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Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages

The problem of finding all four-bar linkages whose coupler curve passes through nine prescribed points has been a longstanding unsolved problem in kinematics. Using a combination of classical elimination, multihomogeneous variables, and numerical polynomial continuation, we show that there are generically 1442 nondegenerate solutions along with their Roberts cognates, for a total of 4326 distinct solutions. Moreover, a computer algorithm that computes all solutions for any given nine points has been developed.

Introduction

The approximate synthesis of a given path by use of four-bar linkages has been studied extensively. Formulations in terms of four or five precision points along with specifications on crank angles or the position of the hinges of the mechanism have been solved (Freudenstein and Sandor, 1959; Shigley and Uicker, 1980; Erdman and Sandor, 1984; Morgan and Wampler, 1989; Subbian and Flugrad, 1989). However, the problem of finding four-bar linkages whose coupler curve passes through nine precision points, which was formulated as early as 1923 (Alt), has until now defied complete solution. Since nine general precision points is the largest number that can be prescribed, this formulation gives a designer maximum control over the shape of the coupler curve.

The first serious attempt to solve the nine-point problem appears to have been conducted by Roth and Freudenstein (1963), as a special case of their treatment of geared five-bar mechanisms. They employed a type of numerical continuation, which they called the "bootstrap procedure," to find some solutions. More recently Tsai and Lu (1989) applied a new continuation method, called the "cheater's homotopy," to increase the reliability of the procedure, but they also did not attempt to find all solutions. Since the problem has many solutions, most of which are either degenerate or have branch or order defects, it is often difficult to find an acceptable solution by trial-and-error procedures. Only by finding *all* nondegenerate solutions can one be sure to find the best mechanism or, in some cases, verify that no acceptable solution exists.

We solve the problem using a combination of analytical and numerical tools. First, we reformulate the problem to analytically reduce the polynomial system of equations that describe the problem. We then use numerical polynomial continuation in multihomogeneous variables (Morgan and Sommese, 1987) to solve a problem with randomly generated precision points, thereby determining the generic structure of the solution set for these problems. In addition to 1442 sets of Robert's cognate triples, the solution set includes several higher-dimensional sets of degenerate solutions. By the theory of "parameter polynomial continuation" (Morgan and Sommese, 1989), we may ignore all the degenerate solutions and use only the nondegenerate ones as start points in subsequent continuations to find all nondegenerate solutions to any other problem of the class. Thus, we have not only established the generic number of nondegenerate solutions to the problem, but also have developed an efficient computer algorithm for finding them.

In any particular example, not all of the $1442 \times 3 = 4326$ solutions are useful. Most give linkages with complex link lengths, whereas others give real linkages that exhibit branch or order defects, or that have poor transmission angles, etc. We discuss these issues in the context of several test problems.

The papers (Wampler, Morgan and Sommese, 1990; Wampler and Morgan, 1990) contain tutorial material on the mathematical techniques we have used. In particular, the reader may wish to consult these papers for discussions of the multihomogeneous Bezout number and the method of numerical reduction (via parameter continuation; also referred to as "the method of the generic case").

Problem Formulation and Reduction

The most concise formulation of the problem is obtained by representing the links as vectors in the complex plane. Our derivation follows that of Roth and Freudenstein (1963), but with a change of variables that allows for subsequent reduction. Referring to Fig. 1, let P_0 be the first precision point, at which the four-bar is given by quadrilateral $ABCD$ with coupler triangle CP_0D. Summing the vectors around the left-hand loop gives

$$u = x - a. \tag{1}$$

Figure 2 shows the four-bar after a displacement of δ_j to precision point P_j, with corresponding angular displacements θ_j, λ_j, μ_j of the coupler triangle, link $\overline{AD}$ and link $\overline{BC}$, respectively. Now, the vector loop equation becomes

$$ue^{i\lambda_j} = xe^{i\theta_j} + \delta_j - a. \tag{2}$$

Contributed by the Mechanisms Committee for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received November 1990.

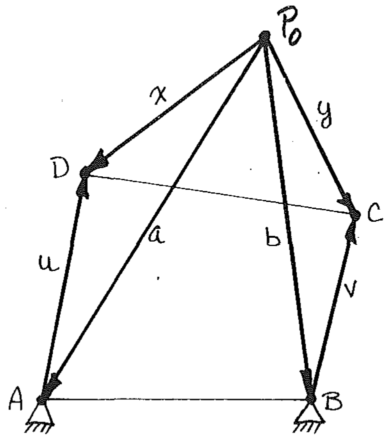

Fig. 1 Four-bar *ABCD* with coupler triangle CP_0D at the initial precision point P_0

