mathbb{X}$ is an ordinary category viewed as a 2-category with trivial 2-cells. An $\mathbb{X}$-*opindexed category* is an X^{op}-indexed category, i.e. a pseudofunctor $\mathbb{X} \to \mathbf{Cat}$. If an (op)indexed category preserves composition on the nose, i.e. it is a (2-)functor, then it is called *strict*.

An *indexed* 1-*cell* $(F, \tau)\colon \mathscr{F} \to \mathscr{G}$ between indexed categories $\mathscr{F}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ and $\mathscr{G}\colon \mathbb{Y}^{\mathrm{op}} \to \mathbf{Cat}$ is a functor $F\colon C \to D$ and a pseudonatural transformation $\tau\colon \mathscr{F} \Rightarrow \mathscr{G} \circ F^{\mathrm{op}}$

$$\begin{tikzcd}
\mathbb{X}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[dr, "\mathscr{F}"] & \\
\arrow[r, phantom, "\tau\Downarrow"] & \mathbf{Cat} \\
\mathbb{Y}^{\mathrm{op}} \arrow[ur, "\mathscr{G}"'] &
\end{tikzcd}$$

with components $\tau_x\colon \mathscr{F}(x) \to \mathscr{G}(Fx)$. For indexed categories with the same base $\mathscr{F}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ and $\mathscr{G}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$, we may consider indexed 1-cells of the form $(1_{\mathbb{X}}, \tau)$

$$\begin{tikzcd}
\mathbb{X}^{\mathrm{op}} \arrow[r, bend left, "\mathscr{F}"] \arrow[r, bend right, "\mathscr{G}"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathbf{Cat}
\end{tikzcd}$$

which are called *indexed functors*. Dually, we have the notion of an opindexed 1-cell and opindexed functor.

An *indexed 2-cell* (β, m) between 1-cells $(F, \tau) \to (G, \sigma)$

$$\begin{tikzcd}
\mathbb{X}^{\mathrm{op}} \arrow[dd, bend right, "F^{\mathrm{op}}"'] \arrow[dd, bend left, "G^{\mathrm{op}}"] \arrow[dr, "\mathscr{F}"] & \\
\arrow[r, phantom, "\tau\Downarrow\,\Downarrow\sigma"] & \mathbf{Cat} \\
\mathbb{Y}^{\mathrm{op}} \arrow[ur, "\mathscr{G}"'] &
\end{tikzcd}$$

is a natural transformation $\beta\colon F \Rightarrow G$ and a modification m

$$\begin{tikzcd}
\mathbb{X}^{\mathrm{op}} \arrow[dd, bend right, "F^{\mathrm{op}}"'] \arrow[dd, bend left] \arrow[dd, phantom, "\overset{\beta^{\mathrm{op}}}{\Longrightarrow}"] \arrow[dr, "\mathscr{F}"] & & & \mathbb{X}^{\mathrm{op}} \arrow[dd, bend right, "G^{\mathrm{op}}"'] \arrow[dr, "\mathscr{F}"] & \\
\arrow[r, phantom, "\Downarrow\tau"] & \mathbf{Cat} & \overset{m}{\Rrightarrow} & \arrow[r, phantom, "\Downarrow\sigma"] & \mathbf{Cat} \\
\mathbb{Y}^{\mathrm{op}} \arrow[ur, "\mathscr{G}"'] & & & \mathbb{Y}^{\mathrm{op}} \arrow[ur, "\mathscr{G}"'] &
\end{tikzcd}$$

given by a family of natural transformations $m_x\colon \mathscr{H}\beta_x \circ \tau_x \Rightarrow \sigma_X$. Between two indexed functors, we may consider *indexed natural transformations*, i.e. / indexed 2-cells of the form (id, m). Dually, we have the notion of an opindexed 2-cell and opindexed natural transformation between opindexed 1-cells and functors respectively.

Thus we obtain a 2-category **ICat** of indexed categories over arbitrary bases, indexed 1-cells, and indexed 2-cells. In particular, there is a 2-category $\mathbf{ICat}(\mathbb{X})$ of indexed categories over a fixed domain category $\mathbb{X}$, indexed functors $(1_X, \tau)$, and indexed natural transformations (id, m). This clearly coincides with the functor 2-category $\mathbf{2\text{-}Cat}_{ps}(\mathbb{X}^{\mathrm{op}}, \mathbf{Cat})$. Dually, we have the 2-categories **OpICat** and $\mathbf{OpICat}(\mathbb{X})$.

Remark 2.2. These categories are also fibred. There is a 2-functor $\mathbf{ICat} \to \mathbf{Cat}$ which sends an indexed category to its domain, and indexed 1-cells and 2-cells to their first components. This is a split fibration, with fibres $\mathbf{ICat}(\mathbb{X})$ above $\mathbb{X}$. Similar is true for for **OpICat**.

2.3. **Grothendieck Construction.** In SGA I [Gro61], Grothendieck introduced a construction for a fibration $P_{\mathscr{F}}\colon \int\mathscr{F} \to \mathbb{X}$ from a given indexed category $\mathscr{F}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ as follows.

If δ and γ are the natural isomorphisms coming with the pseudofunctor $\mathscr{F}$, the total category $\int F$ is defined to have

- objects (x, c) with $x \in \mathbb{X}$, and $c \in \mathscr{F}x$
- morphisms $(f, k)\colon (x, c) \to (y, d)$ with $f\colon x \to y$ a morphism in X, and $k\colon c \to (\mathscr{F}f)(d)$ a morphism in $\mathscr{F}x$
- composition $(g, \ell) \circ (f, k)\colon (x, c) \to (y, d) \to (z, e)$ is given by $(g \circ f, (\delta_{f,g})_e \circ \mathscr{F}g(\ell) \circ k)$ where indeed
$$c \xrightarrow{k} \mathscr{F}f(d) \xrightarrow{\mathscr{F}g(\ell)} (\mathscr{F}g)(\mathscr{F}f)(e) \cong \mathscr{F}(g \circ f)(e)$$
- unit is $(1_x, (\gamma_x)_c)\colon (x, c) \to (x, c)$ where $(\gamma_x)_c\colon c = (1_{\mathscr{F}x}c) \to (\mathscr{F}1_x)(c)$

The functor $P_{\mathscr{F}}\colon \int\mathscr{F} \to \mathbb{X}$ is given by $(x, c) \mapsto x$ on objects, and $(f, k) \mapsto f$ on morphisms; the cartesian lifting of any (y, d) in $\int\mathscr{F}$ along $f\colon x \to y$ in $\mathbb{X}$ is precisely $(f, 1_{(\mathscr{F}f)d})$, hence $P_{\mathscr{F}}$ is a fibration.

On the other hand, given a cloven fibration $P\colon \mathcal{A} \to \mathbb{X}^{\mathrm{op}}$, we can define an $\mathbb{X}$-indexed category $\mathscr{F}_P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$; it sends each object x of $\mathbb{X}$ to its fibre category $\mathcal{A}_x$, and each morphism $f\colon x \to y$ to the corresponding reindexing functor $f^*\colon \mathcal{A}_y \to \mathcal{A}_x$. The isomorphisms $f^* \circ g^* \cong (g \circ f)^*$ and $1_{\mathcal{A}_x} \cong 1_x^*$ by the universal property of cartesian morphisms render this assignments pseudofunctorial.

Details of the above, as well as the correspondence between 1-cells and 2-cells can be found in standard references such as [Bor94, Her93, Jac99]; these can be summarized in the following theorem.

Theorem 2.3.

(1) *Every cloven fibration $P\colon \mathcal{A} \to \mathbb{X}$ gives rise to a pseudofunctor $\mathscr{F}_P\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$.*
(2) *Every indexed category $\mathscr{F}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ gives rise to a cloven fibration $P_{\mathscr{F}}\colon \int\mathscr{F} \to \mathbb{X}$.*
(3) *The above correspondences yield an equivalence of 2-categories*
$$\mathbf{ICat}(\mathbb{X}) \simeq \mathbf{Fib}(\mathbb{X}) \tag{8}$$

so that $\mathscr{F}_{P_{\mathscr{F}}} \cong \mathscr{F}$ *and* $P_{\mathscr{F}_P} \cong P$.

If we combine the above theorem with Remarks Remark 2.1 and Remark 2.2 that point out that the 2-categories **Fib** and **ICat** are fibred over **Cat** with fibres $\mathbf{Fib}(\mathbb{X})$ and $\mathbf{ICat}(\mathbb{X})$ respectively, we obtain the following **Cat**-fibred equivalence

$$\begin{tikzcd} \mathbf{ICat} \arrow[rr, "\simeq"] \arrow[dr, "base"'] & & \mathbf{Fib} \arrow[dl, "cod"] \\ & \mathbf{Cat} & \end{tikzcd} \tag{9}$$

There is an analogous story for opindexed categories and opfibrations, that results into a 2-equivalence between $\mathbf{OpICat}(\mathbb{X})$ and $\mathbf{OpFib}(\mathbb{X})$, which also translates into a **Cat**-fibred equivalence

$$\begin{tikzcd} \mathbf{OpICat} \arrow[rr, "\simeq"] \arrow[dr, "base"'] & & \mathbf{OpFib} \arrow[dl, "cod"] \\ & \mathbf{Cat} & \end{tikzcd}$$

2.4. **Monoidal 2-categories and pseudomonoids.** Recall that a *monoidal 2-category* $\mathcal{K}$ is a 2-category equipped with a pseudofunctor $\otimes\colon \mathcal{K}\times\mathcal{K}\to\mathcal{K}$ and a unit object $I\colon \mathbf{1}\to\mathcal{K}$ which are associative and unital up to coherent equivalence; for an explicit description see [Car95, GPS95].

A *pseudomonoid* (or *monoidale*) in a monoidal 2-category $(\mathcal{K},\otimes,I)$ is an object A equipped with multiplication $m : A\otimes A\to A$, unit $j : I\to A$, and invertible 2-cells

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "A\otimes m"] \arrow[d, "m\otimes A"'] \arrow[dr, phantom, "\overset{\alpha}{\cong}"] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} A\otimes I \arrow[r, "A\otimes j"] \arrow[dr, "\sim"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\overset{\lambda}{\cong}"] \arrow[dr, phantom, "\overset{\rho}{\cong}"] & I\otimes A \arrow[l, "j\otimes A"'] \arrow[dl, "\sim"] \\ & A & \end{tikzcd} \tag{10}$$

expressing assiociativity and unitality up to isomorphism, that satisfy appropriate coherence conditions, see [DS97, McC00]. With the appropriate notions of 1-cells and 2-cells described below, they form 2-categories which are of central importance for what follows.

Definition 2.4. A *lax morphism* between pseudomonoids A, B in a monoidal 2-category $\mathcal{K}$ is a 1-cell $f\colon A\to B$ equipped with 2-cells

$$\begin{tikzcd} A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "m"'] \arrow[dr, phantom, "\overset{\phi}{\swarrow}"] & B\otimes B \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "j"'] \arrow[dr, bend left, "j"] \arrow[dr, phantom, "\overset{\phi_0}{\swarrow}"] & \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{11}$$

such that the following conditions hold, for α, λ, ρ as in (10):

$$\begin{tikzcd}[column sep=small] & B\otimes B\otimes B \arrow[r, "m\otimes 1"] \arrow[d, phantom, "\Downarrow \phi\otimes 1_f"] & B\otimes B \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\phi"] & \\ A\otimes A\otimes A \arrow[ur, "f\otimes f\otimes f"] \arrow[r, "m\otimes 1"] \arrow[dr, "1\otimes m"'] & A\otimes A \arrow[ur, "f\otimes f"] \arrow[dr, "m"] \arrow[d, phantom, "\overset{\alpha}{\cong}"] & & B \\ & A\otimes A \arrow[r, "m"'] & A \arrow[ur, "f"'] & \end{tikzcd} = \begin{tikzcd}[column sep=small] & B\otimes B\otimes B \arrow[r, "m\otimes 1"] \arrow[dr, "1\otimes m"] \arrow[d, phantom, "\Downarrow 1_f\otimes\phi"] & B\otimes B \arrow[dr, "m"] \arrow[d, phantom, "\overset{\alpha}{\cong}"] & \\ A\otimes A\otimes A \arrow[ur, "f\otimes f\otimes f"] \arrow[dr, "1\otimes m"'] & & B\otimes B \arrow[r, "m"] \arrow[d, phantom, "\Downarrow\phi"] & B \\ & A\otimes A \arrow[ur, "f\otimes f"] \arrow[r, "m"'] & B\otimes B \arrow[ur, "f"'] & \end{tikzcd}$$

$$
\begin{tikzcd}
A \cong A\otimes I \arrow[rr, "f\otimes j"] \arrow[rrr, bend left, "f"] \arrow[rrr, phantom, bend left=15, "1_f\otimes\lambda\cong"] \arrow[dr, "1\otimes j"'] \arrow[drr, bend right=50, "1_A"'] \arrow[drr, phantom, bend right=25, "\lambda\cong"] & & B\otimes B \arrow[r, "m"'] & B \\
& A\otimes A \arrow[r, "m"'] \arrow[ur, "f\otimes f"] \arrow[u, phantom, "1_f\otimes\phi_0\ \Downarrow"] & A \arrow[ur, "f"'] \arrow[u, phantom, "\Downarrow\phi"] &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B
\end{tikzcd}
=
\begin{tikzcd}
A \cong I\otimes A \arrow[rr, "j\otimes 1"] \arrow[rrr, bend left, "f"] \arrow[rrr, phantom, bend left=15, "\rho\otimes 1_f\cong"] \arrow[dr, "j\otimes 1"'] \arrow[drr, bend right=50, "1_A"'] \arrow[drr, phantom, bend right=25, "\rho\cong"] & & B\otimes B \arrow[r, "m"'] & B \\
& A\otimes A \arrow[r, "m"'] \arrow[ur, "f\otimes f"] \arrow[u, phantom, "\phi_0\otimes 1_f\ \Downarrow"] & A \arrow[ur, "f"'] \arrow[u, phantom, "\Downarrow\phi"] &
\end{tikzcd}
$$

where the associativity and unity constraints of the monoidal 2-category are suppressed.

