The Low Mach Number Approximation for Multidimensional Modeling of Type I X-ray Bursts

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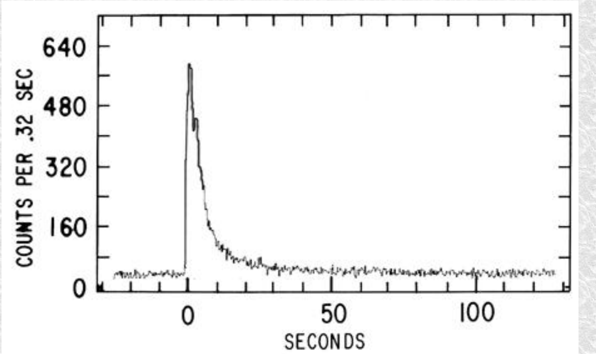
Main Project Goals ...

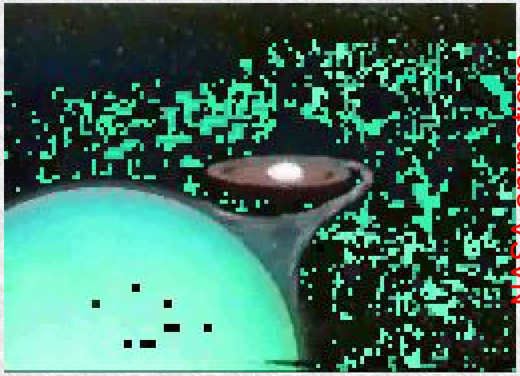
- Develop the *Low Mach Number Approximation* (**LMNA**) to study astrophysical deflagrations:
  - Sub-sonic flows $M < 0.1$
  - Strong gravity $g \sim 10^{14}$ cm s^{-2}
  - Large lateral variations $\delta T / T \sim 0.1$

- Simulate Type I X-ray burst (w/ 2D hydro)
  - ... through a complete burst: 1) heating, 2) peak, 3) cooling
  - Quantify:
    - burning front behavior
    - convective dynamics
    - size of thermodynamic fluctuations
    - compositional mixing
  - Calculate approximate light-curves



MXB 1728-34 Light Curve

(High Energy Astronomy Observatory-1 A2)



NASA animations

LMNA Motivation

- Substantial increase in time-step if M $\ll$ 1.0

$$\Delta t = \lambda_{CFL} \frac{\Delta x}{(v_{flow} + v_{sound})} \rightarrow \lambda_{CFL} \frac{\Delta x}{(v_{flow})}$$ if p-waves are absent

- Speed up calculations by a factor of 10 or more

- Completely avoid artificial acoustic effects at the boundaries of both the computational domain and the convective layer

LMNA Advantages

	Boussinesq	Implicit	Anelastic	LMNA
Key method:	$\nabla \cdot \vec{v} = 0$	n, n-1	$\nabla \cdot \rho \vec{v} = 0$	$\frac{\partial P}{\partial t} = 0$ (EOS, EOE)
Compressible?	NO	YES	weakly	weakly
Acoustic boundary problems?	NO	YES	NO	NO
Allows for large lateral differences?	NO	YES	NO	YES

LMNA Current Implementations

- ***Terrestrial:***

  Fire Dynamics Simulator
  National Institute of Standards and Technology
  (McGratten et al, 2004)

- ***Astrophysical:***
  LBNL Alternative LMNA method:
  1) Evolves density
  2) Currently neglects reaction, composition, & thermal diffusion
  3) Allows for time-dependent base state
  4) We reformulate pressure gradient and buoyancy terms: $\phi = P' - gK$

  Lawrence Berkeley National Laboratory

  (Bell et al., 2004; Zingale, et al., 2005; Almgren et al., 2006a, 2006b)

LMNA Essence

Non-dimensionalizing the Euler momentum equation is instructive:

$$\nabla P = -M^2 \left(\rho \frac{D\vec{v}}{Dt} + F_r^{-1} \rho \hat{g} \right)$$

where $P, \rho, \nabla, t, \vec{v}$ are non-dimensionalized quantities, and

$M = v_o / v_s$ (Mach Number) $F_r = \frac{(v_o^2 / L_o)}{g}$ (Froude Number)

$$P = P_1(t) + M^2 P_2(\vec{r}, t)$$

Perturbatively expand about a time-independent, hydrostatic base state:

$$P(\vec{r}, t) = P_{HS}(z) + P'(\vec{r}, t)$$

$$P' \sim M^2 P$$

Validated
ex-post-facto

	Euler Equations	LMNA Equations
continuity	$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$	$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$
momentum	$\rho\frac{D\vec{v}}{Dt}+\nabla P=\rho\vec{g}$	$\rho\frac{D\vec{v}}{Dt}+\nabla P'=\rho'\vec{g}$
energy	$\rho T\frac{DS}{Dt}=\rho c_p\frac{DT}{Dt}-\delta\frac{DP}{Dt}$	$\rho T\frac{DS}{Dt}=\rho c_p\frac{DT}{Dt}-\delta\frac{DP_{HS}}{Dt}$
state	$\rho=F(T,P,X_l)$	$\rho=F(T,P_{HS},X_l)$
species	$\frac{DX_l}{Dt}=R_l$	$\frac{DX_l}{Dt}=R_l$

Essential approximation: $\frac{\partial P}{\partial t}\rightarrow 0$ in the equations of energy and state

LMNA Evolution Equations

$$\frac{\partial X_l}{\partial t} = -\vec{v} \cdot \nabla X_l + R_l$$

$$\frac{\partial T}{\partial t} = -\vec{v} \cdot \nabla T + \frac{1}{c_p} \left(\dot{s} - \frac{\delta}{\rho} w \rho_{HS} g + \frac{1}{\rho} \nabla \cdot \kappa \nabla T \right)$$

$$\rho, e \Leftarrow EOS(T, P_{HS}, X_l)$$

$$\nabla^2 \phi = \frac{\partial^2 \rho}{\partial t^2} - \nabla \cdot \left\{ \nabla \cdot \left(\rho \vec{v} \vec{v} \right) \right\} - \left\{ \frac{\partial^2 (gK)}{\partial y^2} \right\}$$

$$\phi = P' - gK$$

$$K = \int_0^{z'} \rho'(y, z') \, dz'$$

$$\frac{\partial (\rho \vec{v})}{\partial t} = -\nabla \cdot \left(\rho \vec{v} \vec{v} \right) - \nabla \phi - \left\{ \frac{\partial (gK)}{\partial y} \right\} \hat{j}$$

LMNA Model Specifications

- Split, explicit, finite-difference scheme
- Grid: 2D, uniform (1:1), staggered, Cartesian
- Space: central and upwind differencing
- Time: forward Euler, CFL = 0.5 (2D)
- Input physics:
  - thermal diffusion (opacity routines: Iben, 1975; Christy, 1966; Weaver, 1978)
  - realistic equation of state (tabulated Helmholtz EOS: Timmes & Swesty, 2000)
  - 3α $\left(3\,{}^{4}_{2}He \rightarrow {}^{12}_{6}C\right)$ burner (Fushiki & Lamb, 1987)
  - strong gravity (g = 2 x 10^{14} cm s^{-2})
- Language: Intel Fortran 90
- Parallelism: Message Passing Interface (MPI)
- Computing: *Hydra* (NU Applied Math) 32 node cluster

Neutron Star Initial Model

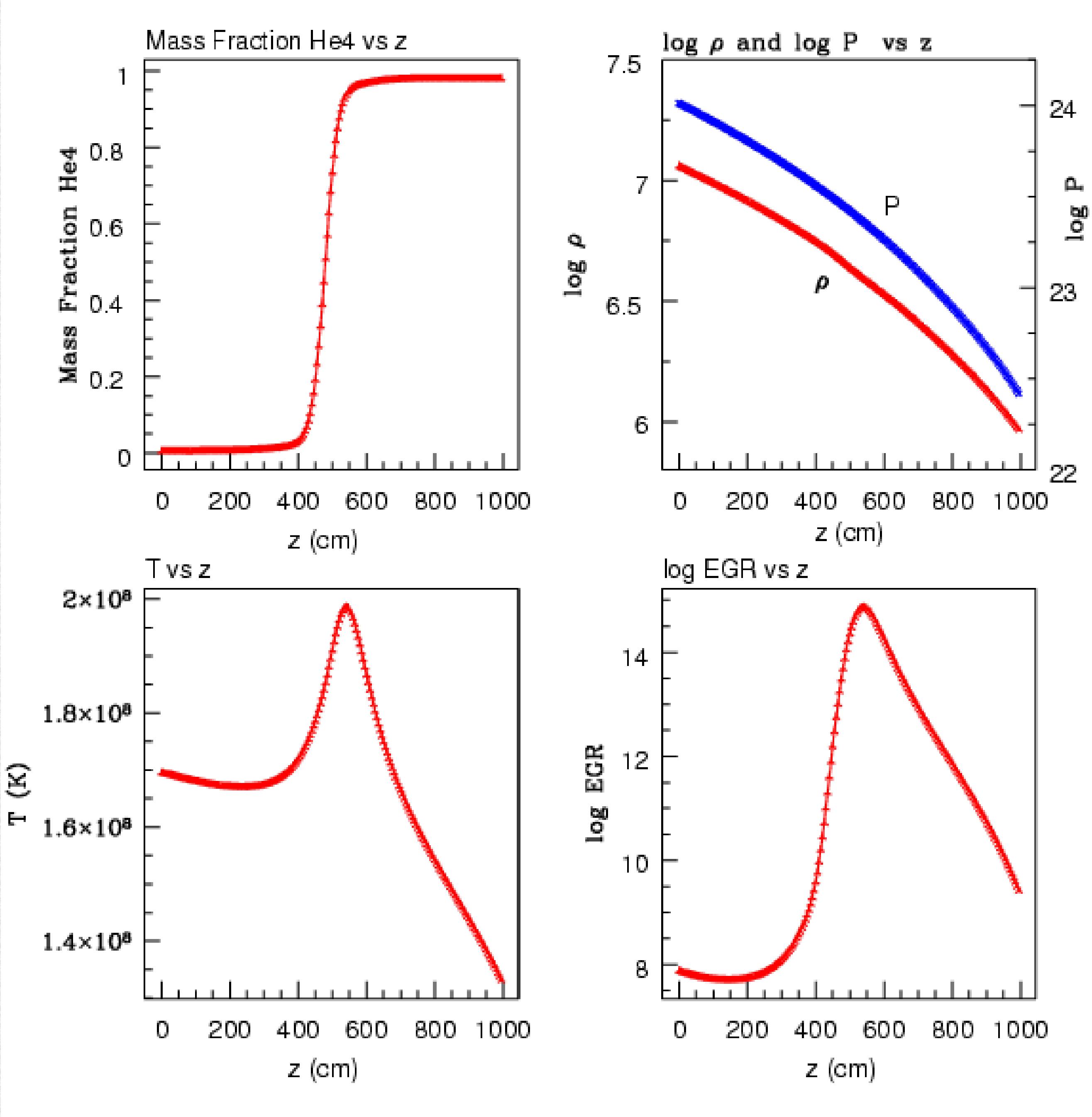
- 10^3 cm of upper neutron star envelope

$M_{NS} = 1.4 M_{\odot}, R_{NS} = 10^6\,cm$

- 1 cm zone^{-1} resolution
- hydrostatic & thermal equilibrium
- mass accretion rate:

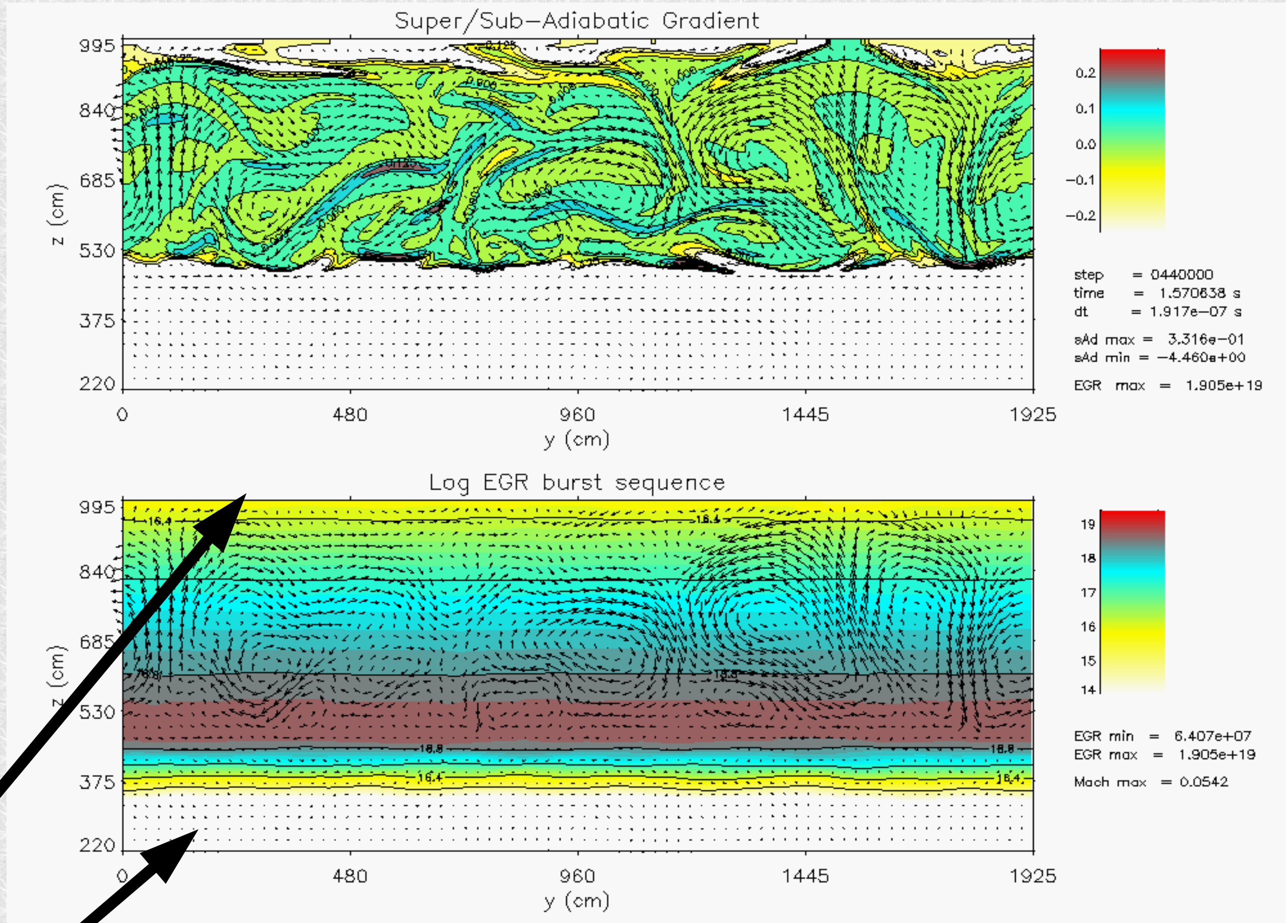
$\dot{m} = 5 \times 10^{-9} M_{\odot}\, yr^{-1}\, cm^{-2}$

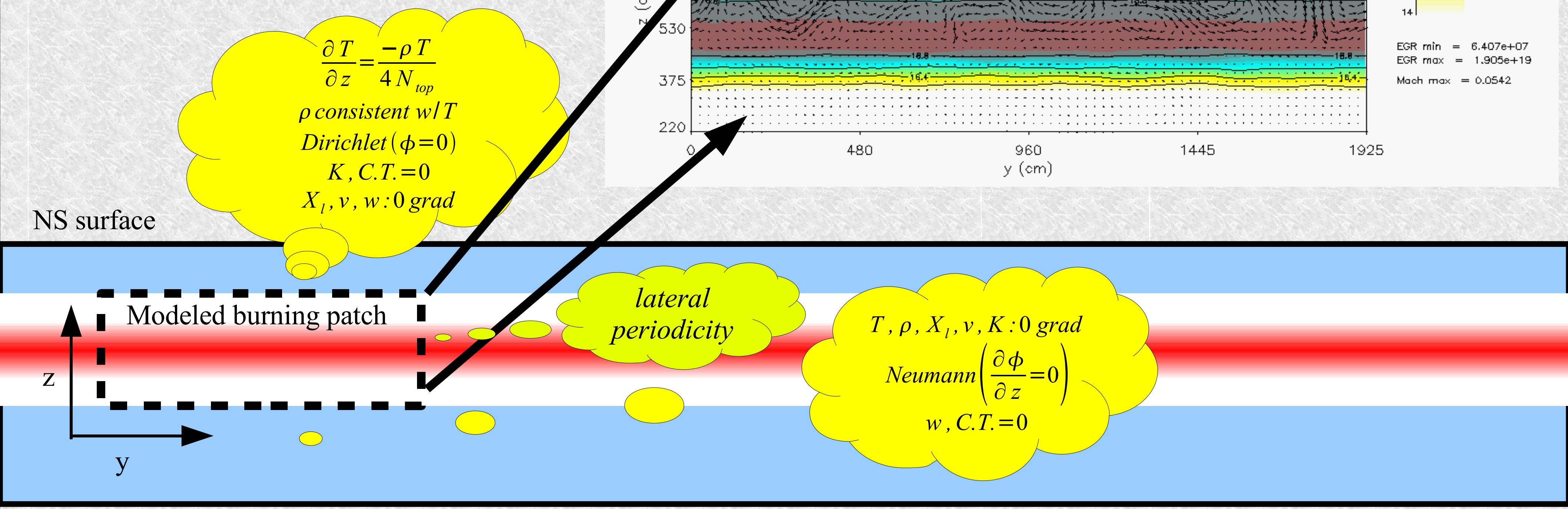
- pressure scale height ~ 200 cm
- 1D diffusional-thermal evolution through multiple burst cycles
- initially sub-adiabatic



2D Complete Burst Model

- 386 x 200 zones
  1930 x 1000 cm
- 5 cm zone^{-1}
- Plane parallel approx.
- Initial ρ perturbation
  Gaussian, 10^{-6} ρ, centered, σ = 50 cm
- $\mathrm{EGR}_{\mathrm{init}} = 7\times10^{14}$ erg g^{-1} s^{-1}