Substituting equation (1) and performing the same derivations for the right-hand side of the mechanism, we obtain

$$(x-a)e^{i\lambda_j}=xe^{i\theta_j}+\delta_j-a \tag{3}$$

$$(y-b)e^{i\mu_j}=ye^{i\theta_j}+\delta_j-b \tag{4}$$

Multiplying each side of these equations by its complex conjugate and letting

$$\gamma_j=e^{i\theta_j}-1, \tag{5}$$

we get

$$(a^*-\delta_j^*)x\gamma_j+(a-\delta_j)x^*\gamma_j^*+\delta_j(a^*-x^*)+\delta_j^*(a-x)-\delta_j\delta_j^*=0 \tag{6}$$

$$(b^*-\delta_j^*)y\gamma_j+(b-\delta_j)y^*\gamma_j^*+\delta_j(b^*-y^*)+\delta_j^*(b-y)-\delta_j\delta_j^*=0 \tag{7}$$

where the asterisk indicates complex conjugation. From equation (5), we also have the identity

$$\gamma_j\gamma_j^*+\gamma_j+\gamma_j^*=0. \tag{8}$$

Since conjugation is not algebraic, some modification of the proceding equations is necessary to make them polynomial, which is a prerequisite to finding all solutions. The usual approach is to explicitly write each variable in terms of its real and imaginary parts, for example, we may replace x with x_1+ix_2, and so on. Of course, x^* is then replaced by x_1-ix_2. If we then treat x_1 and x_2 as complex variables and solve the polynomial system, only those solutions for which x_1 and x_2 are both real have physical meaning. Note that x_1+ix_2 and x_1-ix_2 are complex conjugates of each other if, and only if, x_1 and x_2 are real. A simpler treatment is to retain x as a complex variable and replace x^* by an independent complex variable, say $\hat{x}$, and so on. Now, the condition for a solution to have physical meaning is that $x^*=\hat{x}$, etc.

Although Roth and Freudenstein used the real and imaginary parts of x, a, y, b as variables, a set of synthesis equations essentially the same as theirs is obtained as follows. For each precision point P_j, $j=1,\ldots,8$, treat equations (6, 7) as a pair of linear equations in γ_j and γ_j^*, solve these using Cramer's rule, and substitute into equation (8). This gives eight equations in eight unknowns. Solutions to this set of equations correspond to four-bar linkages whose coupler curve passes through the nine precision points P_j, $j=0,\ldots,8$. Each of the eight equations is a seventh degree polynomial, yielding a total degree for the system of $7^8=5{,}764{,}801$.

Reduction. We reduce the polynomial system by introducing some new variables and using multihomogeneous coordinates. As discussed above, we first replace x^*, a^*, y^*, b^*,

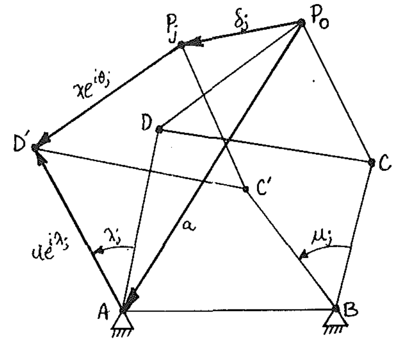

Fig. 2 Displacement of the four-bar to a new precision point P_j

γ_j^* with $\hat{x}$, $\hat{a}$, $\hat{y}$, $\hat{b}$, $\hat{\gamma}_j$, respectively, in equations (6, 7, 8) to make them polynomial. Moreover, we can reduce the degrees of the equations by introducing the new variables n, $\hat{n}$, m, $\hat{m}$ defined as follows

$$n=a\hat{x},\quad \hat{n}=\hat{a}x,\quad m=b\hat{y},\quad \hat{m}=\hat{b}y. \tag{9}$$

Finally, we eliminate some solutions at infinity by partitioning the variables into 10 groups:

$$\{x,\hat{x},a,\hat{a},n,\hat{n}\},\ \{y,\hat{y},b,\hat{b},m,\hat{m}\},\ \{\gamma_j,\hat{\gamma}_j\},\ j=1,\ldots,8$$

and introducing homogeneous coordinates for each group, respectively, as x^0, y^0, γ_j^0, $j=1,\ldots,8$. With all of these modifications, we may rewrite equations (6–9) as

$$nx^0=a\hat{x},\quad \hat{n}x^0=\hat{a}x,\quad my^0=b\hat{y},\quad \hat{m}y^0=\hat{b}y \tag{10}$$

$$(\hat{n}-\hat{\delta}_jx)\gamma_j+(n-\delta_j\hat{x})\hat{\gamma}_j+[\delta_j(\hat{a}-\hat{x})+\hat{\delta}_j(a-x)-\delta_j\hat{\delta}_jx^0]\gamma_j^0=0 \tag{11}$$

$$(\hat{m}-\hat{\delta}_jy)\gamma_j+(m-\delta_j\hat{y})\hat{\gamma}_j+[\delta_j(\hat{b}-\hat{y})+\hat{\delta}_j(b-y)-\delta_j\hat{\delta}_jy^0]\gamma_j^0=0 \tag{12}$$

$$\gamma_j\hat{\gamma}_j+\gamma_j\gamma_j^0+\hat{\gamma}_j\gamma_j^0=0 \tag{13}$$

Equations (10–13) are a set of 28 equations in 28 unknowns.

To reduce the number of variables, we solve equations (11, 12) using Cramer's rule to obtain for $j=1,\ldots,8$

$$\gamma_j=\begin{vmatrix} n-\delta_j\hat{x} & \delta_j(\hat{a}-\hat{x})+\hat{\delta}_j(a-x)-\delta_j\hat{\delta}_jx^0 \\ m-\delta_j\hat{y} & \delta_j(\hat{b}-\hat{y})+\hat{\delta}_j(b-y)-\delta_j\hat{\delta}_jy^0 \end{vmatrix} \tag{14}$$

$$\hat{\gamma}_j=\begin{vmatrix} \delta_j(\hat{a}-\hat{x})+\hat{\delta}_j(a-x)-\delta_j\hat{\delta}_jx^0 & \hat{n}-\hat{\delta}_jx \\ \delta_j(\hat{b}-\hat{y})+\hat{\delta}_j(b-y)-\delta_j\hat{\delta}_jy^0 & \hat{m}-\hat{\delta}_jy \end{vmatrix} \tag{15}$$

$$\gamma_j^0=\begin{vmatrix} \hat{n}-\hat{\delta}_jx & n-\delta_j\hat{x} \\ \hat{m}-\hat{\delta}_jy & m-\delta_j\hat{y} \end{vmatrix} \tag{16}$$

Substituting these expressions into equation (13) and retaining equations (10), we have a system of 12 equations in 12 unknowns.

