REPRESENTATION

→ How do we represent mathematical theory so we can prove theorems abt. it?

- Which logic do we use? — Propositional, FOL, HOL, Hoare Logic?
- Do we axiomatize our theory, or define it in terms of more primitive concepts?
- What style do we use? — e.g. functions vs. relations

AXIOMS vs. DEFINITIONS

→ e.g. Let's say we want to reason using natural numbers $\{0, 1, 2, 3, \ldots\}$

- AXIOMATIZE? Assume a collection of fn symbols and unproven axioms. i.e. the Peano axioms:

$$\forall x.\ \neg(0 = S(x))$$
$$\forall x.\ x + 0 = x$$
$$\forall x.\ x + S(y) = S(x+y)$$
$$\vdots$$

(+) Sometimes less work

(−) How do we know our axiomatization is complete and consistent?

- DEFINE? If our logic has sets as a primitive, then we can define $\mathbb{N}$ via the von Neumann ordinals:

$$0 = \emptyset, \quad 1 = \{\emptyset\}, \quad 2 = \{\emptyset, \{\emptyset\}\}, \ldots$$

Then we can prove the Peano axioms for this definition

→ (+++) If the underlying logic is consistent, then we are guaranteed to be consistent → Relative consistency

(−) Can be a lot of work.

PAST EXAM QN

Explain how a non-functional binary relation r between 2 objects x and y can be represented w/ a fn in FOL?

→ Can be achieved using sets:

- A fn f_r can be defined from x to the set of all y for which $r(x, y)$ holds, i.e. $f_r(x) = \{y \mid r(x, y)\}$.
- Then if there is no such y such tht. $r(x, y)$ for some x, then $f_r(x) = \{\}$

Axiomatisation, an example: Set Theory

Let's take FOL, a binary atomic predicate $\in$ and the following axiom for every formula P with one free variable x:

$$\exists y.\ \forall x.\ x \in y \leftrightarrow P(x)$$

"For every predicate P there is a set y such that its members are exactly those that satisfy P"

We can now define empty set, pairing, union, intersection...

But it is **too powerful!** Let $P(x) \equiv \neg(x \in x)$. Then by the axiom there is a y such that:

$$y \in y \leftrightarrow y \notin y$$

This is **Russell's paradox.**

Background: the axiom is called "unrestricted comprehension", it was replaced by:

$$\forall z.\ \exists y.\ \forall x.\ (x \in y \leftrightarrow (x \in z \wedge P(x)))$$

+ some other axioms to give ZF set theory.

Other Representation Examples

- The rationals $\mathbb{Q}$ can be defined as pairs of integers. Reasoning about the rationals therefore reduces to reasoning about the integers.
- The reals $\mathbb{R}$ can be defined as sets of rationals. Reasoning about the reals therefore reduces to reasoning about the rationals.
- The complex numbers $\mathbb{C}$ can be defined as pairs of reals. Reasoning about the complex numbers therefore reduces to reasoning about the reals.
- In this way, we have **relative consistency.** !!
  - If the theory of natural numbers is consistent, so is the theory of complex numbers.

Building up Definitions: Integers

Starting from the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, we can define:

- each integer $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ as an **equivalence class** of pairs of natural numbers under the relation $(a, b) \sim (c, d) \iff a + d = b + c$;
- For example, -2 is represented by the equivalence class $[(0, 2)] = [(1, 3)] = [(100, 102)] = \ldots$.
- we define the sum and product of two integers as

$$[(a, b)] + [(c, d)] = [(a + c, b + d)]$$
$$[(a, b)] \times [(c, d)] = [(ac + bd, ad + bc)]\,;$$

- we define the set of **negative** integers as the set $\{[(a, b)] \mid b > a\}$.
- Exercise: show that the product of negative integers is non-negative.

Functions or Predicates?