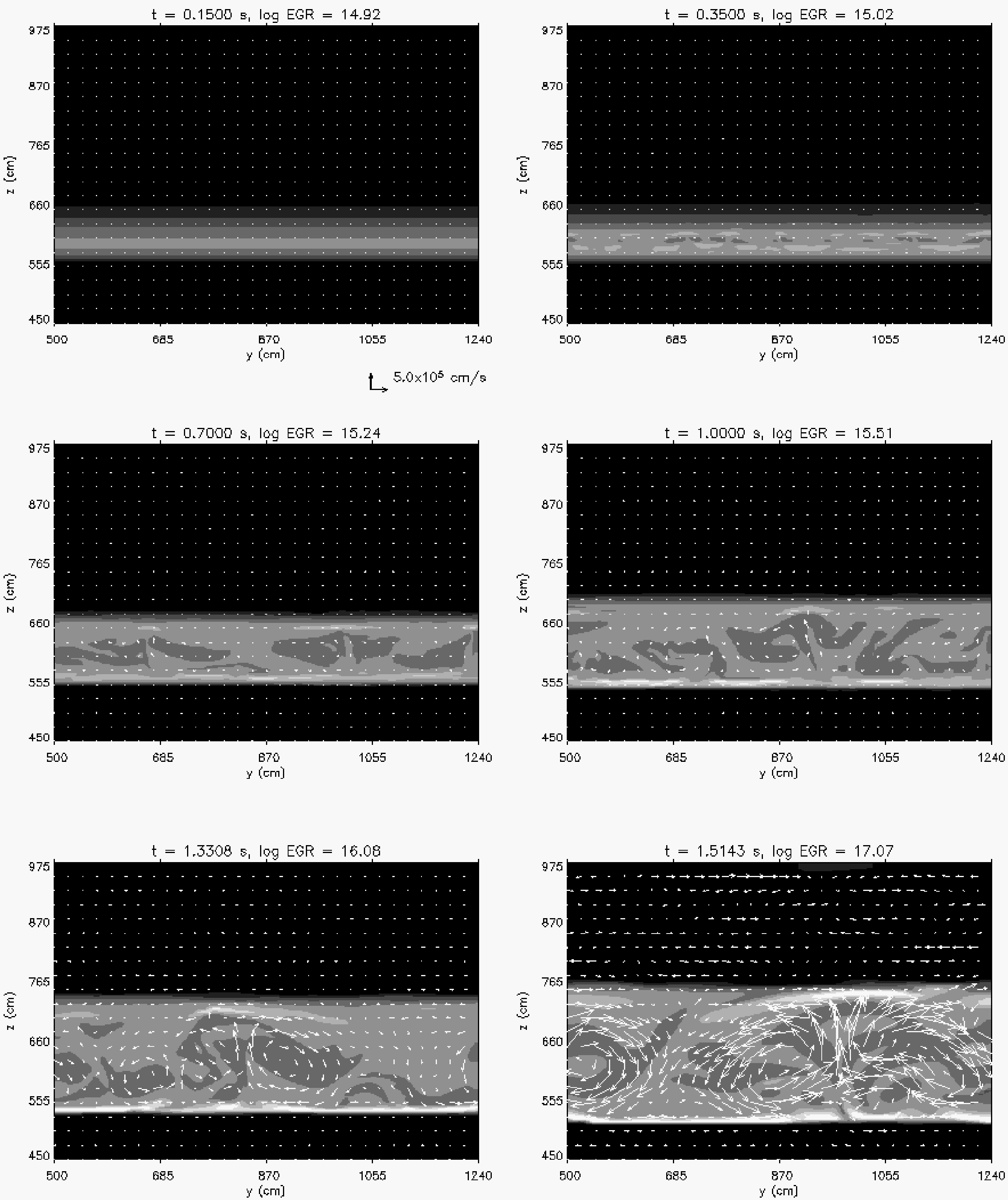


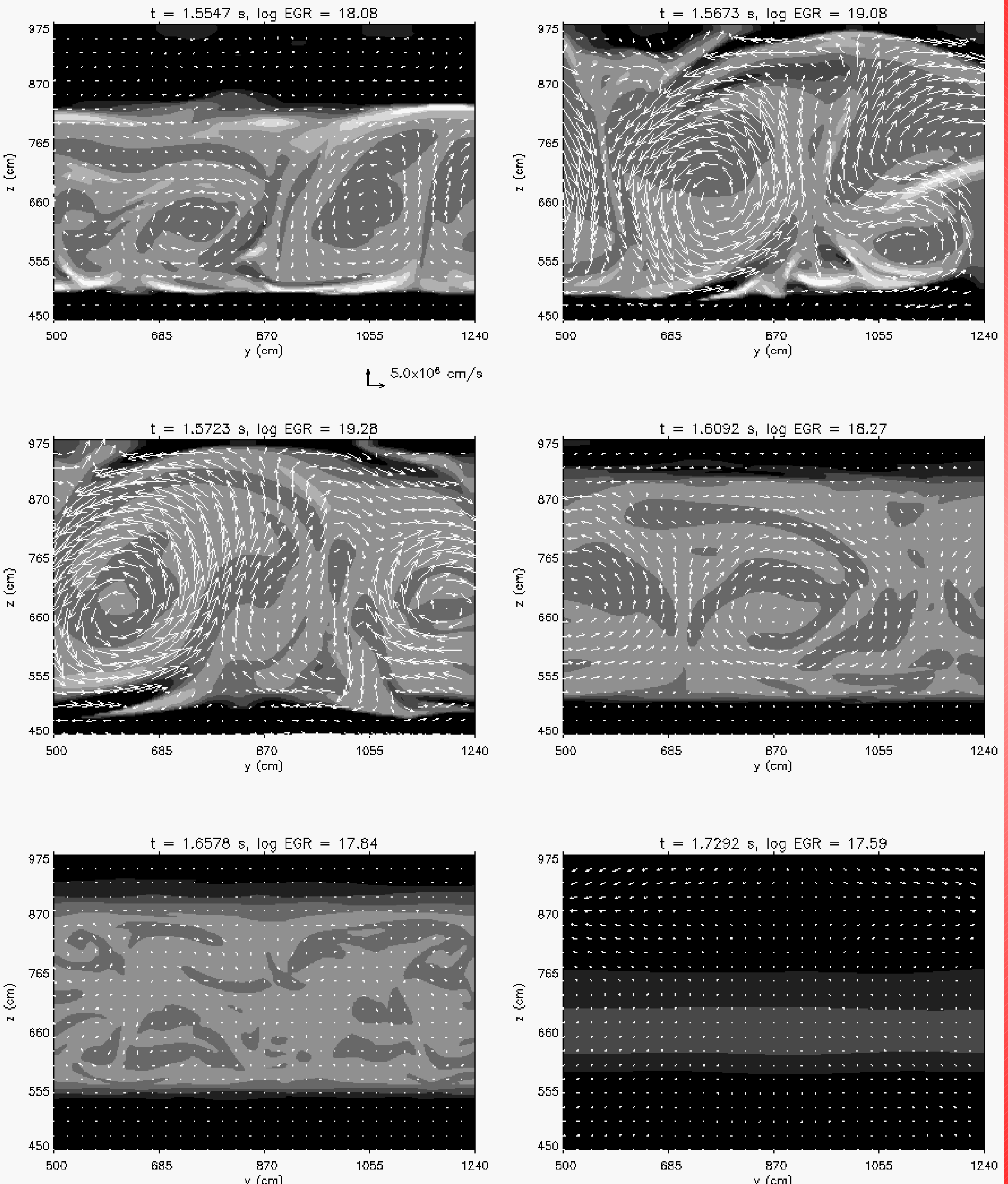


Neutron Star Envelope
Type I X-ray Burst Sequence

386x200
5 cm/zone

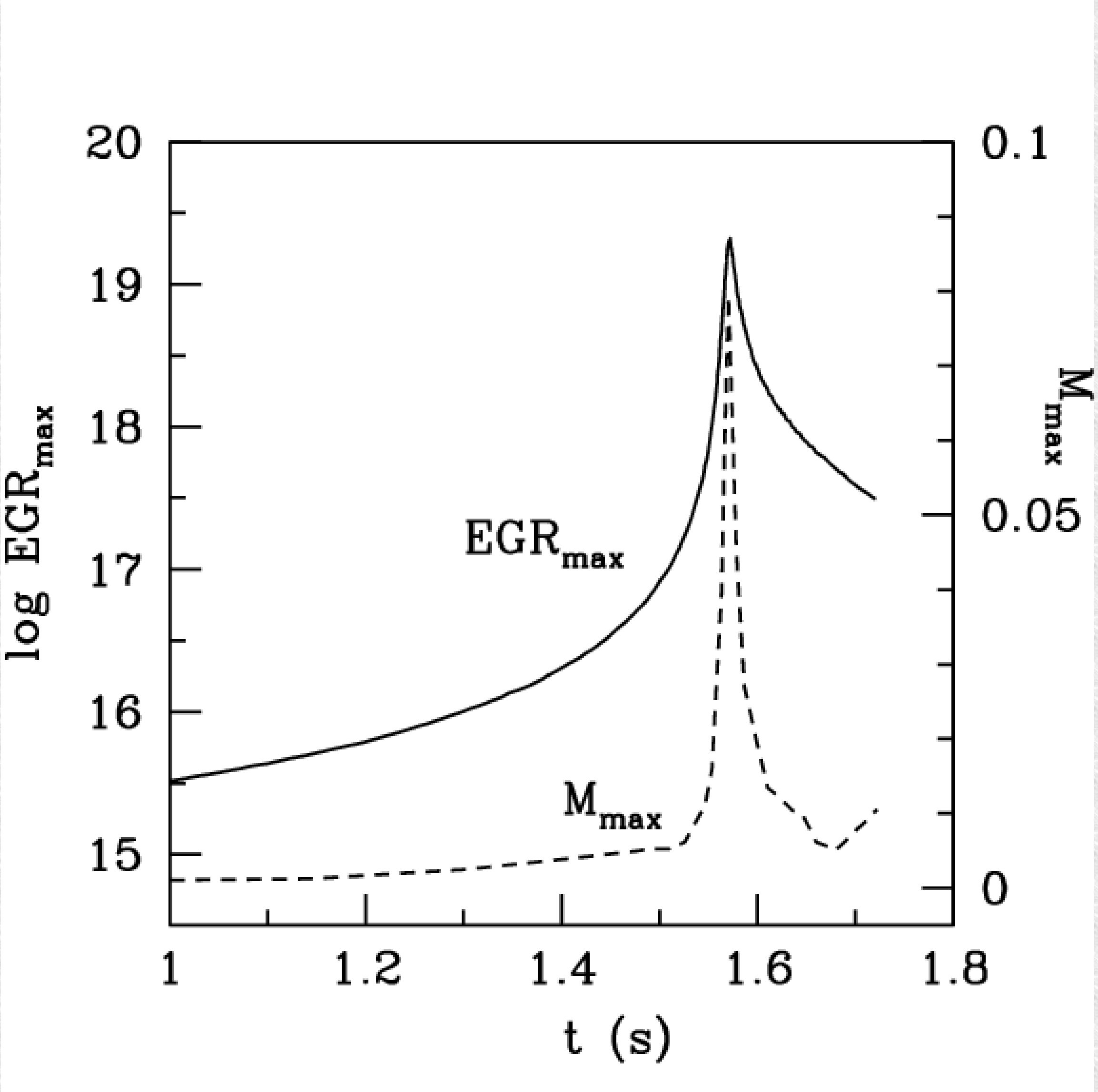
LMNA Model


t = 0.1500 s, log EGR = 14.92
t = 0.3500 s, log EGR = 15.02
t = 0.7000 s, log EGR = 15.24
t = 1.0000 s, log EGR = 15.51
t = 1.3308 s, log EGR = 16.08
t = 1.5143 s, log EGR = 17.07
5.0x10^5 cm/s
z (cm)
y (cm)
975
870
765
660
555
450
500
685
870
1055
1240


t = 1.5547 s, log EGR = 18.08
t = 1.5673 s, log EGR = 19.08
t = 1.5723 s, log EGR = 19.28
t = 1.6092 s, log EGR = 18.27
t = 1.6578 s, log EGR = 17.84
t = 1.7292 s, log EGR = 17.59
5.0x10⁶ cm/s
z (cm)
y (cm)
975
870
765
660
555
450
500
685
870
1055
1240

Burst Evolution: Global Diagnostics

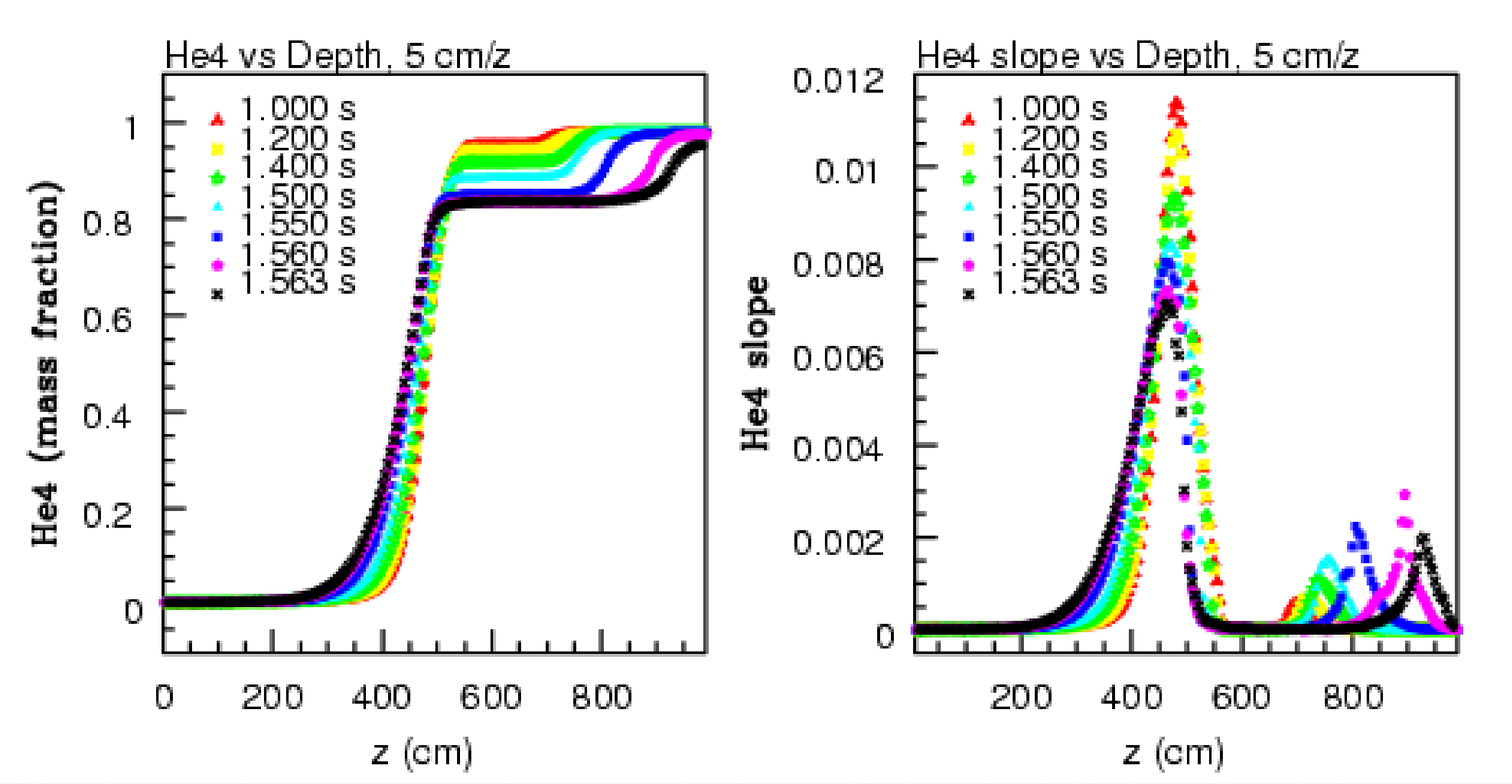
- $EGR_{peak} \sim 2x10^{19}$ erg g^{-1} s^{-1} (t_{peak} = 1.572 s)
- $T_{peak} \sim 1.7x10^{9}$ K
- M_{peak} = 0.085
- Consumed ~75% of fuel



Convective Layer Expansion Speeds

- Convective layer expands due to thermal diffusion of heat from bursting layer

- Lower boundary velocity $\sim 10^2$ cm s^{-1}
- Upper boundary velocity $\sim 10^4$ cm s^{-1}



Velocity Correlation, Gradients

Velocity correlations help quantify the extent and evolution of the convective layer:

$$W_{(corr)} = \frac{\langle w_k w_{ref} \rangle}{\langle w_k \rangle^{1/2} \langle w_{ref} \rangle^{1/2}}$$

$k_{ref} = \textit{center of conv. layer at} \log EGR = 16$

Actual temperature gradient:

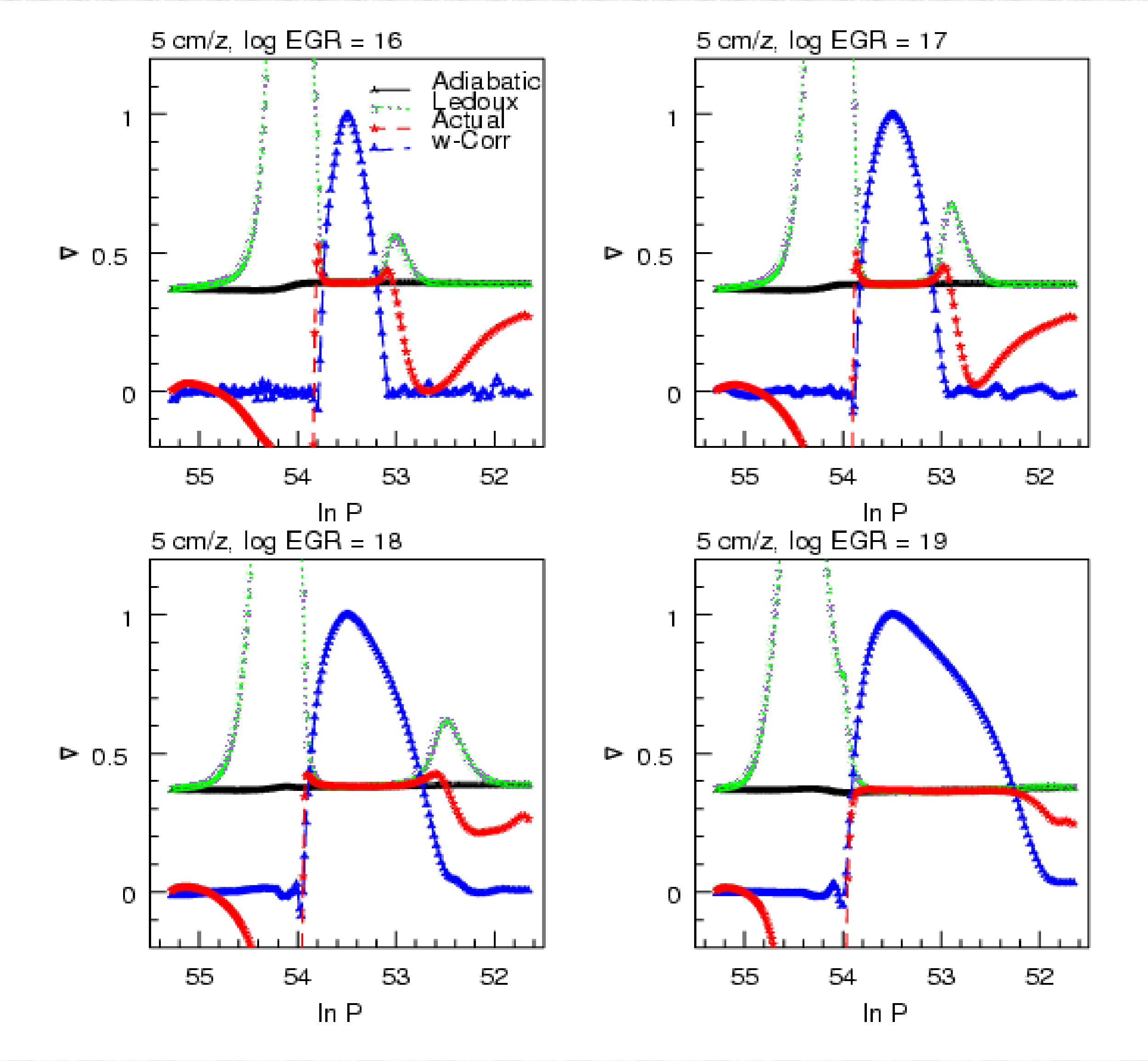
$$\nabla = \left(\frac{\partial \ln T}{\partial \ln P} \right)_{actual}$$

Adiabatic temperature gradient:

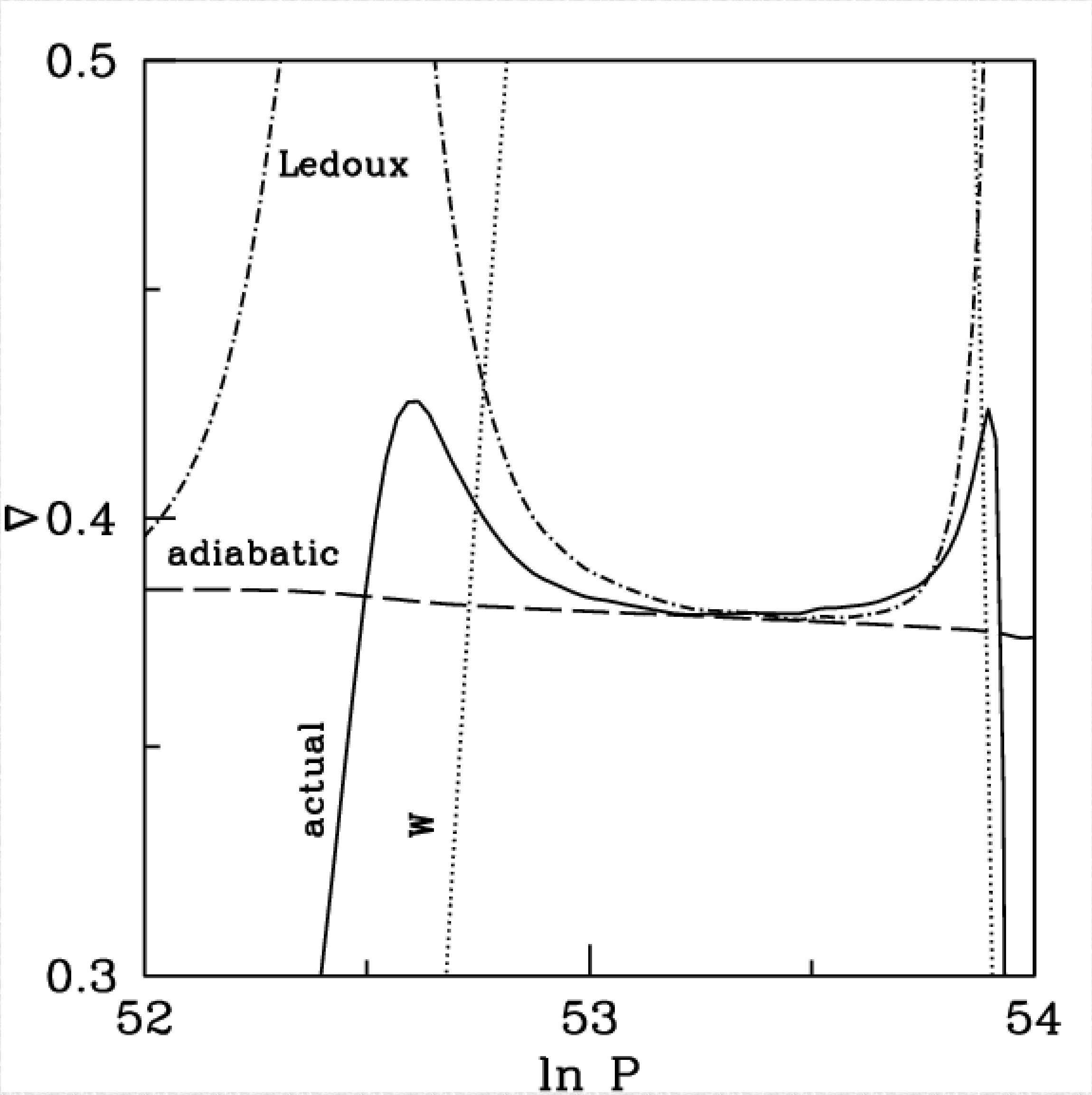
$$\nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln P} \right)_S$$

Ledoux temperature gradient:

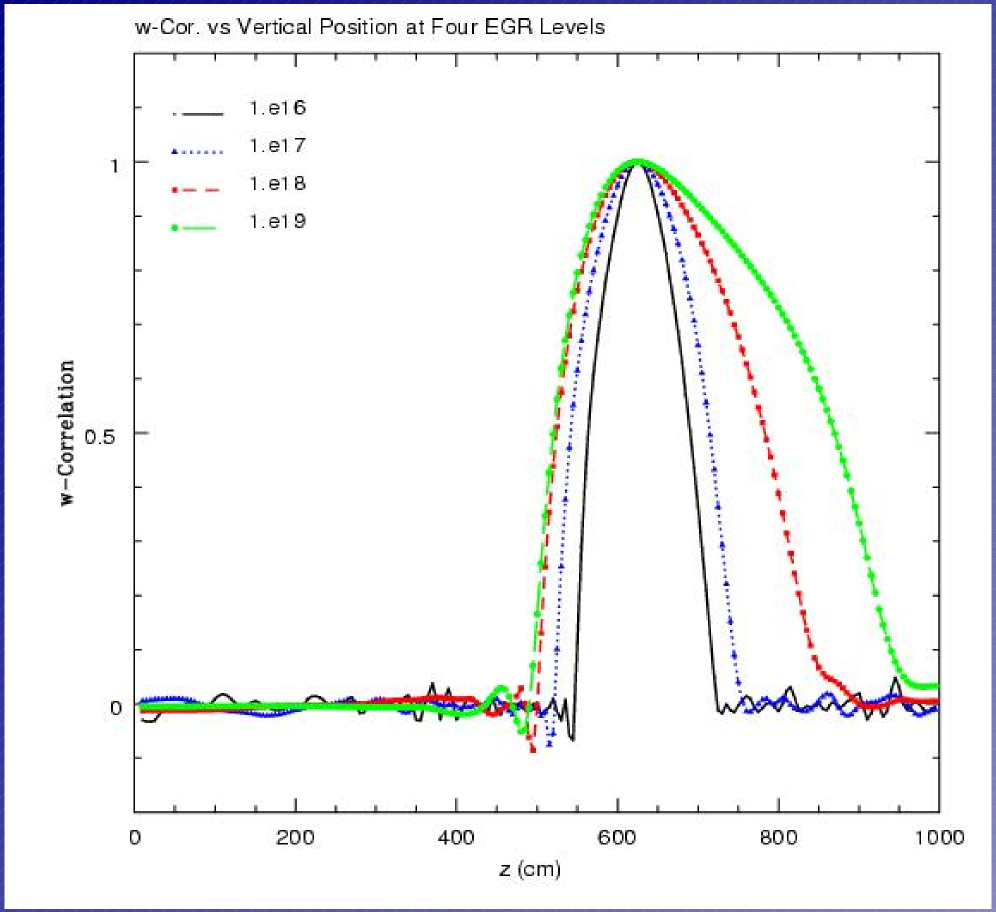
$$\nabla_L = \nabla_{ad} + \frac{c_1}{c_2} \left(\frac{\partial \ln \mu}{\partial \ln P} \right)$$



Velocity Correlation, Gradients

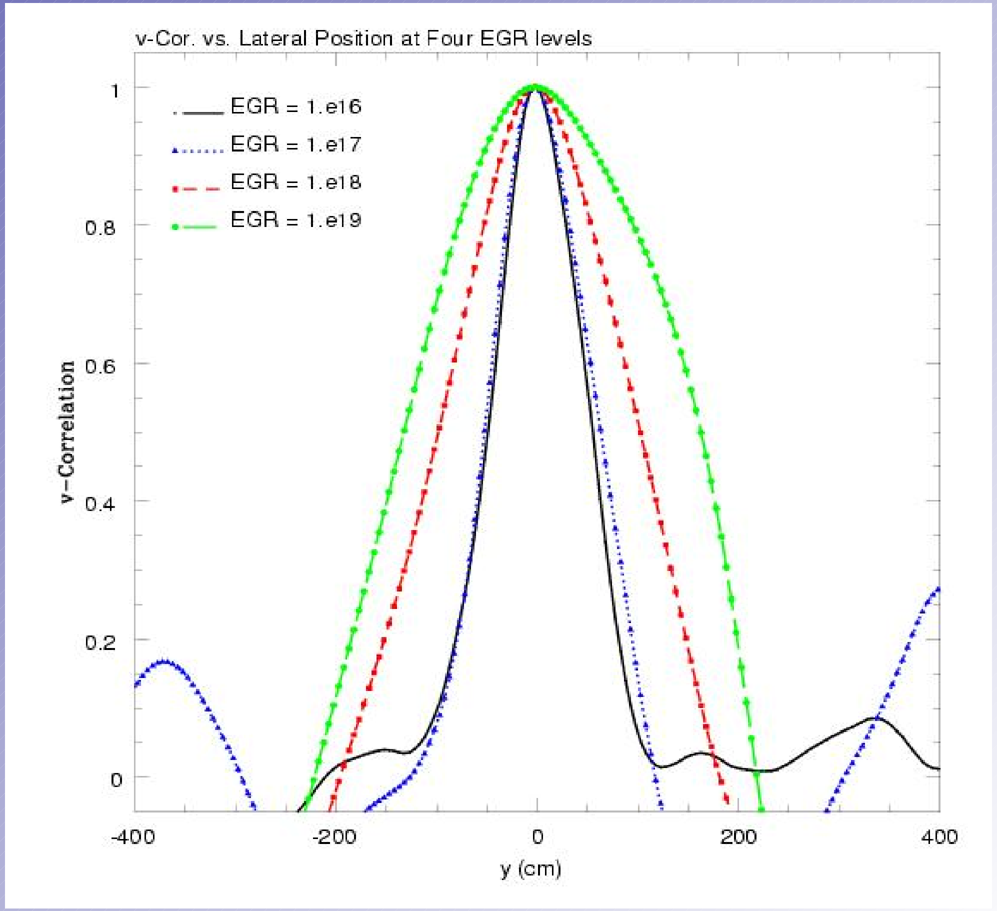


Log EGR 18



Vertical (W) Correlations

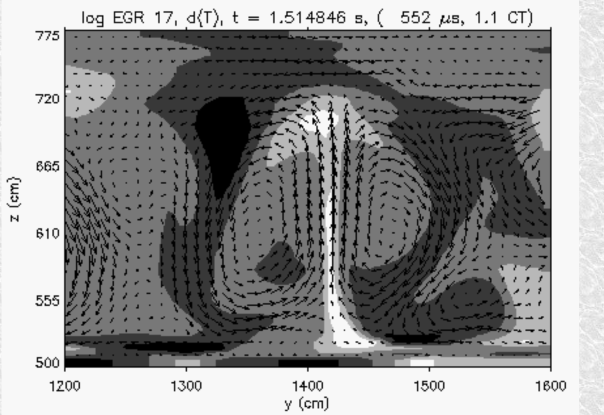
Lateral (V) Correlations



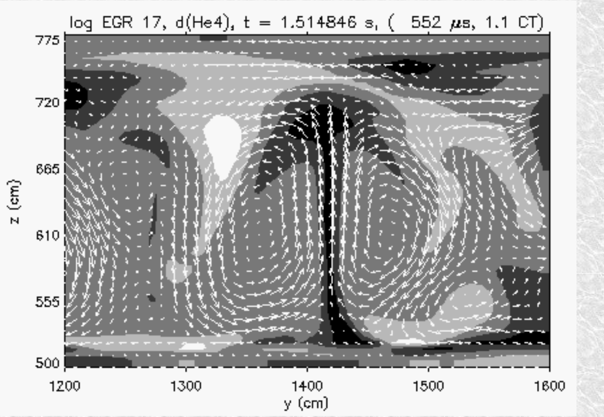
Thermodynamic Fluctuations I

- Fluctuations in temperature T and composition Y are calculated from lateral averages:

$$d(A)_{j,k} = A_{j,k} - A_{k\,ave} \qquad A_{k\,ave} = \frac{\sum_{j=1}^{j_{max}-1} A_{j,k}}{j_{max}-1}$$