The system consists of 4 quadrics and 8 quartics for a total degree of $2^4 4^8=1{,}048{,}576$. However, the 2-homogeneous Bezout number is only 286,720. This number is the result of a certain combinatorial calculation that can be mechanized as follows. First construct Table 1, listing the degree of each equation with respect to each of the two groups of variables, arbitrarily named the α-group and the β-group. Then, form a single term for each equation, being the sum of the degrees times either α or β, as appropriate, and multiply all these terms together to get the polynomial $(2\alpha)^2(2\beta)^2(2\alpha+2\beta)^8$. Now, since there are 6 variables in each group, we want the coefficient of $\alpha^6\beta^6$ in this polynomial, which turns out to be $2^{12}8!/(4!4!)$ = 286,720. (The **ratcoef** command in Macsyma and similar commands in other symbolic manipulation programs can perform this calculation.) Thus, from the original figure of 7^8, our re-

Table 1 Degree table for 2-homogeneous Bezout count

Equation	α-group $\{x, \hat{x}, a, \hat{a}, n, \hat{n}\}$	β-group $\{y, \hat{y}, b, \hat{b}, m, \hat{m}\}$	Term
$nx^0 = a\hat{x}$	2	0	2α
$\hat{n}x^0 = \hat{a}x$	2	0	2α
$my^0 = b\hat{y}$	0	2	2β
$\hat{m}y^0 = \hat{b}y$	0	2	2β
Eq. (13) $j = 1, \ldots, 8$	2	2	$(2\alpha + 2\beta)^8$

duction has cut the number of possible solutions by a factor of slightly more than 20.

We remark as an aside that the 10-homogeneous Bezout number for equations (10–13) is also 286,720, but we prefer the formulation with a fewer number of variables.

Roberts Cognates. A well-known result due to Roberts (1875) is that every four-bar linkage has two four-bar cognates; that is, there are two other four-bar linkages that produce the same coupler curve. Clearly, if a four-bar linkage solves our nine-point problem, then so will both of its cognates. Referring to equations (3, 4), suppose that a linkage and its angular displacement are given by $z = (x, a, y, b, \theta_j, \lambda_j, \mu_j)$ for a displacement of the coupler point of δ_j. Then, the corresponding position of one of its cognates is given by the formula

$$r(x, a, y, b, \theta_j, \lambda_j, \mu_j) = \left(\frac{(x-a)y}{x-y}, \frac{bx-ay}{x-y}, a-x, a, \lambda_j \mu_j, \theta_j\right). \quad (17)$$

This may be confirmed by noting that if we rewrite equations (3, 4) as $f(z) = 0$ and $g(z) = 0$, respectively, then $f(r(z)) = (xg(z) - yf(z))/(x-y)$ and $g(r(z)) = f(z)$, so that if z satisfies the equations then so does $r(z)$. The fact that cognates come in triples can be confirmed by showing that $r(r(r(z))) = z$.

In our polynomial formulation of the problem, the angles have been eliminated and variables $(\hat{x}, \hat{a}, \hat{y}, \hat{b})$ have been introduced. To compute a cognate of a solution in these variables, we drop the angular entries from the formula of equation (17) and apply it to both (x, a, y, b) and $(\hat{x}, \hat{a}, \hat{y}, \hat{b})$ independently. The corresponding values of $n, \hat{n}, m, \hat{m}$ are then calculated from their definitions in equation (9). For brevity, we do not write out the complete formula, but simply refer to it hereafter as $R(z)$.

Symmetry. In addition to the cognate groups, our equations contain a two-fold symmetry. If we simply exchange the variables in the two homogeneous groups, that is, if we replace z by $S(z)$ where

$$S(x, \hat{x}, a, \hat{a}, n, \hat{n}, x^0, y, \hat{y}, b, \hat{b}, m, \hat{m}, y^0) = (y, \hat{y}, b, \hat{b}, m, \hat{m}, y^0, x, \hat{x}, a, \hat{a}, n, \hat{n}, x^0), \quad (18)$$

the new point also satisfies the equations. This operation simply exchanges the rows in each determinant in equations (14–16), thereby changing the sign of each of $\gamma_j, \hat{\gamma}_j, \gamma_j^0$, which leaves equation (13) unchanged. The physical interpretation of this symmetry is that we may exchange the labels on the left and right sides of the four-bar linkages, thereby exchanging the values of x with y, etc., while the linkage itself remains the same.

For a general point z, the Roberts cognate formula and the symmetry formula give a group of six points as follows: z, $R(z)$, $R(R(z))$, $S(z)$, $S(R(z))$, $S(R(R(z)))$. Because of the identity $R(S(z)) = S(R(R(z)))$, the group is closed.

