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Scattering theory approach to electrodynamic Casimir forces

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We give a comprehensive presentation of methods for calculating the Casimir force to arbitrary accuracy, for any number of objects, arbitrary shapes, susceptibility functions, and separations. The technique is applicable to objects immersed in media other than vacuum, nonzero temperatures, and spatial arrangements in which one object is enclosed in another. Our method combines each object's classical electromagnetic scattering amplitude with universal translation matrices, which convert between the bases used to calculate scattering for each object, but are otherwise independent of the details of the individual objects. The method is illustrated by rederiving the Lifshitz formula for infinite half-spaces, by demonstrating the Casimir-Polder to van der Waals crossover, and by computing the Casimir interaction energy of two infinite, parallel, perfect metal cylinders either inside or outside one another. Furthermore, it is used to obtain new results, namely, the Casimir energies of a sphere or a cylinder opposite a plate, all with finite permittivity and permeability, to leading order at large separation.

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I. INTRODUCTION

Materials that couple to the electromagnetic field alter the spectrum of its quantum and thermal fluctuations. The resulting change in energy depends on the relative positions of the objects, leading to a fluctuation-induced force, usually called the Casimir force. This force has been the subject of precision experimental measurements [1–17] and can influence the operation of nanoscale devices [5,18].

Casimir and Polder calculated the fluctuation-induced force on a polarizable atom in front of a perfectly conducting plate and between two polarizable atoms, both to leading order at large separation, and obtained a simple result depending only on the atoms' static polarizabilities [19]. Casimir then extended this result to his famous calculation of the pressure on two perfectly conducting parallel plates [20]. Feinberg and Sucher [21,22] generalized the result of Casimir and Polder to include both electric and magnetic polarizabilities. Lifshitz, Dzyaloshinskii, and Pitaevskii extended Casimir's result for parallel plates by incorporating nonzero temperature, permittivity, and permeability into a general formula for the pressure on two infinite half-spaces separated by a gap [23–27].

In order to study Casimir forces in more general geometries, it turns out to be advantageous to describe the influence of an arrangement of objects on the electromagnetic field by the way they scatter electromagnetic waves. In this article we derive and apply a representation of the Casimir energy, developed in Refs. [28,29], that characterizes each object by its on-shell electromagnetic scattering amplitude. The separations and orientations of the objects are encoded in universal translation matrices, which describe how a solution to the source-free Maxwell's equations in the basis appropriate to one object looks when expanded in the basis appropriate to another. The translation matrices depend on the displacement and orientation of coordinate systems, but not on the nature of the objects themselves. The scattering amplitudes and translation matrices are then combined in a simple algorithm that allows efficient numerical and, in some cases, analytical calculations of Casimir forces and torques for a wide variety of geometries, materials, and external conditions. We will generalize the formalism summarized in [28] further to show how it applies in a wide variety of circumstances, including

(i) n arbitrarily shaped objects, whose surfaces may be smooth or rough or may include edges and cusps;
(ii) objects with arbitrary linear electromagnetic response, including frequency-dependent, lossy electric permittivity and magnetic permeability tensors;
(iii) objects separated by vacuum or by a medium with uniform, frequency-dependent isotropic permittivity and permeability;
(iv) zero or nonzero temperature;
(v) and objects outside of one another or enclosed in each other.

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These ideas build on a range of previous related work, an inevitably incomplete subset of which is briefly reviewed here: Scattering theory methods were first applied to the parallel plate geometry, when Kats reformulated Lifshitz theory in terms of reflection coefficients [30]. Jaekel and Reynaud derived the Lifshitz formula using reflection coefficients for lossless infinite plates [31] and Genet, Lambrecht, and Reynaud extended this analysis to the lossy case [32]. Lambrecht, Maia Neto, and Reynaud generalized these results to include nonspecular reflection [33].

Around the same time as Kats's work, Balian and Duplantier developed a multiple scattering approach to the Casimir energy for perfect metal objects and used it to compute the Casimir energy at asymptotically large separations [34,35] at both zero and nonzero temperature. In their approach, information about the conductors is encoded in a local surface scattering kernel, whose relation to more conventional scattering formalisms is not transparent, and their approach was not pursued further at the time. One can find multiple scattering formulas in an even earlier article by Renne [36], but scattering is not explicitly mentioned, and the technique is only used to rederive older results.

Another scattering-based approach has been to express the Casimir energy as an integral over the density of states of the fluctuating field, using the Krein formula [37–39] to relate the density of states to the $\mathcal{S}$-matrix for scattering from the ensemble of objects. This $\mathcal{S}$-matrix is difficult to compute in general. In studying many-body scattering, Henseler and Wirzba connected the $\mathcal{S}$-matrix of a collection of spheres [40] or disks [41] to the objects' individual $\mathcal{S}$-matrices, which are easy to find. Bulgac, Magierski, and Wirzba combined this result with the Krein formula to investigate the scalar and fermionic Casimir effect for disks and spheres [42–44]. Casimir energies of solitons in renormalizable quantum field theories have been computed using scattering theory techniques that combine analytic and numerical methods [45].

Bordag, Robaschik, Scharnhorst, and Wieczorek [46,47] introduced path integral methods to the study of Casimir effects and used them to investigate the electromagnetic Casimir effect for two parallel perfect metal plates. Li and Kardar used similar methods to study the scalar thermal Casimir effect for Dirichlet, Neumann, and mixed boundary conditions [48,49]. The quantum extension was developed further by Golestanian and Kardar [50,51] and was subsequently applied to the quantum electromagnetic Casimir effect by Emig, Hanke, Golestanian, and Kardar, who studied the Casimir interaction between plates with roughness [52] and between deformed plates [53]. (Techniques developed to study the scalar Casimir effect can be applied to the electromagnetic case for perfect metals with translation symmetry in one spatial direction, since then the electromagnetic problem decomposes into two scalar ones.) Finally, the path integral approach was connected to scattering theory by Emig and Büscher [54].

Closely related to the work we present here is that of Kenneth and Klich, who expressed the data required to characterize Casimir fluctuations in terms of the transition $\mathbb{T}$-operator for scattering of the fluctuating field from the objects [55]. Their abstract representation made it possible to prove general properties of the sign of the Casimir force. In Refs. [28,29], we developed a framework in which this abstract result can be applied to concrete calculations. In this approach, the $\mathbb{T}$-operator is related to the scattering amplitude for each object individually, which in turn is expressed in an appropriate basis of multipoles. While the $\mathbb{T}$-operator is in general ''off-shell,'' meaning it has matrix elements between states with different frequencies, the scattering amplitudes are the ''on-shell'' matrix elements of this operator between states of equal frequency.[1] In this approach, the objects can have any shape or material properties, as long as the scattering amplitude can be computed in a multipole expansion (or measured). The approach can be regarded as a concrete implementation of the proposal emphasized by Schwinger [56] that the fluctuations of the electromagnetic field can be traced back to charge and current fluctuations on the objects. This formalism has been applied and extended in a number of Casimir calculations [57–62].

The basis in which the scattering amplitude for each object is supplied is typically associated with a coordinate system appropriate to the object. Of course a plane, a cylinder, or a sphere would be described in Cartesian, cylindrical, or spherical coordinates, respectively. However, any compact object can be described, for example, in spherical coordinates, provided that the matrix of scattering amplitudes can be either calculated or measured in that coordinate system. There are a limited number of coordinate systems in which such a partial wave expansion is possible, namely, those for which the vector Helmholtz equation is separable. The translation matrices for common separable coordinate systems, obtained from the free Green's function, are supplied in Appendix C. For typical cases, the final computation of the Casimir energy can be performed on a desktop computer for a wide range of separations. Asymptotic results at large separation can be obtained analytically.

The primary limitation of the method is on the distance between objects, since the basis appropriate to a given object may become impractical as two objects approach. For small separations, sufficient accuracy can only be obtained if the calculation is taken to very high partial wave order. In the case of two spheres, the scattering

[1]Because of this relationship, these scattering amplitudes are also referred to as elements of the *T-matrix*. In standard conventions, however, the T-matrix differs from the matrix elements of the $\mathbb{T}$-operator by a basis-dependent constant, so we will use the term ''scattering amplitude'' to avoid confusion.

amplitude is available in a spherical basis, but as the two spheres approach, the Casimir energy is dominated by waves near the point of closest approach [63]. As the spheres come into contact, an infinite number of spherical waves are needed to capture the dominant contribution (see Sec. III for further discussion). A particular basis may also be fundamentally inappropriate at small separations. For instance, if the interaction of two elliptic cylinders is expressed in an ordinary cylindrical basis, when the elliptic cylinders are close enough one may not fit inside the smallest circular cylinder that encloses the other. In that case the cylindrical basis would not "resolve" the two objects (although an elliptic cylindrical basis would; see Sec. III). Finally, for a variety of conceptual and computational reasons, we are limited to linear electromagnetic response.

To illustrate this general formulation, we provide some sample applications, including the closed-form expressions for computing the interaction of a plate and a sphere with finite, uniform, frequency-dependent electric permittivity and magnetic permeability. We present the Casimir interaction energy explicitly at asymptotically large separations in terms of the zero-frequency permittivities and permeabilities of the objects. Although most experiments have centered around the sphere-plate configuration [1–3,5,8–10,12,13,64,65], it is only recently that the force between a dielectric sphere and an idealized metallic plate has been obtained for all distances [66]. Subsequently, this result has been extended to the situation where both objects are described by the plasma model [67]. In addition, we present the Casimir interaction energy of a plate and a cylinder at asymptotically large distances in terms of the two objects' zero-frequency permittivities and permeabilities. Results beyond the leading order using our closed-form formulation are not explicitly included, but all the essential formulas are contained here. These results extend the perfect metal cylinder and plate results presented in Ref. [68].

The article is organized as follows: In Sec. II we review the derivation of the ground state energy of a field theory using path integrals. In Sec. III we expand the free electromagnetic Green's functions in terms of regular and outgoing waves, taking into account that the pairs of waves in the expansion are evaluated with respect to two different coordinate systems. This analysis yields the translation matrices. Section IV provides an overview of elements of scattering theory we will use, including the connection between the $\mathbb{T}$-operator and the scattering amplitude. In Sec. V the path integral expression for the energy is reexpressed in terms of the results of the preceding two sections, yielding the main result, Eq. (5.13). In Sec. VI sample applications are presented: A short derivation of the Lifshitz formula, the crossover between van der Waals and Casimir regimes for two atoms, a general derivation of previous results for cylinders [69,70], and new results for the energy between a dielectric sphere or cylinder and a dielectric plane.

II. CASIMIR ENERGY FROM FIELD THEORY

A. Electromagnetic Lagrangian and action

We consider the Casimir effect for objects without free charges and currents but with nonzero electric and magnetic susceptibilities. The macroscopic electromagnetic Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} - \mathbf{B}\cdot\mathbf{H}). \tag{2.1}$$

The electric field $\mathbf{E}(t, \mathbf{x})$ and the magnetic field $\mathbf{B}(t, \mathbf{x})$ are related to the fundamental four-vector potential A^μ by $\mathbf{E} = -c^{-1}\partial_t\mathbf{A} - \boldsymbol{\nabla}A^0$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. We treat stationary objects whose responses to the electric and magnetic fields are linear. For such materials, the $\mathbf{D}$ and $\mathbf{B}$ fields are related to the $\mathbf{E}$ and $\mathbf{H}$ fields by the convolutions $\mathbf{D}(t, \mathbf{x}) = \int_{-\infty}^{\infty} dt' \boldsymbol{\epsilon}(t', \mathbf{x})\mathbf{E}(t - t', \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x}) = \int_{-\infty}^{\infty} dt' \boldsymbol{\mu}(t', \mathbf{x})\mathbf{H}(t - t', \mathbf{x})$ in time. We consider local, isotropic permittivity and permeability, although our derivation can be adapted to apply to nonlocal and nonisotropic media simply by substituting the appropriate nonlocal and tensor permittivity and permeability functions. A more formal derivation of our starting point Eq. (2.1), which elucidates the causality properties of the permeability and permittivity response functions, is given in Appendix A.

We define the quantum-mechanical energy through the path integral, which sums all configurations of the electromagnetic fields constrained by periodic boundary conditions in time between 0 and T. Outside of this time interval the fields are periodically continued. Substituting the Fourier expansions of the form $\mathbf{E}(t, \mathbf{x}) = \sum_{n=-\infty}^{\infty} \mathbf{E}(\omega_n, \mathbf{x})e^{-i\omega_n t}$ with $\omega_n = 2\pi n/T$, we obtain the action

$$\begin{aligned} S(T) &= \frac{1}{2}\int_0^T dt \int d\mathbf{x}(\mathbf{E}\cdot\mathbf{D} - \mathbf{B}\cdot\mathbf{H}) \\ &= \frac{1}{2}T\sum_{n=-\infty}^{\infty}\int d\mathbf{x}(\mathbf{E}^*\cdot\boldsymbol{\epsilon}\mathbf{E} - \mathbf{B}^*\cdot\boldsymbol{\mu}^{-1}\mathbf{B}), \end{aligned} \tag{2.2}$$

where $\boldsymbol{\epsilon}$, $\mathbf{E}$, $\boldsymbol{\mu}$, and $\mathbf{B}$ on the right-hand side are functions of position $\mathbf{x}$ and frequency ω_n, and we have used $\mathbf{D}(\omega, \mathbf{x}) = \boldsymbol{\epsilon}(\omega, \mathbf{x})\mathbf{E}(\omega, \mathbf{x})$ and $\mathbf{H}(\omega, \mathbf{x}) = \frac{1}{\mu(\omega,\mathbf{x})}\mathbf{B}(\omega, \mathbf{x})$.

From the definition of the fields $\mathbf{E}$ and $\mathbf{B}$ in terms of the vector potential A^μ, we have $\boldsymbol{\nabla}\times\mathbf{E} = i\frac{\omega}{c}\mathbf{B}$, which enables us to eliminate $\mathbf{B}$ in the action,

$$\begin{aligned} S(T) = \frac{1}{2}T\sum_{n=-\infty}^{\infty}\int d\mathbf{x}\Big[\mathbf{E}^*\cdot\Big(\mathbb{I} - \frac{c^2}{\omega_n^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Big)\mathbf{E} \\ - \frac{c^2}{\omega_n^2}\mathbf{E}^*\cdot\mathbb{V}\mathbf{E}\Big], \end{aligned} \tag{2.3}$$

where

$$\mathbb{V} = \mathbb{I}\frac{\omega_n^2}{c^2}(1-\epsilon(\omega_n,\mathbf{x})) + \boldsymbol{\nabla}\times\left(\frac{1}{\mu(\omega_n,\mathbf{x})}-1\right)\boldsymbol{\nabla}\times \tag{2.4}$$

is the potential operator and we have restored the explicit frequency dependence of ϵ and μ. The potential operator is nonzero only at those points in space where the objects are located ($\epsilon \neq 1$ or $\mu \neq 1$). At small frequencies, typical materials have $\epsilon > 1$ and $\mu \approx 1$, and $\mathbb{V}$ can be regarded as an attractive potential.

In the functional integral we will sum over configurations of the field A^μ. This sum must be restricted by a choice of gauge, so that it does not include the infinitely redundant gauge orbits. We will choose to work in the gauge $A^0 = 0$, although of course no physical results depend on this choice.

B. Casimir energy of a quantum field

We use standard tools to obtain a functional integral expression for the ground state energy of a quantum field in a fixed background described by $\mathbb{V}(\omega,\mathbf{x})$. The overlap between the initial state $|\mathbf{E}_a\rangle$ of a system with the state $|\mathbf{E}_b\rangle$ after time T can be expressed as a functional integral with the fields fixed at the temporal boundaries [71],

$$\langle\mathbf{E}_b|e^{-iHT\hbar}|\mathbf{E}_a\rangle = \int \mathcal{D}\mathbf{A}|_{\substack{\mathbf{E}(t=0)=\mathbf{E}_a\\ \mathbf{E}(t=T)=\mathbf{E}_b}} e^{(i/\hbar)S[T]}, \tag{2.5}$$

where $S(T)$ is the action of Eq. (2.2) with the time integrals taken between zero and T, and H is the corresponding Hamiltonian.

If the initial and final states are set equal and summed over, the resulting functional integration defines the Minkowski space functional integral

$$\begin{aligned}\mathcal{Z}(T) &\equiv \sum_a \langle\mathbf{E}_a|e^{-iHT/\hbar}|\mathbf{E}_a\rangle = \mathrm{tr}e^{-iHT/\hbar}\\ &= \int \mathcal{D}\mathbf{A}e^{(i/\hbar)S[T]},\end{aligned} \tag{2.6}$$

which depends on the time T and the background potential $\mathbb{V}(\omega,\mathbf{x})$. The partition function that describes this system at temperature $1/\beta$ is defined by

$$Z(\beta) = \mathcal{Z}(-i\hbar\beta) = \mathrm{tr}e^{-\beta H}, \tag{2.7}$$

and the free energy F of the field is

$$F(\beta) = -\frac{1}{\beta}\log Z(\beta). \tag{2.8}$$

The limit $\beta\to\infty$ projects the ground state energy out of the trace,

$$\mathcal{E}_0 = F(\beta=\infty) = -\lim_{\beta\to\infty}\frac{1}{\beta}\log Z(\beta). \tag{2.9}$$

The unrenormalized energy $\mathcal{E}_0$ generally depends on the ultraviolet cutoff, but cutoff-dependent contributions arise from the objects individually and do not depend on their separations or orientations. Such terms can also arise after renormalization if objects are assumed to constrain electromagnetic waves with arbitrarily high frequencies (for example, if the fields are forced to vanish on a surface). Such boundary conditions should be regarded as artificial idealizations; in reality, when the wavelengths of the electromagnetic waves become shorter than the length scales that characterize the interactions of the material, the influence of the material on the waves vanishes [72]. Accordingly, the potential $\mathbb{V}$ should vanish for real materials in the high-frequency limit. Since we are only interested in energy *differences*, we can remove these divergences by subtracting the ground state energy of the system when the objects are in some reference configuration. In most cases we will take this configuration to have the objects infinitely far apart, but when calculating Casimir energies for one object inside another, some other configuration must be used. We denote the partition function for this reference configuration by $\bar{Z}$. In this way we obtain the Casimir energy,

$$\mathcal{E} = -\lim_{\beta\to\infty}\frac{1}{\beta}\log Z(\beta)/\bar{Z}(\beta). \tag{2.10}$$

Throughout our calculation of $\mathcal{E}$, we will thus be able to neglect any overall factors that are independent of the relative positions and orientations of the objects.

C. Euclidean electromagnetic action

By replacing the time T by $-i\hbar\beta$, we transform the Minkowski space functional integral $\mathcal{Z}(T)$ into the partition function $Z(\beta)$. In $A^0 = 0$ gauge, the result is simply to replace the frequencies $\omega_n = \frac{2\pi n}{T}$ in Eq. (2.4) by $i\frac{2\pi n}{\hbar\beta} = ic\kappa_n$, where κ_n is the nth Matsubara frequency divided by c. (In other gauges the temporal component A^0 of the vector field must be rotated too.)

The Lagrangian is quadratic, so the modes with different κ_n decouple and the partition function decomposes into a product of partition functions for each mode. Since the electromagnetic field is real, we have $\mathbf{E}^*(\omega) = \mathbf{E}(-\omega)$ on the real axis. We can thus further simplify this decomposition on the imaginary axis by considering $\kappa \geq 0$ only, but allowing $\mathbf{E}$ and $\mathbf{E}^*$ to vary independently in the path integral. Restricting to positive κ is possible because the response functions $\epsilon(ic\kappa,\mathbf{x})$ and $\mu(ic\kappa,\mathbf{x})$ are invariant under a change of sign in $ic\kappa$, as shown Appendix A. In the limit $\beta\to\infty$, the sum $\sum_{n\geq 0}$ turns into an integral $\frac{\hbar c\beta}{2\pi}\times\int_0^\infty d\kappa$, and we have

$$\mathcal{E}_0 = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa \log Z(\kappa), \tag{2.11}$$

where

$$Z(\kappa) = \int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{A}^* \exp\Big[-\beta \int d\mathbf{x}\mathbf{E}^* \cdot \Big(\mathbb{I} + \frac{1}{\kappa^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Big)\mathbf{E} + \frac{1}{\kappa^2}\mathbf{E}^* \cdot \mathbb{V}(ic\kappa, \mathbf{x})\mathbf{E}\Big], \tag{2.12}$$

$$\mathbb{V}(ic\kappa, \mathbf{x}) = \mathbb{I}\kappa^2(\boldsymbol{\epsilon}(ic\kappa, \mathbf{x}) - 1) + \boldsymbol{\nabla}\times\Big(\frac{1}{\boldsymbol{\mu}(ic\kappa, \mathbf{x})} - 1\Big)\boldsymbol{\nabla}\times. \tag{2.13}$$

The potential $\mathbb{V}(ic\kappa, \mathbf{x})$ is real for real κ, even though $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ can have imaginary parts for real frequencies ω. Our goal is now to manipulate $Z(\kappa)$ in Eq. (2.12) so that it is computable from the scattering properties of the objects.

