BERNARD ROTH
Assistant Professor, Department of Mechanical Engineering, Stanford University, Stanford, Calif.
Assoc. Mem. ASME

FERDINAND FREUDENSTEIN
Professor, Department of Mechanical Engineering, Columbia University, New York, N. Y. Mem. ASME

Synthesis of Path-Generating Mechanisms by Numerical Methods[1]

Algebraic methods in kinematic synthesis are extended to cases in which the development of iterative numerical procedures are required. Algorithms are developed for the numerical solution of nonlinear, simultaneous, algebraic equations. Convergence is obtained without the need for a "good" initial approximation.

The theory is applied to the nine-point path synthesis of geared five-bar motion, in terms of which four-bar motion may be considered as a special case.

Introduction

The approximate synthesis of a given path by use of hinged mechanisms has been studied extensively in connection with four-bar mechanisms. Analytical [1][2] and graphical [2] solutions have been obtained for the problem specified in terms of five precision points and four crank angles; however, problems specified in terms of nine points (and no angles) have not been previously solved. Two published *formulations* of the nine-point path-synthesis problem are known to the authors [2, 3]. Both are for the four-bar mechanism; however, in the first no attempt is made to solve the equations, and in the second the suggested method of solution seems incomplete.

In this investigation we consider *geared five-bar mechanisms*, Fig. 1. Since they can generate a large variety of coupler curves [4, 5, 6], these linkages can be used for the solution of varied and complex design problems [7]. Their analysis is more involved than that of four-bar mechanisms, which can be considered as a special case of the geared five-bar—both mechanisms have equivalent coupler curves when the gear ratio is plus one [8, 9, 10, 11]. Previous geared five-bar path syntheses consist of a graphical-design procedure based on the two-degree-of-freedom property of the five-bar "loop" [12], and two analytical formulations of the prescribed crank-rotation problem [13, 19].

Four-bar linkages have (single) coupler links whose both hinge points describe a circular path. In contrast, five-bar linkages have two "floating" coupler links, where only one of the two hinge points (on each link) describes a circular path. Therefore, if the four-bars are called "double circle point" mechanisms, the geared five-bar mechanisms are "single circle point" mechanisms. The present work can be interpreted as a generalization of Burmester's theory to the (single circle point) geared five-bar group: By deriving and solving the equations of synthesis we obtain the center and circle points for the *coupled motion of two planes*, in this case determined by eight path-increment vectors and no crank angles.

The geared five-bar problem possesses a degree of complexity which renders closed-form algebraic solutions unattainable. Solutions closed-form have been developed for path-angle syntheses with the aid of complex-number and matrix concepts [1]. In the present investigation, the extension of these concepts is based on two procedures. The first is the transformation of the equations of synthesis into an appropriate form. The "unwanted" parameters are eliminated at the start and the closure equations reduced to one (nonlinear) equation per precision condition [3]. Secondly, mathematical methods were developed in order to obtain convergence of the numerical iterations used in solving these equations. These mathematical methods, which are included in a digital computer program, contain the following new features:

1 The "bootstrap" procedure—this essentially eliminates the need for a "good" initial approximation.

2 The "position interchange" procedure—this reformulates the problem in order to eliminate the cause of nonconvergence.

3 The "quality-index-control" procedure—this assures convergence to solutions characterized by a reasonable ratio of maximum to minimum link length.

The Theory of Path Synthesis

Definition. Dimensional kinematic synthesis is the procedure of determining the dimensions of a mechanism from the required motion. When the synthesis is phrased in terms of generating a given curve, the procedure is called path synthesis.

Usually one does not attempt to generate the given curve exactly. In fact, only a limited class of motions could be so generated [18, 22], and in general it suffices if within a desired interval the generated curve is a good approximation to the given one. In this paper the approximate path-synthesis problem is formulated by specifying the location of the *precision points* (points at which the given curve and the generated path coincide);

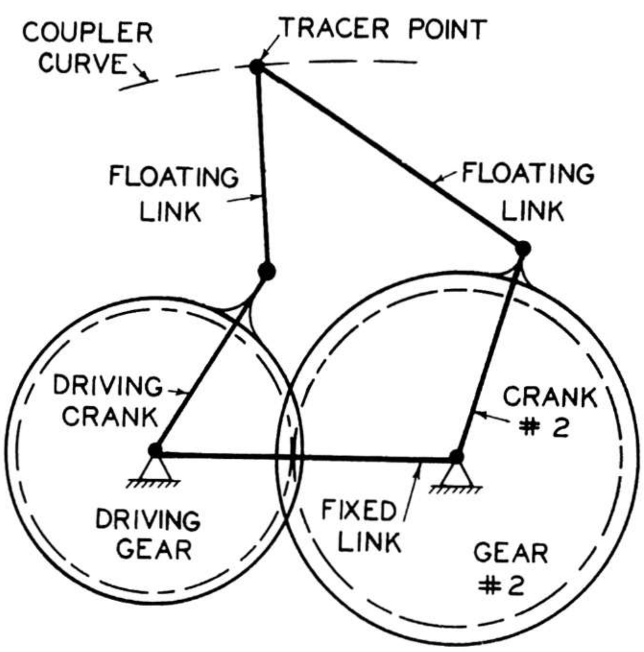

Fig. 1 Geared five-bar mechanism

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[2] Numbers in brackets designate References at end of paper.

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such formulations are called point-approximations to the path-synthesis problem.

Maximum Number of Precision Points. It can be shown [14] that in general the maximum number of precision points for the approximate path-synthesis problem without prescribed crank rotation is equal to the number of (constant) linkage-parameters. The number of linkage-parameters is determined by the type of mechanism to be synthesized. For geared five-bar mechanisms, Fig. 2, there are nine such parameters, namely, the five link-lengths, the inclination of the fixed link, the phase angle, and the vector from the origin of coordinates to the fixed crank-pivot.

