2023

Bayesian statistics: a short introduction

David Makowski

david.makowski@inra.fr

Some references

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Laplace…

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PHILOSOPHICAL
TRANSACTIONS:

An Essay towards Solving a Problem in the Doctrine of Chances. By the Late Rev. Mr. Bayes, F. R. S. Communicated by Mr. Price, in a Letter to John Canton, A. M. F. R. S.

Mr. Bayes and Mr. Price

Phil. Trans. 1763 **53**, 370-418, published 1 January 1763

quodque ſolum, certa nitri ſigna præbere, ſed plura concurrere debere, ut de vero nitro producto dubium non relinquatur.

LII. *An Eſſay towards ſolving a Problem in the Doctrine of Chances. By the late Rev. Mr.* Bayes, *F. R. S. communicated by Mr.* Price, *in a Letter to* John Canton, *A. M. F. R. S.*

Dear Sir,

Read Dec. 23, 1763.

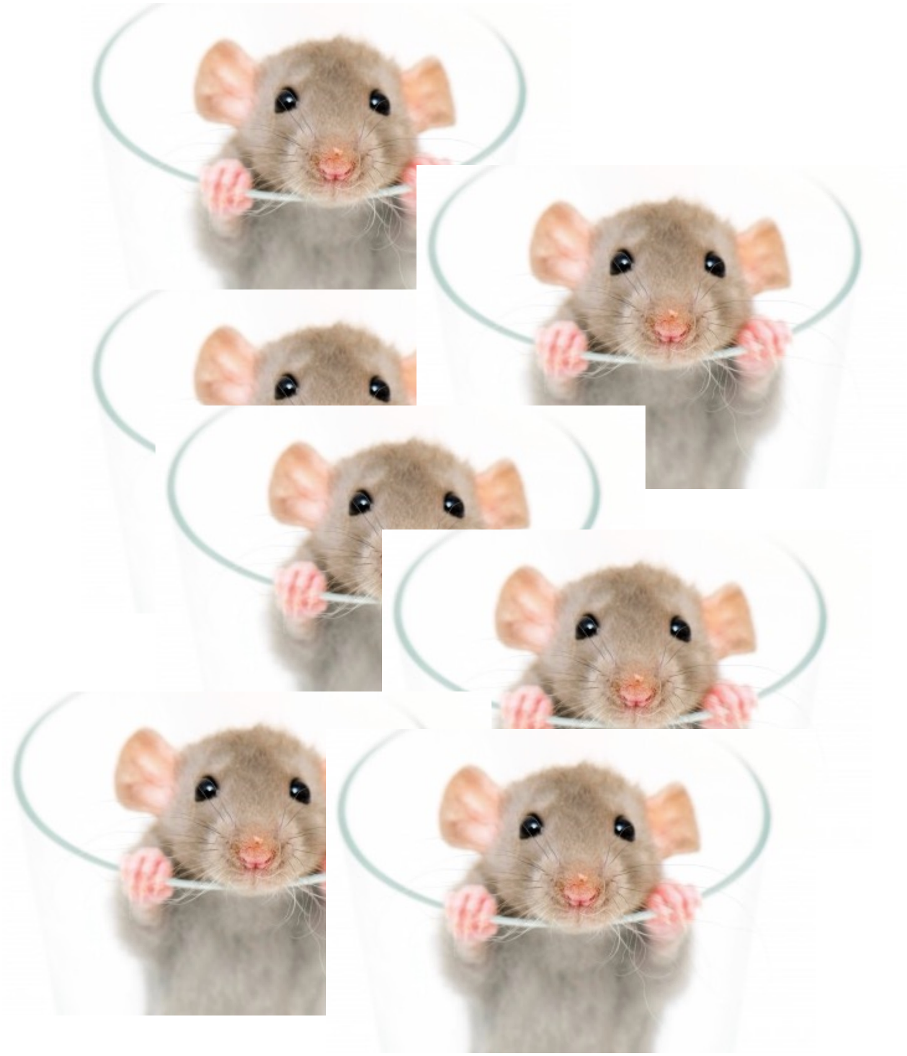
I Now ſend you an eſſay which I have found among the papers of our deceaſed friend Mr. Bayes, and which, in my opinion, has great merit, and well deſerves to be preſerved. Experimental philoſophy, you will find, is nearly intereſted in the ſubject of it; and on this account there ſeems to be particular reaſon for thinking that a communication of it to the Royal Society cannot be improper.

PROBLEM.

Given the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a ſingle trial lies ſomewhere between any two degrees of probability that can be named.

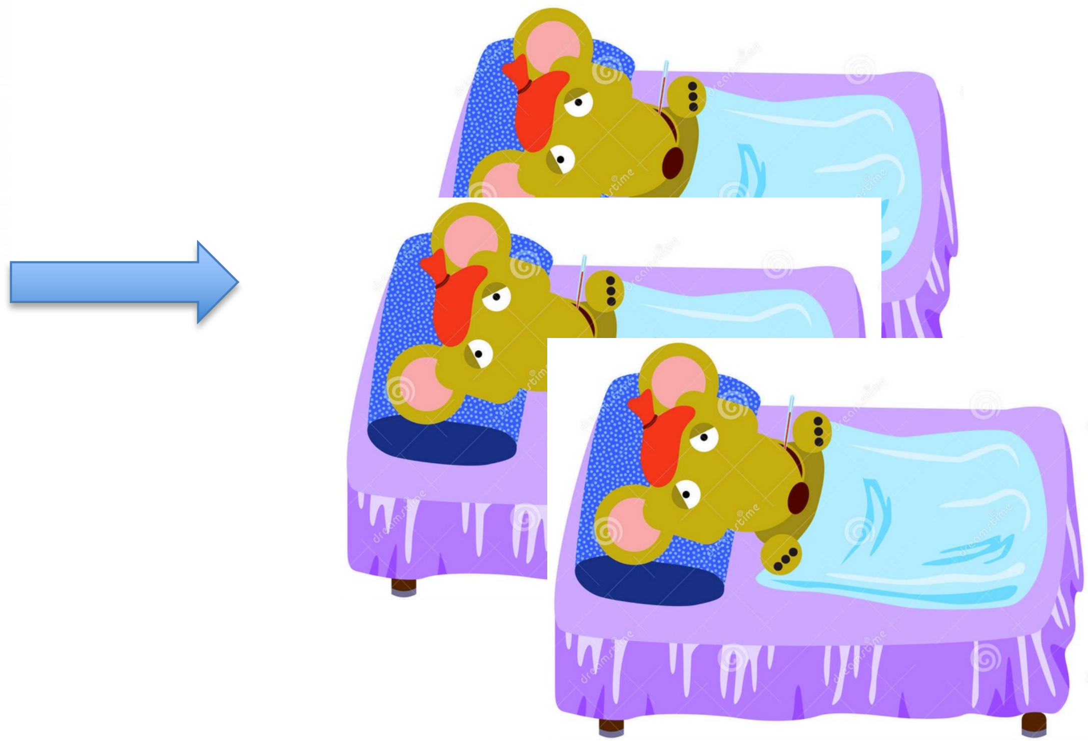
« Given that *n* independent Bernoulli trials with unknown constant success probability *p* have brought *k* successes, what is the probability that the parameter *p* lies between two given bounds? »

n rats



n rats

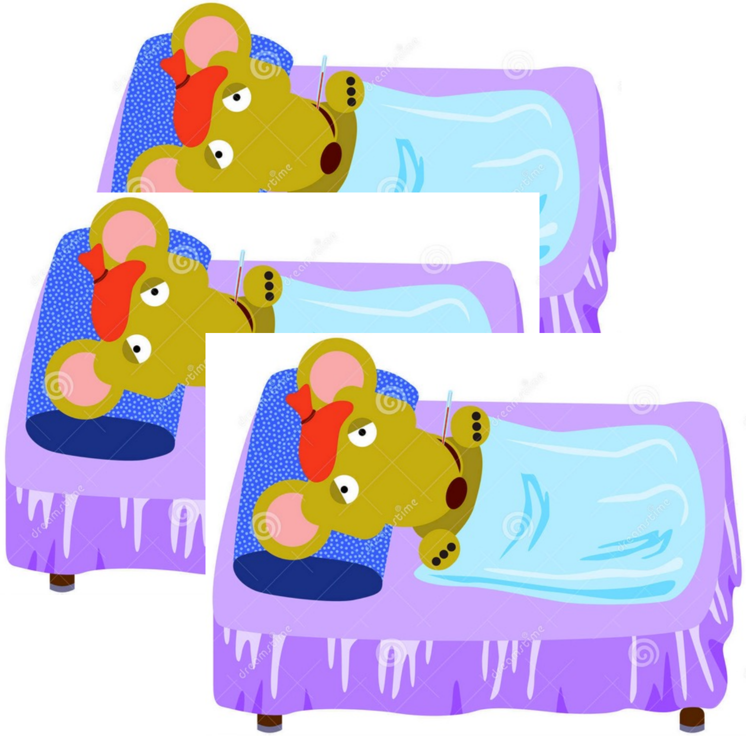
k contaminated rats



Probability p of contamination?

n rats

k contaminated rats



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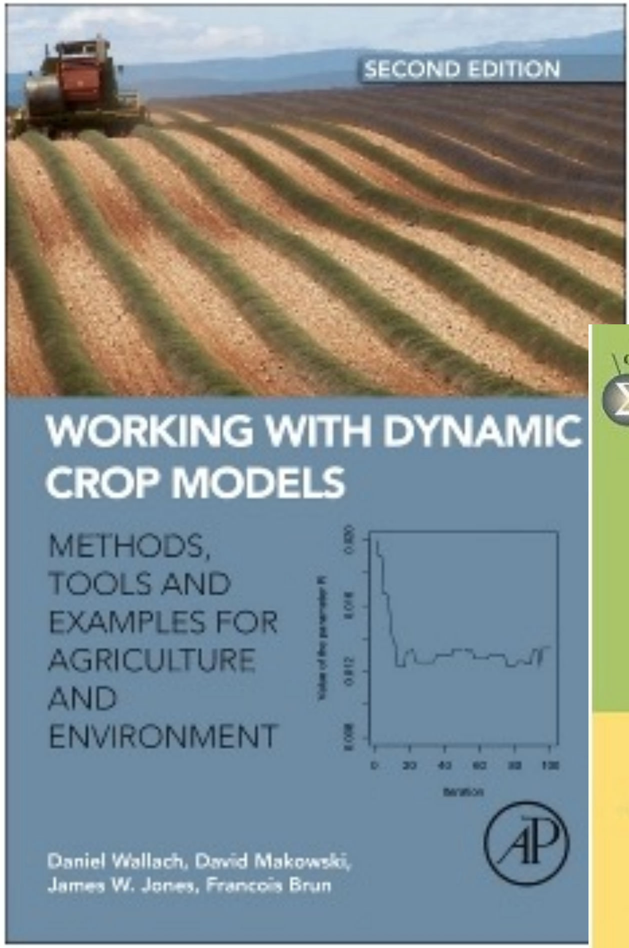
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Préface de Jean-Michel Marin
ellipses

Estimation of parameters (θ)

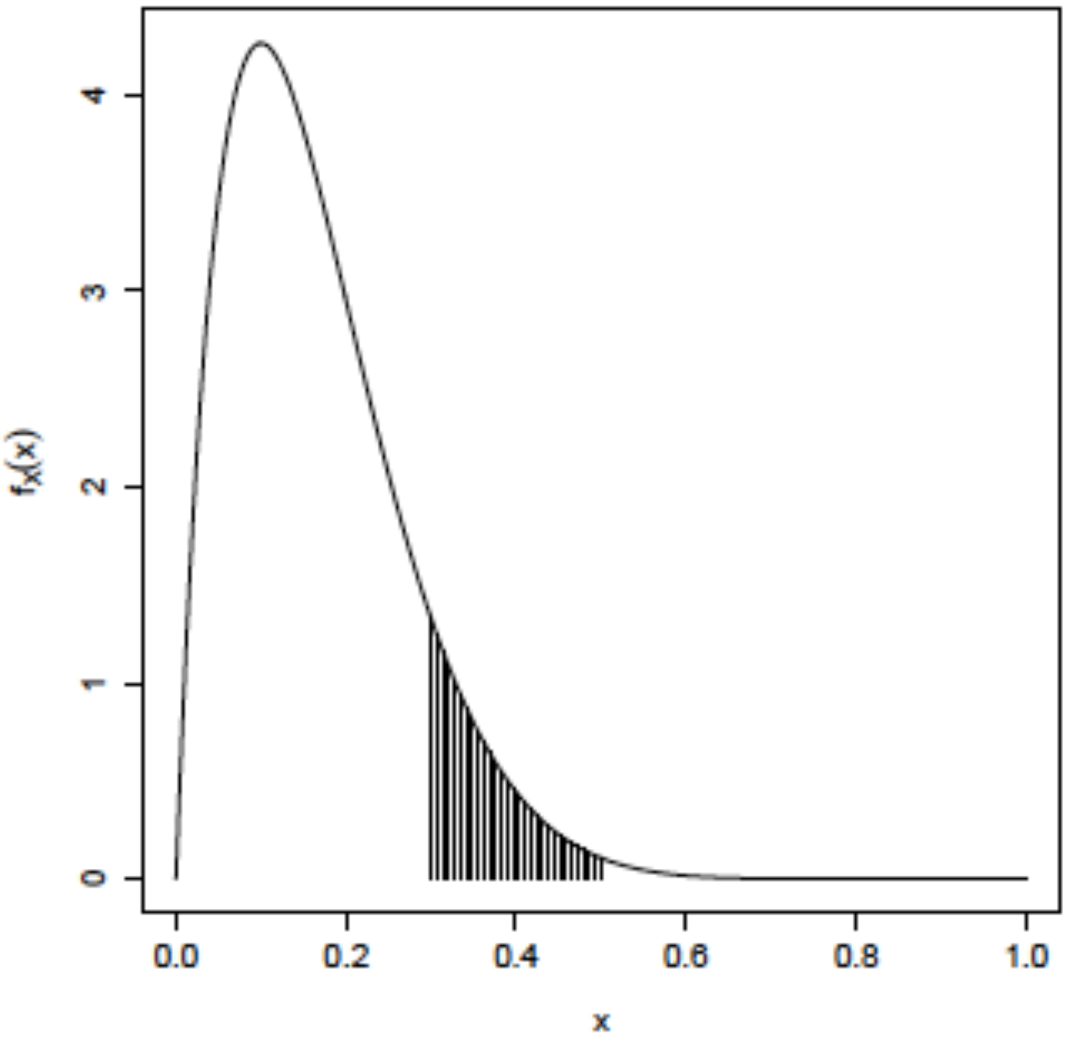
Parameter = Unknow numerical value

Information available to estimate parameters

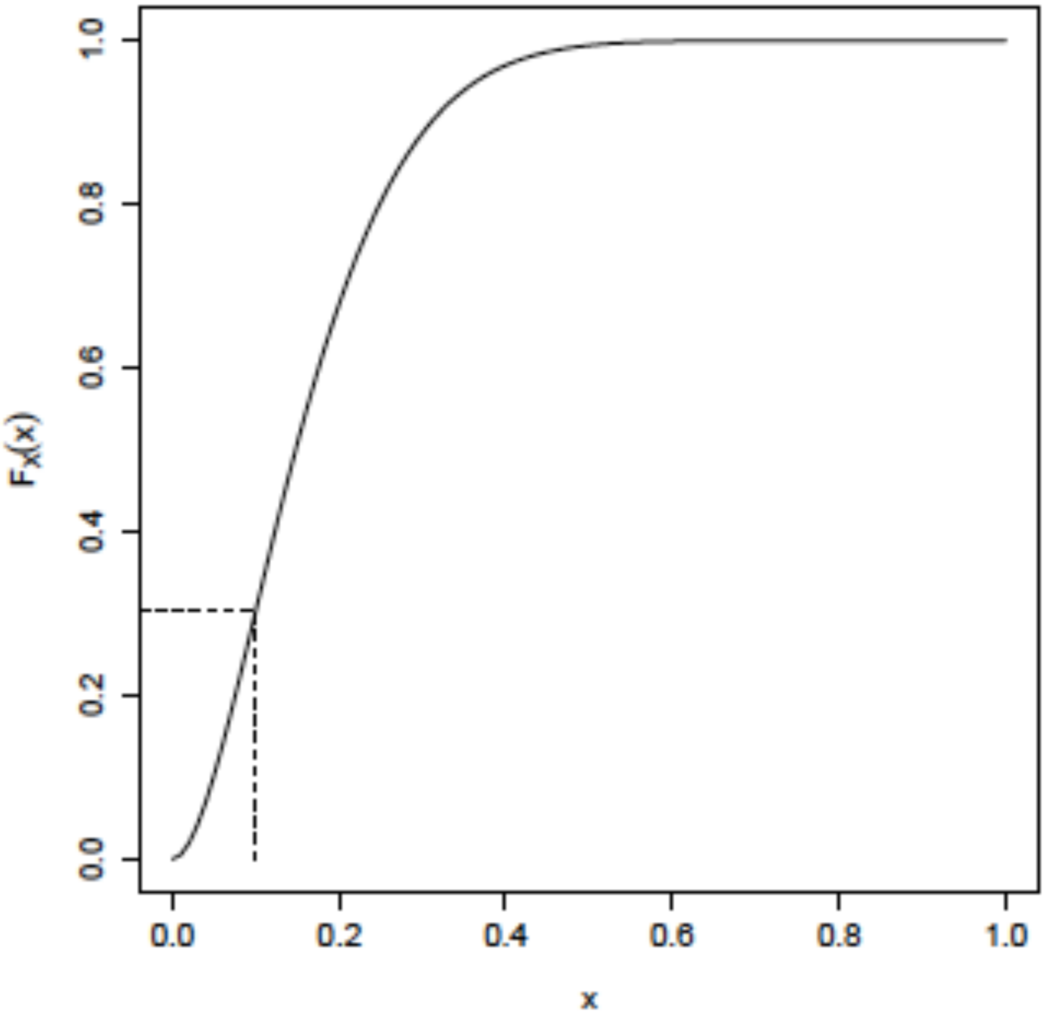
- A dataset (*Y*).