III. GREEN'S FUNCTION EXPANSIONS AND TRANSLATION FORMULAS

A. The free Green's function

The free Green's function and its representations in various coordinate systems are crucial to our formalism. The free electromagnetic field ($\mathbb{V} = 0$) obeys equations of motion obtained by extremizing the corresponding action, Eq. (2.2),

$$\Big(-\mathbb{I}\frac{\omega^2}{c^2} + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Big)\mathbf{E}(\omega, \mathbf{x}) = 0. \tag{3.1}$$

We will employ the electromagnetic dyadic Green's function $\mathbb{G}_0$, defined by

$$\Big(-\mathbb{I}\frac{\omega^2}{c^2} + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Big)\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') = \mathbb{I}\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{3.2}$$

written here in the position space representation. It is easy to express $\mathbb{G}_0$ as a Fourier transform,

$$\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') = \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2 - (\omega/c + i\epsilon)^2}\Big(\mathbb{I} - \frac{c^2}{\omega^2}\mathbf{k}\otimes\mathbf{k}\Big), \tag{3.3}$$

where the displacement of the singularities at $k = \pm\frac{\omega}{c}$ corresponds to outgoing wave boundary conditions at infinity. By replacing the factors of $\mathbf{k}$ by gradients, $\mathbb{G}_0$ may be expressed in terms of elementary functions,

$$\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') = \Big(\mathbb{I} - \frac{c^2}{\omega^2}\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}'\Big)\frac{e^{i\omega|\mathbf{x}-\mathbf{x}'|/c}}{4\pi|\mathbf{x} - \mathbf{x}'|}. \tag{3.4}$$

In this representation it is easy to see that $\mathbb{G}_0$ is transverse, i.e. $\boldsymbol{\nabla}\cdot\mathbb{G}_0(\mathbf{x}, \mathbf{x}', \omega) = \mathbb{G}_0(\mathbf{x}, \mathbf{x}', \omega)\cdot\overleftarrow{\boldsymbol{\nabla}}' = 0$, for $\mathbf{x} \neq \mathbf{x}'$. $\mathbb{G}_0$ is not transverse at $\mathbf{x} = \mathbf{x}'$, as can be seen by taking the divergence of Eq. (3.2).

We work in coordinate systems in which we can use separation of variables and employ a spectral representation of $\mathbb{G}_0(\mathbf{x}, \mathbf{x}', \omega)$. That is, we represent the Green's function through the complete set of nonsingular ("regular"), transverse solutions to the differential equation, Eq. (3.1),

$$\mathbf{E}^{\mathrm{reg}}_\alpha(\omega, \mathbf{x}) = \langle\mathbf{x}|\mathbf{E}^{\mathrm{reg}}_\alpha(\omega)\rangle, \tag{3.5}$$

represented formally by the eigenstate kets $|\mathbf{E}^{\mathrm{reg}}_\alpha(\omega)\rangle$, where the generalized index α labels the scattering channel, including the polarization. For example, for spherical wave functions it represents the angular momentum quantum numbers (l, m) and the polarization E or M. We will choose to normalize these states in accord with standard conventions in electromagnetic scattering theory; as a result they are not necessarily normalized according to the conventions typically used in quantum mechanics. A list of the eigenfunctions for various common bases is given in Appendix B. The Green's functions can be expressed as the coordinate-space matrix element of the operator

$$\mathbb{G}_0(\omega) = \int_0^\infty d\omega' \sum_\alpha \mathcal{C}_\alpha(\omega')\frac{|\mathbf{E}^{\mathrm{reg}}_\alpha(\omega')\rangle\langle\mathbf{E}^{\mathrm{reg}}_\alpha(\omega')|}{(\omega'/c)^2 - (\omega/c + i\epsilon)^2}, \tag{3.6}$$

where the $i\epsilon$ has again been included to implement outgoing wave boundary conditions, so that the Green's function is causal.[2] We use the symbol $\mathbb{G}_0$ to represent both the matrix-valued representation of the Green's function in position space, Eq. (3.2), and the abstract Hilbert space operator, Eq. (3.6). The coefficients $\mathcal{C}_\alpha(\omega')$ are inserted because of our choice of normalization,

$$\int_0^\infty d\omega' \sum_\alpha \mathcal{C}_\alpha(\omega')|\mathbf{E}^{\mathrm{reg}}_\alpha(\omega')\rangle\langle\mathbf{E}^{\mathrm{reg}}_\alpha(\omega')| = \mathbb{I}. \tag{3.7}$$

It is also useful to represent the Green's function in a different way, in which one of the separable coordinates is identified as the "radial" variable and treated differently from the remaining coordinates. We let ξ_1 represent this coordinate and denote the remaining coordinates as ξ_2 and ξ_3. We introduce the "outgoing" solution in ξ_1, which is in the same scattering channel as the corresponding regular solution but obeys outgoing wave boundary conditions as $\xi_1 \to \infty$. It is linearly independent of the regular solution. The full outgoing solution is then obtained by multiplying the outgoing solution for ξ_1 by the regular solutions for ξ_2 and ξ_3. We can then express one of the regular wave functions in the position space representation of Eq. (3.6) as a sum of the outgoing solution for ω and the outgoing solution for $-\omega$. By specifying explicitly which of the two arguments of the Green's function has a greater value of ξ_1,

[2]The coordinate-space matrix element of Eq. (3.6) is transverse for all $\mathbf{x}$ and $\mathbf{x}'$, and therefore differs from the Green's function defined in Eq. (3.4) by terms local at $\mathbf{x} = \mathbf{x}'$. Since we never employ $\mathbb{G}_0$ at coincident points, we ignore this subtlety [73]. The use of the retarded Green's function not only makes sense physically, but is also dictated by the imaginary-frequency formalism, just as is the case for the response functions $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$. It is the *retarded* response functions that are analytically continued in the frequency domain to positive imaginary frequency, as shown in Appendix A.

we can carry out the ω integral for each of these two terms separately by closing the contour in the appropriate half-plane [73], and obtain

$$\mathbb{G}_0(\omega, \mathbf{x}, \mathbf{x}') = \sum_\alpha C_\alpha(\omega) \begin{cases} \mathbf{E}^{\text{out}}_\alpha(\omega, \xi_1, \xi_2, \xi_3) \otimes \mathbf{E}^{\text{reg}*}_\alpha(\omega, \xi_1', \xi_2', \xi_3') & \text{if } \xi_1(\mathbf{x}) > \xi_1'(\mathbf{x}') \\ \mathbf{E}^{\text{reg}}_\alpha(\omega, \xi_1, \xi_2, \xi_3) \otimes \mathbf{E}^{\text{in}*}_\alpha(\omega, \xi_1', \xi_2', \xi_3') & \text{if } \xi_1(\mathbf{x}) < \xi_1'(\mathbf{x}'). \end{cases} \tag{3.8}$$

In this form, the outgoing wave boundary condition is implemented explicitly. Since the Green's function is written as a linear combination of solutions to the free wave equation, it clearly satisfies Eq. (3.2) for $\mathbf{x} \neq \mathbf{x}'$. The normalization constant $C_\alpha(\omega)$, which is determined using the Wronskian of the regular and outgoing solutions and the completeness relationship for the regular solutions in ξ_2 and ξ_3, sets the correct "jump condition" for $\mathbf{x} = \mathbf{x}'$.

The outgoing solution is typically singular at $\xi_1 = 0$, but the Green's function with distinct arguments does not encounter that region, because the outgoing function is always evaluated for the larger argument. For example, in a spherical system the outgoing solution could take the form of a spherical Hankel function $h_l^{(1)}(kr) \sim \frac{e^{ikr}}{kr}$ with $k = \omega/c$, which obeys outgoing wave boundary conditions, is singular at the origin, and is independent of the corresponding regular solution $j_l(kr)$.

We will usually work on the imaginary k axis, in which case we will encounter the corresponding modified special functions. We continue to label these functions as "regular," "outgoing," and "incoming," even though they now increase exponentially for large ξ_1 for incoming and regular waves and decrease exponentially for outgoing waves. We also note that it may be convenient to redefine the wave functions to match the standard form of the corresponding modified functions, and to assign different phases to the two polarizations. The prefactor $C_\alpha(\omega)$ is then correspondingly redefined as $C_\alpha(\kappa)$ to incorporate these changes. A list of Green's function expansions in various common bases is given in Appendix B.

For a Cartesian coordinate system some of the previous statements have to be adapted slightly. We will take one of the Cartesian coordinates, say, z, to be the "radial" coordinate, as required by the context. For example, z might be the direction normal to the planar surface of a dielectric. The solutions are then given in terms of plane waves, $e^{ik_x x + ik_y y \pm i\sqrt{(\omega/c)^2 - \mathbf{k}_\perp^2} z}$, where $\mathbf{k}_\perp$ is the momentum perpendicular to the $\hat{\mathbf{z}}$ direction. All are regular and all contribute in the integral representation of Eq. (3.6). After analytic continuation to imaginary frequency, the free Green's function in Cartesian coordinates is expressed by the above formula if we identify outgoing solutions with plane wave functions that are exponentially decreasing in the $+\hat{\mathbf{z}}$ direction, $e^{ik_x x + ik_y y - \sqrt{\kappa^2 + \mathbf{k}_\perp^2} z}$, and regular solutions with the exponentially growing solutions $e^{ik_x x + ik_y y + \sqrt{\kappa^2 + \mathbf{k}_\perp^2} z}$.

The wave functions that appear in the series expansion of the free Green's functions in Eq. (3.8) satisfy wave equations with frequency ω. The integral representations in Eq. (3.6), on the other hand, contain wave functions of all frequencies. As we will see in Sec. IV, the ability to express the Casimir energy entirely in terms of an on-shell partial wave expansion with fixed ω will greatly simplify our calculations.

B. Translation matrices

We will use the free Green's function described in the previous subsection to combine the scattering amplitudes for two different objects. In this calculation, the one argument of the Green's function will be located on each object. As a result, if Eq. (3.8) is written in the basis appropriate to one object, we will want to "translate" one of the scattering solutions to the basis appropriate to the other object. The configuration of the two objects—either outside of each other, or one inside the other—will determine which object has the larger or smaller value of ξ_1, and therefore which solution needs to be expanded in the other basis.

We will make use of two expansions:

(1) The regular solutions form a complete set no matter what origin is used to define the decomposition into partial waves. Let $\{\mathbf{E}^{\text{reg}}_\beta(\kappa, \mathbf{x}_j)\}$ be the regular solutions expressed with respect to the origin of coordinates appropriate to object j, $\mathcal{O}_j$. It must be possible to expand a regular solution $\mathbf{E}^{\text{reg}}_\alpha(\kappa, \mathbf{x}_i)$, defined with respect to the origin $\mathcal{O}_i$ appropriate to object i, in terms of the $\{\mathbf{E}^{\text{reg}}_\beta(\kappa, \mathbf{x}_j)\}$,

$$\mathbf{E}^{\text{reg}}_\alpha(\kappa, \mathbf{x}_i) = \sum_\beta \mathcal{V}^{ji}_{\beta,\alpha}(\kappa, \mathbf{X}_{ji}) \mathbf{E}^{\text{reg}}_\beta(\kappa, \mathbf{x}_j), \tag{3.9}$$

where $\mathbf{X}_{ij} = -\mathbf{X}_{ji} = \mathbf{x}_i - \mathbf{x}_j$ is shown in Fig. 1. Note that $\mathbf{x}_i$ and $\mathbf{x}_j$ refer to the same space point $\mathbf{x}$, expressed as the displacement from different origins. This expansion will be applicable to the case of one object inside the other.

(2) The same type of expansion must also exist when the original wave obeys outgoing boundary conditions *except in a region that contains the origin* $\mathcal{O}_i$, where $\mathbf{E}^{\text{out}}_\alpha(\kappa, \mathbf{x}_i)$ is singular. We therefore have the expansion

$$\mathbf{E}^{\text{out}}_\alpha(\kappa, \mathbf{x}_i) = \sum_\beta \mathcal{U}^{ji}_{\beta,\alpha}(\kappa, \mathbf{X}_{ji}) \mathbf{E}^{\text{reg}}_\beta(\kappa, \mathbf{x}_j), \quad \text{for } \mathbf{x} \notin N(\mathcal{O}_i), \tag{3.10}$$

where $N(\mathcal{O}_i)$ is a neighborhood of the origin $\mathcal{O}_i$.

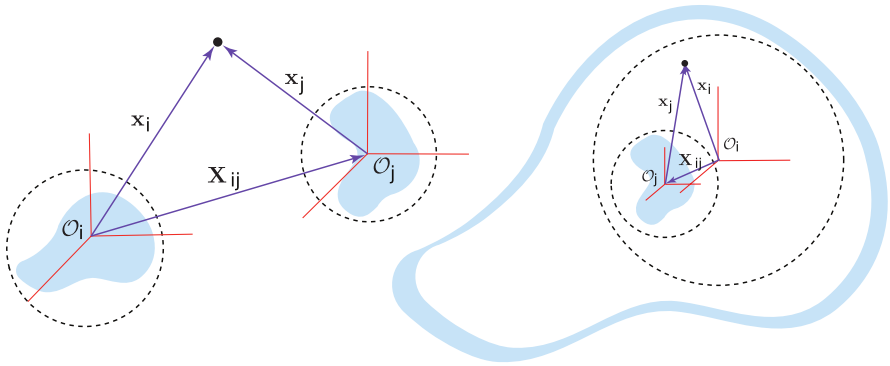

FIG. 1 (color online). Geometry of the outside (left) and inside (right) configurations. The dotted lines show surfaces separating the objects on which the radial variable is constant. The translation vector $\mathbf{X}_{ij} = \mathbf{x}_i - \mathbf{x}_j = -\mathbf{X}_{ji}$ describes the relative positions of the two origins.

This expansion will be applicable to the case where the objects are outside each other.

To apply these results to a given geometry, we must be able to distinguish between regular and outgoing waves over the whole of each object. That is, we require there to exist an origin and a separable coordinate system so that for all points $\mathbf{x}$ in one object and $\mathbf{x}'$ in another object, $\xi_1(\mathbf{x})$ is always greater than $\xi_1(\mathbf{x}')$, or vice versa. Having $\xi_1(\mathbf{x}) > \xi_1(\mathbf{x}')$ ensures that the Green's function is always evaluated by letting $\mathbf{x}$ be the argument of the outgoing wave function and $\mathbf{x}'$ be the argument of the regular wave function. We therefore require that any two objects be separated by a surface defined by the locations $\mathbf{x}$ where $\xi_1(\mathbf{x})$ is constant, as shown in Fig. 1. Depending on the coordinate system, this surface could be a plane, cylinder, sphere, etc.

The case of an elliptic cylinder and a circular cylinder illustrates this requirement. At large distances, the elliptic cylinder object can be separated from the circular cylinder object by a circular cylinder of radius ρ, as shown in Fig. 2(a). All points on the elliptic cylinder object have values of ρ_1 that are smaller than any point on the circular cylinder object, so in this case we could carry out the calculation in ordinary cylindrical coordinates. However, as shown in Fig. 2(b), if the separation becomes small enough, points on the circular cylinder object are closer to the center of the elliptic cylinder object (i.e. they lie at smaller ρ_1 than points on the elliptic cylinder object), and our method cannot be used in ordinary cylindrical coordinates. However, in *elliptic cylindrical* coordinates (see Appendix B 4), the surface of the elliptic cylinder object is itself a surface of constant elliptical radius μ_1, so all points on the elliptic cylinder object have smaller μ_1 than any point on the circular cylinder object, and our method applies. This case is shown in Fig. 2(c).

In a plane wave basis, we would exclude the case of two interlocking combs [74], since each comb has values of z that are both bigger and smaller than points on the other object, so again a single assignment of regular and outgoing solutions cannot be made.

When object j lies wholly outside of object i, as shown in the left panel of Fig. 1, in the basis of object i the point on object j will always have greater ξ_1 than the point on object i. We will therefore need to expand the outgoing wave in the basis for object j. Since the origin $\mathcal{O}_i$ is never encountered when the point $\mathbf{x}$ lies on object j, the outgoing solutions for i can be expanded in terms of the regular solutions for object j using Eq. (3.10). Since i is also wholly outside j, we can also proceed the other way around and expand the outgoing wave functions in the basis of object j in terms of regular solutions in the basis of object i. This implies that the translation matrix satisfies $\mathcal{U}^{ij} = \mathcal{U}^{ji,\dagger}$. When one object is inside another, as shown in the right panel of Fig. 1, in the basis of object i, the point on object j will always have smaller ξ_1 than the point on object i. We will therefore need to expand the regular wave in the basis for object j using Eq. (3.9). In contrast, we cannot use the expansion of the outgoing wave functions, because the origin of the inside object may overlap with the

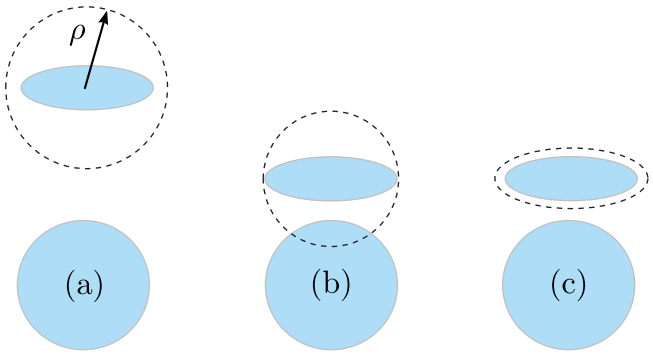

FIG. 2 (color online). An elliptic cylinder approaching another cylinder. When the elliptic cylinder is far (a), every point on the cylinder has smaller radius than any point on the lower cylinder and an expansion using an ordinary cylindrical basis can be used. This expansion breaks down once the elliptic cylinder is close (b), but in that case an expansion using an elliptic cylindrical basis applies (c).

origin of the outside object, in which case the expansion does not converge.

For a Cartesian geometry, the translation matrix is proportional to $e^{-i\mathbf{k}_\perp\cdot\mathbf{X}_{ji,\perp}-\sqrt{\kappa^2+\mathbf{k}_\perp^2}X_{ji,z}}$. It takes this simple form because plane wave functions are eigenfunctions of the translation operator. Then the regular wave function is evaluated on the object whose z coordinates are smaller and the outer and inner objects have larger and smaller z values, respectively.

The criterion for the expansion of the outgoing or regular wave functions is not topological. Instead, the proximity of the objects and their origins determines which expansion to use. In practice, it is usually easy to see which expansion is appropriate for any objects.

After expanding wave functions with respect to another origin using translation matrices, we can convert the wave functions from one basis to another, for example, from plane wave to spherical or cylindrical wave functions. This transformation is useful when the two objects are best described in different coordinate bases. The needed conversion matrices are supplied in Appendix D. Since it is more convenient to describe this conversion as a change of basis of the scattering amplitudes, we will not explicitly consider the combination of translation and conversion in this derivation, but instead we will illustrate the change of basis of the scattering amplitude in the examples.