Restrictions on the Choice of Precision Points. It is known [5] that when the gear ratio is given by $N = \pm p/q$, the degree of the generated locus is $2p + 4q$; the circularity[3] is $p + 2q$ when the gear ratio is positive, and $2q$ when it is negative. Here p and q are integers with no common factors and $0 < p \leq q$.

Thus, the characteristics of the generated locus depend upon the gear ratio. These generic properties are important in consistency considerations, and in determining restrictions on the choice of precision points. For example, consider the problem of generating a locus with nine precision points on a circle (a nine-point approximate circle). Since, in general, a circle intersects a tricircular sextic in six finite points, and a bicircular sextic in eight finite points, both the four-bar and the minus-one geared five-bar are inadequate. However, one could use, for example, either the plus one half or minus-one-half five-bar, whose tenth-degree loci generally intersect a circle in ten and twelve points, respectively.

The foregoing example illustrates two principles: Of the several loci of the same degree the one with the least singularities will allow the widest choice of precision points; when mechanisms with the same maximum number of precision points generate loci of different degree, the highest order locus will admit the widest choice of precision points. Thus, the more complicated the locus, the less restrictive the choice of precision points.

[3] A term used in algebraic geometry. See, for instance, H. Hilton, "Plane Algebraic Curves," Oxford University Press, London, England, 1932.

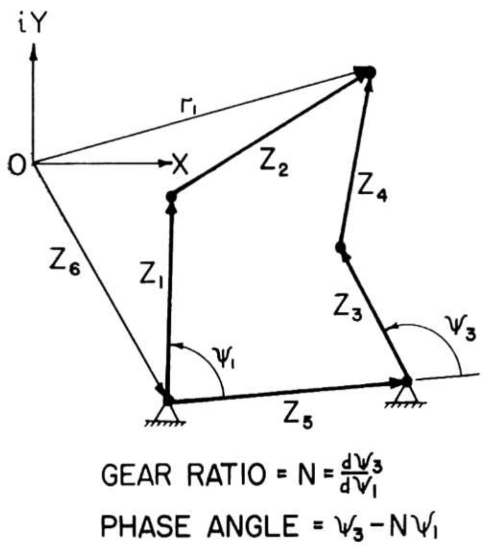

Fig. 2 Plane vectors $\mathbf{Z}_1$ to $\mathbf{Z}_6$ defining the geared five-bar in its reference position

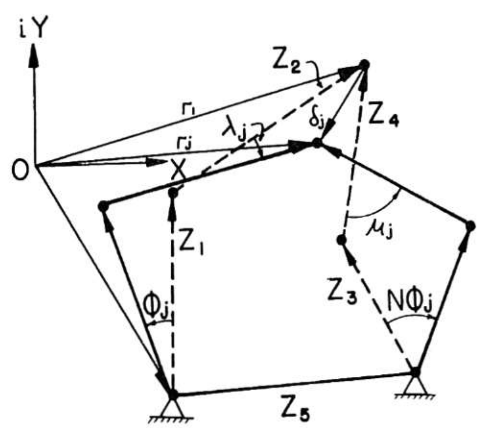

Fig. 3 The jth position (solid lines) defined with respect to the reference position

The Equations of Synthesis

We now derive the equations of synthesis for the geared five-bar mechanism with arbitrary gear ratio.

The geared five-bar with its link represented by directed line segments, Fig. 2, can be defined by two independent loop equations:

$$\left.\begin{aligned} \mathbf{Z}_6 + \mathbf{Z}_1 + \mathbf{Z}_2 &= \mathbf{r}_1 \\ \mathbf{Z}_6 + \mathbf{Z}_5 + \mathbf{Z}_3 + \mathbf{Z}_4 &= \mathbf{r}_1 \end{aligned}\right\} \tag{1}$$

We consider the plane vectors $\mathbf{Z}_1$ to $\mathbf{Z}_6$ as describing the mechanism in its reference position and define the jth position in terms of these, Fig. 3,

$$\left.\begin{aligned} \mathbf{Z}_6 + \mathbf{Z}_1 e^{i\phi j} + \mathbf{Z}_2 e^{i\lambda j} &= \mathbf{r}_j \\ \mathbf{Z}_6 + \mathbf{Z}_5 + \mathbf{Z}_3 e^{iN\phi j} + \mathbf{Z}_4 e^{i\mu j} &= \mathbf{r}_j \end{aligned}\right\} \tag{2}$$

Here, ϕ_j, λ_j, and μ_j are the angular displacements from the reference position of links 1, 2, and 4, respectively, and N is the gear ratio.

By subtracting the reference position equations (1) from those describing the jth position (2), we obtain the path-increment form of the equations of closure:

$$\left.\begin{aligned} \mathbf{Z}_1(e^{i\phi j} - 1) + \mathbf{Z}_2(e^{i\lambda j} - 1) &= \boldsymbol{\delta}_j \\ \mathbf{Z}_3(e^{iN\phi j} - 1) + \mathbf{Z}_4(e^{i\mu j} - 1) &= \boldsymbol{\delta}_j \end{aligned}\right\}; \tag{3}$$

the path-increment vector $\boldsymbol{\delta}_j$ is given by $\boldsymbol{\delta}_j = \mathbf{r}_j - \mathbf{r}_1$. Since, at most, nine precision points may be specified, $j = 2, 3, \ldots, 9$ for the path-synthesis problem.