Definition 2.5. If (f, ϕ, ϕ_0) and (g, ψ, ψ_0) are two lax morphisms between pseudomonoids A and B, a 2-cell between them $\alpha\colon f \Rightarrow g$ in $\mathcal{K}$ which is compatible with multiplications and units, in the sense that

$$
\begin{tikzcd}
& B\otimes B \arrow[dr, "m"] & \\
A\otimes A \arrow[ur, bend left, "f\otimes f"] \arrow[dr, bend right, "m"'] \arrow[rr, phantom, "\phi\Downarrow"] & & B \\
& A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\alpha"] &
\end{tikzcd}
=
\begin{tikzcd}
& B\otimes B \arrow[dr, "m"] & \\
A\otimes A \arrow[ur, bend left, "f\otimes f"] \arrow[ur, bend right, "g\otimes g"'] \arrow[ur, phantom, "\Downarrow\alpha\otimes\alpha"] \arrow[dr, bend right, "m"'] \arrow[rr, phantom, "\psi\Downarrow"] & & B \\
& A \arrow[ur, bend right, "g"'] &
\end{tikzcd}
\tag{12}
$$

$$
\begin{tikzcd}
I \arrow[rr, bend left, "j"] \arrow[dr, bend right, "j"'] \arrow[rr, phantom, "\phi_0\Downarrow"] & & B \\
& A \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\alpha"] &
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[rr, bend left, "j"] \arrow[dr, bend right, "j"'] \arrow[rr, phantom, "\Downarrow\psi_0"] & & B \\
& A \arrow[ur, bend right, "g"'] &
\end{tikzcd}
$$

We obtain a 2-category $\mathsf{PsMon}_{\mathrm{lax}}(\mathcal{K})$ for any monoidal 2-category $\mathcal{K}$, which is sometimes denoted by $\mathbf{Mon}(\mathcal{K})$ [CLS10]. By changing the direction of the 2-cells in (11) and the rest of the axioms appropriately, we have 2-categories $\mathsf{PsMon}_{\mathrm{opl}}(\mathbb{K})$ and $\mathsf{PsMon}(\mathbb{K})$ of pseudomonoids with *oplax* or *strong* morphisms between them.

Example 2.6. The prototypical example is that of the cartesian monoidal 2-category $\mathcal{K} = (\mathbf{Cat}, \times, \mathcal{I})$ of categories, functors and natural transformations with the cartesian product of categories and the unit category with a unique object and arrow. A pseudomonoid in $(\mathbf{Cat}, \times, \mathcal{I})$ is a monoidal category, a lax (resp. oplax, strong) morphism between them is precisely a lax (resp. oplax, strong) monoidal functor, whereas a 2-cell is a monoidal natural transformation. Therefore we obtain the well-known 2-categories $\mathbf{MonCat}_{\mathrm{lax}}$, $\mathbf{MonCat}_{\mathrm{opl}}$ and $\mathbf{MonCat}$.

Now recall that a *weak monoidal pseudofunctor* $\mathscr{F}\colon \mathcal{K} \to \mathcal{L}$ between monoidal 2-categories is a pseudofunctor equipped with pseudonatural transformations

$$
\begin{tikzcd}
\mathcal{K}\times\mathcal{K} \arrow[r, "\mathscr{F}\times\mathscr{F}"] \arrow[d, "\otimes_{\mathcal{K}}"'] \arrow[dr, phantom, "\overset{\chi}{\swarrow}"] & \mathcal{L}\times\mathcal{L} \arrow[d, "\otimes_{\mathcal{L}}"] \\
\mathcal{K} \arrow[r, "\mathscr{F}"'] & \mathcal{L}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I} \arrow[d, "I_{\mathcal{K}}"'] \arrow[dr, bend left, "I_{\mathcal{L}}"] \arrow[dr, phantom, "\overset{\iota}{\swarrow}"] & \\
\mathcal{K} \arrow[r, "\mathscr{F}"'] & \mathcal{L}
\end{tikzcd}
$$

with components $\chi_{a,b}\colon \mathscr{F}a \otimes \mathscr{F}b \to \mathscr{F}(a \otimes b)$, $\iota\colon I_{\mathcal{L}} \to \mathscr{F}I_{\mathcal{K}}$, and invertible modifications subject to coherence conditions, see [GPS95]. A *weak monoidal 2-functor* is such that the underlying homomorphism is really a 2-functor. Moreover, a *monoidal* *pseudonatural transformation* $\alpha\colon \mathscr{F} \Rightarrow \mathscr{G}$ between two weak monoidal pseudofunctors is a pseudonatural transformation, equipped with two invertible modifications satisfying coherence axioms.

Along with the appropriate notion of monoidal modifications, for any monoidal 2-categories $\mathcal{K}, \mathcal{L}$ there are 2-categories $\mathbf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})$ denoted by $\mathbf{WMonHom}(\mathcal{K}, \mathcal{L})$ in [DS97] for bicategories. If we take weak monoidal 2-functors and monoidal 2-transformations, the corresponding sub-2-category is denoted by $\mathbf{Mon2Cat}(\mathcal{K}, \mathcal{L})$.

Remark 2.7. Any pseudomonoid A in $\mathcal{K}$ can be expressed as a weak monoidal *normal* pseudofunctor $A\colon \mathbf{1} \to \mathcal{K}$, namely $A(1_*) = 1_A$. Since every pseudofunctor is equivalent to a normal one, the 2-category of pseudomonoids $\mathsf{PsMon}(\mathcal{K})$ can be precisely viewed as $\mathbf{Mon2Cat}_{(\mathrm{ps})}(\mathbf{1}, \mathcal{K})$, the 2-category of weak monoidal (normal) pseudofunctors $\mathbf{1} \to \mathcal{K}$, monoidal pseudonatural transformations and monoidal modifications.

As was shown in [DS97, Prop. 5] for bicategories and pseudofunctors, a weak monoidal 2-functor $\mathscr{F}\colon \mathcal{K} \to \mathcal{L}$ takes pseudomonoids to pseudomonoids; in fact [McC00], there is a functor $\mathsf{PsMon}(\mathscr{F})$ that commutes with the respective forgetful functors

$$\begin{tikzcd}
\mathsf{PsMon}(\mathcal{K}) \arrow[r, "\mathsf{PsMon}(\mathscr{F})"] \arrow[d] & \mathsf{PsMon}(\mathcal{L}) \arrow[d] \\
\mathcal{K} \arrow[r, "\mathscr{F}"'] & \mathcal{L}.
\end{tikzcd}$$

Moreover, any monoidal 2-transformation $\alpha\colon \mathscr{F} \Rightarrow \mathscr{G}$ can be restricted to a 2-natural transformation $\mathsf{PsMon}(\alpha)\colon \mathsf{PsMon}(\mathscr{F}) \Rightarrow \mathsf{PsMon}(\mathscr{G})$. It is not hard to see that these define a 2-functor, also expressed as the hom-2-functor $\mathsf{PsMon}(-) \simeq \mathbf{Mon2Cat}_{(\mathrm{ps})}(\mathbf{1}, -)$.

Proposition 2.8. *There is a 2-functor* $\mathsf{PsMon}\colon \mathbf{Mon2Cat} \to \mathbf{2}\text{-}\mathbf{Cat}$ *which maps a monoidal 2-category to its 2-category of pseudomonoids.*

The theory in [DS97, McC00] extends the above definitions to the case of *braided* and *symmetric* pseudomonoids in *braided* and *symmetric monoidal 2-categories*. Recall that a *braiding* for $(\mathcal{K}, \otimes, I)$ is a pseudonatural equivalence with components $\sigma\colon A \otimes B \to B \to A$ and invertible modifications, whereas a *syllepsis* is an invertible modification $A \otimes B \xrightarrow{\mathrm{id}} A \otimes B \Rightarrow A \otimes B \xrightarrow{\sigma} B \otimes A \xrightarrow{\sigma} A \otimes B$, which is called *symmetry* if it satisfies extra axioms. With the appropriate notions of *braided* and *sylleptic* weak monoidal pseudofunctors and monoidal pseudonatural transformations (and usual monoidal modifications), we have 3-categories $\mathbf{BrMon2Cat}_{\mathrm{ps}}$ and $\mathbf{SymMon2Cat}_{\mathrm{ps}}$. As earlier, there exist 2-categories of braided and symmetric pseudomonoids with strong morphisms between them, expressed as $\mathsf{BrPsMon}(\mathcal{K}) = \mathbf{BrMon2Cat}_{(\mathrm{ps})}(\mathbf{1}, \mathcal{K})$ and $\mathsf{SymPsMon}(\mathcal{K}) = \mathbf{SymMon2Cat}_{(\mathrm{ps})}(\mathbf{1}, \mathcal{K})$.

Proposition 2.9. *There are 2-functors*

$$\mathsf{BrPsMon}\colon \mathbf{BrMon2Cat} \to \mathbf{2}\text{-}\mathbf{Cat}, \quad \mathsf{SymPsMon}\colon \mathbf{SymMon2Cat} \to \mathbf{2}\text{-}\mathbf{Cat}$$

which map a braided or symmetric monoidal 2-category to its 2-category of braided or symmetric pseudomonoids.

Finally, recall the notion of a monoidal 2-equivalence arising as the equivalence internal to the 2-category $\mathbf{Mon2Cat}$.

Definition 2.10. A *monoidal 2-equivalence* is a 2-equivalence $\mathscr{F}\colon \mathcal{K} \simeq \mathcal{L} : \mathscr{G}$ where both 2-functors are weak monoidal, and the 2-natural isomorphisms $1_{\mathcal{K}} \cong \mathscr{F}\mathscr{G}$, $\mathscr{G}\mathscr{F} \cong 1_{\mathcal{L}}$ are monoidal. Similarly for *braided* and *symmetric* monoidal 2-equivalences.

As is the case for any 2-functor between 2-categories, PsMon as well as $\mathsf{BrPsMon}$ and $\mathsf{SymPsMon}$ map equivalences to equivalences.