- Prior knowledge about parameter values.

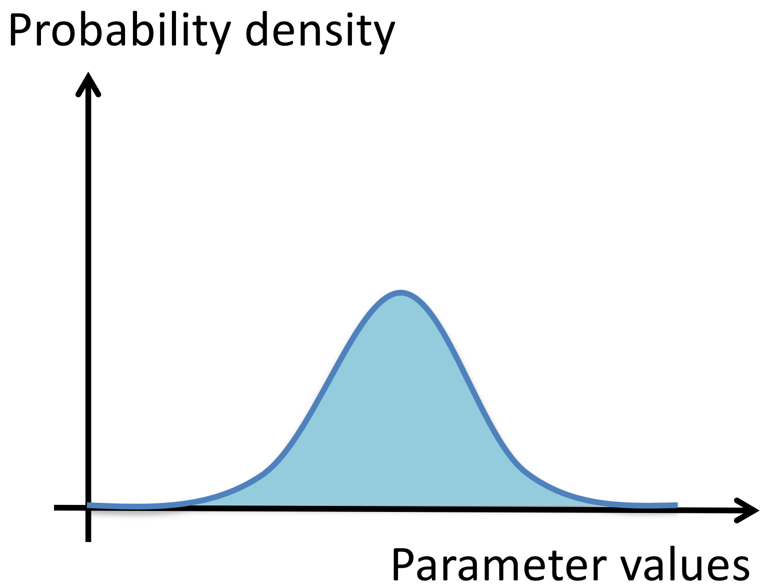
Probability density



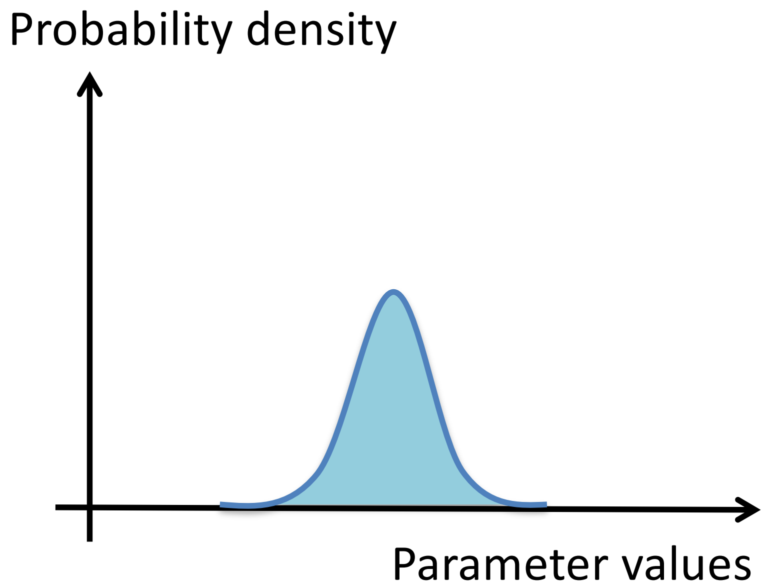
Cumulative probability distribution



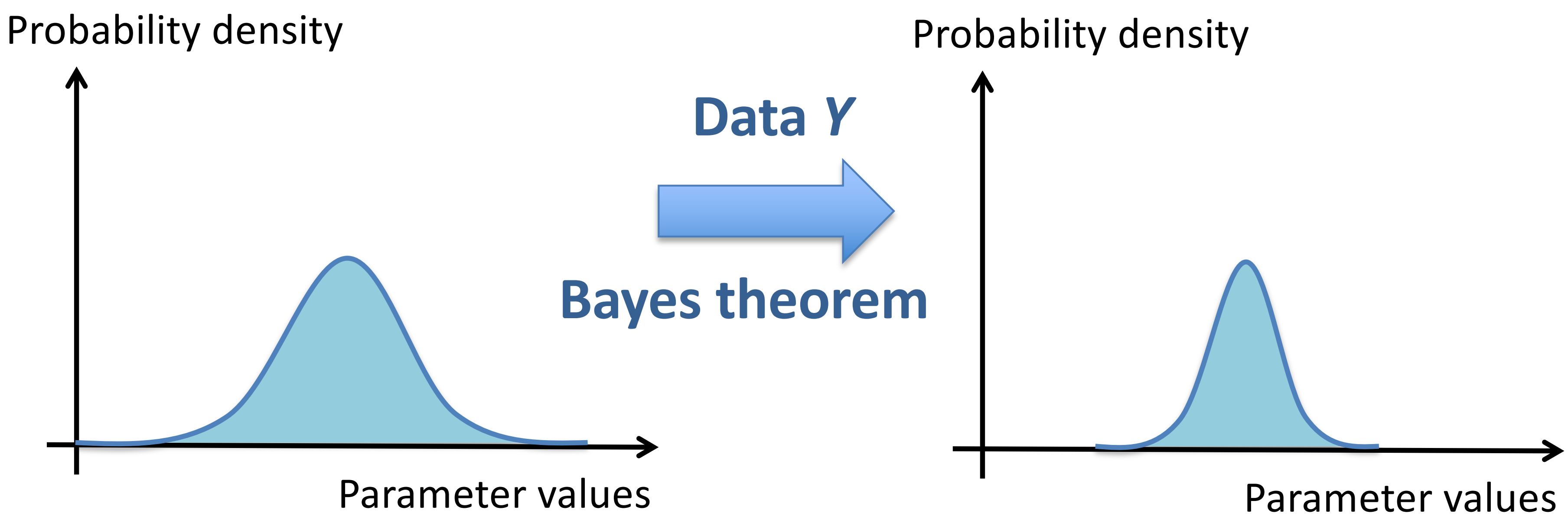
PRIOR probability distribution



POSTERIOR probability distribution



PRIOR probability distribution **POSTERIOR probability distribution**



	Classic methods	Bayesian methods
Parameters	Fixed	Random

	Classic methods	Bayesian methods
Parameters	Fixed	Random
Prior knowledge about parameter values	Not taken into account	Taken into account

	Classic methods	Bayesian methods
Parameters	Fixed	Random
Prior knowledge about parameter values	Not taken into account	Taken into account
Likelihood	Used	Used

	Classic methods	Bayesian methods
Parameters	Fixed	Random
Prior knowledge about parameter values	Not taken into account	Taken into account
Likelihood	Used	Used
Computation	Easier	More difficult

Estimation of parameters (θ)

Parameter = Unknow numerical value

Information available to estimate parameters

- A set of observations (Y).

- Prior knowledge about parameter values.

Two probability distributions

• **Prior parameter distribution = probability distribution describing our initial knowledge about parameter values.**

$$P(\theta)$$

• **Likelihood function = function relating data to parameters.**

$$P(Y|\theta)$$

The Bayes theorem

Posterior ∝ Likelihood x Prior

$$P(\theta \mid Y) = \frac{P(Y \mid \theta)\,P(\theta)}{P(Y)}$$

Measurements

Prior Information about parameter values

Bayesian method

Combined info about parameters

Example 1

Probability of disease in function of a diagnosis test

$$P\left(Diseased \mid Test+\right) = \frac{P\left(Test+ \mid Diseased\right)P\left(Diseased\right)}{P\left(Test+\right)}$$

Example 1

Probability of disease in function of a diagnosis test

Prior

Data

$$P\left(Diseased \mid Test+\right) = \frac{P\left(Test+ \mid Diseased\right)P\left(Diseased\right)}{P\left(Test+\right)}$$

$$P\left(\theta \mid Y\right) = \frac{P\left(Y \mid \theta\right)P\left(\theta\right)}{P\left(Y\right)}$$

Example 1

Probability of disease in function of a diagnosis test

$$P(Diseased \mid Test+) = \frac{P(Test+ \mid Diseased)\,P(Diseased)}{P(Test+ \mid Diseased)\,P(Diseased) + P(Test+ \mid Not\ Diseased)\,P(Not\ Diseased)}$$

Example 1

Probability of disease in function of a diagnosis test

$$P\left(Diseased \mid Test+\right) = \frac{P\left(Test+ \mid Diseased\right) P\left(Diseased\right)}{P\left(Test+ \mid Diseased\right) P\left(Diseased\right) + P\left(Test+ \mid Not\ Diseased\right) P\left(Not\ Diseased\right)}$$

$$P\left(Diseased \mid Test+\right) = \frac{\text{Sensitivity} \times \text{Prevalence}}{\text{Sensitivity} \times \text{Prevalence} + (1\text{-Specificity})(1\text{-Prevalence})}$$

Example 1

Probability of disease in function of a diagnosis test

$$P(Diseased \mid Test+) = \frac{P(Test+ \mid Diseased)P(Diseased)}{P(Test+ \mid Diseased)P(Diseased) + P(Test+ \mid Not\ Diseased)P(Not\ Diseased)}$$

$$P(Diseased \mid Test+) = \frac{\text{Sensitivity} \times \text{Prevalence}}{\text{Sensitivity} \times \text{Prevalence} + (1\text{-Specificity})(1\text{-Prevalence})}$$

Perfect test → $P(Diseased \mid Test+) = \frac{1 \times \text{Prevalence}}{1 \times \text{Prevalence} + (1\text{-}1)(1\text{-Prevalence})} = 1$

Example 1

Probability of disease in function of a diagnosis test

$$P\left(Diseased \mid Test+\right) = \frac{P\left(Test+ \mid Diseased\right)P\left(Diseased\right)}{P\left(Test+ \mid Diseased\right)P\left(Diseased\right) + P\left(Test+ \mid Not\ Diseased\right)P\left(Not\ Diseased\right)}$$

$$P\left(Diseased \mid Test+\right) = \frac{\text{Sensitivity} \times \text{Prevalence}}{\text{Sensitivity} \times \text{Prevalence} + (1\text{-Specificity})(1\text{-Prevalence})}$$

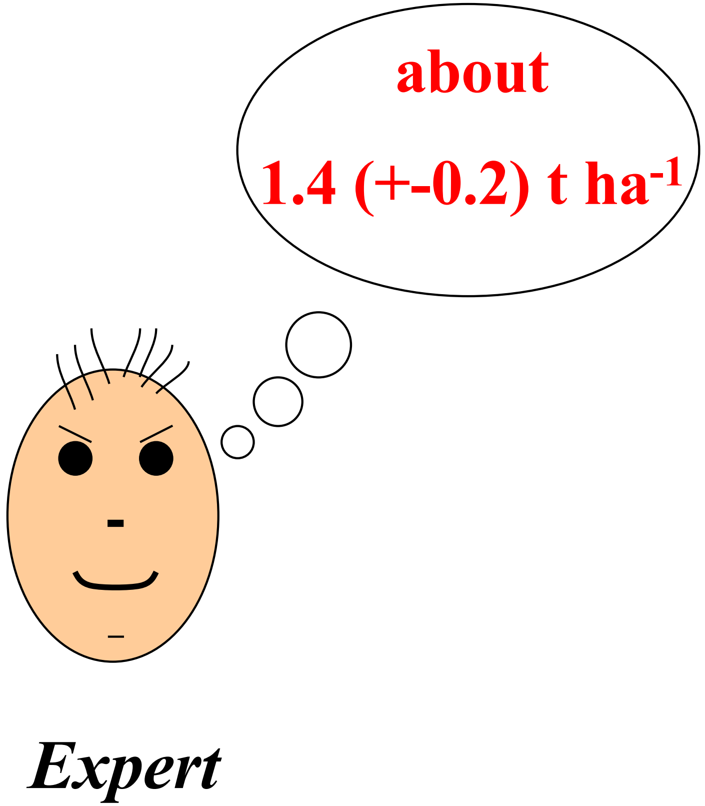
Useless test →

$$P\left(Diseased \mid Test+\right) = \frac{0.5 \times \text{Prevalence}}{0.5 \times \text{Prevalence} + (1\text{-}0.5)(1\text{-Prevalence})}$$

$$= \text{Prevalence}$$

Example 2

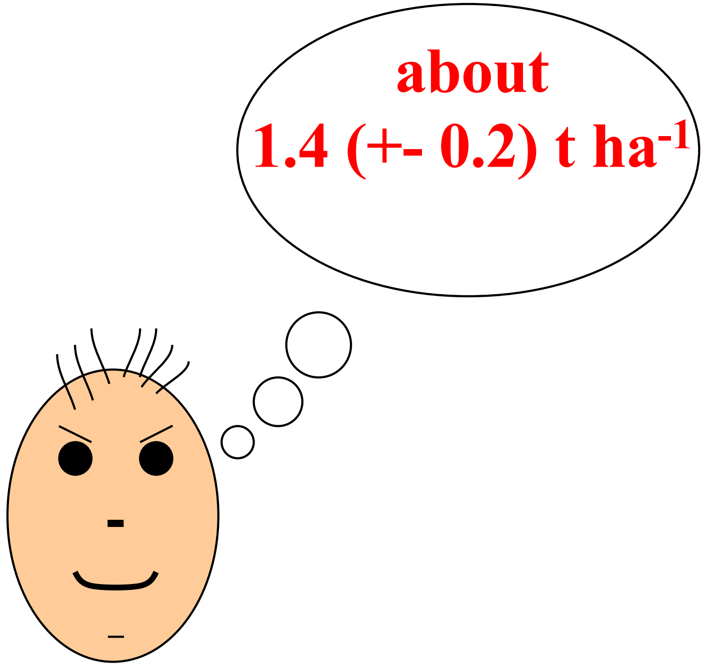
Estimation of yield θ in a crop field by combining a measurement with expert knowledge



Example 2

Estimation of yield θ in a crop field by combining a measurement with expert knowledge

$Y = 1.9\ (+\!-0.2)\ \mathrm{t\ ha^{-1}}$

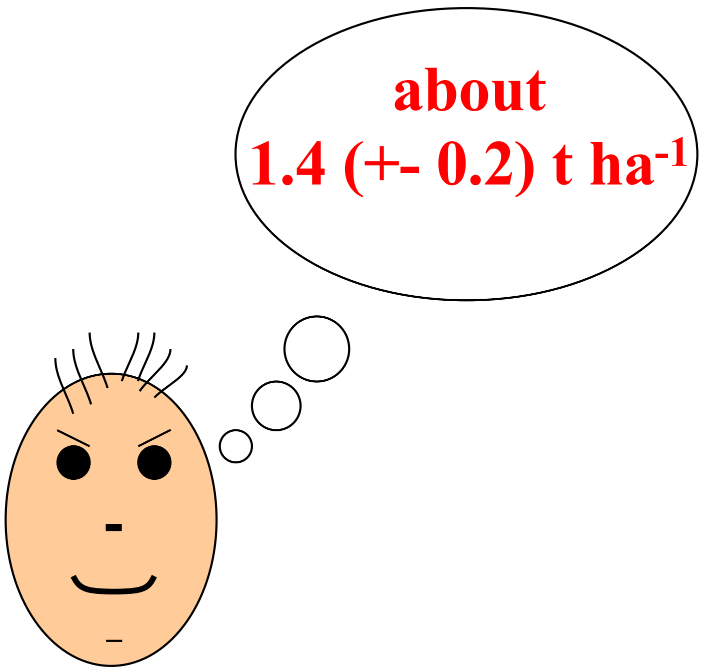


Expert

Prior distribution

- It describes our belief about the parameter values **before** we observe the measurements.

- It is based on past studies, expert knowledge, and litterature.