Rewriting (3), we obtain

$$\left.\begin{aligned} \mathbf{Z}_2 e^{i\lambda j} &= \boldsymbol{\delta}_j - \mathbf{Z}_1(e^{i\phi j} - 1) + \mathbf{Z}_2 \\ \mathbf{Z}_4 e^{i\mu j} &= \boldsymbol{\delta}_j - \mathbf{Z}_3(e^{iN\phi j} - 1) + \mathbf{Z}_4 \end{aligned}\right\} \tag{4}$$

By multiplying each equation in (4) by its complex conjugate, we eliminate λ_j and μ_j and obtain the following quadratics

$$\left.\begin{aligned} \mathbf{a}_{0j}\mathbf{X}_j^2 + \mathbf{a}_{1j}\mathbf{X}_j + a_{2j} &= 0 \quad (5) \\ \mathbf{b}_{0j}\mathbf{W}_j^2 + \mathbf{b}_{1j}\mathbf{W}_j + b_{2j} &= 0 \quad (6) \end{aligned}\right\} j = 2, 3, \ldots, 9$$

where

$$\mathbf{X}_j = e^{i\phi j} - 1, \quad \mathbf{W}_j = e^{iN\phi j} - 1;$$

$$\mathbf{a}_{0j} = -(\mathbf{Z}_1\overline{\mathbf{Z}}_1) - \mathbf{Z}_1(\overline{\mathbf{Z}}_2 + \overline{\boldsymbol{\delta}}_j),$$

$$\mathbf{a}_{1j} = a_{2j} + (\boldsymbol{\delta}_j\overline{\mathbf{Z}}_1 - \overline{\boldsymbol{\delta}}_j\mathbf{Z}_1) + (\mathbf{Z}_2\overline{\mathbf{Z}}_1 - \overline{\mathbf{Z}}_2\mathbf{Z}_1),$$

$$a_{2j} = \boldsymbol{\delta}_j\overline{\boldsymbol{\delta}}_j + (\boldsymbol{\delta}_j\overline{\mathbf{Z}}_2 + \overline{\boldsymbol{\delta}}_j\mathbf{Z}_2),$$

$$\mathbf{b}_{0j} = -(\mathbf{Z}_3\overline{\mathbf{Z}}_3) - \mathbf{Z}_3(\overline{\mathbf{Z}}_4 + \overline{\boldsymbol{\delta}}_j),$$

$$\mathbf{b}_{1j} = b_{2j} + (\boldsymbol{\delta}_j\overline{\mathbf{Z}}_3 - \overline{\boldsymbol{\delta}}_j\mathbf{Z}_3) + (\mathbf{Z}_4\overline{\mathbf{Z}}_3 - \overline{\mathbf{Z}}_4\mathbf{Z}_3),$$

$$b_{2j} = \boldsymbol{\delta}_j\overline{\boldsymbol{\delta}}_j + (\boldsymbol{\delta}_j\overline{\mathbf{Z}}_4 + \overline{\boldsymbol{\delta}}_j\mathbf{Z}_4),$$

and the bar denotes complex conjugation.

Since $\mathbf{X}_j$ and $\mathbf{W}_j$ are functions of the same variable (ϕ_j, the crank angle), we form the eliminant of (5) and (6)—see the Appendix—and obtain the following equation:

$$(\mathbf{G}_j^{2N} + \overline{\mathbf{G}}_j^{2N})(\mathbf{b}_{0j}\overline{\mathbf{b}}_{0j})$$

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$$+ (-2b_{0jx} + b_{1jx})[(\bar{\mathbf{a}}_{0j}\mathbf{a}_{0j})^N(-2b_{0jx} + b_{1jx}) + (\mathbf{G}_j{}^N + \overline{\mathbf{G}}_j{}^N)(\bar{\mathbf{a}}_{0j}{}^N\mathbf{b}_{0j} + \mathbf{a}_{0j}{}^N\bar{\mathbf{b}}_{0j})] \quad (7)$$

$$+ (\bar{\mathbf{a}}_{0j}{}^{2N}\mathbf{b}_{0j}{}^2 + \mathbf{a}_{0j}{}^{2N}\bar{\mathbf{b}}_{0j}{}^2) = 0 \quad j = 2, 3, \ldots, 9$$

where

$$\mathbf{G}_j = \left(a_{0jx} - \frac{a_{1jx}}{2}\right) + i\left(a_{0jy}{}^2 + a_{0jx}a_{1jx} - \frac{a_{1jx}{}^2}{4}\right)^{1/2}.$$

In addition, we have made use of

$$\left.\begin{aligned} \mathbf{a}_{kj} &= a_{kjx} + ia_{kjy} \\ \mathbf{b}_{kj} &= b_{kjx} + ib_{kjy} \end{aligned}\right\} k = 0, 1, 2$$

Equation (7) represents a set of eight real equations in the four unknown linkage vectors $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, and $\mathbf{Z}_4$. We describe these vectors by their Cartesian components Z_{1x}, Z_{1y}, Z_{2x}, Z_{2y}, Z_{3x}, Z_{3y}, Z_{4x}, Z_{4y}, and consider solving (7) for these eight unknown lengths.

The problem is thus converted from complex equations, linear in the $\mathbf{Z}$'s, to nonlinear real equations. The advantage of this transformation is that it yields a single real equation for each (prescribed) path-increment vector.

Although further formal elimination of the unknowns is theoretically possible, it is not practical. A closed form solution is not feasible, since these equations are highly nonlinear; for example, when $N = \pm 1$ (7) becomes a set of seventh degree algebraic equations in the (eight) unknown lengths. Thus, we apply numerical techniques to the solution of (7).

The Four-Bar. The case of $N = +1$ is of special interest, since for this gear ratio there exists an equivalent four-bar mechanism for every five-bar (actually three, if we consider the Roberts cognates). One such equivalent set is shown in Fig. 4.