	$d(T)$	$d(Y)$
Upflows:	> 0	< 0
Downflows:	< 0	> 0



- Complementary behavior between T and Y always holds!

$$-0.10 < d(T)/T < +0.10 \qquad -0.02 < d(Y)/Y < +0.02$$

Neutron Star Envelope
d(T) & d(Y)
log EGR = 18.5 (t = 1.559-1.563)

386x200
5 cm/zone

LMNA Model

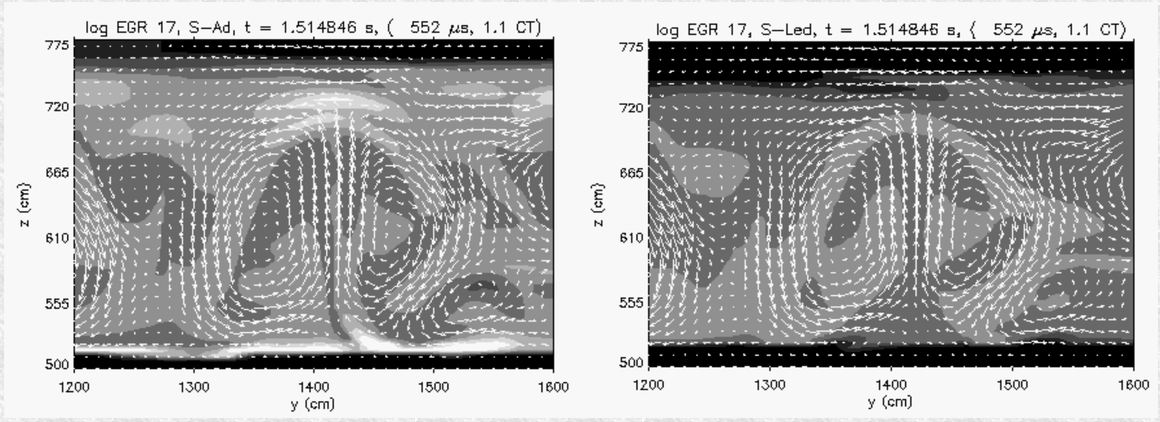
Thermodynamic Fluctuations II

- Fluctuations in adiabaticity SAd and Ledoux excess $SLed$:

$$SAd_{j,k} = \nabla_{(j,k)} - \nabla_{ad(j,k)} \qquad SLed_{j,k} = \nabla_{(j,k)} - \nabla_{L(j,k)}$$

$$-0.25 < SAd < +0.25 \qquad -0.25 < SLed < +0.25$$

- *SAd* is *not* obviously correlated with any instantaneous values
- *SAd* correlations with total time-integrated Δ in $\partial / \partial z$ due to:
  - *Y advection*: **strong** correlation
  - *T advection*: **strong** correlation
  - burning: minor role
  - diffusion: negligible



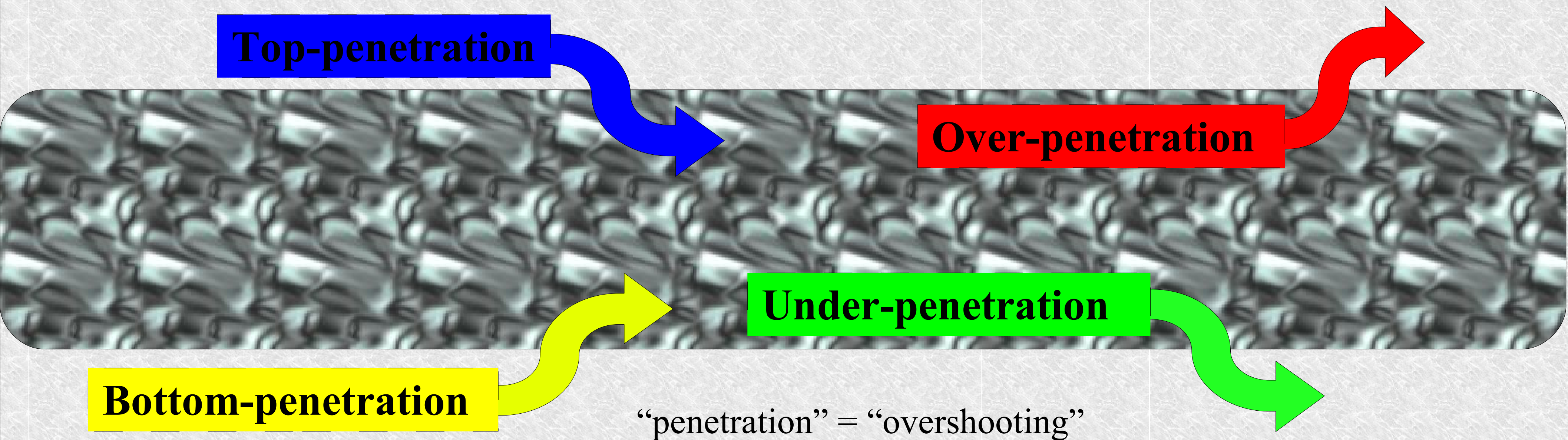
Neutron Star Envelope
SAd & SLed
log EGR = 18.5 (t = 1.559-1.564s)

386x200
5 cm/zone

LMNA Model

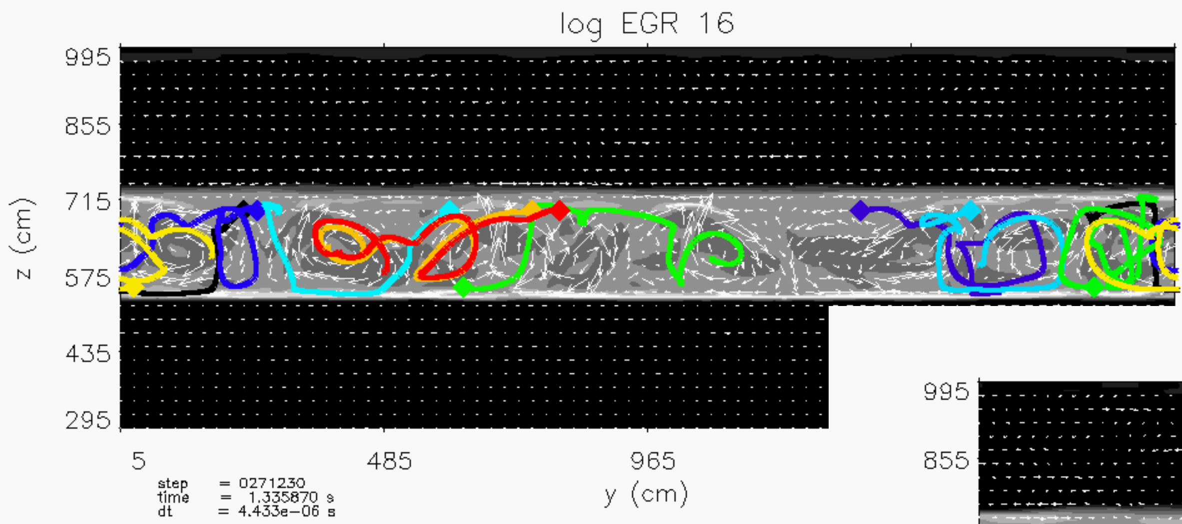
Tracer Particle Analysis

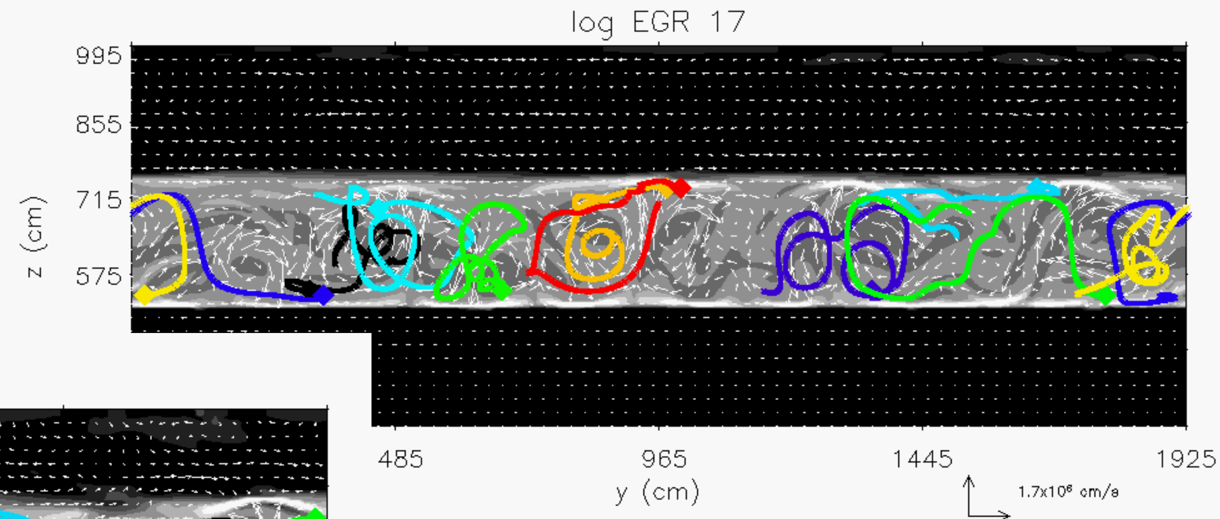
Quantify transport of material through convective boundaries:

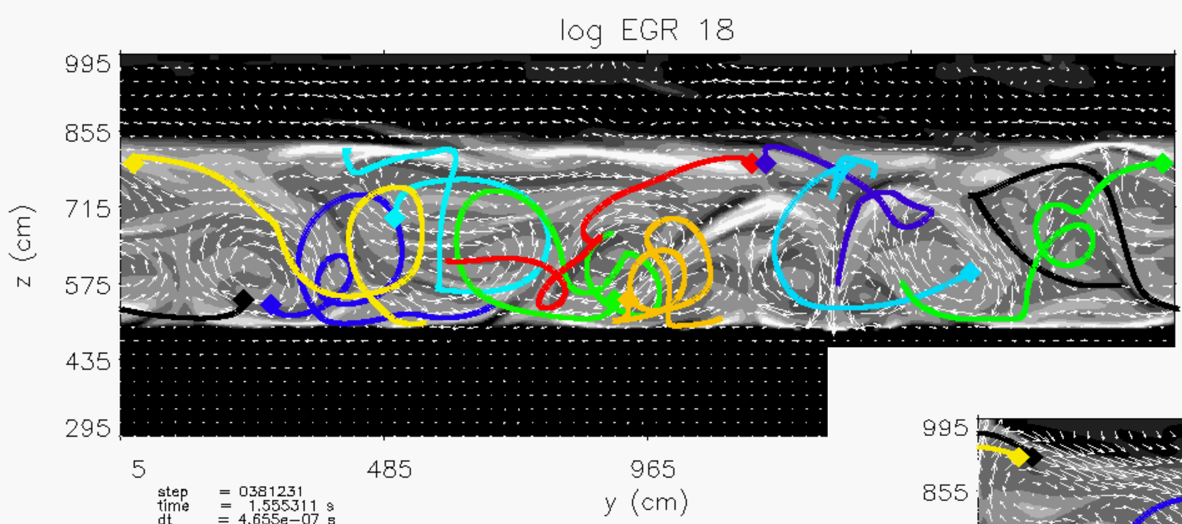


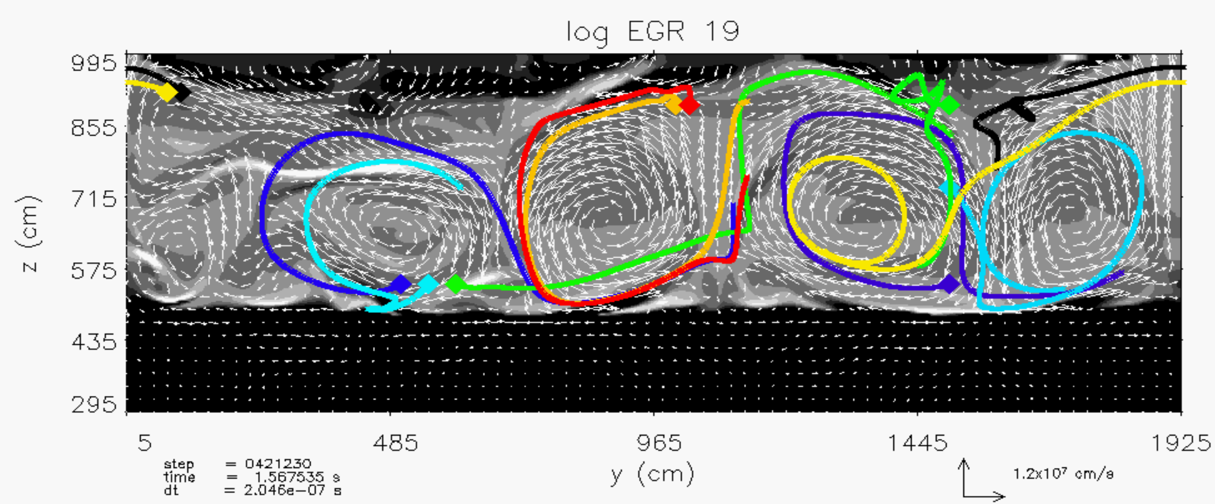
770 tracer particles (2 per lateral position)
5 representative EGR levels (16, 17, 18, 18.5, 19)
Several convective times
Forward Euler with 2D linear interpolation

Under-, *Over-* penetration:
10 representative particle paths
(stream lines)

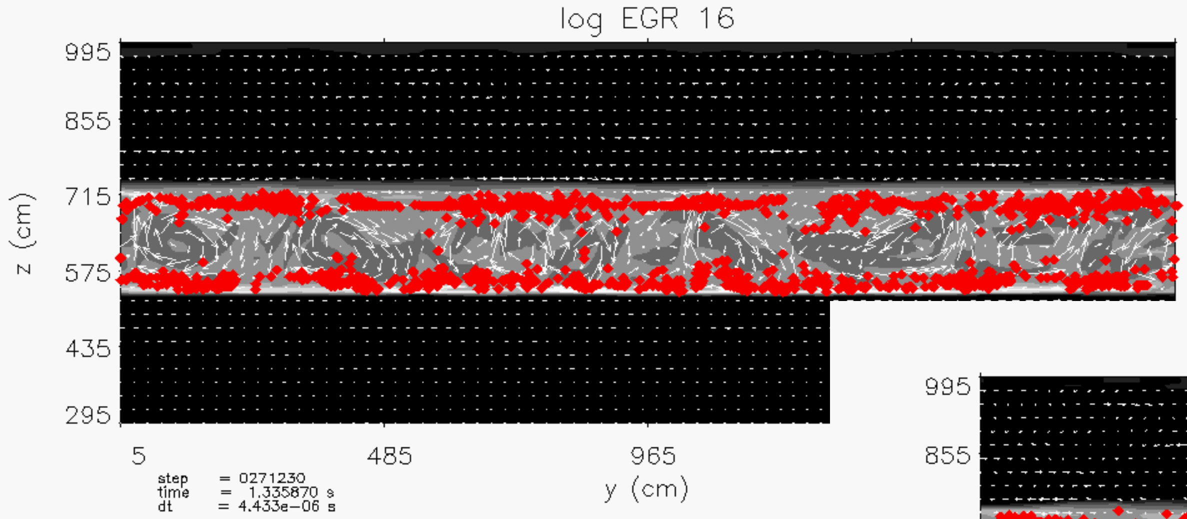


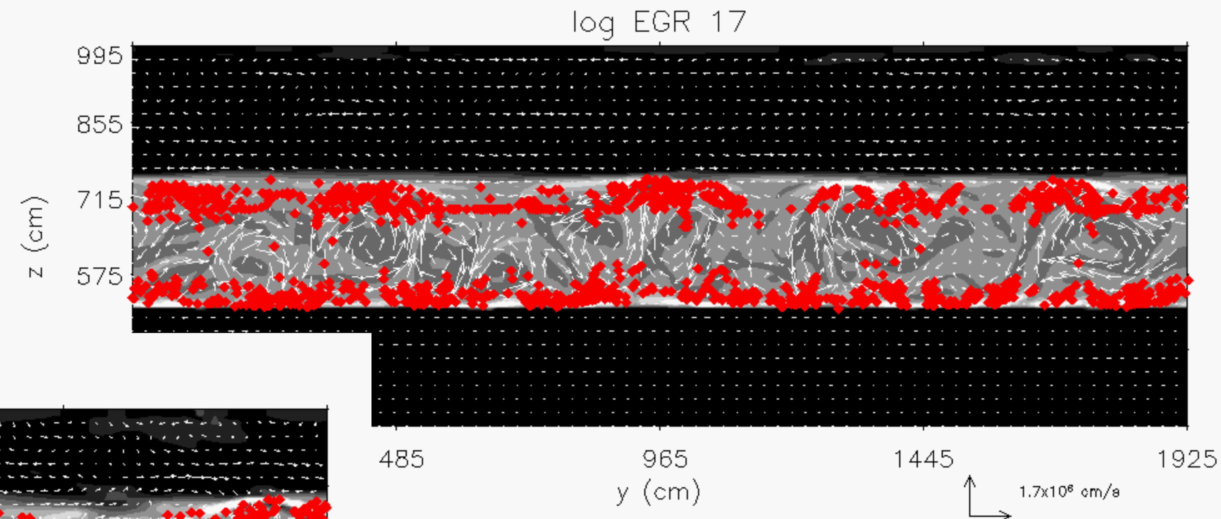


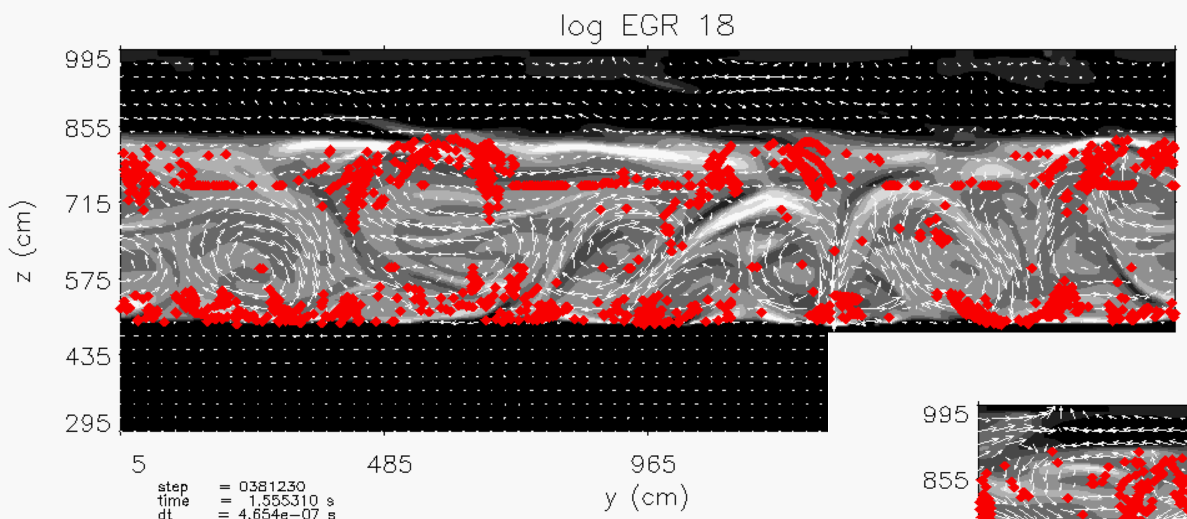


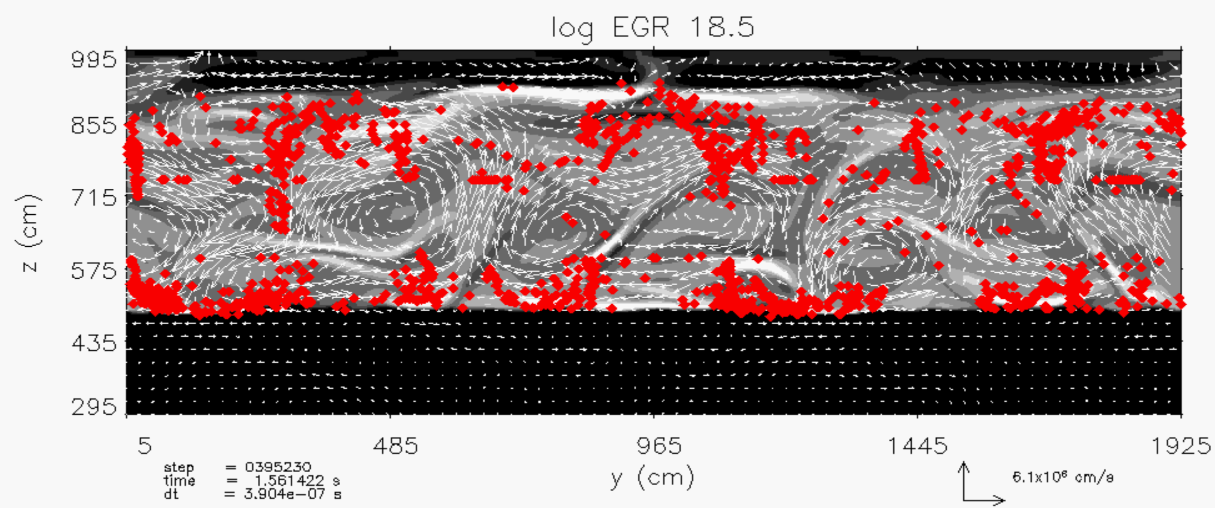


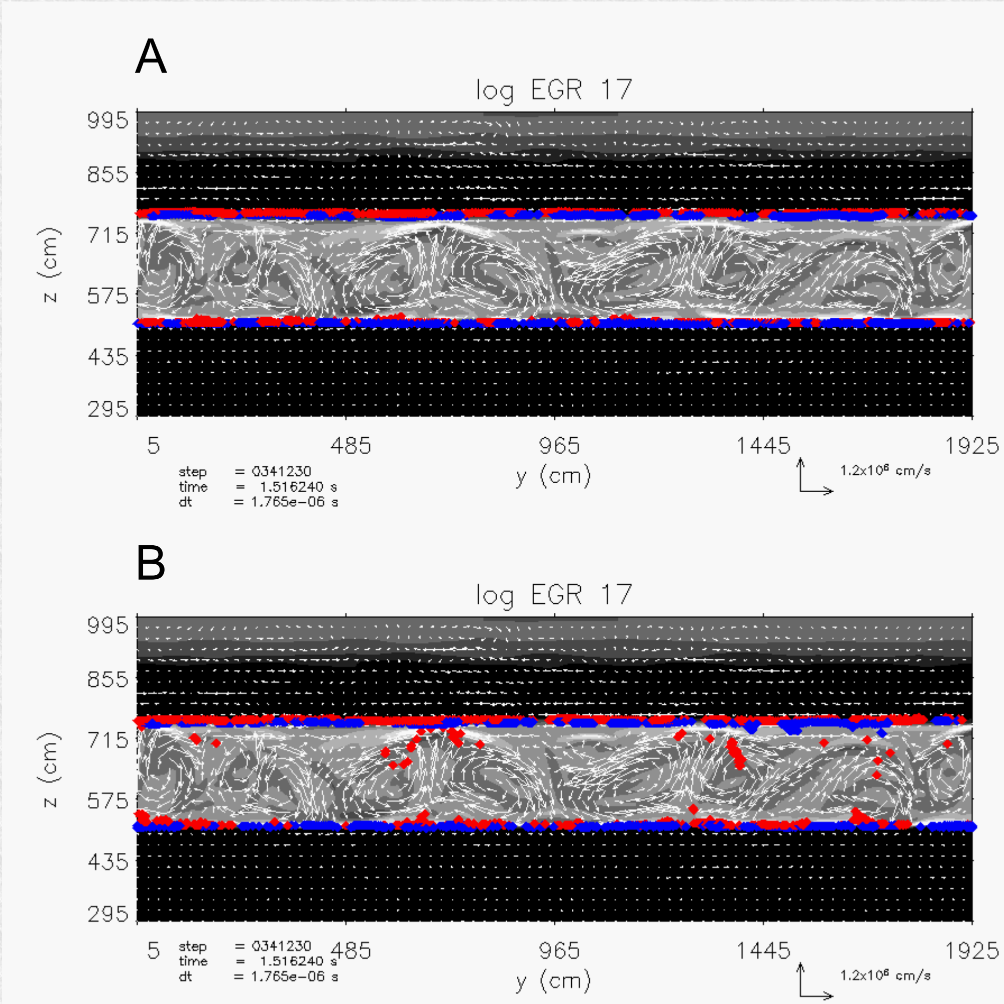
Under-, *Over-* penetration:
770 particle max/min position limits