Example 2 (continued)

Definition of a prior distribution

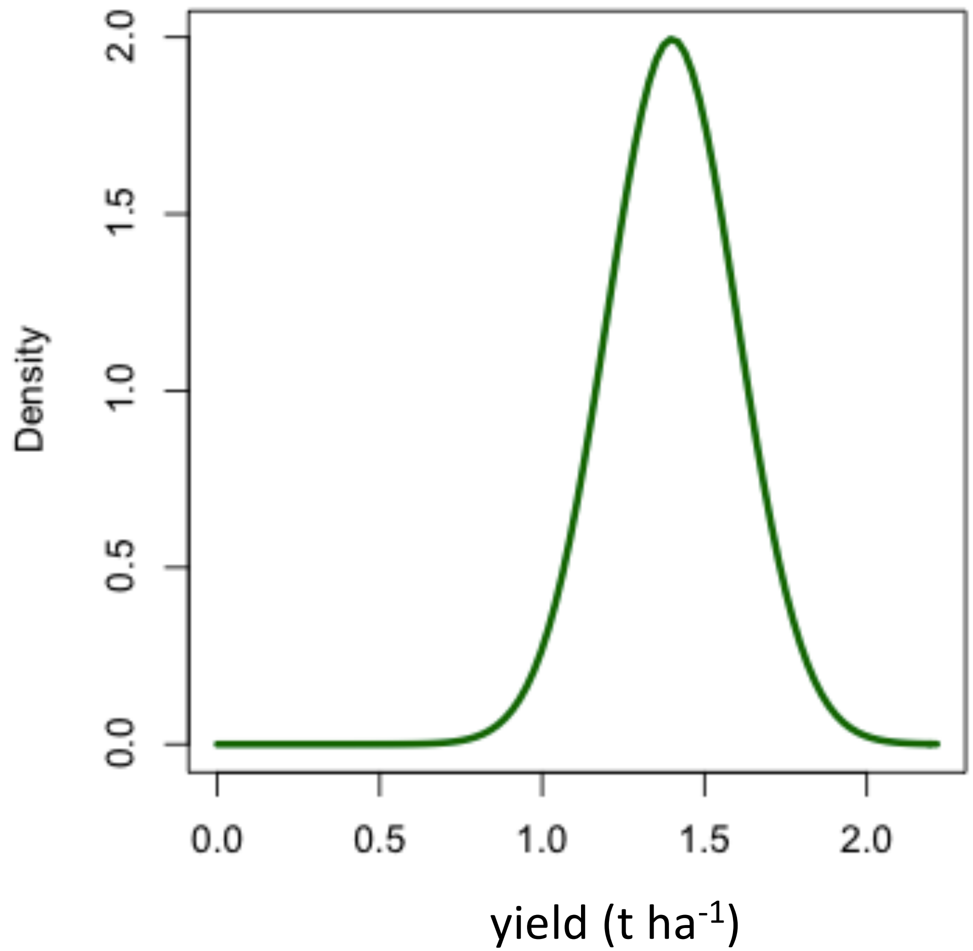
$$\theta \sim N\left(\mu, \tau^2\right)$$

$$P(\theta) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left[-\frac{(\theta-\mu)^2}{2\tau^2}\right] = \frac{1}{\sqrt{2\pi 0.2^2}} \exp\left[-\frac{(\theta-1.4)^2}{2\times 0.2^2}\right]$$

- Normal probability distribution.
- Expected value equal to 1.4
- Standard error equal to 0.2

Example 2 (continued)

Plot of the prior distribution



Likelihood function

- **A likelihood function is a function relating data to parameters.**

- **It is equal to the probability that the measurements would have been observed given some parameter values.**

- **Notation: P(Y | θ)**

Example 2 (continued)

Statistical model

$$Y = \theta + \varepsilon$$

$$\varepsilon \sim N\left(0, \sigma^2\right)$$

$$Y \mid \theta \sim N\left(\theta, \sigma^2\right)$$

Example 2 (continued)

Definition of a likelihood function

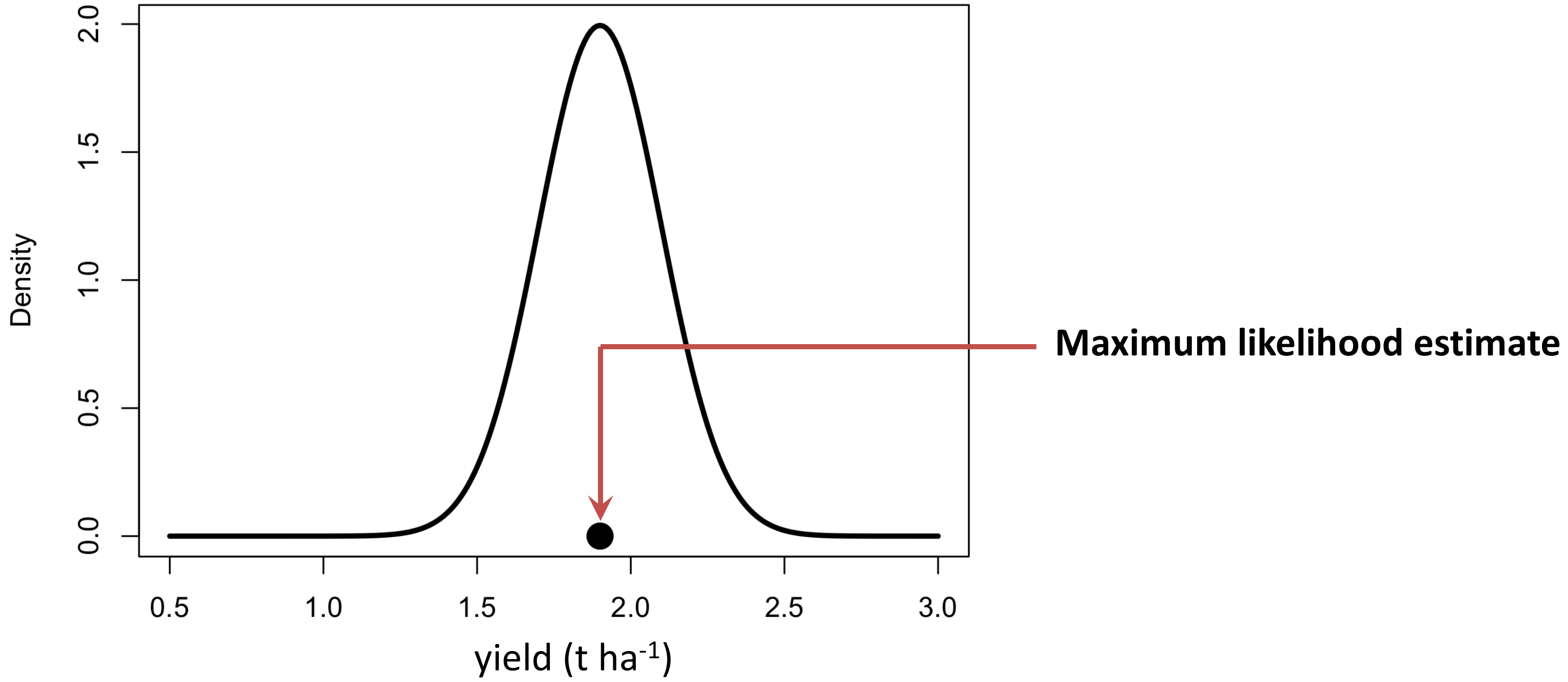
$$Y \mid \theta \sim N\left(\theta, \sigma^2\right)$$

$$P\left(Y \middle| \theta\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(Y-\theta\right)^2}{2\sigma^2}\right] = \frac{1}{\sqrt{2\pi}0.2} \exp\left[-\frac{\left(1.9-\theta\right)^2}{2\times 0.2^2}\right]$$

- Normal probability distribution.
- Measurement *y* assumed unbiaised and equal to 1.9
- Standard error *σ* assumed equal to 0.2

Example 2 (continued)

Definition of a likelihood function

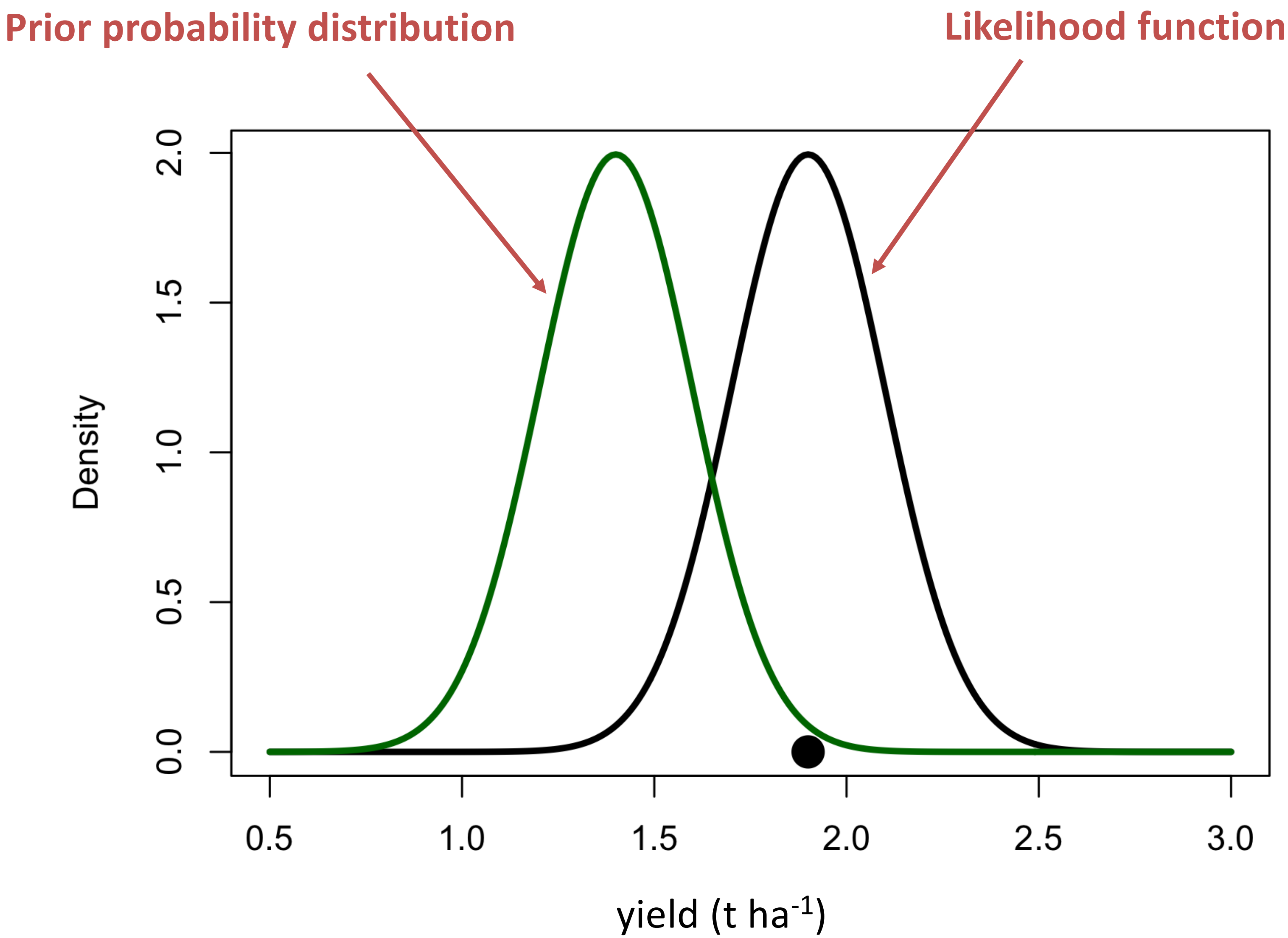


Maximum likelihood

Likelihood functions are also used by frequentist to implement the *maximum likelihood method*.

The maximum likelihood estimator is the value of θ maximizing

P(Y | θ) .



Prior probability distribution
Likelihood function
Density
0.0
0.5
1.0
1.5
2.0
0.5
1.0
1.5
2.0
2.5
3.0
yield (t ha-1)

Example 2 (continued)

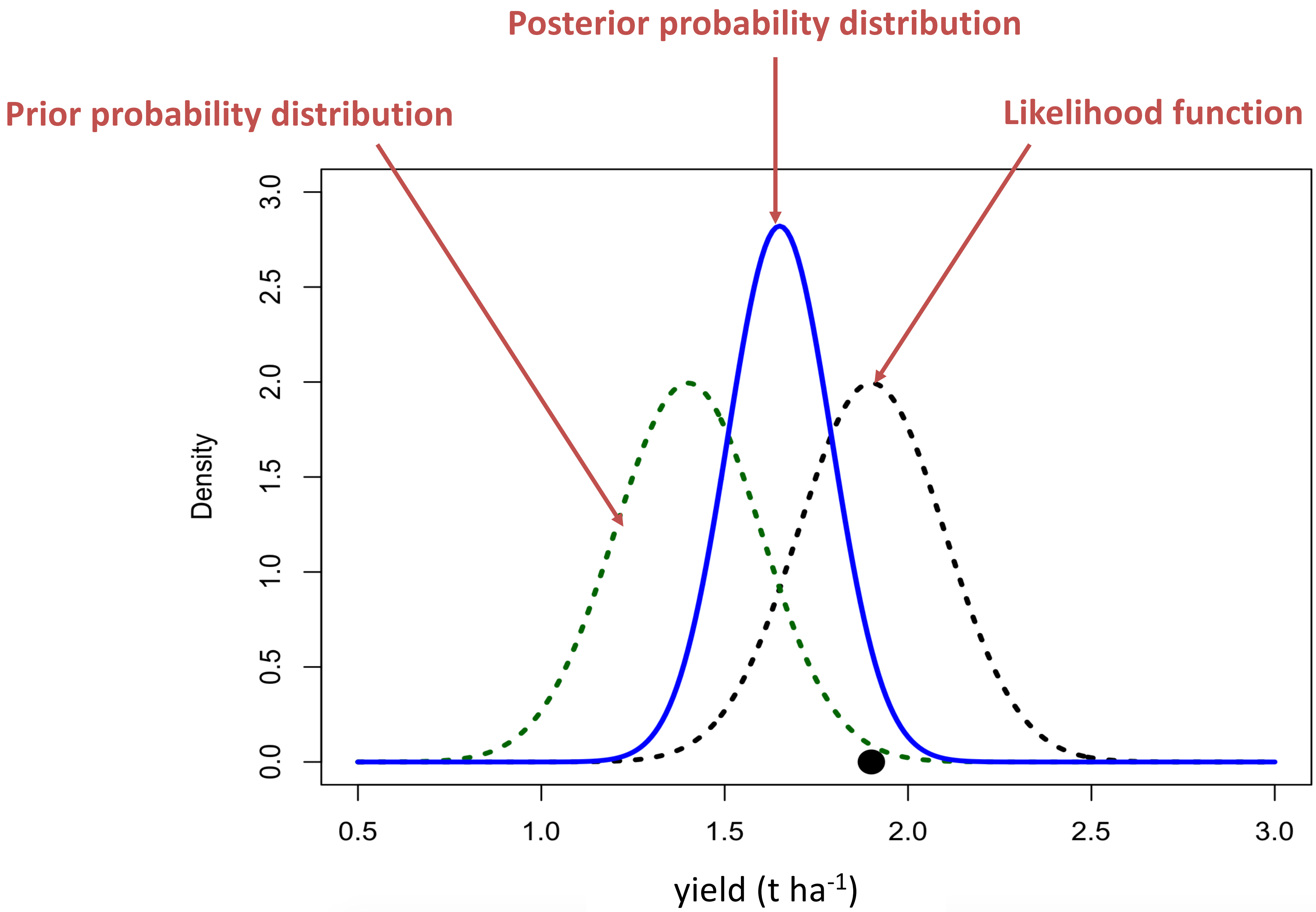
Analytical expression of the posterior distribution

