Fluctuation-induced quantum interactions between compact objects and a plane mirror

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Abstract. The interaction of compact objects with an infinitely extended mirror plane due to quantum fluctuations of a scalar or electromagnetic field that scatters off the objects is studied. The mirror plane is assumed to obey either Dirichlet or Neumann boundary conditions or to be perfectly reflecting. Using the method of images, we generalize a recently developed approach for compact objects in unbounded space to show that the Casimir interaction between the objects and the mirror plane can be accurately obtained over a wide range of separations in terms of charge and current fluctuations of the objects and their images. Our general result for the interaction depends only on the scattering matrices of the compact objects. It applies to scalar fields with arbitrary boundary conditions and to the electromagnetic field coupled to dielectric objects. For the experimentally important electromagnetic Casimir interaction between a perfectly conducting sphere and a plane mirror we present the first results that apply at all separations. We obtain both an asymptotic large-distance expansion and the two lowest-order correction terms to the proximity force approximation. The asymptotic Casimir–Polder potential for an atom and a mirror is generalized to describe the interaction between a dielectric sphere and a mirror, involving higher-order multipole polarizabilities that are important at sub-asymptotic distances.

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Contents

1. Introduction

The interaction between neutral objects is dominated by fluctuation forces due to the coordinated behavior of fluctuating charges or collective modes inside the objects. At zero temperature or at sufficiently small distances, the interactions result from quantum fluctuations. They are important on atomic scales as well as between macroscopic bodies. A prominent example for the latter is the Casimir force between parallel metallic plates due to current or electromagnetic field fluctuations [1]. During the last decade experimental verifications of this effect have been performed with increasing precision. These high-precision measurements were enabled by the use of curved surfaces instead of parallel planar mirrors in order to avoid the problem of parallelism. The most commonly employed geometry is a sphere-plate set-up that was used in the first high-precision tests of the Casimir effect [2, 3]. This geometry has been successfully used ever since in most of the recent experimental studies of Casimir forces between metallic surfaces [4]–[11]. In order to keep the deviations from two parallel plates sufficiently small, spheres with a radius much larger than the surface distance have been used. The effect of curvature has been accounted for by the 'proximity force approximation' (PFA) [12]. This scheme is assumed to describe the interaction for sufficiently small ratios of radius of curvature to distance. However, this an uncontrolled assumption since PFA becomes exact only for infinitesimal separations, and corrections to PFA are generally unknown.

At the other extreme, the interaction between a planar surface and objects that are either very small or at asymptotically large distance is governed by the Casimir–Polder potential that was derived for the case of an atom and a perfectly conducting plane [13]. This limit has been probed experimentally with high precision for a Bose–Einstein condensate that was trapped close to a planar surface [14]. Recently, there have been the first attempts to go beyond the two extreme limits of asymptotically large and

small separations by measuring the Casimir force between a sphere and a plane over a larger range of ratios of sphere radius to distance [15].

So far, no theoretical prediction is available that can describe the electromagnetic Casimir interaction between a compact object and a planar surface at all distances, including the important sphere-plate geometry. Until recently, progress in understanding the geometry dependence of fluctuation forces was hampered by the lack of practical methods that are applicable at all separations. Conceptually, the effect of geometry and shape is difficult to study due to the non-additivity of fluctuation forces. Explicit consequences of this non-additivity and also non-monotonic changes in the force have been recently predicted for a pair of cylinders next to planar walls [16]. This behavior has been interpreted in terms of collective charge fluctuations inside the bodies [17].

For some decades, there has been considerable interest in the theory of Casimir forces between objects with curved surfaces. Two types of approaches have been pursued: attempts to compute the force explicitly in particular geometries and efforts to develop a general framework which yields the interaction in terms of characteristics of the objects like polarizability or curvature. Within the second type of approach, Balian and Duplantier studied the electromagnetic Casimir interaction between compact perfect metals in terms of a multiple reflection expansion and also derived explicit results to leading order at asymptotically large separations [18]. For parallel and partially transmitting plates a connection to scattering theory has been established which yields the Casimir interaction of the plates as a determinant of a diagonal matrix of reflection amplitudes [19]. For non-planar, deformed plates, a general representation of the Casimir energy as a functional determinant of a matrix that describes reflections at the surfaces and free propagation between them has been developed in [20]. Later on, an equivalent representation has been applied to perturbative computations in the case of rough and corrugated plates with finite conductivity [21, 22].

Functional determinant formulas have also been used for open geometries that do not fall into the class of parallel plates with deformations. For the electromagnetic Casimir interaction between a planar plate and infinitely long cylinders, a partial wave expansion of the functional determinant has been developed [23]. The same results have been reobtained and used to compute corrections to the PFA for the cylinder-plate geometry in [24]. Kenneth and Klich identified the inverted Green's function in the functional determinant as a T-matrix and derived a formal result for the Casimir interaction in terms of this matrix in the case of scalar fields in a medium with a space-and frequency-dependent speed of light [25].

Recently we described a new method based on a multipole expansion of fluctuating charges that makes possible accurate and efficient calculations of Casimir forces and torques between any number of compact objects [26, 27]. The method applies to electromagnetic fields and dielectrics as well as perfect conductors. It also applies to other fields, such as scalar and Dirac, and to any boundary conditions. In this approach, the Casimir energy is given in terms of the fluctuating field's scattering amplitudes from the individual objects, which encode the effects of the shape and boundary conditions. An equivalent partial wave expansion has been applied to a scalar model for dielectrics [28].

Casimir interactions due to scalar field fluctuations serve as a simplified model for the full electromagnetic interaction. This model is usually easier to analyze and provides an important tool in developing conceptually new approaches and in estimating geometry

dependences. For the sphere–plate geometry with Dirichlet boundary conditions Bulgac *et al* obtained the Casimir interaction over a wide range of separations from a modified Krein trace formula [29, 30]. Most notably in the context of scalar fields, a versatile numerical world-line algorithm based on Monte Carlo methods has been developed and applied to a number of interesting geometries, including the here studied sphere–plate interaction [31, 32].

In this work, we extend our approach developed in [26, 27] to describe the interaction of compact objects in the presence of a plane mirror. Our general result holds for scalar and electromagnetic fields. In the electromagnetic case the mirror is assumed to be perfectly conducting but the compact objects can be dielectrics or perfect conductors. The derivation of the general result for the Casimir interaction, see equations (2.48) and (2.100), is given in section 2 by combining a functional integral approach and the method of images. In section 3 we apply our approach to compute the interaction of a sphere with a plane mirror over a wide range of separations for the scalar Dirichlet and Neumann problem and for the electromagnetic field that is most relevant to applications. We provide a large-distance expansion of the interaction, generalizing the Casimir–Polder potential to include higher-order multipole polarizabilities. For small separations, we compute the two leading correction terms to the PFA.

2. Interaction between fluctuating sources and their images

In this section we first review the functional integral formulation of Casimir interactions between compact objects for a scalar field and the electromagnetic field. The fluctuating field is integrated out in order to obtain an effective action for the fluctuating sources on the objects. Then we show that the interaction of compact objects with an infinite plane mirror can be described by the equivalent problem of the interaction of compact objects with their mirror images in otherwise empty space, i.e. without the plane mirror. This equivalence holds for Dirichlet or Neumann boundary conditions (at the mirror and the objects) for a scalar field and for a perfectly conducting mirror plane and arbitrary dielectric objects in the case of electromagnetic fluctuations.

2.1. Scalar field

We first consider a real quantum field $\phi(\mathbf{x}, t)$ in order to introduce the method of images in the path integral formulation of Casimir interactions. We assume that the space is divided into two half-spaces by a mirror plane at $z = 0$. N fixed closed surfaces Σ_α, $\alpha = 1, \ldots, N$, are located in the right half-space ($z > 0$). These surfaces are regarded as the boundaries of objects on which either Dirichlet or Neumann boundary conditions are imposed. The action of the unconstrained field in Minkowski space is

$$S[\phi] = \frac{1}{2} \int \mathrm{d}t \int_> \mathrm{d}\mathbf{x} \left\{ \frac{1}{c^2}(\partial_t \phi)^2 - (\nabla \phi)^2 \right\}. \tag{2.1}$$

Here the $\mathbf{x}$ integration runs over the right half-space, indicated by $\int_>$. The free energy $\mathcal{F}$ of the constraint field $\phi(\mathbf{x}, t)$ at inverse temperature β is represented by the Euclidean functional integral

$$\mathrm{e}^{-\beta \mathcal{F}} = Z = \int [\mathcal{D}\phi]_{\mathcal{C}} \exp\left(-S_{\mathrm{E}}[\phi]/\hbar\right), \tag{2.2}$$

where the Euclidean action S_E follows from equation (2.1) after the Wick rotation $ct = -ix^0$:

$$S_E = \frac{1}{2c} \int_0^\Lambda dx^0 \int_> d\mathbf{x} \left\{ (\partial_0 \phi)^2 + (\nabla \phi)^2 \right\}$$ (2.3)

with $\Lambda = \beta\hbar c$. The functional integral is over all fields that are periodic in the x_0 interval from 0 to Λ and that obey the boundary conditions at the surfaces, indicated by the subscript $\mathcal{C}$.

The surfaces are fixed and the boundary conditions are time-independent. Hence each Fourier component of the field with respect to x^0 obeys the constraints at the surfaces separately. To make use of this property, we expand $\phi(\mathbf{x}, x^0)$ as

$$\phi(\mathbf{x}, x^0) = \sum_{n=-\infty}^{\infty} \phi_n(\mathbf{x}) e^{i\kappa_n x^0}$$ (2.4)

with Matsubara frequencies $\kappa_n = 2\pi n/\Lambda$ and $\phi_{-n}(\mathbf{x}) = \phi_n^*(\mathbf{x})$. The functional integral splits into independent functional integrals over the $\phi_n(\mathbf{x})$ and the logarithm of Z can be written as the sum

$$\ln Z = \sum_{n=-\infty}^{\infty} \ln \int [\mathcal{D}\phi_n]_\mathcal{C} \exp\left[-\frac{\Lambda}{2\hbar c} \int_> d\mathbf{x} \left\{ \kappa_n^2 |\phi_n|^2 + |\nabla \phi_n|^2 \right\} \right].$$ (2.5)

In the following we are interested in the limit of zero temperature. Then $\Lambda \to \infty$ and the sum over n can be replaced by the integral $\frac{\Lambda}{2\pi} \int_{-\infty}^{\infty} d\kappa$ and $\phi_n(\mathbf{x})$ is replaced by $\phi(\mathbf{x}, \kappa)$. Combining positive and negative κ, we get

$$\ln Z = \frac{\Lambda}{2\pi} \int_0^\infty d\kappa \ln \mathcal{Z}(\kappa)$$ (2.6)

with

$$\mathcal{Z}(\kappa) = \int [\mathcal{D}\phi(\mathbf{x}, \kappa)]_\mathcal{C} [\mathcal{D}\phi^*(\mathbf{x}, \kappa)]_\mathcal{C} \exp\left[-\frac{\Lambda}{\hbar c} \int_> d\mathbf{x} \left\{ \kappa^2 |\phi|^2 + |\nabla \phi|^2 \right\} \right].$$ (2.7)

The ground state energy is obtained from Z as

$$\begin{aligned} \mathcal{E}_0 &= -\lim_{\beta \to \infty} \frac{1}{\beta} \ln Z \\ &= -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \ln \mathcal{Z}(\kappa). \end{aligned}$$ (2.8)

The Casimir energy is the difference between the ground state energy of the surfaces at their actual distance and the sum of the ground state energies of the separate surfaces which is obtained by removing the surfaces to infinite separation:

$$\mathcal{E} = -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \ln \frac{\mathcal{Z}(\kappa)}{\mathcal{Z}_\infty(\kappa)},$$ (2.9)

where $\mathcal{Z}_\infty(\kappa)$ is the partition function for the surfaces with infinite separation. In the following we suppress the label κ. Next, the constraints at the objects are implemented by functional δ functions [33, 34]. For Dirichlet boundary conditions, $\phi = 0$, on the surfaces

Σ_α, the constraint functional integral can be expressed in terms of an unconstrained integral by using

$$\int [\mathcal{D}\phi]_\mathcal{C}[\mathcal{D}\phi^*]_\mathcal{C} = \int \mathcal{D}\phi\mathcal{D}\phi^* \prod_{\alpha=1}^{\mathrm{N}} \int \mathcal{D}\varrho_\alpha \mathcal{D}\varrho_\alpha^* \exp\left[\mathrm{i}\int_{\Sigma_\alpha} \mathrm{d}\mathbf{x}\left\{\varrho_\alpha^*(\mathbf{x})\phi(\mathbf{x}) + \mathrm{c.c.}\right\}\right], \qquad (2.10)$$

where the δ functions at each position of the surfaces have been written as an integral over a source field $\varrho_\alpha(\mathbf{x})$ that is non-zero on the surfaces Σ_α only. When we use this representation of the constraints in equation (2.7), the now unconstrained functional integral over $\phi(\mathbf{x})$ is Gaussian and yields

$$\mathcal{Z}(\kappa) = \mathcal{Z}_0 \prod_{\alpha=1}^{\mathrm{N}} \int \mathcal{D}\varrho_\alpha \mathcal{D}\varrho_\alpha^* \exp\left[-\frac{\hbar c}{2\Lambda}\sum_{\alpha\beta}\int_{\Sigma_\alpha}\mathrm{d}\mathbf{x}\int_{\Sigma_\beta}\mathrm{d}\mathbf{x}'\left\{\varrho_\alpha^*(\mathbf{x})G_>(\mathbf{x},\mathbf{x}',\kappa)\varrho_\beta(\mathbf{x}') + \mathrm{c.c.}\right\}\right]$$

$$\equiv \mathcal{Z}_0 \prod_{\alpha=1}^{\mathrm{N}} \int \mathcal{D}\varrho_\alpha \mathcal{D}\varrho_\alpha^* \mathrm{e}^{-(\hbar c/\Lambda)\tilde{S}[\varrho]}, \qquad (2.11)$$

where $\mathcal{Z}_0$ is the partition function of the field in the right half-space without the surfaces Σ_α. For Neumann boundary conditions at the surfaces Σ_α the field $\phi(\mathbf{x})$ in the exponential of equation (2.10) is replaced by $\partial_n\phi(\mathbf{x})$ and $G_>(\mathbf{x},\mathbf{x}',\kappa)$ in equation (2.11) is replaced by $\partial_n\partial_{n'}G_>(\mathbf{x},\mathbf{x}',\kappa)$, where ∂_n is the normal derivative pointing out of the objects. Here $G_>(\mathbf{x},\mathbf{x}',\kappa)$ is the free Green's function in the right half-space which is given by

$$G_>(\mathbf{x},\mathbf{x}',\kappa) = G_0(\mathbf{x},\mathbf{x}',\kappa) \mp G_0(\mathbf{x},\mathbf{x}'_R,\kappa), \qquad (2.12)$$

where $\mathbf{x}_R = (\mathbf{x}_\|, -z)$ is the mirror image of $\mathbf{x} = (\mathbf{x}_\|, z)$ and the $-$ (+) sign applies to Dirichlet (Neumann) boundary conditions at the mirror plane at $z = 0$. The Green's function of free unbounded space is given by

$$G_0(\mathbf{x},\mathbf{x}',\kappa) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\Big|_{|k\to\mathrm{i}\kappa}. \qquad (2.13)$$