C. Green's functions and translation matrices

To obtain the Green's function when one argument, say, $\mathbf{x}$, lies on object i and the other argument, say, $\mathbf{x}'$, lies on object j, we expand $\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}')$ in terms of coordinates $\mathbf{x}_i$ and $\mathbf{x}'_j$ that describe each point relative to the origin of the body on which it lies. For the different situations given above we have

$$\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}')=\sum_{\alpha,\beta}C_\beta(\kappa)\begin{cases}\mathbf{E}^{\mathrm{reg}}_\alpha(\kappa,\mathbf{x}_i)\otimes\mathcal{U}^{ji}_{\alpha\beta}(\kappa)\mathbf{E}^{\mathrm{reg}*}_\beta(\kappa,\mathbf{x}'_j) & \text{if } i \text{ and } j \text{ are outside each other}\\ \mathbf{E}^{\mathrm{reg}}_\alpha(\kappa,\mathbf{x}_i)\otimes\mathcal{V}^{ij}_{\alpha\beta}(\kappa)\mathbf{E}^{\mathrm{in}*}_\beta(\kappa,\mathbf{x}'_j) & \begin{cases}\text{if } i \text{ is inside } j, \text{ or}\\ \text{if } i \text{ is below } j \text{ (plane wave basis)}\end{cases}\\ \mathbf{E}^{\mathrm{out}}_\alpha(\kappa,\mathbf{x}_i)\otimes\mathcal{W}^{ji}_{\alpha\beta}(\kappa)\mathbf{E}^{\mathrm{reg}*}_\beta(\kappa,\mathbf{x}'_j) & \begin{cases}\text{if } j \text{ is inside } i, \text{ or}\\ \text{if } j \text{ is below } i \text{ (plane wave basis)}\end{cases}\end{cases}\tag{3.11}$$

where $\mathcal{W}^{ji}_{\alpha\beta}(\kappa)=\mathcal{V}^{ji,\dagger}_{\alpha\beta}(\kappa)\frac{C_\alpha(\kappa)}{C_\beta(\kappa)}$ and C_α is the normalization constant defined in Eq. (3.8). We can express these cases in the consolidated form,

$$\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}')=\sum_{\alpha,\beta}C_\beta(\kappa)(\mathbf{E}^{\mathrm{reg}}_\alpha(\kappa,\mathbf{x}_i)\ \mathbf{E}^{\mathrm{out}}_\alpha(\kappa,\mathbf{x}_i))\otimes\begin{pmatrix}\mathcal{U}^{ji}_{\alpha\beta}(\kappa) & \mathcal{V}^{ij}_{\alpha\beta}(\kappa)\\ \mathcal{W}^{ji}_{\alpha\beta}(\kappa) & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}^{\mathrm{reg}*}_\beta(\kappa,\mathbf{x}'_j)\\ \mathbf{E}^{\mathrm{in}*}_\beta(\kappa,\mathbf{x}'_j)\end{pmatrix},\tag{3.12}$$

where only one of the three submatrices is nonzero for any pair of objects i and j as given in Eq. (3.11). The expansion can be written more formally as

$$\mathbb{G}_0(ic\kappa)=\sum_{\alpha,\beta}(-C_\beta(\kappa))(|\mathbf{E}^{\mathrm{reg}}_\alpha(\kappa)\rangle\ |\mathbf{E}^{\mathrm{out}}_\alpha(\kappa)\rangle)\mathbb{X}^{ij}_{\alpha\beta}(\kappa)\begin{pmatrix}\langle\mathbf{E}^{\mathrm{reg}}_\beta(\kappa)|\\ \langle\mathbf{E}^{\mathrm{in}}_\beta(\kappa)|\end{pmatrix},\tag{3.13}$$

where the bras and kets are to be evaluated in position space in the appropriately restricted domains and the $\mathbb{X}$-matrix is defined, for convenience, as the negative of the matrix containing the translation matrices,

$$\mathbb{X}^{ij}(\kappa)=\begin{pmatrix}-\mathcal{U}^{ji}(\kappa) & -\mathcal{V}^{ij}(\kappa)\\ -\mathcal{W}^{ji}(\kappa) & 0\end{pmatrix}.\tag{3.14}$$

The translation matrices for various geometries are provided in Appendix C.

IV. A REVIEW OF ASPECTS OF THE CLASSICAL SCATTERING OF ELECTROMAGNETIC FIELDS

In this section, we review the key results from scattering theory needed to compute the scattering amplitude of each body individually. Comprehensive derivations can be found in Refs. [75,76]. The approach we will use was first developed by Waterman [77,78], albeit not in the operator form that is used here. In the subsequent section we will then combine these results with the translation matrices of the previous section to compute $Z(\kappa)$.

The Fourier-transformed electromagnetic action of Eq. (2.2) yields the frequency-dependent Maxwell equations:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{E}(\omega,\mathbf{x})&=i\frac{\omega}{c}\mathbf{B}(\omega,\mathbf{x}),\\ \boldsymbol{\nabla}\times\frac{1}{\mu}\mathbf{B}(\omega,\mathbf{x})&=-i\frac{\omega}{c}\boldsymbol{\epsilon}\mathbf{E}(\omega,\mathbf{x}).\end{aligned}\tag{4.1}$$

Combining these two equations, we obtain

$$(\mathbb{H}_0+\mathbb{V}(\omega,\mathbf{x}))\mathbf{E}(\omega,\mathbf{x})=\frac{\omega^2}{c^2}\mathbf{E}(\omega,\mathbf{x}),\tag{4.2}$$

where

$$\mathbb{H}_0 = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times, \tag{4.3}$$
$$\mathbb{V}(\omega, \mathbf{x}) = \mathbb{I}\frac{\omega^2}{c^2}(1 - \epsilon(\omega, \mathbf{x})) + \boldsymbol{\nabla} \times \left(\frac{1}{\mu(\omega, \mathbf{x})} - 1\right)\boldsymbol{\nabla} \times,$$

which is the same potential operator as the one obtained by rearranging the Lagrangian [see Eq. (2.4)]. Since the electromagnetic potential is a differential operator, care must be taken with operator ordering.

The Lippmann-Schwinger equation [79]

$$|\mathbf{E}\rangle = |\mathbf{E}_0\rangle - \mathbb{G}_0\mathbb{V}|\mathbf{E}\rangle \tag{4.4}$$

expresses the general solution to Eq. (4.2). Here $\mathbb{G}_0$ is the free electromagnetic tensor Green's function discussed in Sec. III and the homogeneous solution $|\mathbf{E}_0\rangle$ obeys $(-\frac{\omega^2}{c^2}\mathbb{I} + \mathbb{H}_0)|\mathbf{E}_0\rangle = 0$, which can be chosen to be either a regular or outgoing wave for a particular frequency ω. We can iteratively substitute for $|\mathbf{E}\rangle$ in Eq. (4.4) to obtain the formal expansion

$$\begin{aligned}|\mathbf{E}\rangle &= |\mathbf{E}_0\rangle - \mathbb{G}_0\mathbb{V}|\mathbf{E}_0\rangle + \mathbb{G}_0\mathbb{V}\mathbb{G}_0\mathbb{V}|\mathbf{E}\rangle - \ldots \\ &= |\mathbf{E}_0\rangle - \mathbb{G}_0\mathbb{T}|\mathbf{E}_0\rangle,\end{aligned} \tag{4.5}$$

where the electromagnetic $\mathbb{T}$-operator is defined as

$$\mathbb{T} = \mathbb{V}\frac{\mathbb{I}}{\mathbb{I} + \mathbb{G}_0\mathbb{V}} = \mathbb{V}\mathbb{G}\mathbb{G}_0^{-1}, \tag{4.6}$$

and $\mathbb{G}$ is the Green's function of the full Hamiltonian, $(-\frac{\omega^2}{c^2}\mathbb{I} + \mathbb{H}_0 + \mathbb{V})\mathbb{G} = \mathbb{I}$. We note that $\mathbb{T}$, $\mathbb{G}_0$, and $\mathbb{G}$ are all functions of frequency ω and nonlocal in space. As can be seen from expanding $\mathbb{T}$ in Eq. (4.6) in a power series, $\mathbb{T}(\omega, \mathbf{x}, \mathbf{x}') = \langle\mathbf{x}|\mathbb{T}(\omega)|\mathbf{x}'\rangle$ is zero whenever $\mathbf{x}$ or $\mathbf{x}'$ is not located on an object, i.e., where $\mathbb{V}(\omega, \mathbf{x})$ is zero. This result does not, however, apply to

$$\mathbb{T}^{-1} = \mathbb{G}_0 + \mathbb{V}^{-1}, \tag{4.7}$$

because the free Green's function is nonlocal.

Next we connect the matrix elements of the $\mathbb{T}$-operator between states with equal ω to the scattering amplitude $\mathcal{F}$. In our formalism, only this restricted subset of $\mathbb{T}$-operator matrix elements is needed in the computation of the Casimir energy.

A. Outside scattering

We consider a scattering process in which a regular wave interacts with an object and scatters outward, as depicted in the left panel of Fig. 3.[3] For outside scattering the homogeneous solution $|\mathbf{E}_0\rangle$ in Eq. (4.5) is taken to be the regular wave function $|\mathbf{E}_\alpha^{\text{reg}}(\omega)\rangle$. We choose a convenient "scattering origin" in the inside region, consistent with any symmetries of the problem if possible.

To find the field $\mathbf{E}$ at a coordinate $\mathbf{x}$ far enough outside the object, we use Eq. (4.5) in position space and the expansion in Eq. (3.8) for $\mathbb{G}_0$:

$$\begin{aligned}\mathbf{E}(\omega, \mathbf{x}) = \mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x}) &- \sum_\beta \mathbf{E}_\beta^{\text{out}}(\omega, \mathbf{x}) \\ &\times \int C_\beta(\omega)\mathbf{E}_\beta^{\text{reg}*}(\omega, \mathbf{x}') \\ &\cdot \mathbb{T}(\omega, \mathbf{x}', \mathbf{x}'')\mathbf{E}_\alpha^{\text{reg}}(\omega, \mathbf{x}'')d\mathbf{x}'d\mathbf{x}''.\end{aligned} \tag{4.8}$$

Here "far enough outside" means that $\mathbf{x}$ has larger ξ_1 than any point on the object, meaning that we are always taking the same choice in Eq. (3.8), as described in Sec. III. The equation can be written in Dirac notation, again with the condition that the domain of the functional Hilbert space is chosen appropriately to the type of solution,

$$\begin{aligned}|\mathbf{E}(\omega)\rangle = |\mathbf{E}_\alpha^{\text{reg}}(\omega)\rangle &+ \sum_\beta |\mathbf{E}_\beta^{\text{out}}(\omega)\rangle \\ &\times \underbrace{(-1)C_\beta(\omega)\langle\mathbf{E}_\beta^{\text{reg}}(\omega)|\mathbb{T}(\omega)|\mathbf{E}_\alpha^{\text{reg}}(\omega)\rangle}_{\mathcal{F}_{\beta,\alpha}^{ee}(\omega)},\end{aligned} \tag{4.9}$$

which defines $\mathcal{F}_{\beta,\alpha}^{ee}$ as the exterior/exterior scattering amplitude (the one evaluated between two regular solutions). We will use analogous notation in the other cases below.

At coordinates $\mathbf{x}$ "far enough inside" a cavity of the object, meaning that $\mathbf{x}$ has smaller ξ_1 than any point on the object, we have the opposite case in Eq. (3.8) and the field $\mathbf{E}$ is given by

$$\begin{aligned}|\mathbf{E}(\omega)\rangle = |\mathbf{E}_\alpha^{\text{reg}}(\omega)\rangle &+ \sum_\beta |\mathbf{E}_\beta^{\text{reg}}(\omega)\rangle \\ &\times \underbrace{(-1)C_\beta(\omega)\langle\mathbf{E}_\beta^{\text{in}}(\omega)|\mathbb{T}(\omega)|\mathbf{E}_\alpha^{\text{reg}}(\omega)\rangle}_{\mathcal{F}_{\beta,\alpha}^{ie}(\omega)},\end{aligned} \tag{4.10}$$

where again the free states are only defined over the appropriate domain in position space, and $\mathcal{F}^{ie}$ indicates the interior/exterior scattering amplitude.

B. Inside scattering

In the study of Casimir problems with one object inside the other, it is useful to imagine a situation that would be difficult to realize in actual scattering experiments, in which the wave probing the object originates inside the object and is scattered as a regular wave inside the object and as an outgoing wave outside, as depicted in the right panel of Fig. 3.

The situation is expressed mathematically by letting the homogeneous solution $|\mathbf{E}_0\rangle$ in Eq. (4.5) be an outgoing wave $|\mathbf{E}_\alpha^{\text{out}}(\omega)\rangle$. The equation can be expressed in condensed form as before. Inside the object we have

[3] Alternatively, we can set up asymptotically incoming and outgoing waves on the outside and regular waves inside. The amplitudes of the outgoing waves are then given by the $\mathcal{S}$-matrix, which is related to the scattering amplitude $\mathcal{F}$ by $\mathcal{F} = (\mathcal{S} - \mathcal{I})/2$. Although these two matrices carry equivalent information, the scattering amplitude will be more convenient for our calculation.

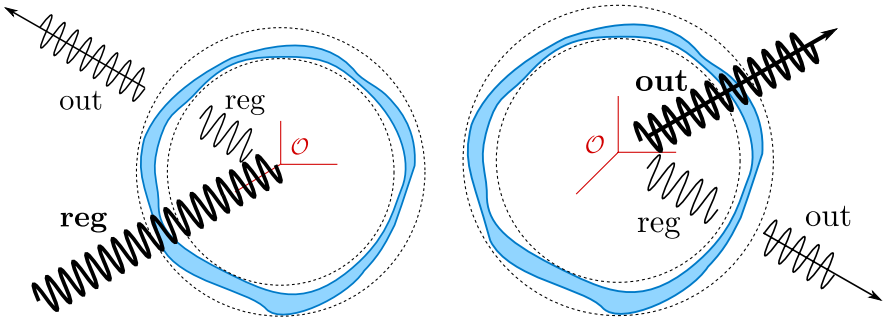

FIG. 3 (color online). The scattering waves for outside scattering (left panel) and inside scattering (right panel). In both cases the homogeneous solution $\mathbf{E}_0(\omega)$ is shown in bold. For outside scattering, the homogeneous solution is a regular wave, which produces a regular wave inside the object and an outgoing wave outside the object. For inside scattering, the homogeneous solution is an outgoing wave, which produces a regular wave inside the object and an outgoing wave outside the object.

$$|\mathbf{E}(\omega)\rangle = |\mathbf{E}^{\text{out}}_\alpha(\omega)\rangle + \sum_\beta |\mathbf{E}^{\text{reg}}_\beta(\omega)\rangle \times \underbrace{(-1)C_\beta(\omega)\langle \mathbf{E}^{\text{in}}_\beta(\omega)|\mathbb{T}(\omega)|\mathbf{E}^{\text{out}}_\alpha(\omega)\rangle}_{\mathcal{F}^{ii}_{\beta,\alpha}(\omega)}, \quad (4.11)$$

and outside the object we have

$$|\mathbf{E}(\omega)\rangle = |\mathbf{E}^{\text{out}}_\alpha(\omega)\rangle + \sum_\beta |\mathbf{E}^{\text{out}}_\beta(\omega)\rangle \times \underbrace{(-1)C_\beta(\omega)\langle \mathbf{E}^{\text{reg}}_\beta(\omega)|\mathbb{T}(\omega)|\mathbf{E}^{\text{out}}_\alpha(\omega)\rangle}_{\mathcal{F}^{ei}_{\beta,\alpha}(\omega)}. \quad (4.12)$$

C. Remarks

We have obtained the scattering amplitude in the basis of free solutions with fixed ω. Since one is normally interested in the scattering of waves outside the object, the scattering amplitude usually refers to $\mathcal{F}^{ee}$. We will use a more general definition, which encompasses all possible combinations of inside and outside. The scattering amplitude is always on-shell, because the frequencies of the bra and ket wave functions are both equal to ω. As a result, it is a special case of the $\mathbb{T}$-operator, which connects wave functions with different ω.

It is usually not practical to calculate the matrix elements by finding the abstract $\mathbb{T}$-operator and taking its inner products with free wave functions. Instead, one typically considers an ansatz for the solutions appropriate for inside or outside scattering in the various regions, with unknown scattering amplitudes, and then solves the wave equation, matching the solutions in different regions at their boundaries.

We will find it convenient to assemble the scattering amplitudes for inside and outside into a single matrix,

$$\mathbb{F}(\kappa) = \begin{pmatrix} \mathcal{F}^{ee}(\kappa) & \mathcal{F}^{ei}(\kappa) \\ \mathcal{F}^{ie}(\kappa) & \mathcal{F}^{ii}(\kappa) \end{pmatrix} = (-1)\begin{pmatrix} [C_\alpha(\kappa)\langle \mathbf{E}^{\text{reg}}_\alpha(\kappa)|\mathbb{T}(ic\kappa)|\mathbf{E}^{\text{reg}}_\beta(\kappa)\rangle] & [C_\alpha(\kappa)\langle \mathbf{E}^{\text{reg}}_\alpha(\kappa)|\mathbb{T}(ic\kappa)|\mathbf{E}^{\text{out}}_\beta(\kappa)\rangle] \\ [C_\alpha(\kappa)\langle \mathbf{E}^{\text{in}}_\alpha(\kappa)|\mathbb{T}(ic\kappa)|\mathbf{E}^{\text{reg}}_\beta(\kappa)\rangle] & [C_\alpha(\kappa)\langle \mathbf{E}^{\text{in}}_\alpha(\kappa)|\mathbb{T}(ic\kappa)|\mathbf{E}^{\text{out}}_\beta(\kappa)\rangle] \end{pmatrix}. \quad (4.13)$$

Here we have written this expression in terms of modified wave functions for $\omega = ic\kappa$, with the corresponding normalization constant, since that is the case we will use. The derivations of the scattering amplitudes carry over directly to this case, with κ replaced by ω; for example, Eq. (4.9) becomes

$$|\mathbf{E}(\kappa)\rangle = |\mathbf{E}^{\text{reg}}_\alpha(\kappa)\rangle + \sum_\beta |\mathbf{E}^{\text{out}}_\alpha(\kappa)\rangle \times \underbrace{(-1)C_\beta(\kappa)\langle \mathbf{E}^{\text{reg}}_\beta(\kappa)|\mathbb{T}(ic\kappa)|\mathbf{E}^{\text{reg}}_\alpha(\kappa)\rangle}_{\mathcal{F}^{ee}_{\beta,\alpha}(\kappa)}. \quad (4.14)$$

V. PARTITION FUNCTION IN TERMS OF THE SCATTERING AMPLITUDE

With the tools of the previous two sections, we are now able to reexpress the Euclidean electromagnetic partition function of Eq. (2.12) in terms of the scattering theory results derived in Sec. IV for imaginary frequency. We will exchange the fluctuating field $\mathbf{A}$, which is subject to the potential $\mathbb{V}(ic\kappa, \mathbf{x})$, for a free field $\mathbf{A}'$, together with fluctuating currents $\mathbf{J}$ and charges $-\frac{i}{\omega}\nabla\cdot\mathbf{J}$ that are confined to the objects. The sequence of two changes of variables that will be performed is often referred to as the Hubbard-Stratonovich transformation in condensed matter physics.

We multiply and divide the partition function Eq. (2.12) by

$$W = \int \mathcal{D}\mathbf{J}\mathcal{D}\mathbf{J}^*|_{\text{obj}} \exp\left[-\frac{T}{\hbar}\int d\mathbf{x}\mathbf{J}^*(\mathbf{x})\cdot\mathbb{V}^{-1}(ic\kappa,\mathbf{x})\mathbf{J}(\mathbf{x})\right] = \det\mathbb{V}(ic\kappa,\mathbf{x},\mathbf{x}'), \tag{5.1}$$

where $|_{\text{obj}}$ indicates that the currents are defined only over the objects, i.e. the domain where $\mathbb{V}$ is nonzero (and therefore $\mathbb{V}^{-1}$ exists), and we have represented the local potential as a matrix in position space, $\mathbb{V}(ic\kappa,\mathbf{x},\mathbf{x}') = \mathbb{V}(ic\kappa,\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{x}')$. Our derivation generalizes straightforwardly to the case of a nonlocal potential $\mathbb{V}(ic\kappa,\mathbf{x},\mathbf{x}')$, assuming it is still confined to each object individually.

