Cosmological parameters estimation

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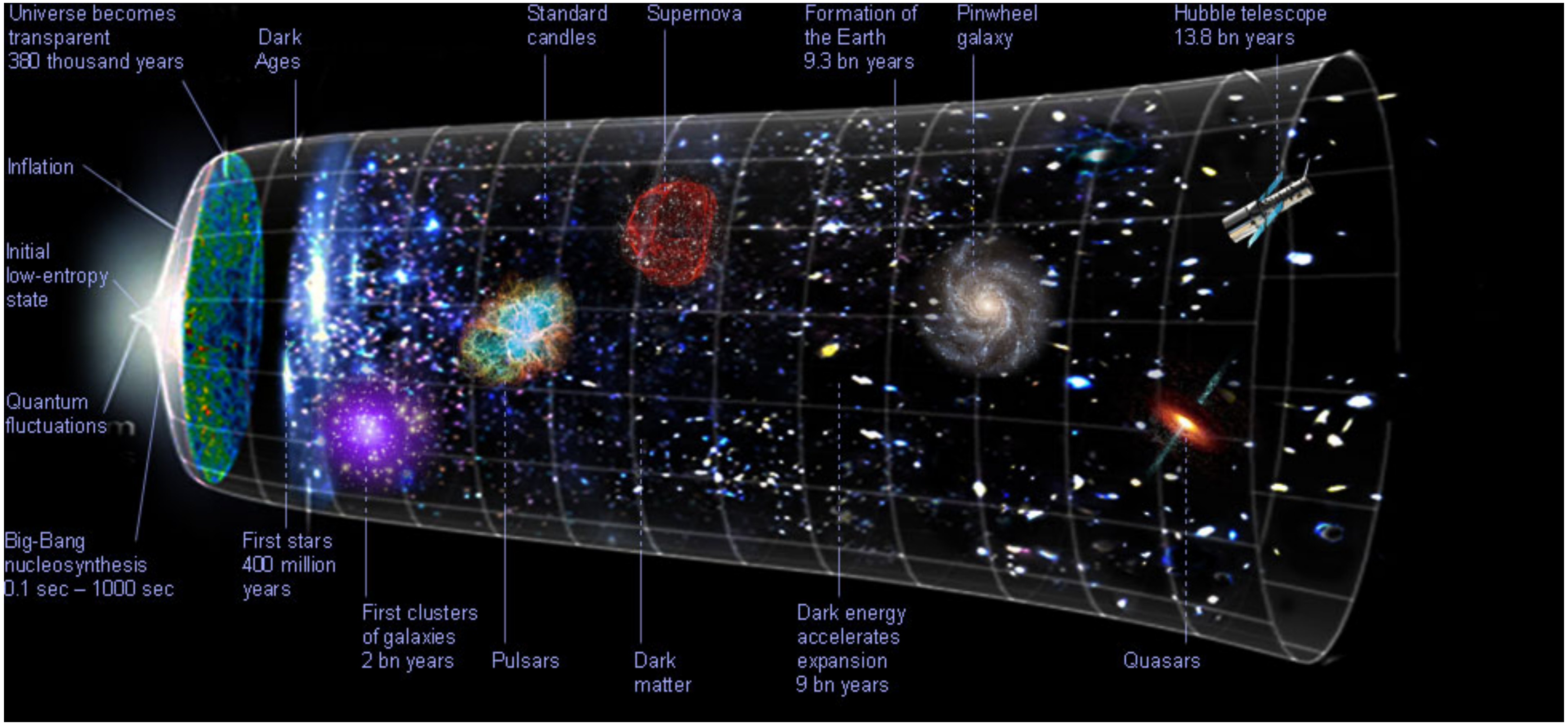
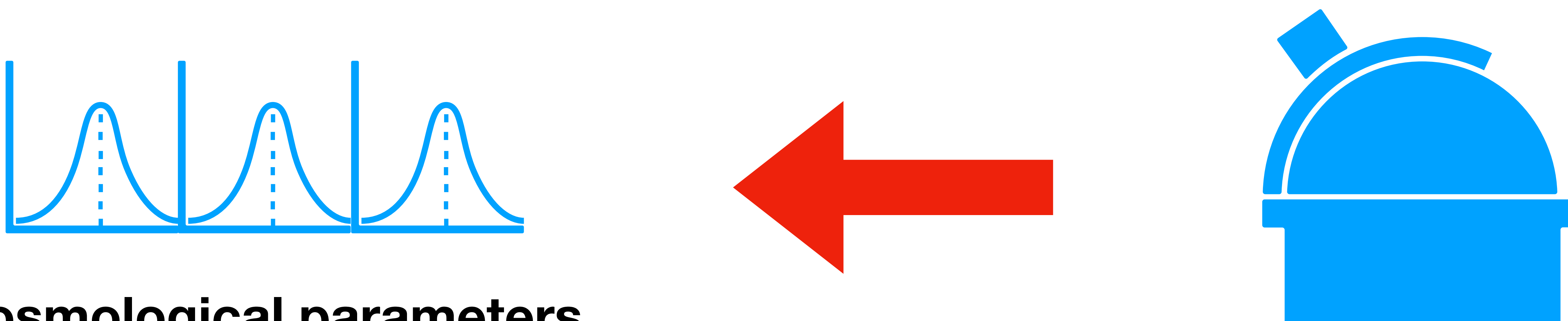
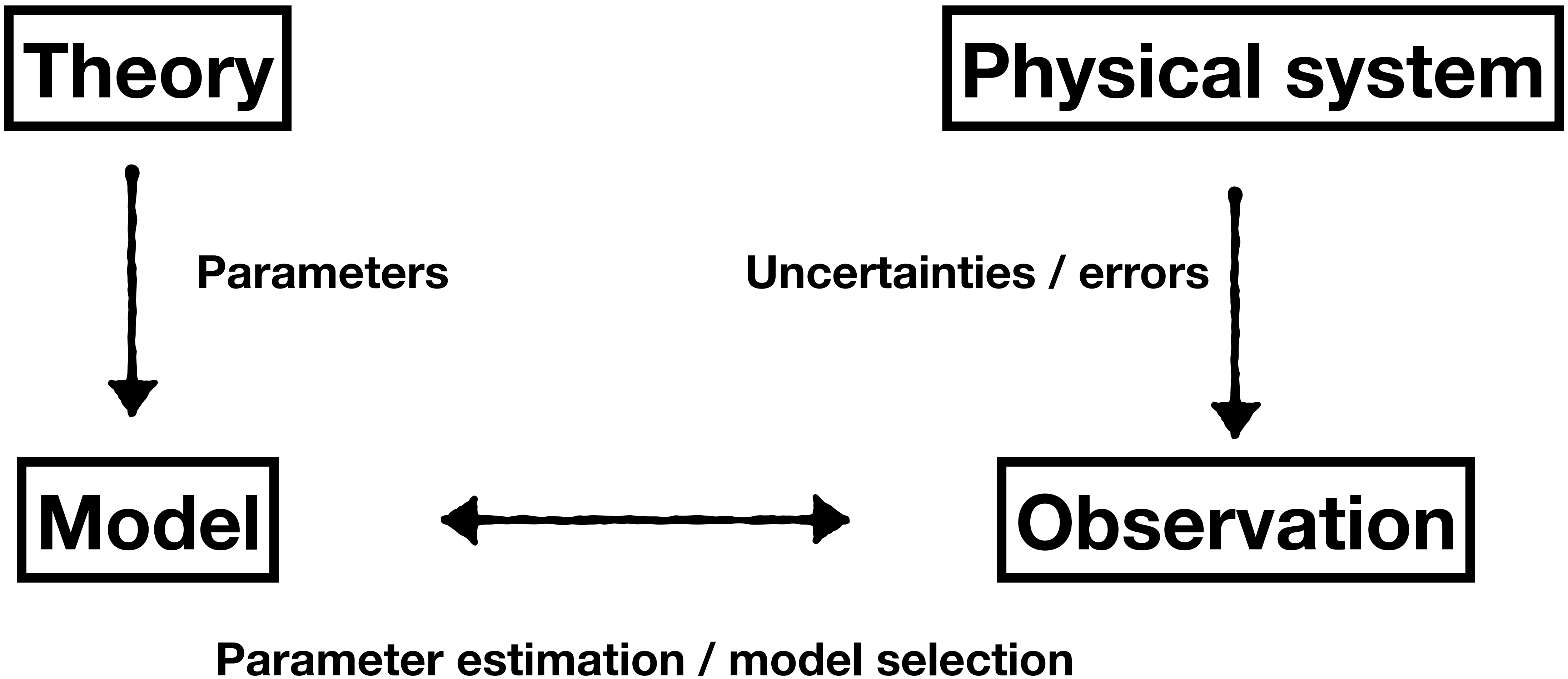
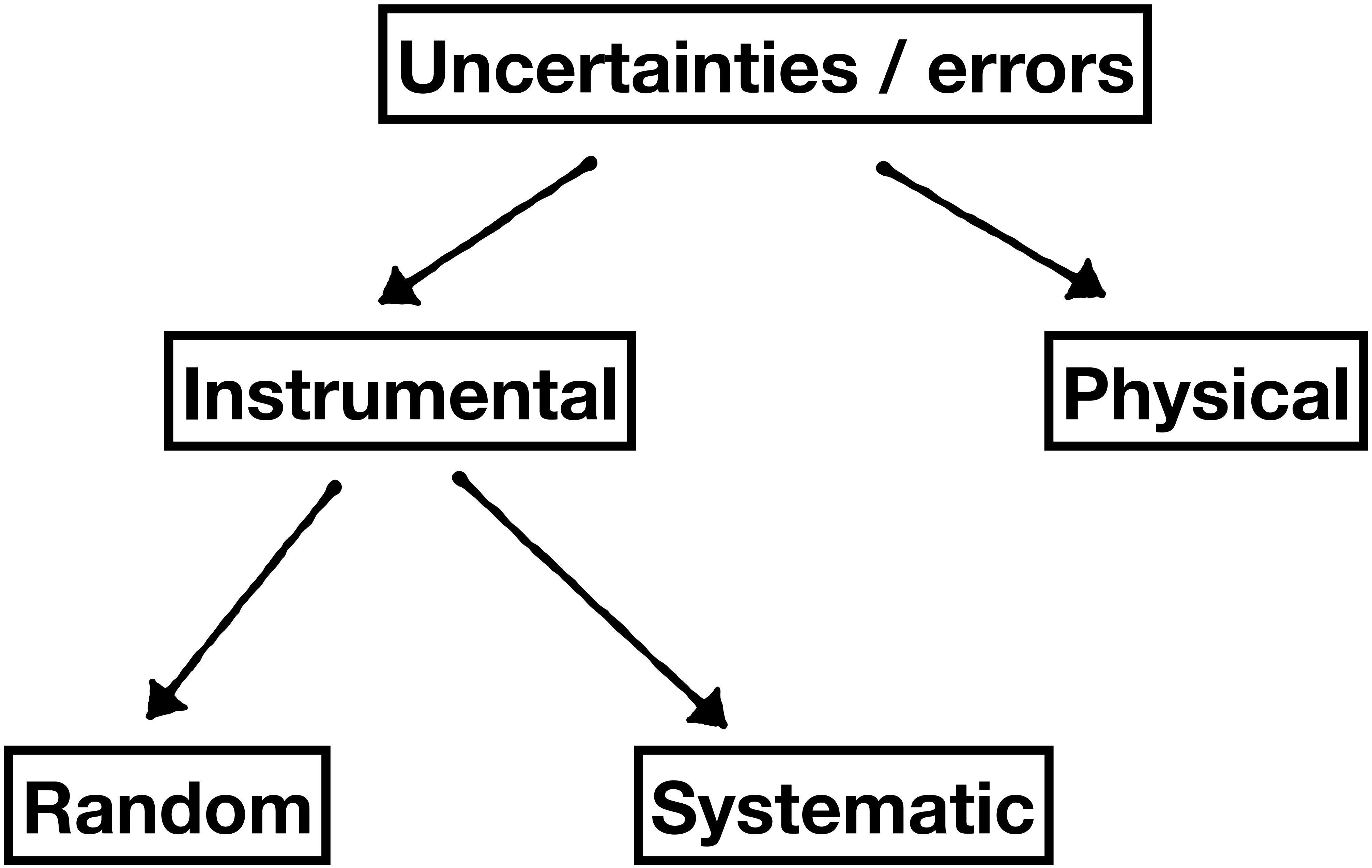




