Optimal State Estimation

Optimal State Estimation

Kalman, H_∞, and Nonlinear Approaches

Dan Simon
Cleveland State University

WILEY-INTERSCIENCE

A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

For general information on our other products and services or for technical support, please contact our Customer Care Department within the U.S. at (800) 762-2974, outside the U.S. at (317) 572-3993 or fax (317) 572-4002.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic format. For information about Wiley products, visit our web site at www.wiley.com.

Library of Congress Cataloging-in-Publication is available.

ISBN-13 978-0-471-70858-2
ISBN-10 0-471-70858-5

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

CONTENTS

Acknowledgments xiii

Acronyms xv

List of algorithms xvii

Introduction xxi

PART I INTRODUCTORY MATERIAL

1 Linear systems theory 3

1.1 Matrix algebra and matrix calculus 4
- 1.1.1 Matrix algebra 6
- 1.1.2 The matrix inversion lemma 11
- 1.1.3 Matrix calculus 14
- 1.1.4 The history of matrices 17

1.2 Linear systems 18

1.3 Nonlinear systems 22

1.4 Discretization 26

1.5 Simulation 27
- 1.5.1 Rectangular integration 29
- 1.5.2 Trapezoidal integration 29
- 1.5.3 Runge–Kutta integration 31

1.6 Stability 33

1.6.1 Continuous-time systems 33
1.6.2 Discrete-time systems 37
1.7 Controllability and observability 38
1.7.1 Controllability 38
1.7.2 Observability 40
1.7.3 Stabilizability and detectability 43
1.8 Summary 45
Problems 45

2 Probability theory **49**

2.1 Probability 50
2.2 Random variables 53
2.3 Transformations of random variables 59
2.4 Multiple random variables 61
2.4.1 Statistical independence 62
2.4.2 Multivariate statistics 65
2.5 Stochastic Processes 68
2.6 White noise and colored noise 71
2.7 Simulating correlated noise 73
2.8 Summary 74
Problems 75

3 Least squares estimation **79**

3.1 Estimation of a constant 80
3.2 Weighted least squares estimation 82
3.3 Recursive least squares estimation 84
3.3.1 Alternate estimator forms 86
3.3.2 Curve fitting 92
3.4 Wiener filtering 94
3.4.1 Parametric filter optimization 96
3.4.2 General filter optimization 97
3.4.3 Noncausal filter optimization 98
3.4.4 Causal filter optimization 100
3.4.5 Comparison 101
3.5 Summary 102
Problems 102

4 Propagation of states and covariances **107**

4.1 Discrete-time systems 107
4.2 Sampled-data systems 111
4.3 Continuous-time systems 114

4.4 Summary 117
Problems 117

PART II THE KALMAN FILTER

5 The discrete-time Kalman filter 123

5.1 Derivation of the discrete-time Kalman filter 124
5.2 Kalman filter properties 129
5.3 One-step Kalman filter equations 131
5.4 Alternate propagation of covariance 135
5.4.1 Multiple state systems 135
5.4.2 Scalar systems 137
5.5 Divergence issues 139
5.6 Summary 144
Problems 145

6 Alternate Kalman filter formulations 149

6.1 Sequential Kalman filtering 150
6.2 Information filtering 156
6.3 Square root filtering 158
6.3.1 Condition number 159
6.3.2 The square root time-update equation 162
6.3.3 Potter's square root measurement-update equation 165
6.3.4 Square root measurement update via triangularization 169
6.3.5 Algorithms for orthogonal transformations 171
6.4 U-D filtering 174
6.4.1 U-D filtering: The measurement-update equation 174
6.4.2 U-D filtering: The time-update equation 176
6.5 Summary 178
Problems 179