Proposition 2.11. *Any monoidal 2-equivalence $\mathcal{K} \simeq \mathcal{L}$ induces a 2-equivalence between the respective 2-categories of pseudomonoids $\mathsf{PsMon}(\mathcal{K}) \simeq \mathsf{PsMon}(\mathcal{L})$. Similarly any braided or symmetric monoidal 2-equivalence induces $\mathsf{BrPsMon}(\mathcal{K}) \simeq \mathsf{BrPsMon}(\mathcal{L})$ or $\mathsf{SymPsMon}(\mathcal{K}) \simeq \mathsf{SymPsMon}(\mathcal{L})$.*

3. Monoidal Grothendieck Construction

In this section, we give explicit descriptions of the 2-categories of pseudomonoids in the cartesian monoidal 2-categories **Fib** and **ICat**, and we exhibit their equivalence indeuced by the *monoidal* Grothendieck construction. We also consider the fixed-base case, namely pseudomononoids in $\mathbf{Fib}(\mathbb{X})$ and $\mathbf{ICat}(\mathbb{X})$ and the corresponding equivalences that arise. These two cases are in general distinct, and can be summed up

$$\begin{tikzcd}[column sep=small]
 & \mathbf{Fib} \simeq \mathbf{ICat} \arrow[dl] \arrow[dr] & \\
\mathbf{MonFib} \simeq \mathbf{MonICat} \arrow[d] & & \mathbf{Fib}(\mathbb{X}) \simeq [\mathbb{X}^{\mathrm{op}}, \mathbf{Cat}]_{\mathrm{ps}} \arrow[d] \\
\mathbf{MonFib}(\mathbb{X}) \simeq \mathbf{MonICat}(\mathbb{X}) & & \mathsf{PsMon}(\mathbf{Fib}(\mathbb{X})) \simeq [\mathbb{X}^{\mathrm{op}}, \mathbf{MonCat}]_{\mathrm{ps}}
\end{tikzcd} \tag{13}$$

These coincide only under certain circumstances, discussed in detail in Section 4.

3.1. Monoidal Fibrations.

The 2-categories of fibrations and opfibrations over arbitrary bases **Fib** and **OpFib**, recalled in Section 2.1, have a natural monoidal structure inherited from $\mathbf{Cat}^{\mathbf{2}}$. For two (op)fibrations P and Q, their cartesian (2-)product

$$P \times Q \colon \mathcal{A} \times \mathcal{B} \to \mathbb{X} \times \mathbb{Y} \tag{14}$$

is also an (op)fibration, where a (co)cartesian lifting is a pair of a P-lifting and a Q-lifting. The monoidal unit is the trivial fibration $1_{\mathbf{1}} \colon \mathbf{1} \to \mathbf{1}$. Since the monoidal structure is cartesian, they are both symmetric monoidal 2-categories.

A pseudomonoid in $(\mathbf{Fib}, \times, 1)$ is called a *monoidal fibration*, a notion first introduced in [Shu08]; a detailed argument of how it captures the required structure can be found in [Vas18].

Definition 3.1. A *monoidal fibration* $T\colon \mathcal{V} \to \mathbb{W}$ is a fibration for which both the total and the base category $\mathcal{V}$ and $\mathbb{W}$ are monoidal categories, T is a strict monoidal functor, and the the tensor product $\otimes_{\mathcal{V}}$ of $\mathcal{V}$ preserves cartesian liftings.

Explicitly, the multiplication and unit 1-cells of the pseudomonoid structure are fibred 1-cells $m = (\otimes_{\mathcal{V}}, \otimes_{\mathbb{W}})\colon T \times T \to T$ and $j = (I_{\mathcal{V}}, I_{\mathbb{W}})\colon 1 \to T$ displayed as

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{V} \arrow[r, "\otimes_{\mathcal{V}}"] \arrow[d, "T\times T"'] & \mathcal{V} \arrow[d, "T"] \\
\mathbb{W}\times\mathbb{W} \arrow[r, "\otimes_{\mathbb{W}}"'] & \mathbb{W}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathcal{I} \arrow[r, "I_{\mathcal{V}}"] \arrow[d, "1"'] & \mathcal{V} \arrow[d, "T"] \\
\mathcal{I} \arrow[r, "I_{\mathbb{W}}"'] & \mathbb{W}
\end{tikzcd} \tag{15}$$

For two monoidal fibrations, $T\colon \mathcal{V} \to \mathbb{W}$ and $S\colon \mathcal{U} \to \mathbb{Z}$, we can describe a lax morphism of pseudomonoids, Definition 2.4, between them. It amounts to a fibred 1-cell, i.e. a commutative

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "H"] \arrow[d, "T"'] & \mathcal{U} \arrow[d, "S"] \\
\mathbb{W} \arrow[r, "F"'] & \mathbb{Z}
\end{tikzcd} \tag{16}$$

where H is cartesian, equipped with two 2-cells (11) in **Fib**, i.e. fibred 2-cells

$$\begin{tikzcd}
& \mathcal{V} \arrow[dr, "\otimes_{\mathcal{U}}"] \arrow[dd, phantom, "\Downarrow\phi"] & \\
\mathcal{V}\times\mathcal{V} \arrow[ur, "H\times H"] \arrow[dr, "\otimes_{\mathcal{V}}"'] \arrow[ddd, "T\times T"'] & & \mathcal{U} \arrow[ddd, "S"] \\
& \mathcal{U}\times\mathcal{U} \arrow[ur, "H"'] & \\
& \mathbb{W} \arrow[dr, "\otimes_{\mathbb{Z}}"] \arrow[dd, phantom, "\Downarrow\psi"] & \\
\mathbb{W}\times\mathbb{W} \arrow[ur, "F\times F"] \arrow[dr, "\otimes_{\mathbb{W}}"'] & & \mathbb{Z} \\
& \mathbb{Z}\times\mathbb{Z} \arrow[ur, "F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I} \arrow[rr, bend left, "I_{\mathcal{U}}"] \arrow[dr, "I_{\mathcal{V}}"'] \arrow[dd, equal, "\mathrm{id}"'] & & \mathcal{U} \arrow[dd, "S"] \\
& \mathcal{V} \arrow[ur, "H"'] \arrow[u, phantom, "\Downarrow\phi_0"] & \\
\mathcal{I} \arrow[rr, bend left, "I_{\mathbb{Z}}"] \arrow[dr, "I_{\mathbb{W}}"'] & & \mathbb{Z} \\
& \mathbb{W} \arrow[ur, "F"'] \arrow[u, phantom, "\Downarrow\psi_0"] &
\end{tikzcd}$$

satisfying certain axioms. Explicitly from (6), ϕ and ψ are natural transformations with components

$$\phi_{x,y}\colon Hx \otimes Hy \to H(x \otimes y), \quad \psi_{u,v}\colon Fu \otimes Fv \to F(u \otimes v)$$

such that ϕ is above ψ, i.e. the following commutes

$$\begin{tikzcd}
FTx \otimes FTy \arrow[r, "\psi_{Tx,Ty}"] \arrow[d, equal, "(15)"'] & F(Tx \otimes Ty) \arrow[d, equal, "(16)"] \\
SHx \otimes SHy \arrow[d, equal, "(16)"'] & FT(x \otimes y) \arrow[d, equal, "(15)"] \\
S(Hx \otimes Hy) \arrow[r, "S\phi_{x,y}"'] & SH(x \otimes y)
\end{tikzcd}$$

Similarly, ϕ_0 and ψ_0 have a single component, namely

$$\phi_0\colon I_{\mathcal{U}} \to H(I_{\mathcal{V}}), \qquad \psi_0\colon I_{\mathbb{Z}} \to F(I_{\mathbb{W}})$$

such that $S(\phi_0) = \psi_0$. These two conditions in fact say that the identity transformation (16) is a monoidal one, as is expressed in [Shu08, 12.5]. The axioms dictate that these maps precisely give F and G the structure of lax monoidal functors.

Definition 3.2. A *lax monoidal fibred 1-cell* between two monoidal fibrations T and S is a fibred 1-cell (H, F) as in (16), where both functors have lax monoidal structures (H, ϕ, ϕ_0) and (F, ψ, ψ_0) such that $S(\phi_{x,y}) = \psi_{Tx,Ty}$ and $S\phi_0 = \psi_0$.

We now describe a 2-cell, Definition 2.5, between two lax morphisms (H, F) and (K, G) as in (16) between pseudomonoids T, S in **Fib**. It is a fibred 2-cell (α, β) (6) of the form

$$\begin{tikzcd} \mathcal{V} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "T"] & \mathcal{U} \arrow[d, "S"] \\ \mathbb{W} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbb{Z} \end{tikzcd}$$

where α, β are natural transformations with components satisfying

$$\begin{tikzcd} SHx \arrow[r, "S\alpha_x"] \arrow[d, equal, "(16)"'] & SKx \arrow[d, equal] \\ FTx \arrow[r, "\beta_{Tx}"] & GTy. \end{tikzcd}$$

Then the required axioms (12) come down to the fact that both α and β are monoidal natural transformations between the respective lax monoidal functors.

Definition 3.3. A *monoidal fibred 2-cell* between two monoidal fibred 1-cells is an ordinary fibred 2-cell (α, β) where both natural transformations are monoidal.

We denote $\mathsf{PsMon}(\mathbf{Fib})_{\text{lax}} = \mathbf{MonFib}_{\text{lax}}$, the 2-category of monoidal fibrations, lax monoidal fibred 1-cells and monoidal fibred 2-cells. Clearly, we have appropriate definitions of $\mathbf{MonFib}_{\text{opl}}$ and **MonFib**, by changing the choice of morphisms between pseudomonoids. Moreover, the 2-categories of respectively braided and symmetric pseudomonoids are denoted by **BrMonFib** and **SymMonFib**.

Dually, we have 2-categories of monoidal opfibrations **MonOpFib**, **BrMonOpFib** and **SymMonFib** of pseudomonoids in $(\mathbf{OpFib}, \times, 1_{\mathbf{1}})$. All the structures are constructed dually, where a monoidal opfibration is a strict monoidal functor such that the tensor product of the total category preserves cocartesian liftings. Moreover, we will write $\mathbf{MonFib}(\mathbb{W})$ and $\mathbf{MonOpFib}(\mathbb{W})$ for the full sub-2-categories of monoidal (op)fibrations over a fixed monoidal base $\mathbb{W}$, strong monoidal (op)fibred 1-cells of the form $(H, 1_{\mathbb{W}})$ (4) and monoidal (op)fibred 2-cells of the form $(\alpha, 1_{1_{\mathbb{W}}})$ (6).

Finally, and in a way separately from the above development, we consider pseudomonoids inside the respective 2-categories of (op)fibrations over a fixed base, $\mathbf{Fib}(\mathbb{X})$ and $\mathbf{OpFib}(\mathbb{X})$. These are both also cartesian monoidal 2-categories, but in a different manner due to the cartesian monoidal structure of $\mathbf{Cat}/\mathbb{X}$. Explicitly, for fibrations $P \colon \mathcal{A} \to \mathbb{X}$ and $Q \colon \mathcal{B} \to \mathbb{X}$, their tensor product $P \otimes Q$ is given by any of the two equal functors to $\mathcal{X}$ from the pullback

$$\begin{tikzcd} \mathcal{A} \times_{\mathbb{X}} \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, dashed, "P\otimes Q"] & \mathcal{A} \arrow[d, "P"] \\ \mathcal{B} \arrow[r, "Q"'] & \mathbb{X} \end{tikzcd} \tag{17}$$

since pulling back along a fibration always gives a fibration. The monoidal unit is $1_{\mathbb{X}}\colon \mathbb{X}\to\mathbb{X}$.

A pseudomonoid in $\mathbf{Fib}(\mathbb{X})$ is a fibration P along with two fibred 1-cells $P\otimes P\to P$ and $1_X\to P$ displayed as

$$\begin{tikzcd} \mathcal{A}\times_{\mathbb{X}}\mathcal{A} \arrow[rr, "m"] \arrow[dr, "P\otimes P"'] & & \mathcal{A} \arrow[dl, "P"] \\ & \mathbb{X} & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{X} \arrow[rr, "j"] \arrow[dr, "1_X"'] & & \mathcal{A} \arrow[dl, "P"] \\ & \mathbb{X} & \end{tikzcd}$$

along with invertible fibred 2-cells satisfying the usual axioms. Explicitly, the pullback $\mathcal{A}\times_{\mathbb{X}}\mathcal{A}$ is the subcategory of the product $\mathcal{A}\times\mathcal{A}$ of objects which are pairs of objects of $\mathcal{A}$ which are in the same fibre of P. The map m sends such a pair to their underlying object defining their fibre, whereas the map j sends an object $x\in\mathbb{X}$ to a chosen one in its fibre. As is also clear in the light of the later Theorem 3.16, these coherence data make each fibre $\mathcal{A}_X$ into a monoidal category, and the reindexing functors into strong monoidal functors. Now a strong morphism between two such fibrations is a fibred 1-cell (4) such that the induced functors $H_X\colon\mathcal{A}_X\to\mathcal{B}_X$ between the fibres as in (5) are strong monoidal, whereas a 2-cell between them is a fibred natural transformation $H\Rightarrow K$ (7) which is monoidal when restricted to the fibres, $H_X\Rightarrow K_X$.