Cosmological parameters


Theory
Physical system
Parameters
Uncertainties / errors
Model
Observation
Parameter estimation / model selection


Uncertainties / errors
Instrumental
Physical
Random
Systematic

Parameter space

- Sampled
  - Physical
  - Nuisance
- Derived

Parameter	*Planck* alone	*Planck* + BAO
$\Omega_b h^2$	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10} A_s)$	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0	67.36 ± 0.54	67.66 ± 0.42
Ω_Λ	0.6847 ± 0.0073	0.6889 ± 0.0056
Ω_m	0.3153 ± 0.0073	0.3111 ± 0.0056
$\Omega_m h^2$	0.1430 ± 0.0011	0.14240 ± 0.00087
$\Omega_m h^3$	0.09633 ± 0.00030	0.09635 ± 0.00030
σ_8	0.8111 ± 0.0060	0.8102 ± 0.0060
$\sigma_8(\Omega_m/0.3)^{0.5}$ . . .	0.832 ± 0.013	0.825 ± 0.011
z_{re}	7.67 ± 0.73	7.82 ± 0.71
Age[Gyr]	13.797 ± 0.023	13.787 ± 0.020
r_*[Mpc]	144.43 ± 0.26	144.57 ± 0.22
$100\theta_*$	1.04110 ± 0.00031	1.04119 ± 0.00029
r_{drag}[Mpc]	147.09 ± 0.26	147.57 ± 0.22
z_{eq}	3402 ± 26	3387 ± 21
k_{eq}[Mpc^{-1}]	0.010384 ± 0.000081	0.010339 ± 0.000063
Ω_K	-0.0096 ± 0.0061	0.0007 ± 0.0019
Σm_ν [eV]	< 0.241	< 0.120
N_{eff}	$2.89^{+0.36}_{-0.38}$	$2.99^{+0.34}_{-0.33}$
$r_{0.002}$	< 0.101	< 0.106