Gear Ratio = ±1. When $N = \pm 1$, equation (7) simplifies to

$$(a_{0jx}b_{1jx} - a_{1jx}b_{0jx})^2 + (a_{0jy}b_{1jx} + Na_{1jx}b_{0jy})^2 - 4(Na_{0jx}b_{0jy} + a_{0jy}b_{0jx})(a_{0jy}b_{1jx} + Na_{1jx}b_{0jy}) = 0 \quad j = 2, 3, \ldots, 9 \quad (8)$$

We note that, since $\mathbf{X}_j = \mathbf{W}_j$ when $N = +1$, and $\mathbf{X}_j = \overline{\mathbf{W}}_j$ when $N = -1$, (8) could also have been obtained by directly forming the quadratic eliminant of (5) and (6).

Equation (8) contains two roots which correspond to degenerate mechanisms. For $N = -1$, the degeneracy is limited to problems in which all precision points are colinear. In this case $|\mathbf{Z}_1| = |\mathbf{Z}_3|$, $|\mathbf{Z}_2| = |\mathbf{Z}_4|$, phase angle = 180 deg, satisfies (8) identically (Fig. 5). For $N = +1$, the root $\mathbf{Z}_1 = \mathbf{Z}_3$, $\mathbf{Z}_2 = \mathbf{Z}_4$ is degenerate. The mechanism corresponding to this root is an indeterminate chain (Fig. 6); although the solution is valid if the link lengths satisfy

$$|(|\mathbf{Z}_1| - |\mathbf{Z}_2|)| \leq |\mathbf{r}_j - \mathbf{Z}_6| \leq |\mathbf{Z}_1| + |\mathbf{Z}_2|,$$

degenerate mechanisms of this type are of no interest.

A More Concise Formulation. By eliminating the $\mathbf{Z}$'s from the path-increment equations of closure (3), it is possible [14] to reduce the nine-point path-synthesis problem to a dimensionless system of six real (nonlinear) equations in the six unknown angles ϕ_2, ϕ_3, λ_2, λ_3, μ_2, μ_3. This formulation has the advantage of being explicitly independent of the choice of origin (as is also (7)), axis orientation, and scale factor. Unfortunately, the elimination procedure introduces certain extraneous roots which make this formulation computationally inferior to (7).

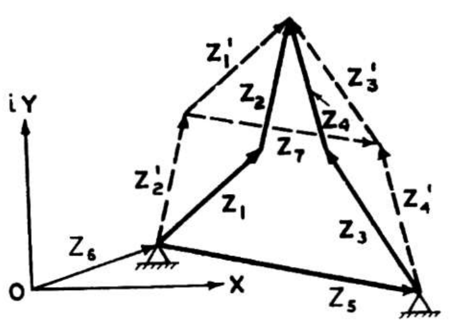

Fig. 4 A plus-one geared five-bar (solid lines), and its equivalent four-bar (broken lines). $\mathbf{Z}_1 = \mathbf{Z}_1{}^1$, $\mathbf{Z}_2 = \mathbf{Z}_2{}^1$, $\mathbf{Z}_3 = \mathbf{Z}_3{}^1$, $\mathbf{Z}_4 = \mathbf{Z}_4{}^1$, **and** $\mathbf{Z}_1{}^1\mathbf{Z}_3{}^1\mathbf{Z}_7$ **is the four-bar coupler triangle.**

Numerical Methods. For simplicity, the equations of synthesis (7) are written as

$$E_j(X_1, X_2, \ldots, X_8) = 0 \quad (9)$$
$$j = 2, 3, \ldots, 9$$

where

$$E_j = (\mathbf{G}_j{}^{2N} + \overline{\mathbf{G}}_j{}^{2N})(\mathbf{b}_{0j}\bar{\mathbf{b}}_{0j}) + (-2b_{0jx} + b_{1jx})[(\bar{\mathbf{a}}_{0j}\mathbf{a}_{0j})^N(-2b_{0jx} + b_{1jx}) + (\mathbf{G}_j{}^N + \overline{\mathbf{G}}_j{}^N)(\bar{\mathbf{a}}_{0j}{}^N\mathbf{b}_{0j} + \mathbf{a}_{0j}{}^N\bar{\mathbf{b}}_{0j})] + (\bar{\mathbf{a}}_{0j}{}^{2N}\mathbf{b}_{0j}{}^2 + \mathbf{a}_{0j}{}^{2N}\bar{\mathbf{b}}_{0j}{}^2) = 0,$$

and

$$X_1 = Z_{1x}, \quad X_2 = Z_{1y}, \quad X_3 = Z_{2x}, \ldots, X_8 = Z_{4y}.$$

(9) is a system of eight simultaneous algebraic equations in the eight unknown lengths $X_1, X_2, \ldots, X_8$. This system is highly nonlinear, and should be solved by iterative techniques.

Of the several possible iteration techniques, we choose the generalized Newton-Raphson procedure [15] because of its second-order convergence. In order to apply this procedure one must first choose an initial approximation to the solution. Then this approximation, $X_1{}^0, X_2{}^0, \ldots, X_8{}^0$, is corrected by ΔX_1, ΔX_2, $\ldots$, ΔX_8; the ΔX's are computed from the linear system

$$\sum_{k=1}^{8} \frac{\partial E_j}{\partial X_k}(-\Delta X_k) = E_j \qquad j = 2, 3, \ldots, 9.$$

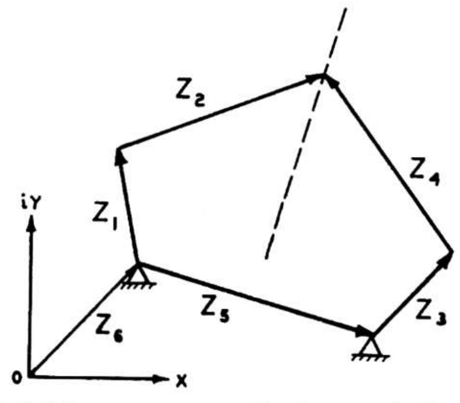

Fig. 5 Limited degeneracy—symmetric minus-one five-bar. $|\mathbf{Z}_1| = |\mathbf{Z}_3|$, $|\mathbf{Z}_2| = |\mathbf{Z}_4|$, **and the phase angle = 180 deg.**