$$\theta \mid Y \sim N\left(\mu_{post}, \sigma^2_{post}\right)$$

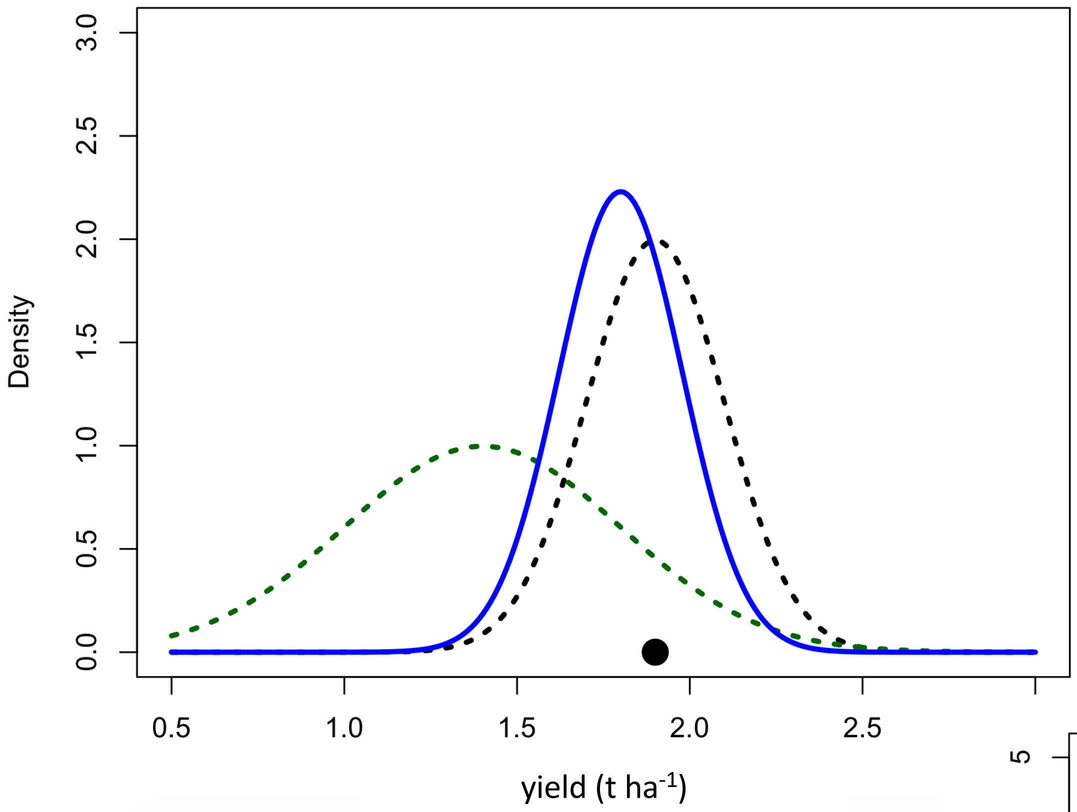
$$\mu_{post} = (1 - B) \times \mu + B \times Y = 1.65$$

$$\tau^2_{post} = (1 - B) \times \tau^2 = 0.02$$

$$B = \frac{\tau^2}{\tau^2 + \sigma^2} = 0.5$$

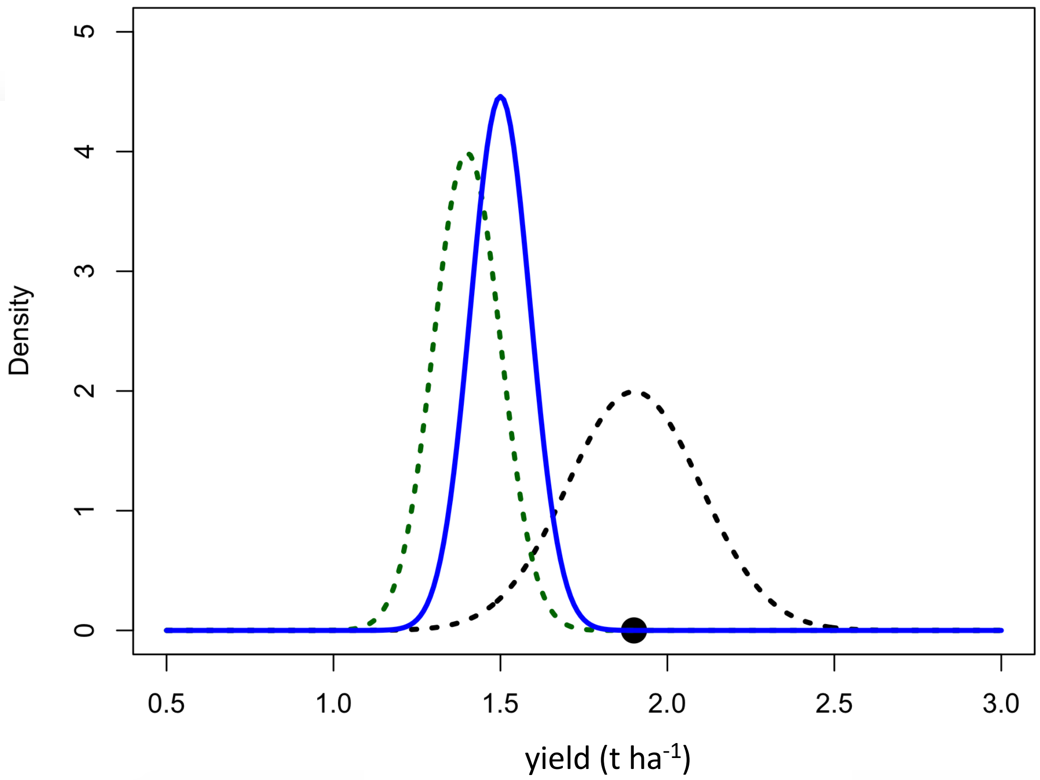


Posterior probability distribution
Prior probability distribution
Likelihood function
Density
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.5
1.0
1.5
2.0
2.5
3.0
yield (t ha-1)



Higher prior uncertainty

Lower prior uncertainty



Example 2 (continued)

Discussion of the posterior distribution

1. Result is a probability distribution (posterior distr.)
2. Posterior mean is intermediate between prior mean and observation.
3. Weight of each depends on prior variance and measurement error.
4. Posterior variance is lower than both prior variance and measurement error variance.
5. Used just one data point and still got estimator.

Frequentist *versus* Bayesian

Bayesian analysis introduces an element of subjectivity:
***the prior distribution*.**

But its representation of the uncertainty is easy to understand

- the uncertainty is assessed conditionally to the observations,

- the calculations are straightforward when the posterior distribution is known.

What is better?

Bayesian methods often lead to

- more realistic estimated parameter values,

- in some cases, more accurate model predictions.

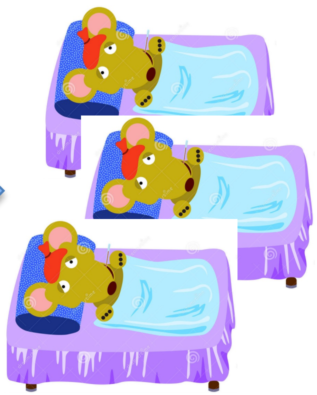
Problems when prior information is wrong and when one has a strong confidence in it.

Example 3

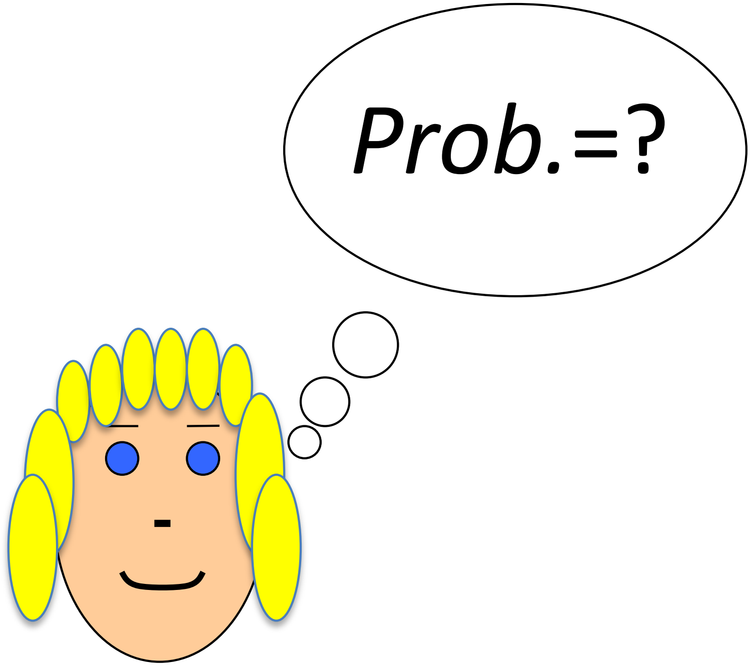
Estimation of a probability of contamination by combining a measurement with expert knowledge

***N* rats**

***Y* diseased rats**

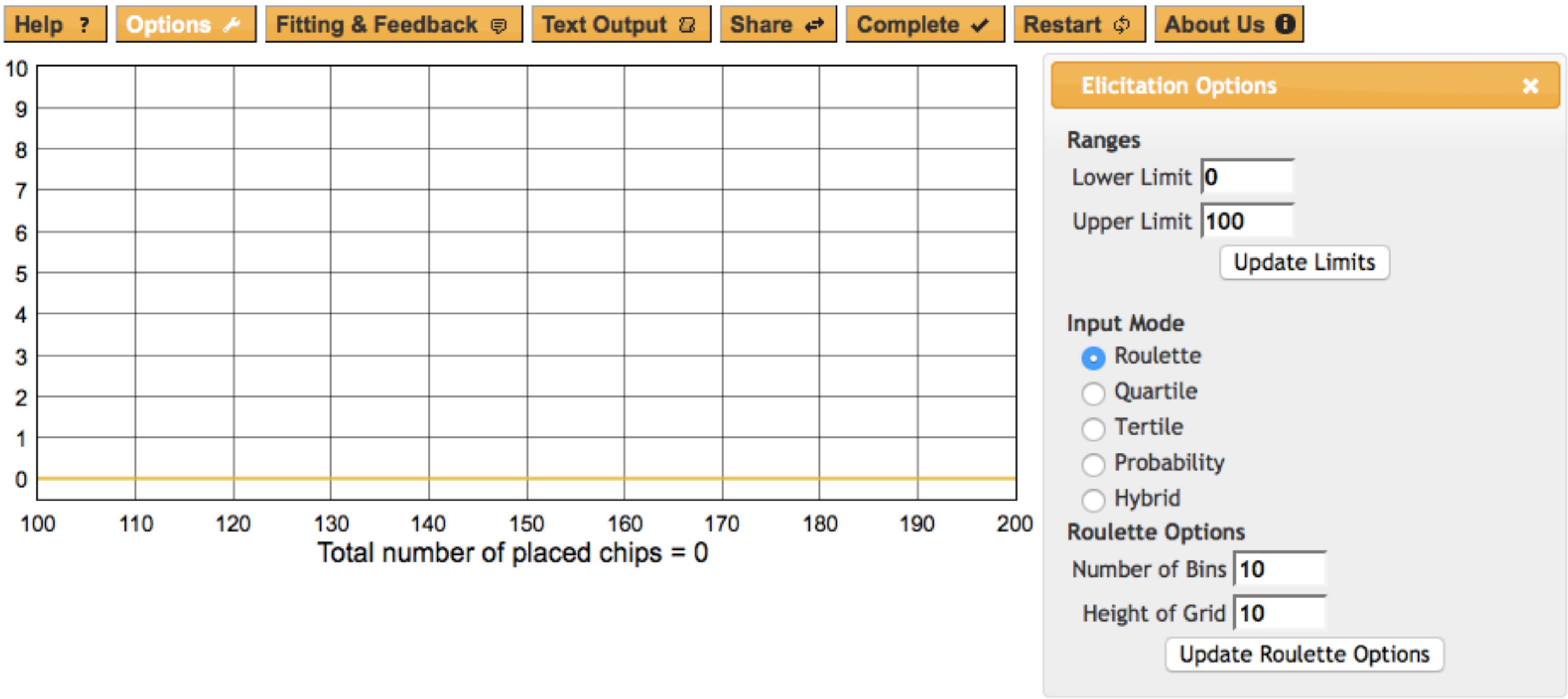


Probability of contamination?