Using equation (2.12) the action $\tilde{S}[\varrho]$ defined by equation (2.11) can be expressed in terms of the original sources $\varrho_\alpha(\mathbf{x})$ and their mirror sources $\varrho_\alpha^R(\mathbf{x})$:

$$\tilde{S}[\varrho] = \tfrac{1}{2}\sum_{\alpha\beta}\left\{\int_{\Sigma_\alpha}\mathrm{d}\mathbf{x}\int_{\Sigma_\beta}\mathrm{d}\mathbf{x}'\varrho_\alpha^*(\mathbf{x})G_0(\mathbf{x},\mathbf{x}',\kappa)\varrho_\beta(\mathbf{x}')\right.$$

$$\left. + \int_{\Sigma_\alpha}\mathrm{d}\mathbf{x}\int_{\Sigma_\beta^R}\mathrm{d}\mathbf{x}'\varrho_\alpha^*(\mathbf{x})G_0(\mathbf{x},\mathbf{x}',\kappa)\varrho_\beta^R(\mathbf{x}') + \mathrm{c.c.}\right\} \qquad (2.14)$$

for Dirichlet boundary conditions at the surfaces Σ_α and with $G_0(\mathbf{x},\mathbf{x}',\kappa)$ replaced by $\partial_n\partial_{n'}G_0(\mathbf{x},\mathbf{x}',\kappa)$ for Neumann boundary conditions at the surfaces Σ_α. Here we have introduced the mirror sources

$$\varrho_\alpha^R(\mathbf{x}) = \mp\varrho_\alpha(\mathbf{x}_R), \qquad (2.15)$$

where the $-$ (+) sign applies to a Dirichlet (Neumann) mirror. The mirror sources are located on the mirror surfaces Σ_α^R that are obtained from the Σ_α by $z \to -z$ for all surface positions, see figure 1. The first term of the action of equation (2.14) describes
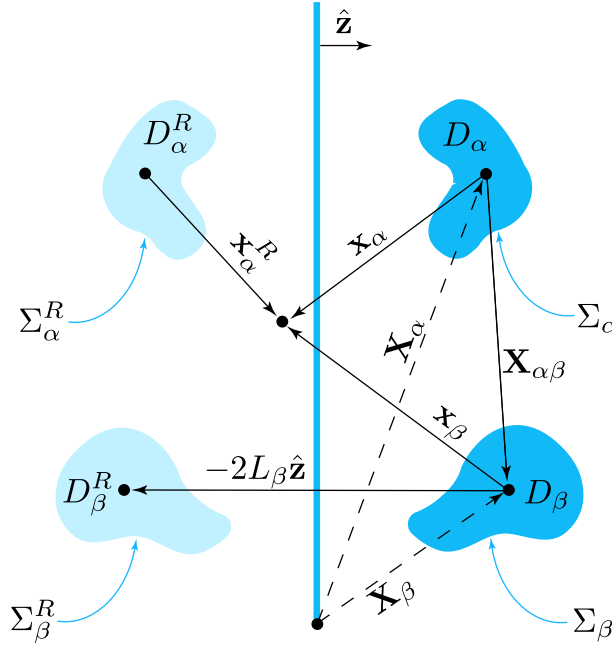
Figure 1. Geometry with original surfaces Σ_α (objects D_α) in the right half-space ($z > 0$) and mirror surfaces Σ_α^R (objects D_α^R). $\mathbf{x}_\alpha$, $\mathbf{x}_\alpha^R$ are local coordinate vectors measured from the object's origins, while L_β is the center-to-mirror distance of surface Σ_β (object D_β).

the interaction of the surface sources in the absence of the mirror plane. The second term couples each surface source to all mirror sources. Since the mirror problem is now described by an action in *free* space with sources and mirror sources, we can apply the concepts of the previously developed approach for Casimir interactions between compact objects in unbounded space [27]. Below, we provide an explicit derivation for the case of Dirichlet boundary conditions at the surfaces Σ_α but we shall also indicate how the derivation has to be modified for Neumann boundary conditions.

The action of equation (2.14) is composed of two qualitatively different terms that we will now consider separately. Firstly, there are terms that couple sources on different surfaces ($\mathbf{x} \neq \mathbf{x}'$) where we use the term 'surface' in the following for the original and the mirror surfaces. We shall call these terms *off-diagonal*. As *diagonal* terms we shall denote those which couple sources on the same surface ($\mathbf{x} = \mathbf{x}'$ possible). Both type of terms can be expressed in terms of the multipole moments of the sources.

Off-diagonal terms. These terms couple sources on different objects:

$$\tilde{S}_{\alpha\beta} = \tfrac{1}{2} \int_{\Sigma_\alpha} \mathrm{d}\mathbf{x}_\alpha \left\{ \varrho_\alpha^*(\mathbf{x}_\alpha) \phi_\beta(\mathbf{x}_\alpha) + \text{c.c.} \right\} \tag{2.16}$$

for $\alpha \neq \beta$ and the original sources to the mirror sources:

$$\tilde{S}_{\alpha\beta}^R = \tfrac{1}{2} \int_{\Sigma_\alpha} \mathrm{d}\mathbf{x}_\alpha \left\{ \varrho_\alpha^*(\mathbf{x}_\alpha) \phi_\beta^R(\mathbf{x}_\alpha) + \text{c.c.} \right\} \tag{2.17}$$

for *all* α, β. Here we have introduced local coordinates $\mathbf{x}_\alpha$ that are measured relative to an arbitrarily chosen origin that is located inside the surface α, see figure 1. We have also defined the fields

$$\phi_\beta(\mathbf{x}) = \int_{\Sigma_\beta} d\mathbf{x}' G_0(\mathbf{x}, \mathbf{x}', \kappa)\varrho_\beta(\mathbf{x}'),$$

$$\phi_\beta^R(\mathbf{x}) = \int_{\Sigma_\beta^R} d\mathbf{x}' G_0(\mathbf{x}, \mathbf{x}', \kappa)\varrho_\beta^R(\mathbf{x}'), \tag{2.18}$$

which are the classical fields generated by the sources. For Neumann boundary conditions on the surfaces Σ_α, $\phi_\beta(\mathbf{x}_\alpha)$ and $\phi_\beta^R(\mathbf{x}_\alpha)$ in equations (2.16) and (2.17) have to be replaced by $\partial_n\phi_\beta(\mathbf{x}_\alpha)$ and $\partial_n\phi_\beta^R(\mathbf{x}_\alpha)$, respectively. Also, $G_0(\mathbf{x}, \mathbf{x}', \kappa)$ in equation (2.18) has to be replaced by $\partial_{n'}G_0(\mathbf{x}, \mathbf{x}', \kappa)$.

Since we can assume that every position on Σ_α is outside a sphere enclosing Σ_β or Σ_β^R, we can use the partial wave expansion of $G_0(\mathbf{x}, \mathbf{x}', \kappa)$ for $|\mathbf{x}'| < |\mathbf{x}|$:

$$G_0(\mathbf{x}, \mathbf{x}', \kappa) = -\kappa \sum_{lm} j_l(i\kappa r')h_l^{(1)}(i\kappa r)Y_{lm}(\hat{\mathbf{x}})Y_{lm}^*(\hat{\mathbf{x}}'). \tag{2.19}$$

When we consider equation (2.18) with coordinates $\mathbf{x}_\beta$ relative to the origin inside surface Σ_β, the field that is generated by the source ϱ_β can be written as

$$\phi_\beta(\mathbf{x}_\beta) = -\kappa \sum_{lm} Q_{\beta,lm} h_l^{(1)}(i\kappa r_\beta)Y_{lm}(\hat{\mathbf{x}}_\beta), \tag{2.20}$$

where we have defined the multipole moments of the source ϱ_β as

$$Q_{\beta,lm} = \int_{\Sigma_\beta} d\mathbf{x}_\beta j_l(i\kappa r_\beta)Y_{lm}^*(\hat{\mathbf{x}}_\beta)\varrho_\beta(\mathbf{x}_\beta). \tag{2.21}$$

The field of the mirror sources can be expressed in the same form:

$$\phi_\beta^R(\mathbf{x}_\beta^R) = -\kappa \sum_{lm} Q_{\beta,lm}^R h_l^{(1)}(i\kappa r_\beta^R)Y_{lm}(\hat{\mathbf{x}}_\beta^R), \tag{2.22}$$

where $\mathbf{x}_\beta^R$ denotes the local coordinates that are measured relative to the origin inside surface Σ_β^R. The multipole moments of the mirror source are given by

$$Q_{\beta,lm}^R = \mp(-1)^{l+m}Q_{\beta,lm} \tag{2.23}$$

due to equation (2.15) and $Y_{lm}(\hat{\mathbf{x}}_R) = (-1)^{l+m}Y_{lm}(\hat{\mathbf{x}})$. For Neumann boundary conditions at the surfaces Σ_α the above expressions remain valid if the multipole moments $Q_{\beta,lm}$ are replaced by Neumann multipoles which have the form of equation (2.21) but with $j_l(i\kappa r_\beta)Y_{lm}^*(\hat{\mathbf{x}}_\beta)$ replaced by $\partial_n[j_l(i\kappa r_\beta)Y_{lm}^*(\hat{\mathbf{x}}_\beta)]$.

In order to express the action of equation (2.16) in terms of multipole moments, we have to write the field generated by the surface source ϱ_β as a function of the local coordinate $\mathbf{x}_\alpha$ that is regular at the origin inside Σ_α. This can be done using translation matrices $\mathbb{U}$ which relate outgoing ($h_l^{(1)}$) and regular (j_l) spherical Bessel functions:

$$h_l^{(1)}(i\kappa r_\beta)Y_{lm}(\hat{\mathbf{x}}_\beta) = \sum_{l'm'} \mathcal{U}_{l'm'lm}(\mathbf{X}_{\alpha\beta})j_{l'}(i\kappa r_\alpha)Y_{l'm'}(\hat{\mathbf{x}}_\alpha). \tag{2.24}$$

The matrix elements $\mathcal{U}_{l'm'lm}$ of $\mathbb{U}$ depend on the vector $\mathbf{X}_{\alpha\beta} = \mathbf{X}_\beta - \mathbf{X}_\alpha = \mathbf{x}_\alpha - \mathbf{x}_\beta$ from the origin inside Σ_α to the origin inside Σ_β, see figure 1, and are given by [27]

$$\mathcal{U}_{l'm'lm}(\mathbf{X}_{\alpha\beta}) = \sqrt{4\pi}(-1)^m i^{l-l'}\sqrt{(2l+1)(2l'+1)}\sum_{l''} i^{l''}\sqrt{2l''+1}$$

$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m' & m'-m \end{pmatrix} h_{l''}^{(1)}(\mathrm{i}\kappa|\mathbf{X}_{\alpha\beta}|)Y_{l''m-m'}(\hat{\mathbf{X}}_{\alpha\beta}), \quad (2.25)$$

where we have assumed that the Cartesian coordinate frames associated with the two origins have identical orientation, i.e. they are related by a translation. The summation over l'' involves only a finite number of terms since the 3-j symbols vanish for $l'' > l + l'$ and $l'' < |l - l'|$.

Using the translation formula of equation (2.24), the field generated by the source on surface Σ_β, given by equation (2.20), can be written as a function of the coordinate $\mathbf{x}_\alpha$ as

$$\phi_\beta(\mathbf{x}_\alpha) = -\kappa \sum_{lm} Q_{\beta,lm} \sum_{l'm'} \mathcal{U}_{l'm'lm}^{\alpha\beta} j_{l'}(\mathrm{i}\kappa r_\alpha)Y_{l'm'}(\hat{\mathbf{x}}_\alpha), \quad (2.26)$$

where we have introduced $\mathcal{U}_{l'm'lm}^{\alpha\beta} \equiv \mathcal{U}_{l'm'lm}(\mathbf{X}_{\alpha\beta})$. Similarly we obtain for the field generated by the mirror sources of equation (2.22), now expressed as a function of $\mathbf{x}_\alpha$:

$$\phi_\beta^R(\mathbf{x}_\alpha) = \pm\kappa \sum_{lm} Q_{\beta,lm} \sum_{l'm'} \mathcal{U}_{l'm'lm}^{R,\alpha\beta} j_{l'}(\mathrm{i}\kappa r_\alpha)Y_{l'm'}(\hat{\mathbf{x}}_\alpha), \quad (2.27)$$

where we defined

$$\mathcal{U}_{l'm'lm}^{R,\alpha\beta} \equiv (-1)^{l+m}\mathcal{U}_{l'm'lm}(\mathbf{x}_\alpha - \mathbf{x}_\beta^R), \quad (2.28)$$

and used equation (2.23). Notice that the latter formula applies also to the case $\alpha = \beta$ which describes the translation between the surface Σ_α and its mirror image so that the argument of the translation matrix becomes $\mathbf{x}_\alpha - \mathbf{x}_\alpha^R = -2L_\alpha\hat{\mathbf{z}}$, where L_α is the normal distance between the origin of surface Σ_α and the mirror plane. For this case of translations along the $\hat{\mathbf{z}}$ direction the translation matrix simplifies to [27]

$$\mathcal{U}_{l'm'lm}(-2L_\alpha\hat{\mathbf{z}}) = \delta_{m'm}(-1)^m i^{l-l'}\sqrt{(2l+1)(2l'+1)}$$

$$\times \sum_{l''} i^{-l''}(2l''+1) \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} h_{l''}^{(1)}(\mathrm{i}\kappa 2L_\alpha). \quad (2.29)$$

When we substitute the result for the fields of equations (2.26) and (2.27) into equations (2.16) and (2.17), we obtain the action in terms of the original multipole moments:

$$\sum_{\alpha\neq\beta} \tilde{S}_{\alpha\beta} = -\kappa \sum_{\alpha\neq\beta} \sum_{lml'm'} Q_{\alpha,l'm'}^* \tilde{\mathcal{U}}_{l'm'lm}^{\alpha\beta} Q_{\beta,lm}, \quad (2.30)$$

$$\sum_{\alpha\beta} \tilde{S}_{\alpha\beta}^R = \pm\kappa \sum_{\alpha\beta} \sum_{lml'm'} Q_{\alpha,l'm'}^* \tilde{\mathcal{U}}_{l'm'lm}^{R,\alpha\beta} Q_{\beta,lm}, \quad (2.31)$$

which applies to Dirichlet as well as Neumann boundary conditions at the surfaces Σ_α.

To simplify notation, we have defined the modified translation matrix elements

$$\tilde{\mathcal{U}}^{\alpha\beta}_{l'm'lm} \equiv (-1)^{l'} \mathcal{U}^{\alpha\beta}_{l'm'lm}, \tag{2.32}$$

$$\tilde{\mathcal{U}}^{R,\alpha\alpha}_{l'mlm} \equiv (-1)^{l'} \mathcal{U}^{R,\alpha\alpha}_{l'mlm}, \tag{2.33}$$

$$\tilde{\mathcal{U}}^{R,\alpha\beta}_{l'm'lm} \equiv \tfrac{1}{2} \left[(-1)^{l'} \mathcal{U}^{R,\alpha\beta}_{l'm'lm} + (-1)^{l} \mathcal{U}^{R,\beta\alpha*}_{lml'm'} \right], \qquad (\alpha \neq \beta). \tag{2.34}$$

Here we have used the definition of equation (2.21) and $[j_l(\mathrm{i}\kappa r)]^* = (-1)^l j_l(\mathrm{i}\kappa r)$. In addition, we applied the symmetry relations $\mathcal{U}^{\alpha\beta*}_{lml'm'} = (-1)^{l+l'} \mathcal{U}^{\beta\alpha}_{l'm'lm}$ and $\mathcal{U}^{R,\alpha\alpha}_{lml'm} = (-1)^{l+l'} \mathcal{U}^{R,\alpha\alpha}_{l'mlm}$ which follow from equations (2.25) and (2.29), symmetry properties of 3-j symbols and spherical harmonics, and $h_l^{(1)}(\mathrm{i}z) = -\mathrm{i}^{-l}\sqrt{2/(\pi z)}K_{l+1/2}(z)$ where $K_{l+1/2}(z)$ is the (real valued) modified Bessel function of second kind. Notice that the actions of equations (2.30) and (2.31) couple sources on different surfaces and hide the dependence on the particular boundary conditions and shape of the surfaces.