We can represent some property r holding between two objects x and y as:

a function with equality	$r(x) = y$
a predicate	$r(x, y)$

Is it better to use functions or predicates to represent properties?

It is not always clear which is best!

Functional Representation

For example, suppose we represent division of real numbers $(/)$ by a function $div : real \times real \Rightarrow real$.

- We define $div(x, y)$ when $y \neq 0$ in normal way
- What about division-by-zero? What is the value of $div(x, 0)$?
- In first-order logic, functions are assumed to be **total**, so we have to pick a value!
- We could *choose* a convenient element: say 0. That way:

$$0 \leq x \rightarrow 0 \leq 1/x.$$

Predicate Representation

Q) Can we represent division of real numbers $(/)$ by a relation $Div : real \times real \times real \Rightarrow bool$ such that $Div(x, y, z)$ is

- $x/y = z$ when $y \neq 0$, and
- $\perp$ when $y = 0$?

A) Yes: $Div(x, y, z) \equiv x = y * z \wedge \forall w.\ x = y * w \rightarrow z = w$

That is, z is that *unique* value such that $x = y * z$.

But now formulas are more complicated.

Nested struct; better not to use predicates in the first place

$$x, y \neq 0 \rightarrow \frac{1}{((x/y)/x)} = y$$

becomes

$$Div(x, y, u) \wedge Div(u, x, v) \wedge Div(1, v, w) \wedge \cancel{x, y \neq 0} \rightarrow w = y$$

$(x \neq 0 \wedge y \neq 0)$

Functional Representation

Can we represent the concept of *square roots* with a function $\surd : real \Rightarrow real$?

- All positive real numbers have *two* square roots, and yet a function maps points to *single* values.
- We can pick one of the values arbitrarily: say, the *positive* (*principal*) square root.
- **Or** we can have the function map every real to a *set* $\surd : real \Rightarrow real\ set$.

$$\sqrt{x} \equiv \{y \mid x = y^2\}$$

- But now we have two kinds of object: reals and sets of reals, and we cannot conveniently express:

$$(\sqrt{x})^2 = x$$

- Our representation of reals is no longer **self-contained**.

Predicate Representation

Q) Can we represent the concept of *square roots* with a relation $Sqrt : real \times real \Rightarrow bool$?

A) Yes. E.g. $Sqrt(x, y) \equiv x = y^2$.

Again drawback of formulas being more complicated

Functions, Predicates and Sets

We can translate back and forth. **But too much translation makes a formalisation hard to use!**

e.g. Any function $f : \alpha \to \beta$ can be represented as a relation $R : \alpha \times \beta \to bool$ or a set $S : (\alpha \times \beta) set$ by defining:

$$R(x, y) \equiv f(x) = y$$
$$S \equiv \{(x, y) \mid f(x) = y\}.$$

e.g. Any predicate $\boxed{P}$ can be represented by a function f or a set S by defining:

$$f(x) \equiv \begin{cases} \mathit{True} & : \; P(x) \\ \mathit{False} & : \; \text{otherwise} \end{cases}$$
$$S \equiv \{x \mid P(x)\}.$$

e.g. Any set $\boxed{S}$ can be represented by a function f or a predicate P by defining:

$$f(x) \equiv \begin{cases} \mathit{True} & : \; x \in S \\ \mathit{False} & : \; \text{otherwise} \end{cases}$$
$$P(x) \equiv x \in S$$

Set Theory, Functions, and HOL

→ In **pure** (without axioms) **FOL**, we **cannot directly represent** the statement:
there is a function that is larger on all arguments than the log function.

→ To formalise it, we would need to quantify over functions:

$$\exists f.\ \forall x.\ f(x) > \log x.$$

→ Likewise we cannot quantify over predicates.

→ **Solutions in FOL:**

- Represent all functions and predicates by **sets**, and quantify over these. This is the approach of first-order set theories such as *ZF*.
- Introduce sorts for predicates and functions. Not so elegant now having 2 kinds of each.