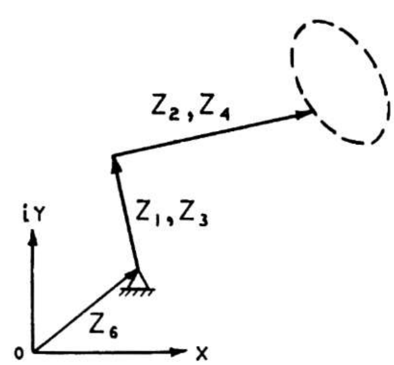

Fig. 6 Complete degeneracy—symmetric plus-one five-bar. $\mathbf{Z}_1 = \mathbf{Z}_2$ **and** $\mathbf{Z}_3 = \mathbf{Z}_4$.

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The corrected approximations

$$X_k^1 = X_k^0 + \Delta X_k \qquad k = 1, 2, \ldots, 8$$

are then used to compute the next group of corrections, and so forth. This procedure is repeated until the corrections become very small (as compared to the unknowns), or it becomes apparent that the iteration will not converge.

For the synthesis problem, the convergence of the Newton-Raphson iteration procedure is strongly dependent upon a good initial approximation to the desired solution [16, 20]. This is a serious limitation which has in the past limited the effectiveness of the iterative procedure. It has been found possible to remove this restriction by dividing each problem into a number of subsidiary problems in accordance with a procedure which we call the "Bootstrap" method.

The "Bootstrap" Method. In general, it is possible to imagine a physical transformation of any mechanism into any other mechanism of the same type, by a continuous sequence of changes in the mechanism proportions (parameters). We may, therefore, seek to obtain a mechanism which generates a desired locus by starting with any other mechanism of the same type, and slowly incrementing its parameters to the desired values. For the synthesis problem it is more convenient to think of the path, rather than the mechanism, as being deformed. Thus we start with any known solution (i.e., a mechanism and nine points on its coupler curve). Next we solve the synthesis problem for a path which differs only slightly from the original one. This new solution is in turn a good first approximation for the solution of a second synthesis problem defined by further small alterations in the path, etc. Eventually, after a sufficient number of alterations, we obtain the required solution. Thus, in contrast to standard numerical procedures, we are continuously altering not only the "trial solution," but also the "constants" of the equations themselves.[4]

By selecting any starting point and solving enough intermediate problems, it should, in general, be possible to synthesize any nine points; however, it is desirable to limit the number of intermediate problems by judiciously choosing the starting position.

Position Interchange. The Newton-Raphson iteration fails to converge when the Jacobian matrix ($\|\partial E_j/\partial X_k\|$) of the system is singular, or the problem is "ill-conditioned" in the neighborhood of the solution or the initial approximation. Physically, this "ill-conditioning" may be due to parallelism of links [5] or other special circumstances. Since $X_1, X_2, \ldots, X_8$ correspond to the projections of the moving links in their reference position, a new set of simultaneous equations and also a new set of initial approximations can be obtained by simply considering one of the eight other precision positions as defining the reference configuration. This change of reference configuration has been found to be effective in dealing with nonconvergent iterations (and also degenerate roots). In this way, the total number of subsidiary problems remains the same, even if the iteration procedure fails to converge. The above process of changing the reference position and thereby redefining the equations of synthesis can be repeated, if necessary, up to eight times for each of the (subsidiary) problems.

Quality-Index Control. Nine precision points do not uniquely determine the linkage parameters, and therefore any solution of the nine-point path-synthesis problem is not unique. Since, in general, a large number of solutions exists,[5] it is possible to obtain a wide variety of linkages whose loci contain the same nine precision points. Which solution is actually obtained depends on the starting linkage and its nine locus points, the number of intermediate problems, and the manner in which the intermediate problems are formulated.

From practical considerations, a large quality index (in this case, ratio of maximum to minimum link length) is undesirable, and therefore a procedure was devised for avoiding solutions with large quality indexes. Briefly, whenever the quality index increases above a predetermined value, the path-increment vectors are "incremented" one at a time, and their effect on the quality-index is noted. The $\boldsymbol{\delta}_j$'s are then selectively "incremented" so as to decrease the quality-index below a predetermined maximum. Thereafter, uniform "incrementing" of the $\boldsymbol{\delta}_j$'s is reintroduced.

Similarly, we could devise procedures to control other characteristics of the synthesized linkage. Specifically, instead of controlling the quality-index (or in addition to it), it is believed possible to develop nonuniform-incrementation techniques to obtain solutions with prescribed crank rotation, a specified minimum transmission angle, all precision points on the same circuit, and/or precision points traced out in a specified order.

An alternate approach would be to reformulate the equations of synthesis: Specifying n_0 (say) instead of n precision points, and adding $n - n_0$ constraint equations involving the characteristics to be controlled [21].

For problems phrased in terms of the generation of a given path, we have additional means of controlling the final solution by varying the location of the precision points. Since we are generally at liberty to select any nine precision points on the prescribed path, it is possible to develop controls based on changing the location of the precision points.

Examples

Table 1 lists the required precision points for seven different nine-point synthesis problems which were solved by applying the computational procedures described in this paper. The computations were performed, on an IBM 7090 digital computer, in accordance with a program written in the Fortran programming system.[6] The 7090 computational time for a nine-point synthesis with one hundred intermediate problems (for instance, Example 1) is from three to ten minutes, approximately, depending upon the number of reference position changes and quality-index-control applications.

Table 2 lists the "arbitrary" starting mechanisms, and Table 3 gives the synthesized linkages.