Therefore we obtain the 2-category $\mathsf{PsMon}(\mathbf{Fib}(\mathbb{X}))$ and dually $\mathsf{PsMon}(\mathbf{OpFib}(\mathbb{X}))$.

Remark 3.4. As is evident from the above descriptions, the 2-categories $\mathbf{MonFib}(\mathbb{X})$ and $\mathsf{PsMon}(\mathbf{Fib}(\mathbb{X}))$, as well as their opfibrations counterparts, are completely different. A monoidal fibration over $\mathbb{X}$ is a strict monoidal functor where the total tensor product is cartesian, whereas a pseudomonoid in fixed-base fibrations is a fibration with monoidal fibres in a coherent way: none of the base or the total category need to be monoidal. Formally, this says that starting with $\mathbf{Fib}$, taking its pseudomonoids and then restricting them over some base is *not* the same as first restricting the base and then taking pseudomonoids. This explains the distinct fibration parts of (13).

3.2. **Monoidal Indexed Categories.** We can use the cartesian monoidal structure of $\mathbf{Cat}$ to define a monoidal structure on the 2-category of indexed categories $\mathbf{ICat}$, recalled in Section 2.2. Explicitly, given two indexed categories $\mathscr{F}\colon\mathbb{X}^{\mathrm{op}}\to\mathbf{Cat}$ and $\mathscr{G}\colon\mathbb{Y}^{\mathrm{op}}\to\mathbf{Cat}$, define their product $\mathscr{F}\otimes\mathscr{G}\colon(\mathbb{X}\times\mathbb{Y})^{\mathrm{op}}\to\mathbf{Cat}$ to be the composite

$$(\mathbb{X}\times\mathbb{Y})^{\mathrm{op}}\simeq\mathbb{X}^{\mathrm{op}}\times\mathbb{Y}^{\mathrm{op}}\xrightarrow{\mathscr{F}\times\mathscr{G}}\mathbf{Cat}\times\mathbf{Cat}\xrightarrow{\times}\mathbf{Cat} \tag{18}$$

i.e. $(\mathscr{F}\otimes\mathscr{G})(x,y)=\mathscr{F}(x)\times\mathscr{G}(y)$. Similarly for indexed 1-cells and 2-cells, e.g. for $\tau\colon\mathscr{F}\Rightarrow\mathscr{F}'F^{\mathrm{op}}\colon\mathbb{X}^{\mathrm{op}}\to\mathbf{Cat}$ and $\sigma\colon\mathscr{G}\Rightarrow\mathscr{G}'G^{\mathrm{op}}\colon\mathbb{Z}^{\mathrm{op}}\to\mathbf{Cat}$ indexed 1-cells, their tensor product has components

$$(\tau\otimes\sigma)_{x,z}\colon\mathscr{F}(x)\times\mathscr{G}(z)\xrightarrow{\tau_x\times\sigma_z}\mathscr{F}'(Fx)\times\mathscr{G}'(Gz)$$

and is a pseudonatural transformation. The monoidal unit indexed category is just $\Delta\mathbf{1}\colon\mathbf{1}^{\mathrm{op}}\to\mathbf{Cat}$ that picks out the terminal category $\mathbf{1}$ in $\mathbf{Cat}$. Notice that this is a cartesian monoidal 2-structure. In a very similary way, the 2-category $\mathbf{OpICat}$ also has a cartesian monoidal structure.

A pseudomonoid in $(\mathbf{ICat},\otimes,\Delta\mathbf{1})$ is an indexed category $\mathscr{M}\colon\mathbb{W}^{\mathrm{op}}\to\mathbf{Cat}$ with an indexed 1-cells $(\otimes_{\mathbb{W}},\mu)\colon\mathscr{M}\otimes\mathscr{M}\to\mathscr{M}$, $(\eta,\mu_0)\colon\Delta\mathbf{1}\to\mathscr{M}$ where $\otimes_{\mathbb{W}}\colon\mathbb{W}\times\mathbb{W}\to\mathbb{W}$

and $\eta\colon \mathbf{1}\to\mathbb{W}$ are functors, and $\mu\colon \mathscr{M}\otimes\mathscr{M}\Rightarrow\mathscr{M}\circ\otimes^{\mathrm{op}}_{\mathbb{W}}$ and $\mu_0\colon \Delta\mathbf{1}\Rightarrow\mathscr{M}$ are pseudonatural transformations

$$
\begin{tikzcd}
\mathbb{W}^{\mathrm{op}}\times\mathbb{W}^{\mathrm{op}} \arrow[d, "\otimes^{\mathrm{op}}_{\mathbb{W}}"'] \arrow[dr, bend left, "\mathscr{M}\otimes\mathscr{M}"] \arrow[dr, phantom, "\Downarrow\mu"] & \\
\mathbb{W}^{\mathrm{op}} \arrow[r, "\mathscr{M}"] & \mathbf{Cat} \\
\mathbf{1}^{\mathrm{op}} \arrow[u, "\eta^{\mathrm{op}}"] \arrow[ur, bend right, "\Delta\mathbf{1}"'] \arrow[ur, phantom, "\Uparrow\mu_0"] &
\end{tikzcd}
$$

with invertible indexed 2-cells

$$
\begin{tikzcd}
\mathscr{M}\otimes\mathscr{M}\otimes\mathscr{M} \arrow[r, "1_{\mathscr{M}}\otimes(\otimes_{\mathbb{W}},\mu)"] \arrow[d, "(M,\mu)\otimes 1_{\mathscr{M}}"'] \arrow[dr, phantom, "\simeq"] & \mathscr{M}\otimes\mathscr{M} \arrow[d, "(\otimes_{\mathbb{W}},\mu)"] \\
\mathscr{M}\otimes\mathscr{M} \arrow[r, "(M,\mu)"'] & \mathscr{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M}\otimes I \arrow[r, "1_{\mathscr{M}}\otimes(\eta,\mu_0)"] \arrow[dr, "\simeq"'] & \mathscr{M}\otimes\mathscr{M} \arrow[d, "(\otimes_{\mathbb{W}},\mu)"] \arrow[dl, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & I\otimes\mathscr{M} \arrow[l, "(\eta,\mu_0)\otimes 1_{\mathscr{M}}"'] \arrow[dl, "\simeq"] \\
& \mathscr{M} &
\end{tikzcd}
$$

The axioms render $\mathbb{W}$ a monoidal category with $\otimes_{\mathbb{W}}\colon \mathbb{W}\times\mathbb{W}\to\mathbb{W}$ its tensor product functor, and the components $\mu_{x,y}\colon \mathscr{M}(x)\times\mathscr{M}(y)\to\mathscr{M}(x,y)$ satisfy axioms giving $(\mathscr{M},\mu)$ the structure of a weak monoidal pseudofunctor, recalled in Section 2.4. We call this a *monoidal indexed category.*

Definition 3.5. A *monoidal indexed category* is a weak monoidal pseudofunctor $(\mathscr{M},\mu,\mu_0)\colon(\mathbb{W}^{\mathrm{op}},\otimes^{\mathrm{op}},I)\to(\mathbf{Cat},\times,\mathbf{1})$, where $(\mathbb{W},\otimes,I)$ is a monoidal category thought of as a monoidal 2-category with trivial 2-cells. When the context makes it clear, we write such a monoidal indexed category in the abbreviated form $\mathscr{M}\colon\mathbb{W}^{\mathrm{op}}\to\mathbf{Cat}$.

For two monoidal indexed categories

$$
\begin{gathered}
(\mathscr{M},\phi,\phi_0)\colon(\mathbb{W},\otimes_{\mathbb{W}},I_{\mathbb{W}})\to(\mathbf{Cat},\times,\mathbf{1})\\
(\mathscr{H},\psi,\psi_0)\colon(\mathbb{X},\otimes_{\mathbb{X}},I_{\mathbb{X}})\to(\mathbf{Cat},\times,\mathbf{1})
\end{gathered}
$$

a lax morphism between them, Definition 2.4, is an indexed 1-cell $(F,\tau)\colon\mathscr{M}\to\mathscr{H}$ with indexed 2-cells

$$
\begin{tikzcd}
\mathbb{W}^{\mathrm{op}}\times\mathbb{W}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}\times F^{\mathrm{op}}"] \arrow[dd, "\otimes^{\mathrm{op}}_{\mathbb{W}}"'] \arrow[dr] & \arrow[d, phantom, "\overset{\tau\times\tau}{\Longrightarrow}"] & \mathbb{X}^{\mathrm{op}}\times\mathbb{X}^{\mathrm{op}} \arrow[dd, "\otimes^{\mathrm{op}}_{\mathbb{X}}"] \arrow[dl] \\
\arrow[r, phantom, "\mu\Downarrow"] & \mathbf{Cat} \arrow[r, phantom, "\Downarrow\eta"] \arrow[d, phantom, "\overset{\tau}{\Longrightarrow}"] & {} \\
\mathbb{W}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"'] \arrow[ur] & {} & \mathbb{X}^{\mathrm{op}} \arrow[ul]
\end{tikzcd}
\overset{m}{\Rrightarrow}
\begin{tikzcd}
\mathbb{W}^{\mathrm{op}}\times\mathbb{W}^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}\times F^{\mathrm{op}}"] \arrow[d, "\otimes^{\mathrm{op}}_{\mathbb{W}}"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathbb{X}^{\mathrm{op}}\times\mathbb{X}^{\mathrm{op}} \arrow[d, "\otimes_{\mathbb{X}}"] \\
\mathbb{W}^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}"'] & \mathbb{X}^{\mathrm{op}}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& \mathbf{1}^{\mathrm{op}} \arrow[dl, bend right, "\eta^{\mathrm{op}}_{\mathbb{W}}"'] \arrow[d, "\Delta\mathbf{1}"] \arrow[dr, bend left, "\eta^{\mathrm{op}}_{\mathbb{X}}"] & \\
\arrow[r, phantom, "\Downarrow\phi_0"] & \mathbf{Cat} \arrow[r, phantom, "\Downarrow\psi_0"] & {} \\
\mathbb{W}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"'] \arrow[ur, "\mathscr{M}"] & \arrow[u, phantom, "\overset{\tau}{\Longrightarrow}"] & \mathbb{X}^{\mathrm{op}} \arrow[ul, "\mathscr{H}"']
\end{tikzcd}
\overset{m_0}{\Rrightarrow}
\begin{tikzcd}
& \mathbf{1}^{\mathrm{op}} \arrow[dl, bend right, "\eta^{\mathrm{op}}_{\mathbb{W}}"'] \arrow[dr, bend left, "\eta^{\mathrm{op}}_{\mathbb{X}}"] \arrow[d, phantom, "\Downarrow\beta_0"] & \\
\mathbb{W}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"'] & {} & \mathbb{X}^{\mathrm{op}}
\end{tikzcd}
$$

The pseudonatural transformations μ and η are part of the structure associated to $\mathscr{M}$ and $\mathscr{H}$ making them weak monoidal pseudofunctors $\mathbb{W} \to \mathbf{Cat}$. The natural transformation ϕ is the laxator that makes $F\colon \mathbb{W} \to \mathbb{X}$ into a lax monoidal functor.

Definition 3.6. A *lax monoidal indexed 1-cell* between two monoidal indexed categories $\mathscr{M}$ and $\mathscr{H}$ is an indexed 1-cell (F, τ), where the functor F is lax monoidal, and the pseudonatural transformation τ is monoidal.

Similarly, describing a 2-cell between (lax) morphisms of pseudomonoids Definition 2.5 produces the definition below.

Definition 3.7. A *monoidal indexed 2-cell* between two monoidal indexed 1-cells (F, τ) and (G, σ) is an indexed 2-cell (β, m) such that m is a monoidal modification and β is a monoidal pseudonatural transformation.

We denote $\mathsf{PsMon}(\mathbf{ICat})_{\mathrm{lax}} = \mathbf{MonICat}_{\mathrm{lax}}$ and also $\mathsf{PsMon}(\mathbf{ICat}) = \mathbf{MonICat}$ when the morphisms are strong. Moreover, the 2-categories of braided and symmetric pseudomonoids are denote by **BrMonICat** and **SymMonICat** respectively. Similarly, we have 2-categories of monoidal opindexed categories **MonOpICat**, braided monoidal opindexed categories **BrMonOpICat**, and symmetric monoidal opindexed categories **SymMonOpICat**.