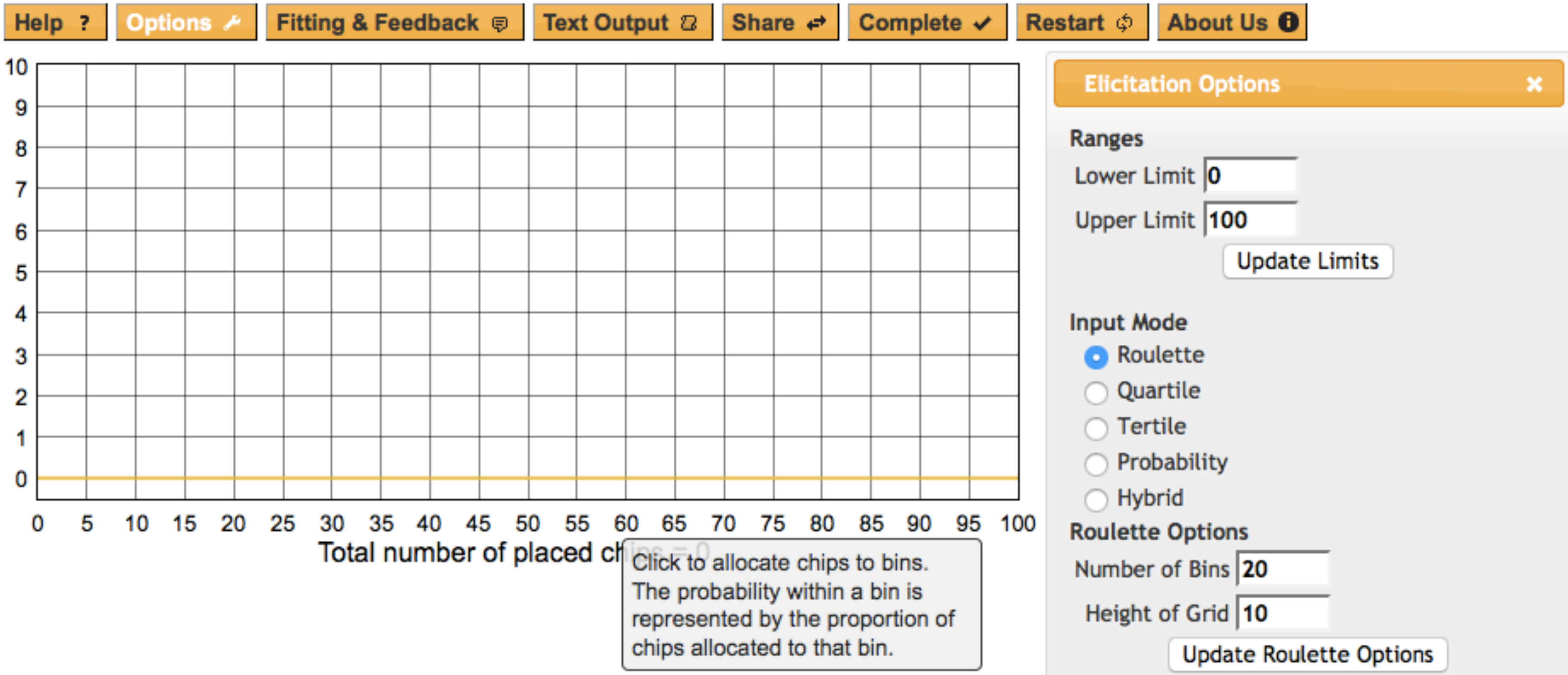


Expert

Expert elicitation



Match Uncertainty Elicitation Tool
Shelf package



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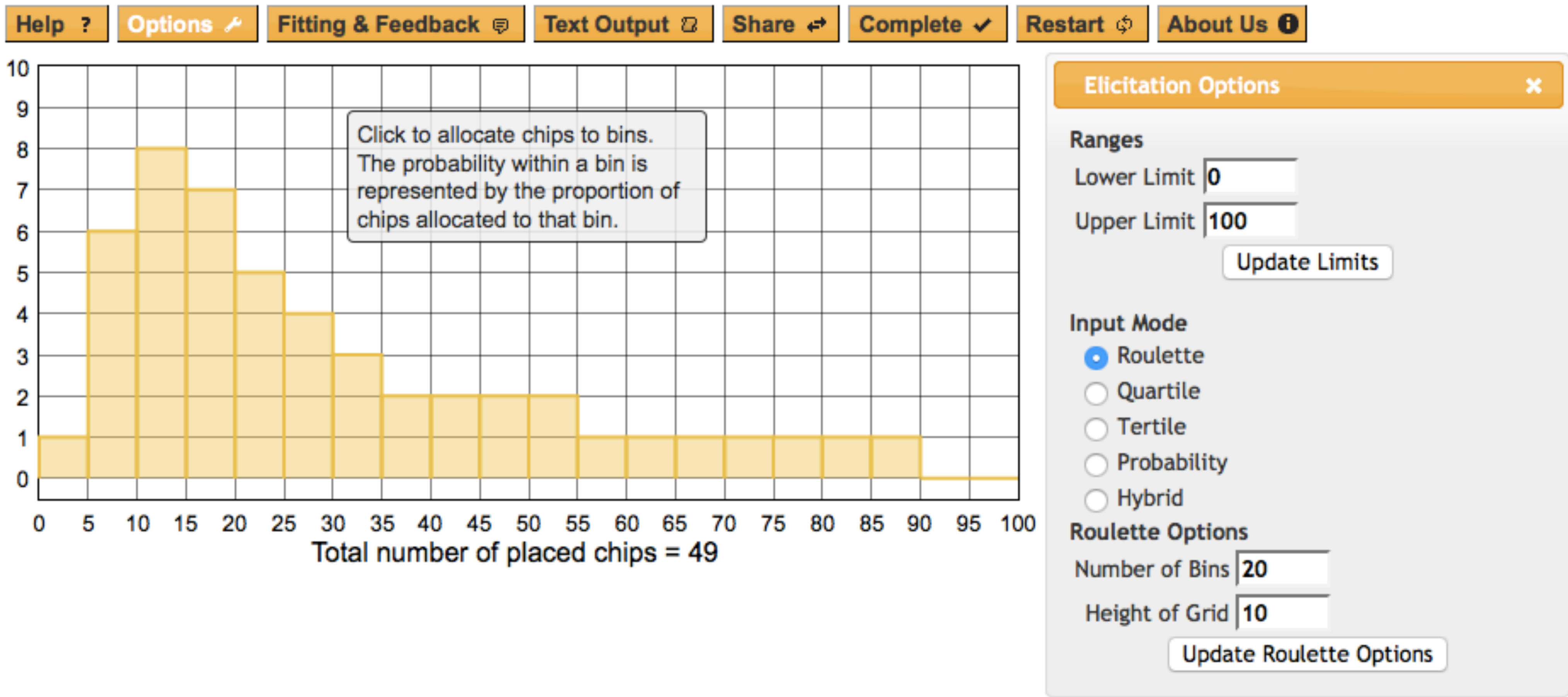
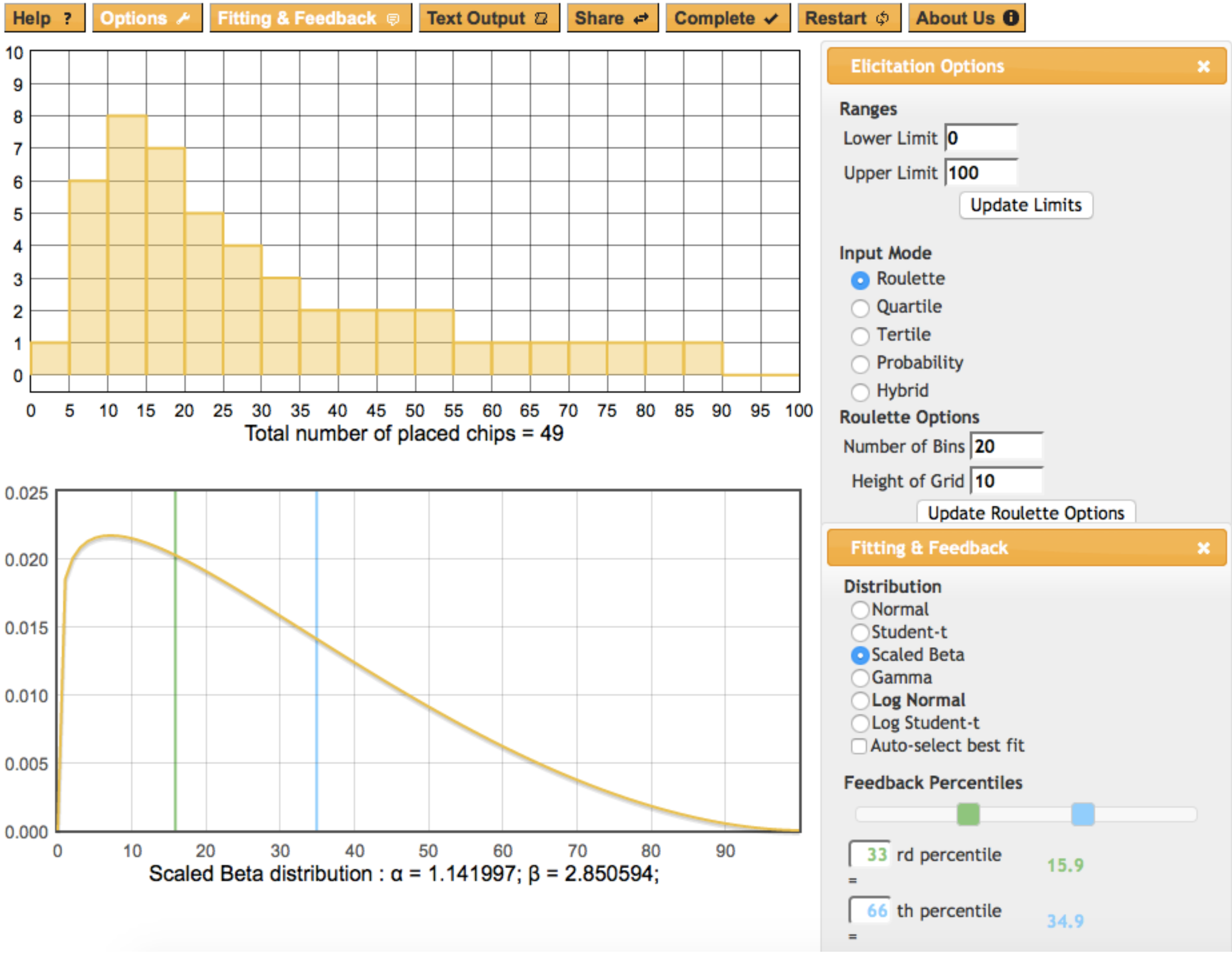
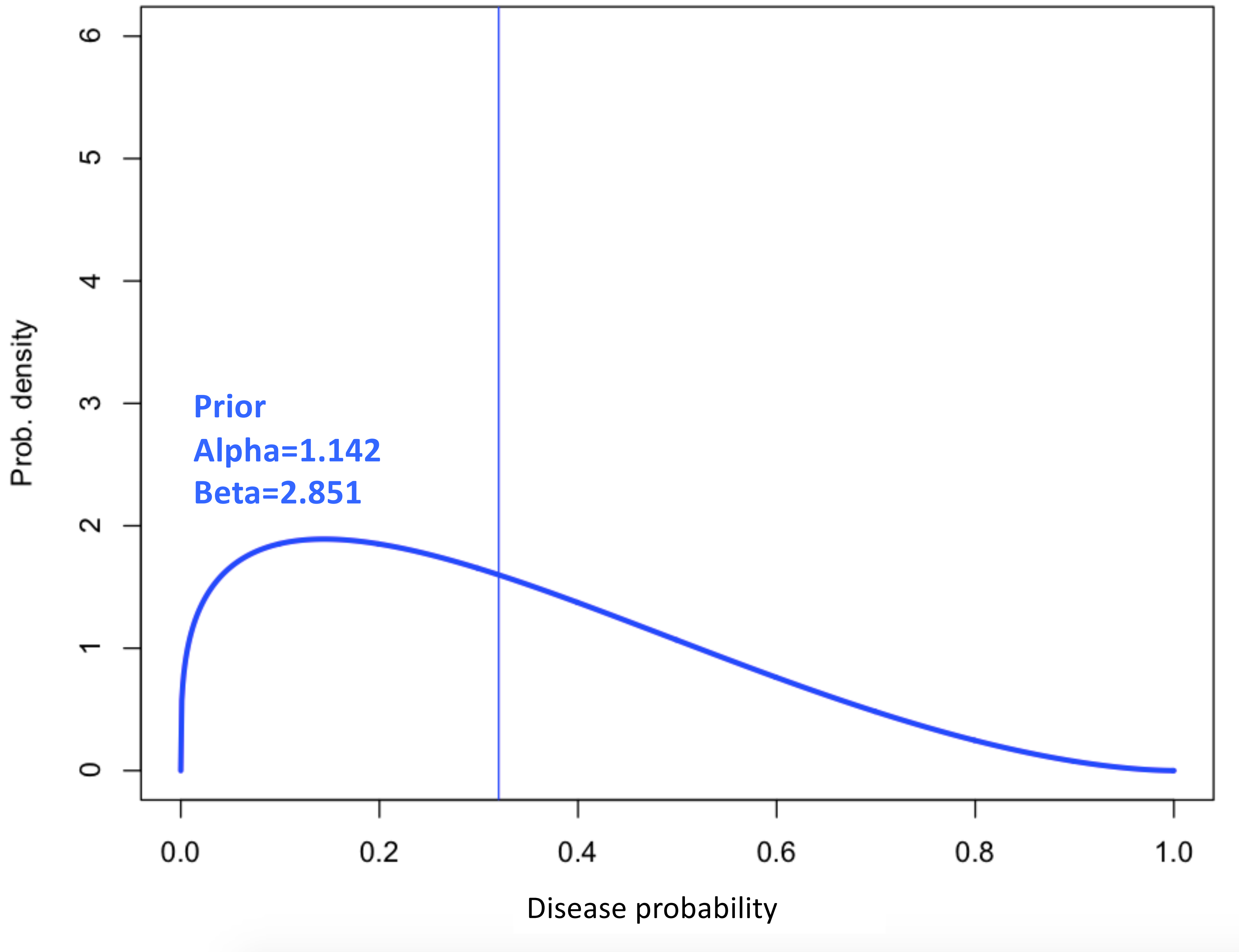
Options
Fitting & Feedback
Text Output
Share
Complete
Restart
About Us
10
9
8
7
6
5
4
3
2
1
0
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100
Total number of placed ch
Click to allocate chips to bins. The probability within a bin is represented by the proportion of chips allocated to that bin.
Elicitation Options
Ranges
Lower Limit 0
Upper Limit 100
Update Limits
Input Mode
Roulette
Quartile
Tertile
Probability
Hybrid
Roulette Options
Number of Bins 20
Height of Grid 10
Update Roulette Options


Help ?
Options
Fitting & Feedback
Text Output
Share
Complete
Restart
About Us
Click to allocate chips to bins. The probability within a bin is represented by the proportion of chips allocated to that bin.
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100
0 1 2 3 4 5 6 7 8 9 10
Total number of placed chips = 49
Elicitation Options
Ranges
Lower Limit 0
Upper Limit 100
Update Limits
Input Mode
Roulette
Quartile
Tertile
Probability
Hybrid
Roulette Options
Number of Bins 20
Height of Grid 10
Update Roulette Options


Help ?
Options
Fitting & Feedback
Text Output
Share
Complete
Restart
About Us
Total number of placed chips = 49
Scaled Beta distribution : α = 1.141997; β = 2.850594;
Elicitation Options
Ranges
Lower Limit 0
Upper Limit 100
Update Limits
Input Mode
Roulette
Quartile
Tertile
Probability
Hybrid
Roulette Options
Number of Bins 20
Height of Grid 10
Update Roulette Options
Fitting & Feedback
Distribution
Normal
Student-t
Scaled Beta
Gamma
Log Normal
Log Student-t
Auto-select best fit
Feedback Percentiles
33 rd percentile =
15.9
66 th percentile =
34.9



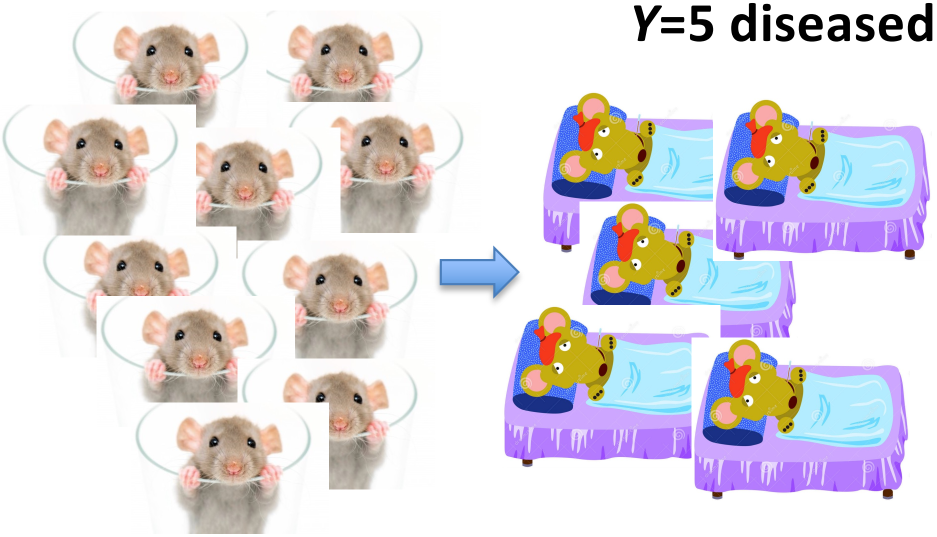
Prior
Alpha=1.142
Beta=2.851
Prob. density
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
Disease probability

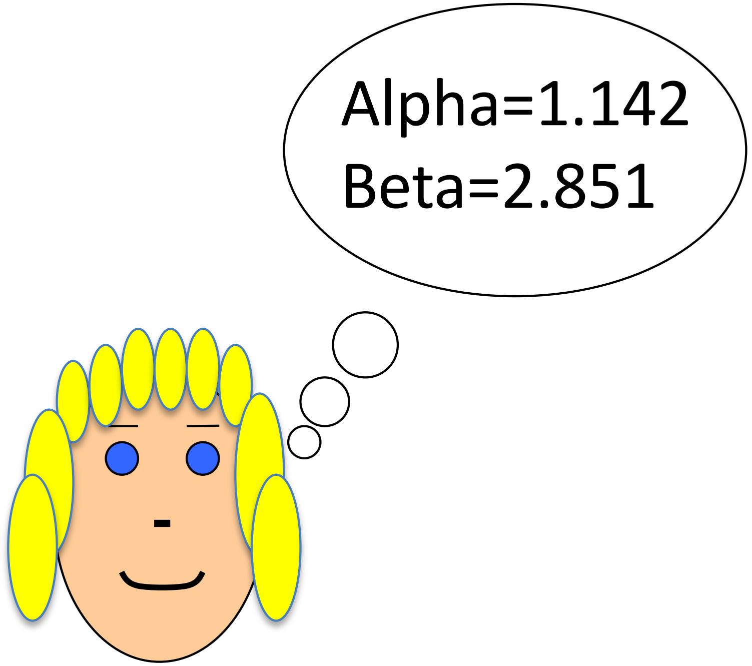
Example 3

Estimation of a probability of contamination by combining a measurement with expert knowledge

***N*=10 rats**

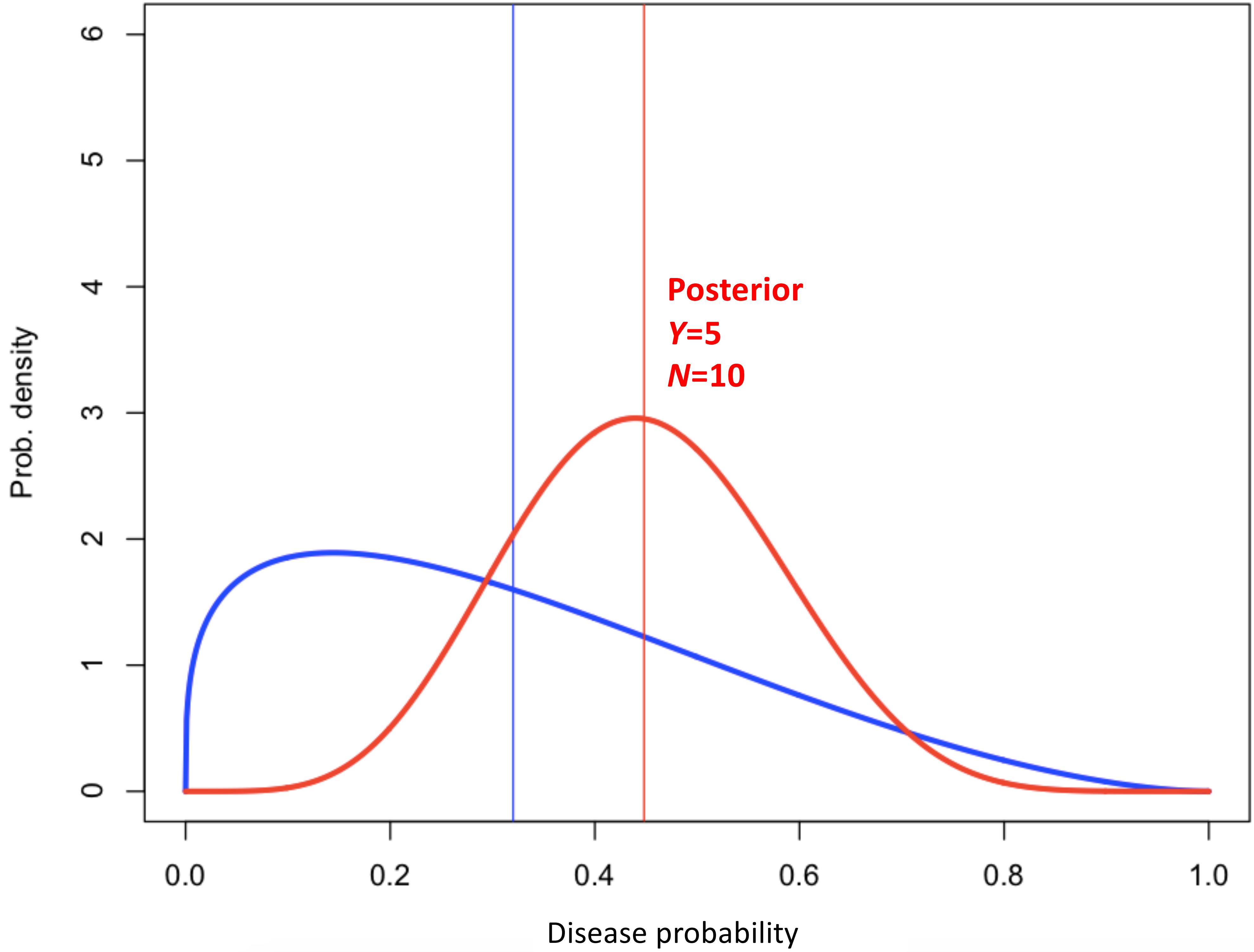
***Y*=5 diseased rats**





Probability of contamination?