Diagonal terms. These are the self-action terms

$$\tilde{S}_\alpha = \tfrac{1}{2} \int_{\Sigma_\alpha} \mathrm{d}\mathbf{x} \left\{ \varrho^*_\alpha(\mathbf{x})\phi_\alpha(\mathbf{x}) + \mathrm{c.c.} \right\} \tag{2.35}$$

in equation (2.14), which we have expressed here in terms of the classical field defined in equation (2.18). For Neumann boundary conditions on the surface Σ_α we have to replace again $\phi_\alpha(\mathbf{x})$ by $\partial_n\phi_\alpha(\mathbf{x})$ in equation (2.35). Here and in the following we only use the coordinate system associated with the origin inside Σ_α, and hence drop the label α on the coordinates. The classical field generated by the source $\varrho_\alpha(\mathbf{x})$ on surface Σ_α obeys the Helmholtz equation

$$(\nabla^2 - \kappa^2)\phi_\alpha(\mathbf{x}) = -\varrho_\alpha(\mathbf{x}). \tag{2.36}$$

For positions $\mathbf{x}$ that are located on the surface Σ_α the *total* field $\phi(\mathbf{x})$ generated by all sources must obey the same Helmholtz equation. The part of the total field that is generated by sources other than $\varrho_\alpha(\mathbf{x})$ can be regarded as the incident field $\phi_{0,\alpha}(\mathbf{x})$ at the surface Σ_α, which obeys in the region around Σ_α, that is free of sources other than $\varrho_\alpha(\mathbf{x})$, the homogeneous Helmholtz equation. Hence the total field can be written as

$$\begin{aligned} \phi(\mathbf{x}) &= \phi_{0,\alpha}(\mathbf{x}) + \phi_\alpha(\mathbf{x}) \\ &= \phi_{0,\alpha}(\mathbf{x}) + \int_{\Sigma_\alpha} \mathrm{d}\mathbf{x}' \, G_0(\mathbf{x}, \mathbf{x}', \kappa) \varrho_\alpha(\mathbf{x}') \end{aligned} \tag{2.37}$$

for all $\mathbf{x}$ located on Σ_α. For Neumann boundary conditions at Σ_α the Green's function in equation (2.37) is replaced again by $\partial_{n'} G_0(\mathbf{x}, \mathbf{x}', \kappa)$. We would like to evaluate the action of equation (2.35) in terms of multipole moments. Hence, we must consider field configurations with a fixed source on surface Σ_α that is characterized by its multipole moments. This implies that we have to find the incident field $\phi_{0,\alpha}(\mathbf{x})$ that induces a prescribed set of multipoles $Q_{\alpha,lm}$ on Σ_α. The multipole moments can be identified as the amplitudes of the scattered field which is given by $\phi_\alpha(\mathbf{x})$ with $\mathbf{x}$ located outside of the

surface Σ_α. Using the partial wave expansion of equation (2.19), we get

$$\phi_\alpha(\mathbf{x}) = -\kappa \sum_{l'm'} Q_{\alpha,l'm'} h_{l'}^{(1)}(\mathrm{i}\kappa r) Y_{l'm'}(\hat{\mathbf{x}}). \tag{2.38}$$

From scattering theory we know that the amplitudes of the scattered field are related to the amplitudes of the regular incident field by the transition matrix $\mathbb{T}$ which is related to the scattering matrix $\mathbb{S}$ by $\mathbb{T} = (\mathbb{S} - 1)/2$. If we expand the incident field as

$$\phi_{0,\alpha}(\mathbf{x}) = \sum_{lm} \phi_{0,\alpha,lm} j_l(\mathrm{i}\kappa r) Y_{lm}(\hat{\mathbf{x}}), \tag{2.39}$$

the amplitudes of the scattered field are given by

$$-\kappa Q_{\alpha,l'm'} = \sum_{lm} \mathcal{T}_{l'm'lm}^\alpha \, \phi_{0,\alpha,lm}, \tag{2.40}$$

where the $\mathcal{T}_{l'm'lm}^\alpha$ denote the matrix elements of the $\mathbb{T}$-matrix of the surface Σ_α. Hence, the amplitudes of the incident field have to be given by

$$\phi_{0,\alpha,lm} = -\kappa \sum_{l'm'} [\mathcal{T}^\alpha]_{lml'm'}^{-1} Q_{\alpha,l'm'}. \tag{2.41}$$

For Dirichlet boundary conditions, the total field $\phi(\mathbf{x})$ of equation (2.37) has to vanish on Σ_α so that $\phi_\alpha(\mathbf{x}) = -\phi_{0,\alpha}(\mathbf{x})$ on the surface. Hence, using equations (2.39) and (2.41), the action of equation (2.35) can be expressed in terms of the multipole moments of $\varrho_\alpha(\mathbf{x})$:

$$\tilde{S}_\alpha = \kappa \sum_{lml'm'} Q_{\alpha,lm}^* [\tilde{\mathcal{T}}^\alpha]_{lml'm'}^{-1} Q_{\alpha,l'm'} \tag{2.42}$$

with the modified $\tilde{\mathbb{T}}$-matrix defined by

$$\tilde{\mathcal{T}}_{l'm'lm}^\alpha \equiv (-1)^l \mathcal{T}_{l'm'lm}^\alpha. \tag{2.43}$$

Here we have used the definition of the multipoles in equation (2.21) to integrate over the surface and applied the relation $\mathcal{T}_{l'm'lm}^{\alpha*} = (-1)^{l+l'} \mathcal{T}_{lml'm'}^\alpha$.

For Neumann boundary conditions on Σ_α, the normal derivative of the total field of equation (2.37) has to vanish on Σ_α so that $\partial_n \phi_\alpha(\mathbf{x}) = -\partial_n \phi_{0,\alpha}(\mathbf{x})$ on the surface. When we use the definition of the multipole moments for Neumann boundary conditions, we obtain again equation (2.42) but with the matrix $\mathbb{T}^\alpha$ for Neumann boundary conditions.

Now we can combine all results for off-diagonal and diagonal terms and express the total action of equation (2.14) in terms of the original multipole moments. Since we have

$$\tilde{S} = \sum_{\alpha \neq \beta} \tilde{S}_{\alpha\beta} + \sum_\alpha \tilde{S}_\alpha + \sum_{\alpha,\beta} \tilde{S}_{\alpha\beta}^R \tag{2.44}$$

we obtain from equations (2.30), (2.31) and (2.42) the total action of the multipole moments:

$$\tilde{S}[Q] = \sum_{\alpha\beta} Q_\alpha^* \mathbb{M}^{\alpha\beta} Q_\beta, \tag{2.45}$$

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where we have suppressed the sum over the indices l, m, l', m' and defined the matrix

$$\mathbb{M}^{\alpha\beta} = \kappa \left\{ [\tilde{\mathbb{T}}^{\alpha}]^{-1}\delta_{\alpha\beta} - \tilde{\mathbb{U}}^{\alpha\beta}(1 - \delta_{\alpha\beta}) \pm \tilde{\mathbb{U}}^{R,\alpha\beta} \right\} \quad (2.46)$$

for Dirichlet $(+)$ or Neumann $(-)$ boundary conditions at the mirror plane. The partition function of equation (2.11) is then obtained by integrating over all multipole moments:

$$\mathcal{Z}(\kappa) = \mathcal{Z}_0 \prod_{\alpha=1}^{N} \int \mathrm{d}Q_\alpha \mathrm{d}Q_\alpha^* \exp\left[-\frac{\hbar c}{\Lambda} \sum_{\alpha\beta} Q_\alpha^* \mathbb{M}^{\alpha\beta} Q_\beta \right]. \quad (2.47)$$

The functional determinant that results from the change of variables from sources $\varrho_\alpha(\mathbf{x})$ to multipoles can be discarded because it is a common factor which cancels between $\mathcal{Z}(\kappa)$ and $\mathcal{Z}_\infty(\kappa)$, see equation (2.9).

The Gaussian integral over the multipoles is proportional to the inverse determinant of $\mathbb{M}^{\alpha\beta}$. Finally, we substitute into equation (2.9) to obtain the Casimir energy:

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty \mathrm{d}\kappa \ln \frac{\det \mathbb{M}}{\det \mathbb{M}_\infty}, \quad (2.48)$$

where the determinant is taken with respect to the partial wave indices l, m and the surface indices α, β. The matrix $\mathbb{M}_\infty$ is the result of moving the surfaces to infinite separation, where the translation matrices $\tilde{\mathbb{U}}$, $\tilde{\mathbb{U}}^R$ vanish so that

$$\mathbb{M}_\infty^{\alpha\beta} = \kappa [\tilde{\mathbb{T}}^{\alpha}]^{-1}\delta_{\alpha\beta}. \quad (2.49)$$

In the special case of one compact surface in front of the mirror plane equation (2.48) simplifies to

$$\mathcal{E}_1 = \frac{\hbar c}{2\pi} \int_0^\infty \mathrm{d}\kappa \ln \det \left(1 \pm \mathbb{T}^1 \mathbb{U}^{R,11} \right) \quad (2.50)$$

for Dirichlet $(+)$ or Neumann $(-)$ boundary conditions at the mirror plane. This expression applies to Dirichlet, Neumann and even more general boundary conditions at the compact surface which enter only via the matrix $\mathbb{T}^1$. Notice that this result depends on the original matrix elements (without a tilde) since the phase factors of equations (2.33) and (2.43) drop out when taking the matrix product of $\tilde{\mathbb{T}}^1$ and $\tilde{\mathbb{U}}^{R,11}$. This general result shows that the Casimir interaction between a mirror and an object with arbitrary shape and boundary condition can be obtained from the transition matrix $\mathbb{T}^1$ of the object and the translation matrix $\mathbb{U}^{R,11}$ that describes the (classical) interaction between the induced source and its mirror image.

2.2. Electromagnetic field

The derivation of the Casimir energy for a scalar field can be extended to electromagnetic field fluctuations in the presence of dielectric objects [26]. A new aspect that arises from the electromagnetic field when it couples to dielectric materials is that we cannot impose boundary conditions on the object's surfaces as in the scalar case since the field can penetrate the material objects. In macroscopic electrodynamics this is usually taken into account by imposing continuity conditions across the surfaces of the objects. Here we employ a different approach and express the interaction energy between the objects not in

terms of the field but obtain the energy from the charges and currents that fluctuate inside the material objects. The result will have the form of an effective action for electric and magnetic multipoles of the current densities $\mathbf{J}_\alpha$ inside the objects. We consider again N objects that are located in the right half-space that is bounded by a perfectly conducting plane at $z = 0$. At his plane the tangential electric field and the normal magnetic field vanish, $\mathbf{E}_\parallel = 0$, $\mathbf{B}_\perp = 0$. The assumption of a perfectly conductivity mirror plane is justified if the object's separations from the plane are sufficiently large compared to the plasma wavelength of a metallic mirror. For shorter separations the dielectric properties of the mirror plane have to taken into account. This can be done by similar methods as those presented here but the method of images can no longer be applied. We leave this problem for a future publication. However, the material objects in front of the mirror are assumed to be dielectrics that are characterized by a frequency-dependent dielectric function $\epsilon_\alpha(\omega)$ and permeability function $\mu_\alpha(\omega)$.

The partition function can be factorized again into a product of factors $\mathcal{Z}(\kappa)$ at a fixed Wick rotated frequency κ. Hence, in the following we consider all expressions at fixed κ and suppress the label κ. The Euclidean action for the electromagnetic field in the presence of macroscopic media without external sources can be expressed as

$$S_0 = \tfrac{1}{2} \int_> d\mathbf{x}\, [\mathbf{E}\mathbf{D}^* + \mathbf{B}\mathbf{H}^* + \text{c.c.}] \tag{2.51}$$

in terms of the macroscopic fields $\mathbf{D} = \epsilon(i\kappa)\mathbf{E}$, $\mathbf{H} = \mu^{-1}(i\kappa)\mathbf{B}$. Here the *energy* density of the field is integrated since under a Wick rotation to imaginary time the Lagrangian in real time is generally transformed to the Hamiltonian in imaginary time. In this description the induced (bound) currents inside the material objects have been absorbed into the definition of the macroscopic fields. The partition function for this action is given by a functional integral over the vector potential $\mathbf{A}$ and the scalar potential Φ (after introducing a Faddeev–Popov gauge fixing term) where the fields are expressed in terms of the potentials as

$$\mathbf{E} = i\kappa\mathbf{A} - \nabla\Phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.52}$$

In order to express the interaction energy in terms of the fluctuating currents inside the objects, we consider an alternative description in terms of the fields $\mathbf{E}$, $\mathbf{B}$ only which is obtained if the bound charges (ρ) and currents ($\mathbf{J}$) density inside the objects are not substituted by the macroscopic fields but considered *explicitly*. Then the Wick rotated action can be written in terms of the potentials as

$$S[\mathbf{J}, \rho] = \int_> d\mathbf{x} \left[\kappa^2(|\mathbf{A}|^2 + |\Phi|^2) + \sum_{j=1}^{3} |\nabla A_j|^2 + |\nabla\Phi|^2 + (\mathbf{A}\mathbf{J}^* + \Phi\rho^* + \text{c.c.}) \right],$$

$$\tag{2.53}$$

where we have chosen the Feynman gauge[1].

The strategy for computing the partition function for the action of equation (2.53) is as follows. The goal is to obtain an effective action for the multipoles of the currents inside the objects. This action will consists of two qualitatively different contributions: one that

---

[1] Choosing a different gauge would change the photon propagator $\mathcal{G}_>$ that couples the currents, see equation (2.54), but would not change the gauge-invariant interaction energy.

describes the 'self-energies' of the separate objects in terms of their inverse T-matrix, and a contribution that accounts for the interaction of the multipoles inside different objects. The effective action is constructed by first integrating over the unconstrained potentials $\mathbf{A}$, Φ in the right half-space. This will yield an action for the sources $\mathbf{J}$, ρ. The sources are then expressed in terms of their multipoles over which we integrate finally to obtain the partition function. However, in the latter integration the multipoles must be weighted according to the energy cost for inducing them on the objects. This energy cost must depend on the shape and material of the objects. We will see below that the inverse T-matrices in the self-energies of the objects account for this energy cost.

We proceed by integrating out the unconstrained fluctuations of the potentials $\mathbf{A}$, Φ in the right half-space. This integration yields the partition function as a *weighted* functional integral over sources which we indicate at this stage by a subscript w on the integration variable. The boundary conditions for the field on the mirror plane are taken into account by constructing the half-space Green's function by the methods of images. When we denote the current density in the interior D_α of object α by $\mathbf{J}_\alpha$, we get

$$
\mathcal{Z}(\kappa) = \mathcal{Z}_0 \prod_{\alpha=1}^{\mathrm{N}} \int [\mathcal{D}\mathbf{J}_\alpha]_w [\mathcal{D}\mathbf{J}_\alpha^*]_w \exp\left[-\tfrac{1}{2} \sum_{\alpha\beta} \int_{D_\alpha} \mathrm{d}\mathbf{x} \int_{D_\beta} \mathrm{d}\mathbf{x}' \{ \mathbf{J}_\alpha^*(\mathbf{x}) \mathcal{G}_>(\mathbf{x}, \mathbf{x}', \kappa) \mathbf{J}_\beta(\mathbf{x}') + \mathrm{c.c.} \} \right]
$$

$$
= \mathcal{Z}_0 \prod_{\alpha=1}^{\mathrm{N}} \int [\mathcal{D}\mathbf{J}_\alpha]_w [\mathcal{D}\mathbf{J}_\alpha^*]_w \, \mathrm{e}^{-\tilde{S}[\mathbf{J}]}, \tag{2.54}
$$

where we have used the continuity equation $\nabla \mathbf{J} = -\mathrm{i}\kappa\rho$ to eliminate the charge density by introducing the tensor Green's function for the half-space

$$
\mathcal{G}_>(\mathbf{x}, \mathbf{x}', \kappa) = G_0(\mathbf{x}, \mathbf{x}', \kappa)\mathbf{I} - G_0(\mathbf{x}, \mathbf{x}'_R, \kappa) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

$$
+ \frac{1}{\kappa^2} \nabla \otimes \nabla' \left[G_0(\mathbf{x}, \mathbf{x}', \kappa) - G_0(\mathbf{x}, \mathbf{x}'_R, \kappa) \right], \tag{2.55}
$$

where the terms depending on $\mathbf{x}'_R$ result from the mirror currents in the left half-space. Here $G_0(\mathbf{x}, \mathbf{x}', \kappa)$ is the free scalar Green's function of equation (2.13), $\mathbf{I}$ is the identity matrix and $\mathbf{x}_R$ is defined below equation (2.12). The action of equation (2.54) is independent of the material and shape of the objects. As explained before, these properties enter the partition function through proper weights on the currents that measure the susceptibility of the objects to current fluctuations. Below we will show that in the evaluation of $\mathcal{Z}(\kappa)$ the weights can be taken into account by interpreting the diagonal terms with $\alpha = \beta$ as the energies that are associated with inducing the currents $\mathbf{J}_\alpha$ inside the objects. These energies can also be expressed as the change in total field energy when an object is placed into the external field that has to act on the object to generate the prescribed current density. The relation between that incident field and the multipole moments of the current density is provided by the T-matrix.