We then change variables in the integration, $\mathbf{J}(\mathbf{x}) = \mathbf{J}'(\mathbf{x}) + \frac{i}{\kappa}\mathbb{V}(ic\kappa,\mathbf{x})\mathbf{E}(\mathbf{x})$ and $\mathbf{J}^*(\mathbf{x}) = \mathbf{J}'^*(\mathbf{x}) + \frac{i}{\kappa}\mathbb{V}(ic\kappa,\mathbf{x})\mathbf{E}^*(\mathbf{x})$, to obtain

$$Z(\kappa) = \frac{1}{W}\int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{A}^*\mathcal{D}\mathbf{J}'\mathcal{D}\mathbf{J}'^*|_{\text{obj}} \times \exp\left[-\frac{T}{\hbar}\int d\mathbf{x} \times\left(\mathcal{H} + \left(\mathbf{J}'^*(\mathbf{x}) + \frac{i}{\kappa}\mathbb{V}(ic\kappa,\mathbf{x})\mathbf{E}^*(\mathbf{x})\right)\cdot\mathbb{V}^{-1}(ic\kappa,\mathbf{x})\left(\mathbf{J}'(\mathbf{x}) + \frac{i}{\kappa}\mathbb{V}(ic\kappa,\mathbf{x})\mathbf{E}(\mathbf{x})\right)\right)\right], \tag{5.2}$$

where

$$\mathcal{H} = \mathbf{E}^*(\mathbf{x})\cdot\left(\mathbb{I} + \frac{1}{\kappa^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\right)\mathbf{E}(\mathbf{x}) + \frac{1}{\kappa^2}\mathbf{E}^*(\mathbf{x})\cdot\mathbb{V}(ic\kappa,\mathbf{x})\mathbf{E}(\mathbf{x}). \tag{5.3}$$

Next we use a second change of variables, $\mathbf{E}(ic\kappa,\mathbf{x}) = \mathbf{E}'(ic\kappa,\mathbf{x}) - i\kappa\int d\mathbf{x}'\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}')\mathbf{J}'(\mathbf{x}')$ and $\mathbf{E}^*(ic\kappa,\mathbf{x}) = \mathbf{E}'^*(ic\kappa,\mathbf{x}) - i\kappa\int d\mathbf{x}'\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}')\mathbf{J}'^*(\mathbf{x}')$, which simplifies Eq. (5.2) to

$$Z(\kappa) = \frac{Z_0}{W}\int \mathcal{D}\mathbf{J}'\mathcal{D}\mathbf{J}'^*|_{\text{obj}} \exp\left[-\frac{T}{\hbar}\int d\mathbf{x}d\mathbf{x}'\mathbf{J}'^*(\mathbf{x})\cdot(\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}') + \mathbb{V}^{-1}(ic\kappa,\mathbf{x},\mathbf{x}'))\mathbf{J}'(\mathbf{x}')\right], \tag{5.4}$$

where

$$Z_0 = \int \mathcal{D}\mathbf{A}'\mathcal{D}\mathbf{A}'^*\exp\left[-\frac{T}{\hbar}\int d\mathbf{x}\mathbf{E}'^*(\mathbf{x})\cdot\left(\mathbb{I} + \frac{1}{\kappa^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\right)\mathbf{E}'(\mathbf{x})\right] \tag{5.5}$$

is the partition function of the free field, which is independent of the objects. The new partition function of Eq. (5.4) contains a sum over current fluctuations in place of the original field fluctuations in Eq. (2.12). The interaction of current fluctuations at different points $\mathbf{x}$ and $\mathbf{x}'$ is described by the free Green's function $\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}')$ alone. [If the potential $\mathbb{V}(ic\kappa,\mathbf{x},\mathbf{x}')$ is nonlocal, this statement still holds for two points $\mathbf{x}$ and $\mathbf{x}'$ on two different objects.] This is the expected interaction term. For example, in the static limit $\kappa = 0$, the free Green's function is just the Coulomb interaction term $\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$. The inverse potential penalizes current fluctuations if the potential is small. In vacuum, the potential vanishes, so current fluctuations are infinitely costly and thus are not permitted. But of course the current fluctuations are already constrained to the objects.

To put the partition function into a suitable form for practical computations, we will use the results of the previous sections to reexpress the microscopic current fluctuations as macroscopic multipole fluctuations, which then can be connected to the individual objects' scattering amplitudes. This transformation comes about naturally once the current fluctuations are decomposed according to the objects on which they occur and the appropriate expansions of the Green's function are introduced. We begin this process by noticing that the operator in the exponent of the integrand in Eq. (5.4) is the negative of the inverse of the $\mathbb{T}$-operator [see Eq. (4.7)], and hence

$$Z(\kappa) = Z_0\det\mathbb{V}^{-1}(ic\kappa,\mathbf{x},\mathbf{x}')\det\mathbb{T}(ic\kappa,\mathbf{x},\mathbf{x}') \tag{5.6}$$

which is in agreement with a more direct calculation: Since $Z_0 = \det\mathbb{G}_0(ic\kappa,\mathbf{x},\mathbf{x}')$ and $Z(\kappa) = \det\mathbb{G}(ic\kappa,\mathbf{x},\mathbf{x}')$, we only need to take the determinant of Eq. (4.6) to arrive at the result of Eq. (5.6).

Both Z_0 and $\det\mathbb{V}^{-1}(ic\kappa,\mathbf{x})$ are independent of the separation of the objects, since the former is simply the free Green's function, while the latter is diagonal in $\mathbf{x}$. Even a nonlocal potential $\mathbb{V}(ic\kappa,\mathbf{x},\mathbf{x}')$ only connects points within the same object, so its determinant is also independent of the objects' separation. Because these determinants do not depend on separation, they will be canceled by a reference partition function in the final result. We are thus left with the task of computing the determinant of the $\mathbb{T}$-operator.

As has been discussed in Sec. IV, the $\mathbb{T}$-operator $\mathbb{T}(ic\kappa,\mathbf{x},\mathbf{x}')$ is not diagonal in the spatial coordinates. Its determinant needs to be taken over the spatial indices $\mathbf{x}$ and $\mathbf{x}'$, which are restricted to the objects because the fluctuating currents $\mathbf{J}(\mathbf{x})$ in the functional integrals are zero away from the objects. This determinant also runs over the ordinary vector components of the electromagnetic $\mathbb{T}$-operator.

A change of basis to momentum space does not help in computing the determinant of the $\mathbb{T}$-operator, even though it does help in finding the determinant of the free Green's function. One reason is that the momentum basis is not orthogonal over the domain of the indices $\mathbf{x}$ and $\mathbf{x}'$, which is restricted to the objects. In addition, a complete momentum basis includes not only all directions of the momentum vector, but also all magnitudes of the momenta. So, in the matrix element $\langle\mathbf{E}_{\mathbf{k}}|\mathbb{T}(\omega)|\mathbf{E}_{\mathbf{k}'}\rangle$ the wave numbers k and k'

would not have to match, and could also differ from ω/c. That is, the matrix elements could be off-shell. Therefore, the $\mathbb{T}$-operator could not simply be treated as if it was the scattering amplitude, which is the on-shell representation of the operator in the subbasis of frequency ω (see Sec. IV), and is significantly easier to calculate. Nonetheless, we will see that it is possible to express the Casimir energy in terms of the on-shell operator only, by remaining in the position basis.

From Eq. (4.6), we know that the inverse of the $\mathbb{T}$-operator equals the sum of the free Green's function and the inverse of the potential. Since the determinant of the inverse operator is the reciprocal of the determinant, it is expedient to start with the inverse $\mathbb{T}$-operator. We then separate the basis involving all the objects into blocks for the n objects. In a schematic notation, we have

$$[\langle \mathbf{x}|\mathbb{T}^{-1}|\mathbf{x}'\rangle] = \begin{pmatrix} [\langle \mathbf{x}_1|\mathbb{T}_1^{-1}|\mathbf{x}_1'\rangle] & [\langle \mathbf{x}_1|\mathbb{G}_0|\mathbf{x}_2'\rangle] & \cdots \\ [\langle \mathbf{x}_2|\mathbb{G}_0|\mathbf{x}_1'\rangle] & [\langle \mathbf{x}_2|\mathbb{T}_2^{-1}|\mathbf{x}_2'\rangle] & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}, \tag{5.7}$$

where the ijth submatrix refers to $\mathbf{x} \in$ object i and $\mathbf{x}' \in$ object j and $\mathbf{x}_i$ represents a point in object i measured with respect to some fixed coordinate system. Unlike the position vectors in Sec. III, at this point the subscript of $\mathbf{x}_i$ does not indicate the origin with respect to which the vector is measured, but rather the object on which the point lies. Square brackets are used to remind us that we are considering the entire matrix or submatrix and not a single matrix element. We note that the operators $\mathbb{T}$ and $\mathbb{G}_0$ are functions of $ic\kappa$, but for simplicity we suppress this argument throughout this derivation. When the two spatial indices lie on different objects, only the free Green's function remains in the off-diagonal submatrices. Even if the potential $\mathbb{V}(ic\kappa, \mathbf{x}, \mathbf{x}')$ is nonlocal in space, it does not connect points on different objects. It follows that the inverse of the potential is block diagonal in position space, where each block involves points on the same object, i.e., $\langle \mathbf{x}_i|\mathbb{V}^{-1}|\mathbf{x}_j'\rangle = 0$ for $i \neq j$.

Next, we multiply $\mathbb{T}^{-1}$ by a reference $\mathbb{T}$-operator $\mathbb{T}_\infty$ without off-diagonal submatrices, which can be interpreted as the $\mathbb{T}$-operator at infinite separation,

$$[\langle \mathbf{x}|\mathbb{T}_\infty\mathbb{T}^{-1}|\mathbf{x}''\rangle] = \begin{pmatrix} [\langle \mathbf{x}_1|\mathbf{x}_1''\rangle] & [\int d\mathbf{x}_1' \langle \mathbf{x}_1|\mathbb{T}_1|\mathbf{x}_1'\rangle\langle \mathbf{x}_1'|\mathbb{G}_0|\mathbf{x}_2''\rangle] & \cdots \\ [\int d\mathbf{x}_2' \langle \mathbf{x}_2|\mathbb{T}_2|\mathbf{x}_2'\rangle\langle \mathbf{x}_2'|\mathbb{G}_0|\mathbf{x}_1''\rangle] & [\langle \mathbf{x}_2|\mathbf{x}_2''\rangle] & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}. \tag{5.8}$$

Each off-diagonal submatrix $[\int d\mathbf{x}_i' \langle \mathbf{x}_i|\mathbb{T}_i|\mathbf{x}_i'\rangle\langle \mathbf{x}_i'|\mathbb{G}_0|\mathbf{x}_j''\rangle]$ is the product of the $\mathbb{T}$-operator of object i, evaluated at two points $\mathbf{x}_i$ and $\mathbf{x}_i'$ on that object, multiplied by the free Green's function, which connects $\mathbf{x}_i'$ to some point $\mathbf{x}_j''$ on object j.

Now we shift all variables to the coordinate systems of the objects on which they lie. As a result, the index on a position vector $\mathbf{x}_i$ now refers to the object i on which the point lies *and* to the coordinate system with origin $\mathcal{O}_i$ in which the vector is represented, in agreement with the notation of Sec. III. The off-diagonal submatrices in Eq. (5.8) can then be rewritten using Eq. (3.13) as

$$\sum_{\alpha,\beta}\Big[(\langle \mathbf{x}_i|\mathbb{T}_i|\mathbf{E}_\alpha^{\mathrm{reg}}(\kappa)\rangle \ \langle \mathbf{x}_i|\mathbb{T}_i|\mathbf{E}_\alpha^{\mathrm{out}}(\kappa)\rangle)\mathbb{X}_{\alpha\beta}^{ij} \begin{pmatrix} \langle \mathbf{E}_\beta^{\mathrm{reg}}(\kappa)|\mathbf{x}_j''\rangle \\ \langle \mathbf{E}_\beta^{\mathrm{in}}(\kappa)|\mathbf{x}_j''\rangle \end{pmatrix} (-C_\beta(\kappa)) \Big]. \tag{5.9}$$

The matrix $[\langle \mathbf{x}|\mathbb{T}_\infty\mathbb{T}^{-1}|\mathbf{x}''\rangle]$ has the structure $\mathbb{I} + \mathbb{A}\mathbb{B}$, where

$$\mathbb{A} = \sum_\alpha \begin{pmatrix} 0 & 0 & [\langle \mathbf{x}_1|\mathbb{T}_1|\mathbf{E}_\alpha^{\mathrm{reg}}(\kappa)\rangle\mathbb{X}_{\alpha\beta}^{12}] & [\langle \mathbf{x}_1|\mathbb{T}_1|\mathbf{E}_\alpha^{\mathrm{out}}(\kappa)\rangle\mathbb{X}_{\alpha\beta}^{12}] & \cdots \\ [\langle \mathbf{x}_2|\mathbb{T}_2|\mathbf{E}_\alpha^{\mathrm{reg}}(\kappa)\rangle\mathbb{X}_{\alpha\beta}^{21}] & [\langle \mathbf{x}_2|\mathbb{T}_2|\mathbf{E}_\alpha^{\mathrm{out}}(\kappa)\rangle\mathbb{X}_{\alpha\beta}^{21}] & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{5.10}$$

and

$$\mathbb{B} = \begin{pmatrix} [-C_\beta(\kappa)\langle \mathbf{E}_\beta^{\mathrm{reg}}(\kappa)|\mathbf{x}_1''\rangle] & 0 & \cdots \\ [-C_\beta(\kappa)\langle \mathbf{E}_\beta^{\mathrm{in}}(\kappa)|\mathbf{x}_1''\rangle] & 0 & \cdots \\ 0 & [-C_\beta(\kappa)\langle \mathbf{E}_\beta^{\mathrm{reg}}(\kappa)|\mathbf{x}_2''\rangle] & \cdots \\ 0 & [-C_\beta(\kappa)\langle \mathbf{E}_\beta^{\mathrm{in}}(\kappa)|\mathbf{x}_2''\rangle] & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \tag{5.11}$$

and the matrix multiplication now encompasses both the object index and the partial wave index β. Although the same symbols are used for each wave function, the bases (spherical, planar, etc.) can be chosen differently for each object.

Using Sylvester's determinant formula $\det(\mathbb{I} + \mathbb{AB}) = \det(\mathbb{I} + \mathbb{BA})$, we see that the determinant is unchanged if we replace the off-diagonal submatrices in Eq. (5.8) by

$$\left[\sum_{\beta}(-1)C_{\alpha}(\kappa)\begin{pmatrix}\langle \mathbf{E}_{\alpha}^{\mathrm{reg}}(\kappa)|\mathbb{T}_i|\mathbf{E}_{\beta}^{\mathrm{reg}}(\kappa)\rangle & \langle \mathbf{E}_{\alpha}^{\mathrm{reg}}(\kappa)|\mathbb{T}_i|\mathbf{E}_{\beta}^{\mathrm{out}}(\kappa)\rangle \\ \langle \mathbf{E}_{\alpha}^{\mathrm{in}}(\kappa)|\mathbb{T}_i|\mathbf{E}_{\beta}^{\mathrm{reg}}(\kappa)\rangle & \langle \mathbf{E}_{\alpha}^{\mathrm{in}}(\kappa)|\mathbb{T}_i|\mathbf{E}_{\beta}^{\mathrm{out}}(\kappa)\rangle\end{pmatrix}\mathbb{X}_{\beta,\gamma}^{ij}\right]. \tag{5.12}$$

With this change, the diagonal submatrices in Eq. (5.8) become diagonal in the partial wave indices rather than in position space. The matrix elements of the $\mathbb{T}$-operator are the scattering amplitudes, which can be obtained from ordinary scattering calculations, as demonstrated in Sec. IV. The first matrix in Eq. (5.12), including the prefactor $(-1)C_{\alpha}(\kappa)$, is $\mathbb{F}_i(\kappa)$, the modified scattering amplitude of object i, defined in Eq. (4.13).

Putting together Eqs. (2.11), (2.12), (5.6), and (5.8), we obtain

$$\mathcal{E} = \frac{\hbar c}{2\pi}\int_0^{\infty} d\kappa \log\,\det(\mathbb{M}\mathbb{M}_{\infty}^{-1}), \tag{5.13}$$

where

$$\mathbb{M} = \begin{pmatrix}\mathbb{F}_1^{-1} & \mathbb{X}^{12} & \mathbb{X}^{13} & \cdots \\ \mathbb{X}^{21} & \mathbb{F}_2^{-1} & \mathbb{X}^{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots\end{pmatrix} \tag{5.14}$$

and $\mathbb{M}_{\infty}^{-1}$ is a block diagonal matrix $\mathrm{diag}(\mathbb{F}_1\mathbb{F}_2\cdots)$.

Using the block determinant identity

$$\begin{aligned}\det\begin{pmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{pmatrix} &= \det(\mathbb{A})\det(\mathbb{D} - \mathbb{C}\mathbb{A}^{-1}\mathbb{B}) \\ &= \det(\mathbb{D})\det(\mathbb{A} - \mathbb{B}\mathbb{D}^{-1}\mathbb{C}),\end{aligned} \tag{5.15}$$

we can simplify this expression for the case of the interaction between two objects,

$$\mathcal{E} = \frac{\hbar c}{2\pi}\int_0^{\infty} d\kappa \log\,\det(\mathbb{I} - \mathbb{F}_a\mathbb{X}^{ab}\mathbb{F}_b\mathbb{X}^{ba}). \tag{5.16}$$

Usually, not all of the submatrices of $\mathbb{F}$ and $\mathbb{X}$ are actually needed for a computation. For example, if all objects are outside of one another, only the submatrices $\mathcal{F}^{ee}$ of the scattering amplitude that describe outside reflection are needed. If there are only two objects, one inside another, then only the inside reflection submatrix $\mathcal{F}^{ii}$ of the outside object and the outside reflection submatrix $\mathcal{F}^{ee}$ of the inside object are needed.

In order to obtain the free energy at nonzero temperature instead of the ground state energy, we do not take the limit $\beta \to \infty$ in Eq. (2.9) [23]. Instead, the integral $\frac{\hbar c}{2\pi}\int_0^{\infty} d\kappa$ is replaced everywhere by $\frac{1}{\beta}\sum_n{}'$, where $c\kappa_n = \frac{2\pi n}{\hbar\beta}$ with $n = 0, 1, 2, 3\ldots$ is the nth Matsubara frequency. A careful analysis of the derivation shows that the zero-frequency mode is weighted by $1/2$ compared to the rest of the terms in the sum; this modification of the sum is denoted by a prime on the summation symbol. The factor of $1/2$ comes about because the fluctuating charges or currents have to be real for zero frequency. Thus, for κ_0, the expressions on the right-hand side of Eq. (5.6) should be placed under a square root. (For a complex field, both signs of the integer n would be included separately, and $n = 0$ would be included once, with the normal weight.)

If the medium between the objects is not vacuum but instead has permittivity $\epsilon_m(ic\kappa)$ and magnetic permeability $\mu_m(ic\kappa)$ different from unity, then the free Green's function is multiplied by $\mu_m(ic\kappa)$, and its argument κ is replaced by $n_m(ic\kappa)\kappa$, where $n_m(ic\kappa) = \sqrt{\epsilon_m(ic\kappa)\mu_m(ic\kappa)}$ is the medium's index of refraction. Effectively, this change just scales all frequency dependencies in the translation matrices $\mathbb{X}(\kappa)$, which become $\mathbb{X}(n_m(ic\kappa)\kappa)$. Furthermore, the scattering amplitudes absorb the factor $\mu_m(ic\kappa)$ from the free Green's function and change nontrivially, i.e. not just by some overall factor or a scaling of the frequency. They have to be computed with the nonzero electric and magnetic susceptibilities of the medium.

VI. APPLICATIONS

Our technique for calculating the Casimir energy applies to a wide range of situations. In this section we demonstrate the method through a variety of examples.