Expert



Posterior
Y=5
N=10
Prob. density
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
Disease probability

Prior

$Beta(\alpha, \beta)$

Posterior

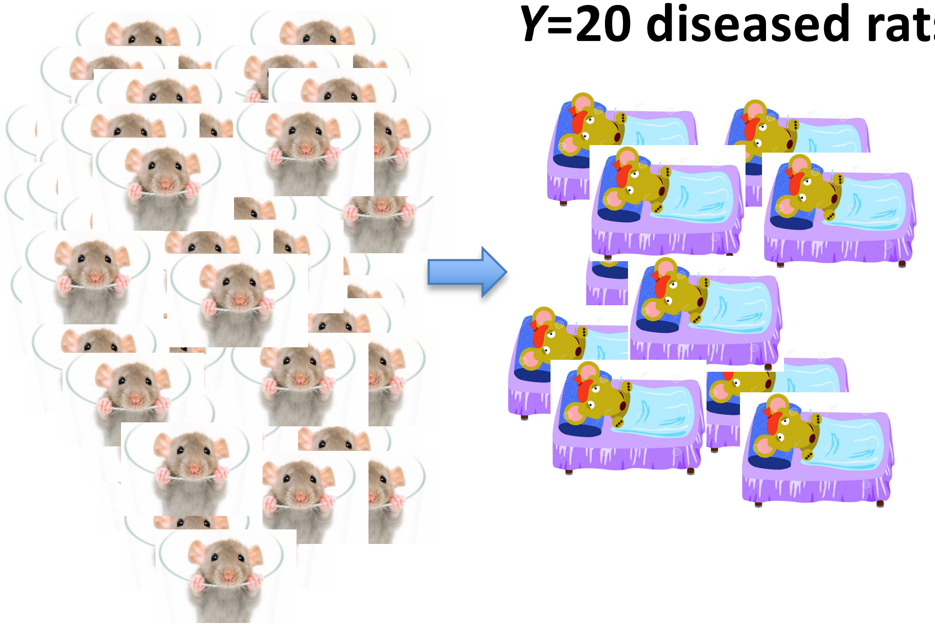
$Beta(\alpha + Y\ \beta + N - Y)$

Example 3

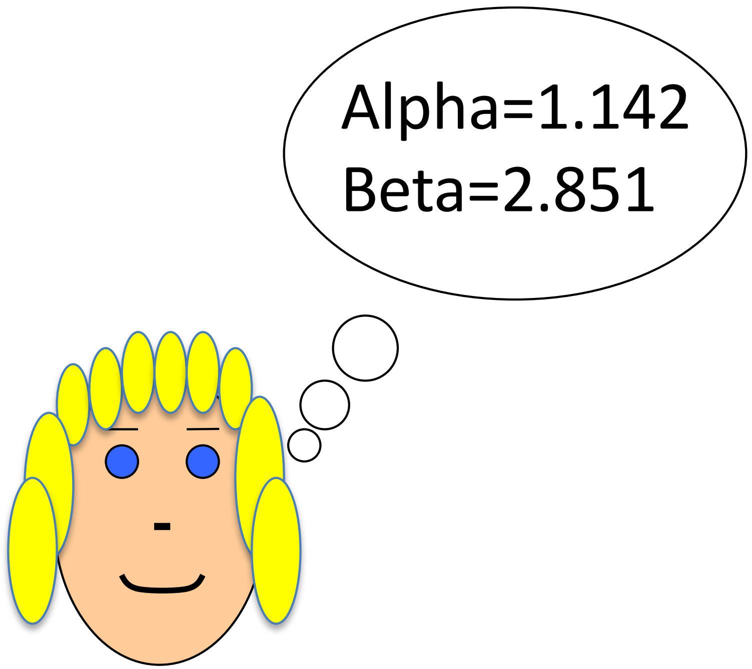
Estimation of a probability of contamination by combining a measurement with expert knowledge

***N*=40 rats**

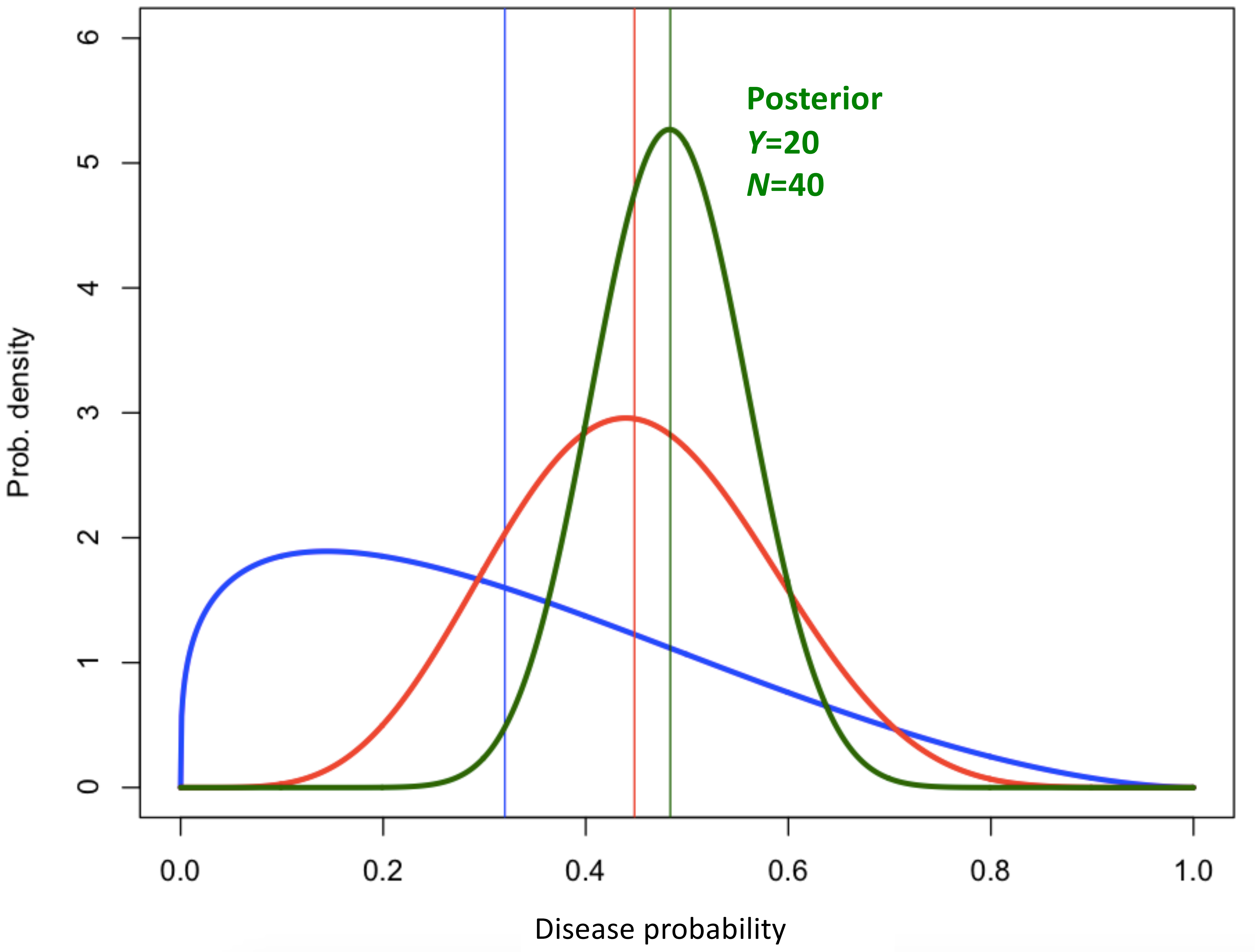
***Y*=20 diseased rats**



Probability of contamination?



Expert



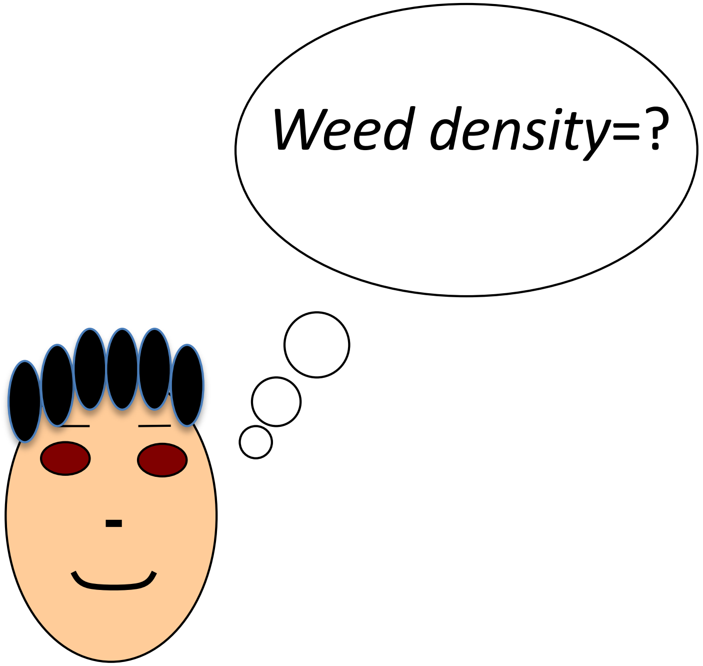
Posterior
Y=20
N=40
Prob. density
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0
Disease probability

Exercise

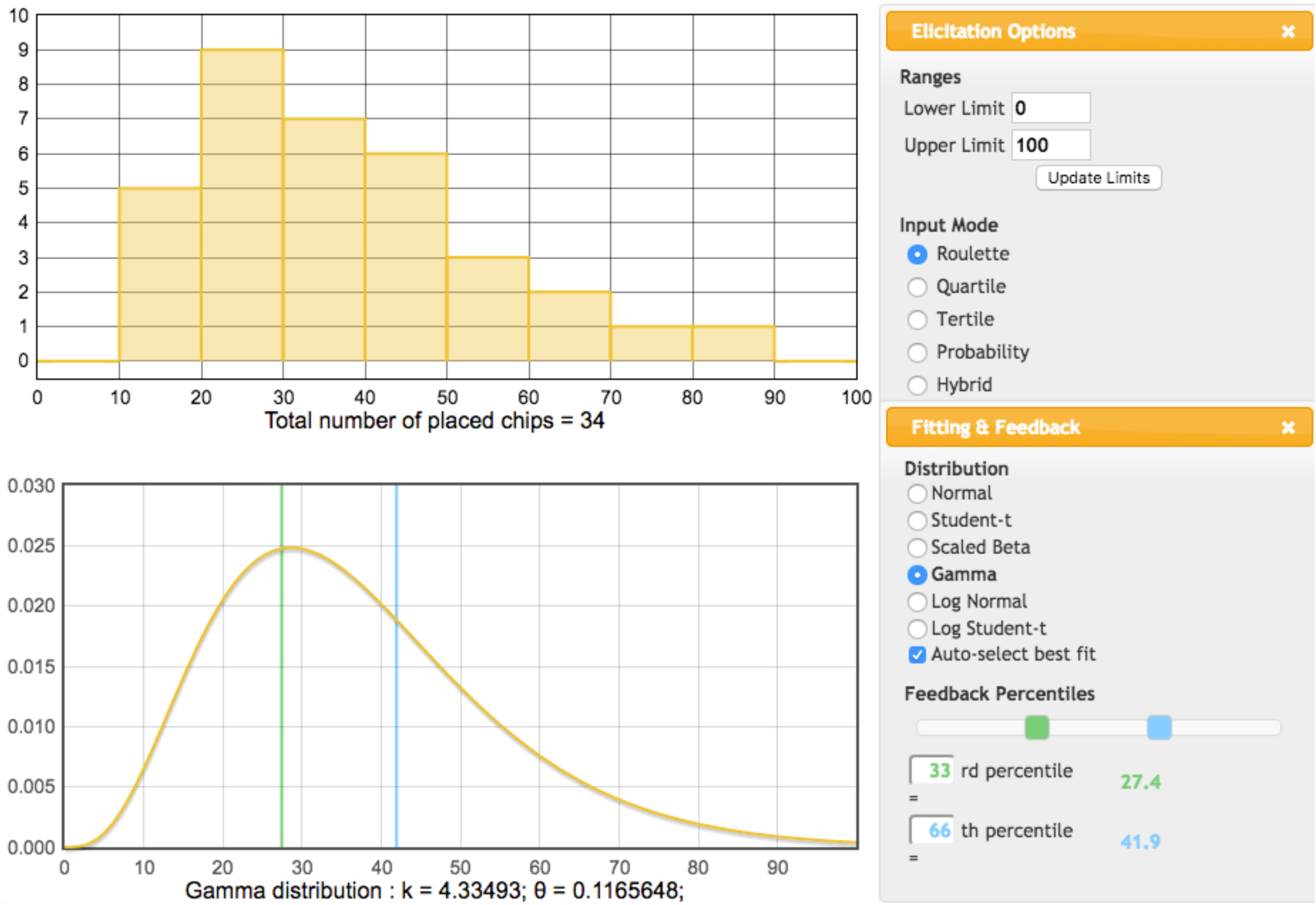
- Compute the posterior distribution for
  - N=10, Y=0
  - N=20, Y=0
- Calculate the posterior mean
- Plot the results
- Conclude

Example 4

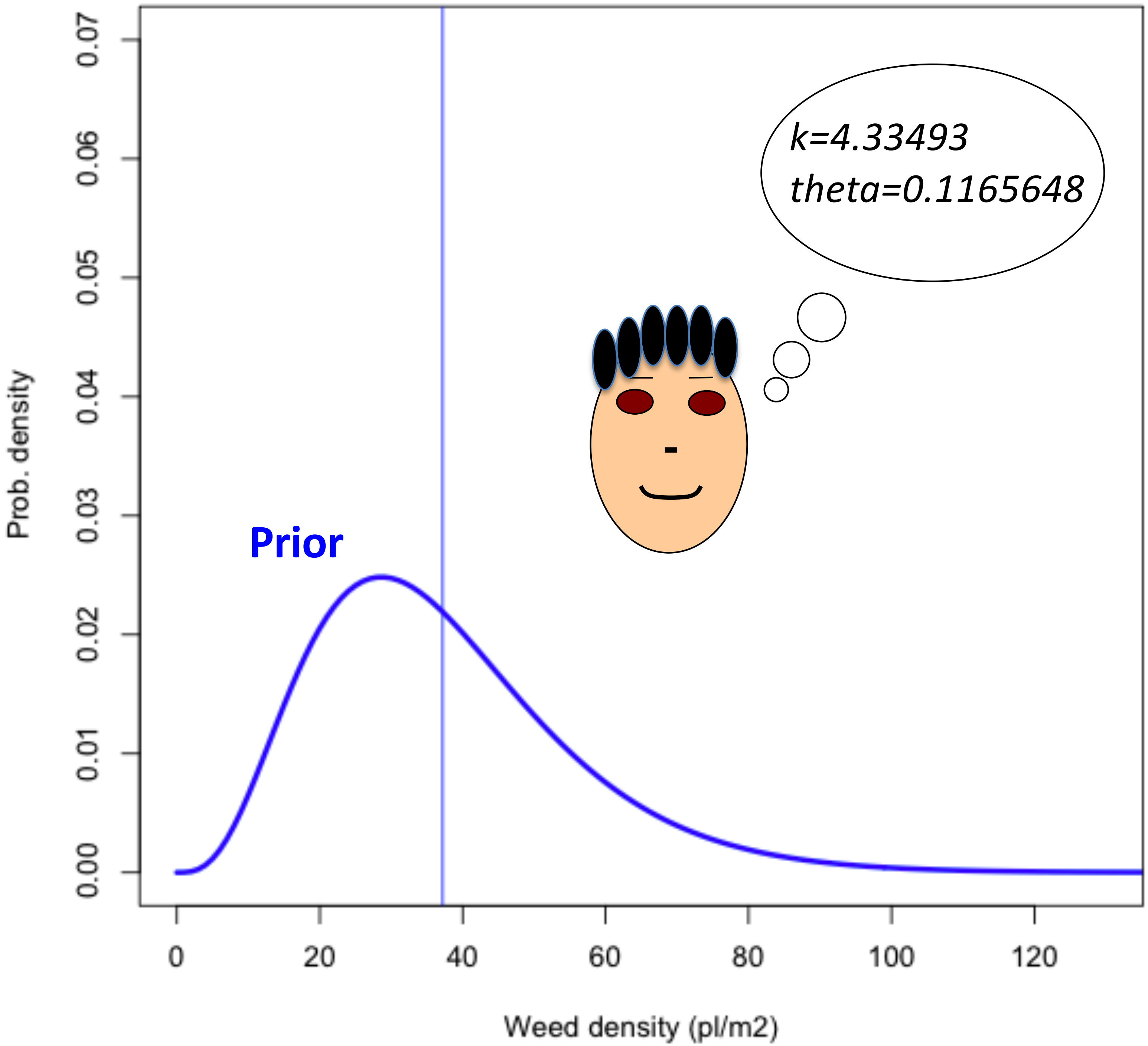
Estimation of a plant density
by combining a measurement with expert knowledge