Degenerate Solutions. Our set of equations admits several types of degenerate solutions. One of these occurs when $z = S(z)$, that is $x = y$, $a = b$, etc. In that case, the mechanism degenerates into a two-degree-of-freedom linkage that can reach any point inside an annulus centered on the fixed pivots, which are coincident. Another degenerate case occurs when $x = y = \hat{n} = \hat{m} = 0$, which makes $\gamma_j = \gamma_j^0 = 0$. This case is purely a mathematical figment, since the coupler triangle becomes a point and the linkage does not move. In either case, we are not interested in such linkages.

In fact, there are several other sets of degenerate solutions, but they all obey one of the following conditions:

$$x = 0 \text{ or } y = 0 \text{ or } \hat{x} = 0 \text{ or } \hat{y} = 0, \quad (19)$$

$$x = y \text{ or } \hat{x} = \hat{y}. \quad (20)$$

Again, we are not interested in linkages satisfying these conditions.

If we try to solve the problem by applying Newton's method from an initial guess, there is a high likelihood that it will converge to one of these degenerate cases. This is one of the motivations for developing a more sophisticated numerical approach.

Precision Points. It is known (Roth and Freudenstein, 1963; Primrose and Freudenstein, 1963) that a four-bar coupler curve can intersect a circle or a line at most 6 times. Therefore, of the nine precision points, no more than 6 can lie on a circle or line. This conclusion can easily be derived from our formulation. After substituting from equations (14–16) into (13), we note that for a given four-bar linkage, equation (13) can be viewed as a polynomial equation in $\delta_j, \hat{\delta}_j$. We then ask how many solutions this equation shares with an arbitrary circle in the plane, which has the form

$$\beta_1 \delta_j \hat{\delta}_j + \beta_2 \delta_j + \beta_3 \hat{\delta}_j + \beta_4 = 0. \quad (21)$$

Treating equations (13, 21) as 2-homogeneous, we note that equation (13) is bi-cubic and equation (21) is bilinear, so that the Bezout number is the coefficient of $\alpha\hat{\alpha}$ in the polynomial $(3\alpha + 3\hat{\alpha})(\alpha + \hat{\alpha})$, which is 6. For a line, we have the special case $\beta_1 = 0$ and the argument still holds.

Summary. Using only analytical means, we have shown that the nine-point problem has at most 286,720 isolated solutions and, because of the two-fold S symmetry, these represent at most 143,360 distinct mechanisms. Moreover, these mechanisms must appear in groups of three according to Roberts theorem. The fact that 143,360 is not a multiple of 3 is not a problem because this Bezout calculation includes many degenerate solutions and for most of these the Roberts cognates are not well-defined. Finally, while we can solve the problem for nine generally placed precision points, no more than six of the points can lie on a circle or a line.

Numerical Reduction by Continuation

If the nine-point problem actually had 143,360 solutions, it would seem impractical to compute all the solutions as a matter of routine in real design problems. However, it turns out that the vast majority of this Bezout number is due to the degenerate solution sets, which can be eliminated numerically. The method of reduction requires a one-time computation of the full set of 143,360 solutions for a generic set of precision points. These solutions are then sorted as to whether or not they satisfy the *side conditions*, equations (19, 20), defining degeneracy. Only the nondegenerate solutions can lead to nondegenerate solutions for other sets of precision points, so the number of these solutions determines the cost of solving subsequent problems. Thus, we can reduce the problem to manageable proportions by a single, albeit expensive, numerical computation. This approach is justified by the theory of parameter continuation (Morgan and Sommese, 1989) and is developed in the context of a robotics problem in Wampler and Morgan (1990).

We proceed as follows. Let $f(p, z) = 0$ represent our polynomial formulation of the nine-point problem, equations (10, 13), where z is the set of 12 variables and p is the set of path increments, δ_j, $\hat{\delta}_j$ $j=1,\ldots,8$. First, we construct a polynomial system $g(z)$ that has the same 2-homogeneous degree structure as $f(p, z)$, such that $g(z)=0$ has a full set of 286,720 nonsingular solutions. Then, to solve the problem for a particular set of precision points p_0, form the homotopy

$$h(p_0, z, t) = \zeta(1-t)g(z) + tf(p_0, z), \qquad (22)$$

where ζ is a random, complex number. To satisfy the "generic" requirement of the reduction procedure, we choose p_0 to be random, complex numbers. Note than any point z_0 satisfying $g(z_0)=0$ also satisfies $h(p_0, z_0, 0)=0$. Starting from each such point, we numerically track the solution to $h(p_0, z, t)=0$ from $t=0$ to $t=1$.

The upshot of this is that every nonsingular nondegenerate solution of $f(p_0, z)=0$ will be the endpoint of one of these continuation paths. Only these solutions, which we expect to be a relatively small proportion of the whole, will be used as start points in a parameter continuation to solve any subsequent problem, say $f(p_1, z)=0$, as discussed further below. This is why we refer to the initial calculation as a numerical reduction.

Start System. To carry out the numerical reduction, we need a two-homogeneous start system $g(z)$ with all nonsingular solutions. Furthermore, we need to be able to compute all of its solutions inexpensively. In addition, we can save half of the computation time by incorporating the two-fold S symmetry. A start system that meets all of these requirements is the following:

$$n^2-1=0,\ \hat{n}^2-1=0,\ m^2-1=0,\ \hat{m}^2-1=0 \qquad (23)$$

$$l_j(x, \hat{x}, a, \hat{a})l_j'(x, \hat{x}, a, \hat{a})\, l_j(y, \hat{y}, b, \hat{b})l_j'(y, \hat{y}, b, \hat{b})=0, \quad j=1,\ldots,8 \qquad (24)$$

where l_j and l_j' are linear expressions whose coefficients have been chosen general enough to yield all nonsingular solutions.