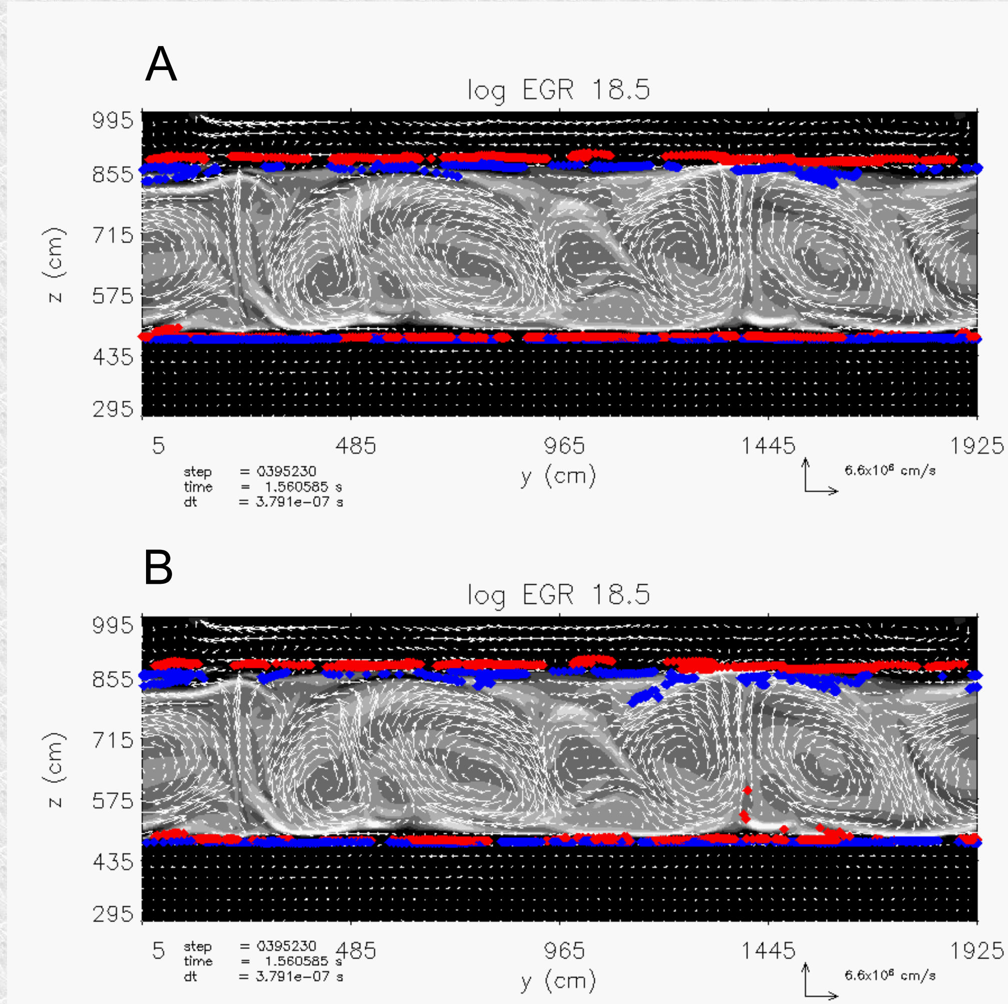




Bottom-, *Top-* penetration:

770 particle max/min position limits

For each EGR level, Study A's initial placement is 1 zone higher than Study B's

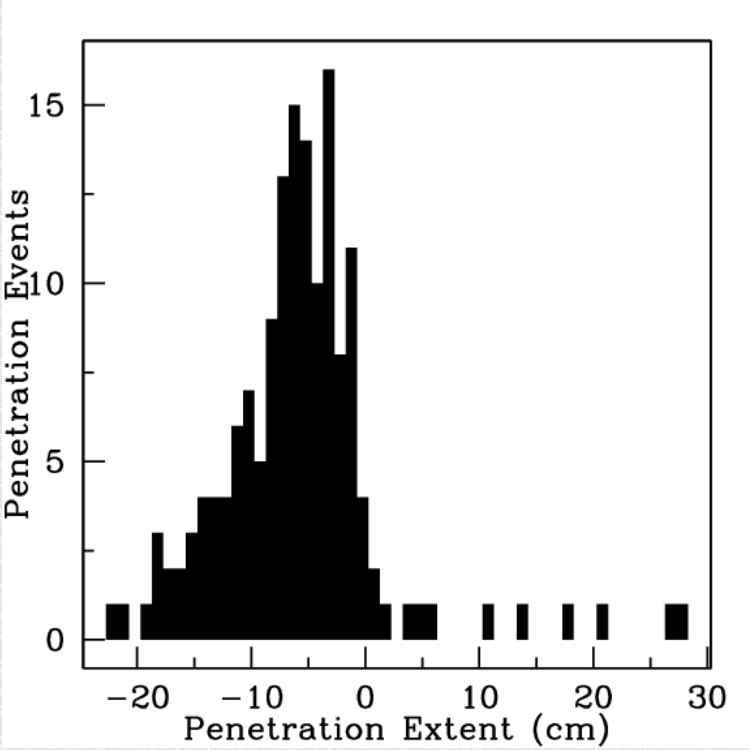


Blue = lowest particle position
Red = highest particle position

Reproducible results with extended domain (386x205)

Tracer Analysis Findings

Log EGR	Under-penetration	Over-penetration
16	None	None
17	5 cm (1/40)	None
18	10 cm (1/20)	None
18.5	20 cm (1/10)	40 cm (1/5)
19	60 cm (1/3)	Indeterminate



- Under-, over-penetration are relatively *rare* events: < 15%
- Under-, over-penetration events are *temporary* and *spatially limited*
- Bottom-, top-penetration are very *sensitive* to initial placement
- *Vertical flows* stop while *lateral flows* dominate at convective boundaries
- *Convective modes* are necessary but not sufficient for penetration to occur

Non-Local Convection

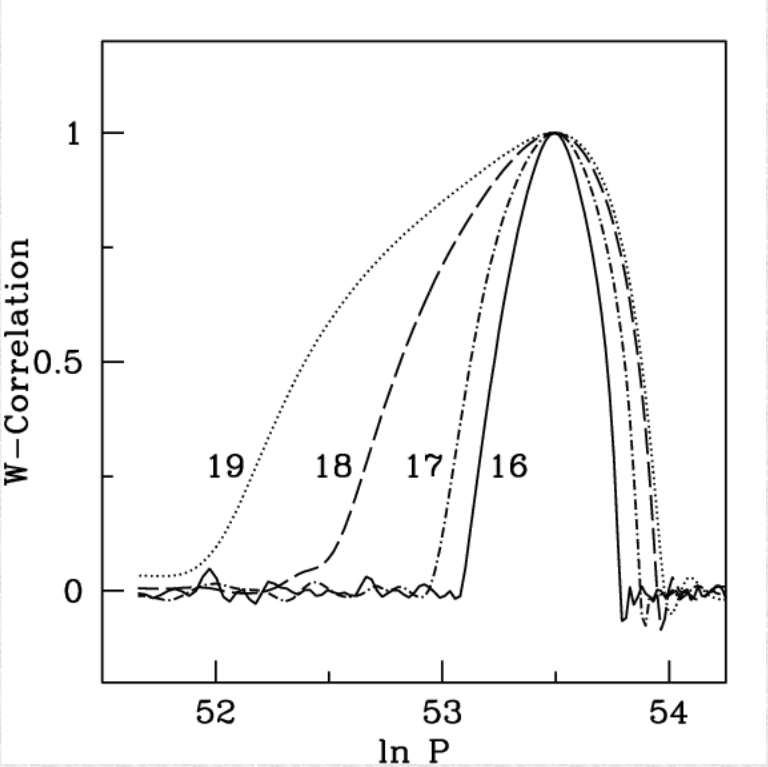
- Mixing Length Theory: a *local* theory of convection commonly used as a model for convection in simulations

$$l = \alpha H_P \qquad w \propto \alpha^2 \Delta\nabla$$

$$\Delta\nabla = \nabla - \nabla_{ad} \qquad T' \propto \alpha T \Delta\nabla$$

$$F_{conv} \propto \frac{w'}{\alpha}$$

$$F_{conv}^2 \propto \alpha T'^3$$



- No consistent value of α can be determined from these relationships
- Practical value of $\alpha \sim 1$ to 2
- Present results suggest our simulated convection is *non-local*

Fluxes

- Fluxes: nuclear (F_{nuc}), radiative (F_{rad}), advective (F_{adv})

$$F_{nuc} = \int_0^{z_{top}} \rho \dot{s}_{3\alpha} dz$$

$$F_{rad} = -\kappa \nabla T$$

$$\frac{\partial F_{adv}}{\partial z} = w \left(\rho c_p \frac{\partial T}{\partial z} - \delta \frac{\partial P}{\partial z} \right)$$

$$F_{adv}(z') = \int_0^{z'} \frac{\partial F_{adv}}{\partial z} dz$$

- Light-curves cannot be accurately calculated wth current model, because the surface of star is not presently modeled

- Significant differences between 1D and 2D flux behavior, attributable to convective energy transport in 2D

- 1D models need to properly account for the effects of convection in order to produce more realistic light-curves

Model	Rise time (s)	Fall time (s)
1D	0.030	0.032
2D	0.103	0.014

Nuclear Flux

2D burst significantly delayed and greater in magnitude due to:

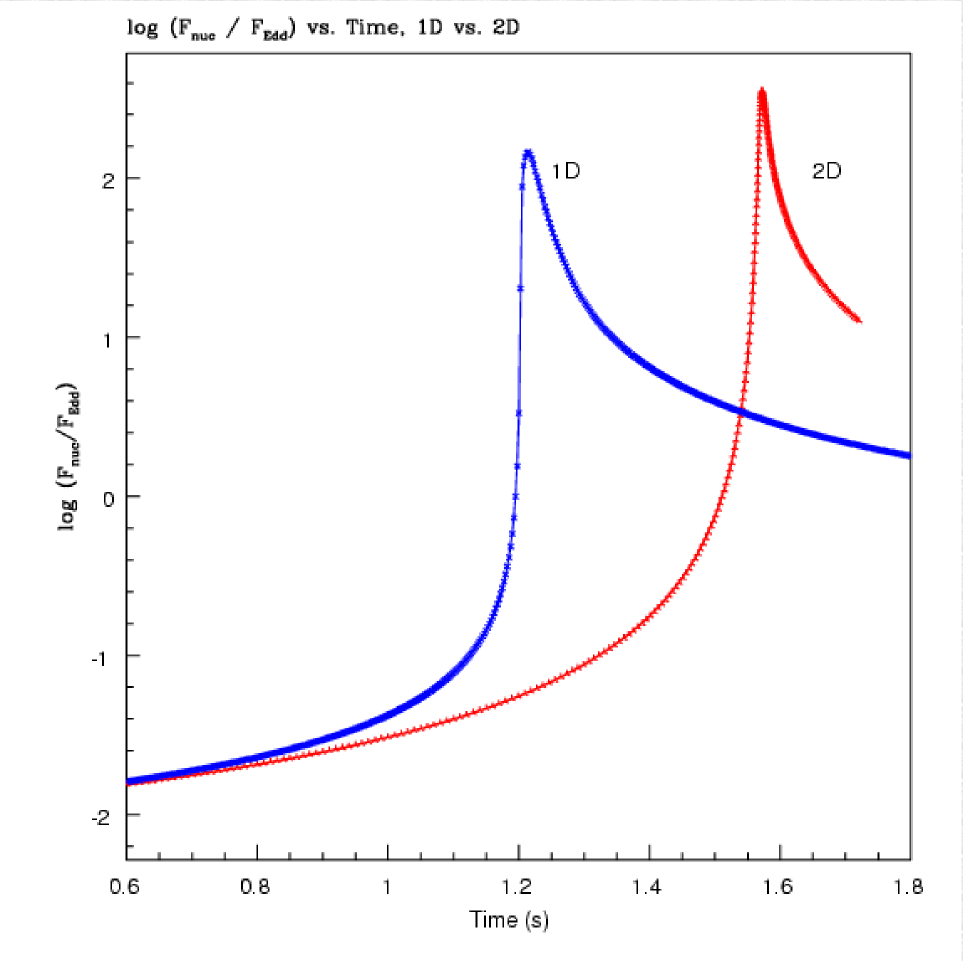
i) convective cooling

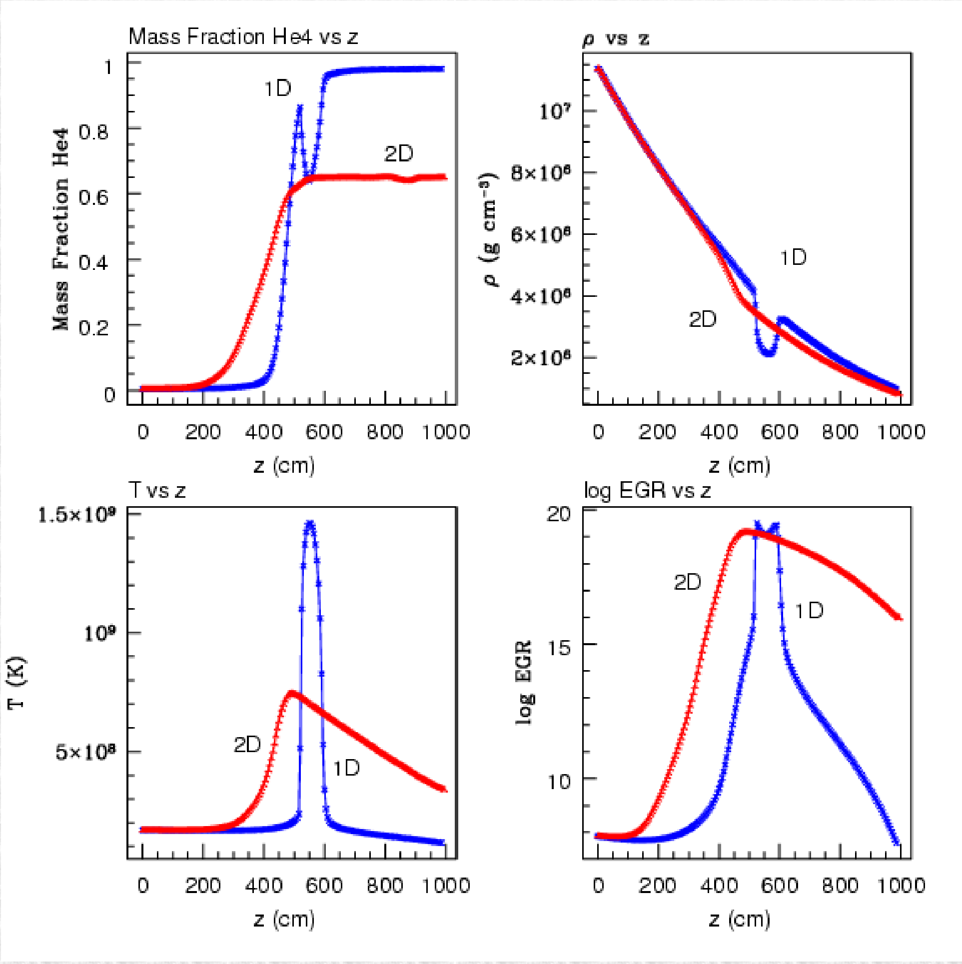
ii) convective modification of ∇

iii) convective mixing of fuel

$F_{Edd} = 2.5\times10^{25}$ erg s^{-1} cm^{-2}

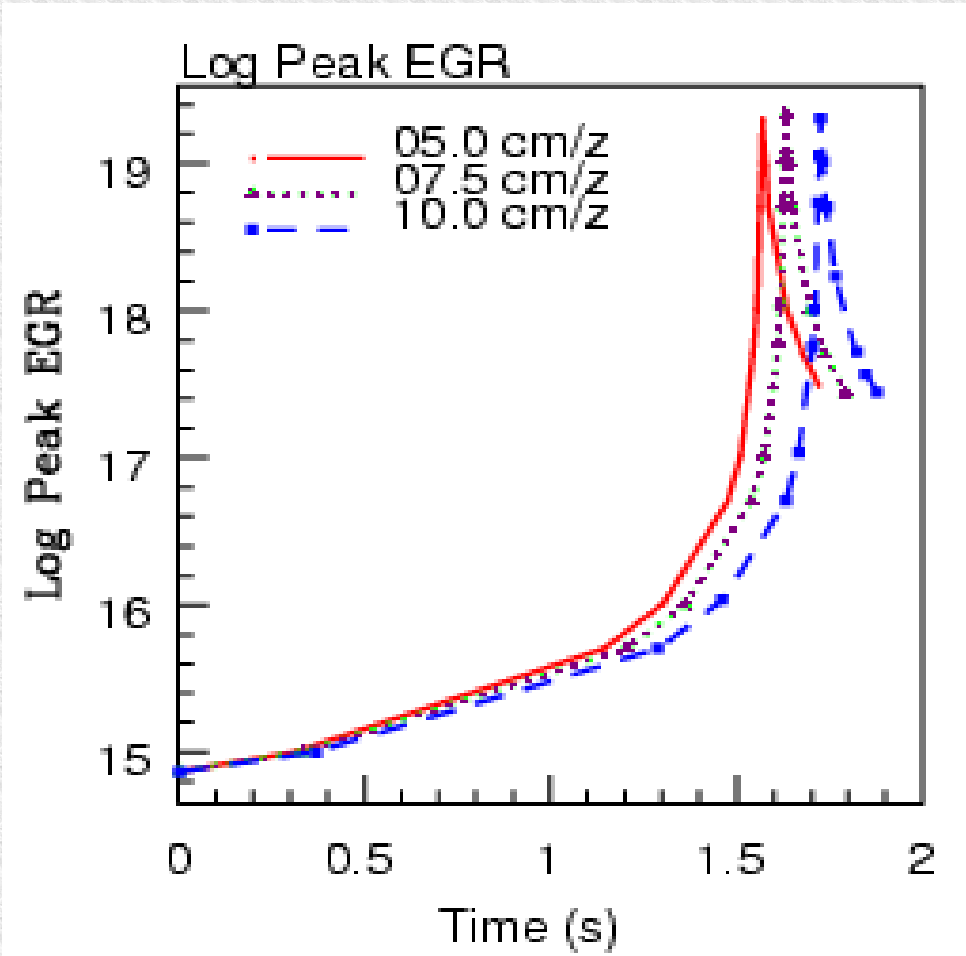
Rise time = t (Edd) to t (peak)
Fall time = t (e^{-1} peak) to t (peak)

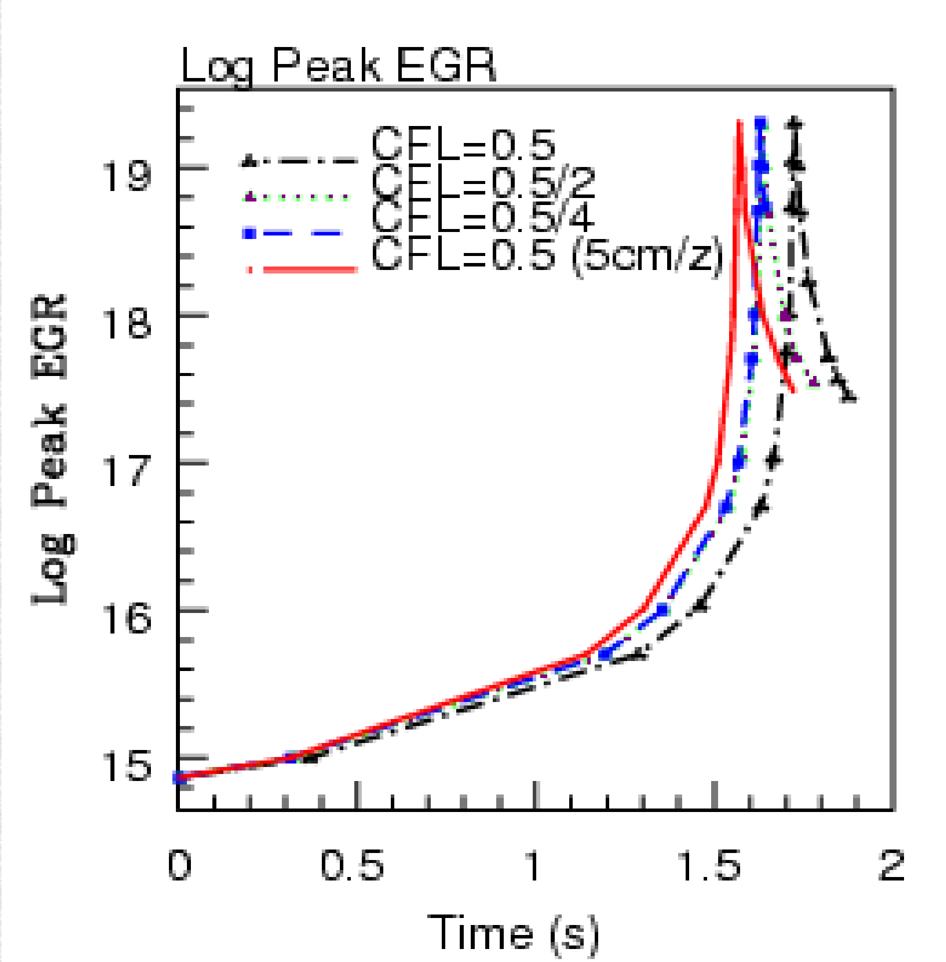




LMNA Model Validation

- Rigorous testing of separate modules for consistency
  (e.g. advection, burning, diffusion, elliptic solver, parallelism, etc)
- Refinements: spatial and temporal resolution, domain sizes:





The Present

Computational

- LMNA model successfully implemented in 2D
- LMNA verified by zone, time, domain refinements
- LMNA applied to model Type I X-ray burst in a 2D patch
- Computational time savings about a factor of 10-100 compared to fully explicit Eulerian methods

Astrophysical

- Burning layer relatively stationary
- Decidedly subsonic ($M < 0.10$)
- Convection self-organizes into Benard-like cells which fill up the convective layer:
  - height of major cells = height of layer
  - superadiabatic on average
  - vertically expands due to thermal diffusion, facilitates mixing from radiative regions
  - mixing is very efficient within it
  - limited penetration through convective boundaries on convective time-scales
  - $\nabla_{ad} < \nabla < \nabla_L$
  - $\nabla \sim \nabla_L$ at convective boundaries
- Convective dynamics significantly affects energy transport

The Future

Computational

- Time-dependent base state
- *3D*
- Rotation
- Additional nuclear burning networks
- Other coordinate systems
- Adaptive gridding
- Turbulence model

Astrophysical

- Astophysical deflagrations:
  - Type I X-ray bursts
  - Pre-ejection stage of classical novae
  - Pre-detonation stage of supernovae
  - Hydrodynamics and burning in cores of main sequence stars