Design of the parameter space

- Independent set of parameter for the observables considered

Reduce the parameter space

- Low correlation among the physical parameters

Change the parameter space

- Non-negligible correlation between the physical and the nuisance parameters

Remove uncorrelated parameters

Bayes inference

Prior

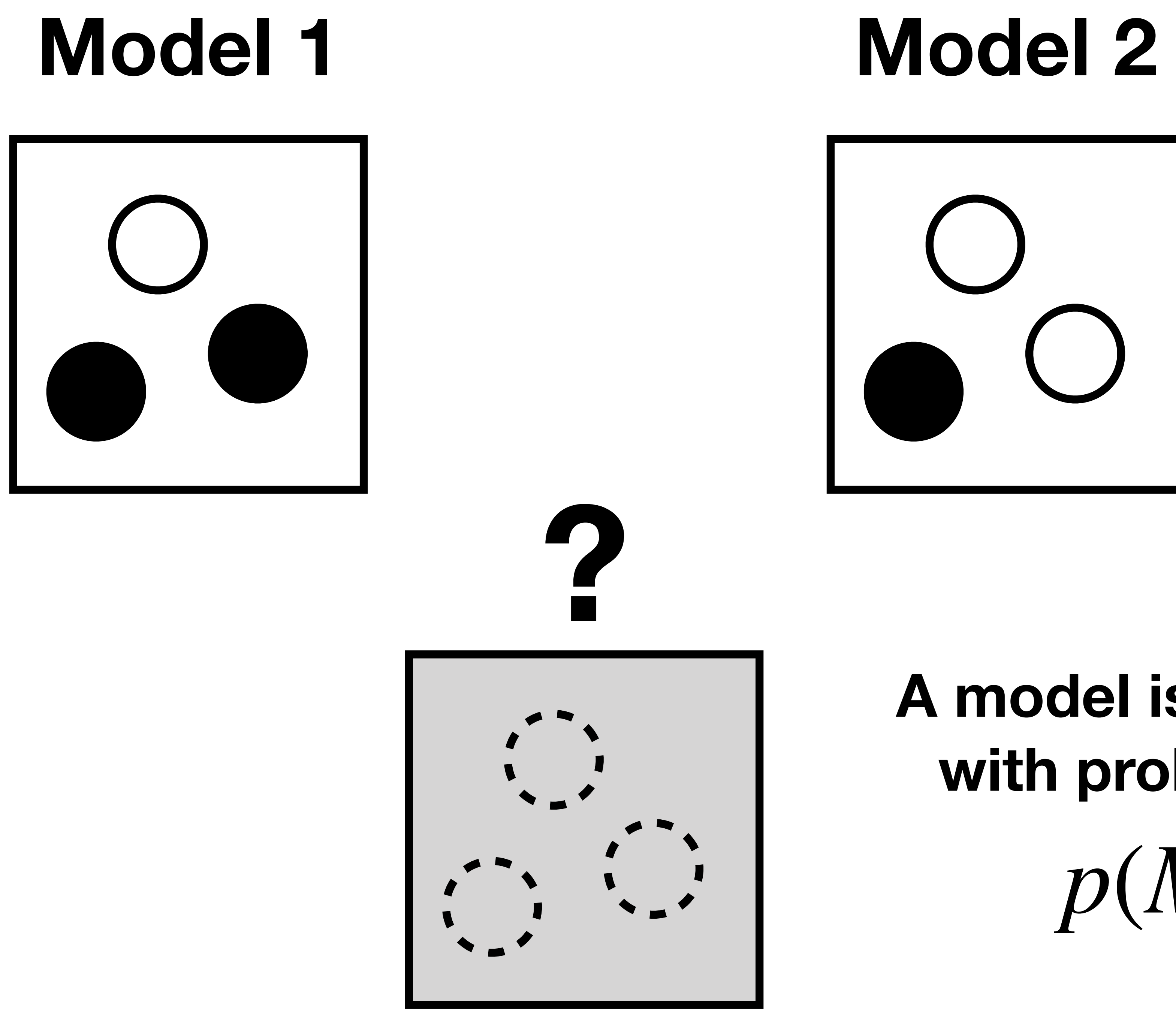
Likelihood

Posterior

$$P(\theta\,|\,d) = \frac{\pi(\theta)\; L(d\,|\,\theta)}{P(d)}$$

Evidence

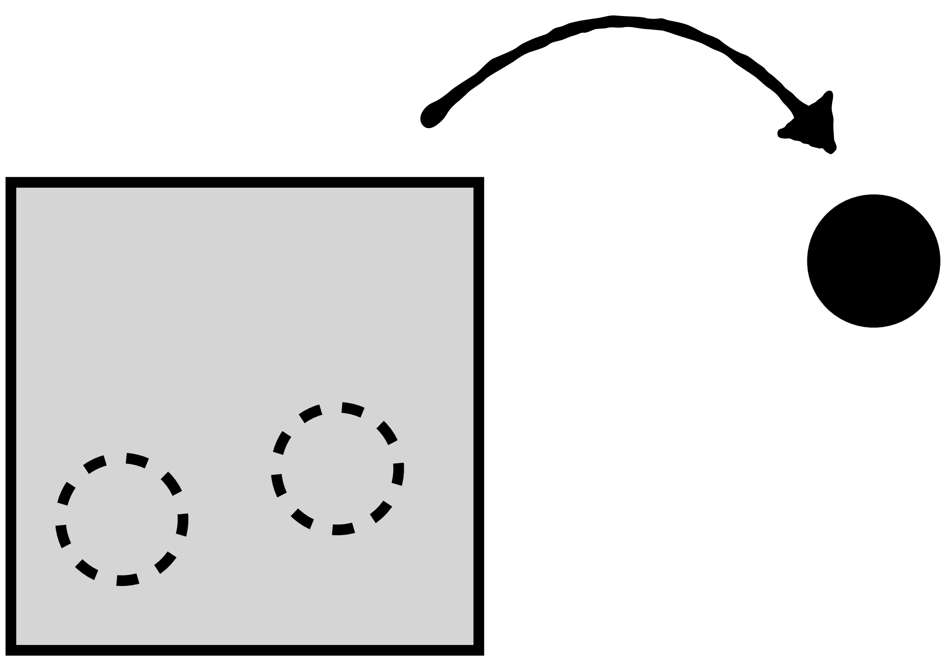
Example: Model selection



A model is chosen with probability

$$p(M)$$

Observation

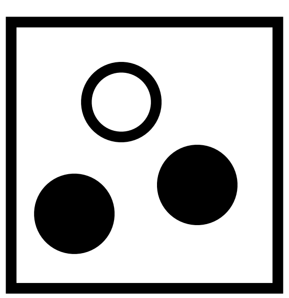


$D = B$

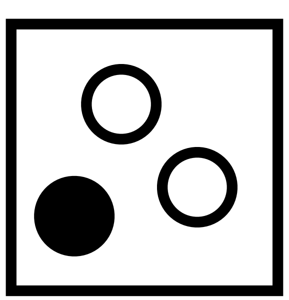
What is the probability of each model?

$$P(M \mid D) = \frac{P(M)\ P(D \mid M)}{P(D)}$$

Prior **Posterior**



$$P(M=1) = \frac{1}{2} \qquad P(M=1 \mid D=B) = \frac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{3}} = \frac{2}{3}$$



$$P(M=2) = \frac{1}{2} \qquad P(M=2 \mid D=B) = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{3}} = \frac{1}{3}$$