The full set of solutions to equations (23, 24) is found as follows. First, equations (23) have 2^4 solutions given by $n=\pm 1$, etc. Then, partition equations (24) into two groups of four, which can be done in $8!/(4!4!)=70$ ways. For each equation in the first group, set either $l_j(x, \hat{x}, a, \hat{a})=0$ or $l_j'(x, \hat{x}, a, \hat{a})=0$ and solve the resulting linear system. Do the same for the second group, but in terms of $(y, \hat{y}, b, \hat{b})$. Since for each partitioning of the equations, there are a total of 2^8 independent choices of l_j versus l_j', this entire procedure yields $2^4 \cdot 2^8 \cdot 70 = 286{,}720$ solutions.

Due to the symmetries in $g(z)$ and $f(p, z)$, it is easy to see that if $h(p, z, t)=0$ then $h(p, S(z), t)=0$, so that our continuation paths appear in symmetric pairs. Thus, we can cut the computational cost in half by tracking only one of each pair.

Path Tracking. Each path of equation (22) is tracked numerically using the method described in Morgan (1987). This is a predictor-corrector path tracker which predicts along the tangent to the continuation path (the length of the prediction being variable and adaptive) and then corrects back to the path using Newton's method. The continuation parameter t is forced to be strictly increasing, as justified by the theory of polynomial continuation, and t is held fixed during correction. For more details, see the description of the program CONSOL8 in Morgan (1987). Also, since some of the paths go to solutions at infinity, we use the projective transformation technique (Morgan, 1986). All of the calculations were done in double precision complex FORTRAN.

The main possibility for failure in polynomial continuation is to take too large a prediction step, resulting in a large enough prediction error that the corrector may converge to one of the other paths. Our path tracker attempts to prevent such path-crossings without sacrificing too much efficiency by adjusting the prediction step to keep the correction steps sufficiently small.

Results. We chose random, complex path increments p_0 with magnitudes around unity (for good scaling). After computing all 143,360 path endpoints of the continuation, we then eliminated the degenerate solutions. Ideally, such a solution should exactly satisfy one of the degeneracy conditions in equations (19, 20), but the numerically computed endpoints are never exact. This is especially true of singular endpoints, which are often difficult to compute accurately. Thus, the sifting process must be handled carefully.

The first step was to recompute in extended precision any path with a suspicious endpoint. Since singular endpoints are often also degenerate, any endpoint simultaneously having a high condition number ($>10^6$) without satisfying one of the degeneracy conditions to within 10^{-3} fell into this category. So did any endpoint for which $\|f(p_0, z)\| > 10^{-6}$. In total, about 1 percent of paths required this treatment.

Next, we estimated the accuracy of each solution by computing a correction to it using Newton's method. If the Newton correction is small, the solution is accurately computed. Typically, all nonsingular solutions quadratically converge to small final Newton corrections. Among all the accurate endpoints (final Newton correction smaller than 10^{-10}), we eliminated those that were both degenerate and singular (degenerate to $<10^{-5}$ and condition number $>10^{12}$). This left 4327 endpoints. These appeared to be both nondegenerate and nonsingular.

To check that there were not any meaningful solutions hiding among the inaccurate endpoints, those with condition numbers $<10^8$ or with all degeneracy conditions $>10^{-3}$ were refined in extended precision. All of these then became clearly degenerate and singular. Further testing using variations on these criteria convinced us that the list of 4327 endpoints was complete.

Now, nonsingular solutions should be distinct and the Roberts cognates of every nondegenerate solution should be present. Checking for distinctness, we found that 2 of the solutions appeared twice. The corresponding paths were recomputed with a tighter path-tracking tolerance, and one of each pair then went to a degenerate endpoint, as expected. This left 4325 distinct solutions (no two agreed to more than 2 digits). Checking for Roberts cognates and declaring a match between two solutions z_1 and z_2 if, for example, $z_2 = R(z_1)$ to at least 8 digits, we found that the 4325 solutions divided into 1441 groups of cognate triples plus one cognate pair. Obviously, the third cognate of the lonely pair was missed due to a path-crossing, but since the missing solution can be reconstructed by applying the cognate formula $R(z)$ to the pair, we have a complete list of 1442 cognate triples. While some path-crossing has taken place, the occurrence of only one incomplete pair and the absence of incomplete singles in the cognate check indicates that its incidence rate is very low. Therefore, it seems unlikely that all three points of a cognate triple were missed. In conclusion, we feel confident that the problem has 4326 nondegenerate solutions appearing in 1442 cognate triples. Furthermore, all nondegenerate solutions are also nonsingular. We will present further evidence below that this is the complete solution.

The entire computational cost of the numerical reduction was 331.9 hours of CPU time on an IBM 3081. (The IBM 3081 is about 1/3 as fast as an IBM 3090.) Fortunately, that is a one-time only expense, and subsequent solutions of the problem cost only a small fraction as much.

