

# Internal Note: The Casimir Force in a Micron-Scale Optical Trap Geometry

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## Abstract

This document explains the recent theory and numerical methods relevant to computing the casimir force in our geometry. I will cover the historical analytical solutions, and then focus on the new scattering theory, discussing various techniques needed in different regimes. I then present the best numerical result from our geometry and discuss uncertainty. The overall purpose of this document is to summarize our knowledge of Casimir forces within the sphere-cantilever geometry, and compile useful literature for robust comparison of our eventual measurements to existing theory. I also try to make explicit the connection between the microscopic and macroscopic descriptions of the Casimir force, corresponding generally to the comparison of the particulate and field descriptions, quantum mechanically.

## 1 Background

The Casimir effect rises from zero-point energy fluctuations between conducting surfaces, which can be thought of as a result of positive feedback between stochastic deviations from the nominally non-polarized state. A good description of this is provided by Rodriguez et al. (2011), reproduced here:

Van der Waals forces are a familiar concept from introductory physics and chemistry: two neutral particles have fluctuating dipole moments resulting from quantum or thermal effects, which, for a particle separation of  $d$ , lead to a  $d^{-6}$  interaction energy that is commonly used, for example, as a long-range attraction term when describing the interactions between atoms and molecules. Whenever one particle acquires a spontaneous dipole moment  $\mathbf{p1}$ , the resulting dipole electric field polarizes the adjacent particle to produce an induced dipole moment  $\mathbf{p2} \sim d^{-3}$ . Assuming positive polarizabilities, the direction of the dipole fields means that these two dipoles are oriented so as to attract each other, with an interaction energy that scales as  $d^{-6}$ . This leads to the van der Waals ‘dispersion’ force, and similar considerations apply to particles with permanent dipole moments that can rotate freely. The key to more general considerations of Casimir physics is to understand that this  $d^{-6}$  picture of van der Waals forces makes two crucial approximations that are not always valid: it employs the quasi-static approximation to ignore wave effects, and also ignores multiple scattering if there are more than two particles.

The quasi-static approximation assumes that the dipole moment  $\mathbf{p1}$  polarizes the second particle instantaneously, which is valid if  $d$  is much smaller than the typical wavelength of the fluctuating fields. However, the finite wave propagation speed of light must be taken into account when  $d$  is much larger than the typical wavelength, and it turns out that the resulting Casimir-Polder interaction energy asymptotically scales as  $d^{-7}$  for large  $d$ . More generally, the interaction is not a simple power law between these

limits, but instead depends on an integral of fluctuations at all frequencies scaled by a frequency-dependent polarizability of the particles.

The presence of multiple particles further complicates the situation because multiple scattering must be considered. For example, with three particles, the initial dipole  $\mathbf{p1}$  will induce polarizations  $\mathbf{p2}$  and  $\mathbf{p3}$  in the other two particles, but  $\mathbf{p2}$  will create its own field that further modifies  $\mathbf{p3}$ , and so on. Thus, the interaction between multiple particles is generally non-additive, and there is no two-body force law that can simply be summed to incorporate all interactions. Multiple scattering is negligible for a sufficiently dilute gas or for weak polarizabilities, but it becomes very significant for interactions between two (or more) solid bodies, which consist of many fluctuating dipole moments that all interact in a complicated way through electromagnetic radiation. When these multiple scattering effects are combined with wave retardation in a complete picture, they yield the Casimir force.

Hendrik Casimir based his prediction on a simplified model involving two parallel perfectly conducting plates separated by a vacuum. Although the Casimir force arises from electromagnetic fluctuations, real photons are not involved. Quantum mechanically, these fluctuations can be described in terms of virtual photons of energy equal to the zero-point energies of the electromagnetic modes of the system. By considering the contribution of the electromagnetic field modes to the zero-point energy of the parallel plate configuration, Casimir predicted an attractive force between the plates. Because only electromagnetic modes that have nodes on both walls can exist within the cavity, the mode frequencies ( $\omega$ ) depend on the separation between the plates, giving rise to a pressure  $P_C$ . The force in this case is attractive because the mode density in free space is larger than that between the plates. Following Casimir's calculation, Lifshitz, Dzyaloshinski and Pitaveski considered the more general case of realistic dielectric plates by exploiting the fluctuation-dissipation theorem, which relates the dissipative properties of the plates (that is, the optical absorption resulting from the many microscopic dipoles in the plates) and the resulting electromagnetic fluctuations at equilibrium. For realistic metallic plates separated by  $d$ , the force again scales as  $d^{-4}$  for large  $d$ . At small  $d$ , the force scales as  $d^{-3}$  (this is the quasi-static limit, where the coefficient is known as the Hamaker constant), with a complicated intermediate  $d$ -dependence that is determined by the frequency-dependent permittivity ( $\epsilon$ ) of the materials. Here, 'small' and 'large'  $d$  are relative to a characteristic wavelength  $\lambda_0$ , which for metals is the plasma wavelength and is typically in the ultraviolet range (a few hundred nanometres). The geometry of the system can be used to greatly modify wave propagation beyond the simple planar regime, but a broad-bandwidth scattering calculation is required to capture the complete physics of such interactions.