Example 1 is a minus-one five-bar synthesis of nine arbitrarily chosen precision points. During the solution of the one-hundred intermediate problems the Newton-Raphson iteration procedure failed to converge eleven times, and, therefore, there were eleven changes of the reference position. The synthesized linkage is shown in Fig. 7 (together with the starting linkage). All nine points lie on one circuit (of the locus), and the synthesized mechanism can perform complete crank rotations.

Example 2 refers to a plus-one five-bar (i.e., a four-bar) synthesis. In order to illustrate the difficulty of directly applying the Newton-Raphson iteration technique (and therefore the usefulness of the bootstrap procedure), the precision points were chosen by slightly displacing the corresponding locus points of an arbitrary starting solution.

If we attempt to solve this problem without using the bootstrap procedure (i.e., no intermediate problems), we find, the Newton-Raphson iteration of the equations of synthesis does not converge. By defining four subsidiary problems, it is possible to obtain the solution shown in Figs. 8 and 9.

Since, in this problem, the initial approximation (the starting linkage) and the final solution are quite similar, it becomes apparent that, due to the highly nonlinear character of the equations of synthesis, the initial approximation must lie very close to the final solution, if one is to successfully apply the Newton-Raphson iteration without the bootstrap procedure.

In Example 3, we synthesize two different minus-one five-bars whose coupler curves contain the same nine arbitrary precision

[4] It is hoped to present the purely mathematical formulation of this method in a separate article.

[5] By applying Bezout's theorem [17] we conclude that there may be as many as $(2p + 4q + 1)^8$ solutions for the nine-point synthesis of a five-bar with gear ratio $N = \pm p/q$; however, the number of practically obtainable linkages will generally be much smaller, since many of these solutions may be imaginary.

[6] This program can be obtained by writing to the authors.

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Table 1 Prescribed performance

Example number	1		2		3	
Precision point	r_x	r_y	r_x	r_y	r_x	r_y
1	3.4716232	4.7789161	0.89618666	−0.098029166	2.6910434	2.7570346
2	5.4303129	4.4036093	1.2156535	−1.1874910	2.4979071	2.9863218
3	5.3169564	3.8423449	1.515143	−0.85449608	2.3366802	3.1144034
4	4.2843019	3.5434635	1.6754775	−0.48768058	2.0574603	3.0627708
5	3.0408324	3.4982918	1.7138690	−0.30099232	1.9270929	2.8688841
6	1.7320434	3.6323337	1.7215236	0.032699525	1.6718905	2.0624215
7	0.40162112	3.6075705	1.6642029	0.33241088	1.4639201	−0.12130798
8	0.03185781	3.5808909	1.4984171	0.74435576	1.4837444	−0.35175213
9	−0.44631289	3.2284120	1.3011834	0.92153806	1.5197819	−0.56148106

Example number	4, 5		6		7	
Precision point	r_x	r_y	r_x	r_y	r_x	r_y
1	−1.2632773 (−1.6500000	4.257256 4.7200000)*	−1.6234615	1.4079174	−0.99958136	2.4499164
2	1.0000000	4.8989795	−1.7853533	1.3110117	−1.1441907	2.5795773
3	2.0000000	4.582757	−2.0376154	1.2726351	−1.2778539	2.7553319
4	3.0000000	4.0000000	−2.195805	1.3634706	−1.3864768	2.9945506
5	4.0000000	3.0000000	−2.2857299	1.4302842	−1.4437859	3.3151627
6	4.0000000	−3.0000000	−2.3556470	1.4991761	−1.4066815	3.723720
7	3.0000000	−4.0000000	−1.6116313	2.0618837	−1.2250824	4.1885100
8	2.0000000	−4.5825757	−1.5962205	1.8513574	−0.89346956	4.1644969
9	1.0000000	−4.8989795	−1.6100027	1.6278040	−0.51782419	4.8895790

* Point 1 for Example 5.

Table 2 Arbitrary starting linkage projections and corresponding locus points

Example number	1, 4		2		3	
Link j	Z_{jx}	Z_{jy}	Z_{jx}	Z_{jy}	Z_{jx}	Z_{jy}
1	1.0000004	−0.00000307	0.02689809	0.61689165	1.2380012	0.84697249
2	−1.1955797	2.1955886	1.2347322	−0.61734615	1.9217513	1.4375938
3	0.00000271	0.99999955	0.96500078	−0.046714761	−0.78583814	0.76967203
4	−2.1955908	1.1955911	−0.82938716	−0.53627264	1.1461263	1.5157119
Locus Point	r_x	r_y	r_x	r_y	r_x	r_y
1	−0.19558261	2.1955825	1.0000000	0.0000000	3.159412	2.2849807
2	1.3481011	3.0093745	1.2101537	−1.1935621	2.8084695	2.7040243
3	2.1072536	2.7223853	1.514419	−0.85681699	2.5592979	2.9058374
4	2.0141473	2.0812546	1.672618	−0.49005225	2.1899271	2.8979799
5	1.6135281	1.5258546	1.7097463	−0.32305998	2.034293	2.7000059
6	1.2093023	1.3298602	1.7357395	0.01730220	1.7389975	1.7825320
7	0.72459940	1.4604242	1.7119624	0.3111159	1.5967087	0.044418894
8	−0.22151931	1.6139378	1.5652307	0.7600353	1.6325876	−0.21812138
9	−1.2848193	1.0935106	1.3947743	0.973082	1.6943856	−0.44224571