The action of equation (2.54) can be expressed in terms of the original current densities $\mathbf{J}_\alpha$ and the mirror current densities. Their components parallel and perpendicular to the mirror plane are

$$
\mathbf{J}_{\alpha,\parallel}^R(\mathbf{x}) = -\mathbf{J}_{\alpha,\parallel}(\mathbf{x}_R), \qquad J_{\alpha,z}^R(\mathbf{x}) = J_{\alpha,z}(\mathbf{x}_R). \tag{2.56}
$$

The action is then

$$\tilde{S}[\mathbf{J}] = \tfrac{1}{2} \sum_{\alpha\beta} \Bigg\{ \int_{D_\alpha} \mathrm{d}\mathbf{x} \int_{D_\beta} \mathrm{d}\mathbf{x}' \mathbf{J}_\alpha^*(\mathbf{x}) \mathcal{G}_0(\mathbf{x}, \mathbf{x}', \kappa) \mathbf{J}_\beta(\mathbf{x}')$$

$$+ \int_{D_\alpha} \mathrm{d}\mathbf{x} \int_{D_\beta^R} \mathrm{d}\mathbf{x}' \mathbf{J}_\alpha^*(\mathbf{x}) \mathcal{G}_0(\mathbf{x}, \mathbf{x}', \kappa) \mathbf{J}_\beta^R(\mathbf{x}') + \mathrm{c.c.} \Bigg\}. \tag{2.57}$$

Now the sources are coupled by the free, infinite space Green's function

$$\mathcal{G}_0(\mathbf{x}, \mathbf{x}', \kappa) = G_0(\mathbf{x}, \mathbf{x}', \kappa) + \frac{1}{\kappa^2} \nabla \otimes \nabla' G_0(\mathbf{x}, \mathbf{x}', \kappa). \tag{2.58}$$

The mirror currents are located on the mirror objects D_α^R that are obtained from the original objects by reflection at the mirror plane. This action has the same structure as in the case of scalar fields. It can be expressed in terms of multipole moments of the current densities very similarly to the scalar case. Again, we consider diagonal and off-diagonal terms separately.

Off-diagonal terms. We introduce again local coordinates $\mathbf{x}_\alpha$ that are measured relative to an origin inside D_α. The terms that couple the original sources on different objects can be written as

$$\tilde{S}_{\alpha\beta} = \frac{1}{2\mathrm{i}\kappa} \int_{D_\alpha} \mathrm{d}\mathbf{x}_\alpha \mathbf{J}_\alpha^*(\mathbf{x}_\alpha) \mathbf{E}_\beta(\mathbf{x}_\alpha) + \mathrm{c.c.} \tag{2.59}$$

for $\alpha \neq \beta$ and the terms involving mirror sources become

$$\tilde{S}_{\alpha\beta}^R = \frac{1}{2\mathrm{i}\kappa} \int_{D_\alpha} \mathrm{d}\mathbf{x}_\alpha \mathbf{J}_\alpha^*(\mathbf{x}_\alpha) \mathbf{E}_\beta^R(\mathbf{x}_\alpha) + \mathrm{c.c.} \tag{2.60}$$

for all α, β. Here we have introduced the electric fields

$$\mathbf{E}_\beta(\mathbf{x}) = \mathrm{i}\kappa \int_{D_\beta} \mathrm{d}\mathbf{x}' \mathcal{G}_0(\mathbf{x}, \mathbf{x}', \kappa) \mathbf{J}_\beta(\mathbf{x}'), \tag{2.61}$$

$$\mathbf{E}_\beta^R(\mathbf{x}) = \mathrm{i}\kappa \int_{D_\beta^R} \mathrm{d}\mathbf{x}' \mathcal{G}_0(\mathbf{x}, \mathbf{x}', \kappa) \mathbf{J}_\beta^R(\mathbf{x}'), \tag{2.62}$$

which are generated by the current densities. For positions $\mathbf{x}_\beta$ that are located outside a sphere that encloses the object D_β, the electric field can be expressed in terms of the electric and magnetic multipole moments of the current density $\mathbf{J}_\beta$ for $l \geq 1$, $|m| \leq l$:

$$Q_{\mathrm{M},lm}^\beta = \int \mathrm{d}\mathbf{x}_\beta \, \mathbf{J}_\beta(\mathbf{x}_\alpha)(-1)^l \mathbf{M}_{lm}^{j*}(\mathbf{x}_\beta), \tag{2.63}$$

$$Q_{\mathrm{E},lm}^\beta = \int \mathrm{d}\mathbf{x}_\beta \, \mathbf{J}_\beta(\mathbf{x}_\alpha)(-1)^{l+1} \mathbf{N}_{lm}^{j*}(\mathbf{x}_\beta), \tag{2.64}$$

where $\mathbf{M}_{lm}^j(\mathbf{x}_\alpha)$, $\mathbf{N}_{lm}^j(\mathbf{x}_\alpha)$ are the regular, divergence-less solutions:

$$\mathbf{M}_{lm}^j(\mathbf{x}) = \frac{1}{\lambda} \nabla \times [\mathbf{x} j_l(\mathrm{i}\kappa r) Y_{lm}(\hat{\mathbf{x}})], \tag{2.65}$$

$$\mathbf{N}_{lm}^j(\mathbf{x}) = \frac{1}{\lambda} \frac{1}{\mathrm{i}\kappa} \nabla \times \nabla \times [\mathbf{x} j_l(\mathrm{i}\kappa r) Y_{lm}(\hat{\mathbf{x}})] \tag{2.66}$$

of the vector Helmholtz equation and $\lambda = \sqrt{l(l+1)}$. The electric fields of the original sources can then be written as

$$\mathbf{E}_\beta(\mathbf{x}_\beta) = -\mathrm{i}\kappa^2 \sum_{lm} \left[Q_{\mathrm{M},lm}^\beta \mathbf{M}_{lm}^h(\mathbf{x}_\beta) + Q_{\mathrm{E},lm}^\beta \mathbf{N}_{lm}^h(\mathbf{x}_\beta) \right], \qquad (2.67)$$

which is an expansion in the outgoing solutions

$$\mathbf{M}_{lm}^h(\mathbf{x}) = \frac{1}{\lambda} \nabla \times [\mathbf{x} h_l^{(1)}(\mathrm{i}\kappa r) Y_{lm}(\hat{\mathbf{x}})], \qquad (2.68)$$

$$\mathbf{N}_{lm}^h(\mathbf{x}) = \frac{1}{\lambda} \frac{1}{\mathrm{i}\kappa} \nabla \times \nabla \times [\mathbf{x} h_l^{(1)}(\mathrm{i}\kappa r) Y_{lm}(\hat{\mathbf{x}})] \qquad (2.69)$$

of the vector Helmholtz equation. The electric fields of the mirror sources can also be expressed in terms of the multipole moments of the original sources:

$$\mathbf{E}_\beta^R(\mathbf{x}_\beta^R) = -\mathrm{i}\kappa^2 \sum_{lm} \left[Q_{\mathrm{M},lm}^{\beta,R} \mathbf{M}_{lm}^h(\mathbf{x}_\beta^R) + Q_{\mathrm{E},lm}^{\beta,R} \mathbf{N}_{lm}^h(\mathbf{x}_\beta^R) \right], \qquad (2.70)$$

where $\mathbf{x}_\beta^R$ denotes the local coordinates of the mirror object D_β^R. Using equation (2.56), the definition of the vector solutions of equation (2.65) and $Y_{lm}(\hat{\mathbf{x}}_R) = (-1)^{l+m} Y_{lm}(\hat{\mathbf{x}})$, the moments of the mirror currents are given by

$$Q_{\mathrm{M},lm}^{\beta,R} = (-1)^{l+m} Q_{\mathrm{M},lm}^\beta, \qquad (2.71)$$

$$Q_{\mathrm{E},lm}^{\beta,R} = -(-1)^{l+m} Q_{\mathrm{E},lm}^\beta. \qquad (2.72)$$

The expansion of the electric fields into outgoing vector waves with respect to the origin of the object that generates the field does not allow us to perform the integrations in equations (2.59) and (2.60). We would like to expand the electric field generated by object D_β in terms of vector waves that are regular at the origin of object D_α. When the coordinate systems associated with the two objects have identical orientation, this can be done by relating the outgoing and regular vector waves by a translation matrix $\mathbb{U}$. Generalizing the result for scalar fields, the translation matrix couples both types of vector solutions:

$$\mathbf{M}_{lm}^h(\mathbf{x}_\beta) = \sum_{l'm'} \left\{ B_{l'm'lm}(\mathbf{X}_{\alpha\beta}) \mathbf{M}_{l'm'}^j(\mathbf{x}_\alpha) + C_{l'm'lm}(\mathbf{X}_{\alpha\beta}) \mathbf{N}_{l'm'}^j(\mathbf{x}_\alpha) \right\}, \quad (2.73)$$

$$\mathbf{N}_{lm}^h(\mathbf{x}_\beta) = \sum_{l'm'} \left\{ B_{l'm'lm}(\mathbf{X}_{\alpha\beta}) \mathbf{N}_{l'm'}^j(\mathbf{x}_\alpha) + C_{l'm'lm}(\mathbf{X}_{\alpha\beta}) \mathbf{M}_{l'm'}^j(\mathbf{x}_\alpha) \right\} \quad (2.74)$$

with $\mathbf{X}_{\alpha\beta} = \mathbf{x}_\alpha - \mathbf{x}_\beta$. The matrix elements are given by [35]

$$B_{l'm'lm}(\mathbf{X}_{\alpha\beta}) = (-1)^m i^{l-l'} \sum_{l''} \frac{i^{l''}}{2} [l(l+1) + l'(l'+1) - l''(l''+1)]$$

$$\times \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{l(l+1)l'(l'+1)}} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m' & m'-m \end{pmatrix}$$

$$\times h_{l''}^{(1)}(\mathrm{i}\kappa|\mathbf{X}_{\alpha\beta}|) Y_{l''m-m'}(\hat{\mathbf{X}}_{\alpha\beta}), \qquad (2.75)$$

$$C_{l'm'lm}(\mathbf{X}_{\alpha\beta}) = \frac{\kappa}{\sqrt{l(l+1)l'(l'+1)}}\mathbf{X}_{\alpha\beta} \cdot \left[\hat{\mathbf{x}}\frac{1}{2}\left(\lambda_{lm}^{+}A_{l'm'lm+1}(\mathbf{X}_{\alpha\beta}) + \lambda_{lm}^{-}A_{l'm'lm-1}(\mathbf{X}_{\alpha\beta})\right)\right.$$
$$+ \hat{\mathbf{y}}\frac{1}{2\mathrm{i}}\left(\lambda_{lm}^{+}A_{l'm'lm+1}(\mathbf{X}_{\alpha\beta}) - \lambda_{lm}^{-}A_{l'm'lm-1}(\mathbf{X}_{\alpha\beta})\right)$$
$$\left.+ \hat{\mathbf{z}}\, m\, A_{l'm'lm}(\mathbf{X}_{\alpha\beta})\right] \tag{2.76}$$

with

$$A_{l'm'lm}(\mathbf{X}_{\alpha\beta}) = (-1)^{m}i^{l-l'}\sum_{l''}i^{l''}\sqrt{(2l+1)(2l'+1)(2l''+1)}$$
$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & -m' & m'-m \end{pmatrix}h_{l''}^{(1)}(\mathrm{i}\kappa|\mathbf{X}_{\alpha\beta}|)Y_{l''m-m'}(\hat{\mathbf{X}}_{\alpha\beta}) \tag{2.77}$$

and $\lambda_{lm}^{\pm} = \sqrt{(l\mp m)(l\pm m+1)}$. It is useful to combine the translation matrix elements with the matrix

$$\mathcal{U}_{l'm'lm}^{\alpha\beta} \equiv \begin{pmatrix} B_{l'm'lm} & C_{l'm'lm} \\ -C_{l'm'lm} & -B_{l'm'lm} \end{pmatrix}(\mathbf{X}_{\alpha\beta}) \tag{2.78}$$

that acts on the vector $\mathbf{Q}_{lm}^{\alpha} = (Q_{\mathrm{M},lm}^{\alpha}, Q_{\mathrm{E},lm}^{\alpha})$ of multipole moments.

Using the translation formula of equations (2.73) and (2.74), we can express the field $\mathbf{E}_{\beta}(\mathbf{x}_{\alpha})$ in equation (2.59) in terms of regular vector solutions and perform the integration. This integration yields the multipole moments defined by equations (2.64) and (2.63). The action of equation (2.59) can then be written as

$$\sum_{\alpha\neq\beta}\tilde{S}_{\alpha\beta} = -\kappa\sum_{\alpha\neq\beta}\sum_{lml'm'}\mathbf{Q}_{l'm'}^{\alpha*}\tilde{\mathcal{U}}_{l'm'lm}^{\alpha\beta}\mathbf{Q}_{lm}^{\beta}, \tag{2.79}$$

where we have defined the modified translation matrix

$$\tilde{\mathcal{U}}_{l'm'lm}^{\alpha\beta} \equiv \tfrac{1}{2}\left[(-1)^{l'}\mathcal{U}_{l'm'lm}^{\alpha\beta} + (-1)^{l}\mathcal{U}_{lml'm'}^{\beta\alpha\dagger}\right], \tag{2.80}$$

where $\dagger$ denotes the conjugate transpose of the matrix of equation (2.78). For translations along the z axis, $\mathbf{X}_{\alpha\beta} \sim \hat{\mathbf{z}}$, the latter expression is diagonal in m and simplifies to $\tilde{\mathcal{U}}_{l'mlm}^{\alpha\beta} = (-1)^{l'}\mathcal{U}_{l'mlm}^{\alpha\beta}$ due to the symmetries $B_{lml'm}^{*}(\mathbf{X}_{\beta\alpha}) = (-1)^{l+l'}B_{l'mlm}(\mathbf{X}_{\alpha\beta})$ and $C_{lml'm}^{*}(\mathbf{X}_{\beta\alpha}) = -(-1)^{l+l'}C_{l'mlm}(\mathbf{X}_{\alpha\beta})$.