The nominal quantity computed in analysis of this effect is the casimir energy as a function of separation between metallic reflectors. In Casimir's classical result (Casimir, 1948) the energy between two parallel perfectly conducting plates was calculated exactly to be (given in nice form by Lambrecht et al., 2006)

$$\mathcal{E} = -\frac{hc\pi^2 A}{720L^3}$$

which of course makes the Casimir pressure between the two plates

$$P_C = \frac{1}{A} \frac{d\mathcal{E}}{dL} = \frac{hc\pi^2}{240L^4}$$

and thus in vacuum, two perfectly conducting plates at  $T = 0K$  will attract each other with a pressure independent of plate area. This geometry has since been studied in various limits, including most notably finite conductivity and temperature (Lambrecht et al., 2006, and references therein) but also finite extent and non-planar geometries (Rahi et al., 2009, and references therein).

Despite the theoretical interest, the difficulty inherent in measuring the Casimir force prevented precise measurement until the landmark work of Lamoreaux (1997), who measured the force between a plate and a curved surface, to confirm the theory based on small approximations to the classical model. In this geometry, the Casimir force became an attraction with a magnitude of tens of micro dynes at separations of a few micrometers, for a plate and a small curved surface with radius of curvature of about 10cm. This work helped to solidify the theory behind finite-conductivity corrections, thermal corrections, and verify a long-used approximation called the Proximity Force Approximation (PFA, see next section) which has been used to calculate the Casimir force in nearly parallel geometries. More recent work performed this verification with parallel planes, a much more difficult measurement due to the requirement of static parallel configuration between planes, the motivation for the curved-planar geometry (Rodriguez et al., 2011).

In geometries which deviate from the non-ideal case, the calculation of the Casimir force becomes highly non-trivial. The PFA, for example, assumes that in the nearly planar limit, the Casimir force can be computed as the superposition of forces from all sets of nearly parallel subsections of the curved surface directly opposite the planar surface. It is widely known that this approximation only holds in the limit of large curvature compared to distance, and thus for a sphere with curvature radius equal to or less than the separation between surfaces, the only analytical expression for this force is no longer strictly valid.

In the context of this experiment, this is the main barrier preventing us from producing a quick prediction for the Casimir force. Our geometry is neither spherically nor cylindrically symmetric, as the cantilever we use as a force probe has a rectangular cross-section with one dimension much longer than the other, and the sphere is much smaller than typical separations. In addition, our cantilever is silicon coated in gold, and our bead is made of silica, whereas the normal materials dealt with are conductors thick enough that only one material correction needs to be made. Finally, we operate the bead at non-zero temperature, which requires temperature corrections which are non-trivial in the non-parallel geometry. Despite the barrier to computation of the force, it should in principle be easy to measure if we can understand the systematic effects inherent to our geometry; this is the motivation behind this note. In addition, it will most likely represent the main background we must compensate for in the measurement of other sub micron forces, namely non-newtonian gravitational corrections and chameleon models of dark energy.

In this note, I will outline the steps taken in attempting to predict the magnitude of the Casimir force we expect to measure in our experiment, beginning with the PFA, expansions to the PFA, and other simple analytical solutions. I will then cover the finite conductivity and temperature corrections, and then discuss the exact analytic and numerical solutions which exist to compensate for these non-idealities. I will present predictions, and discuss other approaches we have attempted and potential limitations of the calculations. The idea will be that it is left up to those more theoretically inclined to compute the exact theoretical prediction to our model, but that we should be able to come up with a prediction to within 10-20% of the true Casimir force as a function of distance. I will conclude by discussing the reason for the limitations and assumptions.

## 2 Analytical Solutions

The only exact solutions come for the parallel plane solution as initially derived by Casimir (Casimir, 1948) for a perfect electrical conductor (PEC), and later for finite conductivity, multilayer films, and extended geometries (Rodriguez et al., 2011). Casimir's original formula was presented in the previous section. To extend this analytical solution to arbitrary situations, one has to correct for a wide range of effects, including

- Non-Planar Geometry
- Compact Geometry
- Finite Conductivity
- Finite Temperature
- Asymmetry

Of these effects, the first, third, and fourth were historically first to be attempted, but until recently asymmetric geometries and compact geometries were not generally explored. The recent theoretical progress in this area has been focused on determining a general framework by which, in any arbitrary configuration, the calculation resulting in the Casimir force can be constructed from the framework of scattering theory, largely within electromagnetism rather than quantum field theory. Good references for this scattering theory approach include Lambrecht et al. (2006), Emig (2008), and Rahi et al. (2009). These papers come from two distinct theory groups, one in France (Lambrecht et. al.) and one at MIT (Kardar, Jaffe, Emig et. al) who have separately developed this theory and contributed synergistically to the effort. The group in France come from a more experimental angle, which the MIT group comes from a more pure theory standpoint.