7 Kalman filter generalizations 183

7.1 Correlated process and measurement noise 184
7.2 Colored process and measurement noise 188
7.2.1 Colored process noise 188
7.2.2 Colored measurement noise: State augmentation 189
7.2.3 Colored measurement noise: Measurement differencing 190
7.3 Steady-state filtering 193
7.3.1 α-β filtering 199
7.3.2 α-β-γ filtering 202
7.3.3 A Hamiltonian approach to steady-state filtering 203
7.4 Kalman filtering with fading memory 208

7.5 Constrained Kalman filtering 212
7.5.1 Model reduction 212
7.5.2 Perfect measurements 213
7.5.3 Projection approaches 214
7.5.4 A pdf truncation approach 218
7.6 Summary 223
Problems 225

8 The continuous-time Kalman filter **229**

8.1 Discrete-time and continuous-time white noise 230
8.1.1 Process noise 230
8.1.2 Measurement noise 232
8.1.3 Discretized simulation of noisy continuous-time systems 232
8.2 Derivation of the continuous-time Kalman filter 233
8.3 Alternate solutions to the Riccati equation 238
8.3.1 The transition matrix approach 238
8.3.2 The Chandrasekhar algorithm 242
8.3.3 The square root filter 246
8.4 Generalizations of the continuous-time filter 247
8.4.1 Correlated process and measurement noise 248
8.4.2 Colored measurement noise 249
8.5 The steady-state continuous-time Kalman filter 252
8.5.1 The algebraic Riccati equation 253
8.5.2 The Wiener filter is a Kalman filter 257
8.5.3 Duality 258
8.6 Summary 259
Problems 260

9 Optimal smoothing **263**

9.1 An alternate form for the Kalman filter 265
9.2 Fixed-point smoothing 267
9.2.1 Estimation improvement due to smoothing 270
9.2.2 Smoothing constant states 274
9.3 Fixed-lag smoothing 274
9.4 Fixed-interval smoothing 279
9.4.1 Forward–backward smoothing 280
9.4.2 RTS smoothing 286
9.5 Summary 294
Problems 294

10 Additional topics in Kalman filtering 297

10.1 Verifying Kalman filter performance 298
10.2 Multiple-model estimation 301
10.3 Reduced-order Kalman filtering 305
10.3.1 Anderson's approach to reduced-order filtering 306
10.3.2 The reduced-order Schmidt–Kalman filter 309
10.4 Robust Kalman filtering 312
10.5 Delayed measurements and synchronization errors 317
10.5.1 A statistical derivation of the Kalman filter 318
10.5.2 Kalman filtering with delayed measurements 320
10.6 Summary 325
Problems 326

PART III THE H_∞ FILTER

11 The H_∞ filter 333

11.1 Introduction 334
11.1.1 An alternate form for the Kalman filter 334
11.1.2 Kalman filter limitations 336
11.2 Constrained optimization 337
11.2.1 Static constrained optimization 337
11.2.2 Inequality constraints 339
11.2.3 Dynamic constrained optimization 341
11.3 A game theory approach to H_∞ filtering 343
11.3.1 Stationarity with respect to x_0 and w_k 345
11.3.2 Stationarity with respect to $\hat{x}$ and y 347
11.3.3 A comparison of the Kalman and H_∞ filters 354
11.3.4 Steady-state H_∞ filtering 354
11.3.5 The transfer function bound of the H_∞ filter 357
11.4 The continuous-time H_∞ filter 361
11.5 Transfer function approaches 365
11.6 Summary 367
Problems 369

12 Additional topics in H_∞ filtering 373

12.1 Mixed Kalman/H_∞ filtering 374
12.2 Robust Kalman/H_∞ filtering 377
12.3 Constrained H_∞ filtering 381
12.4 Summary 388
Problems 389

PART IV NONLINEAR FILTERS

13 Nonlinear Kalman filtering **395**

13.1 The linearized Kalman filter 397
13.2 The extended Kalman filter 400
13.2.1 The continuous-time extended Kalman filter 400
13.2.2 The hybrid extended Kalman filter 403
13.2.3 The discrete-time extended Kalman filter 407
13.3 Higher-order approaches 410
13.3.1 The iterated extended Kalman filter 410
13.3.2 The second-order extended Kalman filter 413
13.3.3 Other approaches 420
13.4 Parameter estimation 422
13.5 Summary 425
Problems 426