The action of equation (2.60) that couples the original and mirror sources can be cast into a form similar to equation (2.79) with a translation matrix $\mathbb{U}^{R}$ that is defined by

$$\mathcal{U}_{l'm'lm}^{R,\alpha\beta} = (-1)^{l+m}\begin{pmatrix} B_{l'm'lm} & -C_{l'm'lm} \\ -C_{l'm'lm} & B_{l'm'lm} \end{pmatrix}(\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}^{R}), \tag{2.81}$$

where the phase factors relating the original and mirror multipole moments in equations (2.71) and (2.72) have been absorbed in the translation matrix. With this definition equation (2.70) becomes

$$\sum_{\alpha\beta}\tilde{S}_{\alpha\beta}^{R} = \kappa\sum_{\alpha\beta}\sum_{lml'm'}\mathbf{Q}_{l'm'}^{\alpha*}\tilde{\mathcal{U}}_{l'm'lm}^{R,\alpha\beta}\mathbf{Q}_{lm}^{\beta} \tag{2.82}$$

with the modified translation matrix

$$\tilde{\mathcal{U}}_{l'm'lm}^{R,\alpha\beta} \equiv \tfrac{1}{2} \left[(-1)^{l'} \mathcal{U}_{l'm'lm}^{R,\alpha\beta} + (-1)^{l} \mathcal{U}_{lml'm'}^{R,\beta\alpha\dagger} \right]. \tag{2.83}$$

For $\alpha = \beta$, the action of equation (2.82) describes the interaction between a source and its mirror image. In this case the translation vector is $\mathbf{x}_\alpha - \mathbf{x}_\alpha^R = -2L_\alpha \hat{\mathbf{z}}$ where L_α is the normal distance between the origin of object D_α and the mirror plane. Hence, the translation is along the z axis and the translation matrix elements of equations (2.75) and (2.76) simplify to

$$B_{l'm'lm}(-2L_\alpha\hat{\mathbf{z}}) = \delta_{m'm}(-1)^m i^{l-l'} \sum_{l''} \frac{i^{-l''}}{2} \left[l(l+1) + l'(l'+1) - l''(l''+1) \right]$$

$$\times \sqrt{\frac{(2l+1)(2l'+1)}{l(l+1)l'(l'+1)}} (2l''+1) \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} h_{l''}^{(1)}(i\kappa 2L_\alpha), \tag{2.84}$$

$$C_{l'm'lm}(-2L_\alpha\hat{\mathbf{z}}) = -\delta_{m'm}\kappa 2L_\alpha m(-1)^m i^{l-l'} \sum_{l''} i^{-l''} \sqrt{\frac{(2l+1)(2l'+1)}{l(l+1)l'(l'+1)}} (2l''+1)$$

$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} h_{l''}^{(1)}(i\kappa 2L_\alpha). \tag{2.85}$$

These matrix elements obey the symmetry relations $B_{l'mlm}^*(-2L_\alpha\hat{\mathbf{z}}) = B_{lml'm}(-2L_\alpha\hat{\mathbf{z}})$, $C_{l'mlm}^*(-2L_\alpha\hat{\mathbf{z}}) = C_{lml'm}(-2L_\alpha\hat{\mathbf{z}})$ so that the matrix of equation (2.83) simplifies for $\alpha = \beta$ to $\tilde{\mathcal{U}}_{l'mlm}^{R,\alpha\alpha} = (-1)^{l'} \mathcal{U}_{l'mlm}^{R,\alpha\alpha}$.

Diagonal terms. So far we have described the interaction between *arbitrary* multipoles inside the material objects. But the functional integral over currents of equation (2.54) contains weights that measure the energy cost for inducing currents on an object with particular shape and material composition. In the following we will show that these weights can be implemented straightforwardly when we express the diagonal terms of the action of equation (2.57):

$$\tilde{S}_\alpha = \tfrac{1}{2} \int_{D_\alpha} d\mathbf{x} \int_{D_\alpha} d\mathbf{x}' \left\{ \mathbf{J}_\alpha^*(\mathbf{x}) \mathcal{G}_0(\mathbf{x}, \mathbf{x}', \kappa) \mathbf{J}_\alpha(\mathbf{x}') + \text{c.c.} \right\}, \tag{2.86}$$

in terms of the incident field which generates a prescribed current that corresponds to the induced polarization and magnetization inside the object. Then the relation between the induced current and the incident field ensures that the currents are weighted properly. To see this we employ the macroscopic formulation of the electromagnetic action, see equation (2.51). The energy of equation (2.86) is associated with the process of building up the induced current $\mathbf{J}_\alpha$ inside the object. This energy must therefore be equal to the change in the total macroscopic field energy that results when the object is placed into an external field. Thus we obtain

$$\tilde{S}_\alpha = \tfrac{1}{2} \int d\mathbf{x} \left[\mathbf{E}\mathbf{D}^* + \mathbf{B}\mathbf{H}^* - (\mathbf{E}_0\mathbf{D}_0^* + \mathbf{B}_0\mathbf{H}_0^*) + \text{c.c.} \right], \tag{2.87}$$

where integration extends over all space. Field vectors with subscript 0 represent the incident field that is generated by some fixed external sources in otherwise empty space

and field vectors without labels stand for the total field from the external and induced sources after adding the object. $\tilde{S}_\alpha$ can also be written as

$$\tilde{S}_\alpha = \tfrac{1}{2} \int d\mathbf{x} \left[\mathbf{E}\mathbf{D}_0^* - \mathbf{D}^*\mathbf{E}_0 + \mathbf{B}\mathbf{H}_0^* - \mathbf{H}^*\mathbf{B}_0 + \text{c.c.}\right] + \tfrac{1}{2} \int d\mathbf{x}[(\mathbf{E} + \mathbf{E}_0)(\mathbf{D}^* - \mathbf{D}_0^*)$$
$$+ (\mathbf{B} + \mathbf{B}_0)(\mathbf{H}^* - \mathbf{H}_0^*) + \text{c.c.}]. \tag{2.88}$$

The second integral of this expression vanishes. This can be seen by setting $\mathbf{E} + \mathbf{E}_0 = i\kappa\mathbf{A} - \nabla\Phi$, $\mathbf{B} + \mathbf{B}_0 = \nabla \times \mathbf{A}$ and using that $\nabla\mathbf{D}^* = \rho_{\text{ext}}^* = \nabla\mathbf{D}_0^*$, $\nabla \times \mathbf{H}^* + i\kappa\mathbf{D}^* = \mathbf{J}_{\text{ext}}^* = \nabla \times \mathbf{H}_0^* + i\kappa\mathbf{D}_0^*$ since the external sources ρ_{ext}, $\mathbf{J}_{\text{ext}}$ are fixed when the object is added. Notice that the bound (induced) sources do not appear explicitly since they are included in the macroscopic fields. Since $\mathbf{D}_0 = \mathbf{E}_0$, $\mathbf{H}_0 = \mathbf{B}_0$ and $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{H} = \mu^{-1}\mathbf{B}$ with $\epsilon = \epsilon_\alpha$, $\mu = \mu_\alpha$ inside the objects and $\epsilon = 1$, $\mu = 1$ outside the objects, we get

$$\tilde{S}_\alpha = \tfrac{1}{2} \int_{D_\alpha} d\mathbf{x} \left[(1 - \epsilon_\alpha)\mathbf{E}\mathbf{E}_0^* + (1 - \mu_\alpha^{-1})\mathbf{B}\mathbf{B}_0^* + \text{c.c.}\right], \tag{2.89}$$

where integration runs only over the interior of the object. The material-dependent functions ϵ_α and μ_α can be expressed in terms of the polarization $\mathbf{P}_\alpha$ and magnetization $\mathbf{M}_\alpha$ of the object. The relations between macroscopic fields yield $\mathbf{P}_\alpha = \mathbf{D} - \mathbf{E} = (\epsilon_\alpha - 1)\mathbf{E}$ and $\mathbf{M}_\alpha = \mathbf{B} - \mathbf{H} = (1 - 1/\mu_\alpha)\mathbf{B}$. When we substitute $\mathbf{B}_0^* = i/\kappa\nabla \times \mathbf{E}_0^*$ in equation (2.89) we can integrate by parts to obtain

$$\tilde{S}_\alpha = -\frac{1}{2} \int_{D_\alpha} d\mathbf{x} \left[\frac{1}{i\kappa}\mathbf{J}_\alpha(\mathbf{x})\mathbf{E}_0^*(\mathbf{x}) + \text{c.c.}\right], \tag{2.90}$$

where we have combined the polarization and magnetization to yield the induced current density:

$$\mathbf{J}_\alpha = i\kappa\mathbf{P}_\alpha + \nabla \times \mathbf{M}_\alpha. \tag{2.91}$$

Notice that the incident field $\mathbf{E}_0$ in equation (2.90) depends on the current density $\mathbf{J}_\alpha$ since $\mathbf{E}_0$ has to induce the prescribed current density. Hence, the problem of expressing the diagonal part of the action in terms of multipole moments has been reduced to computing the incident field that has to act on the object to generate a given current density. In scattering theory one usually encounters the opposite problem. For an incident field one would like to compute the scattered field which can be expanded in outgoing partial vector waves, see equation (2.67). Here the situation is slightly different. We seek to determine the incident field that generates a given set of multipole moments inside the object. In other words, for a given scattered field, which, according to equation (2.67), is given by the multipole moments, we would like to obtain the corresponding incident field. We expand the incident field as

$$\mathbf{E}_0(\mathbf{x}) = -i\kappa^2 \sum_{lm} \left[\phi_{\text{M},lm}\mathbf{M}_{lm}^j(\mathbf{x}) + \phi_{\text{E},lm}\mathbf{N}_{lm}^j(\mathbf{x})\right]. \tag{2.92}$$

The relation between the multipole moments and the amplitudes of the incident field is determined by the T-matrix $\mathbb{T}^\alpha = (\mathbb{S}^\alpha - 1)/2$, where $\mathbb{S}^\alpha$ is the scattering matrix of the object. When we solve this relation for the incident field amplitudes $\phi_{lm} = (\phi_{\text{M},lm}, \phi_{\text{E},lm})$,

we get

$$\phi_{lm} = \sum_{l'm'} [\mathcal{T}^\alpha]^{-1}_{lml'm'} \mathbf{Q}^\alpha_{l'm'}, \tag{2.93}$$

where $\mathcal{T}^\alpha_{lml'm'}$ is a 2×2 matrix acting on magnetic and electric multipoles. This relation together with equation (2.92) yields the incident field that generates the multipoles $\mathbf{Q}^\alpha_{lm}$. With this result we can perform the integration in equation (2.90) which yields $\tilde{S}_\alpha$ in terms of the multipoles:

$$\tilde{S}_\alpha = \kappa \sum_{lml'm'} \mathbf{Q}^{\alpha*}_{lm} \tfrac{1}{2} \left\{ [\tilde{\mathcal{T}}^\alpha]^{-1}_{lml'm'} + \left[[\tilde{\mathcal{T}}^\alpha]^{-1} \right]^\dagger_{l'm'lm} \right\} \mathbf{Q}^\alpha_{l'm'} \tag{2.94}$$

with the modified T-matrix

$$\tilde{\mathcal{T}}^\alpha_{lml'm'} = (-1)^{l'} \begin{pmatrix} -T^{\alpha,\mathrm{MM}}_{lml'm'} & T^{\alpha,\mathrm{ME}}_{lml'm'} \\ -T^{\alpha,\mathrm{EM}}_{lml'm'} & T^{\alpha,\mathrm{EE}}_{lml'm'} \end{pmatrix}, \tag{2.95}$$

where the $T^{\alpha,XY}_{lml'm'}$ are the elements of the matrix $\mathcal{T}^\alpha_{lml'm'}$ with X, $Y = \mathrm{M}$, E labeling magnetic and electric elements, respectively. Due to the symmetry

$$\begin{pmatrix} T^{\alpha,\mathrm{MM}}_{lml'm'} & T^{\alpha,\mathrm{ME}}_{lml'm'} \\ T^{\alpha,\mathrm{EM}}_{lml'm'} & T^{\alpha,\mathrm{EE}}_{lml'm'} \end{pmatrix}^* = (-1)^{l+l'} \begin{pmatrix} T^{\alpha,\mathrm{MM}}_{l'm'lm} & -T^{\alpha,\mathrm{EM}}_{l'm'lm} \\ -T^{\alpha,\mathrm{ME}}_{l'm'lm} & T^{\alpha,\mathrm{EE}}_{l'm'lm} \end{pmatrix} \tag{2.96}$$

of the T-matrix equation (2.94) can be simplified to

$$\tilde{S}_\alpha = \kappa \sum_{lml'm'} \mathbf{Q}^{\alpha*}_{lm} [\tilde{\mathcal{T}}^\alpha]^{-1}_{lml'm'} \mathbf{Q}^\alpha_{l'm'}. \tag{2.97}$$

The total action $\tilde{S}[\mathbf{Q}]$ for the multipole moments is given by the sum of the actions of equations (2.79), (2.82) and (2.97). In compact matrix notation it can be written as

$$\tilde{S}[\mathbf{Q}] = \sum_{\alpha\beta} \mathbf{Q}^*_\alpha \mathbb{M}^{\alpha\beta} \mathbf{Q}_\beta, \tag{2.98}$$

with the matrix

$$\mathbb{M}^{\alpha\beta} = \kappa \left\{ [\tilde{\mathbb{T}}^\alpha]^{-1} \delta_{\alpha\beta} - \tilde{\mathbb{U}}^{\alpha\beta} (1 - \delta_{\alpha\beta}) + \tilde{\mathbb{U}}^{R,\alpha\beta} \right\}, \tag{2.99}$$

where $\tilde{\mathbb{T}}^\alpha$, $\tilde{\mathbb{U}}^{\alpha\beta}$ and $\tilde{\mathbb{U}}^{R,\alpha\beta}$ stand for the matrices with matrix elements given by equations (2.95), (2.80) and (2.83), respectively. In analogy to the scalar case, equation (2.47), the partition function $\mathcal{Z}(\kappa)$ is obtained by integrating over all multipoles. Notice that by construction of the diagonal parts of the action of equation (2.98) the proper weights are assigned to the multipole configurations.

The Gaussian integral over multipoles and equation (2.9) lead to the final result

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty \mathrm{d}\kappa \ln \frac{\det \mathbb{M}}{\det \mathbb{M}_\infty} \tag{2.100}$$

for the electromagnetic Casimir energy where the determinant is taken with respect to the partial wave indices l, m, polarization indices M, E and the object indices α, β. The
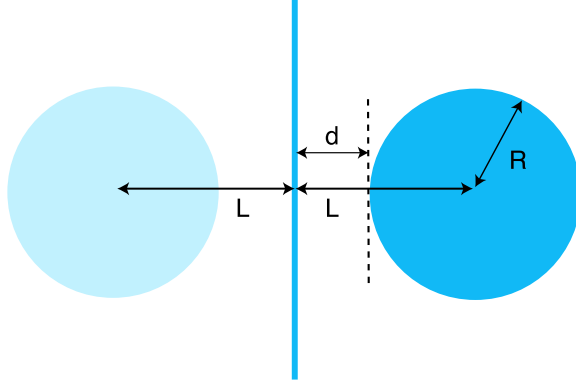
Figure 2. Length scales of the sphere–mirror geometry.

matrix $\mathbb{M}_\infty$ is given by equation (2.99) with the translation matrices $\tilde{\mathbb{U}}^{\alpha\beta}$ and $\tilde{\mathbb{U}}^{R,\alpha\beta}$ set to zero.

For one object in front of a perfectly reflecting mirror plane, the Casimir energy can be written as

$$\mathcal{E}_1 = \frac{\hbar c}{2\pi} \int_0^\infty \mathrm{d}\kappa \ln\det\left(1 + \tilde{\mathbb{T}}^1 \tilde{\mathbb{U}}^{R,11}\right). \tag{2.101}$$

The matrix $\tilde{\mathbb{U}}^{R,11}$ that describes the interaction between fluctuating currents on the object and its mirror image is universal, and the shape and material composition of the object enters through its T-matrix $\tilde{\mathbb{T}}^1$.

3. Plate-sphere geometry

In this section we consider a geometry that is most relevant to a large number of experimental studies of Casimir interactions carried out in the last decade. This geometry, consisting of a plane mirror and a sphere, is experimentally favorable since it avoids the problem of parallelism for plane surfaces facing each other. Despite its experimental importance, this geometry lacks a theoretical description of the electromagnetic Casimir interaction. For a scalar field with Dirichlet boundary conditions at the sphere and the plane, the interaction over a wide range of distances has been obtained recently [29, 30], including an analytic expression for the lowest-order correction to the proximity force approximation [38].