### 2.1 Proximity Force Approximation (PFA)

Consider the geometry shown in figure 1. This is clearly not the case derived by Casimir, and even for the closest separation of sphere from plane, neither object can be thought of as extended. Remembering, however, that Casimir forces scale as  $d^{-4}$ , given one surface much closer to another than the scale length of its physical distortion, one might postulate that, if the separation along the  $z$  axis is small enough, we could approximate the Casimir force as the sum of forces from each pair of opposite faces, approximately stated to be parallel. In integral form, this is (Fosco et al., 2011)

$$E_C \approx E_{PFA} = \int_{\Sigma} E_{pp}(z) d\sigma$$

where  $E_{pp}$  is the Casimir energy per unit area of two parallel planes separated by a distance  $z$ , using the normal Casimir equation from the previous section. Here, the geometric argument that surfaces be approximately parallel, and forces additive, corresponds to the limit  $R \gg d$ . The PFA is based on the more general Derjaguin approximation for distance dependent forces between planar and curved surfaces, as discussed in Bimonte et al. (2012).

### 2.2 Expanded Proximity Force Approximation

Most of this section taken from Fosco et al. (2011), also see Bimonte et al. (2012)

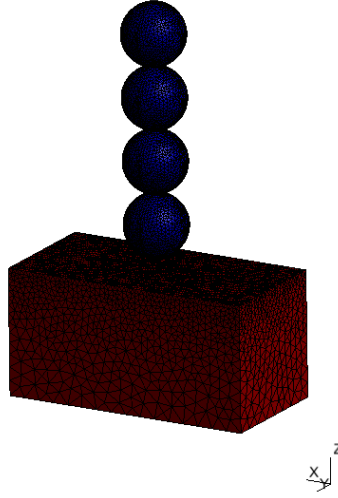


Figure 1: Geometry of cantilever and bead, truncated in the x direction and limited to tens of plasma wavelengths in depth. Here we show four distances of the bead from the cantilever in the z direction.

We don't expect this to hold much past this condition, but can use the derivative expansion to first order to obtain a slightly more accurate term. The formula for the derivative expansion in terms of some surface separation  $z = \psi(\mathbf{x}_{\parallel})$  is

$$E_{DE} = -\frac{\pi^2}{1440} \int \frac{1}{\psi^3} \left( 1 + \frac{2}{3} (\partial_{\alpha} \psi)^2 \right) d\mathbf{x}_{\parallel}$$

and if we add in the correction factor  $\eta_E(z)$  (see next section), we get

$$E_{DE} = -\frac{\pi^2}{1440} \int \frac{\eta_E(\psi)}{\psi^3} \left( 1 + \frac{2}{3} (\partial_{\alpha} \psi)^2 \right) d\mathbf{x}_{\parallel}$$

which, given the complexity of  $\eta_E$ , is best evaluated numerically. We can parameterize the surface of a sphere as

$$\psi_0(r) = R \left( 1 - \sqrt{1 - \frac{r^2}{R^2}} \right)$$

where the surface separation would be  $\psi = a + \psi_0(r)$  and  $a$  the minimal separation between sphere and plane. Given that the derivative with respect to  $\theta$  is 0, the integral simplifies to

$$E_{DE} = -\frac{2\pi^3}{1440} \int \frac{\eta_E(\psi)}{\psi^3} \left( 1 + \frac{2}{3} (\partial_r \psi)^2 \right) dr$$

and we can write, as well,

$$(\partial_r \psi)^2 = R^2 \left[ -\frac{1}{2} \left( 1 - \frac{r^2}{R^2} \right)^{-1/2} \left( \frac{-2r}{R^2} \right) \right]^2 = \frac{r^2}{R^2(1 - \frac{r^2}{R^2})} = \frac{r^2}{R^2 - r^2} = \left( \frac{R^2}{r^2} - 1 \right)^{-1}$$

we can see that the derivative is undefined if we integrate to  $R$ , so as long as we integrate to  $r_{limit} < R$  we should have a convergent integral; this shouldn't affect the accuracy given the  $r^{-3}$