14 The unscented Kalman filter **433**

14.1 Means and covariances of nonlinear transformations 434
14.1.1 The mean of a nonlinear transformation 434
14.1.2 The covariance of a nonlinear transformation 437
14.2 Unscented transformations 441
14.2.1 Mean approximation 441
14.2.2 Covariance approximation 444
14.3 Unscented Kalman filtering 447
14.4 Other unscented transformations 452
14.4.1 General unscented transformations 452
14.4.2 The simplex unscented transformation 454
14.4.3 The spherical unscented transformation 455
14.5 Summary 457
Problems 458

15 The particle filter **461**

15.1 Bayesian state estimation 462
15.2 Particle filtering 466
15.3 Implementation issues 469
15.3.1 Sample impoverishment 469
15.3.2 Particle filtering combined with other filters 477
15.4 Summary 480
Problems 481

Appendix A: Historical perspectives **485**

Appendix B: Other books on Kalman filtering **489**

Appendix C: State estimation and the meaning of life **493**

References 501

Index 521

ACKNOWLEDGMENTS

The financial support of Sanjay Garg and Donald Simon (no relation to the author) at the NASA Glenn Research Center was instrumental in allowing me to pursue research in the area of optimal state estimation, and indirectly led to the idea for this book. I am thankful to Eugenio Villaseca, the Chair of the Department of Electrical and Computer Engineering at Cleveland State University, for his encouragement and support of my research and writing efforts. Dennis Feucht and Jonathan Litt reviewed the first draft of the book and offered constructive criticism that made the book better than it otherwise would have been. I am also indebted to the two anonymous reviewers of the proposal for this book, who made suggestions that strengthened the material presented herein. I acknowledge the work of Sandy Buettner, Joe Connolly, Classica Jain, Aaron Radke, Bryan Welch, and Qing Zheng, who were students in my Optimal State Estimation class in Fall 2005. They contributed some of the problems at the end of the chapters and made many suggestions for improvement that helped clarify the subject matter. Finally I acknowledge the love and support of my wife, Annette, whose encouragement of my endeavors has always been above and beyond the call of duty.

D. J. S.

ACRONYMS

ACR	Acronym
ARE	Algebraic Riccati equation
CARE	Continuous ARE
DARE	Discrete ARE
EKF	Extended Kalman filter
erf	Error function
FPGA	Field programmable gate array
GPS	Global Positioning System
HOT	Higher-order terms
iff	If and only if
INS	Inertial navigation system
LHP	Left half plane
LTI	Linear time-invariant
LTV	Linear time-varying
MCMC	Markov chain Monte Carlo
MIMO	Multiple input, multiple output
$N(a, b)$	Normal pdf with a mean of a and a variance of b
pdf	Probability density function

PDF	Probability distribution function
QED	Quod erat demonstrandum (i.e., "that which was to be demonstrated")
RHP	Right half plane
RMS	Root mean square
RPF	Regularized particle filter
RTS	Rauch–Tung–Striebel
RV	Random variable
SIR	Sampling importance resampling
SISO	Single input, single output
SSS	Strict-sense stationary
SVD	Singular value decomposition
TF	Transfer function
$U(a, b)$	Uniform pdf that is nonzero on the domain $[a, b]$
UKF	Unscented Kalman filter
WSS	Wide-sense stationary

LIST OF ALGORITHMS

Chapter 1: Linear systems theory
Rectangular integration 29
Trapezoidal integration 31
Fourth-order Runge–Kutta integration 32

Chapter 2: Probability theory
Correlated noise simulation 74

Chapter 3: Least squares estimation
Recursive least squares estimation 86
General recursive least squares estimation 88