We consider a sphere of radius R in front of a plane mirror with distance L between the center of the sphere and the mirror, see figure 2. For the scalar field we study Dirichlet or Neumann boundary conditions at the mirror and the sphere. For the electromagnetic field, the sphere consists of a material with arbitrary dielectric function $\epsilon(k)$ and permeability $\mu(k)$ while the mirror is assumed to be perfectly reflecting (implying vanishing of the parallel components of the electric field and the normal component of the magnetic field).

3.1. Large-distance expansion

We start by studying the Casimir interaction at large separations. For separations L that are large compared to the size R of the object, the Casimir energy can be expressed as an

asymptotic series in R/L. Using that $\ln \det = \operatorname{Tr} \ln$ in equations (2.50) and (2.101), we get by expanding the logarithm

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty \mathrm{d}\kappa \, \operatorname{Tr} \ln(1 - \mathbb{N}) \tag{3.1}$$

$$= -\frac{\hbar c}{2\pi} \int_0^\infty \mathrm{d}\kappa \sum_{p=1}^\infty \frac{1}{p} \operatorname{Tr}(\mathbb{N}^p), \tag{3.2}$$

where we have defined the operator $\mathbb{N} = \mp \mathbb{T}^1 \mathbb{U}^{R,11}$ with $-$ ($+$) for a scalar field in the presence of a Dirichlet (Neumann) mirror and $\mathbb{N} = -\tilde{\mathbb{T}}^1 \tilde{\mathbb{U}}^{R,11}$ for the electromagnetic field and a perfectly reflecting mirror. The operator $\mathbb{N}$ describes a wave that travels from the mirror to the object and back, involving one scattering at the object.

3.1.1. Scalar field. All information about the shape of the surface and the boundary conditions is provided by the $\mathbb{T}$-matrix. For a spherically symmetric surface the matrix is diagonal and completely specified by scattering phase shifts $\delta_l(k)$ that do not depend on m:

$$\mathcal{T}_{lml'm'}(k) = \delta_{ll'} \delta_{mm'} \frac{1}{2} \left(\mathrm{e}^{2\mathrm{i}\delta_l(k)} - 1 \right), \tag{3.3}$$

where k is the real frequency. The scattering phase shifts for a sphere of radius R with Dirichlet and Neumann boundary conditions are

$$\cot \delta_l^{\mathrm{D}}(k) = \frac{n_l(\xi)}{j_l(\xi)}, \tag{3.4}$$

$$\cot \delta_l^{\mathrm{N}}(k) = \frac{n_l'(\xi)}{j_l'(\xi)}, \tag{3.5}$$

where $\xi = kR$ and j_l (n_l) are spherical Bessel functions of first (second) kind. To make use of the general result of equation (2.50), we have to evaluate the matrix elements for imaginary frequencies $k = \mathrm{i}\kappa$. This can be done using $j_l(\mathrm{i}z) = \mathrm{i}^l \sqrt{\pi/(2z)} I_{l+1/2}(z)$, $h_l^{(1)}(\mathrm{i}z) = j_l(\mathrm{i}z) + \mathrm{i}n_l(\mathrm{i}z) = -\mathrm{i}^{-l} \sqrt{2/(\pi z)} K_{l+1/2}(z)$, where $I_{l+1/2}$ and $K_{l+1/2}$ are modified Bessel functions of the first and second kind, respectively. We obtain

$$\mathcal{T}_{lmlm}^{\mathrm{D}} = (-1)^l \frac{\pi}{2} \frac{I_{l+1/2}(z)}{K_{l+1/2}(z)}, \tag{3.6}$$

$$\mathcal{T}_{lmlm}^{\mathrm{N}} = (-1)^l \frac{\pi}{2} \frac{I_{l+1/2}(z) - 2z I_{l+1/2}'(z)}{K_{l+1/2}(z) - 2z K_{l+1/2}'(z)} \tag{3.7}$$

for Dirichlet and Neumann boundary conditions where $z = \kappa R$.

The distance dependence of the Casimir energy of equation (2.50) is given by the translation matrix that describes the interaction between the original and mirror multipole moments. Since the origins (centers) of the original and mirror sphere are related by a translation along the z axis, the matrix elements of the translation matrix $\mathbb{U}^{R,11}$ are given by equations (2.28) and (2.29). After a rotation of frequency to the imaginary

axis we obtain

$$\mathcal{U}_{l'mlm}^{R,11} = -(-1)^l i^{-l'+l} \sqrt{(2l+1)(2l'+1)} \sum_{l''} (-1)^{l''}(2l''+1) \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(2\kappa L)}{\sqrt{\pi\kappa L}}. \tag{3.8}$$

With the elements of the transition and translation matrices we can compute the Casimir interaction from equation (2.50). The scaling of the $\mathbb{T}$-matrix for small κ shows that partial waves of order l start to contribute to the energy at order $L^{-2(l+1)}$ if the $\mathbb{T}$-matrix is diagonal in l. Hence, we can truncate the infinite matrix $\mathbb{N}$ in equation (3.1) at finite multipole order to obtain the large-distance expansion of the energy at a given order in R/L. Also, we can truncate the series over p in equation (3.1) since the pth power of $\mathbb{N}$ becomes important only at order $L^{-(p+1)}$. The first p powers of $\mathbb{N}$ describe p scattering off the sphere.

We have used `Mathematica` to perform the matrix operations and to expand the integrand of equation (3.1) in R/L. From this we find that the Casimir energy can be written as

$$\mathcal{E} = \frac{\hbar c}{\pi} \frac{1}{L} \sum_{j=2}^{\infty} b_j \left(\frac{R}{L}\right)^{j-1}, \tag{3.9}$$

where b_j is the coefficient of the term $\sim L^{-j}$. In the following we give the results for a mirror plane with Dirichlet or Neumann boundary conditions separately.

Dirichlet mirror. In this case we have $\mathbb{N} = -\mathbb{T}^1 \mathbb{U}^{R,11}$. If the field obeys Dirichlet boundary conditions at the sphere, we obtain the for leading coefficients

$$b_2 = -\frac{1}{8}, \qquad b_3 = -\frac{5}{64}, \qquad b_4 = -\frac{421}{1152}, \qquad b_5 = -\frac{535}{9216},$$
$$b_6 = -\frac{3083\,041}{4147\,200}, \qquad b_7 = +\frac{2741\,117}{11\,059\,200}, \qquad b_8 = -\frac{557\,222\,415\,727}{292\,626\,432\,000}, \tag{3.10}$$

indicating an attractive force. This result agrees with recent findings of Wirzba [30]. If we impose Neumann boundary conditions at the sphere, we find

$$b_2 = 0, \qquad b_3 = 0, \qquad b_4 = \frac{17}{96}, \qquad b_5 = 0,$$
$$b_6 = \frac{379}{960}, \qquad b_7 = \frac{479}{24576}, \qquad b_8 = \frac{243\,487}{403\,200}. \tag{3.11}$$

This result corresponds to a repulsive force which is expected for unlike boundary conditions. To leading order the energy scales as R^3/L^4 since the first two coefficients vanish. This behavior can be understood from the absence of low-frequency s-wave scattering for a sphere with Neumann boundary conditions.

Neumann mirror. In this case we have $\mathbb{N} = +\mathbb{T}^1 \mathbb{U}^{R,11}$. For a Dirichlet sphere we obtain the coefficients

$$b_2 = \frac{1}{8}, \qquad b_3 = \frac{3}{64}, \qquad b_4 = \frac{403}{1152}, \qquad b_5 = \frac{73}{9216},$$
$$b_6 = \frac{2981\,791}{4147\,200}, \qquad b_7 = -\frac{3636\,227}{11\,059\,200}, \qquad b_8 = \frac{550\,781\,720\,977}{292\,626\,432\,000}, \tag{3.12}$$

and hence the force is again repulsive. The modulus of the coefficients is smaller than those in equation (3.10) for a Dirichlet mirror with the exception of the leading one, b_2 and the one with inverted sign, b_7. Notice that the sum of the Casimir energies for the Dirichlet and Neumann mirrors, both opposite to a Dirichlet sphere, is identical to the Casimir energy between two Dirichlet spheres at a center-to-center distance $2L$ [29]. The reason for this is that any field configuration can be decomposed into symmetric and antisymmetric modes with respect to the mirror plane which obey Neumann and Dirichlet boundary conditions on that plane, respectively. Since the energy between two spheres scales as R^2/L^3 for asymptotically large separations, the modulus of the coefficients b_2 in equations (3.10) and (3.12) must be identical so that the contribution $\sim R/L^2$ cancels. It is easily seen that the higher-order coefficients of equations (3.10) and (3.12) combine to the correct coefficients of the large-distance expansion for two Dirichlet spheres [27].

For a Neumann sphere we get

$$b_2 = 0, \qquad b_3 = 0, \qquad b_4 = -\frac{17}{96}, \qquad b_5 = 0,$$
$$b_6 = -\frac{379}{960}, \qquad b_7 = -\frac{267}{8192}, \qquad b_8 = -\frac{243\,487}{403\,200}. \tag{3.13}$$

For the same reason as before, the first two coefficients vanish. Notice that all given coefficients, with the exception of b_7, are equal to minus the corresponding coefficients for a Neumann sphere and a Dirichlet mirror in equation (3.11). This result is consistent with the observation that the Casimir energy for two Neumann spheres at a center-to-center distance $2L$ is given by the sum of the energies of a Neumann sphere at distance L from a Dirichlet mirror and a Neumann mirror, respectively. Since two Neumann spheres interact at large distance with an energy $\sim R^6/L^7$ with the next-to-leading term $\sim R^8/L^9$, the sum of the corresponding coefficients, except for b_7, must vanish. The coefficients b_7 in equations (3.11) and (3.13) combine to the correct value for two Neumann spheres that was obtained in [27].

3.1.2. Electromagnetic field. We consider a dielectric sphere at a center-to-surface distance L from a perfectly conducting mirror. Due to spherical symmetry, the electric and magnetic multipoles for all l, m are decoupled so that the T-matrix is diagonal [26]:

$$T_{lmlm}^{MM} = (-1)^l \frac{\pi}{2} \frac{\eta I_{l+\frac{1}{2}}(z)[I_{l+\frac{1}{2}}(nz) + 2nz I'_{l+\frac{1}{2}}(nz)] - n I_{l+\frac{1}{2}}(nz)[I_{l+\frac{1}{2}}(z) + 2z I'_{l+\frac{1}{2}}(z)]}{\eta K_{l+\frac{1}{2}}(z)[I_{l+\frac{1}{2}}(nz) + 2nz I'_{l+\frac{1}{2}}(nz)] - n I_{l+\frac{1}{2}}(nz)[K_{l+\frac{1}{2}}(z) + 2z K'_{l+\frac{1}{2}}(z)]}, \tag{3.14}$$

where the sphere radius is R, $z = \kappa R$, $n = \sqrt{\epsilon(i\kappa)\mu(i\kappa)}$ and $\eta = \sqrt{\epsilon(i\kappa)/\mu(i\kappa)}$. T_{lmlm}^{EE} is obtained from equation (3.14) by interchanging ϵ and μ. The limit of a perfectly

conducting sphere is obtained by taking $\epsilon(i\kappa) \to \infty$ at an arbitrarily fixed $\mu(i\kappa)$, which can also vanish. Then the matrix elements become independent of μ:

$$T_{lmlm}^{\mathrm{MM}} = (-1)^l \frac{\pi}{2} \frac{I_{l+1/2}(z)}{K_{l+1/2}(z)}, \tag{3.15}$$

$$T_{lmlm}^{\mathrm{EE}} = (-1)^l \frac{\pi}{2} \frac{I_{l+1/2}(z) + 2z I'_{l+1/2}(z)}{K_{l+1/2}(z) + 2z K'_{l+1/2}(z)}. \tag{3.16}$$

For all partial waves, the *leading* low-frequency contribution is determined by the *static* electric multipole polarizability, $\alpha_l^{\mathrm{E}} = [(\epsilon-1)/(\epsilon+(l+1)/l)]R^{2l+1}$, and the corresponding magnetic polarizability, $\alpha_l^{\mathrm{M}} = [(\mu-1)/(\mu+(l+1)/l)]R^{2l+1}$. Including the next-to-leading terms, the T-matrix has the structure

$$T_{lmlm}^{\mathrm{MM}} = \kappa^{2l} \left[\frac{(-1)^{l-1}(l+1)\alpha_l^{\mathrm{M}}}{l(2l+1)!!(2l-1)!!} \kappa + \gamma_{l3}^{\mathrm{M}} \kappa^3 + \gamma_{l4}^{\mathrm{M}} \kappa^4 + \cdots \right],$$

and T_{lmlm}^{EE} is obtained by $\alpha_l^{\mathrm{M}} \to \alpha_l^{\mathrm{E}}$, $\gamma_{ln}^{\mathrm{M}} \to \gamma_{ln}^{\mathrm{E}}$. The first terms are $\gamma_{13}^{\mathrm{M}} = -[4 + \mu(\epsilon\mu + \mu - 6)]/[5(\mu+2)^2]R^5$, $\gamma_{14}^{\mathrm{M}} = (4/9)[(\mu-1)/(\mu+2)]^2 R^6$ and γ_{13}^{E}, γ_{14}^{E} are obtained again by interchanging μ and ϵ. Higher-order terms can be easily obtained by expanding equation (3.14) for small κ.

The translation matrix elements $\tilde{\mathcal{U}}_{l'mlm}^{R,11} = (-1)^{l'}\mathcal{U}_{l'mlm}^{R,11}$ are obtained from equations (2.81), (2.84) and (2.85). Using $h_l^{(1)}(iz) = -i^{-l}\sqrt{2/(\pi z)}K_{l+1/2}(z)$ we get

$$B_{l'm'lm}(-2L\hat{\mathbf{z}}) = -\delta_{m'm}(-1)^m i^{l-l'} \sum_{l''} \frac{(-1)^{l''}}{2} \left[l(l+1) + l'(l'+1) - l''(l''+1) \right]$$

$$\times \sqrt{\frac{(2l+1)(2l'+1)}{l(l+1)l'(l'+1)}}(2l''+1) \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(2\kappa L)}{\sqrt{\pi\kappa L}}, \tag{3.17}$$

$$C_{l'm'lm}(-2L\hat{\mathbf{z}}) = \delta_{m'm}2\kappa Lm(-1)^m i^{l-l'} \sum_{l''}(-1)^{l''} \sqrt{\frac{(2l+1)(2l'+1)}{l(l+1)l'(l'+1)}}(2l''+1)$$

$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(2\kappa L)}{\sqrt{\pi\kappa L}}. \tag{3.18}$$

Now we can employ the series representation of the Casimir energy in equation (3.1) with $\mathbb{N} = -\tilde{\mathbb{T}}^1 \tilde{\mathbb{U}}^{R,11}$. Since the T-matrix is diagonal in l, partial waves of order l start to contribute to the energy at order $L^{-2(l+1)}$. Also, the pth power of $\mathbb{N}$ becomes important only at order $L^{-(3p+1)}$. Notice the stronger increase of the exponent with p compared to the scalar case where the exponent is $-(p+1)$. This can be understood from the absence of s-waves for the electromagnetic field so that each reflection contributes a factor $1/L^3$ due to p-waves. Matrix operations and the expansion in R/L are performed with `Mathematica`. From this we obtain for a dielectric sphere in front of a perfectly conducting

mirror plane the Casimir energy

$$
\mathcal{E} = -\frac{\hbar c}{\pi} \left\{ \frac{3}{8}(\alpha_1^{\mathrm{E}} - \alpha_1^{\mathrm{M}})\frac{1}{L^4} + \frac{15}{32}(\alpha_2^{\mathrm{E}} - \alpha_2^{\mathrm{M}} + 2\gamma_{13}^{\mathrm{E}} - 2\gamma_{13}^{\mathrm{M}})\frac{1}{L^6} \right.
$$
$$
+ \frac{1}{1024}\left[23(\alpha_1^{\mathrm{M}})^2 - 14\alpha_1^{\mathrm{M}}\alpha_1^{\mathrm{E}} + 23(\alpha_1^{\mathrm{E}})^2 + 2160(\gamma_{14}^{\mathrm{E}} - \gamma_{14}^{\mathrm{M}})\right]\frac{1}{L^7}
$$
$$
\left. + \frac{7}{7200}\left[572(\alpha_3^{\mathrm{E}} - \alpha_3^{\mathrm{M}}) + 675\left(9(\gamma_{15}^{\mathrm{E}} - \gamma_{15}^{\mathrm{M}}) - 55(\gamma_{23}^{\mathrm{E}} - \gamma_{23}^{\mathrm{M}})\right)\right]\frac{1}{L^8} + \cdots \right\}.
$$
(3.19)

The electric contribution to the leading term, $\sim L^{-4}$, was obtained by Casimir and Polder for the interaction of an atom with static polarizability α_1^{E} and a metallic surface [13]. Later Boyer has generalized the leading-order result to include magnetic effects described by α_1^{M} [36]. The higher-order terms are new. They show how higher-order polarizabilities and frequency corrections to the static parameters influence the interaction. There is no $\sim 1/L^5$ term. Notice also that the first three terms of the contribution at order L^{-7} have precisely the structure of the Casimir–Polder interaction between two atoms with static dipole polarizabilities α_1^{M} and α_1^{E} but it is reduced by a factor of $1/2^8$. This factor and the distance dependence $\sim L^{-7}$ of this term suggests that it arises from the interaction of the dipole fluctuations inside the sphere with those inside its image at a distance $2L$. The additional coefficient of $1/2$ in the reduction factor $(1/2)(1/2^7)$ can be traced back to the fact that the forces involved in bringing the dipole in from infinity act only on the dipole and not on its image [37].