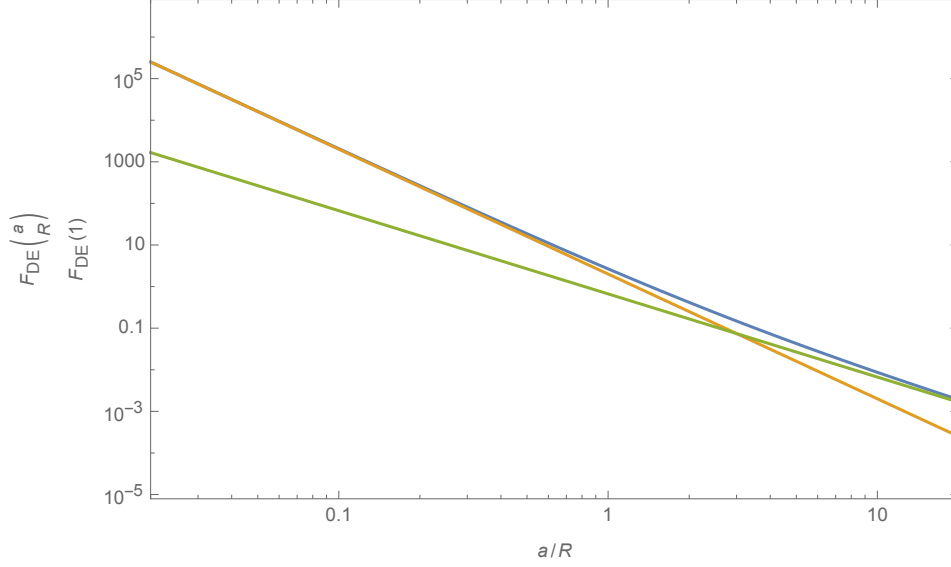


Figure 2: Illustration of the effect of the first order correction on the PFA; blue line shows expansion, orange line is PFA, green line is first order correction.

dependence. The effect of the derivative expansion correction in a perfectly conducting system can be seen below.

Bulgac et al. (2006) calculate the force between plane and sphere exactly, and compare to PFA. Canaguier-Durand et al. (2012) explore various temperature and geometry effects as a function of radius and separation.

### 3 Finite Conductivity Corrections

#### 3.1 Dielectric Correction

Bimonte et al. (2012); Lambrecht and Reynaud (2000): dielectric effects, non-perfect conductivity. The following mostly from Lambrecht and Reynaud (2000)

The dielectric constant is large at frequencies smaller than the plasma frequency  $\omega_p$ , so corrections are more relevant at separations smaller than the plasma wavelength

$$\lambda_p = \frac{2\pi c}{\omega_p}$$

where

$$\omega_p^2 = \frac{Nq^2}{\epsilon_0 m^*} = \frac{ZN_a q^2}{\epsilon_0 m^*} = \frac{\rho q^2}{\epsilon_0 m^* m_z}$$

and in turn,  $N$  is the number of conduction electrons per unit volume,  $Z$  is the number of free electrons per atom (taken to just be  $Z = 1$ ),  $N_a$  is the atomic number density,  $q$  is the electron charge, and  $m^*$  is effective electron mass, also taken to be 1 for Au though not necessarily for  $\text{SiO}_2$ . We in turn calculate  $N_a$  as the density  $\rho$  divided by the atomic mass  $m_z$  for the species.

Thermal corrections relevant at distances larger than

$$\lambda_T = \frac{\hbar c}{k_B T}$$

The reduction factor for the Casimir energy is given by

$$\eta_E = -\frac{180L^3}{c\pi^4} \int_0^\infty \kappa d\kappa \int_0^{c\kappa} \sum_{p=\parallel, \perp} \log(1 - r_{p,1}(i\omega, i\kappa)r_{p,2}(i\omega, i\kappa)e^{-2\kappa L}) d\omega \quad (1)$$

where  $r_{p,i}(i\omega, i\kappa)$  is the reflection amplitude of surface  $i$  for polarization  $p$ , where  $i\omega$  is the imaginary frequency and  $i\kappa$  is the imaginary wave vector along the longitudinal direction of the cavity, or the normal to the plane. The reflection coefficients for thick slabs are given by

$$r_\perp = -\frac{\sqrt{\omega^2(\epsilon(i\omega) - 1) + c^2\kappa} - c\kappa}{\sqrt{\omega^2(\epsilon(i\omega) - 1) + c^2\kappa} + c\kappa} \quad (2)$$

$$r_\parallel = \frac{\sqrt{\omega^2(\epsilon(i\omega) - 1) + c^2\kappa} - c\kappa\epsilon(i\omega)}{\sqrt{\omega^2(\epsilon(i\omega) - 1) + c^2\kappa} + c\kappa\epsilon(i\omega)} \quad (3)$$

Lambrecht and Reynaud (2000) suggest numerical integration over the range  $10^{-4}$  to  $10^3$  eV for distances 0.1 to  $10 \mu\text{m}$

### 3.2 Expected accuracy

We are limited by the validity of the proximity force approximation, which assumes the (roughly) Casimir forces are additive, which is known not to be the case, and assumes the sphere and plane are close enough such that the plane appears infinite and the curvature is relatively small. We include the first order correction to the PFA, which may not extend validity very far and still assumes the plane is long enough to be effectively infinite.

