Lecture 6 and 7

PREDICATE LOGIC

→ Logics are formal languager for:

- representing what we know about the world
- reasoning about this knowledge (draw conclusions frm. it)

↳ 2 COMPONENTS: → Syntax – defines the sentences in the lang.
→ Semantics – defines the meaning of the sentences

MOTIVATION FOR PREDICATE LOGIC

→ Atomic formulas of propositional logic are too atomic

- statements can only be true/false (which is fine bc. it's the same in predicate logic), BUT they have no internal structure

↓

- We overcome this limitation through First-Order logic (FOL), or allowing atomic formulas to be statements about relationships between objects

PREDICATES & CONSTANTS

consider the statements:
- Mary is female
- Mary and John are siblings
- John is male

Propositional Logic → These are just atomic propositions:
- mary-is-female
- john-is-male
- mary-and-john-are-siblings

FOL → We can use predicates as atomic statements with constants as arguments:
- Female(mary)
- Male(john)
- Sibling(mary, john)

VARIABLES & QUANTIFIERS

→ In FOL, predicates may have variables as arguments, whose value may be bounded by quantifiers:

- $\forall x\,(Male(x) \vee Female(x))$ – Everybody is male or female
- $\forall x\,(Male(x) \rightarrow \neg Female(x))$ – A male is not a female

SYNTAX OF FOL

→ Countably infinite supply of

- variable symbols : $x, y, z, \ldots$
- constant symbols : $a, b, c, \ldots$
- predicate symbols : $P, Q, R, \ldots$

Term $t ::= x$ variable
$\quad\quad\quad \mid a$ constant

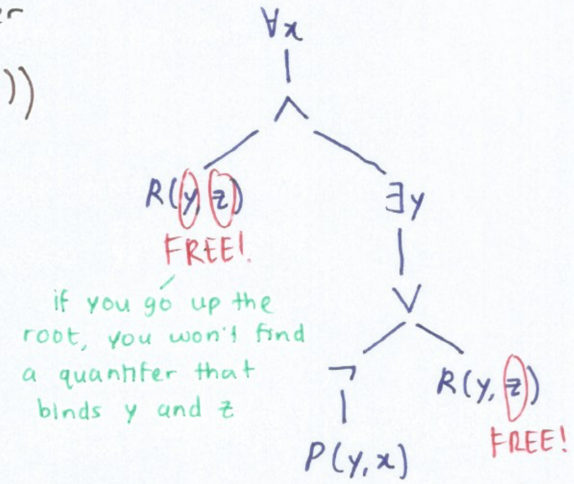
Formula $\phi ::= P(t_1, \ldots, t_n)$ atomic formula
$\mid \neg\phi$ negation
$\mid \phi \wedge \phi$ conjunction
$\mid \phi \vee \phi$ disjunction
$\mid \phi \rightarrow \phi$ implication
$\mid \forall x\, \phi$ universal quantification (if x occurs free in ϕ)
$\mid \exists x\, \phi$ existential quantification (if x occurs free in ϕ)

FREE VARIABLES

→ var. that are not in the scope of any quantifier

→ i.e. $\forall x (R(\underline{y}, \underline{z}) \wedge \exists y (\neg P(y, x) \vee R(y, \underline{z})))$

- Variables in blue are free
- The others are bound



SEMANTICS OF FOL

INTERPRETATIONS

- A formula may be T/F w.r.t a given $\boxed{\text{interpretation}}$
  - an assignment of meaning to the symbols of the lang.

- A $\boxed{\text{first-order structure } \mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle}$ consists of a universe of discourse of Δ and an interpretation $\cdot^{\mathcal{I}}$.
  - Δ: non-empty domain of objects (our universe)
  - $\cdot^{\mathcal{I}}$: gives meaning to constant/predicate symbols

  → Giving meaning to → constant symbols: $a^{\mathcal{I}} \in \Delta$
  → predicate symbols: $R^{\mathcal{I}} \subseteq \Delta^n = \underbrace{\Delta \times \cdots \times \Delta}_{n \text{ times}}$ (R: Predicate of arity n)

  → Interpreting a constant means mapping it to an element of our domain
  Interpreting a predicate of arity n means mapping it to an n-ary relation of elements in domain

- The first-order struct. is not enough to give meaning to formulas in our predicate; we need to assign meaning to variables:

  $\boxed{\text{Variable Assignment } v}$ maps each ~~element~~ variable to an object in Δ

  Notation: $v[x/d]$ is the same as v except that $x \mapsto d$

TERMS

- Interpretation of a term that is a variable x under (I, v) is:

$$x^{I,v} = v(x)$$ – use only v in variables

- Interpretation of a term that is a constant a under (I, v) is:

$$a^{I,v} = a^{I}$$ – use only I in constants

FORMULAS

- $\boxed{(I, v) \models \phi}$ means interpretation (I, v) satisfies formula ϕ

  If ϕ evaluates to true under (I, v), we write $(I, v) \models \phi$

  If ϕ evaluates to false under (I, v), we write $(I, v) \not\models \phi$

  We define the semantics of $\models$ inductively:

BASE CASES

$(I, v) \models \top$, $\quad (I, v) \not\models \bot$

n-ary relation of objects in the domain

Interpretation of predicate P

$$(I, v) \models P(t_1, \dots, t_n) \iff \overbrace{(\underbrace{t_1^{I,v}}, \dots, \underbrace{t_n^{I,v}})}^{\text{n-tuple containing interpretations of each term}} \in P^{I}$$

Each of these is an object of the domain; an element of Δ

$$I, v \models \neg\phi \iff I, v \not\models \phi$$

$$I, v \models \phi \wedge \psi \iff I, v \models \phi \text{ and } I, v \models \psi$$

$$I, v \models \phi \vee \psi \iff I, v \models \phi \text{ or } I, v \models \psi$$

$$I, v \models \phi \to \psi \iff \text{if } I, v \models \phi \text{ then } I, v \models \psi$$

$$I, v \models \forall x\, \phi \iff \text{for every } d \in \Delta : I, v[x/d] \models \phi$$

We go through the entire universe Δ, take each object and plug it into x and no matter which val. we plug in, the formula ϕ is satisfied.

$$I, v \models \exists x\, \phi \iff \text{there exists } d \in \Delta \text{ s.t. } I, v[x/d] \models \phi$$

It's enough to find one elem. of the domain that satisfies the formula ϕ

EQUALITY

→ is a special binary predicate, whose semantic is defined in this way:

We don't interpret the predicate itself but we need to interpret the terms

(Term) $t_1 = t_2$ is true under a given interpretation

IF AND ONLY IF

t_1 and t_2 refer to the same object

That is...

$$I, \nu \models t_1 = t_2 \iff t_1^{I,\nu} = t_2^{I,\nu}$$

When we interpret t_1, we'll end up w/ an element $\boxed{t_1^{I,\nu}} \in \Delta$ and same goes w/ $t_2 \rightarrow \boxed{t_2^{I,\nu}} \in \Delta$

If these 2 elems. are the same object, then the atomic formula $t_1 = t_2$ is satisfied.

EXAMPLE

i.e. Take any first order structure I such that:

$\Delta = \{d_1, \dots, d_n\}$ for $n > 1$ → at least 2 elems. in domain

$a^I = d_1$ → The interpretation of const. symbol a is the elem. d_1

$b^I = d_1$

$Block^I = \{d_1\}$ → Interpretation of Block is the set consisting of elem. d_1, hence Block is a unary predicate

$Red^I = \Delta$ → Interpretation of Red is the whole domain (also a unary predicate)

and any assignment ν such that $x \mapsto d_1$ and $y \mapsto d_2$

1) Does $(I, \nu) \models Block(a) \wedge Block(b) \wedge \neg(a = b)$? NO

$$\iff \underbrace{I, \nu \models Block(a)}_{} \text{ and } \underbrace{I, \nu \models Block(b)}_{} \text{ and } \underbrace{I, \nu \models \neg(a=b)}_{}$$

$\Updownarrow$

$\underbrace{a^{I,\nu} \in Block^I, \quad a^{I,\nu} = a^I}_{}$ (Interpretation of a constant)

$\Updownarrow$

$d_1 \in \{d_1\}$

✓SATISFIED!

$\Updownarrow$

$\underbrace{b^{I,\nu} \in Block^I, \quad b^{I,\nu} = b^I}_{}$

$\Updownarrow$

$d_1 \in \{d_1\}$

✓SATISFIED!