Similarly to the fixed-base case for fibrations,we now consider pseudomonoids in a different monoidal 2-category, namely indexed categories with a fixed domain $\mathbb{X}$. Explicitly, we can describe products in $\mathbf{ICat}(\mathbb{X}) = \mathbf{2\text{-}Cat}(\mathbb{X}^{\mathrm{op}}, \mathbf{Cat})_{ps}$ by modifying the product in **ICat**, using the diagonal functor. For two $\mathbb{X}$-indexed categories $\mathscr{F}, \mathscr{G}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$, their monoidal product is

$$\mathbb{X}^{\mathrm{op}} \xrightarrow{\Delta} \mathbb{X}^{\mathrm{op}} \times \mathbb{X}^{\mathrm{op}} \xrightarrow{\mathscr{F}\times\mathscr{G}} \mathbf{Cat} \times \mathbf{Cat} \xrightarrow{\times} \mathbf{Cat} \tag{19}$$

with 'pointwise' components $(\mathscr{F} \otimes_{\mathbb{X}} \mathscr{G})(x) = \mathscr{F}(x) \times \mathscr{G}(x)$, and the monoidal unit is just $\mathbb{X}^{\mathrm{op}} \xrightarrow{!} \mathbf{1} \xrightarrow{\Delta\mathbf{1}} \mathbf{Cat}$, which we will also call $\Delta\mathbf{1}$.

A pseudomonoid inside the cartesian monoidal 2-category $\mathbf{ICat}(\mathbb{X})$ is some $\mathscr{F}: \mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ equipped with indexed functors, i.e. pseudonatural transformations

$$\begin{tikzcd} \mathbb{X}^{\mathrm{op}} \arrow[r, bend left, "\mathscr{F}\otimes\mathscr{F}"] \arrow[r, bend right, "\mathscr{F}"'] \arrow[r, phantom, "\Downarrow m"] & \mathbf{Cat} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{X}^{\mathrm{op}} \arrow[r, bend left, "\Delta\mathbf{1}"] \arrow[r, bend right, "\mathscr{F}"'] \arrow[r, phantom, "\Downarrow j"] & \mathbf{Cat} \end{tikzcd} \tag{20}$$

whose components $m_x\colon \mathscr{F}x \times \mathscr{F}x \to \mathscr{F}x$ and $j_x\colon \mathbf{1} \to \mathscr{F}x$ along with the invertible modifications make each $\mathscr{F}x$ into a monoidal category. Notably, this has been called an *indexed monoidal category* already in [HM06].

Regarding the 1-cells between such objects, given two pseudomonoids in $\mathbf{ICat}(\mathbb{X})$, $(\mathscr{F}, m_1, j_1)$ and $(\mathscr{G}, m_2, j_2)$, a lax morphism between them is an indexed functor $f\colon \mathscr{F} \to \mathscr{G}$ equipped with indexed natural transformations which have components

$$\begin{tikzcd} \mathscr{F}x \times \mathscr{F}x \arrow[rr, "f_x \times f_x"] \arrow[d, "(m_1)_x"'] & \arrow[d, phantom, "\Downarrow \phi_x"] & \mathscr{G}x \otimes \mathscr{G}x \arrow[d, "(m_2)_x"] \\ \mathscr{F}x \arrow[rr, "f_x"] & \arrow[d, phantom, "\Uparrow (\phi_0)_x"] & \mathscr{G}x \\ & \mathbf{1} \arrow[ul, "(j_1)_x"] \arrow[ur, "(j_2)_x"'] & \end{tikzcd}$$

making each component of f a lax monoidal functor. For a strong morphism between pseudomonoids, each f ends up being a (strong) monoidal functor.

Definition 3.8. An *indexed (lax) monoidal functor* between two $\mathbb{X}$-indexed monoidal categories is an indexed functor $f\colon \mathscr{F} \to \mathscr{G}$ such that each component $f_x\colon \mathscr{F}x \to \mathscr{G}x$ is a (lax) monoidal functor.

Similarly, a 2-cell between lax morphisms of pseudomonoids has the following description.

Definition 3.9. An *indexed monoidal natural transformation* is an indexed natural transformation such that each component is a monoidal natural transformation.

From the above definitions, it is clear that $\mathsf{PsMon}(\mathbf{ICat}(\mathbb{X})) = \mathbf{2\text{-}Cat}(\mathbb{X}^{\mathrm{op}}, \mathbf{MonCat})_{\mathrm{ps}}$. More on such considerations can be found in Section 4.

Remark 3.10. Similar to what was noted in Remark 3.4, from the above descriptions, the 2-categories $\mathbf{MonICat}(\mathbb{X})$ and $\mathsf{PsMon}(\mathbf{ICat}(\mathbb{X}))$ are completely different. A monoidal indexed category with base $\mathbb{X}$ is a weak monoidal pseudofunctor into $\mathbf{Cat}$, whereas a pseudomonoid in $\mathbf{ICat}(\mathbb{X})$ is a pseudofunctor into $\mathbf{MonCat}$. Formally, this says that starting with $\mathbf{ICat}$, taking its pseudomonoids and then restricting them over some base is *not* the same as first choosing a base and then taking pseudomonoids. This is also highlighted by the indexed category part of (13).

3.3. **The equivalence MonFib $\cong$ MonICat.** In Section 2.3, we recalled the equivalence between fibrations and indexed categories. We will now lift this equivalence to their monoidal versions, using general results regarding pseudomonoids in arbitrary monoidal 2-categories described in Section 2.4

Lemma 3.11. *The 2-equivalence* $\mathbf{Fib} \simeq \mathbf{ICat}$ *between the cartesian monoidal 2-categories of fibrations and indexed categories is (symmetric) monoidal.*

Proof. Since they form an equivalence, both 2-functors from Theorem 2.3 preserve limits, therefore are monoidal 2-functors. Moreover, it is not hard to verify that the natural isomorphisms with components $\mathscr{F} \cong \mathscr{F}_{P_{\mathscr{F}}}$ and $P \cong P_{\mathscr{F}_P}$ are monoidal with respect to the cartesian structure, due to universal properties of products. □

As a result, we obtain the following equivalence as a special case of Proposition 2.11.

Theorem 3.12. *There is a 2-equivalence* $\mathbf{MonFib} \cong \mathbf{MonICat}$ *between the 2-categories of monoidal fibrations and monoidal indexed categories. Moreover, this restricts to equivalences* $\mathbf{BrMonFib} \simeq \mathbf{BrMonICat}$ *and* $\mathbf{SymMonFib} \simeq \mathbf{SymMonICat}$*. Dually we have 2-equivalences* $\mathbf{MonOpFib} \simeq \mathbf{MonOpICat}$*.*

Corollary 3.13. *The above 2-equivalences restrict to the sub-2-categories of fixed bases, namely* $\mathbf{MonFib}(\mathbb{X}) \simeq \mathbf{MonICat}(\mathbb{X})$*. Similarly, in the dual case we have* $\mathbf{MonOpFib}(\mathbb{X}) \simeq \mathbf{MonOpICat}(\mathbb{X})$*.*

Although this directly follows from abstract reasoning, we briefly describe this equivalence on the level of objects; more details can be found also in [BFMP17, §6]. Independently and much earlier, in his thesis [Shu09] Shulman explores such an equivalence on the level of double categories (of monoidal fibrations and monoidal pseudofunctors over the same base).

Suppose that $(\mathscr{M},\mu,\mu_0)\colon(\mathbb{W}^{\mathrm{op}},\otimes_{\mathbb{W}},I_{\mathbb{W}})\to(\mathbf{Cat},\times,\mathbf{1})$ is a monoidal indexed category, i.e. a weak monoidal pseudofunctor. The Grothendieck category $\int\mathscr{M}$ constructed as in Section 2.3 obtains a monoidal structure in the following way. Its tensor product $\otimes_\mu\colon\int\mathscr{M}\times\int\mathscr{M}\to\int\mathscr{M}$ is defined by

$$(x,c)\otimes_\mu(x',c')=(x\otimes_{\mathbb{X}}x',\mu_{x,x'}(c,c')) \tag{21}$$

on objects and

$$(f,g)\otimes_\mu(f',g')=(f\otimes_{\mathbb{X}}f',\delta_{f,f'}\circ\mu_{x,x'}(g,g'))$$

on morphisms, where $\delta_{f,f'}\colon\mu_{x,x'}\circ(\mathscr{M}f\times\mathscr{M}f)\to\mathscr{M}(f\otimes_{\mathbb{X}}f')\circ\mu_{y,y'}$. The unit object is $(I_{\mathbb{X}},\mu_0)$.

If $(\mathbb{W},\otimes_{\mathbb{W}},I_{\mathbb{W}})$ is a braided monoidal category with braiding $b_{x,x'}\colon x\otimes x'\to x'\otimes x$, and $(\mathscr{M},\mu,\mu_0)\colon(\mathbb{X},\otimes,I)\to(\mathbf{Cat},\times,1)$ is a braided weak monoidal pseudofunctor, i.e. a braided monoidal indexed category, then we define a braiding

$$B_{(x,c),(x',c')}\colon(x,c)\otimes_\mu(x',c')\to(x',c')\otimes_\mu(x,c)$$

by $B_{(x,c),(x',c')}=(b_{x,x'},1)$. If b satisfies the identity $b_{x',x}\circ b_{x,x'}=1_{x\otimes x'}$, then so does B:

$$\begin{aligned}B_{(x',c'),(x,c)}\circ B_{(x,c),(x',c')}&=(b_{x',x},1)\circ(b_{x,x'},1)\\&=(b_{x',x}\circ b_{x,x'},1\circ\mathscr{M}b_{x',x}(1))\\&=(1_{x\otimes x'},1).\end{aligned}$$

So if $\mathscr{M}$ is weak symmetric monoidal, then $\int\mathscr{M}$ with the above structures is a symmetric monoidal category.

Remark 3.14. Based on an observation made by Mike Shulman in private correspondence with the authors, this 'monoidal version' of the Grothendieck construction may in fact be further generalized to the context of double categories. More specifically, there is evidence of a correspondence between discrete fibrations of double categories, and lax double functors into the double category $\mathbb{S}\mathbf{pan}$ of sets and spans. If such a result was also true for arbitrary fibrations of double categories, Theorem 3.12 would be a special case for double categories with one object and one vertical arrow, namely monoidal categories.

The equivalence of Theorem 3.12 restricts in a straightforward way to the split context.

Theorem 3.15. *There is an equivalence* $\mathbf{MonFib}_{sp}\simeq\mathbf{MonICat}_{sp}$ *between monoidal split fibrations and strict indexed categories, as well as an equivalence* $\mathbf{MonOpFib}_{sp}\simeq\mathbf{MonOpICat}_{sp}$ *between monoidal split opfibrations and strict op-indexed categories.*

Finally, we could also apply Proposition 2.11 directly to the cartesian monoidal 2-categories $\mathbf{Fib}(\mathbb{X})$ and $\mathbf{ICat}(\mathbb{X})$, rather than restricting the general result which induces Corollary 3.13. As already noted in Remarks 3.4 and 3.10, the categories of pseudomonoids in the fixed base 2-categories are of a very different flavor compared to monoidal fibrations and monoidal indexed categories. Since $\mathbf{Fib}(\mathbb{X})\simeq\mathbf{ICat}(\mathbb{X})$ is also a monoidal 2-equivalence, the following is true.

Theorem 3.16. *There is a 2-equivalence* $\mathsf{PsMon}(\mathbf{Fib}(\mathbb{X}))\simeq\mathsf{PsMon}(\mathbf{ICat}(\mathbb{X}))$ *between fibrations with monoidal fibres and strong monoidal reindexing functors, and pseudofunctors* $\mathbb{X}^{\mathrm{op}}\to\mathbf{MonCat}$*. Dually,* $\mathsf{PsMon}(\mathbf{OpFib}(\mathbb{X}))\simeq\mathsf{PsMon}(\mathbf{OpICat}(\mathbb{X}))$.

Theorems 3.12 and 3.16 establish the two different cases of the monoidal Grothendieck construction, depicted in (13). In the following section, we compare them in a special case where they actually coincide.

