C. W. Wampler

A. P. Morgan

Mathematics Department,
General Motors Research Laboratories,
Warren, MI 48090

A. J. Sommese
Department of Mathematics,
University of Notre Dame,
Notre Dame, IN 46556

Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics

Many problems in mechanism design and theoretical kinematics can be formulated as systems of polynomial equations. Recent developments in numerical continuation have led to algorithms that compute all solutions to polynomial systems of moderate size. Despite the immediate relevance of these methods, they are unfamiliar to most kinematicians. This paper attempts to bridge that gap by presenting a tutorial on the main ideas of polynomial continuation along with a section surveying advanced techniques. A seven position Burmester problem serves to illustrate the basic material and the inverse position problem for general six-axis manipulators shows the usefulness of the advanced techniques.

1 Introduction

Many kinematic problems lead naturally to systems of polynomial equations. A typical equation in the system might state, for example, that two points of a rigid link must remain a fixed distance apart. Commonly, a polynomial system is solved by providing initial guesses to an iterative numerical method, usually Newton's method. Since many of the start points will diverge and many others will converge to the same solution, it is generally difficult to find more than a few of the solutions. In contrast, recently developed methods in numerical continuation can reliably compute all solutions to polynomial systems to moderate size. When applied to mechanism synthesis, such a method would provide the engineer all choices that meet the motion specifications. In displacement analysis of mechanisms, all possible configurations of the mechanism for a given input displacement would be found.

The purpose of this paper is to familiarize the kinematicians with polynomial continuation and to demonstrate its usefulness on some challenging kinematics problems. First, we will consider Burmester's problem for guidance of a body through five planar positions and a related seven-position spatial problem. After a short tutorial on polynomial theory and the basics of polynomial continuation, these problems are completely solved in a straightforward manner. Then, we present a quick survey of more advanced polynomial continuation techniques and illustrate their applicability to the inverse position problem for general six-revolute manipulators. (This latter problem is equivalent to the displacement analysis of general 7R mechanisms.) We do not endeavor to survey the extensive literature on polynomial continuation that exists in the numerical analysis journals, but rather concentrate on developments which we judge to be most useful for kinematics problems.

The solution of a system of nonlinear equations by numerical continuation is suggested by the idea that small changes in the parameters of the system usually produce small changes in the solutions. Suppose then that we know the solutions to problem A and we want to solve a similar problem B. By tracking solutions of the problem as we slowly change the parameters from those of A to those of B, we will find some solutions to problem B. Thus, a solution by continuation consists of three elements: a "start system" (A) with known solutions, a schedule for changing the parameters from those of the start system to those of the target system (B), and a method for tracking the solutions as the scheduled transformation proceeds. In a computer implementation, the continuation process is carried out in discrete steps, thereby generating a series of closely related problems. Solutions are tracked by using the solutions at the previous step to predict the solutions at the next.

To our knowledge, the first application of numerical continuation to kinematics was the "bootstrap" technique of Roth and Freudenstein (1963a, 1963b), which was applied to the synthesis of geared five-bar mechanisms for path generation. Continuation methods for general nonlinear systems, including the bootstrap technique, track a single solution from the start system to the target system, and the final solution so obtained generally depends on both the start system and the schedule for transforming it into the target system. Moreover, numerous difficulties can arise, such as divergence or bifurcation of a solution path (Allgower and Georg, 1980; Garcia and Zangwill, 1981; Morgan, 1987). Roth and Freudenstein invented several heuristic techniques to avoid nonconvergence and to direct the continuation procedure toward a desirable solution.

Polynomial continuation differs from general numerical continuation in that it exploits special properties of polynomial systems to find *all* solutions, while eliminating problems like divergence and bifurcation. The advanced techniques allow increases in efficiency by eliminating the computation of certain kinds of nonphysical solutions. The first example of the application of polynomial continuation in mechanism problems appears to be the work of Tsai and Morgan (1985) on the inverse position problem for general six-axis robots. This problem has stimulated many of our subsequent improvements

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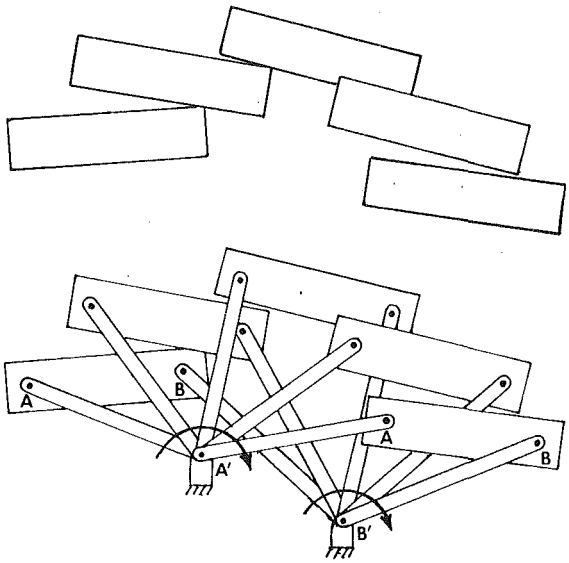

Fig. 1 Planar Burmester problem: Given 5 positions of a body in the plane (*a*), find points of the body that lie on circles. Any two such points, such as points A and B in (*b*), can be used to construct a four-bar passing through the given positions.

in polynomial continuation methods (Morgan and Sommese, 1987a; Morgan and Sommese, 1987b; Wampler and Morgan, 1989) and will be used as an example in this paper.

2 Example Problems

To motivate our discussions of polynomial continuation we first introduce three example problems from kinematics. For any such problem, we wish to do two things: establish the number of solutions to the problem, and develop an algorithm to compute all of the solutions. In some cases, the mathematical formulation of the kinematics problem has solutions that are not physically meaningful. In such cases, we wish to properly adjust the solution count and improve the algorithm to ignore the nonphysical solutions. In this section we define several problems for which solution counts have already been established, usually by difficult and specialized means. In later sections we show how these counts can be found in a more straightforward manner and describe polynomial continuation algorithms for finding all solutions.

2.1 Burmester Problems. Suppose we wish to design a mechanism to carry a rigid body through a series of given positions. We might do so by finding points of the body that lie on a special curve or surface that can be easily implemented in linkwork, such as a sphere, circle, or line. The study of such special points is called Burmester theory, after the 19th century kinematician who first solved the following problem. Suppose we are given a series of positions of a rigid body in the plane. If we can find a point of the body that stays on a circle through each of the positions, then we could pin a rigid link to that point and the center of the circle and still pass through the given positions. If we do the same for a second point, we will have found a four-bar linkage that carries the body through the specified positions (see Fig. 1). Burmester's classical result (circa 1876) showed that for five given positions there are at most 4 such points, any two of which yield a possible four-bar mechanism. The classical method of determining the number of solutions involves an intricate study of intersections between curves of points which fall on circles for a subset of the given positions, and so on. [See, for example, (Bottema and Roth, 1979, pg. 249).] The result can also be derived using changes of variables to reduce the system to two quadric and four linear equations (Bottema and Roth, 1979, pg. 252). In

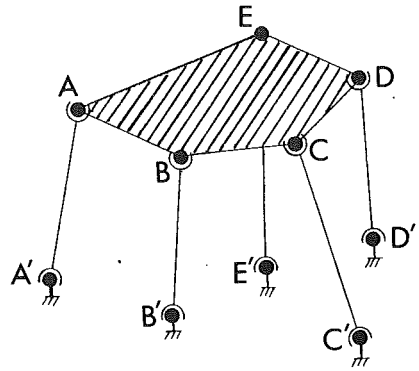

Fig. 2 A spatial 7-bar mechanism that constrains points $A, \ldots, E$ to travel on spheres centered at $A', \ldots, E'$, respectively

Section 5 we will show how elements of modern polynomial theory can give equivalent results in a more direct manner. As we shall see, the same methods apply equally well to a related spatial Burmester problem.