Evidence

Probability of the data independently on the value of the parameters

$$P(d) = \int L(d\,|\,\theta)\ \pi(\theta)\ \mathrm{d}\theta$$

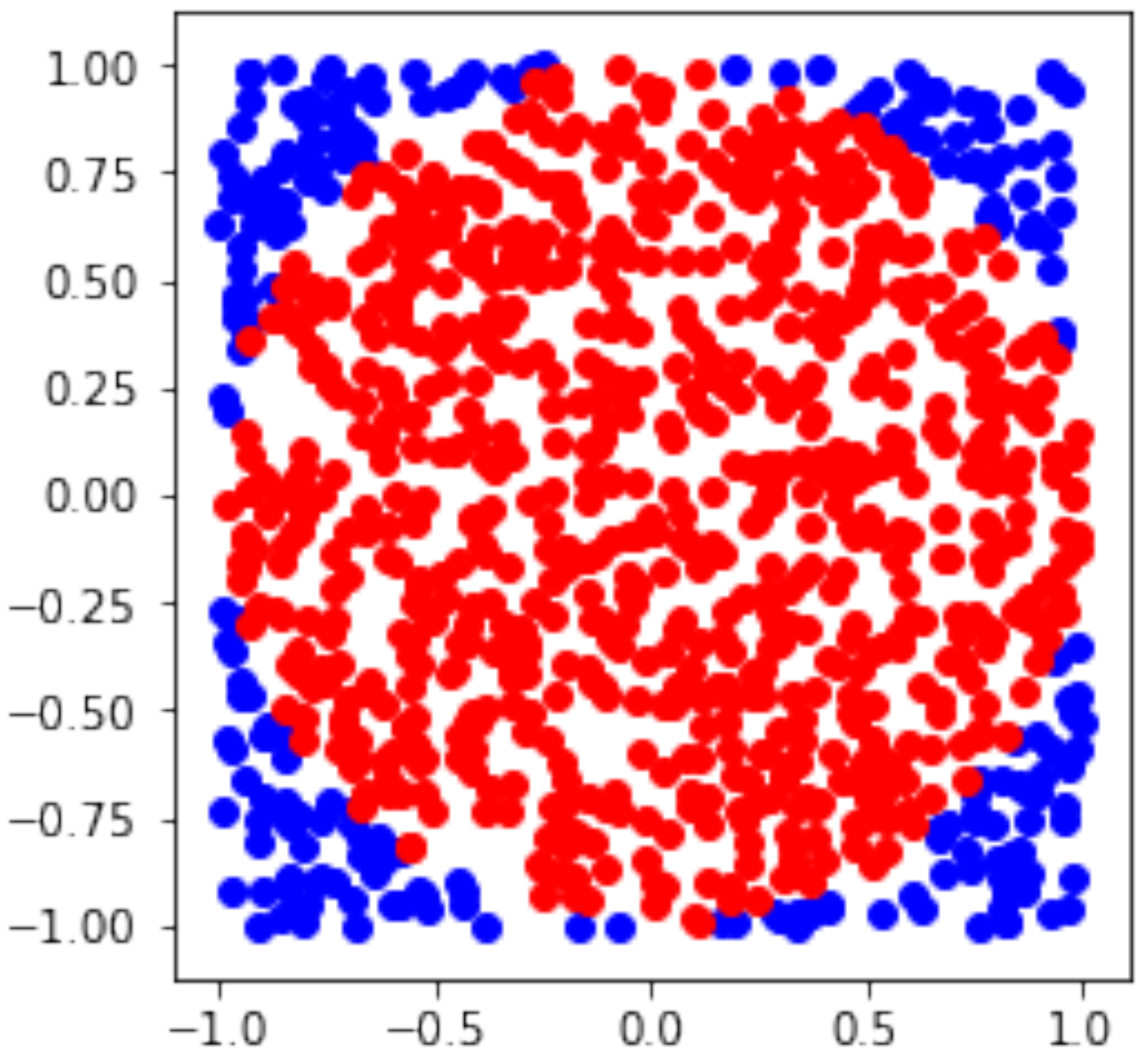
Bayes Factor

$$K = \frac{P(d\,|\,M_1)}{P(d\,|\,M_2)}$$