A. London and Casimir-Polder interaction between two atoms

As a simple example, we rederive the interaction between two identical neutral atoms in the ground state [19]. The atoms are described in a two-state approximation. Within this approximation, the electric dipole polarizability of the atoms is given by

$$\alpha^E = \frac{e^2}{m}\frac{f_{10}}{\omega_{10}^2 - \omega^2}, \tag{6.1}$$

where e is the electron charge, m is the mass, f_{10} is the oscillator strength of the $0 \to 1$ transition, and ω_{10} is the frequency of that transition. We perform a Wick rotation $\omega \to ic\kappa$ and set $\kappa = u/d$, where d is the distance between the atoms. By introducing the characteristic length scale $d_{10} = c/\omega_{10}$ and the static electric polarizability

$$\alpha_0 = f_{10}r_0 d_{10}^2, \tag{6.2}$$

where $r_0 = e^2/(mc^2) \approx 10^{-15}$ m is the classical electron radius, the polarizability can be written as

$$\alpha^E(u) = \frac{(d/d_{10})^2 \alpha_0}{(d/d_{10})^2 + u^2}. \tag{6.3}$$

In the isotropic-dipole approximation, the only nonzero element of the scattering amplitude of the atom is given in terms of the electric dipole polarizability as

$$\mathcal{F}^{ee}_{1mE,1mE} = \tfrac{2}{3}\alpha^E \kappa^3 \tag{6.4}$$

for $m = -1, 0, 1$. The atoms are assumed to have no magnetic polarizability. Using the general expression of Eq. (5.16) for the interaction energy between two objects, we get

$$\mathcal{E} = \frac{\hbar c}{2\pi d}\int_0^\infty du \log\left[\left(1 - 4(1+u)^2 e^{-2u}\frac{(\alpha^E)^2}{d^6}\right) \times \left(1 - (1+u+u^2)^2 e^{-2u}\frac{(\alpha^E)^2}{d^6}\right)\right] \tag{6.5}$$

$$= -\frac{\hbar c}{\pi d^7}\int_0^\infty du (\alpha^E(u))^2 (3 + 6u + 5u^2 + 2u^3 + u^4) e^{-2u}, \tag{6.6}$$

where we have expanded the logarithm assuming $\alpha^E(u) \ll d^3$, so that the interaction energy is proportional to the product of the polarizabilities of the atoms. It is instructive to consider two limits. First, assume that $d \ll d_{10}$. By the change of variable $u = (d/d_{10})z$, one easily finds that in Eq. (6.6) only the leading constant term of the polynomial in u has to be considered, and the exponential factor can be ignored. The integral is convergent at large u due to the behavior of the polarizability $\alpha^E(u)$ at large u. The integral over z yields, to leading order in d/d_{10}, the energy

$$\mathcal{E}_{\mathrm{L}} = -\frac{3}{4}\hbar\omega_{10}\frac{\alpha_0^2}{d^6}, \tag{6.7}$$

which is the well-known London interaction [80].

In the opposite limit $d \gg d_{10}$ retardation is important. From Eq. (6.3) we see that the frequency (u) dependence of the polarizability now can be neglected, so that $\alpha \approx \alpha_0$. In this retarded limit, the polynomial and exponential in u in Eq. (6.6) are important, and integration yields the energy

$$\mathcal{E}_{\mathrm{CP}} = -\frac{23}{4\pi}\hbar c\frac{\alpha_0^2}{d^7}, \tag{6.8}$$

which is known as the Casimir-Polder interaction [19].

For general distances d, the interaction between the atoms can be computed numerically; to quadratic order in the polarizability it is given by the integral of Eq. (6.6). The numerical result and the two limiting forms of the interaction are shown in Fig. 4.

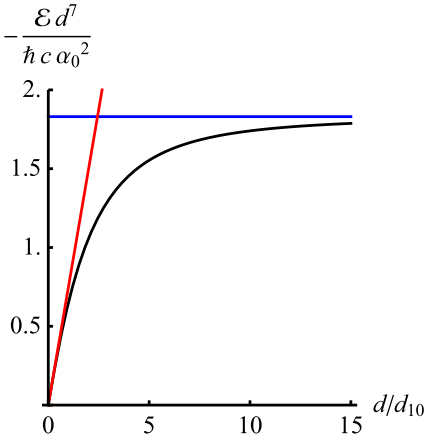

FIG. 4 (color online). Interaction energy $\mathcal{E}$ of two identical atoms, Eq. (6.6), as a function of their separation d. The curve shows the crossover between the London interaction ($d \ll d_{10} = c/\omega_{10}$), Eq. (6.7), and the Casimir-Polder interaction ($d \gg d_{10}$), Eq. (6.8).

B. Derivation of the Lifshitz formula

Next we consider two semi-infinite half-spaces with uniform electric and magnetic susceptibility, as depicted in Fig. 5 [23–27]. We choose a plane wave basis oriented along the $\hat{\mathbf{z}}$ axis.

We decompose the scattering amplitude into magnetic (transverse electric) modes **M** and electric (transverse magnetic) modes **N**.

For the upper object a the scattering solution is [81]

$$\mathbf{E}(\kappa,\mathbf{x}) = \mathbf{M}^{\mathrm{out}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) + \int\frac{L^2 d\mathbf{k}'_\perp}{(2\pi)^2}[\mathbf{M}^{\mathrm{reg}}_{\mathbf{k}'_\perp}(\kappa,\mathbf{x})\mathcal{F}^{ii}_{a,\mathbf{k}'_\perp M,\mathbf{k}_\perp M} + \mathbf{N}^{\mathrm{reg}}_{\mathbf{k}'_\perp}(\kappa,\mathbf{x})\mathcal{F}^{ii}_{a,\mathbf{k}'_\perp E,\mathbf{k}_\perp M}],$$
$$\mathbf{E}(\kappa,\mathbf{x}) = \mathbf{N}^{\mathrm{out}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) + \int\frac{L^2 d^2\mathbf{k}'_\perp}{(2\pi)^2}[\mathbf{M}^{\mathrm{reg}}_{\mathbf{k}'_\perp}(\kappa,\mathbf{x})\mathcal{F}^{ii}_{a,\mathbf{k}'_\perp M,\mathbf{k}_\perp E} + \mathbf{N}^{\mathrm{reg}}_{\mathbf{k}'_\perp}(\kappa,\mathbf{x})\mathcal{F}^{ii}_{a,\mathbf{k}'_\perp E,\mathbf{k}_\perp E}]. \tag{6.9}$$

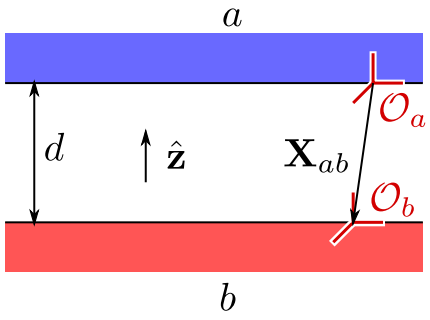

FIG. 5 (color online). The upper infinite half-space a is located a distance d above the half-space b. This is the original configuration considered by Lifshitz. Each half-space has its own uniform electric permittivity $\epsilon_i(ic\kappa)$ and magnetic permeability $\mu_i(ic\kappa)$. We note that our calculation holds even if the two origins $\mathcal{O}_a$ and $\mathcal{O}_b$ are displaced horizontally from one another, as shown here.

Here L is the length of each side of the plates, $\mathbf{k}_\perp$ is the momentum perpendicular to the $\hat{\mathbf{z}}$ direction, and the subscripts M and E on the scattering amplitudes denote the magnetic and electric polarizations, respectively. We consider the limit $L \to \infty$. The scattering amplitudes are given by

$$\begin{aligned}
\mathcal{F}^{ii}_{a,\mathbf{k}'_\perp E,\mathbf{k}_\perp M} &= \mathcal{F}^{ii}_{a,\mathbf{k}'_\perp M,\mathbf{k}_\perp E} = 0, \\
\mathcal{F}^{ii}_{a,\mathbf{k}'_\perp M,\mathbf{k}_\perp M} &= \frac{(2\pi)^2}{L^2}\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp) \\
&\quad \times r_a^M(ic\kappa, (1+\mathbf{k}_\perp^2/\kappa^2)^{-\frac{1}{2}}), \\
\mathcal{F}^{ii}_{a,\mathbf{k}'_\perp E,\mathbf{k}_\perp E} &= \frac{(2\pi)^2}{L^2}\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp) \\
&\quad \times r_a^E(ic\kappa, (1+\mathbf{k}_\perp^2/\kappa^2)^{-\frac{1}{2}}),
\end{aligned} \tag{6.10}$$

in terms of the Fresnel coefficients

$$\begin{aligned}
r_a^M(ic\kappa, x) &= \frac{\mu_a(ic\kappa) - \sqrt{1+(n_a^2(ic\kappa)-1)x^2}}{\mu_a(ic\kappa) + \sqrt{1+(n_a^2(ic\kappa)-1)x^2}}, \\
r_a^E(ic\kappa, x) &= \frac{\epsilon_a(ic\kappa) - \sqrt{1+(n_a^2(ic\kappa)-1)x^2}}{\epsilon_a(ic\kappa) + \sqrt{1+(n_a^2(ic\kappa)-1)x^2}}.
\end{aligned} \tag{6.11}$$

Here, n_a is the index of refraction, $n_a(ic\kappa) = \sqrt{\epsilon_a(ic\kappa)\mu_a(ic\kappa)}$. In the literature the Fresnel coefficients are also sometimes labeled with s instead of M and p in place of E.

The lower object b has the same scattering properties. The relevant scattering equation is the same as in Eq. (6.9), with "reg" and "out" exchanged and $\mathcal{F}^{ii}_a$ replaced by $\mathcal{F}^{ee}_b$, which is obtained from $\mathcal{F}^{ii}_a$ simply by substituting the permittivity $\epsilon_b(ic\kappa)$ and permeability $\mu_b(ic\kappa)$ in place of those of object a.

Using the appropriate $\mathbb{X}$ submatrices as specified in Eq. (3.11) and the corresponding submatrices of $\mathbb{F}$, the energy (5.16) for two objects can be expressed as

$$\mathcal{E} = \frac{\hbar c}{2\pi}\int_0^\infty d\kappa \log\det(\mathcal{I} - \mathcal{F}^{ii}_a \mathcal{W}^{ba}\mathcal{F}^{ee}_b \mathcal{V}^{ba}). \tag{6.12}$$

Since the matrix in the determinant is diagonal in $\mathbf{k}_\perp$, the determinant factors into a product of determinants, each with fixed $\mathbf{k}_\perp$. The logarithm of the product is then given by an integral over the two-dimensional space of $\mathbf{k}_\perp$. Since the integrand is invariant under rotations in $\mathbf{k}_\perp$, we can write this integral in polar coordinates as

$$\begin{aligned}
\mathcal{E} &= \frac{\hbar c}{2\pi}\int_0^\infty d\kappa \int_0^\infty \frac{L^2}{2\pi} k_\perp dk_\perp \\
&\quad \times \log\prod_{i=E,M}(1 - r_a^i r_b^i e^{-2\kappa d\sqrt{1+\mathbf{k}_\perp^2/\kappa^2}}),
\end{aligned} \tag{6.13}$$

where $k_\perp = |\mathbf{k}_\perp|$.

After a change of variable $p = \sqrt{1+\mathbf{k}_\perp^2/\kappa^2}$ we obtain the Lifshitz formula for the energy,

$$\begin{aligned}
\mathcal{E} &= \frac{\hbar c L^2}{(2\pi)^2}\int_0^\infty \kappa^2 d\kappa \int_1^\infty p dp \\
&\quad \times \log[(1 - r_a^M r_b^M e^{-2\kappa p d})(1 - r_a^E r_b^E e^{-2\kappa p d})].
\end{aligned} \tag{6.14}$$

C. Two cylinders

We now rederive the Casimir energy for two perfectly conducting, infinitely long cylinders, depicted in Fig. 6. The result for one cylinder inside the other has been presented in Ref. [70], and the result for both outside each other was presented in Refs. [69,82].

For scattering from outside cylinder a, we have the scattering solutions

$$\begin{aligned}
\mathbf{E}(\kappa, \mathbf{x}) &= \mathbf{M}^{\text{reg}}_{k_z,n}(\kappa, \mathbf{x}) + \int \frac{L dk'_z}{2\pi}\sum_{n'}[\mathbf{M}^{\text{out}}_{k'_z,n'}(\kappa, \mathbf{x}) \\
&\quad \times \mathcal{F}^{ee}_{a,k'_z n' M,k_z n M} + \mathbf{N}^{\text{out}}_{k'_z,n'}(\kappa, \mathbf{x})\mathcal{F}^{ee}_{a,k'_z n' E,k_z n M}], \\
\mathbf{E}(\kappa, \mathbf{x}) &= \mathbf{N}^{\text{reg}}_{k_z,n}(\kappa, \mathbf{x}) + \int \frac{L dk'_z}{2\pi}\sum_{n'}[\mathbf{M}^{\text{out}}_{k'_z,n'}(\kappa, \mathbf{x}) \\
&\quad \times \mathcal{F}^{ee}_{a,k'_z n' M,k_z n E} + \mathbf{N}^{\text{out}}_{k'_z,n'}(\kappa, \mathbf{x})\mathcal{F}^{ee}_{a,k'_z n' E,k_z n E}],
\end{aligned} \tag{6.15}$$

with boundary conditions $\mathbf{E}^\parallel = 0$ and $\mathbf{B}^\perp = 0$ on the cylinder surface. L is the length of the cylinders, and we are considering the limit $L \to \infty$. We have

$$\begin{aligned}
\mathcal{F}^{ee}_{a,k'_z n' E,k_z n M} &= \mathcal{F}^{ee}_{a,k'_z n' M,k_z n E} = 0, \\
\mathcal{F}^{ee}_{a,k'_z n' M,k_z n M} &= -\frac{2\pi}{L}\delta(k_z - k'_z)\delta_{n,n'}\frac{I'_n(R_a p)}{K'_n(R_a p)}, \\
\mathcal{F}^{ee}_{a,k'_z n' E,k_z n E} &= -\frac{2\pi}{L}\delta(k_z - k'_z)\delta_{n,n'}\frac{I_n(R_a p)}{K_n(R_a p)},
\end{aligned} \tag{6.16}$$

and analogous equations hold for scattering from cylinder b. The energy in Eq. (5.16) is given in terms of exterior scattering amplitudes only,

$$\mathcal{E} = \frac{\hbar c}{2\pi}\int_0^\infty d\kappa \log\det(\mathcal{I} - \mathcal{F}^{ee}_a \mathcal{U}^{ba}\mathcal{F}^{ee}_b \mathcal{U}^{ab}). \tag{6.17}$$

The matrix inside the determinant is diagonal in k_z, so the log determinant over this index turns into an overall integral. A change of variable to polar coordinates converts the integrals over κ and k_z to a single integral over $p = \sqrt{k_z^2 + \kappa^2}$, yielding

$$\mathcal{E} = \frac{\hbar c L}{4\pi}\int_0^\infty p dp(\log\det\mathcal{N}^M + \log\det\mathcal{N}^E), \tag{6.18}$$

where

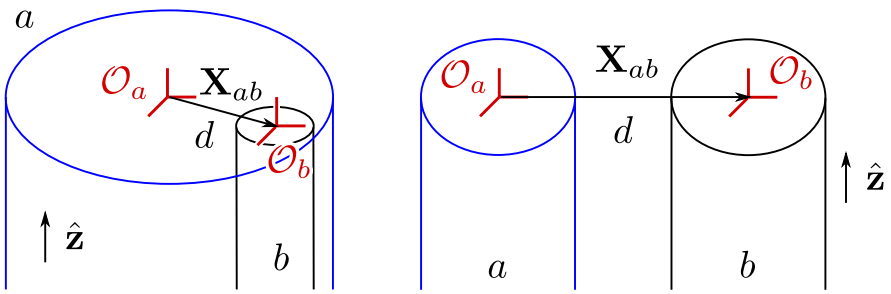

FIG. 6 (color online). Two perfectly conducting infinite cylinders with radii R_a and R_b are separated by a center-to-center distance d. They can be outside one another, or one may be inside the other.

$$\begin{aligned}\mathcal{N}^M_{n,n''} &= \delta_{n,n''} - \sum_{n'} \frac{I'_n(pR_a)}{K'_n(pR_a)} K_{n+n'}(pd) \frac{I'_{n'}(pR_b)}{K'_{n'}(pR_b)} \\ &\quad \times K_{n'+n''}(pd), \\ \mathcal{N}^E_{n,n''} &= \delta_{n,n''} - \sum_{n'} \frac{I_n(pR_a)}{K_n(pR_a)} K_{n+n'}(pd) \frac{I_{n'}(pR_b)}{K_{n'}(pR_b)} \\ &\quad \times K_{n'+n''}(pd).\end{aligned} \tag{6.19}$$

For scattering from inside cylinder a we have the scattering solutions

$$\begin{aligned}\mathbf{E}(\kappa, \mathbf{x}) &= \mathbf{M}^{\text{out}}_{k_z,n}(\kappa, \mathbf{x}) + \int \frac{L dk'_z}{2\pi} \sum_{n'} [\mathbf{M}^{\text{reg}}_{k'_z,n'}(\kappa, \mathbf{x}) \\ &\quad \times \mathcal{F}^{ii}_{a,k'_z n' M,k_z nM} + \mathbf{N}^{\text{reg}}_{k'_z,n'}(\kappa, \mathbf{x}) \mathcal{F}^{ii}_{a,k'_z n' E,k_z nM}], \\ \mathbf{E}(\kappa, \mathbf{x}) &= \mathbf{N}^{\text{out}}_{k_z,n}(\kappa, \mathbf{x}) + \int \frac{L dk'_z}{2\pi} \sum_{n'} [\mathbf{M}^{\text{reg}}_{k'_z,n'}(\kappa, \mathbf{x}) \\ &\quad \times \mathcal{F}^{ii}_{a,k'_z n' M,k_z nE} + \mathbf{N}^{\text{reg}}_{k'_z,n'}(\kappa, \mathbf{x}) \mathcal{F}^{ii}_{a,k'_z n' E,k_z nE}],\end{aligned} \tag{6.20}$$

yielding

$$\begin{aligned}\mathcal{F}^{ii}_{a,k'_z n' E,k_z nM} &= \mathcal{F}^{ii}_{a,k'_z n' M,k_z nE} = 0, \\ \mathcal{F}^{ii}_{a,k'_z n' M,k_z nM} &= -\frac{2\pi}{L} \delta(k_z - k'_z) \delta_{n,n'} \frac{K'_n(R_a p)}{I'_n(R_a p)}, \\ \mathcal{F}^{ii}_{a,k'_z n' E,k_z nE} &= -\frac{2\pi}{L} \delta(k_z - k'_z) \delta_{n,n'} \frac{K_n(R_a p)}{I_n(R_a p)}.\end{aligned} \tag{6.21}$$

We note that the inside scattering amplitude matrix is the inverse of the corresponding outside result. The energy, expressed in Eq. (5.16), now becomes

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \log \det(\mathcal{I} - \mathcal{F}^{ii}_a \mathcal{W}^{ba} \mathcal{F}^{ee}_b \mathcal{V}^{ba}), \tag{6.22}$$

which contains the appropriate scattering amplitudes for the inside problem, $\mathcal{F}^{ii}_a$ for interior scattering of object a and $\mathcal{F}^{ee}_b$ for exterior scattering of object b. Using the same simplifications as in the outside case, we have

$$\mathcal{E} = \frac{\hbar c L}{4\pi} \int_0^\infty p dp (\log \det \mathcal{N}^M + \log \det \mathcal{N}^E), \tag{6.23}$$

where

$$\begin{aligned}\mathcal{N}^M_{n,n''} &= \delta_{n,n''} - \sum_{n'} \frac{K'_n(pR_a)}{I'_n(pR_a)} I_{n+n'}(pd) \frac{I'_{n'}(pR_b)}{K'_{n'}(pR_b)} \\ &\quad \times I_{n'+n''}(pd), \\ \mathcal{N}^E_{n,n''} &= \delta_{n,n''} - \sum_{n'} \frac{K_n(pR_a)}{I_n(pR_a)} I_{n+n'}(pd) \frac{I_{n'}(pR_b)}{K_{n'}(pR_b)} \\ &\quad \times I_{n'+n''}(pd).\end{aligned} \tag{6.24}$$

D. Sphere and plate

In this section we investigate the Casimir interaction of an infinitely thick plate a opposite a sphere b, each with frequency-dependent permittivity and permeability. The geometry is depicted in Fig. 7.