For a perfectly conducting sphere the coefficients of the expansion of the Casimir energy in R/L become universal numbers. These coefficients can be either obtained from equation (3.19) by using the appropriate values of the parameters for a perfect metal or it can be computed directly from the T-matrix elements given in equations (3.15) and (3.16). Following the latter route, we get the series

$$
\mathcal{E} = \frac{\hbar c}{\pi}\frac{1}{L}\sum_{j=4}^{\infty} b_j \left(\frac{R}{L}\right)^{j-1},
$$
(3.20)

where the coefficients up to order $1/L^{11}$ are

$$
b_4 = -\frac{9}{16}, \qquad b_5 = 0, \qquad b_6 = -\frac{25}{32}, \qquad b_7 = -\frac{3023}{4096} \qquad b_8 = -\frac{12\,551}{9600},
$$
$$
b_9 = \frac{1282\,293}{163\,840}, \qquad b_{10} = -\frac{32\,027\,856\,257}{722\,534\,400}, \qquad b_{11} = \frac{39\,492\,614\,653}{412\,876\,800}.
$$
(3.21)

This and the corresponding results for a scalar field appear to be asymptotic series. Therefore, the series cannot be summed to obtain the interaction at small separations. In the next section we use a numerical implementation of equation (2.101) to compute the interaction at all separations.

3.2. Non-perturbative result at all separations

To obtain the Casimir interaction over a broad range of distances, equation (2.101) has to be evaluated numerically. As we employ a partial wave expansion, the computational work increases with decreasing separation between the objects. However, we shall see
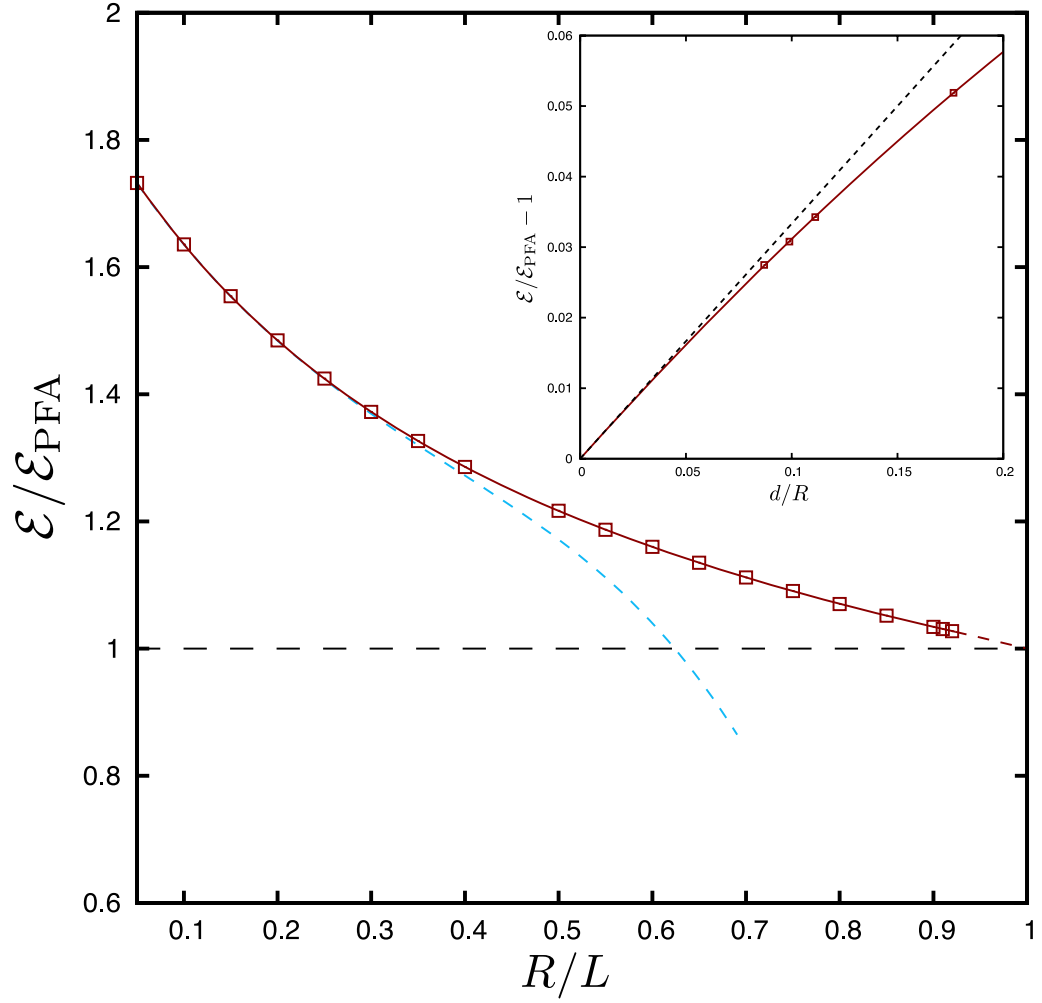
Figure 3. Sphere–plane Casimir interaction of a scalar field with Dirichlet boundary conditions on both surfaces. The energy is scaled by the PFA estimate of equation (3.22). The dashed curve represents the large-distance expansion of equation (3.9) with the coefficients of equation (3.10). Inset: corrections to the PFA at small separations as a function of the surface-to-surface distance $d = L - R$. The dashed curve corresponds to the lowest-order correction to the PFA obtained by Bordag [38].

below that, even at small separations, our method yields sufficient precision to obtain the leading corrections to the proximity force approximation (PFA). The numerical approach is based on the technique presented in [26, 27]. Using the analytic expressions for the matrix elements of the translation and transition matrices, we compute the determinant and the integral over frequency κ in equation (2.101) numerically. The matrices are truncated at a finite multipole order l which yields a series of estimates $\mathcal{E}^{(l)}$ for the Casimir energy. The exact Casimir energy is then obtained by extrapolating the series to $l \to \infty$. For the geometry considered here, we observe an exponentially fast convergence that allows us to obtain accurate results even at small separations from a moderate multipole order.
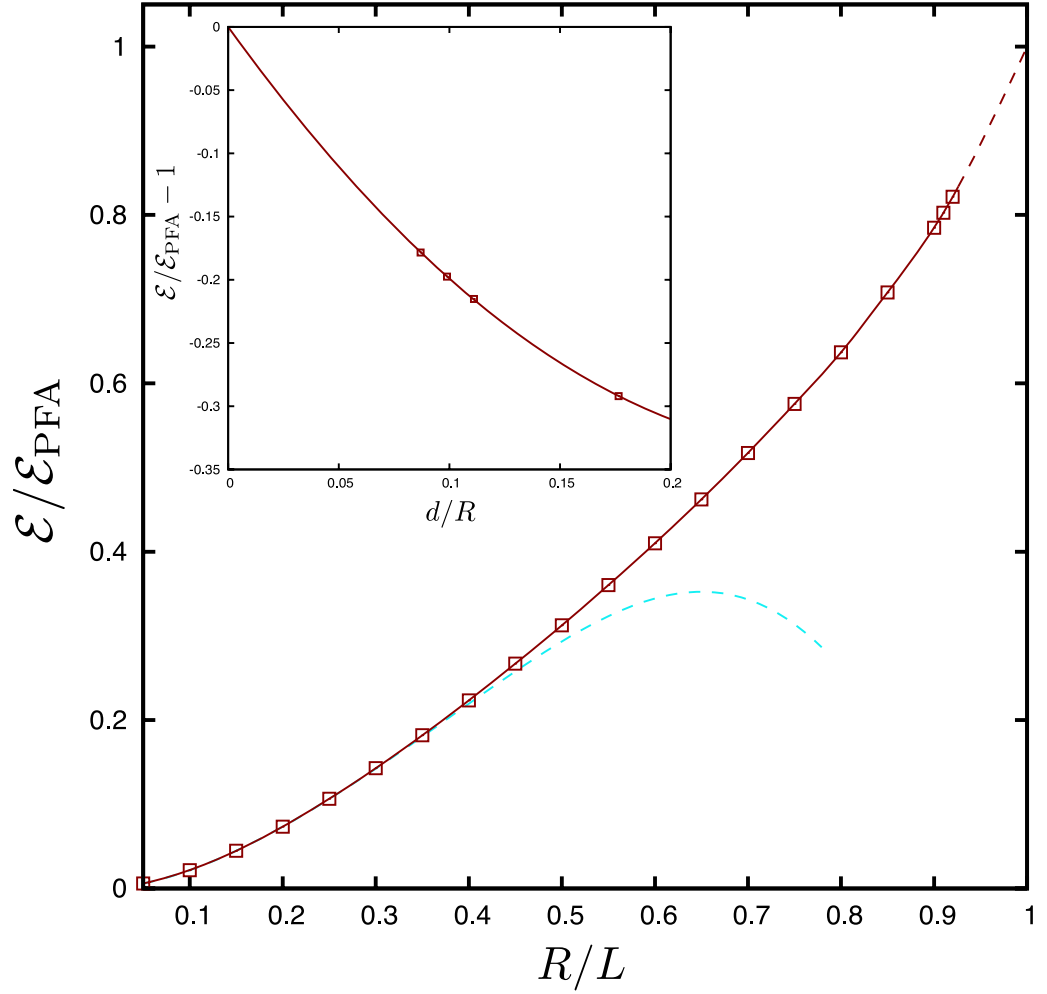
Figure 4. Analog of figure 3 for Neumann boundary conditions at the sphere and the plane. The dashed curve represents the large-distance expansion of equation (3.9) with the coefficients of equation (3.13).

We compare our results for the interaction energy to the estimate that follows from PFA. This is important since PFA is often used over a range of separations although its accuracy is unknown even at small distances for most geometries, including the electromagnetic Casimir interaction between a plane and a sphere. The PFA estimate for the latter geometry is given by

$$\mathcal{E}_{\mathrm{PFA}} = \pi \Phi_0^{\pm/\mathrm{EM}} \frac{\hbar c R}{d^2}, \tag{3.22}$$

where $d = L - R$ is the surface-to-surface distance. The amplitudes follow from the result for two parallel plates and are given by

$$\Phi_0^- = -\frac{\pi^2}{1440}, \tag{3.23}$$
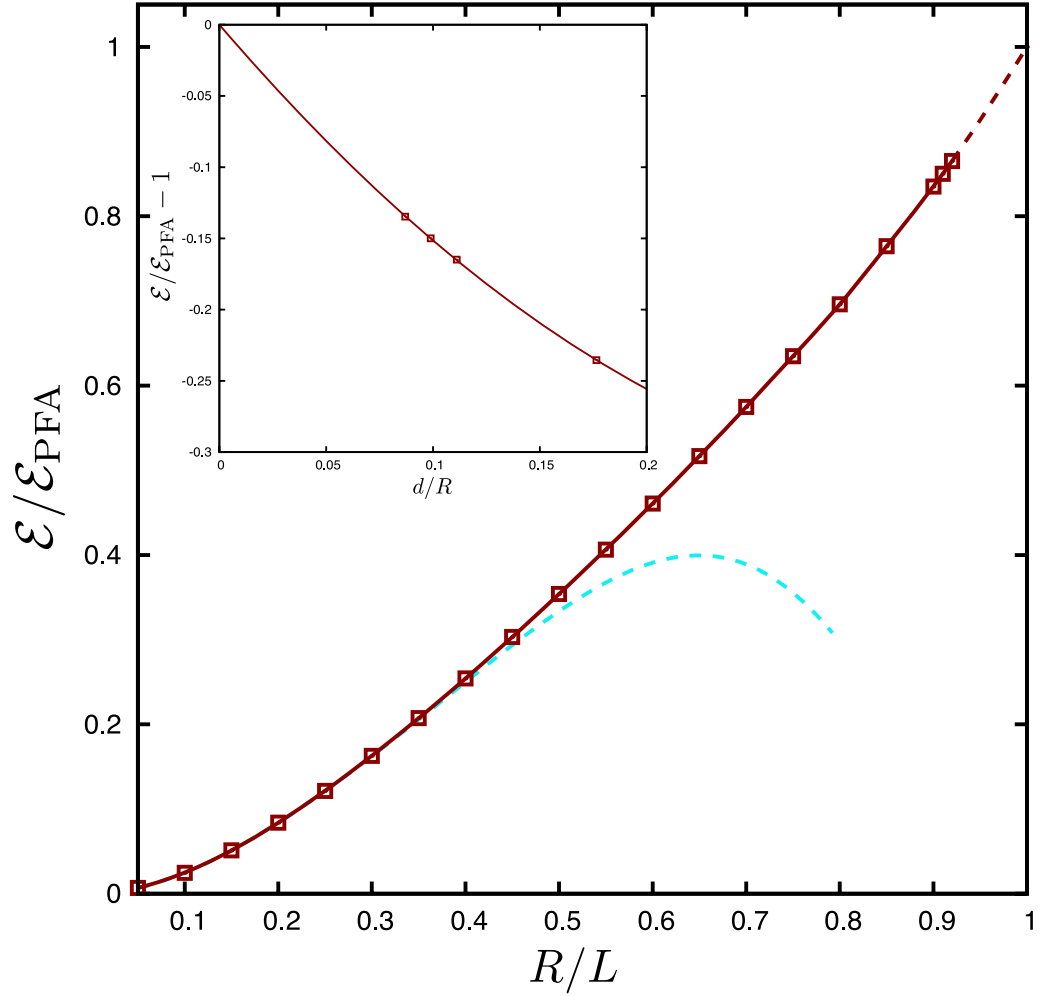
$$\Phi_0^+ = +\frac{7\pi^2}{11\,520}, \tag{3.24}$$

Figure 5. Analog of figure 3 for Neumann boundary conditions at the sphere and Dirichlet boundary conditions at the plane. The dashed curve represents the large-distance expansion of equation (3.9) with the coefficients of equation (3.11).

$$\Phi_0^{\mathrm{EM}} = -\frac{\pi^2}{720} \tag{3.25}$$

for a scalar field with like $(-)$ and unlike $(+)$ boundary conditions and the electromagnetic field, respectively. In the following we present our results for the plane–sphere interaction over a wide range of separations for scalar fields with Dirichlet and Neumann boundary conditions and for the electromagnetic field.