Consider first the planar problem. Let x_j be the position of a point when the body is in position j. Taking the initial position of the body as our reference, subsequent positions are given by a rotation matrix R_j and a translation d_j. Then the position of a point at the jth location of the body is given in terms of its initial position as

$$\mathbf{x}_j = R_j \mathbf{x}_0 + \mathbf{d}_j \tag{1}$$

Suppose that for each of the given positions the point lies on a circle whose center is $\mathbf{p}$ and whose radius is r. Then we have

$$(\mathbf{x}_j - \mathbf{p})^T (\mathbf{x}_j - \mathbf{p}) = r^2, \tag{2}$$

where "T" denotes the transpose operator. Substituting from equation (1) in to equation (2), we have a system of equations in 5 variables: r and two components each in $\mathbf{x}_0$ and $\mathbf{p}$. Thus, barring some unforseen degeneracy, we expect five positions to determine a finite number of solutions. We can eliminate r by subtracting equation (2) for $j=0$ from the remaining four ($j=1, \ldots, 4$) to obtain

$$(\mathbf{x}_j - \mathbf{p})^T (\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_0 - \mathbf{p})^T (\mathbf{x}_0 - \mathbf{p}) = 0 \quad (j = 1, \ldots, 4). \tag{3}$$

Substituting for $\mathbf{x}_j$ from equation (1) and noting that $R_j^T R_j = I$, where I is the identity matrix, we find that terms in $\mathbf{x}_0^2$ and $\mathbf{p}^2$ cancel to give

$$\mathbf{x}_0^T (I - R_j^T)\mathbf{p} + \mathbf{d}_j^T R_j \mathbf{x}_0 - \mathbf{d}_j^T \mathbf{p} + \mathbf{d}_j^2/2 = 0 \quad (j = 1, \ldots, 4). \tag{4}$$

A spatial generalization of the preceding planar problem is to find points of a body that fall on spheres for a series of spatial positions of the body. If five such points can be found, a one-degree-of-freedom 7-bar mechanism can be built that carries the body through the specified positions. The spherical motions are obtained using rigid links with ball-and-socket joints to connect each of the five points to their corresponding centers (see Fig. 2). The spatial problem can be analyzed exactly as its planar equivalent, except now $\mathbf{x}_0$ and $\mathbf{p}$ are points in three-space, with concomitant modifications in the dimensions of R_j and $\mathbf{d}_j$. We have initially seven variables r, $\mathbf{x}_0$, and $\mathbf{p}$, so seven positions will determine isolated points. In 1886, Shoenflies first showed that there are at most 20 points with seven positions on a sphere (Roth, 1967). Both that original proof and the revised one reported by Roth (1967) depend on a detailed examination of curves and surfaces of points that lie on spheres for fewer than seven positions. We will give a much simpler proof of this fact and provide an algorithm for computing all 20 solutions.

2.2 General Six-Revolute Manipulators. The 6-revolute-joint inverse kinematics problem is a good example of how the more advanced homotopies can be applied in a kinematics

context. We wish to find all sets of joint angles that place the end-effector of a general six-revolute joint manipulator at some given position and orientation.

Although the numerical computations of Duffy and Crane (1980) and of Tsai and Morgan (1985) indicated that this problem had at most 16 solutions, Primrose (1986) was the first to prove this conclusively by showing that the 32nd degree Duffy-Crane polynomial always has 16 solutions with pure imaginary parts. Subsequently, Lee and Liang (1988) and Raghavan and Roth (1989) have presented reductions of the problem to a single 16th degree polynomial. We will discuss how advanced polynomial continuation techniques can be used to construct an algorithm to compute the 16 relevant solutions of the Tsai-Morgan formulation.

The Tsai-Morgan formulation, which is described in detail in (Morgan, 1987, Chap. 10), eliminates two of the joint angles, keeping as variables the sines and cosines of the remaining four angles. The resulting system consists of eight second-degree equations in eight unknowns of the form:

$$\begin{aligned} f_l = {} & c_{l,1}x_1x_3 + c_{l,2}x_1x_4 + c_{l,3}x_2x_3 + c_{l,4}x_2x_4 \\ & + c_{l,5}x_5x_7 + c_{l,6}x_5x_8 + c_{l,7}x_6x_7 + c_{l,8}x_6x_8 \\ & + c_{l,9}x_1 + c_{l,10}x_2 + c_{l,11}x_3 + c_{l,12}x_4 \\ & + c_{l,13}x_5 + c_{l,14}x_6 + c_{l,15}x_7 + c_{l,16}x_8 \\ & + c_{l,17} \quad (l = 1, \ldots, 4) \end{aligned}$$

$$f_l = x_{2l-9}^2 + x_{2l-8}^2 - 1 \quad (l = 5, \ldots, 8)$$

However, the coefficients are generated by rather complicated formulas in the physical parameters. These formulas are given in (Morgan, 1987, Table 10-2, with expressions for $R_{3,3}$ and $R_{3,4}$ on p. 253 and for $S_{4,3}$ and $S_{4,4}$ on p. 254).

3 Bezout Numbers

As mentioned above, our first task is to establish the number of solutions to our polynomial formulation of the kinematic problem. We will outline three theorems for this purpose, each one a generalization of its predecessor. Although the first two are widely known to kinematicians, the third, most powerful, one is not commonly known outside the field of algebraic geometry. In reviewing these theorems we will introduce the concepts and terminology that are required for a solid understanding of polynomial continuation. Our key concept is the "Bezout number," named after the 18th century algebraic geometer who did early work in this field.

3.1 Single Polynomials. First, consider a single polynomial equation in one complex variable:

$$\sum_{j=0}^{n} a_j x^j = 0, \tag{5}$$

where a_j are the (constant) coefficients and x is the complex variable. The familiar *fundamental theorem of algebra* states that this equation has exactly n solutions, counting multiplicities as in the example $(x+1)^2(x+2)=0$ for which the solution $x=-1$ counts twice. The definition of multiplicity for a solution of a single polynomial is simply the number of times the solution shows up in the factoring of the polynomial into linear factors. An alternative approach is to define the "Bezout number" of a solution as the number of solutions in its neighborhood for small perturbations in the coefficients of the polynomial. Thus, the solution $x=-1$ in $x^2+2x+1=0$ has Bezout number 2 because $x^2+2x+(1-\epsilon)=0$ has two solutions $x=-1\pm\sqrt{\epsilon}$. For a single polynomial, the Bezout number and the multiplicity of each solution are identical, so the fundamental theorem of algebra can be restated to say that the sum of the Bezout numbers over all the solutions of an nth-degree polynomial is n. We call this sum of Bezout numbers over solutions the *Bezout number of the polynomial.*

