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UNIFICATION

Motivation

Unification: **finding a common instance of two terms**

Informally: we want to make two terms **identical** by finding the **most general substitution** of terms for variables.

Why?

- Applying rules in Isabelle: working out what $?P$, $?Q$, $?x$ are
- Heavily used in automated first-order theorem proving to postpone decisions during proof search: PROLOG, tableau provers, resolution provers
- Also used in most type inference algorithms (Haskell, OCaml, SML, Scala, ...)

A First Look at Unification — SYNTACTIC MATCHING

Unification: **finding a common instance of two terms**

Informally: we want to make two terms **identical** by finding the **most general substitution** of terms for variables.

Q7.

Example

Can we make these pairs of terms equal by finding a common instance (assuming X, Y are variables and a, b are constants)?

replace X w/ a and Y w/ b

Cannot assign a to X and b to X at the same time

$f(X, b)$ and $f(a, Y)$	Yes: $[a/X, b/Y]$	instance: $f(a, b)$
$f(X, X)$ and $f(a, b)$	No	
$f(X, X)$ and $f(Y, g(Y))$	No	

Only (meta-)variables ($X, Y, Z, \ldots$) can be replaced by other terms.

Matching

Problem

Given ***pattern*** *and* ***target*** *find a* ***substitution*** *such that:*

$$pattern[substitution] \equiv target$$

where $\equiv$ *means that the terms are identical.*

Example Q9.

$$(s(X) + Y)[0/X, s(0)/Y] \equiv (s(0) + s(0))$$

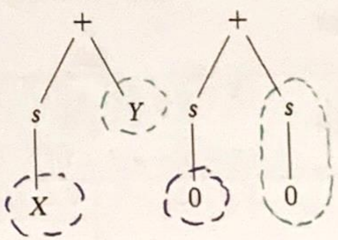
How we do find an adequate substitution?

We view matching as equation solving.

Matching (continued)

Discover a substition by decomposing the equation to be solved along the term trees:

$$(s(X) + Y) \equiv (s(0) + s(0))$$
$$\downarrow$$
$$(s(X) \equiv s(0)) \wedge (Y \equiv s(0))$$
$$\downarrow$$
$$(X \equiv 0) \wedge (Y \equiv s(0))$$



Some Abbreviations

Term	Meaning
$\overrightarrow{t}$	$t_1, \ldots, t_n \quad (t \geq 1)$
$\bigwedge_i t_i$	$t_1 \wedge \ldots \wedge t_n$
vars(t)	the set of free variables in t
Vars	the set of (all) free variables

$\text{vars}(f(X, Y, g(a, Z, X))) = \{X, Y, Z\}$

$\text{vars}(f(\underbrace{a, b, c}_{\text{all constants}})) = \{\}$

Matching as Equation Solving

→ Start with the *pattern* and *target* **standardised apart**:

$$\text{vars}(\textit{pattern}) \cap \text{vars}(\textit{target}) = \{\}$$

Goal is to solve for *vars(pattern)* in equation *pattern* $\equiv$ *target*.

→ Strategy is to use transformation rules:

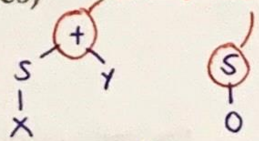
$$\begin{array}{c} \textit{pattern} \equiv \textit{target} \\ \downarrow \\ \vdots \\ \downarrow \\ X_1 \equiv t_1 \wedge \ldots \wedge X_n \equiv t_n \end{array}$$

Resulting substitution is $[t_1/X_1, \ldots, t_n/X_n]$.

→ Transformations end in failure if no match is possible.

Transformation Rules for Matching (Examples)

cannot match



Decompose

$$\begin{array}{c} s(X) + Y \equiv s(0) + s(0) \\ \downarrow \\ s(X) \equiv s(0) \wedge Y \equiv s(0) \end{array}$$

Conflict

$$\begin{array}{c} s(X) + y \equiv s(0) \\ \downarrow \\ \text{fail} \end{array}$$

Cannot match: $s \not\equiv +$

Eliminate

$$\begin{array}{c} (X + \boxed{Y} \equiv s(0) + 0) \wedge \underline{(Y \equiv 0)} \\ \downarrow \\ (X + \boxed{0} \equiv s(0) + 0) \wedge (Y \equiv 0) \end{array}$$

Delete

$$\begin{array}{c} X \equiv 0 \wedge \underline{(s(0) + 0 \equiv s(0) + 0)} \\ \downarrow \\ X \equiv 0 \end{array}$$

↳ There are NO VARIABLES!

Transformation Rules for Matching

Assumptions: s and t are arbitrary terms and are standardised apart.

