$\lambda a\, b.\ b(\lambda x.\ x\ a)$ 'a

AR FINAL EXAM

1. THIS QUESTION IS COMPULSORY

   (a) Represent the following sentences in first-order logic. [3 marks]

   i. If Alice learns programming, she either likes programming or takes it as a subject.

   ii. The only person who got the best score in programming is Alice.

   (b) Using natural deduction, prove the theorem:

   $$(\forall x.P(x) \to Q(x)) \to (\neg\exists x.Q(x)) \to (\neg\exists x.P(x))$$

   For full marks, you should label all the reasoning steps clearly. You may use your preferred notation (the tree-style, the sequent-style etc.) to prove the theorem. [7 marks]

   (c) Consider the following set S of rewrite rules:

   $$\begin{aligned} (x \cdot y) \cdot z &\Rightarrow x \cdot (y \cdot z) & (1) \\ i(x \cdot y) &\Rightarrow i(y) \cdot i(x) & (2) \end{aligned}$$

   where x, y and z are variables.

   i. Give all the critical pairs that arise from the rules in S. For full marks, you need to justify your answer. [6 marks]

   ii. Is S locally confluent? Explain your answer. [3 marks]

   (d) Consider the term $\lambda a\, b.\ b(\lambda x.\ x\ a)$. What is its type? [3 marks]

   (e) Consider a datatype `'a listx` that is similar to the usual list datatype except that it allows the addition of an element *both* to the front *and* to the end of the list. Give an Isabelle definition for the datatype `'a listx`. [3 marks]

datatype 'a listx = Nil | Cons "'a" "'a

$\exists x.$ Best-Score(x, Programming) $\wedge$ x = Alice

$\forall y.\ y \neq x \to \neg$ Best-Score(x, Programming) $\wedge$

$b(x(a))$

$'a \Rightarrow 'x \Rightarrow 'b$

2. ANSWER EITHER THIS QUESTION OR QUESTION 3

(a) Consider the following Isar proof:

```
1  lemma
2    "x mod n = y mod n ⟷
3     (∃q1 q2. x + n * q1 = y + n * q2)"       (is "?lhs ⟷ ?rhs")
4  proof
5    assume H: "x mod n = y mod n"
6    {
7      assume xy: "x ≤ y"
8      from H have th: "y mod n = x mod n"
9      by simp
10     from modeq [OF th xy] have ?rhs
11       apply clarify
12       apply (rule_tac x="q" in exI)
13       by (rule exI[where x="0"], simp)
14   }
15   moreover
16   {
17     assume xy: "y ≤ x"
18     from modeq [OF H xy] have ?rhs
19       apply clarify
20       apply (rule_tac x="0" in exI)
21       by (rule_tac x="q" in exI, simp)
22   }
23   ultimately show ?rhs using linear[of x y]
24   by blast
25 next
26   assume ?rhs
27   then obtain q1 q2 where q12: "x + n * q1 = y + n * q2"
28     by blast
29   hence "(x + n * q1) mod n = (y + n * q2) mod n"
30     by simp
31   thus ?lhs by simp
32 qed
33
```

Give brief answers to each of the following:

i. What are the Isabelle instantiations for the abbreviations `?lhs` and `?rhs` on line 3? Briefly state why such abbreviations are useful to this proof? *[2 marks]*

ii. What is Isabelle's response to the command `proof` on line 4? *[1 mark]*

iii. What is the part of the proof between curly braces {...} on lines 6-14 called? Explain its general purpose and, more concretely, what it does for the above proof. [3 marks]

iv. What are the roles of the keywords `moreover` and `ultimately` on line 15 and line 23 respectively? Illustrate your answer using the given proof. [4 marks]

(b) Consider a simple geometry where the plane is a *set* of points and lines are *subsets* of the plane such that the following axioms A1-A5 hold:

i. The plane is not empty (**A1**).

ii. Every line is a non-empty subset of the plane (**A2**).

iii. For every pair of points in the plane there is a line that contains both points (**A3**).

iv. Two different lines intersect in no more than one point (**A4**).

v. For every line l there is a point in the plane outside of l (**A5**).

Represent this simple geometry as an Isabelle locale named `SimpleGeometry`. You may assume that the following notions from Isabelle's `Set` theory are available: the empty set ({}), set membership (∈), set intersection (∩) and the subset relation (⊆). [12 marks]

(c) Give an Isabelle interpretation for `SimpleGeometry` that is satisfied by a plane with 3 distinct points. For full marks, you must use the appropriate Isabelle syntax. [3 marks]

3. ANSWER EITHER THIS QUESTION OR QUESTION 2

(a) Consider the following first-order logic formula:

$$(\forall x.P(x) \rightarrow \neg Q(x)) \rightarrow \neg(\exists x.P(x) \wedge Q(x))$$

Draw the parse tree for this formula. For each variable in your parse tree, state whether the variable is free or bound. Justify your answer based on the parse tree in each case. [4 marks]

(b) Consider the first-order logic formulas below. What is the result of applying the given substitution in each case?

$$(1)\ (\exists z.P(z, y) \rightarrow \forall x.Q(x, z))[x/y]$$
$$(2)\ (\forall x.(\exists y.(x = y + z)))[g(y)/z]$$

Justify your response for each formula. [4 marks]

(c) Explain the main difference between matching and unification. Provide an expression where matching fails while unification succeeds. Justify your response. [4 marks]

(d) Compute the most general unifier for the following pair of terms; where X, Y and Z are variables:

$$f(X, g(a, Y)) \text{ and } f(g(Z, b), X)$$

Show all the steps of the unification algorithm. [6 marks]

(e) Consider the following set of rewrite rules, where X and Y are variables:

$$\begin{array}{lrl} (1) & X + 0 & \Rightarrow X \\ (2) & X + s(Y) & \Rightarrow s(X + Y) \\ (3) & X \cdot 0 & \Rightarrow 0 \\ (4) & X \cdot s(Y) & \Rightarrow X \cdot Y + X \end{array}$$

Rewrite $s(s(0)) \cdot s(0)$ to normal form by applying these rules. For full marks, you need to show all your steps and underline the redexes. [3 marks]

(f) Consider the following set of rewrite rules; where X, Y and Z are variables:

$$\begin{array}{rl} g(X, X) & \Rightarrow X \\ g(f(X, Y), f(Y, Z)) & \Rightarrow g(X, Z) \end{array}$$

Prove that this set of rewrite rules is terminating. [4 marks]