Chapter 5: The discrete-time Kalman filter
The discrete-time Kalman filter 128

Chapter 6: Alternate Kalman filter formulations
The sequential Kalman filter 151
The information filter 156
The Cholesky matrix square root algorithm 160
Potter's square root measurement-update algorithm 166
The Householder algorithm 171
The Gram–Schmidt algorithm 172
The U-D measurement update 175
The U-D time update 177

Chapter 7: Kalman filter generalizations
The general discrete-time Kalman filter 186
The discrete-time Kalman filter with colored measurement noise 191
The Hamiltonian approach to steady-state Kalman filtering 207
The fading-memory filter 210

Chapter 8: The continuous-time Kalman filter
The continuous-time Kalman filter 235
The Chandrasekhar algorithm 244
The continuous-time square root Kalman filter 247
The continuous-time Kalman filter with correlated noise 249
The continuous-time Kalman filter with colored measurement noise 251

Chapter 9: Optimal smoothing
The fixed-point smoother 269
The fixed-lag smoother 278
The RTS smoother 293

Chapter 10: Additional topics in Kalman filtering
The multiple-model estimator 302
The reduced-order Schmidt–Kalman filter 312
The delayed-measurement Kalman filter 324

Chapter 11: The H_∞ filter
The discrete-time H_∞ filter 353

Chapter 12: Additional topics in H_∞ filtering
The mixed Kalman/H_∞ filter 374
The robust mixed Kalman/H_∞ filter 378
The constrained H_∞ filter 385

Chapter 13: Nonlinear Kalman filtering
The continuous-time linearized Kalman filter 399
The continuous-time extended Kalman filter 401
The hybrid extended Kalman filter 405
The discrete-time extended Kalman filter 409
The iterated extended Kalman filter 411
The second-order hybrid extended Kalman filter 416
The second-order discrete-time extended Kalman filter 419
The Gaussian sum filter 421

Chapter 14: The unscented Kalman filter
The unscented transformation 446
The unscented Kalman filter 448
The simplex sigma-point algorithm 454
The spherical sigma-point algorithm 455

Chapter 15: The particle filter

The recursive Bayesian state estimator 465
The particle filter 468
Regularized particle filter resampling 473
The extended Kalman particle filter 478

INTRODUCTION

This book discusses mathematical approaches to the best possible way of estimating the state of a general system. Although the book is firmly grounded in mathematical theory, it should not be considered a mathematics text. It is more of an engineering text, or perhaps an *applied* mathematics text. The approaches that we present for state estimation are all given with the goal of eventual implementation in software.[1] The goal of this text is to present state estimation theory in the most clear yet rigorous way possible, while providing enough advanced material and references so that the reader is prepared to contribute new material to the state of the art. Engineers are usually concerned with eventual implementation, and so the material presented is geared toward discrete-time systems. However, continuous-time systems are also discussed for the sake of completeness, and because there is still room for implementations of continuous-time filters.

Before we discuss optimal state estimation, we need to define what we mean by the term *state*. The states of a system are those variables that provide a complete representation of the internal condition or status of the system at a given instant of time.[2] This is far from a rigorous definition, but it suffices for the purposes of

[1]I use the practice that is common in academia of referring to a generic third person by the word *we*. Sometimes, I use the word *we* to refer to the reader and myself. Other times, I use the word *we* to indicate that I am speaking on behalf of the control and estimation community. The distinction should be clear from the context. However, I encourage the reader not to read too much into my use of the word *we*; it is more a matter of personal preference and style rather than a claim to authority.

[2]In this book, we use the terms *state* and *state variable* interchangably. Also, the word *state* could refer to the entire collection of state variables, or it could refer to a single state variable. The specific meaning needs to be inferred from the context.

this introduction. For example, the states of a motor might include the currents through the windings, and the position and speed of the motor shaft. The states of an orbiting satellite might include its position, velocity, and angular orientation. The states of an economic system might include per-capita income, tax rates, unemployment, and economic growth. The states of a biological system might include blood sugar levels, heart and respiration rates, and body temperature.