3.2 Total Degree of a Polynomial System. The situation for a system of polynomial equations is more complicated. The *degree* of a multivariable polynomial term is the sum of its exponents; for example, the degree of $4x^3yz^2$ is $3+1+2=6$. The degree of a polynomial is the largest degree of any of its terms. The *total degree* of a system of n polynomial equations in n unknowns is the product $\Pi_{j=1}^{n} d_j$, where d_j is the degree of the jth equation. Bezout's theorem (van der Waerden, 1970) states that the total number of solutions of such a polynomial system is in general equal to the total degree of the system. For example, a quadratic (degree 2) and a cubic (degree 3) will in general intersect 6 times. However, to state this result more precisely we must properly account for two phenomena that do not occur for single polynomials: positive-dimensional solution sets and solutions at infinity. These possibilities are familiar to us in the case of linear systems. Two lines in the plane intersect in a single point (total degree $1\cdot1=1$), unless the lines are parallel. If the two lines are coincident, their intersection is the entire line, a positive-dimensional solution set (dimension 1). If the lines are distinct but parallel, there are no finite solutions, but we may consider the lines to intersect at infinity. Note that in either case, any small perturbation of the lines that upsets their parallelism results in a single point of intersection.

To treat the general case, we must first define "nonsingular solutions" and "geometrically isolated solutions." We call a solution to a polynomial system *nonsingular* if the Jacobian matrix of partial derivatives evaluated at the solution is nonsingular. If every solution of a system is nonsingular, then we have a particularly simple case: there are *exactly* the total degree number of solutions, counting solutions at infinity. A solution x^0 *is geometrically isolated* if there are no other solutions arbitrarily close to it. If x^0 is not geometrically isolated then it is contained in a positive-dimensional solution set. If it is nonsingular, then it is geometrically isolated, but the converse is not necessarily true.

Since systems of polynomials do not factor as do polynomials in one variable, extending the definition of multiplicity to systems is difficult. In contrast, our definition of Bezout numbers is easy to generalize. It turns out that if we perturb the system by arbitrarily small random perturbations, all the solutions to the perturbed system will be nonsingular. Further, near any isolated solution x^0 of the original system will be a number of solutions into which x^0 has been *resolved.* This number is the Bezout number of x^0, which will be 1 for each nonsingular solution and 2 or more for each geometrically isolated singular solution. If the system has a positive-dimensional solution set, then the perturbed system will have a finite number of solutions into which this set resolves. We define this number to be the Bezout number of the positive-dimensional solution set. Proving that these Bezout numbers are independent of the perturbation requires algebraic geometry, but the application of the result is straightforward. If we define the Bezout number of a system to be the sum of the Bezout numbers for all of its geometrically isolated solutions and positive-dimensional solution sets, then Bezout's theorem states that the Bezout number for the system equals the total degree, counting solutions at infinity.

An example will help clarify these concepts:

$$(x+1/2)(x+y-1)=0$$
$$(x^2+y^2-1)(x+y-1)=0.$$

Since the two polynomials share the common factor $(x+y-1)$, any point on the line $x+y-1=0$ is a solution. In addition,

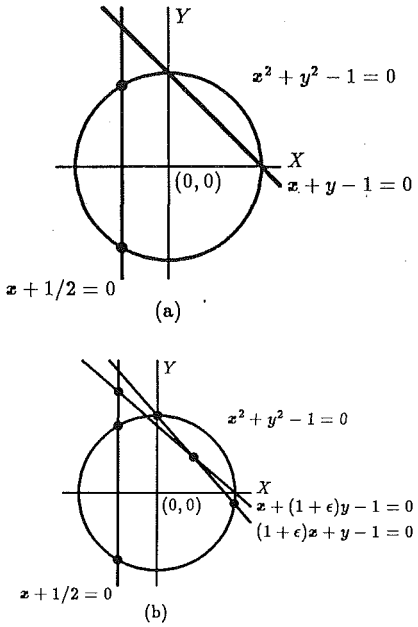

Fig. 3 Upon perturbation, the line of solutions $x+y-1=0$ in (*a*) resolves into four isolated solutions in (*b*)

there are two isolated solutions, $(x, y) = (-1/2, \pm\sqrt{3}/2)$, at the intersections of the line $(x+1/2)$ and the circle (x^2+y^2-1), see Fig. 3(*a*). Since the total degree of the system is 6, the Bezout number of the line must be 4. Indeed, if the equations are perturbed to

$$(x+1/2)((1+\epsilon)x+y-1)=0$$
$$(x^2+y^2-1)(x+(1+\epsilon)y-1)=0,$$

then four new isolated solutions near the original line of solutions appear, as shown in Fig. 3(*b*).

Solutions at infinity can be captured by introducing homogeneous coordinates. If the original n variables are $(x_1, \ldots, x_n)$, we expand the system to $n+1$ variables $(y_0, \ldots, y_n)$ by substituting $x_j = y_j/y_0$ and clearing the denominators by multiplying each equation through by $y_0^{d_j}$, where d_j is the degree of the equation. A finite solution in the original variables is given in terms of a homogeneous solution by dividing $(y_1, \ldots, y_n)$ by y_0. Solutions at infinity are those nonzero solutions with $y_0 = 0$. The simplest example is two distinct, parallel lines such as

$$x_1 + x_2 - 1 = 0$$
$$x_1 + x_2 - 2 = 0,$$

which when homogenized become

$$y_1 + y_2 - y_0 = 0$$
$$y_1 + y_2 - 2y_0 = 0.$$

It is easy to verify that these equations have the solution $(y_0, y_1, y_2) = (0, \alpha, -\alpha)$ for any scalar α. Since $y_0 = 0$, this is a solution at infinity.

In the example just cited, the solution given is actually an entire line of solutions parameterized by α. In fact, it is always true that if $(y_0, \ldots, y_n)$ is a solution to a homogeneous system then so is $(\alpha y_0, \ldots, \alpha y_n)$, where α is any scalar, since the ratios y_j/y_0 do not change. Accordingly, each solution of the original system corresponds to a line through the origin in the homogeneous coordinates, so we consider each such line to be a single solution. Any line through the origin is uniquely determined by a nonzero point on the line, the remainder of the line being obtained by radial projection of the point. For this reason, the homogeneous coordinates are said to represent a "projective space." Now Bezout's theorem can be accurately stated to say that *the Bezout number in complex projective space of a system of polynomial equations is the total degree of the system*. It is easy to see that for one polynomial equation in one complex variable, this Bezout number is the same as given by the fundamental theorem of algebra.

Solutions at infinity not only make the Bezout number exact, but they also can have practical significance as solutions to a mechanical design problem. A straight line in the plane is a special case of a circle of infinite radius with a center at infinity. Thus a problem formulated for rotational joints might yield a center point at infinity that can be implemented in linkwork as a sliding joint.

3.3 Multi-homogeneous Polynomial Systems. A useful generalization of the Bezout theorem applies to polynomial systems with special structures. For example, consider the system (Morgan and Sommese, 1987a, p. 105)

$$x^2 - 1 = 0$$
$$xy - 1 = 0, \tag{6}$$

which is the intersection of two vertical lines, $x = \pm 1$, with a hyperbola. The system has a total degree of 4, but only two finite solutions $(x, y) = (1, 1)$ and $(-1, -1)$. Introducing the homogeneous variable w via the substitutions $x \leftarrow x/w$, $y \leftarrow y/w$, we obtain

$$x^2 - w^2 = 0$$
$$xy - w^2 = 0, \tag{7}$$

which in addition to the original solutions $(w, x, y) = (1, 1, 1)$, $(1, -1, -1)$ has a solution at infinity $(0, 0, 1)$ of multiplicity two. To see that the latter solution has multiplicity two, we may perturb the first equation to $x^2 + \epsilon y^2 - 1 = 0$, thereby changing the two vertical lines into an ellipse with a very long major axis along the y-axis. The ellipse intersects the hyperbola in one point near both the positive and negative y axis, far from the origin.