Name	Before	After	Condition
Decompose	$P \wedge f(\overrightarrow{s}) \equiv f(\overrightarrow{t})$	$P \wedge \bigwedge_i s_i \equiv t_i$	
Conflict	$P \wedge f(\overrightarrow{s}) \equiv g(\overrightarrow{t})$	fail	$f \neq g$
Eliminate	$P \wedge X \equiv t$	$P[t/X] \wedge X \equiv t$	$X \in \text{vars}(P)$
Delete	$P \wedge t \equiv t$	P	

→ Algorithm terminates when no further rules apply and fail has not occurred.

→ The algorithm terminates with a match iff there is one.

→ The algorithm may terminate without a match: e.g., $X \equiv a \wedge b \equiv Y$

Unification

more powerful than what we see earlier

Unification is $\boxed{\text{two-way}}$ matching (there is no distinction between pattern and target).

$$term_1[substitution] \equiv term_2[substitution]$$

Example

What substitution makes $(s(X) + s(0))$ and $(s(0) + Y)$ identical?

$$\theta = [0/X, s(0)/Y]$$

We need to add **extra rules** to the matching algorithm:

$$\begin{array}{cl}
(s(X) + s(0)) \equiv (s(0) + Y) & \\
\downarrow & \text{Decompose} \\
s(X) \equiv s(0) \wedge s(0) \equiv Y & \\
\downarrow & \text{Decompose} \\
X \equiv 0 \wedge s(0) \equiv Y & \\
\downarrow & \text{Switch} \\
X \equiv 0 \wedge Y \equiv s(0) &
\end{array}$$

Unification Algorithm

Assumptions: s and t are arbitrary terms and $Vars = vars(s) \cup vars(t)$.

Name	Before	After	Condition
Decompose	$P \wedge f(\vec{s}) \equiv f(\vec{t})$	$P \wedge \bigwedge_i s_i \equiv t_i$	
Conflict	$P \wedge f(\vec{s}) \equiv g(\vec{t})$	fail	$f \not\equiv g$
NEW ✓ Switch	$P \wedge s \equiv X$	$P \wedge X \equiv s$	$X \in Vars$ $s \notin Vars$
Delete	$P \wedge s \equiv s$	P	
Eliminate	$P \wedge X \equiv s$	$P[s/X] \wedge X \equiv s$	$X \in vars(P)$ $X \notin vars(s)$ $s \notin Vars$
NEW ✓ Occurs Check	$P \wedge X \equiv s$	fail	$X \in vars(s)$ $s \notin Vars$
NEW ✓ Coalesce	$P \wedge X \equiv Y$	$P[Y/X] \wedge X \equiv Y$	$X, Y \in vars(P)$ $X \not\equiv Y$

- Conditions ensure that at most one rule applies to each conjunct
- Algorithm terminates with success when no further rules apply.

New Transformation Rules

Switch

$$t \equiv X \;\downarrow\; X \equiv t$$

Switch rule applies only if *lhs* is not originally a variable

otherwise, algo. will NOT terminate!

Coalesce

$$X \equiv Y + 1 \wedge Y \equiv X \;\downarrow\; X \equiv X + 1 \wedge Y \equiv X$$

Similar to Eliminate, except both *lhs* and *rhs* are variables

Occurs Check

$$X \equiv X + 1 \;\downarrow\; \text{fail}$$

lhs cannot occur in *rhs*

Example

e.g.

$$\begin{array}{c}
f(X, X) \equiv f(Y, Y+1) \\
\downarrow \text{Decompose} \\
X \equiv Y \wedge X \equiv Y + 1 \\
\downarrow \text{Coalesce} \\
X \equiv Y \wedge Y \equiv Y + 1 \\
\downarrow \text{Occurs check} \\
\text{fail}
\end{array}
\qquad
\begin{array}{c}
p(X) \wedge X \equiv X + 1 \\
\downarrow \text{Eliminate} \\
p(X+1) \wedge X \equiv X + 1 \\
\downarrow \text{Eliminate} \\
p((X+1)+1) \wedge X \equiv X + 1 \\
\downarrow \text{Eliminate} \\
\ldots
\end{array}$$

Non-termination can result without the occurs check.

Composition of Unifiers (Substitutions)

Definition

If ϕ and θ are substitutions then their *composition* $\phi \circ \theta$ is also a substitution which, for any term t, satisfies the following property:

$$t[\phi \circ \theta] \equiv (t[\phi])[\theta]$$

Examples:

e.g.

$$[a/x] \circ [b/y] = [a/x, b/y]$$

replace y with b

$$[g(y)/x] \circ [b/y] = [g(b)/x, b/y]$$

$$[a/x] \circ [b/x] = [a/x]$$

no need to write this anymore, as there'll be no more x left after [a/x]

- $\boxed{\text{Equality of substitutions:}}$ $\phi = \theta$ if $x[\phi] = x[\theta]$ for any variable x.
- Properties: $\boxed{(\phi \circ \theta) \circ \sigma = \phi \circ (\theta \circ \sigma)}$ $\boxed{\phi \circ [] = \phi \text{ and } [] \circ \phi = \phi.}$
- Composition is needed to define the notion of a *most general unifier*.

whatever other substitutions you find,
whenever you apply this substitution, then
you will get this $\phi = \theta \circ \psi$ if exist

Properties of the Unification Algorithm

- The algorithm will find a unifier, if it exists.
- It returns the **most general unifier** (mgu) θ.