State estimation is applicable to virtually all areas of engineering and science. Any discipline that is concerned with the mathematical modeling of its systems is a likely (perhaps inevitable) candidate for state estimation. This includes electrical engineering, mechanical engineering, chemical engineering, aerospace engineering, robotics, economics, ecology, biology, and many others. The possible applications of state estimation theory are limited only by the engineer's imagination, which is why state estimation has become such a widely researched and applied discipline in the past few decades. State-space theory and state estimation was initially developed in the 1950s and 1960s, and since then there have been a huge number of applications. A few applications are documented in [Sor85]. Thousands of other applications can be discovered by doing an Internet search on the terms "state estimation" and "application," or "Kalman filter" and "application."

State estimation is interesting to engineers for at least two reasons:

- Often, an engineer needs to estimate the system states in order to implement a state-feedback controller. For example, the electrical engineer needs to estimate the winding currents of a motor in order to control its position. The aerospace engineer needs to estimate the attitude of a satellite in order to control its velocity. The economist needs to estimate economic growth in order to try to control unemployment. The medical doctor needs to estimate blood sugar levels in order to control heart and respiration rates.

- Often an engineer needs to estimate the system states because those states are interesting in their own right. For example, if an engineer wants to measure the health of an engineering system, it may be necessary to estimate the internal condition of the system using a state estimation algorithm. An engineer might want to estimate satellite position in order to more intelligently schedule future satellite activities. An economist might want to estimate economic growth in order to make a political point. A medical doctor might want to estimate blood sugar levels in order to evaluate the health of a patient.

There are many other fine books on state estimation that are available (see Appendix B). This begs the question: Why yet another textbook on the topic of state estimation? The reason that this present book has been written is to offer a pedagogical approach and perspective that is not available in other state estimation books. In particular, the hope is that this book will offer the following:

- A straightforward, bottom-up approach that assists the reader in obtaining a clear (but theoretically rigorous) understanding of state estimation. This is reminiscent of Gelb's approach [Gel74], which has proven effective for many state estimation students of the past few decades. However, many aspects of Gelb's book have become outdated. In addition, many of the more recent books on state estimation read more like research monographs and are not entirely accessible to the average engineering student. Hence the need for the present book.

- Simple examples that provide the reader with an intuitive understanding of the theory. Many books present state estimation theory and then follow with examples or problems that require a computer for implementation. However, it is possible to present simple examples and problems that require only paper and pencil to solve. These simple problems allow the student to more directly see how the theory works itself out in practice. Again, this is reminiscent of Gelb's approach [Gel74].

- MATLAB-based source code[3] for the examples in the book is available at the author's Web site.[4] A number of other texts supply source code, but it is often on disk or CD, which makes the code subject to obsolescence. The author's e-mail address is also available on the Web site, and I enthusiastically welcome feedback, comments, suggestions for improvements, and corrections. Of course, Web addresses are also subject to obsolescence, but the book also contains algorithmic, high-level pseudocode listings that will last longer than any specific software listings.

- Careful treatment of advanced topics in optimal state estimation. These topics include unscented filtering, high-order nonlinear filtering, particle filtering, constrained state estimation, reduced-order filtering, robust Kalman filtering, and mixed Kalman/H_∞ filtering. Some of these topics are mature, having been introduced in the 1960s, but others of these topics are recent additions to the state of the art. This coverage is not matched in any other books on the topic of state estimation.

Some of the other books on state estimation offer some of the above features, but no other books offer *all* of these features.

Prerequisites

The prerequisites for understanding the material in this book are a good foundation in linear systems theory and probability and stochastic processes. Ideally, the reader will already have taken a graduate course in both of these topics. However, it should be said that a background in linear systems theory is more important than probability. The first two chapters of the book review the elements of linear systems and probability that are essential for the rest of the book, and also serve to establish the notation that is used during the remainder of the book.