The scattering amplitude for the plate is easy to express in the plane wave basis using Eq. (6.10). We can apply our result from the plane geometry, Eq. (6.12), changing only $\mathcal{F}^{ee}_b$, which now becomes the scattering amplitude for vector plane wave functions outside a sphere. To express the scattering amplitude of the sphere in the spherical vector wave basis, we use Eqs. (D1) and (D2) and obtain

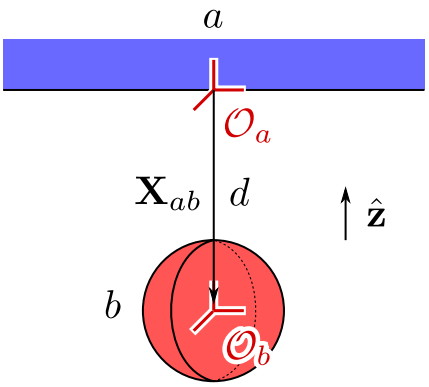

FIG. 7 (color online). A sphere b of radius R is located opposite a plate a, separated by a center-to-surface distance d.

$$\begin{aligned}\mathcal{F}^{ee}_{b,\mathbf{k}_\perp P,\mathbf{k}'_\perp P'} &= (-1)C_{\mathbf{k}_\perp P}(\kappa)\langle \mathbf{E}^{\mathrm{reg}}_{\mathbf{k}_\perp P}(\kappa)|\mathbb{T}_b|\mathbf{E}^{\mathrm{reg}}_{\mathbf{k}'_\perp P'}(\kappa)\rangle \\ &= \sum_{lmQ,l'm'Q'} (-1)\frac{C_{\mathbf{k}_\perp P}(\kappa)}{C_Q(\kappa)} D^\dagger_{\mathbf{k}_\perp P,lmQ} C_Q(\kappa) \\ &\quad \times \langle \mathbf{E}^{\mathrm{reg}}_{lmQ}(\kappa)|\mathbb{T}_b|\mathbf{E}^{\mathrm{reg}}_{l'm'Q'}(\kappa)\rangle D_{l'm'Q',\mathbf{k}'_\perp P'} \\ &= \sum_{lmQ,l'm'Q'} \frac{C_{\mathbf{k}_\perp P}(\kappa)}{C_Q(\kappa)} D^\dagger_{\mathbf{k}_\perp P,lmQ} \\ &\quad \times \mathcal{F}^{ee}_{b,lmQ,l'm'Q'} D_{l'm'Q',\mathbf{k}'_\perp P'}, \end{aligned} \tag{6.25}$$

where P and P' label the polarizations in the planar basis and Q and Q' label the polarizations in the spherical basis. The normalization factors $C_{\mathbf{k}_\perp P}(\kappa)$ and $C_Q(\kappa)$, defined below Eqs. (B1) and (B7), arise from the definition of the scattering amplitude [see, for example, Eq. (4.9)]. For a sphere with uniform permittivity and permeability, we compute the scattering amplitude by solving Eq. (4.9) in the spherical vector wave basis, which yields

$$\begin{aligned}\mathcal{F}^{ee}_{b,lmE,l'm'M} &= \mathcal{F}^{ee}_{b,lmM,l'm'E} = 0, \\ \mathcal{F}^{ee}_{b,lmM,l'm'M} &= -\delta_{l,l'}\delta_{m,m'} \frac{i_l(\kappa R)\partial_R(Ri_l(n_b\kappa R)) - \mu_b \partial_R(Ri_l(\kappa R))i_l(n_b\kappa R)}{k_l(\kappa R)\partial_R(Ri_l(n_b\kappa R)) - \mu_b \partial_R(Rk_l(\kappa R))i_l(n_b\kappa R)}, \\ \mathcal{F}^{ee}_{b,lmE,l'm'E} &= -\delta_{l,l'}\delta_{m,m'} \frac{i_l(\kappa R)\partial_R(Ri_l(n_b\kappa R)) - \epsilon_b \partial_R(Ri_l(\kappa R))i_l(n_b\kappa R)}{k_l(\kappa R)\partial_R(Ri_l(n_b\kappa R)) - \epsilon_b \partial_R(Rk_l(\kappa R))i_l(n_b\kappa R)}, \end{aligned} \tag{6.26}$$

where n_b is the index of refraction, $n_b(ic\kappa) = \sqrt{\epsilon_b(ic\kappa)\mu_b(ic\kappa)}$. The modified spherical Bessel functions i_l and k_l are defined in Appendix B.

Plugging into Eq. (6.12) and using $\det(\mathbb{I} + \mathbb{A}\mathbb{B}) = \det(\mathbb{I} + \mathbb{B}\mathbb{A})$, the energy simplifies to

$$\mathcal{E} = \frac{\hbar c}{2\pi}\int_0^\infty d\kappa \log\det(\mathcal{I} - \mathcal{N}), \tag{6.27}$$

where

$$\begin{aligned}\mathcal{N}_{lmP,l'm'P'} &= \delta_{m,m'}\mathcal{F}^{ee}_{b,lmP,lmP}\int_0^\infty \frac{k_\perp dk_\perp}{2\pi}\frac{e^{-2d\sqrt{\mathbf{k}_\perp^2+\kappa^2}}}{2\kappa\sqrt{\mathbf{k}_\perp^2+\kappa^2}} \\ &\quad \times \sum_Q D_{lmP,\mathbf{k}_\perp Q} r^Q_a(ic\kappa, (1+\mathbf{k}_\perp^2/\kappa^2)^{-\frac{1}{2}}) \\ &\quad \times D^\dagger_{\mathbf{k}_\perp Q,l'mP'}(2\delta_{Q,P'} - 1). \end{aligned} \tag{6.28}$$

Here $k_\perp = |\mathbf{k}_\perp|$ and r^Q_a, defined in Eq. (6.11), is the Fresnel coefficient for reflection of a wave with polarization Q. The ratio of $C_{\mathbf{k}_\perp P}(\kappa)$ to $C_Q(\kappa)$ in Eq. (6.25) has opposite signs depending on whether P and Q represent the same polarization or the opposite polarization, which we have implemented through the term $(2\delta_{P,Q} - 1)$. The integration over all angles of $\mathbf{k}_\perp$ has already been carried out, which makes $\mathcal{N}$ diagonal in m and m'. (Although $D_{lmp,\mathbf{k}_\perp Q}$ seems to depend on the angle of $\mathbf{k}_\perp$, the multiplication with its Hermitian conjugate cancels this dependence.)

To leading order for large d/R, the $l = 1$ components of the sphere's scattering amplitude and the $\kappa \to 0$ limit of the permittivities and permeabilities contribute. The scattering amplitude can be expanded to lowest order in terms of the sphere's electric and magnetic polarizabilities, $\mathcal{F}^{ee}_{b,1mM,1mM} \to \frac{2}{3}\alpha^M\kappa^3$ and $\mathcal{F}^{ee}_{b,1mE,1mE} \to \frac{2}{3}\alpha^E\kappa^3$, where the polarizabilities $\alpha^M = \frac{\mu_{b0}-1}{\mu_{b0}+2}R^3$ and $\alpha^E = \frac{\epsilon_{b0}-1}{\epsilon_{b0}+2}R^3$ are given in terms of the zero-frequency permittivity $\epsilon_{b0} = \epsilon_b(0)$ and permeability $\mu_{b0} = \mu_b(0)$. To leading order the energy is given by

$$\mathcal{E} = -\frac{3\hbar c}{8\pi d^4}(\alpha^M\phi^M + \alpha^E\phi^E), \tag{6.29}$$

where

$$\begin{aligned}\phi^M &= \int_0^1 dx\left[\left(1-\frac{x^2}{2}\right)r^M_a(0,x) - \frac{x^2}{2}r^E_a(0,x)\right], \\ \phi^E &= \int_0^1 dx\left[\left(1-\frac{x^2}{2}\right)r^E_a(0,x) - \frac{x^2}{2}r^M_a(0,x)\right] \end{aligned} \tag{6.30}$$

can be expressed in terms of elementary functions, but the expressions are too complicated to be worth reproducing here. The two functions are plotted in Fig. 8.

The expression for the energy in Eq. (6.29) agrees with the results in Ref. [66] for a perfect metal plate $\epsilon_a \to \infty$ and a sphere with general ϵ_b. It also agrees with the results in Ref. [83] for a perfect metal plate and a perfect metal sphere, $\epsilon_a \to \infty$, and $\epsilon_b \to \infty$. Both of these works arrive at similar general expressions for the energy to what we have found here; Ref. [66] combines scattering theory techniques we have used here with the method of images, while Ref. [83] uses Wigner rotation matrices.

In the calculations of Refs. [66,83], when $\epsilon \to \infty$ the corresponding μ is set to zero to reproduce perfect metal boundary conditions within a low-frequency expansion. (Reference [26] contains the asymptotic Casimir energy formula Eq. (6.29) in the case where the magnetic permeabilities are set equal to one.) As the plots in Fig. 8 show, however, the perfect reflectivity limit of the plate is approached slowly with increasing ϵ_a. To compare with experiments it is thus important to compute the energy (6.27) using the actual permittivities and permeabilities of the material instead of perfect metal limits.

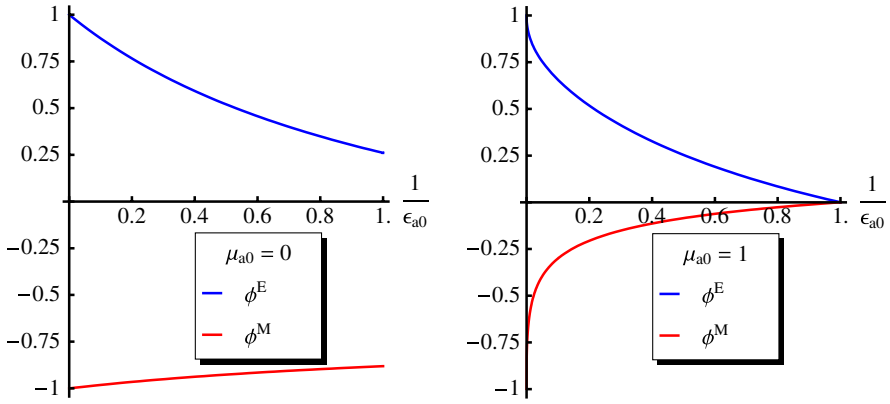

FIG. 8 (color online). Plots of ϕ^E (blue, positive) and ϕ^M (red, negative) as functions of $1/\epsilon_{a0}$ for fixed $\mu_{a0} = 0$ (left) or fixed $\mu_{a0} = 1$ (right). For $\mu_{a0} = 1$ the two functions ϕ^E and ϕ^M approach 1 rather slowly from the right (perfect metal limit). So, for comparison with experiments, it may not be justified to use the perfect metal limit $\epsilon_a \to \infty$ of the plate to compute the Casimir energy.

E. Cylinder and plate

As a final example, we investigate the Casimir interaction energy between an infinitely thick plate a opposite a cylinder b, each with frequency-dependent permittivity and permeability (Fig. 9). We will focus on presenting asymptotic ($d/R \to \infty$) results here, but the derivation is straightforward to extend to intermediate separations, for which the evaluation of the final expression can be performed easily on a computer. We choose the $\hat{\mathbf{z}}$ axis as the axis of symmetry of the cylinder and let $\mathbf{k}_\perp$ denote the vector (k_y, k_z).

Just as in the sphere-plate case, it is most convenient to express the scattering amplitude of the cylinder b in a plane wave basis by

$$\begin{aligned}\mathcal{F}^{ee}_{b,\mathbf{k}_\perp P,\mathbf{k}'_\perp P'} &= (-1)C_{\mathbf{k}_\perp P}(\kappa)\langle \mathbf{E}^{\mathrm{reg}}_{\mathbf{k}_\perp P}(\kappa)|\mathbb{T}_b|\mathbf{E}^{\mathrm{reg}}_{\mathbf{k}'_\perp P'}(\kappa)\rangle \\ &= \sum_{nQ,n'Q'} (-1)\frac{C_{\mathbf{k}_\perp P}(\kappa)}{C_Q} D^\dagger_{\mathbf{k}_\perp P,k_z nQ} \\ &\quad \times C_Q\langle \mathbf{E}^{\mathrm{reg}}_{k_z nQ}(\kappa)|\mathbb{T}_b|\mathbf{E}^{\mathrm{reg}}_{k_z n'Q'}(\kappa)\rangle D_{k_z n'Q',\mathbf{k}'_\perp P'} \\ &= \sum_{nQ,n'Q'} \frac{C_{\mathbf{k}_\perp P}(\kappa)}{C_Q} D^\dagger_{\mathbf{k}_\perp P,k_z nQ}\mathcal{F}^{ee}_{b,k_z nQ,k_z n'Q'} \\ &\quad \times D_{k_z n'Q',\mathbf{k}'_\perp P'}, \end{aligned} \tag{6.31}$$

where $C_{\mathbf{k}_\perp P}(\kappa)$ and C_Q are defined below Eq. (B1) and (B4), respectively.

By solving Eq. (4.9) in a cylindrical wave basis, it is straightforward to find the scattering amplitude of the cylinder, $\mathcal{F}^{ee}_{b,k_z nQ,k'_z n'Q'}$. For uniform permittivity and permeability, the matrix elements are diagonal in k_z and the cylindrical wave index n, but not in TE and TM polarization. The expressions are somewhat complicated; since we are presenting asymptotic results here, we only need the small-radius expansion,

$$\begin{aligned}\mathcal{F}^{ee}_{b,k_z nP,k'_z n'P'} &= \frac{2\pi}{L}\delta(k_z - k'_z)\delta_{n,n'} f_{k_z nPP'} + O(R^4), \\ f_{k_z 0MM} &= \frac{1}{2}(\kappa^2 + k_z^2)R^2(1-\mu_b), \\ f_{k_z 0EE} &= \frac{1}{2}(\kappa^2 + k_z^2)R^2(1-\epsilon_b), \\ f_{k_z \pm 1MM} &= \frac{k_z^2(1+\epsilon_b)(1-\mu_b) - \kappa^2(1-\epsilon_b)(1+\mu_b)}{2(1+\epsilon_b)(1+\mu_b)}R^2, \\ f_{k_z \pm 1EE} &= \frac{k_z^2(1-\epsilon_b)(1+\mu_b) - \kappa^2(1+\epsilon_b)(1-\mu_b)}{2(1+\epsilon_b)(1+\mu_b)}R^2, \\ f_{k_z 1ME} &= f_{k_z -1EM} = \frac{\kappa k_z(\epsilon_b\mu_b - 1)}{(1+\epsilon_b)(1+\mu_b)}R^2, \\ f_{k_z 1EM} &= f_{k_z -1ME} = -f_{k_z 1ME}. \end{aligned} \tag{6.32}$$

All other matrix elements ($|n| > 1$) contribute at higher order in R. It is assumed here that $\epsilon_b(ic\kappa)$ is finite. In the infinite conductivity limit ($\epsilon_b \to \infty$) only one of these scattering amplitudes contributes; this case is discussed below.

We next plug into Eq. (6.12). As in the case of two cylinders, the matrix inside the determinant is diagonal in

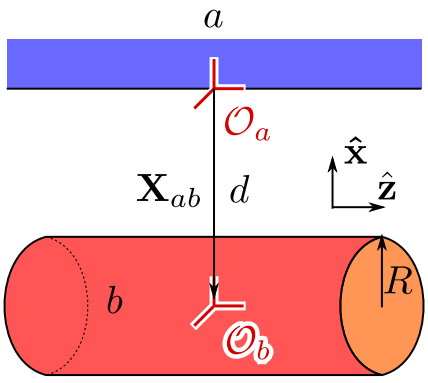

FIG. 9 (color online). A cylinder b of radius R is located opposite a plate a, separated by a center-to-surface distance d.

k_z, so the log determinant over this index turns into an integral. We obtain for the Casimir energy

$$\mathcal{E} = \frac{\hbar c L}{4\pi^2} \int_0^\infty d\kappa \int_{-\infty}^\infty dk_z \log \det(\mathcal{I} - \mathcal{N}), \tag{6.33}$$

where

$$\mathcal{N}_{k_z n P, n' P'} = \sum_{P'', Q} f_{k_z n P P''} \int_{-\infty}^\infty dk_y \frac{e^{-2d\sqrt{\mathbf{k}_\perp^2 + \kappa^2}}}{2\sqrt{\mathbf{k}_\perp^2 + \kappa^2}} \times D_{n k_z P'', \mathbf{k}_\perp Q} r_a^Q(ic\kappa, (1 + \mathbf{k}_\perp^2/\kappa^2)^{-\frac{1}{2}}) \times D^\dagger_{\mathbf{k}_\perp Q, n' k_z P'}(1 - 2\delta_{Q,P'}). \tag{6.34}$$

To find the interaction energy at separations outside of the asymptotic limit, $f_{k_z n P P'}$ must be replaced by the appropriate scattering amplitude expressions for all n, valid to all orders in R. Expanding the log det in Eq. (6.33) to first order in $\mathcal{N}$, we obtain for the interaction in the large distance limit $d/R \to \infty$,

$$\mathcal{E} = -\frac{3\hbar c L R^2}{128\pi d^4} \int_0^1 dx \frac{\epsilon_{b0} - 1}{\epsilon_{b0} + 1} [(7 + \epsilon_{b0} - 4x^2) r^E(0, x) - (3 + \epsilon_{b0}) x^2 r^M(0, x)], \tag{6.35}$$

if the zero-frequency magnetic permeability μ_{b0} of the cylinder is set to one. If we do not set μ_{b0} equal to one, but instead take the perfect reflectivity limit for the plate, we obtain

$$\mathcal{E} = -\frac{\hbar c L R^2}{32\pi d^4} \frac{(\epsilon_{b0} - \mu_{b0})(9 + \epsilon_{b0} + \mu_{b0} + \epsilon_{b0}\mu_{b0})}{(1 + \epsilon_{b0})(1 + \mu_{b0})}. \tag{6.36}$$

Finally, if we let ϵ_b be infinite from the beginning (the perfect metal limit for the cylinder), only the $n = 0$ TM mode of the scattering amplitude, $\mathcal{F}^{ee}_{b,k_z 0E,k_z' 0E} = \frac{2\pi}{L}\delta(k_z - k_z')\frac{1}{\log R/d} + O(\log^{-2}(R/d))$, contributes at lowest order; the previous expansions of the cylinder's scattering amplitude in Eq. (6.32) are not valid. For a plate with zero-frequency permittivity ϵ_{a0} and permeability μ_{a0}, we obtain for the Casimir energy

$$\mathcal{E} = \frac{\hbar c L}{16\pi d^2 \log(R/d)} \phi^E, \tag{6.37}$$

where

$$\phi^E = \int_0^1 \frac{dx}{1 + x} [r_a^E(0, x) - x r_a^M(0, x)]. \tag{6.38}$$

In Fig. 10, ϕ^E is plotted as a function of the zero-frequency permittivity of the plate, ϵ_{a0}, for various zero-frequency permeability values, μ_{a0}.

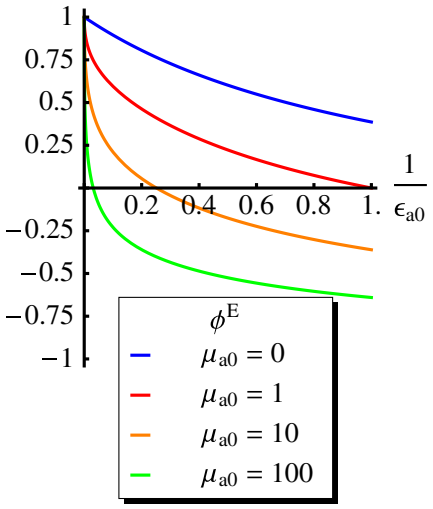

FIG. 10 (color online). Plots of ϕ^E versus $1/\epsilon_{a0}$ for fixed values of μ_{a0}. ϕ^E decreases both with increasing $1/\epsilon_{a0}$ and increasing μ_{a0}. The perfect metal limit ($\phi^E = 1$) is approached slowly for large μ_{a0}, as in the case of a sphere opposite a plate. For large μ_{a0} the interaction becomes repulsive, which is expected given similar results for two infinite plates [92].

