PROPOSITIONAL LOGIC & NATURAL DEDUCTION

Connectives:		
	$\neg$	PRECEDENCE ↑
	$\wedge$	
	$\vee$	
	$\rightarrow$	
	$\leftrightarrow$	

- By contraposition:

$$\boxed{P \rightarrow Q \quad \equiv \quad \neg Q \rightarrow \neg P}$$

→ Treat all binary connectives as $\boxed{\text{right associative}}$

e.g. $A \vee B \wedge C$ denotes $A \vee (B \wedge C)$

$P \rightarrow Q \rightarrow R$ denotes $P \rightarrow (Q \rightarrow R)$

INTERPRETATION

→ is a truth assignment to the symbols in the alphabet $\mathcal{L}$:

it is a fn V from $\mathcal{L}$ to $\{T, F\}$

$[[A]]v = V(A)$

$[[P \wedge Q]]v = [[P]]v$ and $[[Q]]v$

$\vdots$

→ An interpretation $\boxed{\text{satisfies}}$ a wff P if $[[P]]v = T$

→ A wff is $\boxed{\text{satisfiable}}$ if there exists an interpretation tht. satisfies it.

→ A wff is a $\boxed{\text{tautology}}$ / $\boxed{\text{valid}}$ if every interpretation satisfies it (e.g. $P \vee \neg P$)

→ The wffs $P_1, P_2, \dots, P_n$ $\boxed{\text{entails}}$ Q if for any interpretation which satisfies all of $P_1, P_2, \dots, P_n$ also satisfies Q → $[P_1, P_2, \dots, P_n \models Q]$

- Contradictory assumptions entail everything
- Everything entails a tautology → [We write $\models Q$ when Q is a tautology]

e.g. Is $\underbrace{\neg P, Q}_{T} \models \underbrace{Q \wedge (P \rightarrow Q)}_{T}$ a valid entailment?

Assign P to False and Q to True

∴ Yes

→ A formal deductive system is a set of valid inference rules tht. tell us what conclusions we can draw from some premises

INFERENCE RULES

→ tell us how one wff. can be derived in one step from 0, 1/more other wffs

We write:

$$\frac{P_1 \quad P_2 \quad \cdots \quad P_n}{Q}\ (R)$$

if wff Q is derived from wffs $P_1, P_2, \ldots, P_n$ using the rule R

e.g.

$$\frac{P \quad Q}{P \wedge Q}\ (\text{conj I})$$

If we have P and Q, we can conclude $P \wedge Q$ using the rule conjI

$$\frac{P \quad P \to Q}{Q}\ (\text{mp})$$

If we have P and $P \to Q$, we can conclude Q using the rule mp

→ P and Q here are meta-variables (?P and ?Q in Isabelle)

→ Inference rules must be valid.

→ For all instances of $\frac{P_1 \quad P_2 \quad \cdots \quad P_n}{Q}\ (R)$ of the rule R,

we must have $P_1, P_2, \ldots, P_n \models Q$

→ Inference is transitive.

→ In Natural Deduction (ND), rules are split into two groups:

Introduction rules — how to derive $P \square Q$

Elimination rules — what can be derived from $P \square Q$

e.g. $\dfrac{P \quad Q}{P \wedge Q}$ (conj I)

e.g. $\dfrac{P \wedge Q}{P}$ (conjunct 1)

e.g. $\dfrac{P}{P \vee Q}$ (disj I1)

e.g. $\dfrac{P \wedge Q}{Q}$ (conjunct 2)

e.g. $\dfrac{Q}{P \vee Q}$ (disj I2)

e.g. $\dfrac{P \vee Q \quad \begin{matrix}[P] \\ \vdots \\ R\end{matrix} \quad \begin{matrix}[Q] \\ \vdots \\ R\end{matrix}}{R}$ (disjE)

PROOF

→ A proof that $\boxed{P \wedge (Q \vee R) \vDash (P \wedge Q) \vee (P \wedge R)}$

$$\dfrac{\dfrac{P \wedge (Q \vee R)}{Q \vee R} \qquad \dfrac{\dfrac{\dfrac{P \wedge (Q \vee R)}{P} \quad \overset{\text{Assume Q is True}}{[Q]}}{P \wedge Q}}{(P \wedge Q) \vee (P \wedge R)} \qquad \dfrac{\dfrac{\dfrac{P \wedge (Q \vee R)}{P} \quad \overset{\text{Assume R is True}}{[R]}}{P \wedge R}}{(P \wedge Q) \vee (P \wedge R)}}{(P \wedge Q) \vee (P \wedge R)}$$

Note Each proof step will normally be annotated w/ name of its inference rule [e.g. disjE for the bottom most step]

RULES FOR IMPLICATION

1. $$\dfrac{\begin{matrix}[P] \\ \vdots \\ Q\end{matrix}}{P \to Q}\ (\text{impI})$$

impI forward — If on assumption tht. P is true, Q can be shown to hold, then we can conclude $P \to Q$

impI backward — To prove $P \to Q$, assume P is true and prove that Q follows.

