# Calculating limits on Yukawa Interactions

Alexander Rider

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## Potential Integral

A general Yukawa interaction leads to the interaction between two point masses given by

$$V(r) = \frac{-Gm_1m_2}{r}(1 + \alpha e^{-r/\lambda}) \tag{1}$$

where $G$ is the gravitational constant, $m_1$ and $m_2$ are the respective point masses, $\alpha$ is the dimensionless strength of the interaction, and $\lambda$ is the length scale of the interaction. The potential due to the Yukawa interaction between a sphere of density $\rho$ and point mass $m$ can be written as the integral

$$V(s, r_b, \rho) = m\int_s^{s+2r_b}\int_0^{\pi}\int_0^{2\pi} \rho V(r) r^2 sin(\theta) d\phi d\theta dr \tag{2}$$

The geometry of this setup is shown in figure 1. Where $s$ is the separation between the point mass and the closest point on the surface of the sphere, and $r_b$ is the radius of the sphere. Symmetry can be used to perform the integration over the $\theta$ and $\phi$ coordinates and sum up spherical caps weighted by $V(r)$. The

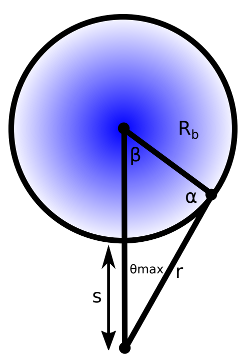


Figure 1: The geometry for integrating a function that is constant of radius between a point and a sphere.

area of the spherical cap intersecting the sphere at a distance $r$ from the point mass is given by

$$A(r) = r^2 \int_0^{\theta_{max}} \int_0^{2\pi} sin(\theta) d\phi d\theta = 2\pi r^2 (1 - cos(\theta_{max})) \tag{3}$$

where $\theta_{max}$ is the maximum azimuthal angle at which the spherical cap centered on the point mass intersects the sphere. $\theta_{max}$ can be calculated from the law of cosines as

$$\theta_{max} = Arccos(\frac{(s + r_b)^2 + r^2 - r_b^2}{2r(s + r_b)}) \tag{4}$$

Inserting this into 3 we obtain

$$A(r) = \pi r \frac{(r - s)(s + 2r_b - r)}{s + r_b} \tag{5}$$

So we can write the total potential between a point mass and a sphere as

$$V(s, r_b, \rho) = \int_s^{s + r_b} V(r) A(r) dr \tag{6}$$

To integrate over the potential given in 1 we note that

$$\int r^n e^r dr = \int [\partial_\beta^n e^{r\beta}]_{\beta=1} dr = [\partial_\beta^n \int e^{\beta r} dr]_{\beta=1} \tag{7}$$

For the relevant cases of $n = 1$ and $n = 3$ we have that

$$\int r e^r dr = (r - 1) e^r \tag{8}$$

$$\int r^3 e^r dr = (r^3 - 3r^2 + 6r - 6) e^r \tag{9}$$

Now we can write

$$\begin{aligned} V(s, r_b, \rho, \lambda, \alpha, m) = \\ -\frac{Gm\rho}{r_b + s}(v(r_b) + 3/2\alpha e^{-s/\lambda} v(\lambda) f(r_b/\lambda)) \end{aligned} \tag{10}$$

where

$$v(r) = 4/3\pi r^3 \tag{11}$$

and

$$f(r_b/\lambda) = e^{-2r_b/\lambda}(1 + r_b/\lambda) + r_b/\lambda - 1 \tag{12}$$

We can now examine limiting cases to investigate the validity of this result. In the limit $\alpha \to 0$ we see that $V \to \frac{-Gm_1m_2}{r}$ and standard Newtonian gravity is obtained. If we take the limit $\alpha >> G$ and $\frac{r_b}{\lambda} << 1$ $f(r/\lambda) \to 2/3(r_b/\lambda)^3$ so $3/2v(\lambda)f(r_b/\lambda) \to v(r_b)$ and we get

$$V(s, r_b, \rho, \lambda, \alpha, m) \approx \alpha \frac{m\rho v(r_b) e^{-s/\lambda}}{r_b + s} \tag{13}$$

The sphere effectively interacts with the Yukawa potential as a point mass in this limit. The potential in (10) can be integrated over an attractor in 3 dimensions to efficiently calculate the interaction between a microsphere and an attractor.

## Nuclear Couplings

Yukawa violations of the inverse square law between two point masses of mass $M_a$ and $M_b$ are generally parameterized with the potential [1]

$$V_{ab}(r) = -\alpha G \frac{M_a M_b}{r} \exp\left(-r/\lambda\right) \tag{14}$$

For the special case of exchange of scalar or vector bosons $\phi$ with mass $m$ between two non-relativistic fermions generically produces produces a potential

$$V_{ab}(r) = \pm \frac{g^a_{S,V} g^b_{S,V}}{4\pi r} \exp\left(-r/\lambda\right) \tag{15}$$

where the $-$ and $+$ signs refer to scalar and vector interactions, respectively, ans $\lambda = \hbar/mc$. For arbitrary vector interactions between electrically neutral atoms with proton number Z, and neutron number N,

$$g_V = g^0_V (Z \cos\tilde{\psi} + N \sin\tilde{\psi}) \tag{16}$$

where $\tilde{\psi} = \arctan \frac{\tilde{q}^n_V}{\tilde{q}^p_V + \tilde{q}^e_V}$. By comparing the generic Yukawa potential in 15 to the potential in 16 for scalar or vector boson exchange we can identify

$$\frac{g^0_a g^0_b}{4\pi} = \alpha G u^2 \left( \left[\frac{\tilde{q}}{\mu}\right]_a \left[\frac{\tilde{q}}{\mu}\right]_b \right)^{-1} \tag{17}$$

where $\mu = M/u$ with $M$ being the atomic mass of the interacting nuclei and $u$ being the atomic mass unit. Furthermore,

$$\left[\frac{\tilde{q}}{\mu}\right] = \left[\frac{Z}{\mu}\right] \cos\tilde{\psi} + \left[\frac{N}{\mu}\right] \sin\tilde{\psi}. \tag{18}$$

For silicon, $\left[\frac{Z}{\mu}\right] = 0.504$ and $\left[\frac{N}{\mu}\right] = 0.496$ weighted by isotopic abundance. For oxygen, $\left[\frac{Z}{\mu}\right] = 0.500$ and $\left[\frac{N}{\mu}\right] = 0.500$ so that

$$\left[\frac{\tilde{q}}{\mu}\right] \approx 0.50 \left( \cos\tilde{\psi} + \sin\tilde{\psi} \right). \tag{19}$$

Thus if we correct for the fact that 15 has been integrated over a mass distribution to correct for the fact that the nuclear potential is per nucleon we can write the correspondence between $\alpha$ and $\frac{g^0_a g^0_b}{4\pi}$

$$\frac{g^0_a g^0_b}{4\pi} = \alpha G u^2 \rho_{n,a} \rho_{n,b} 4.0 \left( \cos\tilde{\psi} + \sin\tilde{\psi} \right)^{-2} \tag{20}$$

where $\rho_n$ is the density of nuclei per unit volume.

## References

[1] E. G. Adelberger, B. R. Heckel, S. Hoedl, C. D. Hoyle, D. J. Kapner, and A. Upadhye. Particle-physics implications of a recent test of the gravitational inverse-square law. *Phys. Rev. Lett.*, 98:131104, Mar 2007.