Expert



10
9
8
7
6
5
4
3
2
1
0
0
10
20
30
40
50
60
70
80
90
100
Total number of placed chips = 34
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0
10
20
30
40
50
60
70
80
90
Gamma distribution : k = 4.33493; θ = 0.1165648;
Elicitation Options
Ranges
Lower Limit 0
Upper Limit 100
Update Limits
Input Mode
Roulette
Quartile
Tertile
Probability
Hybrid
Fitting & Feedback
Distribution
Normal
Student-t
Scaled Beta
Gamma
Log Normal
Log Student-t
Auto-select best fit
Feedback Percentiles
33 rd percentile
=
27.4
66 th percentile
=
41.9

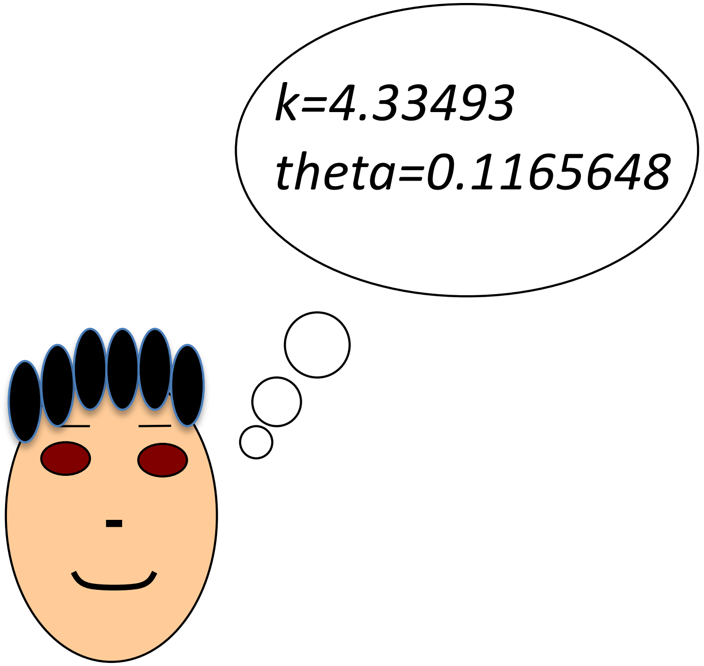


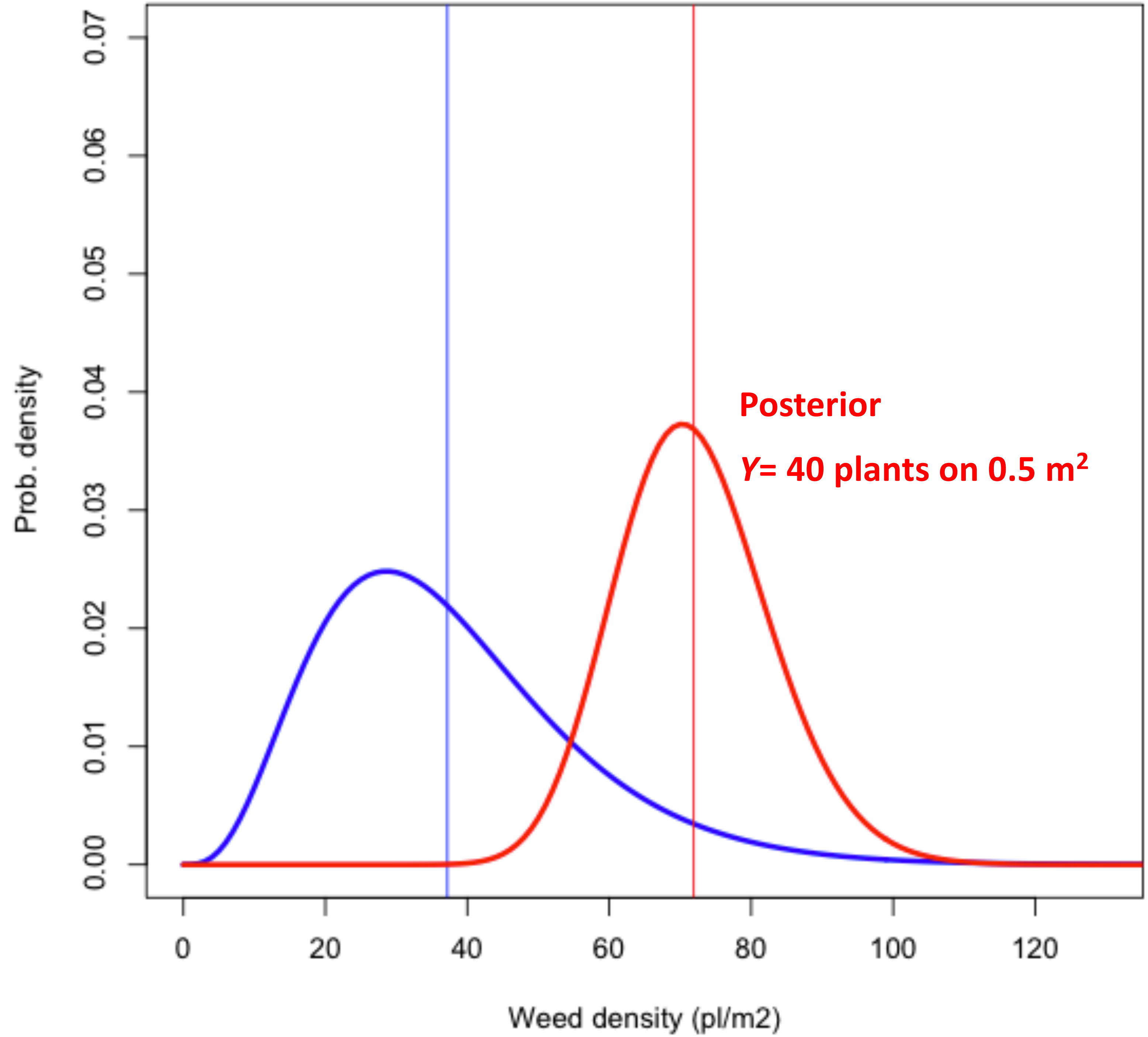
k=4.33493
theta=0.1165648
Prior
Prob. density
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0
20
40
60
80
100
120
Weed density (pl/m2)

Example 4

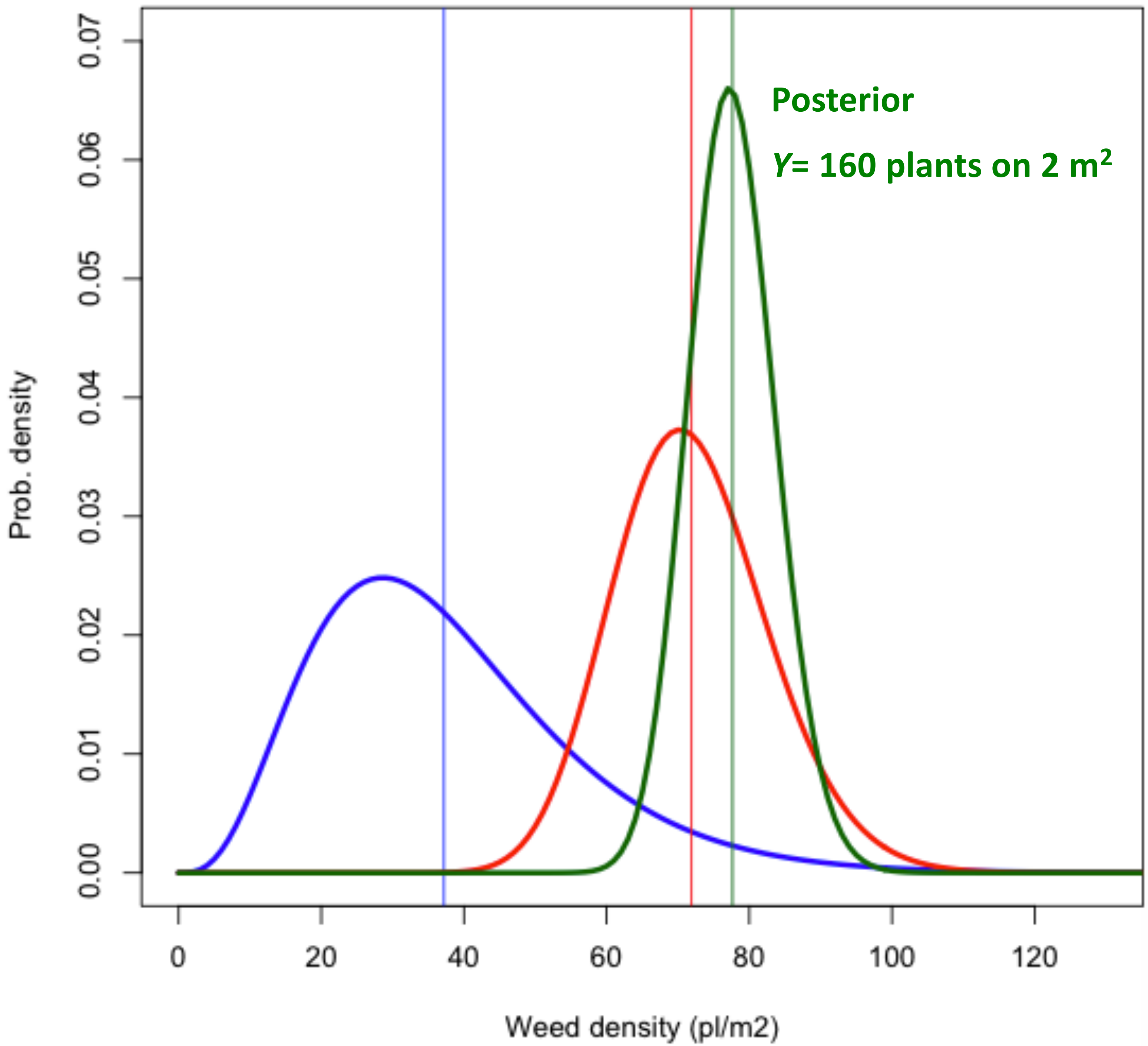
Estimation of a plant density
by combining a measurement with expert knowledge

40 plants on 0.5 m^2





Posterior
Y= 40 plants on 0.5 m2
Prob. density
Weed density (pl/m2)
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0
20
40
60
80
100
120



Posterior
Y= 160 plants on 2 m^2
Prob. density
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0
20
40
60
80
100
120
Weed density (pl/m2)

Practical considerations

• The analytical expression of the posterior distribution can be derived for simple applications :

Prior	Likelihood	Posterior
Gaussian	Gaussian	Gaussian
Beta	Binomial	Beta
Gamma	Poisson	Gamma

Practical considerations

• The analytical expression of the posterior distribution can be derived for simple applications :

Prior	Likelihood	Posterior
Gaussian	Gaussian	Gaussian
Beta	Binomial	Beta
Gamma	Poisson	Gamma

For complex problems, the posterior distribution must be approximated.

Algorithms

- Importance sampling
- Metropolis-Hastings
- Gibbs sampling
- Integrated Nested Laplace approximation

…

Markov chain Monte Carlo (MCMC)

Algorithms

- Importance sampling
- Metropolis-Hastings
- Gibbs sampling
- Integrated Nested Laplace approximation

Markov chain Monte Carlo (MCMC)

…

The importance sampling algorithm

- **Step 0**. Choose a « proposal » distribution $g(\theta)$. This can in principle be ANY (well, almost any) distribution.
- **Step 1**. Generate a sample of size N (for example, N=50000) from $g(\theta)$. This gives $\theta_1, \theta_2, \ldots, \theta_N$.
- **Step 2**. Calculate weights $w_i = P(\theta_i|y) / g(\theta_i)$
- **Step 3**. Calculate normalized weights

$$w_i^* = \frac{w_i}{\sum_{i=1}^{N} w_i}$$

- **Step 4**. The weighted sample $(\theta_1, w^*_1), (\theta_2, w^*_2), \ldots, (\theta_N, w^*_N)$ provides an approximation to the posterior distribution
- **Step 5**. Resampling

Algorithms

- Importance sampling
- Metropolis-Hastings
- Gibbs sampling
- Integrated Nested Laplace approximation

…

The Metropolis-Hastings algorithm

Step 0. Choose a starting value θ_1. Define a proposal distribution $P_p(\theta_c|\theta_i)$. (For example, use a normal distribution with mean equal to θ_i).

Repeat steps 1-3 for i=1,…,N

Step 1. Generate a candidate parameter value θ_c from $P_p(\theta_c|\theta_i)$.

Step 2. Calculate

$$T = \frac{P(Y|\theta_c)P(\theta_c)P_p(\theta_i|\theta_c)}{P(Y|\theta_i)P(\theta_i)P_p(\theta_c|\theta_i)}$$

Step 3. If $T \geq 1$, then $\theta_{i+1} = \theta_c$. If $T<1$, then draw u from a uniform distribution on the interval (0, 1). If $u<T$ then $\theta_{i+1} = \theta_c$ otherwise $\theta_{i+1} = \theta_i$.

The result of the algorithm is a list of N parameter values. The same value may be repeated several times.

The Metropolis-Hastings algorithm

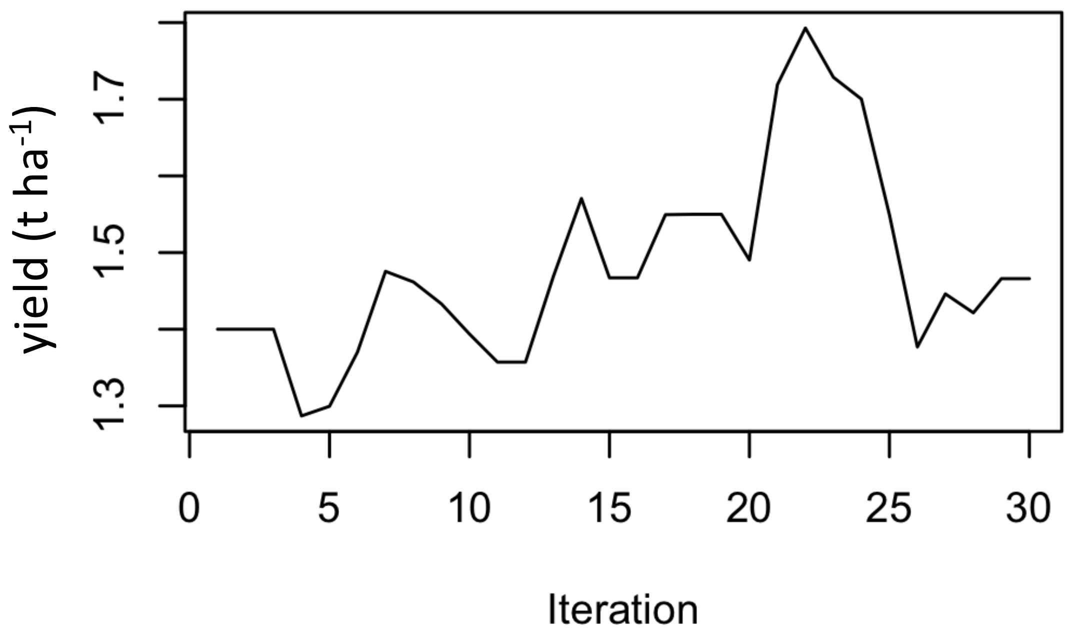
with symmetric proposal distribution

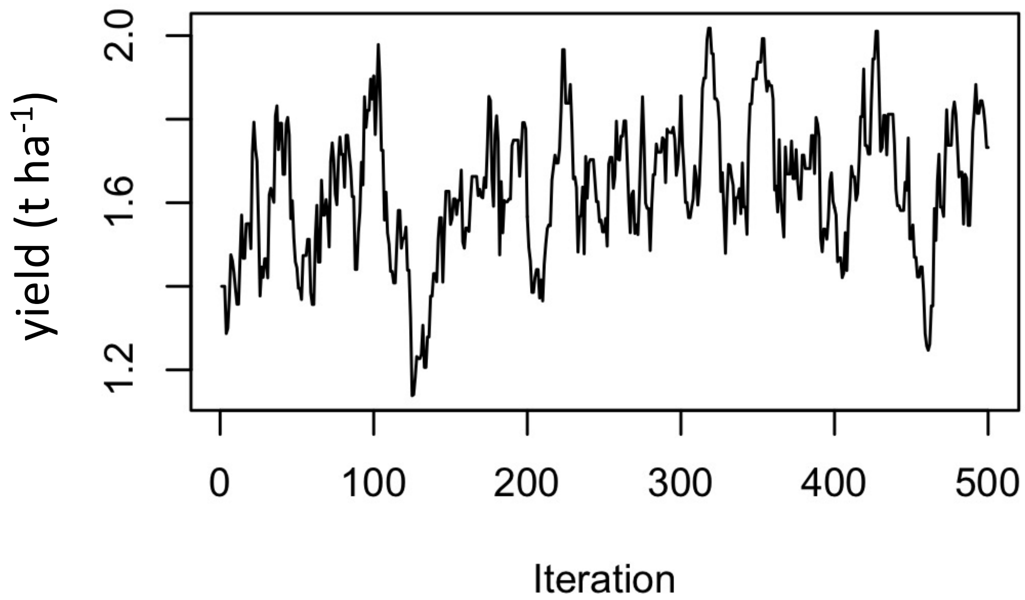
- A common choice for the proposal distribution $P(\theta_c|\theta_i)$ is a normal distribution with mean equal to θ_i and constant variance.