Now, consider what happens if we introduce *two* homogeneous variables via the substitutions $x \leftarrow x/w_1$, $y \leftarrow y/w_2$ obtaining the system

$$x^2 - w_1^2 = 0$$
$$xy - w_1 w_2 = 0. \tag{8}$$

Disallowing any solution where $(w_1, x) = (0,0)$ or $(w_2, y) = (0, 0)$, one may confirm that the only solutions are the original finite solutions $(w_1, x; w_2, y) = (1, 1; 1, 1)$ and $(1, -1; 1, -1)$. This is because of our different treatment of infinity, due to the introduction of more than one homogeneous variable. (The system is called "2-homogeneous" because there are two homogeneous variables.) Thus, we see that the use of multiple homogeneous variables can sometimes reduce the number of solutions at infinity, which will reduce the computational load when we calculate all solutions of the system.

The theory for multihomogeneous polynomial systems (sometimes shortened to m-homogeneous) is very similar to the theory outlined above for 1-homogeneous systems. The principal difference is a new formula for the Bezout number. The general procedure for multihomogenizing a polynomial system is to first divide the variables into m groups, say $\{x_{11}, \ldots, x_{1k_1}\}$, $\{x_{21}, \ldots, x_{2k_2}\}$, $\ldots$, $\{x_{m1}, \ldots, x_{mk_m}\}$, where k_j is the number of variables in group j. Now, introduce a homogeneous variable for each group, say $y_{01}, \ldots, y_{0m}$, via the substitutions $x_{ij} = y_{ij}/y_{0j}$, and clear the denominators. As before, solutions in the homogeneous coordinates can be multiplied by an arbitrary scalar, but now we can scale each group independently. We will count the entire projection of a solution point as just one solution because any point on the projection yields the same values for the original variables $x_{11}, \ldots, x_{mk_m}$.

We saw in equations (7, 8) that a 2-homogeneous treatment of the problem gave a different number of solutions than the 1-homogeneous treatment due to the elimination of two solutions at infinity. The formula for the Bezout number of a multihomogeneous system is as follows. Note that every term in a multihomogeneous polynomial has the same degree with respect to each of the m groups. Let the multihomogeneous degree of equation l with respect to group j to be d_{jl}: it is computed as the sum of the exponents on variables from group j in any term from polynomial l. Our final version of Bezout's theorem is that *the Bezout number in complex projective space of a multihomogeneous system of polynomial equations is equal to the coefficient of* $\Pi_{j=1}^{m} \alpha_j^{k_j}$ in the product

$$\prod_{l=1}^{n} \left(\sum_{j=1}^{m} d_{jl}\, \alpha_j \right). \tag{9}$$

Applying this formula to equation (8), we find that the coefficient of $\alpha_1\alpha_2$ in $2\alpha_1(\alpha_1+\alpha_2)$ is 2, as expected. It is straightforward to confirm that for a 1-homogeneous system, this formula yields the total degree.

Bezout numbers for geometrically isolated solutions and positive-dimensional solution sets of a multihomogeneous system are defined in a fashion similar to that outlined above for 1-homogeneous systems. The only difference is that now we consider only perturbations that do not change the multihomogeneous degree of the polynomials. To find Bezout numbers in equation (7) we introduce the term ϵy^2 into the first polynomial, which is allowed because the polynomial is quadric in the only group, (w, x, y). However, such a perturbation is not allowed in the 2-homogeneous treatment of the same problem, since the first polynomial in equation (8) has degree 0 in group (w_2, y).

4 Basic Polynomial Continuation

So far we have concentrated on establishing the number of solutions to a system of polynomial equations. This is a prerequisite to finding all the solutions, which is the objective of polynomial continuation. We have already indicated that a general numerical continuation method consists of a start system with known solutions, a schedule for transforming the start system into the target system, and a method for tracking solution paths as the transformation proceeds. In this section we discuss how to carry out each of these steps so that by tracking all the solutions of the start system, we will find all the geometrically isolated solutions of the target system and account for the Bezout count of any positive-dimensional solution sets.

4.1 Start Systems. Our basic methodology places three restrictions on the start system: all of its solutions must be known, each solution must be nonsingular, and the system must have the same multihomogeneous structure as the target system. First, we note that for any 1-homogeneous target system, an acceptable start system is

$$x_j^{d_j} - 1 = 0, \qquad (j = 1, \ldots, n) \tag{10}$$

where d_j is the degree of the jth equation of the target system. Each equation yields d_j distinct solution values for x_j, and the entire set of $\Pi_{j=1}^{n} d_j$ solutions are found by taking all possible combinations of these.

Now suppose we wish to treat the system as multihomogeneous rather than as 1-homogeneous, because it reduces the computational load by eliminating some solutions at infinity as in equations (7, 8). We can always generate a start system that can be completely solved using only a linear equation solver, as follows. Let d_{jl} be the degree of equation l with respect to group j. Then, let the corresponding start equation be a product of factors, $\Pi_{j=1}^{m} f_{jl}(x_{j1}, \ldots, x_{jk_j}) = 0$, where the degree of f_{jl} is d_{jl}. This yields a start system with an identical multihomogeneous structure as the target system. If we choose factors that are sufficiently generic, perhaps by choosing random coefficients, we are assured of having the proper number of nonsingular solutions. Solutions to the system are found by setting one factor from each equation equal to zero. Not all choices of factors set to zero yield solutions. In fact, we must choose exactly k_j distinct factors f_{jl} for each group j. All solutions to all such choices produce the entire solution set, with exactly the number of elements as the Bezout number as given in equation (9). If the factors are all linear, each solution is found by solving m systems of linear equations, i.e., one for each homogeneous group. This approach can be extended to higher-degree factors by forming them as products of linear expressions. This method is illustrated below by the start system for the spatial Burmester problem.

4.2 Homotopies. The schedule for transforming the start system into the target system is called a homotopy. The requirement for an acceptable homotopy is that as the transformation proceeds there should be a constant number of solutions which trace out smooth paths and which are always nonsingular until we reach the target system. It has been shown (Morgan and Sommese, 1987a) that the following homotopy suffices:

$$H(y, t) = (1-t)e^{i\theta}G(y) + tF(y), \tag{11}$$

where $G(y)$ is the start system and $F(y)$ is the target system. Here y represents the new variables after homogenizing the system. Note that at $t=0$, the solutions of $H(y, t)=0$ are those of $G(y)=0$, where at $t=1$ its solutions are those of $F(y)=0$. If F and G have the same multihomogeneous structure, so will a linear combination of them, so the Bezout number of H remains constant as t varies from 0 to 1. The simplest case, which we assume for this discussion, is that $G(y)=0$ has only nonsingular solutions. Then the number of solutions of $H(y, t)=0$ equals the Bezout number until (perhaps) when $t=1$ (Morgan and Sommese, 1987a, Theorem 1). According to this theorem, given F and G, a finite number of θ will yield singular paths. Thus we must not choose θ to be one of this finite but unknown set of "bad" θ. Since there are at most a finite number, we will almost certainly avoid them by picking θ at random. In fact, after several years of using this homotopy for computations, we have yet to encounter a numerical problem that was attributable to an unfortunate choice of θ.