$$P(d\,|\,M) = \int L(d\,|\,\theta, M)\ \pi(\theta\,|\,M)\ \mathrm{d}\theta$$

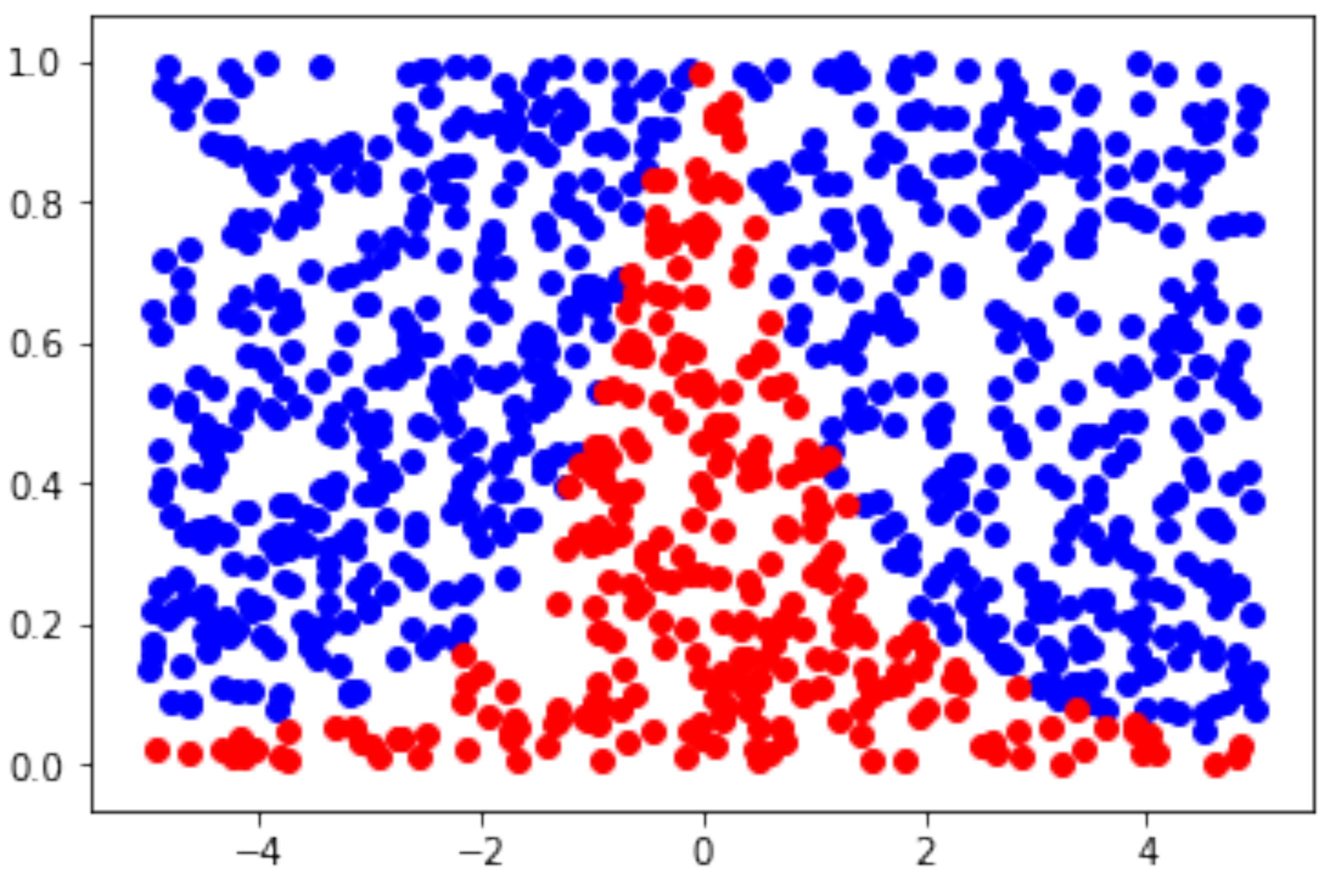
Monte Carlo methods

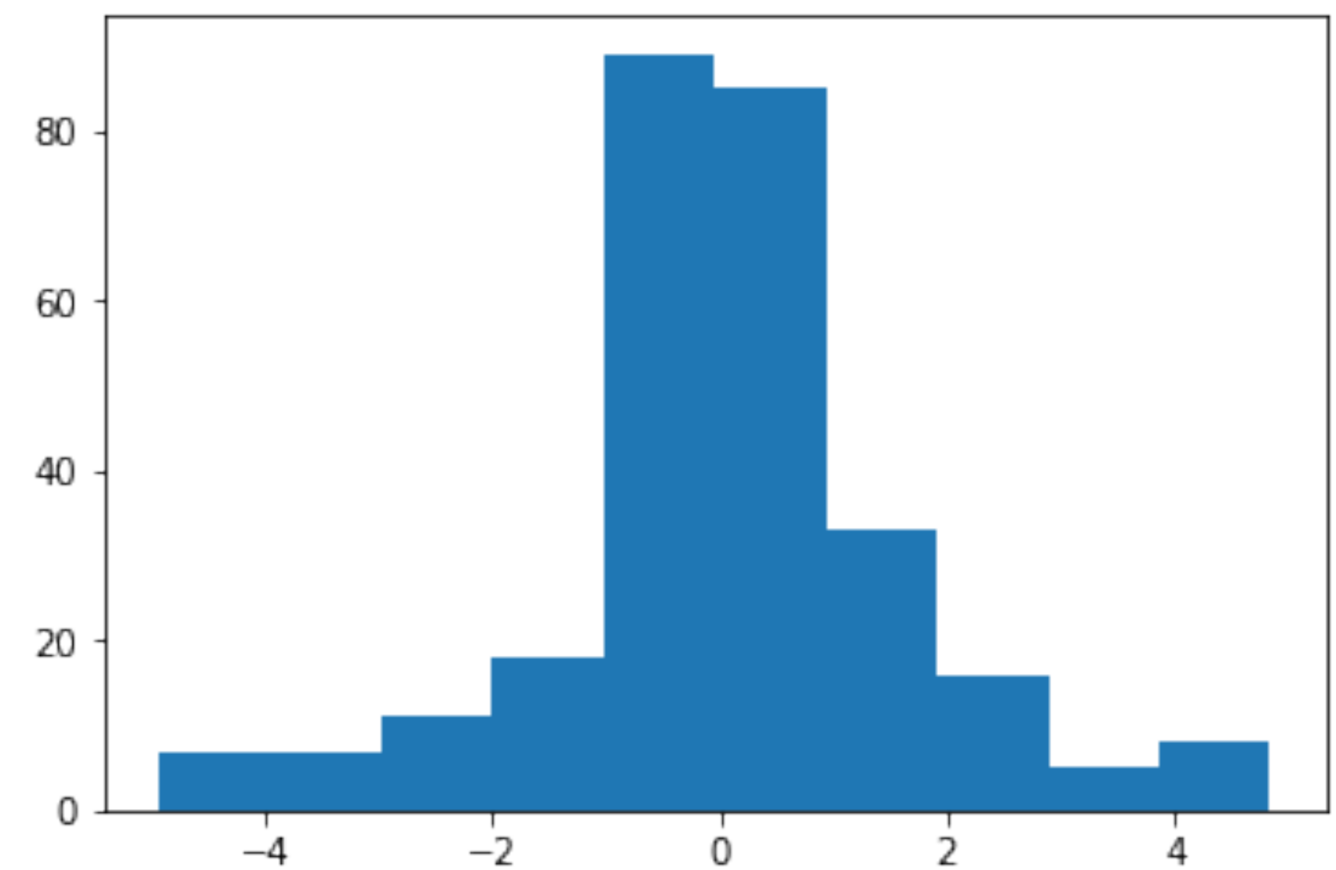
Monte-Carlo methods



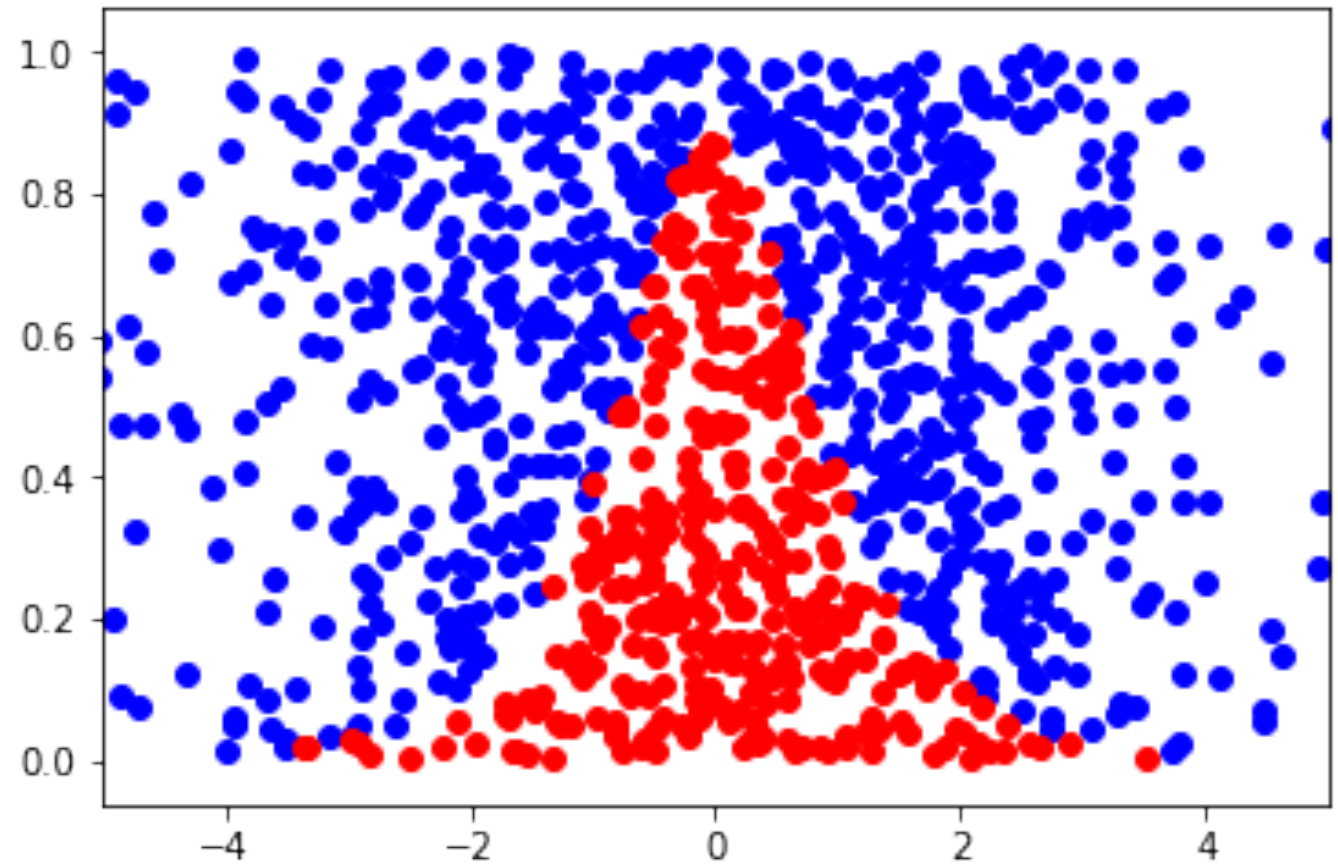
$$\pi = 4\frac{A_{\text{circle}}}{A_{\text{square}}}$$