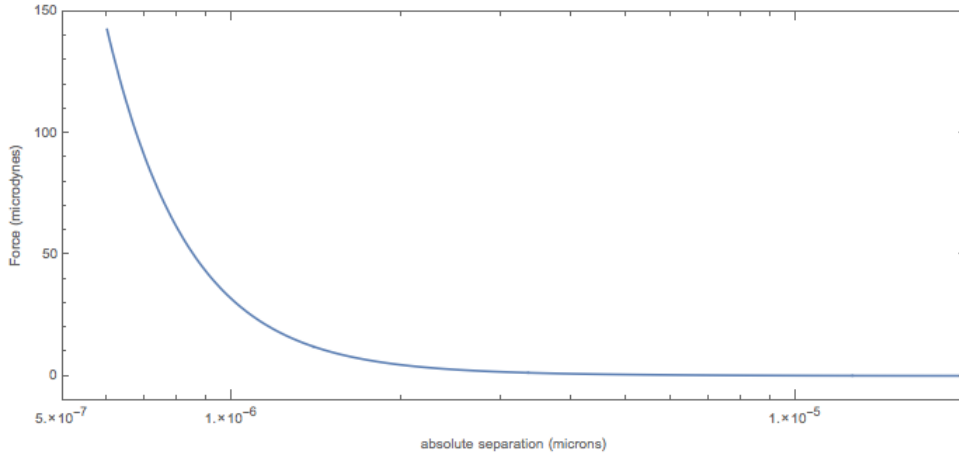


Figure 3: Reproduced calculation of PFA from Lamoreaux (1997)

## 4 Numerical Computation

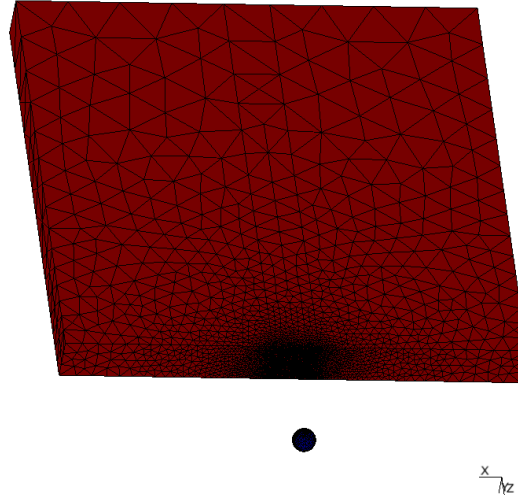


Figure 4: Geometry of cantilever and bead, representing the full x-y shape of the cantilever and extended to 100 plasma wavelengths in depth. Here we show the bead 20 microns from the cantilever in the z direction.