The necessity of the factor $e^{i\theta}$ is most easily demonstrated by the trivial example of a start system which is the negative of the target system, $G(y) = -F(y)$. Then, $(1-t)G(y) + tF(y)$ would vanish at $t=1/2$, while for nonzero θ, $H(y, t)$ would not.

A further property of the homotopy is that as t approaches 1, the number of paths converging to any solution point or positive-dimensional solution set is exactly the Bezout number of that solution or solution set. Thus, every nonsingular solution has exactly one path converging to it. A singular solution with $m_0 > 1$ paths converging to it is either an isolated solution of multiplicity m_0 or it is a point on a positive-dimensional solution set of Bezout number at least m_0. A singular solution with only one path converging to it must be a point on a positive-dimensional solution set. By checking the condition number (Forsythe, Malcolm and Moler, 1977, p. 42) of the Jacobian at each solution point, we can check a computer solution for consistency with the theory. If a computer solution yields the full Bezout number of distinct solutions, then it is a proper run and any singular solution is on a positive-dimensional solution set. If more than one path converges to a nonsingular solution, then an error has occurred. This usually means that two paths passed close to each other at some intermediate value of t and the path tracker jumped paths. This is most easily fixed by re-running the paths in question with tighter tolerances on the path tracking error. An additional

check is that if the coefficients of the final polynomial system are all real, then the geometrically isolated complex solutions should appear in complex conjugate pairs.

4.3 Path Tracking. Path tracking is the process of following the solutions of $H(y, t)=0$ as t is increased from 0 to 1. These solutions form *d continuation paths*, where d is the Bezout number of the system. The first problem we face is that, for a m-homogeneous system, $H(y, t)$ at any given value of t is a system of n equations in $n+m$ unknowns. We have seen earlier that the indeterminancy is in the arbitrary scaling factors associated with homogeneous variables. How shall we resolve the indeterminancy so that the solutions have unique representations? A convenient method due to Morgan (1986) [see also (Morgan and Sommese, 1987b)] is to introduce an inhomogeneous linear equation for each homogeneous group, i.e.,

$$c_{j0}\, y_{j0} + \ldots + c_{jkj}\, y_{jkj} - 1 = 0 \qquad (j=1, \ldots, m), \quad (12)$$

where y_{j0} is the homogeneous variable for group j, and the coefficients c are random. Since a homogeneous solution is of the form $y=\alpha v$ $(v\neq 0)$, the additional linear equation determines a unique scaling factor $\alpha=1/(c^T v)$, unless $c^T v=0$. (We temporarily drop subscripts and use a vector formulation for simplicity.) Since the coefficients are chosen randomly the degenerate condition $c^T v=0$ happens with zero probability (Morgan and Sommese, 1987b, Theorems 2 and 3). Again, our computational experience bears this out.

The new linear equations can be used to eliminate the homogeneous variables from $H(y, t)$, yielding n equations in n unknowns and t. After finding a solution at $t=1$, we can evaluate the homogeneous variables from the linear equations and find the solutions for the original unhomogenized problem by division. For simplicity, let us continue to use $H(y, t)$ to denote the system after the elimination of the homogeneous variables.

To track a path from a known solution (y^0, t^0), we first predict the solution for $t=t^0+\Delta t$ and then correct the prediction using Newton's method with t fixed. Specifically, for small Δy and Δt, the Taylor expansion for H gives

$$H(y+\Delta y, t+\Delta t) \approx H(y, t) + J_y \Delta y + J_t \Delta t, \quad (13)$$

where J_y and J_t are the Jacobians of H with respect to y and t. For the prediction step, we have $H(y^0, t^0)=0$, so setting $H(y^0+\Delta y, t^0+\Delta t)=0$ gives

$$\Delta y = -J_y^{-1} J_t \Delta t. \quad (14)$$

Since this is only approximate, we correct the solution at the new value of t by setting $\Delta t=0$ to give a correction step of

$$\Delta y = -J_y^{-1} H(y, t). \quad (15)$$

The correction step may be repeated several times before taking the next prediction step. A robust and efficient path tracker can be obtained by halving the time step whenever too many correction iterations are required to stay within a given tolerance of the continuation path and doubling it when the prediction step has been sufficiently accurate several consecutive times. More precise recommendations concerning these and other path tracking subleties can be found in (Allgower, 1981; Garcia and Zangwill, 1981; Morgan, 1987; Watson, Billups and Morgan, 1987).

5 Solution of the Burmester Problems

With the methods of the last two sections we can solve the planar and spatial Burmester problems. Looking first at the planar case, we see that equations (1) are quadric $(j=0, \ldots, 4)$, so there are at most $2^5=32$ isolated solutions. After eliminating r, equations (2) are four quadrics, so the total degree is reduced to 16. However, these equations are linear in both $\mathbf{x}_0$ and $\mathbf{p}$, so it is beneficial to treat the system as 2-homogeneous. Introducing homogeneous variables w and q associated with $\mathbf{x}_0$ and $\mathbf{p}$, respectively, we get

$$\mathbf{x}_0^T(I-R_j^T)\mathbf{p} + \mathbf{d}_j^T R_j \mathbf{x}_0 q - \mathbf{d}_j^T \mathbf{p} w + wq\mathbf{d}_j^2/2 = 0 \quad (i=1, \ldots, 4). \quad (16)$$

Each of the four equations has degree 1 in each group, and each group originally had 2 variables, so the 2-homogeneous Bezout number is the coefficient of $\alpha_1^2\alpha_2^2$ in $(\alpha_1+\alpha_2)^4$, which is 6. There is still a discrepancy of 2 between this and the classical result of 4, but the difference can be resolved by looking for solutions at infinity. It turns out that since a 2×2 rotation matrix always has the form

$$R_j = \begin{pmatrix} c_j & -s_j \\ s_j & c_j \end{pmatrix}, \quad (17)$$

the system always has the pair of complex conjugate solutions $(w, x_{01}, x_{02}; q, p_1, p_2) = (0, 1, i; 0, 1, i)$ and $(0, 1, -i, 0, 1, -i)$, independent of the actual positions R_i, $\mathbf{d}_j$. Therefore, there can be at most 4 isolated real solutions, the so-called Burmester points. We remark that if one had not noticed the constant pair of solutions at infinity, it would become apparent after solving the problem using polynomial continuation.

For our spatial Burmester problem, recall that the number of solutions is known to be 20. We can verify this by computing the 2-homogeneous Bezout number for equation (16), noting that now there are 3 variables in each group and the equations run $j=1, \ldots, 6$. Accordingly, the proper count is the coefficient of $\alpha_1^3\alpha_2^3$ in $(\alpha_1+\alpha_2)^6$, which is 20.