Parameter Continuation

The theory of parameter polynomial continuation (Morgan and Sommese, 1989; Wampler, Morgan, and Sommese, 1990) implies that we can find every nonsingular nondegenerate solution for any nine-point problem, say $f(p_1, z)=0$, by starting

Table 2 Precision points for the example problems

j	Problem 1		Problem 2		Problem 3		Problem 4	
0	0.8961867	-0.09802917	0.00	0.0	0.25	0.00	1.000	0.00000000
1	1.2156535	-1.18749100	0.30	-0.1	0.52	0.10	0.875	0.96824583
2	1.5151435	-0.85449608	0.70	0.0	0.80	0.70	0.750	1.32287565
3	1.6754775	-0.48768058	0.82	0.2	1.20	1.00	0.625	1.56124949
4	1.7138690	-0.30099232	0.90	0.4	1.40	1.30	0.500	1.73205080
5	1.7215236	0.03269953	0.90	0.7	1.10	1.48	0.375	1.85404962
6	1.6642029	0.33241088	0.60	0.7	0.70	1.40	0.250	1.93649167
7	1.4984171	0.74435576	0.10	0.5	0.20	1.00	0.125	1.98431348
8	1.3011834	0.92153806	0.00	0.3	0.02	0.40	0.000	2.00000000

Table 3 Program statistics for the example problems

Problem Number	CPU (min.) IBM 3090	CPU (min.) Transputer*	Function Evaluations	No. Real Mechanisms
1	123	46	2.8×10^6	21
2	74	28	1.8×10^6	45
3	69	25	1.7×10^6	64
4	321	181	9.5×10^6	120

*T800, 48-nodes

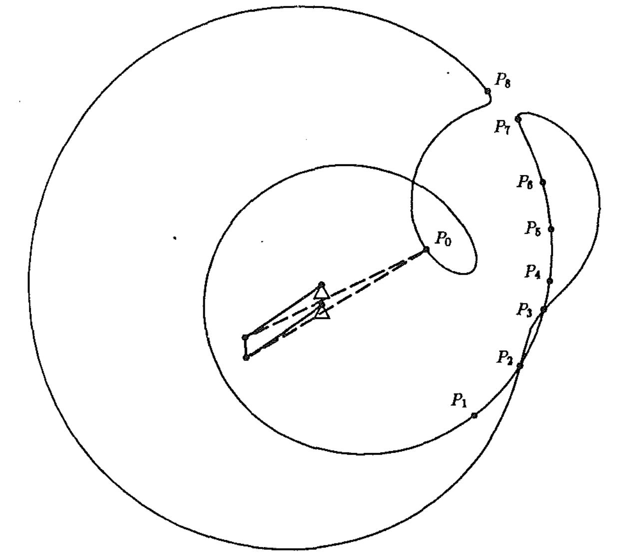

Fig. 3 A new solution to Roth and Freudenstein's problem. (See the discussion of Problem 1.)

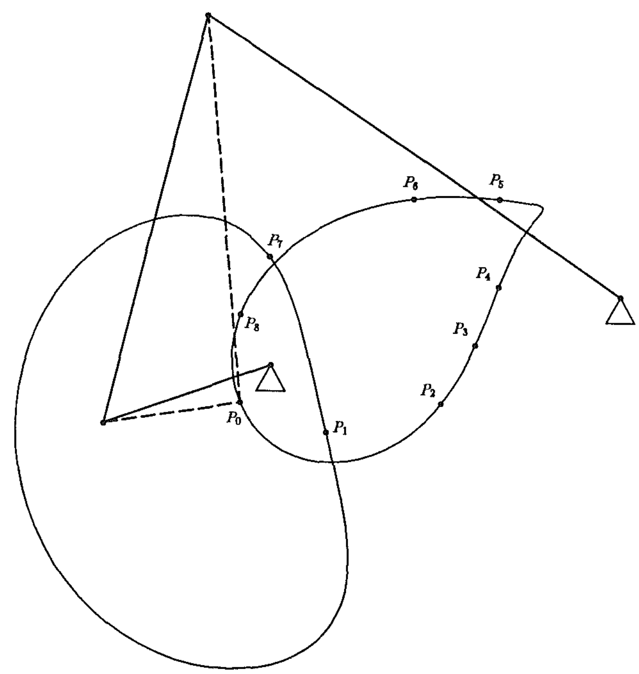

Fig. 4 A solution to Problem 2. All solutions had branch or order errors.

at the nonsingular nondegenerate solutions of $f(p_0, z) = 0$ (p_0 random, complex) and following the continuation paths of

$$f((1-t)p_0 + tp_1, z) = 0 \tag{25}$$

from $t=0$ to $t=1$. Since, as discussed in the previous section, every nondegenerate solution of $f(p_0, z) = 0$ turned out to also be nonsingular, we will by this method find every nondegenerate solution for any set of generally placed precision points. For some special cases, such as Problem 4 below, we will have nondegenerate solutions that lie on positive-dimensional solution sets. The method will then find sample solutions in such sets.

By the Roberts cognate and symmetry relations, if $f(p, z) = 0$ then $f(p, R(z)) = 0$ and $f(p, S(z)) = 0$, that is, the solution paths of equation (25) will appear in groups of 6 according to the cognate and symmetry relations. Therefore, by tracking one of each of the 1442 cognate triples, we will find all the solutions of our target problem. By applying the cognate and symmetry formulas, these 1442 representative solutions yield 4326 distinct mechanisms and 8652 actual solutions. (Recall that z and $S(z)$ give the same mechanism.)