4. (Co)cartesian case: fibrewise and global monoidal structures

In the above section, we obtain two different equivalences between fixed-base fibrations and indexed categories of monoidal flavor. Corollary 3.13 established a correspondence between weak monoidal pseudofunctors $\mathscr{M}\colon \mathbb{W}^{\mathrm{op}} \to \mathbf{Cat}$ and monoidal fibrations $\int \mathscr{M} \to \mathbb{W}$, where the induced monoidal structure on the fibration is 'global': both total and base categories are monoidal, and the fibration strictly preserves the structure. On the other hand, Theorem 3.16 establishes a correspondence between ordinary pseudofunctors $\mathscr{F}\colon \mathbb{X}^{\mathrm{op}} \to \mathbf{MonCat} \hookrightarrow \mathbf{Cat}$ and ordinary fibrations $\int \mathscr{F} \to \mathbb{X}$, with a 'fibrewise' monoidal structure: none of the base or total categories are monoidal, but each fibre is, and the reindexing functors strongly preserve the structure.

Clearly, neither of these two cases implies the other in general. The 'global' monoidal structure as defined in (21) sends two objects in arbitrary fibres to a new object lying in the fibre of the tensor of their underlying objects in the base. Therefore multiplying within the same fibre $\mathcal{V}_x$ gives an object in the fibre $\mathcal{V}_{x\otimes x}$. On the other hand, having a fibrewise tensor products does not, of course, give a way of multiplying objects in different fibres of the total category.

The above two different equivalences are formally expressed as

$$\mathbf{MonFib}(\mathbb{W}) \simeq \mathbf{Mon2Cat}(\mathbb{W}^{\mathrm{op}}, \mathbf{Cat})_{\mathrm{ps}} \tag{22}$$

$$\mathsf{PsMon}(\mathbf{Fib}(\mathbb{X})) \simeq \mathbf{2\text{-}Cat}(\mathbb{X}^{\mathrm{op}}, \mathbf{MonCat})_{\mathrm{ps}} \tag{23}$$

where $\mathbb{W}$ is a monoidal category and $\mathbb{X}$ is an ordinary category, and these correspond to the two different legs of (13).

In [Shu08], Shulman introduces monoidal fibrations (Definition 3.1) as a building block to construct framed bicategories (fibrant double categories). The entire approach for these fibrations is that they are a 'parameterized family of monoidal categories': due to the nature of the examples, the results restrict to the case where the base of the monoidal fibration $T\colon \mathcal{V} \to \mathbb{W}$ is equipped with a cartesian or cocartesian monoidal structure. A central result therein lifts the Grothendieck construction to the monoidal setting, by showing an equivalence between monoidal fibrations over a fixed (co)cartesian base and ordinary pseudofunctors into $\mathbf{MonCat}$.

Theorem 4.1. [Shu08, Thm. 12.7] *If $\mathbb{X}$ is cartesian monoidal,*

$$\mathbf{MonFib}(\mathbb{X}) \simeq \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathbb{X}^{\mathrm{op}}, \mathbf{MonCat}) \tag{24}$$

Dually, if $\mathbb{X}$ is cocartesian monoidal, $\mathbf{MonOpFib}(\mathbb{X}) \simeq \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathbb{X}, \mathbf{MonCat})$.

This result evidently provides an equivalence between the left part of (22) and the right part of (23). The proof of the theorem heavily relies on the cartesianness of the base, and employs the reindexing functors Δ^* and π^* corresponding to the diagonal and projections to move between the appropriate fibres and build the required structures. The 'global' monoidal structure is called *external* and the 'fibrewise' is called *internal*.

Theorem 4.2. *If $\mathbb{X}$ is a cartesian monoidal category,*

$$\begin{tikzcd}
\mathbf{MonFib}(\mathbb{X}) \arrow[r, "\simeq"] \arrow[d, "\wr"'] & \mathbf{Mon2Cat}_{\mathrm{ps}}(\mathbb{X}^{\mathrm{op}}, \mathbf{Cat}) \arrow[d, "\wr"] \\
\mathsf{PsMon}(\mathbf{Fib}(\mathbb{X})) \arrow[r, "\simeq"] & \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathbb{X}^{\mathrm{op}}, \mathbf{MonCat})
\end{tikzcd}$$

Dually, if $\mathbb{X}$ is a cocartesian monoidal category,

$$\begin{tikzcd}
\mathbf{MonOpFib}(\mathbb{X}) \arrow[r, "\simeq"] \arrow[d, "\wr"'] & \mathbf{Mon2Cat}_{\mathrm{ps}}(\mathbb{X}, \mathbf{Cat}) \arrow[d, "\wr"] \\
\mathsf{PsMon}(\mathbf{OpFib}(\mathbb{X})) \arrow[r, "\simeq"] & \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathbb{X}, \mathbf{MonCat})
\end{tikzcd}$$

As mentioned above, this can be established directly using Theorem 4.1 and focusing more on the fibrations side. On the other hand, the equivalence between weak monoidal pseudofunctors $\mathbb{X}^{\mathrm{op}} \to \mathbf{Cat}$ and ordinary pseudofunctors $\mathbb{X}^{\mathrm{op}} \to \mathbf{MonCat}$, which essentially provides a way of transferring the monoidal structure between the target category and the functor itself, provides a new perspective on the behavior of such objects. The following lemma sets to clarify that context.

Lemma 4.3. *For any two monoidal 2-categories $\mathcal{K}$ and $\mathcal{L}$, the following are true.*

(1) For an arbitrary 2-category $\mathcal{A}$,

$$\mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathbb{A}, \mathbf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathbf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathbb{A}, \mathcal{L})) \tag{25}$$

(2) For a cocartesian 2-category $\mathcal{A}$,

$$\mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathbb{A}, \mathbf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathbf{Mon2Cat}_{\mathrm{ps}}(\mathbb{A} \times \mathcal{K}, \mathcal{L}) \tag{26}$$

Proof. First of all, recall [Str80, 1.34] that there are equivalences

$$\mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathcal{A}, \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathcal{A} \times \mathcal{K}, \mathcal{L}) \simeq \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathcal{K}, \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathcal{A}, \mathcal{L}))$$

which underlie (25) and (26) for the respective pseudofunctors. So we only need to establish the correspondence between the respective monoidal structures. Notice that $\mathcal{A} \times \mathcal{K}$ is a monoidal 2-category since both $\mathcal{A}$ and $\mathcal{K}$ are, and also $\mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathcal{A}, \mathcal{L})$ is monoidal since $\mathcal{L}$ is: define $\otimes_{[]}$ and $I_{[]}$ by $(\mathscr{F} \otimes_{[]} \mathscr{G})(a) = \mathscr{F}a \otimes_{\mathcal{L}} \mathscr{G}a$ (similarly to (19)) and $I_{[]} \colon \mathcal{A} \xrightarrow{!} \mathbf{1} \xrightarrow{I_{\mathcal{L}}} \mathcal{L}$.

Take a pseudofunctor $\mathscr{F} \colon \mathcal{A} \to \mathbf{Mon2Cat}_{\mathrm{ps}}(\mathcal{K}, \mathcal{L})$. For every $a \in \mathcal{A}$, its image pseudofunctor $\mathscr{F}a$ is weak monoidal, i.e. comes equipped with morphisms in $\mathcal{L}$

$$\phi^a_{x,y} \colon (\mathscr{F}a)(x) \otimes_{\mathcal{L}} (\mathscr{F}a)(y) \to (\mathscr{F}a)(x \otimes_{\mathcal{K}} y), \quad \phi^a_0 \colon I_{\mathcal{L}} \to (\mathscr{F}a)I_{\mathcal{K}} \tag{27}$$

for every $x, y \in \mathcal{K}$, satisfying coherence axioms.

Now define the pseudofunctor $\bar{\mathscr{F}} \colon \mathcal{K} \to \mathbf{2\text{-}Cat}_{\mathrm{ps}}(\mathcal{A}, \mathcal{L})$, with $(\bar{\mathscr{F}}x)(a) := \mathscr{F}(a)(x)$. Its weak monoidal structure is given by pseudonatural transformations

$$\bar{\mathscr{F}}x \otimes_{[]} \bar{\mathscr{F}}y \Rightarrow \bar{\mathscr{F}}(x \otimes_{\mathcal{K}} y), \quad I_{[]} \Rightarrow \bar{\mathscr{F}}(I_{\mathcal{K}})$$

whose components evaluated on some $a \in \mathcal{A}$ are defined to be (27). Pseudonaturality and weak monoidal axioms follow, and in a similar way we can establish the opposite direction and verify the equivalence.

On the other hand, if $\mathcal{A}$ is a cocartesian monoidal 2-category, a weak monoidal pseudofunctor $\mathscr{F}$ like above induces a pseudofunctor $\tilde{\mathscr{F}}\colon \mathcal{A}\times\mathcal{K}\to\mathcal{L}$ by $\tilde{\mathscr{F}}(a,x) := (\mathscr{F}a)(x)$. Its weak monoidal structure is given by the composite

$$\begin{tikzcd}
\tilde{\mathscr{F}}(a,x)\otimes_{\mathcal{L}}\tilde{\mathscr{F}}(b,y) \arrow[rr, dashed, "\psi_{(a,x),(b,y)}"] \arrow[d, equal] & & \tilde{\mathscr{F}}(a+b,x\otimes_{\mathcal{K}}y) \arrow[d, equal] \\
(\mathscr{F}a)(x)\otimes_{\mathcal{L}}(\mathscr{F}b)(y) \arrow[dr, "(\mathscr{F}\iota_a)_x\otimes(\mathscr{F}\iota_b)_y"'] & & (\mathscr{F}(a+b))(x\otimes_{\mathcal{K}}y) \\
& (\mathscr{F}(a+b))(x)\otimes_{\mathcal{L}}(\mathscr{F}(a+b))(y) \arrow[ur, "\phi^{a+b}_{x,y}"'] &
\end{tikzcd}$$

where $a\xrightarrow{\iota_a}a+b\xleftarrow{\iota_b}b$ are the inclusions, and $\psi_0\colon I_{\mathcal{L}}\xrightarrow{\phi^0_0}\tilde{\mathscr{F}}(0,I_{\mathcal{K}})$; the respective axioms follow. In the opposite direction, starting with some pseudofunctor $\mathscr{G}$ equipped with $\psi_{(a,x),(b,y)}$ and ψ_0, we can build $\hat{\mathscr{G}}$ for which every $\hat{\mathscr{G}}a$ is a weak monoidal pseudofunctor, via

$$\begin{tikzcd}
(\hat{\mathscr{G}}a)(x)\otimes_{\mathcal{L}}(\hat{\mathscr{G}}b)(y) \arrow[rr, dashed, "\phi^a_{(x,y)}"] \arrow[d, equal] & & (\hat{\mathscr{G}}a)(x\otimes_{\mathcal{K}}y) \arrow[d, equal] \\
\mathscr{G}(a,x)\otimes_{\mathcal{L}}\mathscr{G}(a,y) \arrow[r, "\psi_{(a,x),(a,y)}"'] & \mathscr{G}(a+a,x\otimes_{\mathcal{K}}y) \arrow[r, "G(\nabla,1)"'] & \mathscr{G}(a,x\otimes_{\mathcal{K}}y)
\end{tikzcd}$$

$$\phi^a_0\colon I_{\mathcal{L}}\xrightarrow{\psi_0}G(0,I_{\mathcal{K}})\xrightarrow{G(!,1)}G(a,I_{\mathcal{K}})$$

The equivalence follows, using the universal properties of coproducts and initial object. □

In a very similar way, for split indexed categories we can prove appropriate isomorphisms involving **Cat** and $\mathbf{MonCat}_{\text{lax}}$ that exhibit the existence of cotensors and tensors in the latter. For these weighted (co)limit notions, see [Kel05, §3.7]. Below we denote by $[-,-]$ the usual functor category.

Corollary 4.4. *For any two monoidal categories $\mathcal{V}$, $\mathcal{W}$, we have isomorphisms*

$$\mathbf{MonCat}_{\text{lax}}(\mathcal{A}\times\mathcal{V},\mathcal{W})\cong[\mathcal{A},\mathbf{MonCat}_{\text{lax}}(\mathcal{V},\mathcal{W})]\cong\mathbf{MonCat}_{\text{lax}}(\mathcal{V},[\mathcal{A},\mathcal{W}])$$

where the right one holds of any ordinary category $\mathcal{A}$, whereas the left one holds only for a cocartesian $\mathcal{A}$. Therefore $\mathbf{MonCat}_{\text{lax}}$ *is a cotensored 2-category with $\mathcal{A}\pitchfork\mathcal{W}=[\mathcal{A},\mathcal{W}]$ and tensored only by cocartesian categories with $\mathcal{A}\otimes\mathcal{V}=\mathcal{A}\times\mathcal{V}$.*

Using these theoretical results, we can prove Theorem 4.2 through the functor 2-categories side, rather than considering the fibrations side.