We have developed a computer program to solve the spatial Burmester problem. The first step is to construct a start system, which we do in accordance with the procedure outlined in Sec. 4.1. Since each equation in (16) is linear in both $\mathbf{x}=(x, y, z)$ and $\mathbf{p}=(u, v, w)$ the following system has the proper 2-homogeneous structure:

$$\begin{gathered} xu=0 \\ yv=0 \\ zw=0 \\ (x+2y+3z-6)(u+2v+3w-6)=0 \\ (3x+y+2z-6)(3u+v+2w-6)=0 \\ (2x+3y+z-6)(2u+3v+w-6)=0 \end{gathered} \quad (18)$$

We choose three equations to solve for x, y, z, and the remaining three to solve for u, v, w, and we do this in every possible way. Each time, we solve a set of linear equations with a unique solution. The number of combinations of 6 things taken 3 at a time is 20, as expected. Ten solutions are listed in Table 1; the other ten are given by interchanging the values of (x, y, z) with those of (u, v, w).

Now, using the homotopy and path tracker as described in the previous section, we proceed to compute the 20 solutions. Roth (1967) published a sample problem of this type and listed five real solutions. Using the same data, we found 20 distinct solutions to the polynomial system: 7 complex conjugate pairs, Roth's five real solutions, and one additional real solution at $\mathbf{x}=(4.343, 5.117, 32.00)$, $\mathbf{p}=(-3.084, -2.014, 0.0396)$. Since we may choose any 5 of the 6 real solutions to form the legs of the 7-bar mechanism, there are 6 such mechanisms that pass through the seven specified positions. The entire solution set is listed in Table 2. Since all 20 solutions were finite and distinct, it is conceivable that for some set of seven positions there could be 20 distinct real solutions. This problem is solved on an IBM 370-3090 in double-precision FORTRAN in an average of less than 5 sec of CPU time.

Table 1 Solutions to start system, equations (13)

No.	Equations	(x,y,z)	Equations	(u,v,w)
1	1,2,3	(0,0,0)	4,5,6	(1,1,1)
2	1,2,4	(0,0,2)	3,5,6	(12/7,6/7,0)
3	2,3,5	(2,0,0)	1,4,6	(0,12/7,6/7)
4	1,3,6	(0,2,0)	2,4,5	(6/7,0,12/7)
5	1,2,5	(0,0,3)	3,4,6	(−6,6,0)
6	2,3,6	(3,0,0)	1,4,5	(0,−6,6)
7	1,3,4	(0,3,0)	2,5,6	(6,0,−6)
8	1,2,6	(0,0,6)	3,4,5	(6/5,12/5,0)
9	2,3,4	(6,0,0)	1,5,6	(0,6/5,12/5)
10	1,3,5	(0,6,0)	2,4,6	(12/5,0,6/5)

Table 2 Solutions to Roth's spatial Burmester problem

No.	x			p		
1	−3.416	0.421	−4.673	−0.846	0.198	−8.313
2	0.214	−0.446	−2.898	−1.018	1.097	−2.427
3	0.059	1.026	0.298	0.042	0.008	0.256
4	4.343	5.117	32.00	−3.084	−2.104	0.040
5	−0.002	1.000	−0.006	−0.002	−0.001	−0.006
6	−0.494	0.591	−5.740	−4.213	0.970	−5.252
8	$8.272 \pm 8.529i$	$9.880 \mp 8.388i$	$-3.720 \mp 3.523i$	$2.505 \pm 6.229i$	$7.641 \mp 2.367i$	$-1.393 \mp 2.964i$
10	$0.584 \pm 0.327i$	$1.294 \mp 1.074i$	$-1.282 \pm 0.433i$	$0.073 \pm 0.852i$	$1.266 \mp 0.025i$	$-0.434 \pm 0.537i$
12	$0.311 \pm 0.097i$	$0.921 \pm 0.058i$	$0.135 \pm 0.713i$	$0.400 \pm 0.112i$	$0.305 \mp 0.030i$	$0.423 \pm 0.623i$
14	$0.692 \pm 0.648i$	$1.546 \mp 0.849i$	$0.077 \pm 0.176i$	$0.438 \pm 1.311i$	$1.475 \pm 0.241i$	$1.234 \pm 0.531i$
16	$-0.211 \mp 1.150i$	$0.220 \pm 0.359i$	$-1.034 \mp 3.075i$	$0.039 \mp 1.588i$	$-0.088 \pm 0.150i$	$-0.610 \mp 2.796i$
18	$2.749 \pm 1.527i$	$1.766 \mp 2.603i$	$0.214 \pm 3.506i$	$0.475 \pm 51.18i$	$35.77 \pm 2.290i$	$19.90 \pm 33.21i$
20	$0.391 \pm 0.310i$	$0.315 \pm 0.641i$	$-1.759 \mp 1.945i$	$1.000 \mp 1.238i$	$1.133 \pm 1.217i$	$-0.662 \mp 1.877i$

6 Advanced Polynomial Continuation

Often, we can increase the speed of continuation by acknowledging more of the system's structure than just the multihomogeneous Bezout number. The price we must pay for this extra efficiency is a more complicated framework and theory, but in many cases the practical benefits are great. In this section we will outline what is involved. The details are given in Morgan and Sommese (1989).

We have already seen how a multihomogeneous formulation can increase the efficiency of continuation by eliminating the computation of some solutions at infinity. For the planar Burmester problem, equation (4), we saw how a 2-homogeneous formulation lowered the Bezout number from total degree 16 to only 6. This is because the multihomogeneous theory accounts for simplifications due to the absence of certain terms from the polynomial system; in the Burmester example we have quadric equations, but terms in $\mathbf{x}^2$ or $\mathbf{p}^2$ are absent. However, there may exist further simplifications that cannot be captured by a multihomogeneous treatment alone. The two complex solutions at infinity for the Burmester problem, which depend on a special relationship between the coefficients of the system, are an example. One way to eliminate such solutions is to find an appropriate change of variables, as was done by Bottema (Bottema and Roth, 1979, p. 252) in reducing the system to one with total degree 4. For more complicated problems, such an approach becomes infeasible. Fortunately, by modifying our continuation method, we can reveal and exploit much of the special structure of a polynomial system using purely numerical means. This leads not only to more efficient computations, but also to greater understanding, much like the insight classically obtained by reducing a system by hand to a simpler algebraic form.

To apply the basic 2-homogeneous continuation of Section 4 to the planar Burmester problem, we would begin with a start system that has 6 nonsingular solutions unrelated to the target system. Since there is no way to tell *a priori* which start solutions will lead to the unknown target solutions and which will lead to the known solutions at infinity, we must track all six. Suppose that we could arrange a start system and a homotopy that had the same solutions at infinity for any value of t. Then, we could ignore the unwanted solutions from the start and track only the remaining four. Such a homotopy occurs if instead of varying the coefficients of the system with t, we start with a solved Burmester problem and vary the physical parameters, R_j, $\mathbf{d}_j$, with t. This is the essence of the homotopies presented below. Using these methods, we will not only be able to eliminate some solutions at infinity but also certain types of singular solutions. We will also be able to eliminate extraneous solutions which are sometimes generated in reducing a polynomial system. At the same time, we will do this using a completely rigorous theory, avoiding the "hit-or-miss" uncertainty associated with heuristic versions of this general approach. Before describing the methodology for eliminating solution paths, let us introduce several closely related homotopy formulations.

6.1 Modified Homotopies. In this section we describe a series of homotopies, each one more intimately related to the target system than its predecessor. Accordingly, each successive method tends to require fewer solution paths to be tracked, although this efficiency is purchased at the cost of greater programming difficulty.