Our parameter continuation program used the same path tracking method as described above. Typically, about 1 percent to 5 percent of the paths presents some numerical difficulty (depending, for example, on the singularity of the endpoint set) in which case the program automatically tries one of the path's cognates. This strategy has proven to be very effective.

Real Problems. Given nine precision points, we compute the 8 path increment vectors δ_j and let $\hat{\delta}_j = \delta_j^*$. Among the solutions we compute, only those with $\hat{x} = x^*$, $\hat{a} = a^*$, etc., are physically meaningful. (This is analogous to choosing only the real solutions in more typical formulations where the coefficients of the polynomial system are real.) Furthermore, by taking the complex conjugate of equations (10–13), one may verify that if $\hat{\delta}_j = \delta_j^*$ then paired with any solution (x, $\hat{x}$, a, $\hat{a}$,...) is another solution ($\hat{x}^*$, x^*, $\hat{a}^*$, a^*,...), which we call its conjugate. Note that physical solutions are self-conjugate in this sense, while all other solutions must appear in conjugate pairs.

We note that our original problem p_0 was nonphysical in that $\hat{\delta}_j \neq \delta_j^*$. Accordingly, its solutions do not appear in conjugate pairs nor do the continuation paths of equation (25), but for a physical problem p_1 the conjugate of each path endpoint should appear. Just as the number of physical (i.e., self-conjugate) solutions changes from problem to problem, the pairing between the nonphysical solutions also changes.

For physical problems, our program checks for the presence of the conjugate of each solution in the fully expanded list of 8652 solutions. After solving several such problems, we have yet to discover a missing conjugate. Since the pairings change from problem to problem, this is further strong evidence that we have the complete nondegenerate solution list for problem p_0.

Examples. The precision points for four example problems are listed in Table 2. The results of our program are summarized in Table 3, which lists the number of physical cognate triples found for each problem, the CPU times for two machines, and the number of function evaluations. This last figure is equal to the total number of prediction and correction steps used in tracking all 1442 paths. The two CPU times are for an IBM 3090 (mainframe) and a 48-node Transputer network (desktop). Because each path is tracked independently, the multiple nodes of the Transputer network operate very efficiently in parallel to exceed the speed of the mainframe.

Our Problem 1 is Example 2 from Roth and Freudenstein (1963). The same problem was studied by Tsai and Lu (1989). Each of these papers presented one linkage for this problem. Our complete solution yielded 21 physical cognate triples (i.e.,

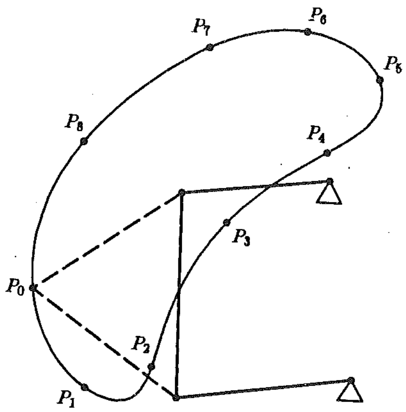

Fig. 5 This mechanism and its cognates are the only viable solutions to Problem 3.

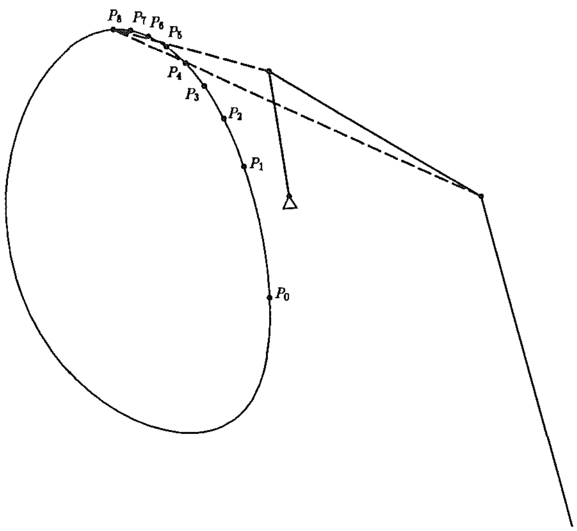

Fig. 6 One of many viable solutions to Problem 4. The long link extending off the page is about 5.5 times longer than the crank.

63 mechanisms), although most of them have branch or order errors. The problem is difficult because eight of the nine precision points lie near a circle. This means that it is nearly degenerate, so it is not surprising that this problem took a greater amount of CPU time: 46 min. As expected, the two previously published solutions were found, along with several others, such as the one shown in Fig. 3, that pass through the precision points in the proper order. However, assuming the intention is a coupler curve with a smooth arc through $P_1, \ldots, P_8$ and a simple return through P_0, none of the mechanisms would be acceptable. If this problem was part of an actual design, some reconsideration of the placement of the precision points would be in order.

Problem 2 has points in an oval shape. It took 28 min and yielded 45 physical cognate triples. In this case, every coupler curve had either a branch or order defect. Figure 4 shows one solution with a branch defect. Having found all solutions, we know that an adjustment of the precision points is necessary. Otherwise, if we were using trial-and-error, we might attempt many trials before giving up, without knowing even then whether the problem was feasible.