Example number	5, 6		7	
Link j	Z_{jx}	Z_{jy}	Z_{jx}	Z_{jy}
1	1.5000000	−0.0000000	1.5000000	0.0000000
2	−2.4995813	2.4499164	−2.4995813	2.4499164
3	0.0000000	2.5000000	0.0000000	2.5000000
4	−2.9995816	−0.0500806	2.9995816	−0.0500836
Locus point	r_x	r_y	r_x	r_y
1	−0.99958136	2.4499164	−0.99958136	2.4499164
2	−1.1803248	2.0171196	−1.1441907	2.5795773
3	−1.3273189	1.6686124	−1.2778539	2.7553319
4	−1.467843	1.3934436	−1.3864768	2.9945506
5	−1.6176268	1.1795093	−1.4437859	3.3151627
6	−3.2299823	1.5185489	−1.4066815	3.7237520
7	−3.1616127	1.8993259	−1.2250824	4.1885100
8	−2.8627830	2.3812472	−0.89346956	4.6144969
9	−2.1638086	2.9047298	−0.51782419	4.8895790

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Table 3 Synthesized linkages

Example number	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Phase angle	Gear ratio
1	1.85967	5.14214	0.89996	4.99243	3.99966	0.10869	50.6023	-1
2	0.42264	1.74247	0.67988	1.15042	1.65565	0.48706	207.822	$+1$
3	0.90962	1.85033	0.62880	1.77338	1.59836	1.11594	173.203	-1
3	1.44810	2.39120	1.05884	2.10113	2.70371	0.06936	169.756	-1
4	5.20643	14.4603	7.35028	5.39559	14.1235	15.2427	-15.9973	-1
5	3.59306	9.87948	6.15158	22.4433	34.5311	10.6588	-273.850	$-\frac{1}{2}$
5	8.05961	13.4056	7.64037	12.1870	11.0521	12.7336	-345.471	$-\frac{1}{2}$
6	0.55955	1.32597	1.07832	1.18996	1.66547	2.17190	-6.99899	$-\frac{1}{2}$
7	3.17106	4.54668	2.35068	3.44326	4.40097	2.73004	237.614	$+\frac{1}{3}$

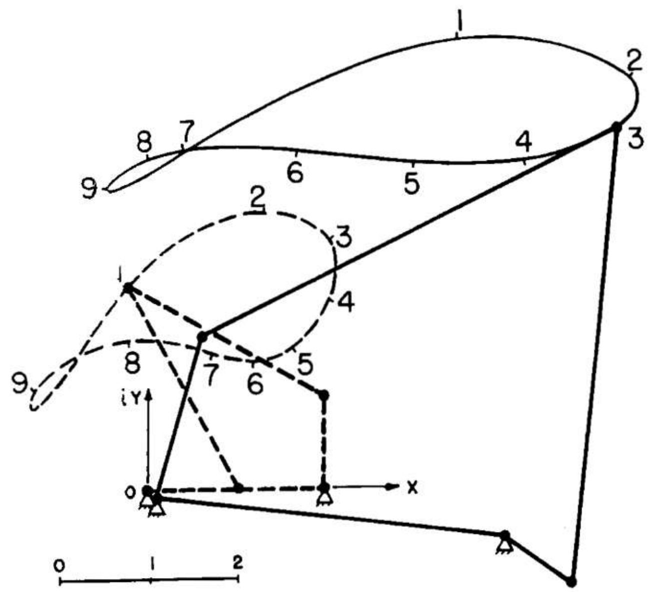

Fig. 7 Example 1, minus-one geared five-bar. The synthesized linkage and its generated path (solid lines); the starting linkage and its locus (broken lines).

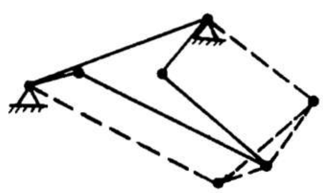

Fig. 8 Solution to Example 2. Synthesized plus-one five-bar (solid) and its derived four-bar.

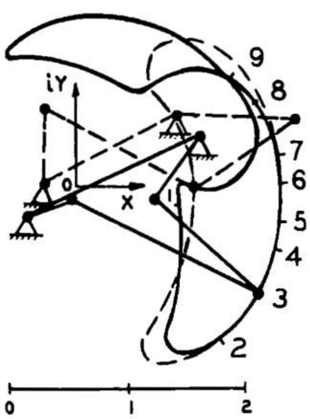

Fig. 9 Solution to Example 2. Synthesized linkage (solid), starting linkage (dashed), and their generated loci.

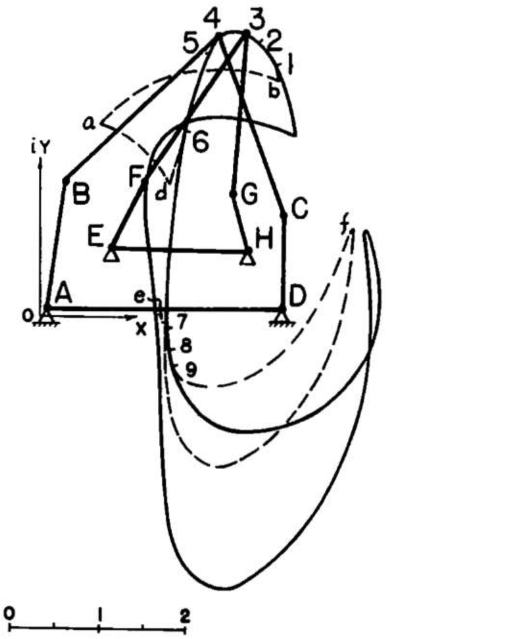

Fig. 10 Two solutions to Example 3: linkage AB4CD and its generated path (solid), linkage EF3GH and its locus (dashed)

points. The two different roots were obtained by solving the synthesis twice, using the same starting point, but a different number of intermediate problems.

The two solutions are shown in Fig. 10. The synthesized linkage EF3GH was obtained in ten intermediate steps. Its coupler curve has two circuits: the upper dashed curve a, b, 1, 2, 3, 4, 5, 6, d, a, and the lower dashed curve e, 7, 8, 9, f, e. By using one-hundred intermediate problems, we obtain AB4CD. This linkage generates the single-circuited (solid line) curve which, of course, intersects the two-circuited locus at the nine precision points.