Proof of Theorem 4.2. The top and bottom right 2-categories of the first square are equivalent as follows, where $\mathbb{X}^{\text{op}}$ is cocartesian.

$$\begin{aligned}
\mathbf{2\text{-}Cat}_{\text{ps}}(\mathbb{X}^{\text{op}},\mathbf{MonCat}) &\simeq \mathbf{2\text{-}Cat}_{\text{ps}}(\mathbb{X}^{\text{op}},\mathsf{PsMon}(\mathbf{Cat})) \\
&\simeq \mathbf{2\text{-}Cat}_{\text{ps}}(\mathbb{X}^{\text{op}},\mathbf{Mon2Cat}_{\text{ps}}(\mathbf{1},\mathbf{Cat})) && (2.7) \\
&\simeq \mathbf{Mon2Cat}_{\text{ps}}(\mathbb{X}^{\text{op}}\times\mathbf{1},\mathbf{Cat}) && (26) \\
&\simeq \mathbf{Mon2Cat}_{\text{ps}}(\mathbb{X}^{\text{op}},\mathbf{Cat})
\end{aligned}$$

□

Finally, recall by Remarks 2.1 and 2.2 that the categories **Fib** and **ICat** are themselves fibred over **Cat**, with fibres $\mathbf{Fib}(\mathbb{X})$ and $\mathbf{ICat}(\mathbb{X})$ respectively. The base category in both cases is the cartesian monoidal category $(\mathbf{Cat}, \times, 1)$, therefore Theorem 4.2 applies for $\mathbb{X} = \mathbf{Cat}$. The following results shows that the connection between the monoidal structures of **Fib**, **ICat** and $\mathbf{Fib}(\mathbb{X})$, $\mathbf{ICat}(\mathbb{X})$, fundamental for inducing the global and fibrewise monoidal structures, follows exactly from the same abstract pattern.

Proposition 4.5. *The fibrations* $\mathbf{Fib} \to \mathbf{Cat}$ *and* $\mathbf{ICat} \to \mathbf{Cat}$ *are monoidal, and moreover their fibres* $\mathbf{Fib}(\mathbb{X})$ *and* $\mathbf{ICat}(\mathbb{X})$ *are monoidal and the reindexing functors are strong monoidal.*

Proof. The pseudofunctors inducing $\mathbf{Fib} \to \mathbf{Cat}$ and $\mathbf{ICat} \to \mathbf{Cat}$ are

$$\begin{tikzcd} \mathbf{Cat}^{\mathrm{op}} \arrow[r] & \mathbf{Cat} \\ \mathbb{X} \arrow[r, mapsto] \arrow[d, "F"'] & \mathbf{Fib}(\mathbb{X}) \\ \mathbb{Y} \arrow[r, mapsto] & \mathbf{Fib}(\mathbb{Y}) \arrow[u, "F^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Cat}^{\mathrm{op}} \arrow[r] & \mathbf{Cat} \\ \mathbb{X} \arrow[r, mapsto] \arrow[d, "F"'] & \mathbf{ICat}(\mathbb{X}) \\ \mathbb{Y} \arrow[r, mapsto] & \mathbf{ICat}(\mathbb{Y}) \arrow[u, "-\circ F^{\mathrm{op}}"'] \end{tikzcd}$$

where F^* takes pullbacks along F and $-\circ F^{\mathrm{op}}$ precomposes with the opposite of F. These are both weak monoidal, with the respective structures essentially being (14) and (18).

Since the base of both monoidal fibrations is cartesian, the global monoidal structure is equivalent to a fibrewise monoidal structure, as per the theme of this whole section. The induced monoidal structure on each $\mathbf{Fib}(\mathbb{X})$ is given by (17) and on each $\mathbf{ICat}(\mathbb{X})$ by (19), and F^*, $-\circ F^{\mathrm{op}}$ are strong monoidal respectively. $\square$

The above essentially lifts the global and fibrewise monoidal structure development one level up, exhibiting fibrations and indexed categories as examples of the monoidal Grothendieck construction themselves.

5. Applications

In this section, we explore certain settings where the equivalence between monoidal fibrations and monoidal indexed categories naturally arises.

5.1. **Global categories of modules and comodules.** For any monoidal category $(\mathcal{V}, \otimes, I)$, there exist "global" categories of modules and comodules, denoted by **Mod** and **Comod** [Vas14, §6.2]. Their objects are all (co)modules over (co)monoids in $\mathcal{V}$, whereas for example a morphism between an A-module M and a B-module N is given by a monoid map $f \colon A \to B$ along with a morphism $k \colon M \to N$ in $\mathcal{V}$ satisfying the commutativity of

$$\begin{tikzcd} A \otimes M \arrow[rr, "\mu"] \arrow[d, "1\otimes k"'] & & M \arrow[d, "k"] \\ A \otimes N \arrow[r, "f\otimes 1"] & B \otimes N \arrow[r, "\mu"] & N \end{tikzcd}$$

Both these categories arise as the total categories of the (split) Grothendieck construction on the functors

$$\begin{tikzcd} \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[rr] & & \mathbf{Cat} & \mathbf{Comon}(\mathcal{V}) \arrow[rr] & & \mathbf{Cat} \\ A \arrow[d, "f"'] \arrow[rr, mapsto, dotted] & & \mathbf{Mod}_{\mathcal{V}}(A) & C \arrow[d, "g"'] \arrow[rr, mapsto, dotted] & & \mathbf{Comod}_{\mathcal{V}}(C) \arrow[d, "g_!"] \\ B \arrow[rr, mapsto, dotted] & & \mathbf{Mod}_{\mathcal{V}}(B) \arrow[u, "f^*"'] & D \arrow[rr, mapsto, dotted] & & \mathbf{Comod}_{\mathcal{V}}(D) \end{tikzcd} \tag{28}$$

where f^* and $g_!$ are (co)restriction of scalars. For example, if M is a B-module, $f^*(M)$ is an A-module via the action

$$A \otimes N \xrightarrow{f\otimes 1} B \otimes N \xrightarrow{\mu} N.$$

The induced split fibration and opfibration, $\mathbf{Mod} \to \mathbf{Mon}(\mathcal{V})$ and $\mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$, map a (co)module to its respective (co)monoid.

Recall that when $(\mathcal{V}, \otimes, I, \sigma)$ is braided monoidal, its categories of monoids and comonoids inherit the monoidal structure: if A and B are monoids, then $A \otimes B$ has also a monoid structure via

$$A \otimes B \otimes A \otimes B \xrightarrow{1\otimes\sigma\otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m\otimes m} A \otimes B$$

$$I \cong I \otimes I \xrightarrow{\eta\otimes\eta} A \otimes B$$

In that case, the induced split fibration and opfibration are both monoidal; this can be deduced by directly checking the conditions of Definition 3.1, or using Theorem 3.15 since both functors (28) are lax monoidal. For example, for any $A, B \in \mathbf{Mon}(\mathcal{V})$ there are natural maps

$$\phi_{A,B} \colon \mathbf{Mod}_{\mathcal{V}}(A) \times \mathbf{Mod}_{\mathcal{V}}(B) \to \mathbf{Mod}_{\mathcal{V}}(A \otimes B)$$
$$\phi_0 \colon \mathbf{1} \to \mathbf{Mod}_{\mathcal{V}}(I)$$

with $\phi_{A,B}(M, N) = M \otimes N$, with the $A \otimes B$-module structure being

$$A \otimes B \otimes M \otimes N \xrightarrow{1\otimes\sigma\otimes 1} A \otimes M \otimes B \otimes N \xrightarrow{\mu\otimes\mu} M \otimes N$$

and $\phi_0(*) = I$. Moreover, since the bases $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{Comon}(\mathcal{V})$ are not (co)cartesian, it is clear that the fibres are not monoidal in this example.

In fact, the monoidal opfibration $\mathbf{Comod} \to \mathbf{Comon}(\mathcal{V})$ serves as the monoidal base of an *enriched fibration* structure on $\mathbf{Mod} \to \mathbf{Mon}(\mathcal{V})$, as explained in [Vas18].

5.2. **Zunino and Turaev categories.** In [CDL06], the authors introduce categories $\mathcal{T}$ of *Turaev* and $\mathcal{Z}$ of *Zunino* R-modules, for a commutative ring R. These serve as the symmetric monoidal categories where group-(co)algebras and Hopf group-(co)algebras, [Tur00], live as (co)monoids and Hopf monoids respectively.

The objects of both $\mathcal{T}$ and $\mathcal{Z}$ are defined to be pairs (X, M) where X is a set and $\{M_x\}_{x\in X}$ is an X-indexed family of R-modules. Their morphisms are pairs

$$(\mathcal{T}) \begin{cases} s \colon M_{f(y)} \to N_y \text{ in } \mathbf{Mod}_R \\ f \colon Y \to X \text{ in } \mathbf{Set} \end{cases} \qquad (\mathcal{Z}) \begin{cases} t \colon M_x \to N_{g(x)} \text{ in } \mathbf{Mod}_R \\ g \colon X \to Y \text{ in } \mathbf{Set} \end{cases}$$

and the monoidal structure is essentially a combination of the cartesian product in $\mathbf{Set}$ and the tensor product of R-modules. Finally, there are monoidal forgetful functors $\mathcal{T} \to \mathbf{Set}^{\mathrm{op}}$, $\mathcal{Z} \to \mathbf{Set}$. It is therein shown that comonoids in $\mathcal{T}$ are monoid-coalgebras and monoids in $\mathcal{Z}$ are monoid-algebras, i.e. indexed families of R-modules together with respective families of linear maps

$$(\mathcal{T})\quad \begin{array}{c} C_{g*h} \to C_g \otimes C_h \\ C_e \to R \end{array} \qquad (\mathcal{Z})\quad \begin{array}{c} A_g \otimes A_h \to A_{g*h} \\ R \to A_e \end{array}$$

satisfying appropriate axioms.

Since these two categories have a typical Grothendieck category flavor, reminiscent of the standard *family fibration*, their structure can be clarified and in fact further generalized as follows.

Suppose $\mathcal{V}$ is a monoidal category. Consider the (strict) functor

$$[-,\mathcal{V}]\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat} \tag{29}$$

which maps every set X (i.e. discrete category) to the usual functor category $[X,\mathcal{V}]$ and every function $f\colon X \to Y$ to $f^* = [f,1]$, i.e. pre-composition with f. The objects of $[X,\mathcal{V}]$ are $M\colon X \to \mathcal{V}$ given by X-indexed objects in $\mathcal{V}$ and morphisms $a\colon M \Rightarrow N$ are arrows $a_x\colon M_x \to N_x$ in $\mathcal{V}$.

The functor (29) has a canonical lax monoidal structure between $\mathbf{Set}^{\mathrm{op}}$ viewed as a cocartesian monoidal category (the opposite of the cartesian $\mathbf{Set}$) and $(\mathbf{Cat},\times,\mathbf{1})$ as follows:

$$\phi_{X,Y}\colon [X,\mathcal{V}]\times[Y,\mathcal{V}] \to [X\times Y,\mathcal{V}]$$
$$\phi_0\colon \mathbf{1} \xrightarrow{I_{\mathcal{V}}} [1,\mathcal{V}] \cong \mathcal{V}$$

where $\phi_{X,Y}$ corresponds, under the tensor-hom adjunction in $\mathbf{Cat}$, to

$$[X,\mathcal{V}]\times[Y,\mathcal{V}]\times X\times Y \xrightarrow{\sim} [X,\mathcal{V}]\times X\times[Y,\mathcal{V}]\times Y \xrightarrow{\mathrm{ev}\times\mathrm{ev}} \mathcal{V}\times\mathcal{V} \xrightarrow{\otimes} \mathcal{V}.$$

By Theorem 3.15, this lax monoidal functor gives rise to a monoidal split fibration over $\mathbf{Set}$, denoted as $\mathbf{Fam}(\mathcal{V}) \to \mathbf{Set}$, which coincides with the forgetful $\mathcal{Z} \to \mathbf{Set}$ for $\mathcal{V} = \mathbf{Mod}_R$. On the other hand, we could use the dual part of the same theorem, and instead consider the induced monoidal opfibration denoted $\mathbf{Maf}(\mathcal{V}) \to \mathbf{Set}^{\mathrm{op}}$ corresponding to the same functor; this coincides with $\mathcal{T} \to \mathbf{Set}^{\mathrm{op}}$ for $\mathcal{V} = \mathbf{Mod}_R$.