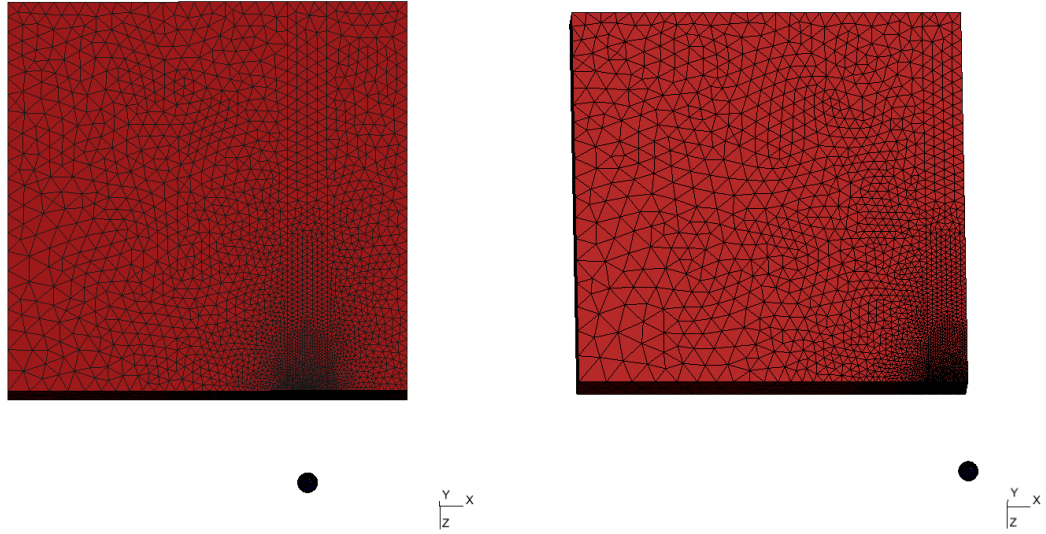


Figure 5: Geometry of cantilever and bead, representing the full x-y shape of the cantilever and extended to 100 plasma wavelengths in depth. Here we show the bead 20 microns from the cantilever in the z direction, offset in the x direction by 25 and 50 microns. The radial meshing pattern has been made to follow to preserve convergence properties of the simulation/maximize accuracy nearest the sphere.

## A Scuff-EM

## B Casimir Force Code

## References

G. Bimonte, T. Emig, R. L. Jaffe, and M. Kardar. Casimir forces beyond the proximity approximation. *EPL (Europhysics Letters)*, 97:50001, March 2012. doi: 10.1209/0295-5075/97/50001.



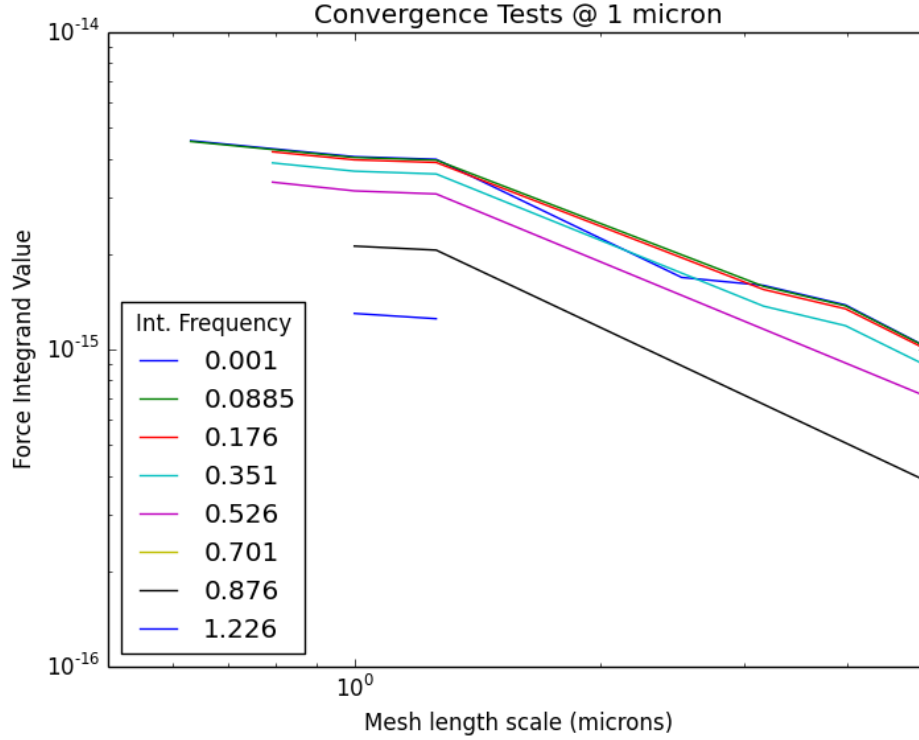


Figure 6: Earlier Casimir force estimates based on different geometries.

Giuseppe Bimonte, Thorsten Emig, and Mehran Kardar. Material dependence of casimir forces: Gradient expansion beyond proximity. *Applied Physics Letters*, 100(7):074110, 2012. doi: <http://dx.doi.org/10.1063/1.3686903>. URL <http://scitation.aip.org/content/aip/journal/apl/100/7/10.1063/1.3686903>.

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Antoine Canaguier-Durand, Romain Guerot, Paulo A. Maia Neto, Astrid Lambrecht, and Serge Reynaud. The casimir effect in the sphere-plane geometry. *International Journal of Modern Physics: Conference Series*, 14:250–259, 2012. doi: 10.1142/S2010194512007374. URL <http://www.worldscientific.com/doi/abs/10.1142/S2010194512007374>.

H.B.G. Casimir. On the Attraction Between Two Perfectly Conducting Plates. *Indag.Math.*, 10: 261–263, 1948.

Thorsten Emig. Fluctuation induced quantum interactions between compact objects and a plane mirror. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(04):P04007, 12 2008. URL <http://stacks.iop.org/1742-5468/2008/i=04/a=P04007>.

César D. Fosco, Fernando C. Lombardo, and Francisco D. Mazzitelli. Proximity force approximation for the casimir energy as a derivative expansion. *Phys. Rev. D*, 84:105031, Nov 2011. doi: 10.1103/PhysRevD.84.105031. URL <http://link.aps.org/doi/10.1103/PhysRevD.84.105031>.