Other material could also be considered prerequisite to understanding this book, such as undergraduate advanced calculus, control theory, and signal processing. However, it would be more accurate to say that the reader will require a moderately high level of mathematical and engineering maturity, rather than trying to identify a list of required prerequisite courses.

[3]MATLAB is a registered trademark of The MathWorks, Inc.
[4]http://academic.csuohio.edu/simond/estimation – if the Web site address changes, it should be easy to find with an internet search.

Problems

The problems at the end of each chapter have been written to give a high degree of flexibility to the instructor and student. The problems include both written exercises and computer exercises. The written exercises are intended to strengthen the student's grasp of the theory, and deepen the student's intuitive understanding of the concepts. The computer exercises are intended to help the student learn how to apply the theory to problems of the type that might be encountered in industrial or government projects. Both types of problems are important for the student to become proficient at the material. The distinction between written exercises and computer exercises is more of a fuzzy division rather than a strict division. That is, some of the written exercises include parts for which some computer work might be useful (even required), and some of the computer exercises include parts for which some written analysis might be useful (even required).

A solution manual to all of the problems in the text (both written exercises and computer exercises) is available from the publisher to instructors who have adopted this book. Course instructors are encouraged to contact the publisher for further information about out how to obtain the solution manual.

Outline of the book

This book is divided into four parts. The first part of the book covers introductory material. Chapter 1 is a review of the relevant areas of linear systems. This material is often covered in a first-semester graduate course taken by engineering students. It is advisable, although not strictly required, that readers of this book have already taken a graduate linear systems course. Chapter 2 reviews probability theory and stochastic processes. Again, this is often covered in a first-semester graduate course. In this book we rely less on probability theory than linear systems theory, so a previous course in probability and stochastic processes is not required for the material in this book (although it would be helpful). Chapter 3 covers least squares estimation of constants and Wiener filtering of stochastic processes. The section on Wiener filtering is not required for the remainder of the book, although it is interesting both in its own right and for historical perspective. Chapter 4 is a brief discussion of how the statistical measures of a state (mean and covariance) propagate in time. Chapter 4 provides a bridge from the first three chapters to the second part of the book.

The second part of the book covers Kalman filtering, which is the workhorse of state estimation. In Chapter 5, we derive the discrete-time Kalman filter, including several different (but mathematically equivalent) formulations. In Chapter 6, we present some alternative Kalman filter formulations, including sequential filtering, information filtering, square root filtering, and U-D filtering. In Chapter 7, we discuss some generalizations of the Kalman filter that make the filter applicable to a wider class of problems. These generalizations include correlated process and measurement noise, colored process and measurement noise, steady-state filtering for computational savings, fading-memory filtering, and constrained Kalman filtering. In Chapter 8, we present the continuous-time Kalman filter. This chapter could be skipped if time is short since the continuous-time filter is rarely implemented in practice. In Chapter 9, we discuss optimal smoothing, which is a way to estimate the state of a system at time τ based on measurements that extend beyond time τ. As part of the derivation of the smoothing equations, the first section of Chapter 9 presents another alternative form for the Kalman filter. Chapter 10 presents some additional, more advanced topics in Kalman filtering. These topics include verification of filter performance, estimation in the case of unknown system models, reduced-order filtering, increasing the robustness of the Kalman filter, and filtering in the presence of measurement synchronization errors. This chapter should provide fertile ground for students or engineers who are looking for research topics or projects.

The third part of the book covers H_∞ filtering. This area is not as mature as Kalman filtering and so there is less material than in the Kalman filtering part of the book. Chapter 11 introduces yet another alternate Kalman filter form as part of the H_∞ filter derivation. This chapter discusses both time domain and frequency domain approaches to H_∞ filtering. Chapter 12 discusses advanced topics in H_∞ filtering, including mixed Kalman/H_∞ filtering and constrained H_∞ filtering. There is a lot of room for further development in H_∞ filtering, and this part of the book could provide a springboard for researchers to make contributions in this area.