Since we have obtained solutions for the same nine arbitrarily chosen points, this problem verifies that nine points do not uniquely determine the curve. These solutions are in fact quite different. The linkage EF3GH has total crank rotation, but the precision points do not lie on the same circuit. On the other hand, AB4CD has only partial crank rotation, and all the precision points lie on the same circuit.

In Example 4 (Fig. 11) we synthesize an eight-point circle by using the same starting point as in Example 1, and one-hundred subsidiary problems. If we choose the ninth point on the circle, two link lengths tend toward infinity, since generally no minus-one five-bar solution exists. Two different minus-one-half geared five-bar syntheses of this nine point circle are listed in Example 5.

Examples 6 and 7 represent syntheses for gear ratio $-1/2$ and $+1/3$, respectively.

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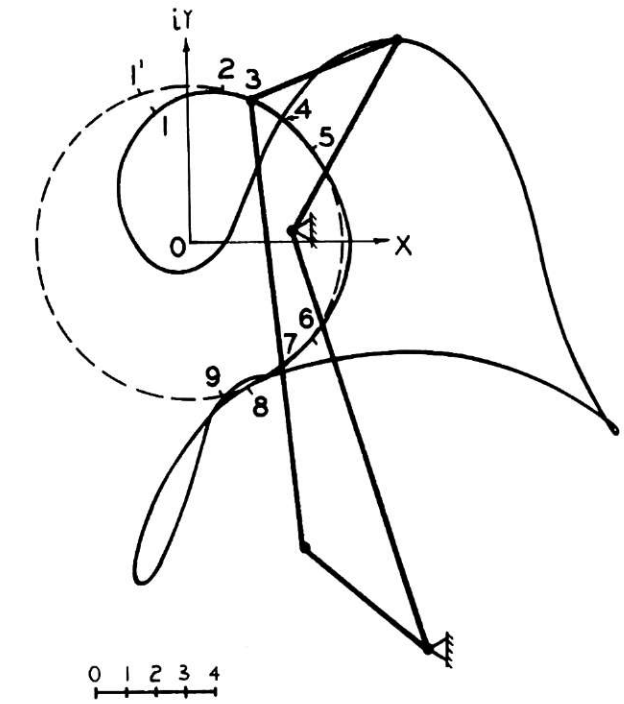

Fig. 11 A solution to the eight-point (symmetric) circle synthesis, Example 4. The dashed curve is the circle on which eight of the nine precision points lie. For Example 5, point 1 is replaced by 1′ thereby defining a nine-point circle.

General Applicability

Other Syntheses. Although this development was restricted to geared five-bars, the computational procedure is general and applicable to a variety of other synthesis problems.

For more complicated linkages the computational scheme remains unchanged, but the equations of synthesis (and analysis) are different. A preliminary investigation has indicated that for six-bars, at least, these equations can be developed in a form which would make the synthesis computationally comparable to the geared five-bar. In addition, it is possible to apply these techniques to the synthesis of mechanisms with other types of joints, multidegree-of-freedom mechanisms, and spatial mechanisms.

It is also possible to apply these computational procedures to function generation and higher-order contact syntheses.

Generalizations of Roberts' Theorem. This study makes available a new tool for the experimental search for cognates of all types of mechanisms: By applying these procedures to syntheses formulated in terms of the path (no crank angles), we might obtain multiple solutions which may be cognates and lead to generalizations of Roberts' theorem.

Acknowledgments

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APPENDIX

The Eliminant of (5) and (6)

We solve (5) for $e^{i\phi j}$ and obtain two roots; for convenience we define these roots as $\mathbf{C}_j$ and $\mathbf{D}_j$ as follows:

$$\mathbf{C}_j = [e^{i\phi j}]_+ = \frac{2\mathbf{a}_{0j} - \mathbf{a}_{1j} + [\mathbf{a}_{1j}^2 - 4\mathbf{a}_{0j}a_{2j}]^{1/2}}{2\mathbf{a}_{0j}}$$

$$\mathbf{D}_j = [e^{i\phi j}]_- = \frac{2\mathbf{a}_{0j} - \mathbf{a}_{1j} - [\mathbf{a}_{1j}^2 - 4\mathbf{a}_{0j}a_{2j}]^{1/2}}{2\mathbf{a}_{0j}}$$

Substituting first $\mathbf{C}_j$ and then $\mathbf{D}_j$ for $e^{i\phi j}$ in (6) yields

$$\mathbf{b}_{0j}\mathbf{C}_j^{2N} + (-2\mathbf{b}_{0j} + \mathbf{b}_{1j})\mathbf{C}_j^{N} + (\mathbf{b}_{0j} - \mathbf{b}_{1j} + b_{2j}) = 0 \quad (10)$$

and

$$\mathbf{b}_{0j}\mathbf{D}_j^{2N} + (-2\mathbf{b}_{0j} + \mathbf{b}_{1j})\mathbf{D}_j^{N} + (\mathbf{b}_{0j} - \mathbf{b}_{1j} + b_{2j}) = 0 \quad (11)$$

which when multiplied together yield the eliminant of (5) and (6). The form of the eliminant given by (7) is obtained by substituting

$$\mathbf{C}_j\mathbf{D}_j = \frac{\bar{\mathbf{a}}_{0j}}{\mathbf{a}_{0j}},$$

$$\mathbf{b}_{0j} - \mathbf{b}_{1j} + b_{2j} = \bar{\mathbf{b}}_{0j},$$

$$-2\mathbf{b}_{0j} + \mathbf{b}_{1j} = -2b_{0jx} + b_{1jx},$$

and then multiplying by $\mathbf{a}_{0j}^{2N}$.

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