Rejection sampling

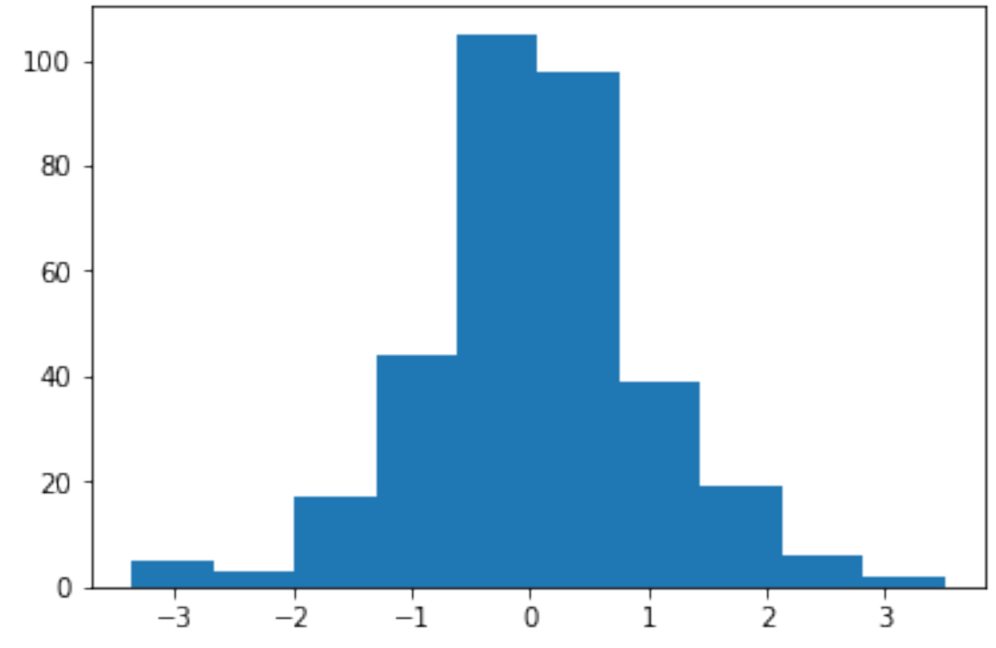




Rejection sampling



$$P_{\text{acceptance}} = \frac{f(x)}{M\ g(x)}$$



Monte-Carlo integration

$$\int_a^b h(x)\, f(x)\ \mathrm{d}x \sim \frac{1}{N}\sum_{i=1}^{N} h(x_i) \qquad x_i \sim f$$

$$\int_a^b h(x)\ \mathrm{d}x \sim \frac{b-a}{N}\sum_{i=1}^{N} h(x_i) \qquad x_i \sim \mathcal{U}(a,b)$$

Importance sampling

$$\langle h(x) \rangle = \int h(x)\, f(x)\ \mathrm{d}x = \int h(x)\ \frac{f(x)}{g(x)}\ g(x)\ \mathrm{d}x$$

- **Sample with a simpler distribution**
- **Reduce the variance of the estimation**

$$g(x) \sim |\,h(x)\,|\,f(x)$$

Markov Chains

Markov process

Transition probability

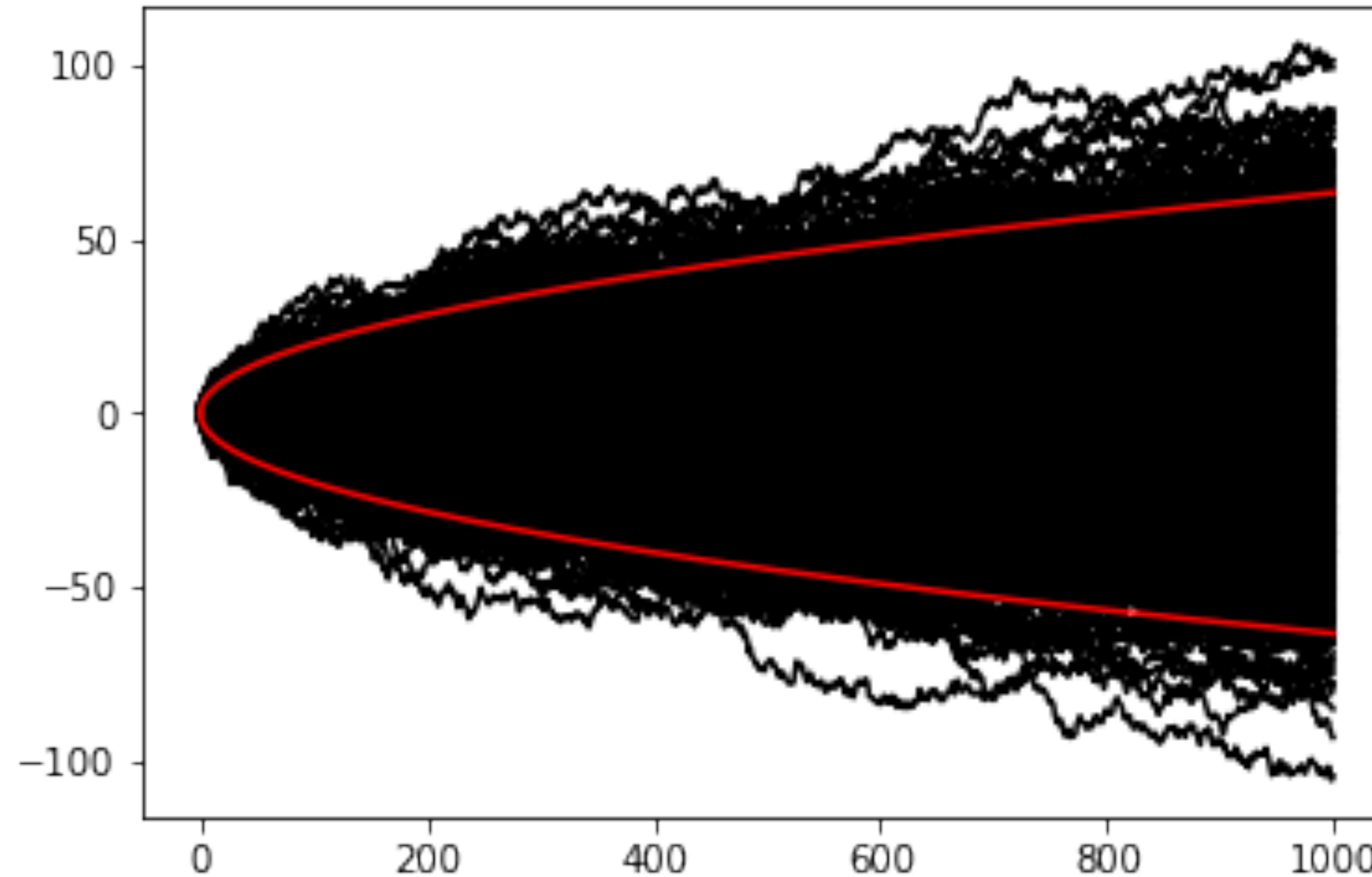
$$P(X_{t+1} \mid X_0, \ldots, X_t) = P(X_{t+1} \mid X_t)$$