Problem 3 has three points on one circle and four on another so that by the addition of a suitably located dyad connected to the coupler point one would have a double-dwell six-bar linkage. This problem required 25 min and gave 64 physical cognate triples. Only the mechanism shown in Fig. 5 (and its cognates) give a simple curve through the points.

Problem 4 has 9 points on the ellipse $x^2 + y^2/4 = 1$ taken at evenly spaced values of x in the first quadrant. Unlike the other three problems, this one generated singular path endpoints, which accounts for the large CPU time: 181 min. Among the singular endpoints were various Cardan (elliptic) mechanisms of both crank-slider and slider-slider types. Neglecting these, we found 120 physical cognate triples, one of which is shown in Fig. 6.

Comparison to Alternative Methods

We wish to briefly contrast our method to two alternatives that have been proposed, both based on continuation. The "bootstrap" method (Roth and Freudenstein, 1963) used real variables. This had the advantage that it would never yield a non-physical complex answer. However, a continuation path between two real solutions to the nine point problem can have segments that branch into the complex domain, and thus the real portions of the path do not connect. Roth and Freudenstein proposed two heuristics to circumvent such difficulties: interchanging which precision point is used as P_0, and redirecting the continuation path according to a quality index. Another aspect of their method is that they used a fixed step in the interpolation parameter t, which makes the procedure prone to path-crossing unless the step is very small. In fact, they reported obtaining different linkages by changing only the step size. Of course, the intent of their effort was only to find some linkages, not all of them, and the scope of the work included geared five-bars, not just the four-bar equivalent case. In this context, the work was very successful.

Tsai and Lu (1989) studied only the four-bar case, as we also have done. Their principal improvement was to work in complex space. To solve a nine point problem, they proposed first selecting a subset of five points, solving a synthesis with prescribed crank angles, and picking four other points on the resulting coupler curves. This gave several starting linkages, all satisfying five of the desired nine precision points. Then, the precision points on the starting coupler curve were given a small random, complex perturbation to move the problem out of the reals. As justified by the "Cheater's Homotopy" (Li, Sauer, and Yorke, 1989), this guaranteed that the continuation path from start to target would be nonsingular and would yield a nonsingular final solution. However, the final solution could be complex and therefore nonphysical. As in the bootstrap method, this method can find several different linkages, but because each starting linkage solves a different nine-point problem, the approach does not provide an organized way to find all solutions. Also, in either method, the final linkage often bears little resemblance to the initial one, and so it may have branch or order errors, bad transmission angle, etc. with no recourse except to try another initial guess. Tsai and Lu used the multiple solutions to their five-point problem, along with their Roberts cognates, to generate new initial guesses.

By finding all solutions, we can pick the one with the most desirable characteristics. Any numerical method for finding all nondegenerate solutions will have to solve the entire problem, including many degenerate solutions, at least once. An alternative to our parameter continuation approach is the closely related Cheater's Homotopy (Li, Sauer, and Yorke, 1989), which could also be used to find all solutions. However, the numerical reduction proposed there is to solve $f(p*, z) + b* = 0$, where both the parameters $p*$ and the additional constants $b*$ are random, complex. Then, for a subsequent problem p_1, use the homotopy

$$f((1-t)p* + tp_1, z) + (1-t)b* = 0, \qquad (26)$$

using as start points only the finite solutions from the original calculation. This approach eliminates only the solutions at infinity and does not eliminate degenerate finite solutions. In the case of the nine-point problem, only about one-third of the degenerate solutions lie at infinity. Also, the additional constants $b*$ destroy the Roberts cognate simplification. Consequently, the end result of the Cheater's Homotopy would be a program that tracked at least 90,000 continuation paths, rather than our 1442.

Summary and Conclusion

We have established that the nine point path synthesis problem for four-bar linkages generically has 4326 distinct nondegenerate linkages occurring in 1442 cognate triples, and we have developed a computer program to find all of them by tracking 1442 continuation paths. This is a reduction by a factor of nearly 4000 from the best total degree (7^8) previously reported for the problem. The reduction is accomplished in two stages: an analytical reduction based on both classical elimination and a 2-homogeneous formulation, and a numerical reduction based on parameter polynomial continuation. The numerical reduction step tracked 143,360 solution paths and established that only 4326 of the endpoints were nondegenerate. Numerical checks for Roberts cognates and complex conjugates provided strong evidence that the solution list is complete.

Of the 1442 solution triples for a set of physical precision points, the number of physically meaningful linkages (i.e., real link lengths) varies. In the examples, this number ranged from 21 to 120 cognate triples.

While the numerical reduction was quite expensive (332 CPU hours, IBM 3081), the problem can now be solved in a manageable time (0.5–3 hours by Transputer, 1–6 CPU hours by IBM 3090). Although these times might be reduced for these particular problems by fine tuning the code's run parameters, we believe it is more representative of the expected behavior not to do so.

The rapid advances in recent years in techniques for polynomial continuation and the fact that mainframe speeds can now be exceeded by a desktop computer assembled from commerically available parallel technology have brought about a great leap in the feasibility and affordability of solving this and other similarly difficult mechanism synthesis problems.

Acknowledgments

We would like to acknowledge the University of Notre Dame for providing computer time on their IBM 3081. The third author would like to thank the National Science Foundation (DMS 87-22330) and the University of Notre Dame for their support.

We would also like to acknowledge the work of Scott Townsend in developing computer codes.

This work represents a cooperative interdisciplinary effort. The order of the names given on this report has no significance. There is no primary author.

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