The fourth part of the book covers filtering for nonlinear systems. Chapter 13 discusses nonlinear filtering based on the Kalman filter, which includes the widely used extended Kalman filter. Chapter 14 covers the unscented Kalman filter, which is a relatively recent development that provides improved performance over the extended Kalman filter. Chapter 15 discusses the particle filter, another recent development that provides a very general solution to the nonlinear filtering problem. It is hoped that this part of the book, especially Chapters 14 and 15, will inspire researchers to make further contributions to these new areas of study.

The book concludes with three brief appendices. Appendix A gives some historical perspectives on the development of the Kalman filter, starting with the least squares work of Roger Cotes in the early 1700s, and concluding with the space program applications of Kalman filtering in the 1960s. Appendix B discusses the many other books that have been written on Kalman filtering, including their distinctive contributions. Finally, Appendix C presents some speculations on the connections between optimal state estimation and the meaning of life.

Figure I.1 gives a graphical representation of the structure of the book from a prerequisite point of view. For example, Chapter 3 builds on Chapters 1 and 2. Chapter 4 builds on Chapter 3, and Chapter 5 builds on Chapter 4. Chapters 6–11 each depend on material from Chapter 5, but are independent from each other. Chapter 12 builds on Chapter 11. Chapter 13 depends on Chapter 8, and Chapter 14 depends on Chapter 13. Finally, Chapter 15 builds on Chapter 3. This structure can be used to customize a course based on this book.

A note on notation

Three dots between delimiters (parenthesis, brackets, or braces) means that the quantity between the delimiters is the same as the quantity between the previous set of identical delimiters in the same equation. For example,

$$\begin{aligned} (A+BCD)+(\cdots)^T &= (A+BCD)+(A+BCD)^T \\ A+\left[B(C+D)\right]^{-1}E\left[\cdots\right] &= A+\left[B(C+D)\right]^{-1}E\left[B(C+D)\right] \end{aligned} \tag{I.1}$$

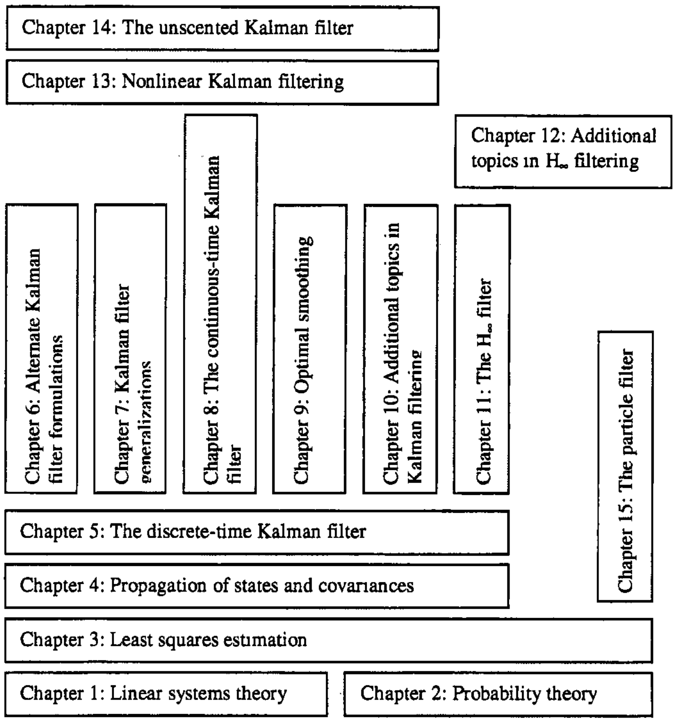

Figure I.1 Prerequisite structure of the chapters in this book.