  Definition

  Given any two terms s and t, θ is their mgu if:

$$s[\theta] \equiv t[\theta] \wedge \forall \phi.\ s[\phi] \equiv t[\phi] \rightarrow \exists \psi.\ \phi = \theta \circ \psi.$$

Consider $g(g(X))$ and $g(Y)$. Is $[g(3)/Y, 3/X]$ a unifier? Is it the mgu? $g(g(3))$ $g(g(3))$

- mgu is **unique** up to alphabetic variance;
- the algorithm can easily be extended to simultaneous unification on n expressions.

The substitution $[g(3)/Y,\ 3/X]$ is a unifier but NOT the mgu. If we replace Y with $g(X)$, it'll be more general than replacing it w/ $g(3)$.

Unification Algorithm for Commutativity

Name	Before	After	Condition
Decompose	$P \wedge f(\vec{s}) = f(\vec{t})$	$P \wedge \bigwedge_i s_i = t_i$	
Conflict	$P \wedge f(\vec{s}) = g(\vec{t})$	fail	$f \neq g$
Switch	$P \wedge s = X$	$P \wedge X = s$	$X \in Vars$ $s \notin Vars$
Delete	$P \wedge s = s$	P	
Eliminate	$P \wedge X = s$	$P[s/X] \wedge X = s$	$X \in vars(P)$ $X \notin vars(s)$ $s \notin Vars$
Check	$P \wedge X = s$	fail	$X \in vars(s)$ $s \notin Vars$
Coalesce	$P \wedge X = Y$	$P[Y/X] \wedge X = Y$	$X, Y \in vars(P)$ $X \neq Y$
NEW! ✓ Mutate	$P \wedge f(s_1, t_1) = f(s_2, t_2)$	$P \wedge s_1 = t_2 \wedge t_1 = s_2$	f is commutative

Decompose and Mutate rules overlap.

Building-in Axioms

General Scheme:

$$(Ax_1 \cup Ax_2) + unif \Longrightarrow Ax_1 + unif_{Ax_2}.$$

Some axioms of the theory become built into unification.

e.g.

Example

Commutative-Unification

$$X + 2 = Y + 3$$
$$\downarrow$$
$$Y = 2 \wedge X = 3$$

Unification works!

We no longer use $\equiv$ but $=$

How do we deal with this?

We can add a new transformation rule (**Mutate rule**).

Most General Unifiers

For ordinary unification, the mgu is unique, but what happens when new rules are built-into the unification algorithm?

Multiple mgus: Commutative unification

$$X + Y = a + b \longrightarrow \begin{cases} X = a \wedge Y = b \\ X = b \wedge Y = a \end{cases}$$

Both are equally general.

Infinitely many mgus: Associative unification $X + (Y + Z) = (X + Y) + Z$.

$$X + a = a + X \longrightarrow \begin{cases} X = a \\ X = a + a \\ X = a + a + a \\ \ldots \end{cases}$$

All independent (not unifiable).

No mgus: Build in $f(0, X) = X$ and $g(f(X, Y)) = g(Y)$:

$$g(X) = g(a) \longrightarrow \begin{cases} X = a \\ X = f(Y_1, a) \\ X = f(Y_1, f(Y_2, a)) \end{cases}$$

Many unifiers but no mgu.

Types of Unification

1) **Unitary** A single unique mgu, or none (predicate logic).
2) **Finitary** Finite number of mgus (predicate logic with commutativity).
3) **Infinitary** Possibly infinite number of mgus (predicate logic with associativity).
4) **Nullary** No mgus exist, although unifiers may exist.
5) **Undecidable** Unification not decidable — no algorithm.

Types of Unification

Axioms	Type	Decidable
nil	unitary	yes
commutative	finitary	yes
associative	infinitary	yes
assoc. + dist.	infinitary	yes
lambda calculus	infinitary	no
λ-calculus pattern fragment	unitary	yes

Summary

- Unification (Bundy Ch. 17.1 - 17.4)
  - Algorithms for matching and unification.
  - Unification as equation solving.
  - Transformation rules for equation solving.
  - Building-in axioms.(E-Unification/Semantic Unification)
  - Most general unifiers and classification.
- Next time: Proof by rewriting