$\Updownarrow$

$I, \nu \not\models a = b$

$\iff a^{I,\nu} \neq b^{I,\nu}$

$= a^I \neq b^I$

$= d_1 \neq d_1$

✗ NOT SATISFIED!

2) Does $(I, \nu) \models \forall x\,(Block(x) \rightarrow Red(x))$? YES

$\iff$ for every $d \in \Delta$: $\boxed{\underbrace{I, \nu[x/d] \models Block(x) \rightarrow Red(x)}}$

The whole implication will be satisfied

Since RHS will always be satisfied

$\Updownarrow$

if $\underbrace{I, \nu[x/d] \models Block(x)}$, then $\underbrace{\boxed{I, \nu[x/d] \models Red(x)}}$

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$\underbrace{\nu[x/d](x)} \in Block^I = \{d_1\}$ $\qquad$ $\underbrace{\nu[x/d](x)}_{d} \in Red^I = \Delta = \langle d_1, \ldots, d_n \rangle$

Since in the variable assignment we are mapping $x \mapsto d$ and we apply this var. assignment to x, this expression is precisely d.

d ranges over the whole domain

✓SATISFIED!

3) Does $(I, \nu) \models Block(c) \vee \neg Block(c)$? YES

$\iff$ $\underbrace{I, \nu \models Block(c)}$ OR $\underbrace{I, \nu \not\models Block(c)}$

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$c^{I,\nu} \in Block^I$ $\qquad$ $c^{I,\nu} \notin Block^I$

$\| \qquad\qquad \|$

$c^I \qquad\qquad c^I$

We don't say anything about const. c in the description of $q\underline{n}$ but it does NOT matter, because one of the two disjuncts will be satisfied.

$\boxed{c^I \in \Delta}$ The interpretation of const. c is an element in Δ

SATISFIABILITY & VALIDITY

→ An interpretation (I, ν) is a model of ϕ if $(I, \nu) \models \phi$

→ A formula is

- satisfiable — if it has a model
- unsatisfiable — if it has no models
- falsifiable — if there is some interpretation that is not a model
- valid — (i.e. a tautology) if every interpretation is a model

  Block(c) ∨ ¬Block(c) is a tautology; no matter which interpretation is considered, the formula will always be true.

EQUIVALENCE

→ Two formulas are logically equivalent ($\phi \equiv \psi$) if they have the same models

That is, for all interpretations (I, ν):

$$I, \nu \models \phi \iff I, \nu \models \psi$$

→ i.e. Are $P(x)$ and $P(y)$ logically equivalent?

→ NO! Consider this variable assignment $\nu = \{x \mapsto 1,\ y \mapsto 2\}$ and domain $\Delta = \{1, 2\}$ and interpretation $P^I = \{1\}$:

- Then $\underbrace{P(x) \text{ will be true under } (I, \nu)}$ & $\underbrace{P(y) \text{ will be false under } (I, \nu)}$

$$I, \nu \models P(x) \iff \underset{\substack{\| \\ 1}}{x^{I,\nu}} \in \underset{\substack{\| \\ \{1\}}}{P^I} \quad \checkmark \text{SATISFIED!}$$

$$I, \nu \models P(y) \iff \underset{\substack{\| \\ 2}}{y^{I,\nu}} \in \underset{\substack{\| \\ \{1\}}}{P^I} \quad 2 \notin \{1\} \quad \times \text{ NOT SATISFIED}$$

i.e. What about $\forall x\, P(x)$ and $\forall y\, P(y)$?

→ YES! The universal quantification removes the dependence from the variable assignment.

PROPERTIES OF QUANTIFIERS

- $\forall x \forall y \, \phi \; = \; \forall y \forall x \, \phi$
- $\exists x \exists y \, \phi \; = \; \exists y \exists x \, \phi$
- $\exists x \forall y \, \phi \; \neq \; \forall x \exists y \, \phi$

QUANTIFIER DUALITY

- $\forall x \; Likes(x, cake) \;\equiv\; \neg \exists x \, \neg Likes(x, cake)$

  Everybody likes caker — There is nobody who does not like cake

- $\exists x \; Likes(x, cake) \;\equiv\; \neg \forall x \, \neg Likes(x, cake)$

  Somebody likes bake — Not everybody does not like cake