D ( $\mu\text{m}$ )	F (aN)	D ( $\mu\text{m}$ )	F (aN)
3	7.37e+01	16	2.34e-01
4	2.90e+01	17	1.85e-01
5	1.38e+01	18	1.47e-01
6	7.61e+00	19	1.19e-01
7	4.61e+00	20	9.71e-02
8	2.96e+00	21	7.99e-02
9	1.97e+00	22	6.63e-02
10	1.35e+00	23	5.54e-02
11	9.53e-01	24	4.66e-02
12	6.92e-01	25	3.95e-02
13	5.14e-01	26	3.36e-02
14	3.89e-01	27	2.87e-02
15	3.00e-01	28	2.47e-02
		29	2.13e-02

Table 1: Force as function of distance, silica bead and gold cantilever, in microns and attonewtons respectively. This points interpolated between slightly sparser points (5,10,15,20,25,30 are exact calculations) and have relative uncertainty on the 10% level to high confidence.

A. Lambrecht and S. Reynaud. Casimir force between metallic mirrors. *European Physical Journal D*, 8:309–318, July 2000. doi: 10.1007/s100530050041.

A. Lambrecht, P. A. M. Neto, and S. Reynaud. The Casimir effect within scattering theory. *New Journal of Physics*, 8:243, October 2006. doi: 10.1088/1367-2630/8/10/243.

S. K. Lamoreaux. Demonstration of the casimir force in the  $0.6$  to  $6\mu$  m range. *Phys. Rev. Lett.*, 78:5–8, Jan 1997. doi: 10.1103/PhysRevLett.78.5. URL <http://link.aps.org/doi/10.1103/PhysRevLett.78.5>.

Sahand Jamal Rahi, Thorsten Emig, Noah Graham, Robert L. Jaffe, and Mehran Kardar. Scattering theory approach to electrodynamic casimir forces. *Physical Review D*, 80(8), 2009. doi: 10.1103/PhysRevD.80.085021.

Alejandro W. Rodriguez, Federico Capasso, and Steven G. Johnson. The casimir effect in microstructured geometries. *Nat Photon*, 5(4):211–221, 04 2011. URL <http://dx.doi.org/10.1038/nphoton.2011.39>.

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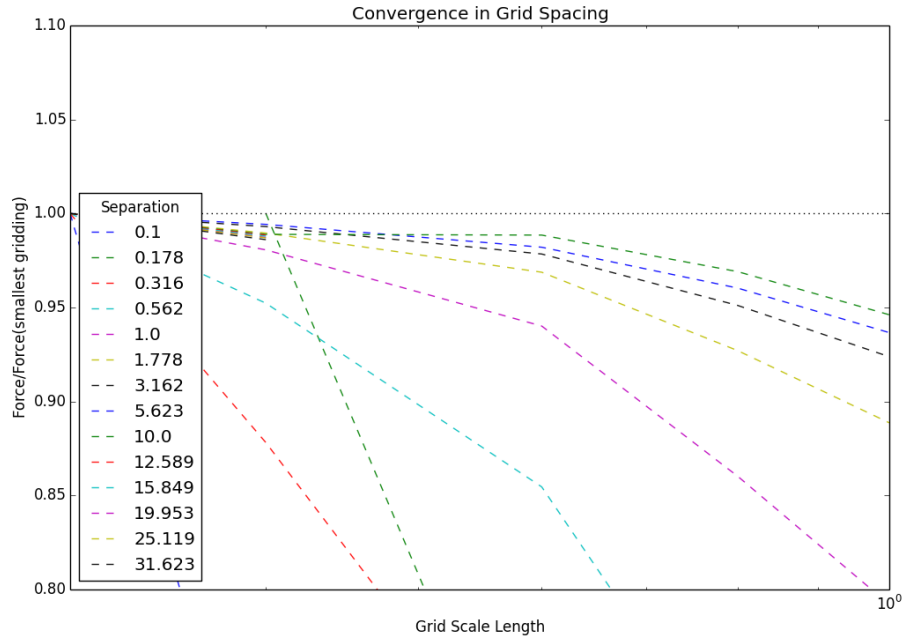
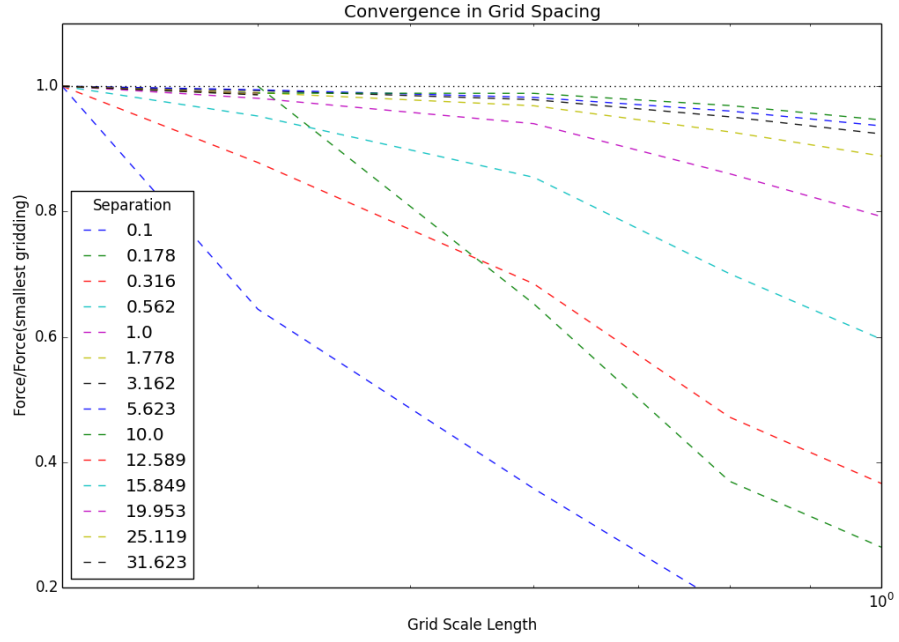