It will be helpful to establish some notation. Let $F(c, x)$ denote the system we want to solve, where c denotes the coefficients and x denotes the variables. Note that a polynomial system is linear in its coefficients, so for any scalars λ, μ and coefficients c^0, c^1 we have

$$\lambda F(c^0, x) + \mu F(c^1, x) = F(\lambda c^0 + \mu c^1, x). \tag{19}$$

Suppose that there is a particular set of coefficients, c^1, giving a system $F(x) = F(c^1, x) = 0$ that we want to solve. Further, assume that c^0 generates the start system, $G(x) = F(c^0, x)$. Now with $\gamma = e^{i\theta}$, the homotopy of equation (11) can be rewritten as

$$H(x,t) = (1-t)\gamma G(x) + tF(x) = F((1-t)\gamma c^0 + tc^1, x). \tag{20}$$

We call any homotopy that has the form of the right hand side of equation (20) a "coefficient homotopy." Recall that in the previous section we required the system to have all nonsingular solutions at $t=0$. We call this the "traditional coefficient homotopy." With it we can exploit the multihomogeneous structure of $F(c, x)$.

In many cases, the target system we want to solve is missing terms whose presence would not alter its multihomogeneous structure. Suppose $F(c, x)$ represents the most general polynomial system with the same multihomogeneous structure as the target system. Then the coefficient list c^1 for the target system $F(c^1, x)$ has some zero entries. There may also be entries in c^1 that are the same for every target system that we want to solve. Let us pick a start system $F(c^0, x)$ with coefficients c^0 that match all the zero and constant entries in c^1, but with the remaining entries chosen at random. We call this our "special coefficient homotopy."

Now we generalize by supposing that there are parameters, $q = (q_1, \ldots, q_s)$ so that $c = c(q)$. That is, the coefficients are given by formulas in the parameters. Further, c can be a very general (complex analytic) function of the q. Trigonometric, exponential, and logarithmic functions are all acceptable. ($F(x)$ must be polynomial, but $c(q)$ can be more general.) Our next homotopy, called the "secant homotopy," is given by

$$H(x, t) = \mathrm{F}((1-t)\gamma c(q^0) + tc(q^1), x), \tag{21}$$

where q^0 and q^1 are parameters of the start and the target systems, respectively. For this homotopy, we must take special measures to verify that the start system is acceptable, as discussed below.

Our final method is called the "parameter homotopy," because t interpolates directly between the parameters q^0, q^1 of the start system and the target system, as follows:

$$H(x,t) = F(c[(1-t)q^0 + tq^1], x). \tag{22}$$

Note that the randomizing factor γ, which serves to avoid

singularities in the preceding homotopies, has been omitted. Accordingly, we must pick the start parameters q^0 as random complex numbers.

6.2 Generic Systems and Side Conditions. It is helpful to have a concept of a "generic system." We are considering a universe of systems of polynomials indexed by some set of parameters: coefficients for homotopies like (20), parameters q for (22). For simplicity, let us suppose that all the systems $f(c[q],x)=0$ under consideration are given by $q\epsilon Q$, where Q is the space of all parameters. (As a special case, q could be the coefficients themselves.) Then a "generic system" is one given by a random choice of $q\epsilon Q$. Now, suppose that $\alpha(q,x)=0$ is another system of equations indexed by $q\epsilon Q$; we call such a system the "side conditions" for the problem. If the side conditions were unrelated to $f(c[q],x)=0$, there would generally be no solutions that satisfy both systems. However, there are several common cases where we expect solutions that satisfy side conditions. For these cases, we have the following important fact: if some isolated solutions to a generic system $f(c[q^0],x)=0$ satisfy $\alpha(q^0, x)=0$, then all generic systems $f(c[q],x)$ will have exactly the same number of isolated solutions satisfying $\alpha(q, x)=0$ (Morgan and Sommese, 1989). Further, if we arrange our homotopies so that $H(x, t)$ is generic for $0\le t<1$ (by choosing appropriate random parameters), then a start point that satisfies the additional equations will yield a path that also satisfies them for $0\le t<1$. By sorting the start points with respect to whether they satisfy the side conditions or not and tracking only paths that have start points with desired properties, we will be able to reduce the amount of computation.

Two examples of side conditions that arise frequently are singular solutions and solutions at infinity. Singular solutions are those solutions that satisfy

$$\det J_x(q, x)=0, \tag{23}$$

where J_x is the Jacobian of $f(c[q], x)$ with respect to x. Also, if $x_{01}, \ldots, x_{0m}$ are the homogeneous variables of the system, solutions at infinity are given by the side condition

$$\prod_{j=1}^{m} x_{0j}=0. \tag{24}$$

Since any singular start point must remain singular, we can eliminate such start points when we are only interested in nonsingular solutions. The same logic applies to eliminating start points at infinity when we are interested only in finite solutions. In general, if we are uninterested in solutions obeying a particular generic property, we may omit start points that have this property.

Following this strategy, we may track some nonsingular (finite) solution paths that become singular (infinite) at $t=1$. Such solutions may nonetheless be physically meaningful: for example, we have already mentioned that a planar rotational joint at infinity is equivalent to sliding joint. We define the *generically nonsingular solutions* of a system to be solutions that are limits of nonsingular solutions of nearby generic systems. These are exactly the solutions that are found by tracking paths from all the nonsingular start points. A well-defined family of kinematics models has the property that, for a *generic* (random) choice from the family, all physically meaningful solutions are nonsingular. It then follows that for *any* choice of model, the physical solutions are *generically* nonsingular; that is, they are either nonsingular or the limiting case of a nonsingular solution. Thus, it makes sense to choose only the nonsingular solutions to the generic start system as start points for the homotopy paths, since all the generically nonsingular solutions will then be found. A similar argument can be made for generically finite solutions.

There is one other way in which side conditions commonly arise. This is the reduction of a system to eliminate one or more variables. By eliminating variables we not only tend to decrease the computational burden, but often we also decrease the degree of the system, as happened in eliminating r from the Burmester problem [cf. equations (2, 3)]. However, sometimes the reduced system has additional *extraneous* solutions that do not satisfy the original system. In such a case, we can use the original system as a side condition when solving the reduced system; that is, we can ignore any start points of the reduced system that are extraneous. Note that it is possible for a generically extraneous solution to become an actual solution at $t=1$. However, such a solution is unstable in that almost any perturbation in the parameters of the problem make the solution extraneous again. Accordingly, we can safely disregard the extraneous start points.

6.3 Path Reduction. To summarize the previous section, the solutions to a generic system can be divided according to whether or not they satisfy given side conditions. Using these as start points in a homotopy that remains generic for $0\le t<1$, we can divide the solutions of the target system between those that always satisfy the side condition and those that generically do not. In a physical problem, we are normally only interested in generically nonsingular finite solutions that satisfy the original unreduced system of equations. If so, we can speed up the computation by only tracking the corresponding start points. Theoretically, one could track the complement of any of these, such as generically singular solutions, even though they are generally meaningless to a physical problem. (Singular solution paths are also difficult to track numerically.) Accordingly, consider the following very general methodology.