Notice that these categories are uniquely induced by the same monoidal indexed category (29), so there exists a correspondence between them. However they are not opposite; in fact $\mathbf{Maf}(\mathcal{V}) = \mathbf{Fam}(\mathcal{V}^{\mathrm{op}})^{\mathrm{op}}$.

5.3. **Graph over Set.** Consider the functor $\mathbf{Set} \to \mathbf{Cat}$ which maps every set X to the category $\mathbf{Grph}_X$ of graphs $V \underset{t}{\overset{s}{\rightrightarrows}} X$ with that set of vertices. Formally, this is the slice category $\mathbf{Set}/X\times X$ of objects $V \to X\times X$ and morphisms functions $k\colon V \to V'$ such that

$$\begin{tikzcd} V \arrow[rr, "k"] \arrow[dr, "{(s,t)}"'] & & V' \arrow[dl, "{(s',t')}"] \\ & X\times X & \end{tikzcd} \quad \text{or equivalently} \quad \begin{tikzcd} V \arrow[r, "k"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & V' \arrow[dl, shift right, "s'"'] \arrow[dl, shift left, "t'"] \\ X & \end{tikzcd}$$

For any function $f\colon X \to Y$, there is a post-composition functor $\mathbf{Set}/X\times X \xrightarrow{(f\times f)\circ -} \mathbf{Set}/Y\times Y$ that maps an X-graph $V \underset{t}{\overset{s}{\rightrightarrows}} X$ to the Y-graph $V \underset{f\circ t}{\overset{f\circ s}{\rightrightarrows}} Y$.

Now, if we consider **Set** with its coproduct, the above described functor $\mathbf{Set}/(-\times -)$ is lax monoidal $(\mathbf{Set}, +, 0) \to (\mathbf{Cat}, \times, 1)$, with structure maps

$$\phi_{X,Y} \colon \mathbf{Grph}_X \times \mathbf{Grph}_Y \to \mathbf{Grph}_{X+Y}$$
$$\phi_0 \colon \mathbf{1} \to \mathbf{Grph}_0$$

where $\phi_{X,Y}(\ V \overset{s}{\underset{t}{\rightrightarrows}} X\ ,\ W \overset{s}{\underset{t}{\rightrightarrows}} Y\ ) =\ V + W \overset{s+s}{\underset{t+t}{\rightrightarrows}} X + Y$ and $\phi_0(*) = 0 \rightrightarrows 0$.

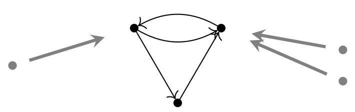
It is standard that the ordinary Grothendieck construction gives **Grph**, the category of graphs and graph morphisms between them, as the total category; the opfibration $\mathbf{Grph} \to \mathbf{Set}$ gives the set of objects. The fact that we can take disjoint unions of graphs and consider them as a graph over the disjoint union of the set of vertices is precisely what gives the induced monoidal opfibration $\mathbf{Grph} \to \mathbf{Set}$ its global monoidal structure.

The fibres $\mathbf{Grph}_X$ have coproducts, given by

$$\begin{tikzcd}
E \arrow[dr] \arrow[ddr, bend right=50, "s"'] \arrow[ddr, bend right=30, "t"] & & E' \arrow[dl] \arrow[ddl, bend left=50, "t'"] \arrow[ddl, bend left=30, "s'"'] \\
& E+E' \arrow[d, shift right, "\exists! s''"'] \arrow[d, shift left, "\exists! t''"] & \\
& X &
\end{tikzcd}$$

This coproduct of graphs with the same vertex set can be computed by first taking the disjoint union of the two graphs, and then identifying all the vertices which come from the same element of X. Another way of looking at this is that you 'overlay' the two graphs so as to line up corresponding vertices: $E + E' \overset{s+s'}{\underset{t+t'}{\rightrightarrows}} X + X' \xrightarrow{\nabla} X$ This is the same as computing the pushout over the obvious inclusions of the graph with vertex set X and no vertices into each of the given graphs.

In [Fon15], Fong considers *decorated cospans* in order to model networks. One of the motivating examples for the general construction is cospans of sets decorated with graphs. Using the above notation, where in fact the lax monoidal functor used was $(\mathbf{FinSet}, +, 0) \to (\mathbf{Set}, \times, 1)$, a decorated cospan is a graph G with finite set of vertices N, along with a cospan $X \to N \leftarrow Y$ of finite sets, equipping the graph with a certain notion of input and output.

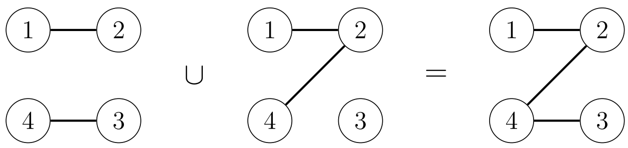


In fact, the apex of the cospan can be viewed as an object of the (discrete) Grothendieck construction on that functor, since it always comes equipped with an element of the set of all its possible decorations.

In that direction, there is evidence that the monoidal Grothendieck construction is the 'bridge' that connects Fong's approach to networks via decorated cospans, with an alternative approach to network theory via *structured cospans* [BC18]. Their explicit relation is aimed to be completely clarified in future work.

5.4. **Network Models.** A large complex network can be built from smaller ones by gluing them together with the right recipe. Such recipes can be written as combinations of a few basic operations. We start by looking at these operations as they arise naturally in the context of simple graphs.

Let X be a finite set with $|X| = n$. Let $\mathbf{SG}(X)$ be the set of simple graphs with vertex set X. Two simple graphs with the same vertex set can be 'overlaid' by identifying the coresponding vertices and simplifying the edge sets. We denote this overlay operation by $\cup$.



This operation defines a monoid structure on $\mathbf{SG}(X)$ with the edgeless graph acting as unit. This monoid admits an action of the symmetric group S_n by permuting the vertices. Thus we have a functor $\mathbf{SG}\colon \mathbf{FB} \to \mathbf{Mon}$, where $\mathbf{FB}$ is the category of finite sets and bijections, equivalent to the permutation groupoid.

Restricting the cocartesian monoidal structure on $\mathbf{FinSet}$ gives the category $\mathbf{FB}$ a symmetric monoidal strucure (no longer cocartesian). Our functor is lax symmetric monoidal with the laxator given by disjoint union of graphs.

$$\sqcup_{X,Y}\colon \mathbf{SG}(X) \times \mathbf{SG}(Y) \to \mathbf{SG}(X+Y)$$

$$(\mathbf{SG}, \sqcup, 1_1)\colon (\mathbf{FB}, +, \emptyset) \to (\mathbf{Mon}, \times, \mathbf{1}).$$

This formalism captures much of the combinatorial information about simple graphs. We can then generalize this in order to describe things with a combinatorial structure similar to that of simple graphs. A *one-colored network model* is a lax symmetric monoidal functor $(F, \Phi)\colon (\mathbf{FB}, +) \to (\mathbf{Mon}, \times)$. Notice that $\mathbf{FB}$ is equivalent to the free symmetric monoidal category on one generating object. For a set C, let $\mathbf{S}C$ denote the free symmetric monoidal category on C, with its tensor denoted by $+$. A *network model* (with set of colors C) is a lax symmetric monoidal functor $(F, \Phi)\colon (\mathbf{S}C, +) \to (\mathbf{Mon}, \times, \mathbf{1})$.

Since monoids can be viewed as one-object categories, $(\mathbf{Mon}, \times, 1) \hookrightarrow (\mathbf{Cat}, \times, 1)$ via delooping, then we can apply the symmetric monoidal grothendieck construction and get a monoidal fibration $(\int F, \otimes_F, \emptyset) \to (\mathbf{S}C, +, 0)$. The *network operad* associated to a network model is the underlying operad of the symmetric monoidal category $(\int F, \otimes_F)$.

References

[BC18] John Baez and Kenny Courser. Structured cospans. To appear, 2018. (Referred to on page 24.)

[BFMP17] John Baez, John Foley, Joseph Moeller, and Blake Pollard. Network models. arXiv:1711.00037 [math.CT], 2017. (Referred to on page 16.)

[Bor94] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures. (Referred to on page 3, 6.)

[BW18] Jonathan Beardsley and Liang Ze Wong. The enriched grothendieck construction. arXiv:1804.03829v2 [math.CT], 2018. (Referred to on page 2.)

[Car95] Sean Carmody. *Cobordism categories*. PhD thesis, University of Cambridge, 1995. (Referred to on page 7.)

[CDL06] S. Caenepeel and M. De Lombaerde. A categorical approach to Turaev's Hopf group-coalgebras. *Comm. Algebra*, 34(7):2631–2657, 2006. (Referred to on page 22.)

[CLS10] D. Chikhladze, S. Lack, and R. Street. Hopf monoidal comonads. *Theory and Applications of Categories*, 24:No. 19, 554–563, 2010. (Referred to on page 8.)

[DS97] Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. *Adv. Math.*, 129(1):99–157, 1997. (Referred to on page 7, 9.)

[Fon15] Brendan Fong. Decorated cospans. *Theory and Applications of Categories*, 30(33), 2015. (Referred to on page 24.)

[GPS95] R. Gordon, A. J. Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995. (Referred to on page 7, 8.)

[Gra66] John W. Gray. Fibred and cofibred categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 21–83. Springer, New York, 1966. (Referred to on page 3.)

[Gro61] Alexander Grothendieck. Catégories fibrées et descente, 1961. Seminaire de géometrie algébrique de l'Institut des Hautes Études Scientifiques (SGA 1), Paris. (Referred to on page 1, 6.)

[Gur13] Nick Gurski. *Coherence in three-dimensional category theory*, volume 201 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. (Referred to on page 2.)

[Her93] Claudio Alberto Hermida. *Fibrations, logical predicates and indeterminates*. PhD thesis, University of Edinburgh, 1993. (Referred to on page 6.)

[Her94] Claudio Hermida. On fibred adjunctions and completeness for fibred categories. In *Recent trends in data type specification (Caldes de Malavella, 1992)*, volume 785 of *Lecture Notes in Comput. Sci.*, pages 235–251. Springer, Berlin, 1994. (Referred to on page 3.)

[HM06] Pieter Hofstra and Federico De Marchi. Descent for monads. *Theory Appl. Categ.*, 16:No. 24, 668–699, 2006. (Referred to on page 15.)

[Jac99] Bart Jacobs. *Categorical logic and type theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999. (Referred to on page 6.)

[Joh02] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 1*, volume 43 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 2002. (Referred to on page 5.)

[Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. (Referred to on page 20.)

[KS74] G. Kelly and Ross Street. Review of the elements of 2-categories. In Gregory Kelly, editor, *Category Seminar*, volume 420 of *Lecture Notes in Mathematics*, pages 75–103. Springer Berlin / Heidelberg, 1974. 10.1007/BFb0063101. (Referred to on page 2.)

[Lac10] Stephen Lack. A 2-categories companion. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 105–191. Springer, New York, 2010. (Referred to on page 2.)

[McC00] Paddy McCrudden. Balanced coalgebroids. *Theory Appl. Categ.*, 7:No. 6, 71–147, 2000. (Referred to on page 7, 9.)

[Shu08] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory Appl. Categ.*, 20:No. 18, 650–738, 2008. (Referred to on page 1, 10, 11, 18.)

[Shu09] Michael Shulman. *Double Categories and Base Change in Homotopy Theory*. PhD thesis, University of Chicago, 2009. (Referred to on page 16.)

[Str80] Ross Street. Fibrations in bicategories. *Cahiers Topologie Géom. Différentielle*, 21(2):111–160, 1980. (Referred to on page 19.)

[Tur00] V.G. Turaev. Homotopy field theory in dimension 3 and crossed group-categories. arXiv:math/0005291 [math.GT], 2000. (Referred to on page 22.)

[Vas14] Christina Vasilakopoulou. *Generalization of Algebraic Operations via Enrichment*. PhD thesis, University of Cambridge, 2014. arxiv:1411.3038 [math.CT]. (Referred to on page 21.)

[Vas18] Christina Vasilakopoulou. On enriched fibrations. *Cahiers Topologie Géom. Différentielle Catég. (in press)*, 2018. arXiv:1801.01386 [math.CT]. (Referred to on page 10, 22.)

Departement of Mathematics, University of California, Riverside, 900 University Avenue, 92521, USA

E-mail address: moeller@math.ucr.edu, vasilak@ucr.edu