3.2.1. Scalar field. We consider the four cases that correspond to Dirichlet or Neumann boundary conditions at the sphere and the mirror plane. The results for the Casimir energy are shown in figures 3–6. At large separations the results of the numerical evaluation of the determinant agree nicely with the large-distance expansions of the previous section. For a sphere with Dirichlet boundary conditions, the energy scales both at large and small separations as R/d^2 so that the curves for $\mathcal{E}/\mathcal{E}_{\mathrm{PFA}}$ in figures 3 and 6 tend also to a constant for $R/L \to 0$. The PFA underestimates the actual interaction energy at all
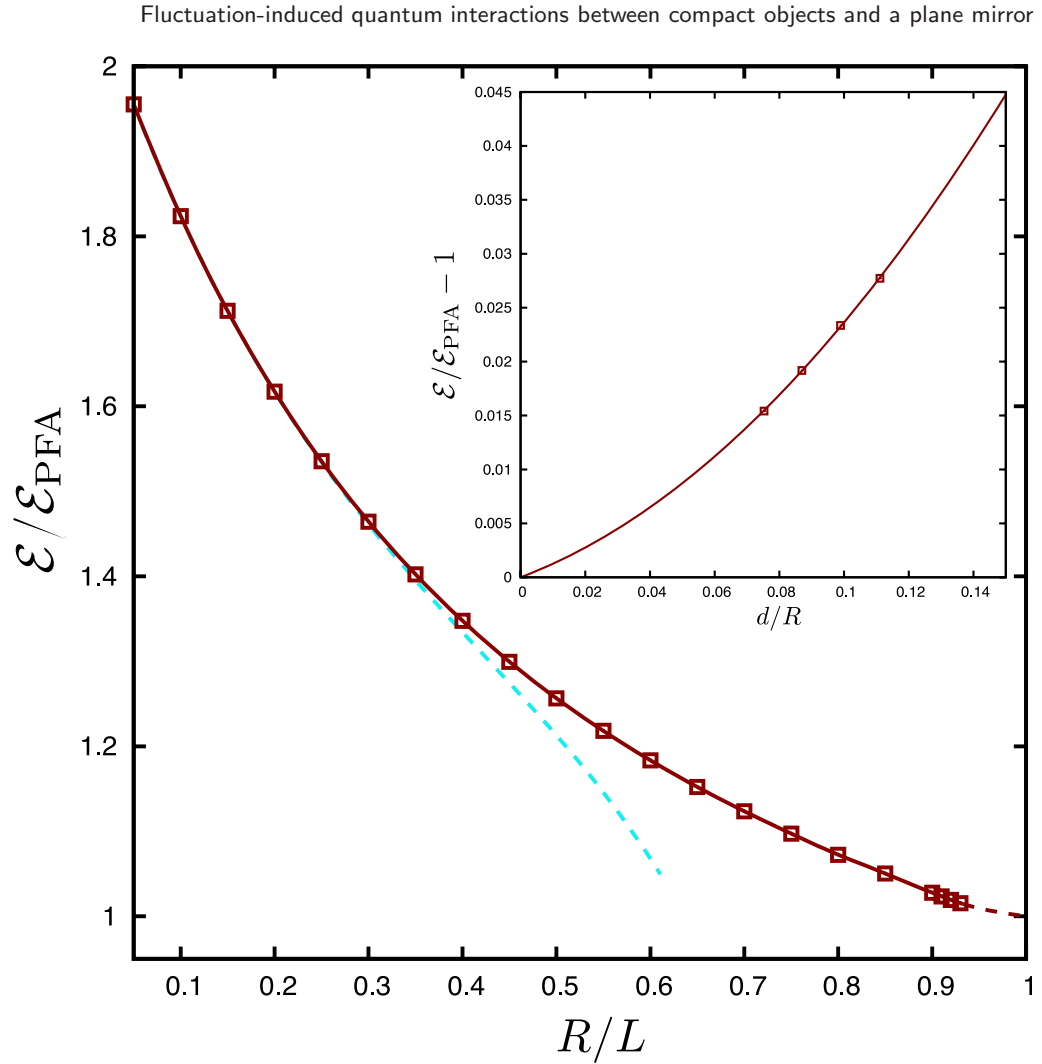
Figure 6. Analog of figure 3 for Dirichlet boundary conditions at the sphere and Neumann boundary conditions at the plane. The dashed curve represents the large-distance expansion of equation (3.9) with the coefficients of equation (3.12).

separations. For a Neumann sphere the interaction scales at large separations as R^3/L^4 so that the curves for $\mathcal{E}/\mathcal{E}_{\mathrm{PFA}}$ in figures 4 and 5 tend to zero for $R/L \to 0$. Here the PFA overestimates the actual interaction energy at all separations.

The non-perturbative approach allows us to study also the case of small separations. In that limit our results can be fitted to a power law of the form

$$\mathcal{E} = \mathcal{E}_{\mathrm{PFA}} \left[1 + \theta_1 \frac{d}{R} + \theta_2 \left(\frac{\mathrm{d}}{R} \right)^2 + \cdots \right]. \tag{3.26}$$

The coefficients θ_j obtained from a fit of the function of equation (3.26) to the data points for the four smallest studied separations are summarized in table 1. The fitted curves are shown as insets in figures 3–6. The result for θ_1 for the case of Dirichlet boundary conditions at the sphere and the mirror agrees with the analytical result $\theta_1 = 1/3$ presented in [38]. Again for Dirichlet conditions, for the second-order coefficient
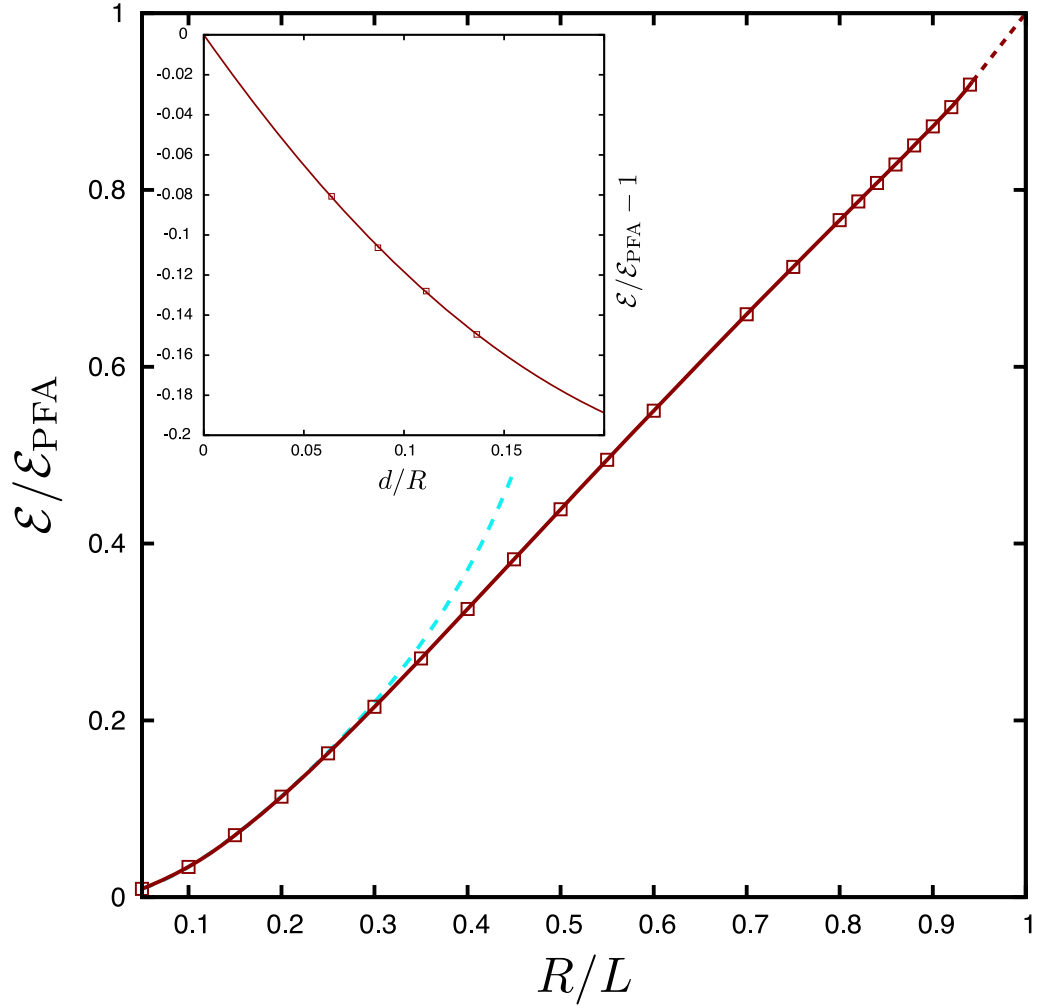
Figure 7. Electromagnetic Casimir energy for the sphere-plate geometry. The energy is scaled by the PFA estimate of equation (3.22). The asymptotic expansion of equation (3.20) to order $1/L^{10}$ is shown as a dashed line. Inset: corrections to the PFA at small distances as a function of $d = L - R$.

a considerably larger numerical estimate of $\theta_2 = 1.92$ has been obtained from world-line Monte Carlo sampling [31, 32]. For the other combinations of boundary conditions our findings represent the first results for the corrections to the PFA.

3.2.2. Electromagnetic field. We focus on a perfectly conducting sphere and mirror plane. The Casimir energy resulting from a numerical computation of the determinant of equation (2.101) is shown for a wide range of separations in figure 7. At large separations the interaction is described by the asymptotic result of equation (3.20). At small distances the Casimir energy approaches to PFA estimate and corrections to PFA can be described again by equation (3.26). A corresponding fit to the data points for the four smallest separations is shown as the inset in figure 7. The corresponding amplitudes of the correction terms are listed in table 1.

Table 1. Coefficients describing corrections to PFA, see equation (3.26). D = Dirichlet, N = Neumann (scalar field) and EM = perfectly conducting boundary conditions (electromagnetic field).

Boundary condition plane/sphere	θ_1	θ_2
D/D	0.3346 ± 0.0017	-0.231 ± 0.012
N/N	-2.4298 ± 0.0070	4.394 ± 0.049
D/N	-1.7493 ± 0.0061	2.354 ± 0.042
N/D	0.1124 ± 0.0043	1.240 ± 0.044
EM/EM	-1.42 ± 0.02	2.39 ± 0.14

This result is important for a number of recent measurements of the Casimir force between almost flat surfaces. These experiments have been performed for a plane mirror and a sphere with a radius that is much larger than the distance between the surfaces in order to avoid difficulties from parallelism control. Our results at short distances indicate that for a sphere of radius $R = 100$ μm the corrections to PFA are below 1% only for surface-to-surface distances d that are smaller than $\approx$700 nm. For a sphere of that size, PFA fails already by 10% at a distance of $\approx$8 μm. From this we conclude that the currently achieved experimental accuracy for measurements of Casimir forces in the sphere-plate geometry with a sphere of $R = 100$ μm at distances below $d = 1$ μm is within the range or slightly less than the corrections to PFA.

However, deviations from PFA become severe when smaller objects interact with surfaces. Hence it is important to understand the crossover of the Casimir interaction between macroscopic objects and the eventual Casimir–Polder interaction between single atoms and a surface. A description of this crossover is provided by our results in figure 7. Corresponding results can also be obtained for a dielectric sphere or less symmetric objects by the methods presented here. Finally, we note that it should also be possible to compute correction amplitudes like the θ_j in equation (3.26) analytically by applying methods similar to those used in [24]. However, the validity range of the corresponding series would be limited to small separations and for an overall description of the interaction one should resort to the device of a numerical evaluation along the lines presented here.

Acknowledgments

References

[1] Casimir H B G, 1948 *Proc. K. Ned. Akad. Wet.* **51** 793
    Casimir H B G, 1948 *Indag. Math.* **10** 261 (Transl.)
[2] Lamoreaux S K, 1997 *Phys. Rev. Lett.* **78** 5
[3] Mohideen U and Roy A, 1998 *Phys. Rev. Lett.* **81** 4549
[4] Roy A *et al*, 1999 *Phys. Rev.* D **60** 111101(R)
[5] Ederth T, 2000 *Phys. Rev.* A **62** 062104
[6] Chan H B *et al*, 2001 *Science* **291** 1941

[7] Chen F *et al*, 2002 *Phys. Rev. Lett.* **88** 101801
[8] Chen F, Klimchitskaya G L, Mostepanenko V M and Mohideen U, 2006 *Phys. Rev. Lett.* **97** 170402
[9] Decca R S, López D, Fischbach E, Klimchitskaya G L, Krause D E and Mostepanenko V M, 2007 *Phys. Rev.* D **75** 077101
[10] Chen F, Klimchitskaya G L, Mostepanenko V M and Mohideen U, 2007 *Phys. Rev.* B **76** 035338
[11] Munday J N and Capasso F, 2007 *Phys. Rev.* A **75** 060102(R)
[12] Parsegian V A, 2005 *van der Waals Forces* (Cambridge: Cambridge University Press)
[13] Casimir H B G and Polder D, 1948 *Phys. Rev.* **73** 360
[14] Harber D M, Obrecht J M, McGuirk J M and Cornell E A, 2005 *Phys. Rev.* A **72** 033610
[15] Krause D E, Decca R S, López D and Fischbach E, 2007 *Phys. Rev. Lett.* **98** 050403
[16] Rodriguez A, Ibanescu M, Iannuzzi D, Capasso F, Joannopoulos J D and Johnson S G, 2007 *Phys. Rev. Lett.* **99** 080401
Rodriguez A, Ibanescu M, Iannuzzi D, Joannopoulos J D and Johnson S G, 2007 *Phys. Rev.* A **76** 032106
[17] Rahi S J, Rodriguez A W, Emig T, Jaffe R L, Johnson S G and Kardar M, 2007 *Preprint* 0711.1987
[18] Balian R and Duplantier B, 1977 *Ann. Phys., NY* **104** 300
Balian R and Duplantier B, 1978 *Ann. Phys., NY* **112** 165
[19] Jaeckel M T and Reynaud S, 1991 *J. Physique* I **1** 1395
[20] Emig T and Büscher R, 2004 *Nucl. Phys.* B **696** 468
[21] Lambrecht A, Maia Neto P A and Reynaud S, 2006 *New J. Phys.* **8** 243
[22] Maia Neto P A, Lambrecht A and Reynaud S, 2005 *Europhys. Lett.* **69** 924
Maia Neto P A, Lambrecht A and Reynaud S, 2005 *Phys. Rev.* A **72** 012115
[23] Emig T, Jaffe R L, Kardar M and Scardicchio A, 2006 *Phys. Rev. Lett.* **96** 080403
[24] Bordag M, 2006 *Phys. Rev.* D **73** 125018
[25] Kenneth O and Klich I, 2006 *Phys. Rev. Lett.* **97** 160401
[26] Emig T, Graham N, Jaffe R L and Kardar M, 2007 *Phys. Rev. Lett.* **99** 170403
[27] Emig T, Graham N, Jaffe R L and Kardar M, 2008 *Phys. Rev.* D **77** 025005
[28] Kenneth O and Klich I, 2007 *Preprint* 0707.4017
[29] Bulgac A, Magierski P and Wirzba A, 2006 *Phys. Rev.* D **73** 025007
[30] Wirzba A, 2007 *Preprint* 0711.2395
[31] Gies H and Klingmüller K, 2006 *Phys. Rev. Lett.* **97** 220401
[32] Gies H and Klingmüller K, 2006 *Phys. Rev.* D **74** 045002
[33] Bordag M, Robaschik D and Wieczorek E, 1985 *Ann. Phys., NY* **165** 192
[34] Li H and Kardar M, 1991 *Phys. Rev. Lett.* **67** 3275
Li H and Kardar M, 1992 *Phys. Rev.* A **46** 6490
[35] Wittmann R C, 1988 *IEEE Trans. Antennas Propag.* **36** 1078
[36] Boyer T H, 1969 *Phys. Rev.* **180** 19
[37] Barnett S, Aspect A and Milonni P W, 2000 *J. Phys. B: At. Mol. Opt. Phys.* **33** L143
[38] Bordag M, 2007 *Talk Presented at the Workshop QFEXT07 (Leipzig, Sept.)*