Markov property: future state only depends on the present state (not the past)

Example: Random walk

$$X_t = X_0 + \sum_{i=1}^{t} z_i$$

$$z_i \sim \mathcal{N}(0, \sigma)$$



$$X_t \;\sim\; \mathcal{N}(X_{t-1}, \sigma) \;\sim\; \mathcal{N}(X_0, \sqrt{t}\sigma)$$

Equilibrium distribution

If the distribution at a given time does not change as the process evolves, then the Markov chain is at equilibrium

$t+1$ $\quad y \quad P_{t+1}(y)$

$P(x\,|\,y)$

t $\quad x \quad P_t(x)$

$$P_{t+1}(x) = \int P_t(y)\ P(x\,|\,y)\ \mathrm{d}y$$

Equilibrium distribution $P(x) = \int P(y)\ P(x\,|\,y)\ \mathrm{d}y$

Reversibility: detailed balance condition

$$P(x)\ P(y\,|\,x) = P(y)\ P(x\,|\,y)$$

If this condition is satisfied, then there exists an equilibrium distribution for the Markov process

It is reversible because the "present" and "future" states can be interchanged

Equilibrium distribution

The Markov process converges to the equilibrium distribution, if the process is

- **Reversibility (detailed balance): there exists an equilibrium distribution**
- **Irreducible: any state can be reached from any arbitrary starting state**
- **Aperiodic: the set of times in which is possible to coming back to the initial state is aperiodic**

$$\lim_{t \to \infty} P_t(X) = P(X)$$

Markov Chains Monte Carlo (MCMC)

Markov Chain Monte Carlo (MCMC)

Objective transition probability

$$P(x\,|\,y) = A(x, y)\ Q(x\,|\,y) + \delta(x - y)\int\left[1 - A(x', y)\right]\ Q(x'\,|\,y)\ \mathrm{d}x'$$

Acceptance probability

Proposed transition probability

Metropolis-Hasting algorithm

$$\frac{P(x)}{P(y)} = \frac{P(x \mid y)}{P(y \mid x)} = \frac{A(x, y)\ Q(x \mid y)}{A(y, x)\ Q(y \mid x)}$$

$$A(x, y) = \min\{\frac{P(x)}{P(y)} \frac{Q(y \mid x)}{Q(x \mid y)}, 1\}$$

Problems in the MCMC sampling

- **Highly correlated parameters**
- **Multimodal distributions**
- **Low or high acceptance ratio**
- **Non-Independent samples**
- **Large correlation time**

Possible solutions:

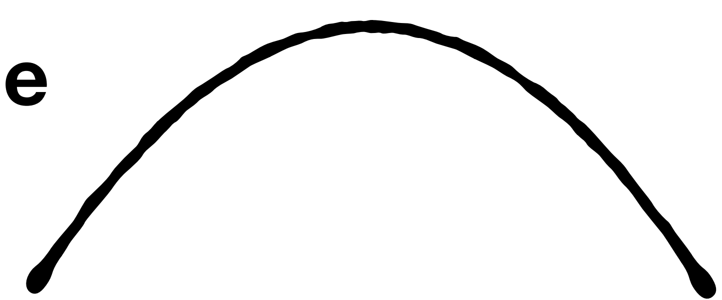
- **Change the parameter space**
- **Appropriate proposed transition probability**
- **Increase the number of chains**
- **Discard the burnin**
- **Thin the Markov chain**
- **Gibbs sampling, multimodal nested sampling algorithm, affine invariant sampler…**

Fisher information matrix

Entropy

$$S(\theta) \equiv \langle -\ln L(x\,|\,\theta)\rangle = -\int \ln L(x\,|\,\theta)\; L(x\,|\,\theta)\;\mathrm{d}x$$

Maximum entropy principle



$$S(\theta) = S(\theta_0) + \frac{\partial S}{\partial \theta^i}(\theta^i - \theta^i_0) + \frac{1}{2}F_{ij}(\theta^i - \theta^i_0)(\theta^j - \theta^j_0) + \dots$$

$$F_{ij} = -\left\langle \frac{\partial^2 \ln L}{\partial \theta^i \partial \theta^j} \right\rangle$$

Fisher matrix measures the curvature of the entropy function

Fisher matrix ~ inverse covariance matrix

$$-\ln L \sim -\ln L_0 + \frac{1}{2}(\theta - \theta_0)\ \mathbf{F}\ (\theta - \theta_0)$$

$$L(\theta) \sim L_0 \exp^{-\frac{1}{2}(\theta-\theta_0)\ \mathbf{F}\ (\theta-\theta_0)} \qquad \mathbf{C} = \mathbf{F}^{-1}$$