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APPENDIX A: DERIVATION OF THE MACROSCOPIC FIELD THEORY

In this appendix we justify the starting point of our derivation, the effective or "macroscopic" field theory in Eq. (2.1), in order to clarify the causality properties of the permittivity ϵ and permeability μ and incorporate the dissipative properties of the materials naturally into our formalism. We also show that both nonlocal and nonisotropic permittivity and permeability *tensors* can be used.

In place of Eq. (2.1), we begin with the Lagrangian density of the free electromagnetic field plus a coupled system of particles. To be concrete, we imagine that the

electromagnetic field couples, say, to electrons of charge $-e$ described by the Lagrangian density operator

$$\hat{\mathcal{L}} = \frac{1}{2}(\hat{\mathbf{E}}^2 - \hat{\mathbf{B}}^2) + i\hbar\hat{\psi}^\dagger\dot{\hat{\psi}} - \frac{\hbar^2}{2m}\left[\left(\nabla - \frac{ie}{\hbar}\hat{\mathbf{A}}\right)\hat{\psi}^\dagger\right] \times \left[\left(\nabla + \frac{ie}{\hbar}\hat{\mathbf{A}}\right)\hat{\psi}\right], \tag{A1}$$

where $\hat{\psi}$ and $\hat{\psi}^\dagger$ are the fermion field annihilation and creation operators, which are spinor functions of space $\mathbf{x}$ and time t. We implicitly sum over spins and suppress the spin index and we continue to work in $A^0 = 0$ gauge. The electrons' coupling to the lattice, their coupling to each other, and their spin-magnetic field interaction can be explicitly taken into account by adding, for example,

$$\begin{aligned}\hat{\mathcal{L}}_{\text{lattice}} &= -\hat{\psi}^\dagger(t,\mathbf{x})u(\mathbf{x})\hat{\psi}(t,\mathbf{x}),\\ \hat{\mathcal{L}}_{\text{int}} &= -\frac{1}{2}\int d\mathbf{x}'\hat{\psi}^\dagger(t,\mathbf{x})\hat{\psi}^\dagger(t,\mathbf{x}')v(\mathbf{x}-\mathbf{x}')\hat{\psi}(t,\mathbf{x}') \times \hat{\psi}(t,\mathbf{x}),\\ \hat{\mathcal{L}}_{\text{spin}} &= \frac{e\hbar g_e}{2mc}\hat{\mathbf{B}}(\mathbf{x})\cdot\hat{\psi}^\dagger(t,\mathbf{x})\boldsymbol{\sigma}\hat{\psi}(t,\mathbf{x}).\end{aligned} \tag{A2}$$

We imagine that such systems, confined to the various regions of space, represent the objects whose Casimir interactions we are calculating. Since the following procedure is quite general, we are not limited to systems described by these particular Lagrangians, but modifications to our approach may be needed in some situations.

The electron-lattice and electron-electron interactions are mediated via the quantum electrodynamic field, but the relevant wavelengths are substantially shorter than the ones dominating the Casimir interaction of different objects, so we can safely approximate the short wavelength interactions by effective potentials $u(\mathbf{x})$ and $v(\mathbf{x}-\mathbf{x}')$.

To compute the total partition function, we exponentiate the time integral of the Lagrangian and analytically continue the time coordinate t to $-i\tau$, yielding

$$Z(\beta) = \int \mathcal{D}\mathbf{A}\, e^{-(1/\hbar)S_{\text{EM}}} Z_{\text{elec}}[\mathbf{A}], \tag{A3}$$

where

$$S_{\text{EM}} = \frac{1}{2}\int_0^{\hbar\beta} d\tau \int d\mathbf{x}(\mathbf{E}^2 + \mathbf{B}^2) \tag{A4}$$

is the free electromagnetic action and

$$Z_{\text{elec}}[\mathbf{A}] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-(1/\hbar)S_{\text{elec}}} \tag{A5}$$

is the electronic partition function. The Euclidean electronic action is

$$S_{\text{elec}} = \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\Big(\hbar\bar{\psi}\partial_\tau\psi + \frac{\hbar^2}{2m}\left[\left(\nabla - \frac{ie}{\hbar}\mathbf{A}\right)\bar{\psi}\right] \times \left[\left(\nabla + \frac{ie}{\hbar}\mathbf{A}\right)\psi\right] + \text{interactions}\Big), \tag{A6}$$

where the interaction terms are taken from Eq. (A2), including at least $\hat{\mathcal{L}}_{\text{lattice}}$ to keep the electrons confined to the various objects. Here the creation and annihilation operators $\hat{\psi}^\dagger$ and $\hat{\psi}$ go over to Grassman path integral variables $\bar{\psi}$ and ψ.

Next, we expand the partition function of the electronic system,

$$Z_{\text{elec}}[\mathbf{A}] \approx Z_{\text{elec}}[0]\Big(1 + \frac{1}{2}\sum_{i,j}\int_0^{\hbar\beta} d\tau d\tau' \int d\mathbf{x}d\mathbf{x}' \times K_{ij}(\tau-\tau',\mathbf{x},\mathbf{x}')A^i(\tau,\mathbf{x})A^j(\tau',\mathbf{x}')\Big), \tag{A7}$$

where the second derivative of Z_{elec},

$$K_{ij}(\tau-\tau',\mathbf{x},\mathbf{x}') = \frac{1}{Z_{\text{elec}}[0]}\frac{\delta^2 Z_{\text{elec}}[\mathbf{A}]}{\delta A^i(\tau,\mathbf{x})\delta A^j(\tau',\mathbf{x}')}\Big|_{\mathbf{A}=0}, \tag{A8}$$

only depends on the difference in imaginary time $\tau - \tau'$. The linear term has been omitted in this expansion because it vanishes for systems with no mean currents. We then obtain the well-known Kubo formula for electrical conductivity [84],

$$\sigma^T_{ij}(ic\kappa_n,\mathbf{x},\mathbf{x}') = -\frac{1}{c\kappa_n}\int_0^{\hbar\beta} d\tau K_{ij}(\tau,\mathbf{x},\mathbf{x}')e^{ic\kappa_n\tau}. \tag{A9}$$

The T superscript indicates that this is the imaginary-time ordered response function. The retarded response function can be obtained by the substitution $ic\kappa_n \to \omega + i0^+$. The conductivity is related to the permittivity by

$$\sigma^T_{ij}(ic\kappa_n,\mathbf{x},\mathbf{x}') = c\kappa_n(\epsilon^T_{ij}(ic\kappa_n,\mathbf{x},\mathbf{x}') - \delta_{ij}\delta(\mathbf{x}-\mathbf{x}')). \tag{A10}$$

After substituting into Eq. (A7), we obtain

$$Z_{\text{elec}} \approx Z_{\text{elec}}[0]\Big(1 + \frac{\beta}{2}\sum_{n=-\infty}^{\infty}\int d\mathbf{x}d\mathbf{x}'\mathbf{E}^*(ic\kappa_n,\mathbf{x}) \cdot (\mathbb{I}\delta(\mathbf{x}-\mathbf{x}') - \boldsymbol{\epsilon}^T(ic\kappa_n,\mathbf{x},\mathbf{x}'))\mathbf{E}(ic\kappa_n,\mathbf{x}')\Big), \tag{A11}$$

and finally, after reexponentiating and plugging into Eq. (A3) we obtain

$$Z(\beta) \approx Z_{\text{elec}}[0] \int \mathcal{D}\mathbf{A} \times \exp\Big[-\beta \sum_{n=-\infty}^{\infty} \Big(\int d\mathbf{x} d\mathbf{x}' \mathbf{E}^*(ic\kappa_n, \mathbf{x}) \cdot \boldsymbol{\epsilon}^T(ic\kappa_n, \mathbf{x}, \mathbf{x}') \mathbf{E}(ic\kappa_n, \mathbf{x}') + \int d\mathbf{x} \mathbf{B}^*(ic\kappa_n, \mathbf{x}) \cdot \mathbf{B}(ic\kappa_n, \mathbf{x}) \Big) \Big]. \tag{A12}$$

The imaginary-time ordered response function can be obtained from the retarded real time response function, which is experimentally accessible, by analytic continuation to imaginary frequencies. The relationship $\epsilon^T_{ij}(ic\kappa_n, \mathbf{x}, \mathbf{x}') = \epsilon^R_{ij}(ic|\kappa_n|, \mathbf{x}, \mathbf{x}')$ between the two response functions and the symmetry of the retarded response function in the indices $(i, \mathbf{x})$ and $(j, \mathbf{x}')$ is discussed in Refs. [85,86].

To be able to consider two objects as separate and distinct, we assume that $\epsilon^T(ic\kappa_n, \mathbf{x}, \mathbf{x}')$ is zero when $\mathbf{x}$ and $\mathbf{x}'$ are on different objects, separated by the vacuum or a medium. This is justified even for small gaps between the objects, since tunneling probabilities decay exponentially in the gap distance.

Although the dissipative part of the electric response (the conductivity) appears in the above equations, there is no actual dissipation at zero temperature. Rather, Eqs. (A7)–(A9) merely show that the conductivity σ^R, which can be measured in an experiment, can be related to the fluctuations that exist in a system in the absence of an external perturbation. The size of these fluctuations is represented by the second derivative of the partition function, Eq. (A8).

Although the magnetic susceptibility is usually negligible compared to the electric susceptibility, we have kept the permeability function μ in our derivations of Casimir interactions. If the spin-magnetic field coupling term $\hat{\mathcal{L}}_{\text{spin}}$ replaces or dominates the standard coupling between charge and electromagnetic field in Eq. (A1), an analogous procedure introduces the magnetic permeability function into the action. In that case, the partition function of the matter field, the analogue of Eq. (A7), has to be expanded in $\mathbf{B}^i = (\boldsymbol{\nabla} \times \mathbf{A})^i$ instead of $\mathbf{A}^i$.

APPENDIX B: GREEN'S FUNCTION EXPANSIONS AND MODIFIED EIGENFUNCTIONS

In this section we supply the Green's function expansions for imaginary-frequency $ic\kappa$ and the associated modified eigenfunctions in various bases. After analytically continuing the wave functions in Eq. (3.8), it is convenient to redefine them by multiplication by an overall factor in order to obtain the conventional definitions of the modified wave functions. Since $\mathbf{E}^{\text{in}*}(\kappa, \mathbf{x}) = \mathbf{E}^{\text{out}}(\kappa, \mathbf{x})$, it suffices to supply the modified regular and outgoing wave functions.

Electromagnetic vector waves are typically divided into TE and TM modes. It is customary to name the TE wave functions $\mathbf{M}(\omega, \mathbf{x})$ and the TM waves $\mathbf{N}(\omega, \mathbf{x})$. Often TE modes are referred to as magnetic modes, labeled by M, and TM modes are referred to as electric modes, labeled by E.

1. Green's function and eigenfunctions—plane wave basis

We choose the $\hat{\mathbf{z}}$ axis as a convenient symmetry axis and let $\mathbf{k}_\perp$ denote momentum perpendicular to this axis. The free dyadic Green's function is

$$\mathbb{G}_0(ic\kappa, \mathbf{x}, \mathbf{x}') = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \begin{cases} C_{\mathbf{k}_\perp M}(\kappa) \mathbf{M}^{\text{out}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \otimes \mathbf{M}^{\text{reg}*}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}') + C_{\mathbf{k}_\perp E}(\kappa) \mathbf{N}^{\text{out}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \otimes \mathbf{N}^{\text{reg}*}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}') & \text{if } z > z' \\ C_{\mathbf{k}_\perp M}(\kappa) \mathbf{M}^{\text{reg}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \otimes \mathbf{M}^{\text{in}*}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}') + C_{\mathbf{k}_\perp E}(\kappa) \mathbf{N}^{\text{reg}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \otimes \mathbf{N}^{\text{in}*}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}') & \text{if } z < z' \end{cases}. \tag{B1}$$

Here, $C_{\mathbf{k}_\perp M}(\kappa) = \frac{1}{2\sqrt{\mathbf{k}_\perp^2 + \kappa^2}} = (-1) C_{\mathbf{k}_\perp E}(\kappa)$, and the modified vector plane wave functions are given by

$$\begin{aligned} \mathbf{M}^{\text{reg}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) &= \frac{1}{|\mathbf{k}_\perp|} \boldsymbol{\nabla} \times \phi^{\text{reg}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \\ \mathbf{M}^{\text{out}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) &= \frac{1}{|\mathbf{k}_\perp|} \boldsymbol{\nabla} \times \phi^{\text{out}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \\ \mathbf{N}^{\text{reg}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) &= \frac{1}{\kappa |\mathbf{k}_\perp|} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi^{\text{reg}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \\ \mathbf{N}^{\text{out}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) &= \frac{1}{\kappa |\mathbf{k}_\perp|} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi^{\text{out}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \end{aligned} \tag{B2}$$

in terms of the modified scalar plane wave functions,

$$\begin{aligned} \phi^{\text{reg}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) &= e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp + \sqrt{\mathbf{k}_\perp^2 + \kappa^2} z}, \\ \phi^{\text{out}}_{\mathbf{k}_\perp}(\kappa, \mathbf{x}) &= e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp - \sqrt{\mathbf{k}_\perp^2 + \kappa^2} z}. \end{aligned} \tag{B3}$$

As discussed in the text, the labels "reg," "out," and "in" are not really appropriate for plane wave functions, but the mathematical results all carry over.

2. Green's function and eigenfunctions—cylindrical wave basis

Again, let us take the $\hat{\mathbf{z}}$ axis as a convenient symmetry axis and let $\rho = \sqrt{x^2 + y^2}$ be the distance to the $\hat{\mathbf{z}}$ axis. The free dyadic Green's function is given by

$$\mathbb{G}_0(ic\kappa, \mathbf{x}, \mathbf{x}') = \int \frac{dk_z}{2\pi} \sum_n \begin{cases} C_M \mathbf{M}^{\text{out}}_{k_z n}(\kappa, \mathbf{x}) \otimes \mathbf{M}^{\text{reg}*}_{k_z n}(\kappa, \mathbf{x}') + C_E \mathbf{N}^{\text{out}}_{k_z n}(\kappa, \mathbf{x}) \otimes \mathbf{N}^{\text{reg}*}_{k_z n}(\kappa, \mathbf{x}') & \text{if } \rho > \rho' \\ C_M \mathbf{M}^{\text{reg}}_{k_z n}(\kappa, \mathbf{x}) \otimes \mathbf{M}^{\text{in}*}_{k_z n}(\kappa, \mathbf{x}') + C_E \mathbf{N}^{\text{reg}}_{k_z n}(\kappa, \mathbf{x}) \otimes \mathbf{N}^{\text{in}*}_{k_z n}(\kappa, \mathbf{x}') & \text{if } \rho < \rho'. \end{cases} \tag{B4}$$

Here $C_E = \frac{1}{2\pi} = (-1)C_M$, the vector cylindrical wave functions are given by

$$\begin{aligned} \mathbf{M}^{\text{reg}}_{k_z n}(\kappa, \mathbf{x}) &= \frac{1}{\sqrt{k_z^2 + \kappa^2}} \boldsymbol{\nabla} \times \phi^{\text{reg}}_{k_z n}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \\ \mathbf{M}^{\text{out}}_{k_z n}(\kappa, \mathbf{x}) &= \frac{1}{\sqrt{k_z^2 + \kappa^2}} \boldsymbol{\nabla} \times \phi^{\text{out}}_{k_z n}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \\ \mathbf{N}^{\text{reg}}_{k_z n}(\kappa, \mathbf{x}) &= \frac{1}{\kappa\sqrt{k_z^2 + \kappa^2}} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi^{\text{reg}}_{k_z n}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \\ \mathbf{N}^{\text{out}}_{k_z n}(\kappa, \mathbf{x}) &= \frac{1}{\kappa\sqrt{k_z^2 + \kappa^2}} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi^{\text{out}}_{k_z n}(\kappa, \mathbf{x}) \hat{\mathbf{z}}, \end{aligned} \tag{B5}$$

and the modified cylindrical wave functions are

$$\begin{aligned} \phi^{\text{reg}}_{k_z n}(\kappa, \mathbf{x}) &= I_n(\rho\sqrt{k_z^2 + \kappa^2}) e^{ik_z z + in\theta}, \\ \phi^{\text{out}}_{k_z n}(\kappa, \mathbf{x}) &= K_n(\rho\sqrt{k_z^2 + \kappa^2}) e^{ik_z z + in\theta}, \end{aligned} \tag{B6}$$

where I_n is the modified Bessel function of the first kind and K_n is the modified Bessel function of the third kind.

3. Green's function and eigenfunctions—spherical wave basis

In spherical coordinates the free dyadic Green's function is given by

$$\mathbb{G}_0(ic\kappa, \mathbf{x}, \mathbf{x}') = \sum_{lm} \begin{cases} C_M(\kappa) \mathbf{M}^{\text{out}}_{lm}(\kappa, \mathbf{x}) \otimes \mathbf{M}^{\text{reg}*}_{lm}(\kappa, \mathbf{x}') + C_E(\kappa) \mathbf{N}^{\text{out}}_{lm}(\kappa, \mathbf{x}) \otimes \mathbf{N}^{\text{reg}*}_{lm}(\kappa, \mathbf{x}') & \text{if } |\mathbf{x}| > |\mathbf{x}'| \\ C_M(\kappa) \mathbf{M}^{\text{reg}}_{lm}(\kappa, \mathbf{x}) \otimes \mathbf{M}^{\text{in}*}_{lm}(\kappa, \mathbf{x}') + C_E(\kappa) \mathbf{N}^{\text{reg}}_{lm}(\kappa, \mathbf{x}) \otimes \mathbf{N}^{\text{in}*}_{lm}(\kappa, \mathbf{x}') & \text{if } |\mathbf{x}| < |\mathbf{x}'|. \end{cases} \tag{B7}$$

Here $C_M(\kappa) = \kappa = (-1)C_E(\kappa)$, the vector spherical wave functions are

$$\begin{aligned} \mathbf{M}^{\text{reg}}_{lm}(\kappa, \mathbf{x}) &= \frac{1}{\sqrt{l(l+1)}} \boldsymbol{\nabla} \times \phi^{\text{reg}}_{lm}(\kappa, \mathbf{x}) \mathbf{x}, \\ \mathbf{M}^{\text{out}}_{lm}(\kappa, \mathbf{x}) &= \frac{1}{\sqrt{l(l+1)}} \boldsymbol{\nabla} \times \phi^{\text{out}}_{lm}(\kappa, \mathbf{x}) \mathbf{x}, \\ \mathbf{N}^{\text{reg}}_{lm}(\kappa, \mathbf{x}) &= \frac{1}{\kappa\sqrt{l(l+1)}} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi^{\text{reg}}_{lm}(\kappa, \mathbf{x}) \mathbf{x}, \\ \mathbf{N}^{\text{out}}_{lm}(\kappa, \mathbf{x}) &= \frac{1}{\kappa\sqrt{l(l+1)}} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi^{\text{out}}_{lm}(\kappa, \mathbf{x}) \mathbf{x}, \end{aligned} \tag{B8}$$

and the modified spherical wave functions are

$$\begin{aligned} \phi^{\text{reg}}_{lm}(\kappa, \mathbf{x}) &= i_l(\kappa|\mathbf{x}|) Y_{lm}(\hat{\mathbf{x}}), \\ \phi^{\text{out}}_{lm}(\kappa, \mathbf{x}) &= k_l(\kappa|\mathbf{x}|) Y_{lm}(\hat{\mathbf{x}}), \end{aligned} \tag{B9}$$

where $i_l(z) = \sqrt{\frac{\pi}{2z}} I_{l+1/2}(z)$ is the modified spherical Bessel function of the first kind, and $k_l(z) = \sqrt{\frac{2}{\pi z}} K_{l+1/2}(z)$ is the modified spherical Bessel function of the third kind.