2. $$\dfrac{P \to Q \qquad P}{Q}\ (\text{mp})$$

Modus Ponens rule.

3. $$\dfrac{P \to Q \qquad P \qquad \begin{matrix}[Q] \\ \vdots \\ R\end{matrix}}{R}\ (\text{impE})$$

RULES FOR $\leftrightarrow$

$$\dfrac{\begin{matrix}[Q] \\ \vdots \\ P\end{matrix} \qquad \begin{matrix}[P] \\ \vdots \\ Q\end{matrix}}{P \leftrightarrow Q}\ (\text{iffI})$$

$$\dfrac{P \leftrightarrow Q \qquad P}{Q}\ (\text{iffD1})$$

$$\dfrac{P \leftrightarrow Q \qquad Q}{P}\ (\text{iffD2})$$

In Isabelle, $\leftrightarrow$ is also denoted by $=$.

RULES FOR FALSE & NEGATION

- Introduce a 0-ary connective $\bot$ to represent false:

$\bot$ is written False in Isabelle

$$\dfrac{\bot}{P}\ (\text{FalseE})$$

- Define $\neg P$ to be $P \to \bot$:

$$\dfrac{\begin{matrix}P \\ \vdots \\ \bot\end{matrix}}{\neg P}\ (\text{notI})$$

$$\dfrac{P \qquad \neg P}{\bot}\ (\text{notE})$$

In Isabelle, this is different

$$\dfrac{P \qquad \neg P}{R}\ (\text{notE})$$

e.g. $((S \vee R) \wedge \neg S) \rightarrow R$

$$\dfrac{\dfrac{\dfrac{[(S\vee R)\wedge\neg S]_1}{S \vee R} \qquad \dfrac{[S]_2 \qquad \dfrac{[(S\vee R)\wedge \neg S]_1}{\neg S}}{R}\,(\text{notE}) \qquad \dfrac{[R]_2}{R}}{R}\,(\text{disjE}_2)}{((S\vee R)\wedge\neg S)\rightarrow R}\,(\text{impI}_1)$$

The subscripts $[\cdot]_1$ and $[\cdot]_2$ on the assumptions refer to the rule instances where they are discharged

SOUNDNESS

→ If Q is provable from assumptions $P_1, \dots, P_n$, then $P_1, \dots, P_n \vDash Q$. This follows bc. all our rules are valid.

→ Is the converse true ?

- No. Can't prove Pierce's law: $((A \rightarrow B) \rightarrow A) \rightarrow A$
- Can prove it using law of excluded middle: $P \vee \neg P$ ← rejects the law
- So far, our proof system is sound & complete for Intuitionistic logic.

$$\boxed{\dfrac{}{\neg P \vee P}\,(\text{excluded\_middle})}$$

$$\boxed{\dfrac{\begin{array}{c}[\neg P] \\ \vdots \\ \bot\end{array}}{P}\,(\text{ccontr})}$$

COMPLETENESS

→ If $P_1, \dots, P_n \vDash Q$, then Q is provable from the assumptions $P_1, \dots, P_n$

SEQUENTS

→ Another notation is sequent-style / Fitch-style:

$$P_1, P_2, \ldots, P_n \vdash Q$$

The assumptions are usually collectively referred using Γ:

$$\Gamma \vdash Q$$

New Rule: $\boxed{\dfrac{P \in \Gamma}{\Gamma \vdash P} \text{(assumption)}}$

NATURAL DEDUCTION IN ISABELLE

→ Isabelle represents $P_1, P_2, \ldots, P_n \vdash Q$ w/ the ff. notation:

$$P_1 \Rightarrow (P_2 \Rightarrow \ldots \Rightarrow (P_n \Rightarrow Q) \ldots)$$

↳ META-IMPLICATION
used to represent the relationship between premises & conclusions of rules

which is also written as

$$\boxed{[[P_1 ; P_2 ; \ldots ; P_n]] \Rightarrow Q}$$

e.g.

$\dfrac{\begin{matrix}[P] \\ \vdots \\ Q\end{matrix}}{P \to Q}$ is written as $(?P \Rightarrow ?Q) \Rightarrow (?P \to ?Q)$

$\dfrac{P \wedge Q}{P}$ is written as $[[?P \wedge ?Q]] \Rightarrow ?P$

```
theory MyTh
imports T1 ... Tn
begin
  ...
end
```

name of theory MyTh.thy

names of imported theories

Using the command

$\boxed{\text{apply (rule disjI1)}}$

on the goal

$[[A; B; C]] \Rightarrow (A \wedge B) \vee D$

yields the subgoal

$[[A; B; C]] \Rightarrow A \wedge B$

INFERENCE RULE

$\boxed{\text{theorem K: "}A \to B \to A\text{"}}$

↳ to enter proof mode

$\boxed{\text{apply assumption}}$

↳ can solve subgoal of form $[[A; B]] \Rightarrow A$