Cramér-Rao bound $\mathrm{cov}(\theta^i, \theta^j) \geq F_{ij}^{-1}$

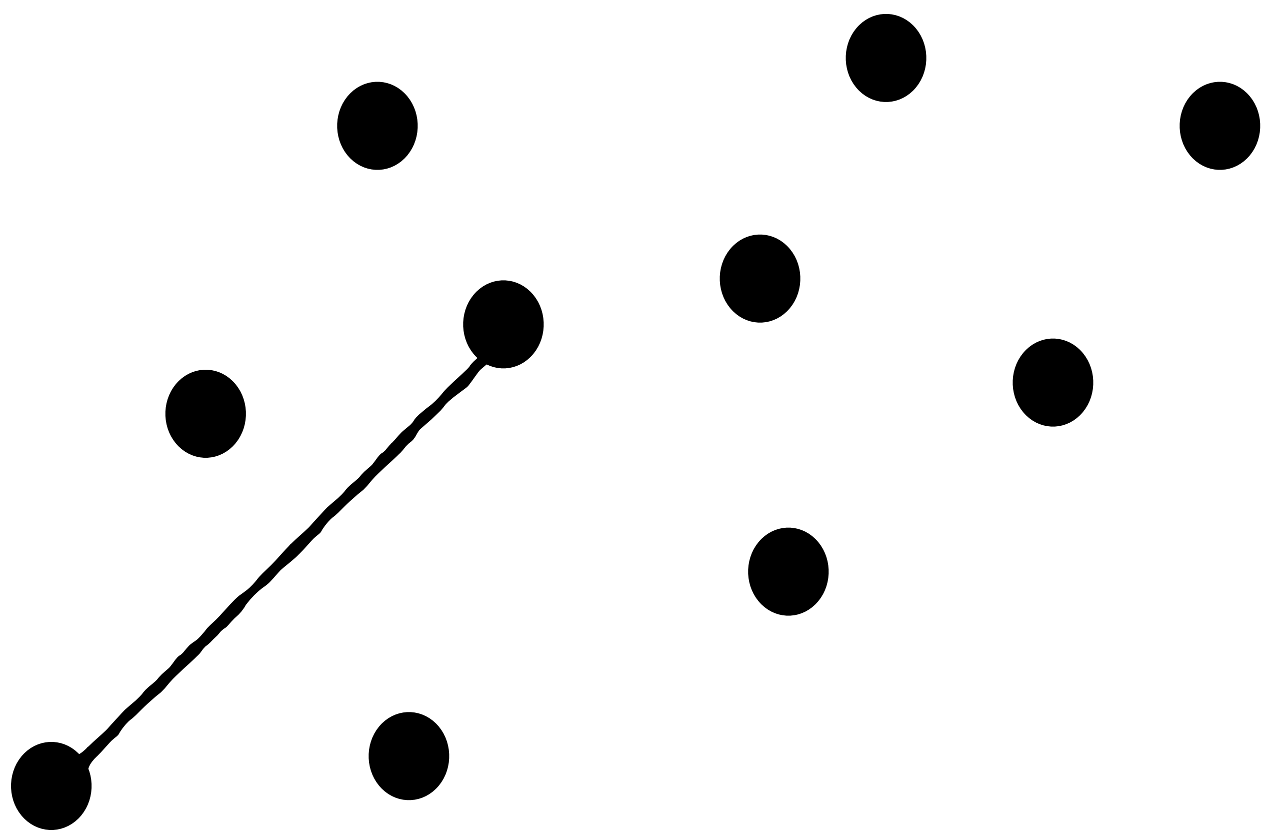
Fisher matrix

- **Forecasting the constrains on the model parameters**
- **Defining (non-informative) prior: Jeffreys prior**
- **Finding uncorrelated parametrisation**
- **Optimising the MCMC sampling**

Optimising the MCMC sampling

1. Find the best fit by an optimisation method
2. Compute the Fisher matrix at the best fit
3. Consider the transition probability according to the Fisher matrix. Typically, a Gaussian distribution whose covariance matrix is the inverse Fisher matrix

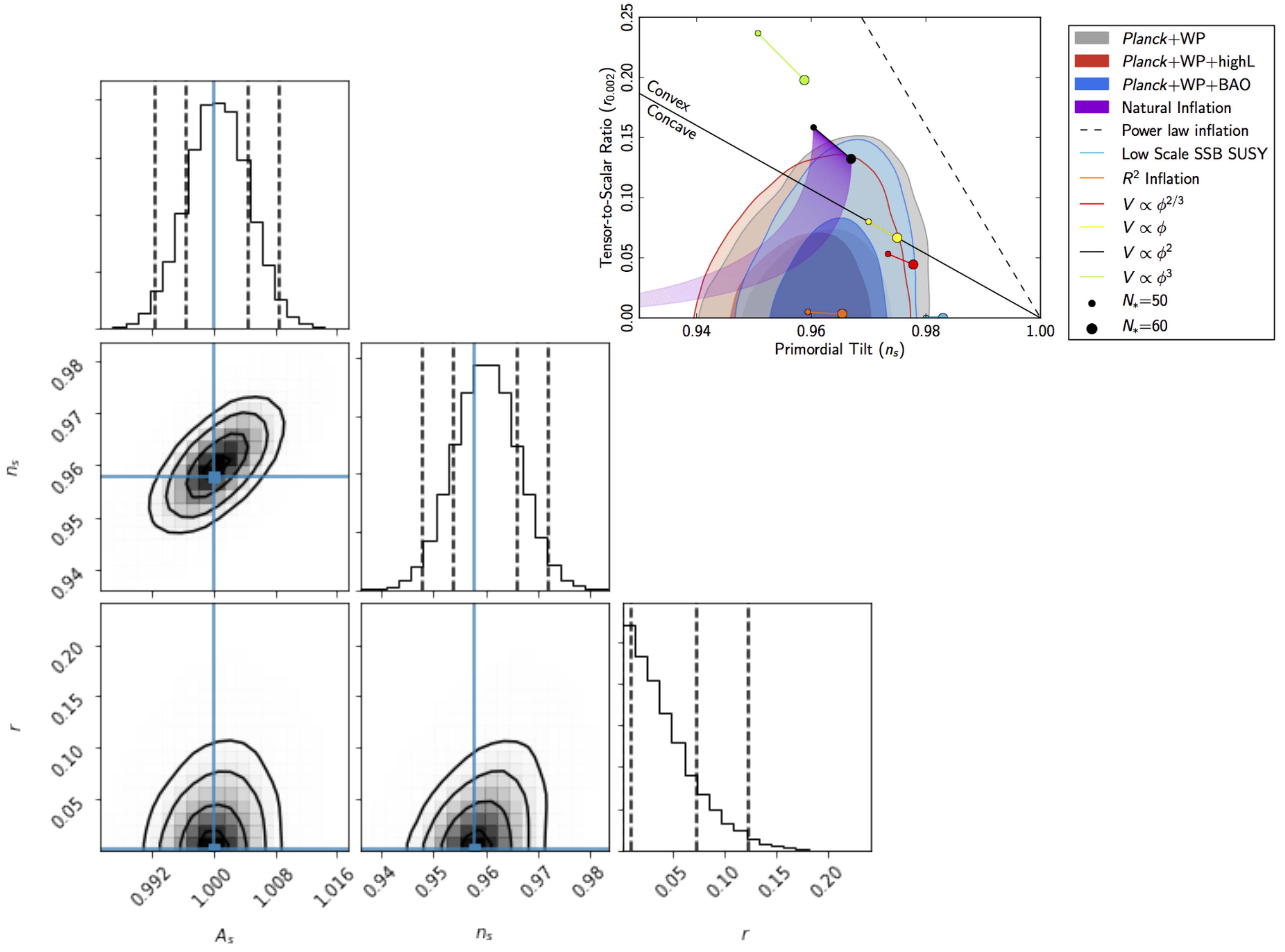
Affine invariant MCMC ensemble sampler



$$X_{t+1}^k = X^j + Z\left(X_t^k - X^j\right)$$

Cosmological parameters estimation

1. Programming the posterior function (likelihood + prior) given the theoretical model and the data
2. Running the MCMC algorithm for the posterior
3. Checking the convergence of the chains
4. Calculation of the different statistics of the parameters
5. Computation of the evidence of the model



Planck+WP
Planck+WP+highL
Planck+WP+BAO
Natural Inflation
Power law inflation
Low Scale SSB SUSY
R^2 Inflation
$V \propto \phi^{2/3}$
$V \propto \phi$
$V \propto \phi^2$
$V \propto \phi^3$
N_*=50
N_*=60
Convex
Concave
Tensor-to-Scalar Ratio ($r_{0.002}$)
Primordial Tilt (n_s)
0.00
0.05
0.10
0.15
0.20
0.25
0.94
0.96
0.98
1.00
n_s
r
A_s
0.992
1.000
1.008
1.016
0.94
0.95
0.96
0.97
0.98
0.05
0.10
0.15
0.20

Python modules

- **emcee: MCMC sampler implementing an affine invariant algorithm (Goodman & Weare, 2010)**
- **corner: representing parameter space**
- **tqdm: toolkit for including progress bar**