4. Green's function—elliptic cylindrical basis

In order to study geometry and orientation dependence of Casimir interactions, it is helpful to be able to study objects with reduced symmetry. In Ref. [87], this formalism was applied to spheroids in scalar field theory. Unfortunately, the vector Helmholtz equation is not separable in spheroidal coordinates as it is in spherical coordinates. While the analogous vector spheroidal harmonics can still be constructed, the scattering matrix for a perfectly conducting spheroid is not diagonal, although it can be obtained from a more elaborate calculation [88]. For a perfectly conducting elliptic cylinder, however, the vector scattering problem is separable, so we describe that case here. Throughout this section, we use the same normalization and conventions as in Ref. [89], in which all functions in elliptic cylindrical coordinates have the same normalization as their circular analogs. As a result, all the functions inherit the usual completeness and orthonormality relations and approach their circular analogs in the limit of long wavelength.

In elliptic cylindrical coordinates, the z coordinate is unchanged, while the components of $\mathbf{x}_\perp$ become $x = a\cosh\mu\cos\theta$ and $y = a\sinh\mu\sin\theta$, where the interfocal separation of the ellipse is $2a$. Far away, θ approaches the ordinary angle in cylindrical coordinates and $|\mathbf{x}_\perp| \approx \frac{a}{2} e^{\mu}$. Separation of variables in these coordinates yields angular and radial Mathieu functions for θ and μ, respectively. The even and odd angular Mathieu functions are $ce_n(\theta, \gamma)$ with $n \geq 0$ and $se_n(\theta, \gamma)$ with $n > 0$, which are the analogs of $\cos n\theta$ and $\sin n\theta$ in the circular case. (We used a complex exponential basis for the circular case, but it could equally well be represented in terms of sines and cosines.) The angular functions now depend on the wave number through the combination $\gamma = -(k_z^2 + \kappa^2)a^2/2$. The corresponding radial functions are now different for the even and odd cases and depend on γ and μ separately rather than through a single product of the two. The even and odd modified radial Mathieu functions of the first kind are denoted $Ie_m(\mu, \gamma)$ and $Io_m(\mu, \gamma)$, respectively, and the even and odd modified radial Mathieu functions of the

third kind are denoted $Ke_m(\mu, \gamma)$ and $Ko_m(\mu, \gamma)$, respectively.

We then obtain the same results as in cylindrical coordinates, but now with

$$\begin{aligned}\phi^{\mathrm{reg}}_{k_z ne}(\kappa, \mathbf{x}) &= Ie_n(\mu, \gamma)ce_n(\theta, \gamma)e^{ik_z z},\\ \phi^{\mathrm{out}}_{k_z ne}(\kappa, \mathbf{x}) &= Ke_n(\mu, \gamma)ce_n(\theta, \gamma)e^{ik_z z},\\ \phi^{\mathrm{reg}}_{k_z no}(\kappa, \mathbf{x}) &= Io_n(\mu, \gamma)se_n(\theta, \gamma)e^{ik_z z},\\ \phi^{\mathrm{out}}_{k_z no}(\kappa, \mathbf{x}) &= Ko_n(\mu, \gamma)se_n(\theta, \gamma)e^{ik_z z}.\end{aligned} \tag{B10}$$

For numerical calculation the required Mathieu functions can be efficiently computed using the C + + package of Alhargan [90,91]. Analogous replacements convert the translation matrices and wave conversion matrices described below into this basis.

APPENDIX C: TRANSLATION MATRICES

In the following, we list the translation matrices that make up $\mathbb{X}^{ij}$, defined in Eq. (3.14). The definition of the vector $\mathbf{X}_{ij}$, which points from the origin of object i to the origin of object j, is illustrated in Fig. 1.

1. Plane wave basis

Plane waves are eigenfunctions of the translation operator, which does not mix TE and TM vector plane wave functions.

If the z coordinates of object i are smaller than those of object j, then $-\mathcal{V}^{ij}$ is the only nonzero entry in $\mathbb{X}^{ij}$. Taking $\mathbf{X}_{ij}$ to point from the origin of object i, $\mathcal{O}_i$, to the origin of object j, $\mathcal{O}_j$ (that is, upward), we obtain

$$\begin{aligned}\mathcal{V}^{ij}_{\mathbf{k}_\perp P, \mathbf{k}'_\perp P'} &= e^{-i\mathbf{k}_\perp\cdot\mathbf{X}_{ij,\perp} - \sqrt{\mathbf{k}_\perp^2+\kappa^2}X_{ij,z}}\\ &\quad\times\frac{(2\pi)^2}{L^2}\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\delta_{P,P'}.\end{aligned} \tag{C1}$$

If i is located above j, then $-\mathcal{W}^{ji}$ is the only nonzero entry in $\mathbb{X}^{ij}$. The vector $\mathbf{X}_{ji}$ points upward from $\mathcal{O}_j$ to $\mathcal{O}_i$, and we have

$$\begin{aligned}\mathcal{W}^{ji}_{\mathbf{k}_\perp P, \mathbf{k}'_\perp P'} &= \mathcal{V}^{ji*}_{\mathbf{k}'_\perp P', \mathbf{k}_\perp P}\frac{C_{\mathbf{k}_\perp P}(\kappa)}{C_{\mathbf{k}'_\perp P'}(\kappa)}\\ &= e^{i\mathbf{k}_\perp\cdot\mathbf{X}_{ji,\perp} - \sqrt{\mathbf{k}_\perp^2+\kappa^2}X_{ji,z}}\\ &\quad\times\frac{(2\pi)^2}{L^2}\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\delta_{P,P'}.\end{aligned} \tag{C2}$$

Since the matrix is diagonal in $\mathbf{k}_\perp$ and P, the factor $\frac{C_{\mathbf{k}_\perp P}(\kappa)}{C_{\mathbf{k}'_\perp P'}(\kappa)}$ cancels.

2. Cylindrical wave basis

Translations do not mix the TE and TM modes of vector cylindrical wave functions. They are constructed by taking the scalar cylindrical wave function, multiplying by the unit vector $\hat{\mathbf{z}}$, and performing one or two curl operations. A TE vector cylindrical wave function is perpendicular to $\hat{\mathbf{z}}$, while the curl of a TM vector cylindrical wave function is perpendicular to $\hat{\mathbf{z}}$. Expanding any of the two vector wave functions around any other point in space must preserve its orthogonality property with respect to the constant vector $\hat{\mathbf{z}}$. So the two are not mixed by the translation matrix.

If two objects i and j are outside of one another, $-\mathcal{U}^{ji}$ is the only nonzero submatrix of $\mathbb{X}^{ij}$. Again, let $\mathbf{X}_{ji}$ point from $\mathcal{O}_j$ to $\mathcal{O}_i$. We have

$$\begin{aligned}\mathcal{U}^{ji}_{k_z nP, k'_z n'P'} &= K_{n-n'}(|\mathbf{X}_{ji,\perp}|\sqrt{k_z^2+\kappa^2})\\ &\quad\times e^{-ik_z X_{ji,z} - i(n-n')\theta_{ji}}(-1)^{n'}\\ &\quad\times\delta_{P,P'}\frac{2\pi}{L}\delta(k_z - k'_z),\end{aligned} \tag{C3}$$

where $|\mathbf{X}_{ji,\perp}|$ is the distance of $\mathbf{X}_{ji}$ to the $\hat{\mathbf{z}}$ axis, i.e. the length of the projection onto the x-y plane, and θ_{ji} is the angle of $\mathbf{X}_{ji}$ in the x-y plane.

When object i is enclosed inside the surface of an infinite cylinder, inside object j, submatrix $-\mathcal{V}^{ij}$ is the only nonzero entry in $\mathbb{X}^{ij}$. We have

$$\begin{aligned}\mathcal{V}^{ij}_{k_z nP, k'_z n'P'} &= I_{n-n'}(|\mathbf{X}_{ij,\perp}|\sqrt{k_z^2+\kappa^2})\\ &\quad\times e^{-ik_z X_{ij,z} - i(n-n')\theta_{ij}}(-1)^{n+n'}\\ &\quad\times\delta_{P,P'}\frac{2\pi}{L}\delta(k_z - k'_z),\end{aligned} \tag{C4}$$

where $\mathbf{X}_{ij}$ points from $\mathcal{O}_i$ to $\mathcal{O}_j$.

If the roles of i and j are reversed, then $-\mathcal{W}^{ji}$ is the nonzero submatrix of $\mathbb{X}^{ij}$, with

$$\begin{aligned}\mathcal{W}^{ji}_{k_z nP, k'_z n'P'} &= \mathcal{V}^{ji*}_{k'_z n'P', k_z nP}\frac{C_P}{C_{P'}}\\ &= I_{n-n'}(|\mathbf{X}_{ji,\perp}|\sqrt{k_z^2+\kappa^2})\\ &\quad\times e^{+ik_z X_{ji,z} - i(n-n')\theta_{ji}}(-1)^{n+n'}\\ &\quad\times\delta_{P,P'}\frac{2\pi}{L}\delta(k_z - k'_z).\end{aligned} \tag{C5}$$

Since the matrix is diagonal in P, the factor $\frac{C_P}{C_{P'}}$ cancels.

3. Spherical wave basis

The TE vector wave functions are orthogonal to the radius vector $\mathbf{x}$. Since the same vector wave function cannot also be orthogonal everywhere to the radius vector of a shifted coordinate system, TE and TM polarizations mix under translation.

Suppose object i and its origin are outside a spherical separating surface, which encloses j. The nonzero submatrix of $\mathbb{X}^{ij}$ is $-\mathcal{U}^{ji}$, with

$$
\begin{aligned}
\mathcal{U}^{ji}_{l'm'M,lmM} &= (-1)^{m+l}\sum_{l''}[l(l+1)+l'(l'+1)-l''(l''+1)]\sqrt{\frac{\pi(2l+1)(2l'+1)(2l''+1)}{l(l+1)l'(l'+1)}} \\
&\quad\times\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & -m' & m'-m\end{pmatrix}k_{l''}(\kappa|\mathbf{X}_{ji}|)Y_{l''m-m'}(\hat{\mathbf{X}}_{ji}), \\
\mathcal{U}^{ji}_{l'm'E,lmM} &= -\frac{i\kappa}{\sqrt{l(l+1)l'(l'+1)}}\mathbf{X}_{ji}\cdot\Big[\hat{\mathbf{x}}\frac{1}{2}(\lambda^{+}_{lm}A_{l'm'lm+1}(\mathbf{X}_{ji})+\lambda^{-}_{lm}A_{l'm'lm-1}(\mathbf{X}_{ji})) \\
&\quad+\hat{\mathbf{y}}\frac{1}{2i}(\lambda^{+}_{lm}A_{l'm'lm+1}(\mathbf{X}_{ji})-\lambda^{-}_{lm}A_{l'm'lm-1}(\mathbf{X}_{ji}))+\hat{\mathbf{z}}mA_{l'm'lm}(\mathbf{X}_{ji})\Big], \\
\mathcal{U}^{ji}_{l'm'M,lmE} &= -\mathcal{U}^{ji}_{l'm'E,lmM}, \\
\mathcal{U}^{ji}_{l'm'E,lmE} &= \mathcal{U}^{ji}_{l'm'M,lmM},
\end{aligned}
$$

where

$$
A_{l'm'lm}(\mathbf{X}_{ji}) = (-1)^{m+l}\sum_{l''}\sqrt{4\pi(2l+1)(2l'+1)(2l''+1)}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & -m' & m'-m\end{pmatrix}k_{l''}(\kappa|\mathbf{X}_{ji}|)Y_{l''m-m'}(\hat{\mathbf{X}}_{ji}) \tag{C6}
$$

and $\lambda^{\pm}_{lm}=\sqrt{(l\mp m)(l\pm m+1)}$.

The translations between regular waves are described by the matrix elements

$$
\begin{aligned}
\mathcal{V}^{ij}_{l'm'M,lmM} &= (-1)^{m}\sum_{l''}[l(l+1)+l'(l'+1)-l''(l''+1)]\sqrt{\frac{\pi(2l+1)(2l'+1)(2l''+1)}{l(l+1)l'(l'+1)}} \\
&\quad\times\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & -m' & m'-m\end{pmatrix}i_{l''}(\kappa|\mathbf{X}_{ij}|)(-1)^{l''}Y_{l''m-m'}(\hat{\mathbf{X}}_{ij}), \\
\mathcal{V}^{ij}_{l'm'E,lmM} &= -\frac{i\kappa}{\sqrt{l(l+1)l'(l'+1)}}\mathbf{X}_{ij}\cdot\Big[\hat{\mathbf{x}}\frac{1}{2}(\lambda^{+}_{lm}B_{l'm'lm+1}(\mathbf{X}_{ij})+\lambda^{-}_{lm}B_{l'm'lm-1}(\mathbf{X}_{ij})) \\
&\quad+\hat{\mathbf{y}}\frac{1}{2i}(\lambda^{+}_{lm}B_{l'm'lm+1}(\mathbf{X}_{ij})-\lambda^{-}_{lm}B_{l'm'lm-1}(\mathbf{X}_{ij}))+\hat{\mathbf{z}}mB_{l'm'lm}(\mathbf{X}_{ij})\Big], \\
\mathcal{V}^{ij}_{l'm'M,lmE} &= -\mathcal{V}^{ij}_{l'm'E,lmM}, \\
\mathcal{V}^{ij}_{l'm'E,lmE} &= \mathcal{V}^{ij}_{l'm'M,lmM},
\end{aligned}
$$

where

$$
B_{l'm'lm}(\mathbf{X}_{ij}) = (-1)^{m}\sum_{l''}\sqrt{4\pi(2l+1)(2l'+1)(2l''+1)}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & -m' & m'-m\end{pmatrix}i_{l''}(\kappa|\mathbf{X}_{ij}|)(-1)^{l''}Y_{l''m-m'}(\hat{\mathbf{X}}_{ij}) \tag{C7}
$$

and $\lambda^{\pm}_{lm}=\sqrt{(l\mp m)(l\pm m+1)}$.

The matrix $\mathcal{W}^{ji}$ is related to $\mathcal{V}^{ji}$,

$$
\mathcal{W}^{ji}_{l'm'P',lmP} = \mathcal{V}^{ji\dagger}_{l'm'P',lmP}\frac{C_{P'}(\kappa)}{C_{P}(\kappa)}. \tag{C8}
$$

$\mathcal{V}^{ji}$, of course, is the same as $\mathcal{V}^{ij}$ with $\mathbf{X}_{ij}$ replaced by $\mathbf{X}_{ji}$. To be more specific, the elements correspond in the following way,

$$
\begin{aligned}
\mathcal{W}^{ji}_{l'm'M,lmM} &= \mathcal{V}^{ji*}_{lmM,l'm'M}, \\
\mathcal{W}^{ji}_{l'm'E,lmM} &= -\mathcal{V}^{ji*}_{lmM,l'm'E}, \\
\mathcal{W}^{ji}_{l'm'M,lmE} &= -\mathcal{V}^{ji*}_{lmE,l'm'M}, \\
\mathcal{W}^{ji}_{l'm'E,lmE} &= \mathcal{V}^{ji*}_{lmE,l'm'E}.
\end{aligned} \tag{C9}
$$

APPENDIX D: WAVE CONVERSION MATRICES

It is not necessary to express all the objects' scattering amplitudes in the same basis. Here, we supply the matrices

that convert modified vector plane wave functions to spherical or cylindrical vector wave functions.

1. Vector plane wave functions to spherical vector wave functions

$$\begin{aligned}\mathbf{M}^{\mathrm{reg}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) &= \sum_{lm} D_{lmM,\mathbf{k}_\perp M}\mathbf{M}^{\mathrm{reg}}_{lm}(\kappa,\mathbf{x}) \\ &\quad + D_{lmE,\mathbf{k}_\perp M}\mathbf{N}^{\mathrm{reg}}_{lm}(\kappa,\mathbf{x}), \\ \mathbf{N}^{\mathrm{reg}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) &= \sum_{lm} D_{lmM,\mathbf{k}_\perp E}\mathbf{M}^{\mathrm{reg}}_{lm}(\kappa,\mathbf{x}) \\ &\quad + D_{lmE,\mathbf{k}_\perp E}\mathbf{N}^{\mathrm{reg}}_{lm}(\kappa,\mathbf{x}).\end{aligned} \tag{D1}$$

The conversion matrices are obtained from the decomposition of a plane wave in spherical coordinates,

$$\begin{aligned}D_{lmM,\mathbf{k}_\perp M} &= \sqrt{\frac{4\pi(2l+1)(l-m)!}{l(l+1)(l+m)!}}\frac{|\mathbf{k}_\perp|}{\kappa}e^{-im\phi_{\mathbf{k}_\perp}} \\ &\quad \times P_l^{lm}(\sqrt{\mathbf{k}_\perp^2+\kappa^2}/\kappa), \\ D_{lmE,\mathbf{k}_\perp M} &= \sqrt{\frac{4\pi(2l+1)(l-m)!}{l(l+1)(l+m)!}}im\frac{\kappa}{|\mathbf{k}_\perp|}e^{-im\phi_{\mathbf{k}_\perp}} \\ &\quad \times P_l^{m}(\sqrt{\mathbf{k}_\perp^2+\kappa^2}/\kappa), \\ D_{lmE,\mathbf{k}_\perp E} &= D_{lmM,\mathbf{k}_\perp M}, \\ D_{lmM,\mathbf{k}_\perp E} &= -D_{lmE,\mathbf{k}_\perp M},\end{aligned} \tag{D2}$$

where P_l^m is the associated Legendre polynomial and prime indicates the derivative of P_l^m with respect to its argument.

2. Vector plane wave functions to cylindrical vector wave functions

The cylindrical vector wave functions are defined as before, but now we consider regular vector plane wave functions that decay along the $-\hat{\mathbf{x}}$ axis instead of the $-\hat{\mathbf{z}}$ axis,

$$\begin{aligned}\mathbf{M}^{\mathrm{reg}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) &= \frac{1}{\sqrt{k_y^2+k_z^2}}\boldsymbol{\nabla}\times e^{\sqrt{\kappa^2+k_y^2+k_z^2}x+ik_y y+ik_z z}\hat{\mathbf{x}}, \\ \mathbf{N}^{\mathrm{reg}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) &= \frac{1}{\kappa\sqrt{k_y^2+k_z^2}}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times e^{\sqrt{\kappa^2+k_y^2+k_z^2}x+ik_y y+ik_z z}\hat{\mathbf{x}}.\end{aligned} \tag{D3}$$

The vector plane wave functions can be decomposed in vector cylindrical wave functions,

$$\begin{aligned}\mathbf{M}^{\mathrm{reg}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) &= \sum_n D_{k_z nM,\mathbf{k}_\perp M}\mathbf{M}^{\mathrm{reg}}_{k_z n}(\kappa,\mathbf{x}) \\ &\quad + D_{k_z nE,\mathbf{k}_\perp M}\mathbf{N}^{\mathrm{reg}}_{k_z n}(\kappa,\mathbf{x}), \\ \mathbf{N}^{\mathrm{reg}}_{\mathbf{k}_\perp}(\kappa,\mathbf{x}) &= \sum_n D_{k_z nM,\mathbf{k}_\perp E}\mathbf{M}^{\mathrm{reg}}_{k_z n}(\kappa,\mathbf{x}) \\ &\quad + D_{k_z nE,\mathbf{k}_\perp E}\mathbf{N}^{\mathrm{reg}}_{k_z n}(\kappa,\mathbf{x}),\end{aligned} \tag{D4}$$

using the conversion matrix elements

$$\begin{aligned}D_{k_z nM,\mathbf{k}_\perp M} &= -i\frac{k_z}{\sqrt{k_y^2+k_z^2}}\sqrt{1+\xi^2}(\sqrt{1+\xi^2}+\xi)^n, \\ D_{k_z nE,\mathbf{k}_\perp M} &= i\frac{\kappa}{\sqrt{k_y^2+k_z^2}}\xi(\sqrt{1+\xi^2}+\xi)^n, \\ D_{k_z nE,\mathbf{k}_\perp E} &= D_{k_z nM,\mathbf{k}_\perp M}, \\ D_{k_z nM,\mathbf{k}_\perp E} &= -D_{k_z nE,\mathbf{k}_\perp M},\end{aligned} \tag{D5}$$

where $\xi = \frac{k_y}{\sqrt{\kappa^2+k_z^2}}$ and $\mathbf{k}_\perp = (k_y, k_z)$.

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