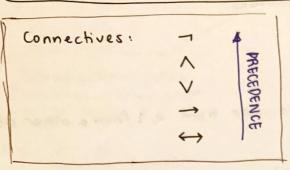
PROPOSITIONAL LOGIC & NATURAL DEDUCTION



· By contraposition:

$$P \rightarrow Q \equiv 7Q \rightarrow 7P$$

→ Treat all binary connectives as right associative

$$P \rightarrow Q \rightarrow R$$
 denotes $P \rightarrow (Q \rightarrow R)$

INTERPRETATION

 \rightarrow is a truth assignment to the symbols in the alphabet L: it is a fn V from $\{$ to $\{T, F\}$

$$[[A]] \lor = \lor(A)$$

$$[[PAQ]]V = [[P]]V$$
 and $[[Q]]V$

- -> An interpretation satisfies a wff P if [[P]]v = T
- -> A wff is satisfiable if there exists an interpretation this satisfies it.
- > A wff is a tautology / valid if every interpretation satisfies it (e.g. PV-P)
- The wffs P_1 , P_2 ,..., P_n entails Q if for any interpretation which satisfies all of P_1 , P_2 ,..., P_n also satisfies $Q \rightarrow [P_1, P_2, ..., P_n \models Q]$
 - · Contradictory assumptions entail everything
 - · Everything entails a tautology → [We write +Q when Q is a tautology]



- A formal deductive system is a set of valid inference rules that tell us what conclusions we can draw from some premues

INFERENCE RULES

-> tell us how one wff. can be derived in one step from 0, 1/more other wffs We write:

$$\frac{P_1}{Q} \qquad P_2 \qquad P_n \qquad (R)$$

if wff Q is derived from wffs P1, P2,..., Pn using the rule R

$$\frac{P}{P \wedge Q}$$
 (con) I)

If we have P and Q, we can

$$\frac{P \qquad P \rightarrow Q}{Q} \qquad (mp)$$

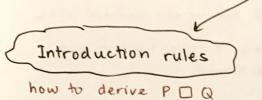
-> P and Q here are meta-variables (?P and ?Q in Isabelle)

- Inference rules must be valid.
- For all instances of P_1 $P_2 \cdots P_n$ (R) of the rule R,

we must have P, P2,..., Pn = Q

Inference is transitive.

- In Natural Deduction (ND), rules are split into two groups:



$$\frac{P \wedge Q}{P}$$
 (conjunct!)

$$\frac{P}{PVQ}$$
 (disj11)

eg.
$$\frac{P \wedge Q}{Q}$$
 (consunct 2)

$$\frac{Q}{PVQ}$$
 (disj12)

PROOF

| | PN(QVR) P | Assume Q is True [Q] | PN(QVR) | Assume Ris True [R] |
|----------|---------------|----------------------------|---------------|---------------------------|
| PA (QVR) | PAQ | | PAR | |
| QVR | (PAQ) V (PAR) | | (PAQ) V (PAR) | |

(PAQ) V(PAR)

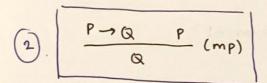
Note Each proof step will normally be annotated w/ name of its inference rule [e.g. disjE for the bottom most step]

RULES FOR IMPLICATION

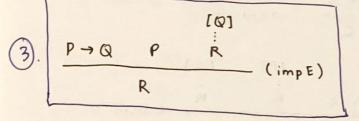
$$\begin{array}{c|c}
\hline
 & & \\
\hline$$

imp I forward - If on assumption tht. P is true, Q can be shown to hold, then we can conclude P-Q

imp I backward - To prove P + Q, assume P is true and prove that Q follows.



Modus Ponens rule.



RULES FOR \

$$\begin{array}{c|c}
[Q] & [P] \\
\vdots & \vdots \\
P & Q \\
\hline
P \leftrightarrow Q
\end{array}$$
(iffI)

$$\frac{P \leftrightarrow Q}{Q} \qquad P \qquad \text{(iff D1)}$$

$$\frac{P \longleftrightarrow Q \quad Q}{P} \quad (iff D2)$$

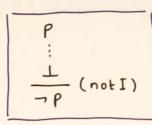
In Isabelle, \leftrightarrow is also denoted by = .

RULES FOR FALSE & NEGATION

_ is written False in Isabelle

→ Introduce a 0-ary connective ⊥ to represent false:

1 (False E)



9. ((SVR) AJS) - R

$$\frac{[(S \vee R) \wedge 1S]_{1}}{S \vee R} = \frac{[S]_{2}}{R} = \frac{[R]_{2}}{R}$$

$$\frac{[R]_{2}}{R} = \frac{[R]_{2}}{R} = \frac{[R]_{2}}{R}$$

$$\frac{[R]_{2}}{R} = \frac{[R]_{2}}{R} = \frac{[R]_{2}}{R}$$

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The subscripts [.], and [.] on the assumptions refer to the rule instances where they are discharged

SOUNDNESS

 \rightarrow If Q is provable from assumptions P_1, \ldots, P_n , then $P_1, \ldots, P_n \models Q$. This follows bc. all our rules are valid.

+) Is the converse true ?

Lo No. Can't prove Pierce's law: ((A→B) → A) → A

Can prove it using law of excluded middle: PV-P = rejects the law

L. So far, our proof system is sound & complete for Intuitionistic logic,

COMPLETENESS

- If P1,..., Pn = Q, then Q is provable from the assumptions P1..., Pn

SEQUENTS

- Another notation is sequent-style / Fitch-style:

The assumptions are usually collectively referred using [:

New Rule:
$$P \in \Gamma$$
 (assumption)

NATURAL DEDUCTION IN ISABELLE

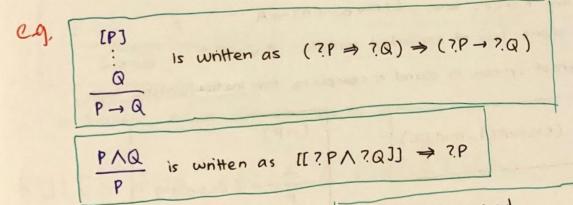
- Isabelle represents P. P. ..., P. + Q w/ the ff. notation:

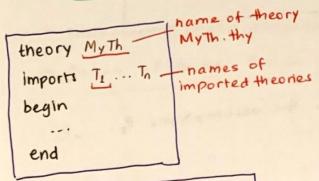
$$P_1 \Rightarrow (P_2 \Rightarrow ... \Rightarrow (P_n \Rightarrow Q) -..)$$

which is also written as

$$[[P_1; P_2; \dots; P_n]] \Rightarrow Q$$

used to represent the relationship between premises & conclusions of rules





Using the command

[apply (rule disjII)]

on the goal

[[A; B; C]] ⇒ (AAB) VD

yields the subgoal

[[A; B; C]] ⇒ AAB

INFERENCE RULE

apply assumption

can solve subgoal of form [[A; B]] > A