This procedure, which applies to all four homotopies described above, consists of three steps:

Step 1. Solve the start system (perhaps by using a traditional coefficient homotopy) and sort the solutions by whether or not they are singular versus nonsingular, finite versus infinite, and whether they satisfy other side conditions. Select the subset S of solutions that have the properties desired in the solutions to be computed for the target system, such as the set of all nonsingular finite solutions. If the structure of the solution set is known to be that of a generic system in the homotopy (see Note 3 below), proceed to Step 3.

Step 2. Establish the generic structure of the solution set for the homotopy. This may be done by choosing at random γ, t, c^0, c^1, q^0, q^1, as applicable, and solving the resulting system, usually by using the traditional coefficient homotopy. If the solution set has the same structure as the start system of Step 1, proceed to Step 3. If not, pick a new start system and return to Step 1, or choose a different homotopy.

Step 3. For each start point $x^0\epsilon S$, track the path beginning at x^0 to find a solution to the target system.

Notes:

(*1*) If we want to solve just the one system $F(c(q^1), x)=0$, then a modified homotopy will not (generally) be efficient. Steps 1 and 2 will often be as computationally expensive as solving the system we are interested in. But it is rarely the case that we want to solve just one system of a particular form. It is more common that we want to solve many systems, all of the same form but with different choices of parameters. Then, Steps 1 and 2 are performed just once, and we proceed directly to Step 3 for each target system.

(*2*) In Step 1 we reduce the number of start points that are used in Step 3. This means that we can track fewer paths than were used to solve the start system. In some cases this is a significant number.

(*3*) Step 2 is usually necessary only for the secant homotopy.

Table 3 Summary: Number of paths vs. homotopy for the example problems

Homotopy	Planar Burmester	6-Revolute Manipulator
1-Homogeneous Traditional	16	256
2-Homogeneous Traditional	6	96
2-Homogeneous Special	6	64
2-Homogeneous Secant	4	64
2-Homogeneous Parameter	4	32
2-Homogeneous Parameter with side conditions	N.A.	16

In the traditional coefficient homotopy we require that the start system have all nonsingular finite solutions, which is as generic as possible and does not allow path reduction. In the special coefficient homotopy and the parameter homotopy, the random selection of c^0 and q^0 cause the start system to be generic for its homotopy. If the reader invents a new homotopy, or selects a non-generic start system for the traditional homotopy, Step 2 must be considered.

The proofs that the procedure above is valid are given in (Morgan and Sommese, 1989), along with other technical details.

Let us briefly review how the different homotopies affect the start system and the number of solution paths for the planar Burmester example (see Table 3). The 1-homogeneous traditional homotopy uses a start system as in equation (10) with degrees $d_j = 2$ ($j = 1, \ldots, 4$), which gives 16 nonsingular, finite start points, all of which must be tracked. The 2-homogeneous traditional homotopy uses a start system similar to equation (18) that has 6 nonsingular, finite solutions, all of which must be tracked. The 2-homogeneous special homotopy uses a start system of the form

$$\mathbf{x}_0^T E_j \mathbf{p} = f_j^T \mathbf{x}_0 + g_j^T \mathbf{p} + h_j = 0, \quad (j = 1, \ldots, 4), \tag{25}$$

where the coefficients E_j, f_j, g_j, h_j are randomly chosen matrices of the appropriate dimension. It turns out that such a start system has 6 nonsingular, finite solutions, so for this problem the special coefficient homotopy offers no improvement over the traditional 2-homogeneous homotopy.

The start system for the secant homotopy is of the same form as the target system, equation (4). Accordingly, the two fixed solutions at infinity will appear in the solution list for both the start system and the target system. However, before we can eliminate these solutions as start points, we must establish that they will be solutions for all intermediate values of the homotopy parameter t. This is easily done by applying the linearity property, equation (19), to the secant homotopy, equation (21), to get

$$H(x,t) = (1-t)\gamma F(c(q^0),x) + tF(c(q^1),x). \tag{26}$$

Since $F(c(q^0),x)$ and $F(c(q^1),x)$ are both zero for the two fixed solutions, $H(x,t)$ will also be zero, independent of t. Accordingly, with the secant homotopy we need only track the 4 finite solutions.

Since the secant homotopy has already eliminated the two spurious solutions, there is no need to consider a parameter homotopy. Nevertheless, the start system for a parameter homotopy would again be of the form of equation (4), with R_j, d_j chosen as random and complex with R_j satisfying the algebraic conditions for a rotation matrix $R^T R = \mathrm{I}$. At intermediate values of t, we would interpolate the parameters, for example,

$$\mathbf{d}_j(t) = (1-t)\mathbf{d}_j^0 + t\mathbf{d}_j^1, \tag{27}$$

where the superscripts indicate the start and target values of $\mathbf{d}_j$. Coefficients such as $\mathbf{d}_j^T(t)\mathbf{d}_j(t)/2$ would be evaluated from these interpolated parameters. Note that this gives different intermediate systems than direct interpolation of the coefficients as in the secant method.

7 Solution of the 6-Revolute Manipulator Problem

Let us review both the basic and the advanced polynomial continuation techniques by discussing some of the options for solving the general six-axis manipulator inverse position problem. Over the years, the attack on this problem has motivated many of the developments in polynomial continuation that have been discussed in this report. Table 3 summarizes the effect of the various homotopies. It was first solved using a traditional 1-homogeneous homotopy, which has $2^8 = 256$ solution paths (Tsai and Morgan, 1985). This was followed by a 2-homogeneous traditional homotopy (Morgan and Sommese, 1987a; Morgan and Sommese, 1987b), which reduced the number of paths to 96. Solving a random example with this homotopy showed that there are only 32 finite solutions, and of these, only 16 are actual solutions to the unreduced problem. Accordingly, much improvement was expected from the modified homotopies of the previous section. In fact, a 2-homogeneous special coefficient homotopy, which has 64 nonsingular paths, proved to be very numerically stable (Morgan and Sommese, 1987b). Although the secant homotopy gives no further improvement, the parameter homotopy has only 32 paths [see also (Li, Sauer and Yorke, 1988) on this]. Of these, we can use side conditions to eliminate 16 paths. The 2-homogeneous special coefficient homotopy has been our "workhorse" code, and consequently it has been extensively tested. Its reliability is excellent, and its speed acceptable for off-line analysis—averaging less than 20 sec of CPU time on an IBM 370-3090 in double-precision FORTRAN. Using a different polynomial formulation of this problem, we have developed a 16-path parameter continuation code that averages less than 10 sec on the same machine (Wampler and Morgan, 1989).

8 Summary and Conclusions

Polynomial continuation is a powerful numerical technique, applicable to a broad range of kinematics problems. By computing the full solution set, it provides the engineer with all possible design alternatives that meet the kinematic specifications of the problem. The technique can also provide insight into the mathematical structure of the problem, for example, by determining the number of generically nonsingular solutions. These benefits have been illustrated by using polynomial continuation to solve 2- and 3-dimensional Burmester problems and the inverse kinematics problem for 6-revolute-joint manipulators.

Although polynomial continuation tends to be computationally expensive, the path reductions possible with a multi-homogeneous formulation or the more advanced homotopies discussed in this paper mitigate this. The computation times reported here are short enough for interactive mechanism design and off-line robot programming. Of course, a less powerful computer will require more time, but to compensate for this, one could use a separate processor to independently track each continuation path. Hence, the applicability of the method will be extended by the growing development of parallel computing.

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