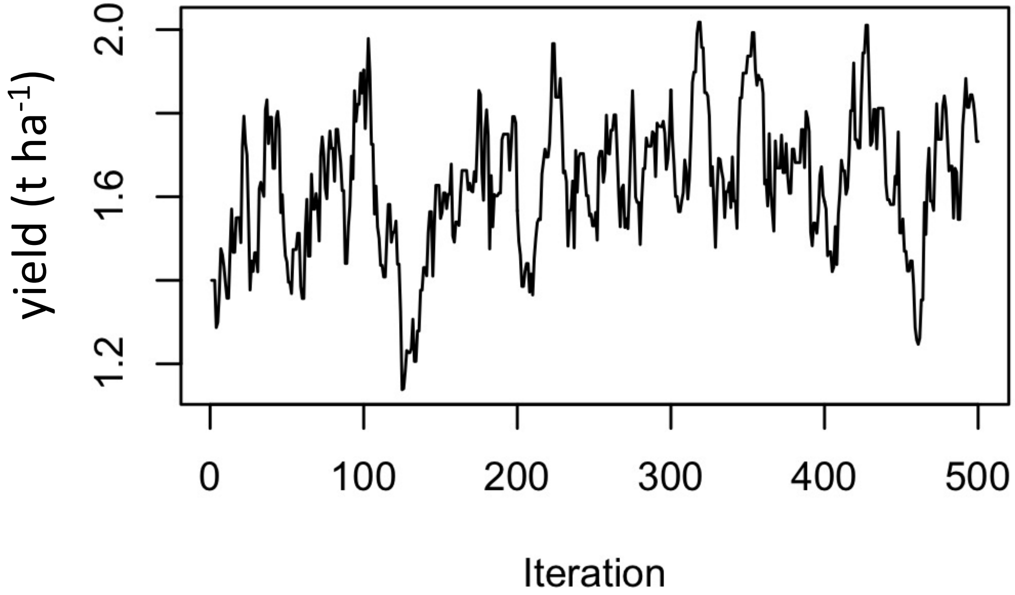
- In this case $P(\theta_c|\theta_i) = P(\theta_i|\theta_c)$ and the expression for T simplifies:

$$T = \frac{P(Y|\theta_c)P(\theta_c)}{P(Y|\theta_i)P(\theta_i)}$$

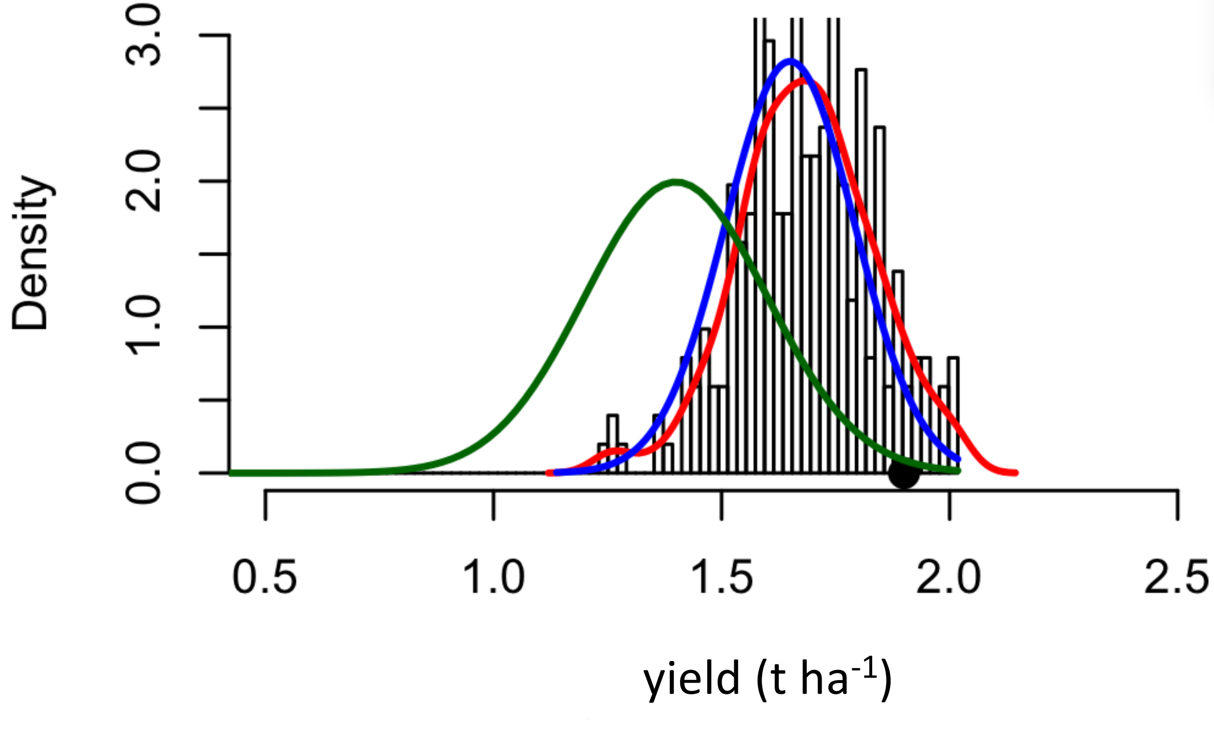
Likelihood **Prior density**


yield (t ha-1)
1.3
1.5
1.7
0
5
10
15
20
25
30
Iteration

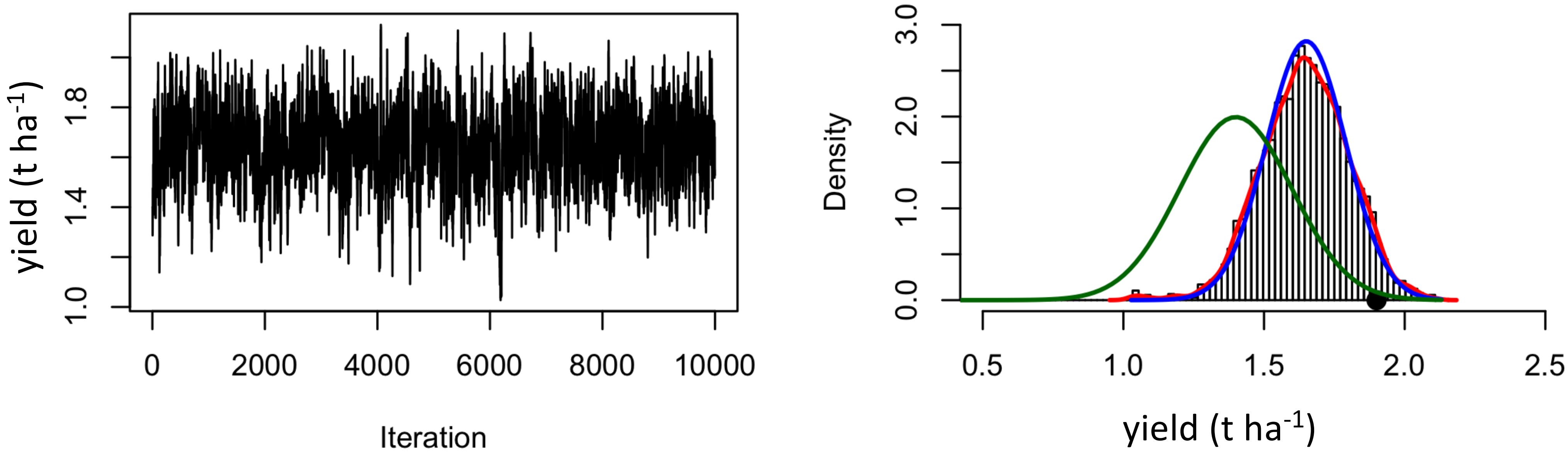

yield (t ha-1)
1.2
1.6
2.0
0
100
200
300
400
500
Iteration



yield (t ha-1)
2.0
1.6
1.2
0
100
200
300
400
500
Iteration



Density
3.0
2.0
1.0
0.0
0.5
1.0
1.5
2.0
2.5
yield (t ha-1)


yield (t ha^{-1})
1.0
1.4
1.8
0
2000
4000
6000
8000
10000
Iteration
Density
0.0
1.0
2.0
3.0
0.5
1.0
1.5
2.0
2.5
yield (t ha^{-1})

Exercise

- Open the file MHyield.R
- Run the algorithm with 100 iterations
- Increase the number of iterations to 10000
- Double the standard deviation of the measurement and compute the posterior.
- What is the effect of reducing the accuracy of the measurement?

The Bayesian boom

- New methods for estimating posterior probability distributions

✓ Markov chain Monte Carlo (MCMC)	Late 1990s
✓ Importance sampling	Late 1990s
✓ Approximate Bayesian Computation (ABC)	Early 2000s
✓ Bayesian melding	Early 2000s
✓ Bayesian Model Averaging (BMA)	Early 2000s
✓ Integrated Nested Laplace Approximations	~ 2010

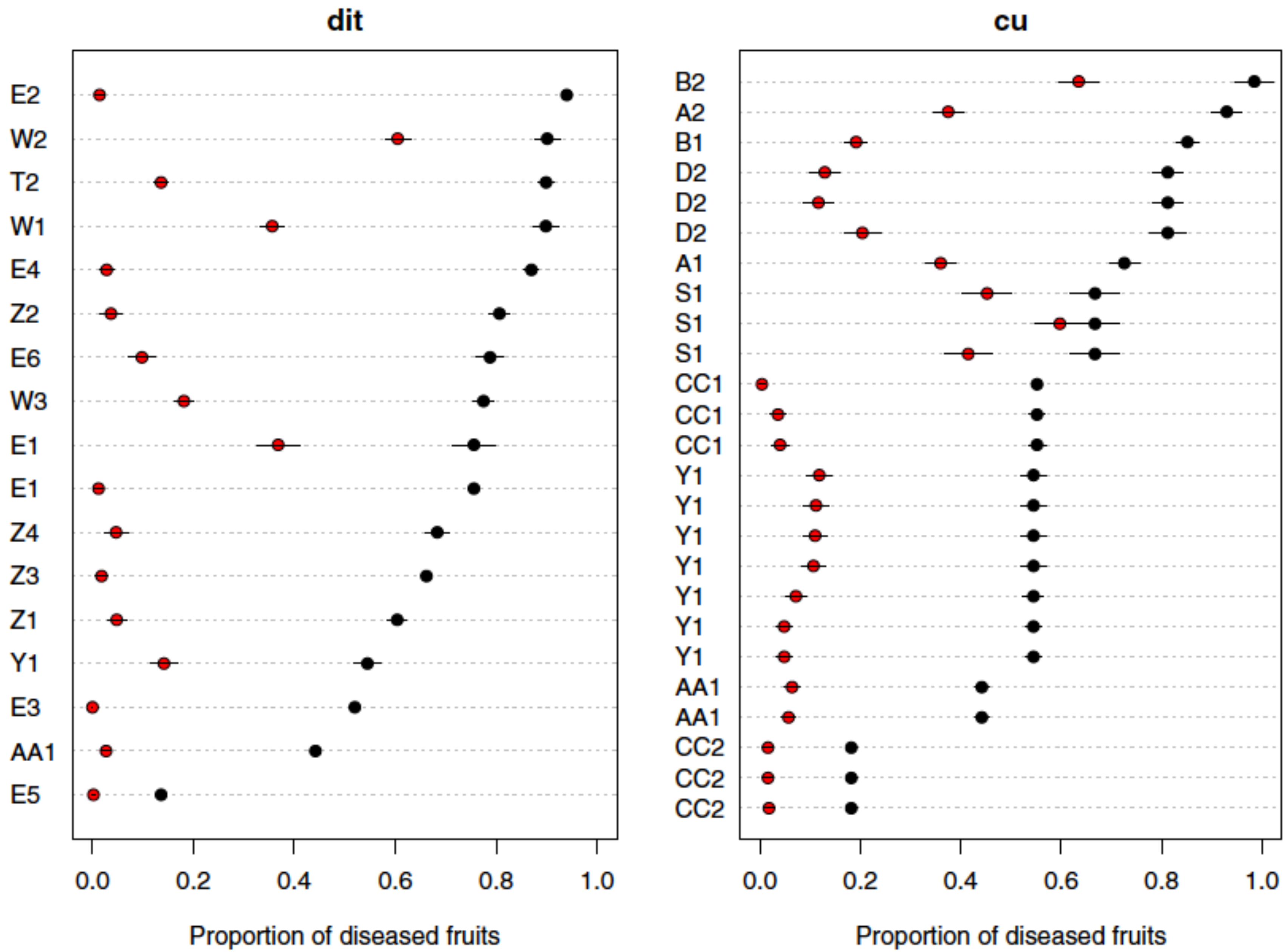
- Strong decrease of computational time

- Dedicated packages (Rjags, MCMCglmm, brms, etc.)

→ **It is now possible to apply Bayesian techniques to complex problems**

Comparison of statistical models in a meta-analysis of fungicide treatments for the control of citrus black spot caused by *Phyllosticta citricarpa*

D. Makowski · A. Vicent · M. Pautasso · G. Stancanelli · T. Rafoss


dit
cu
E2
W2
T2
W1
E4
Z2
E6
W3
E1
E1
Z4
Z3
Z1
Y1
E3
AA1
E5
B2
A2
B1
D2
D2
D2
A1
S1
S1
S1
CC1
CC1
CC1
Y1
Y1
Y1
Y1
Y1
Y1
Y1
AA1
AA1
CC2
CC2
CC2
0.0
0.2
0.4
0.6
0.8
1.0
Proportion of diseased fruits
Proportion of diseased fruits

$$Y_{ij} \sim Binomial \quad (n_{ij}, \pi_{ij})$$

i = study index
j = treatment index

$$\text{logit}(\pi_{ij}) = log\left(\frac{\pi_{ij}}{1-\pi_{ij}}\right) = \alpha_{0i} + \alpha_{Ti} X_{ij}$$

$$\begin{pmatrix} \alpha_{0i} \\ \alpha_{Ti} \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_0 \\ \mu_T \end{pmatrix}, \Sigma\right] \qquad \Sigma = \begin{bmatrix} \sigma_0^2 & c \\ c & \sigma_T^2 \end{bmatrix}$$

$$\mu_0, \mu_T \sim N(0, 10^6).$$

$$\Sigma \sim InvWish(\psi, v)$$

Type of fungicide	Disease incidence	
	Untreated fruits	Treated fruits
“cu”	0.74 (0.53-0.87)	0.16 (0.072-0.32)
“dit”	0.74 (0.62-0.83)	0.053 (0.022-0.12)

Summary

- Concepts are relatively simple
- Bayesian methods are useful for
  - combining expert knowledge and data
  - analyzing uncertainty
  - dealing with complex data structure
- Software are available: many R packages
- Computation can be long and complex