Figure 7: Earlier Casimir force estimates based on different geometries.

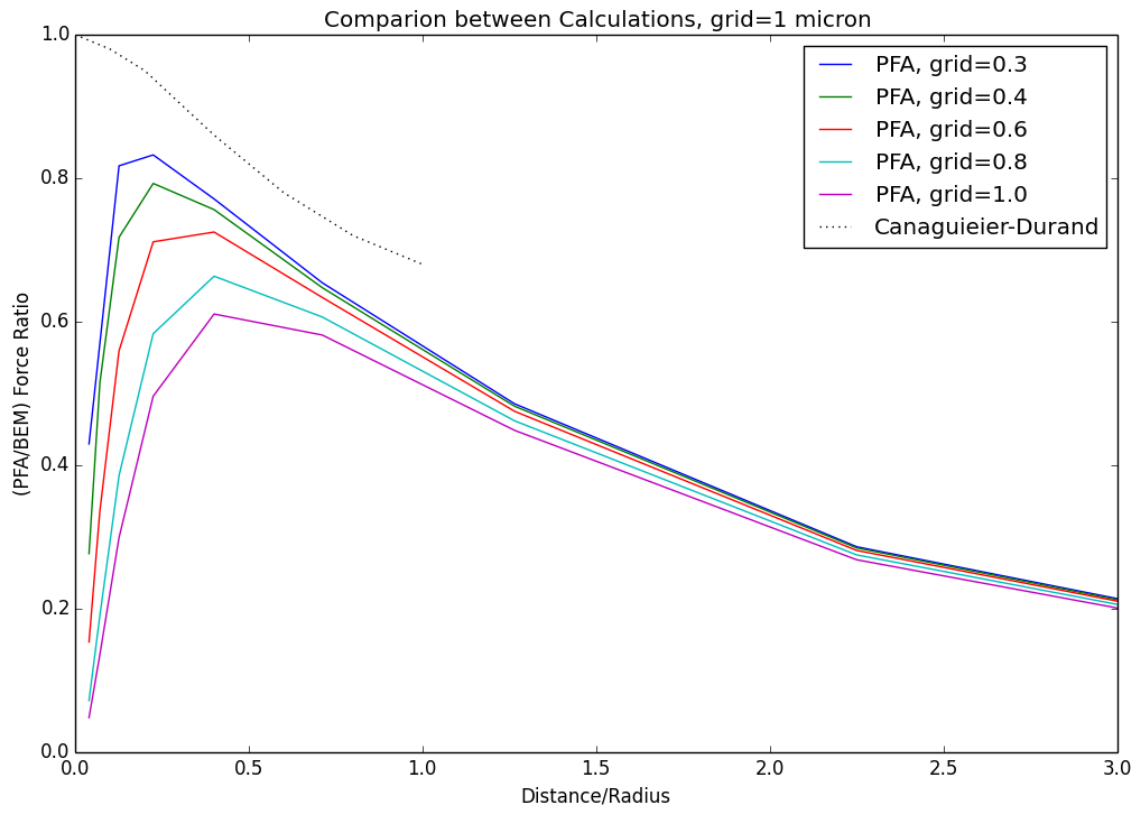


Figure 8: Earlier Casimir force estimates based on different geometries.

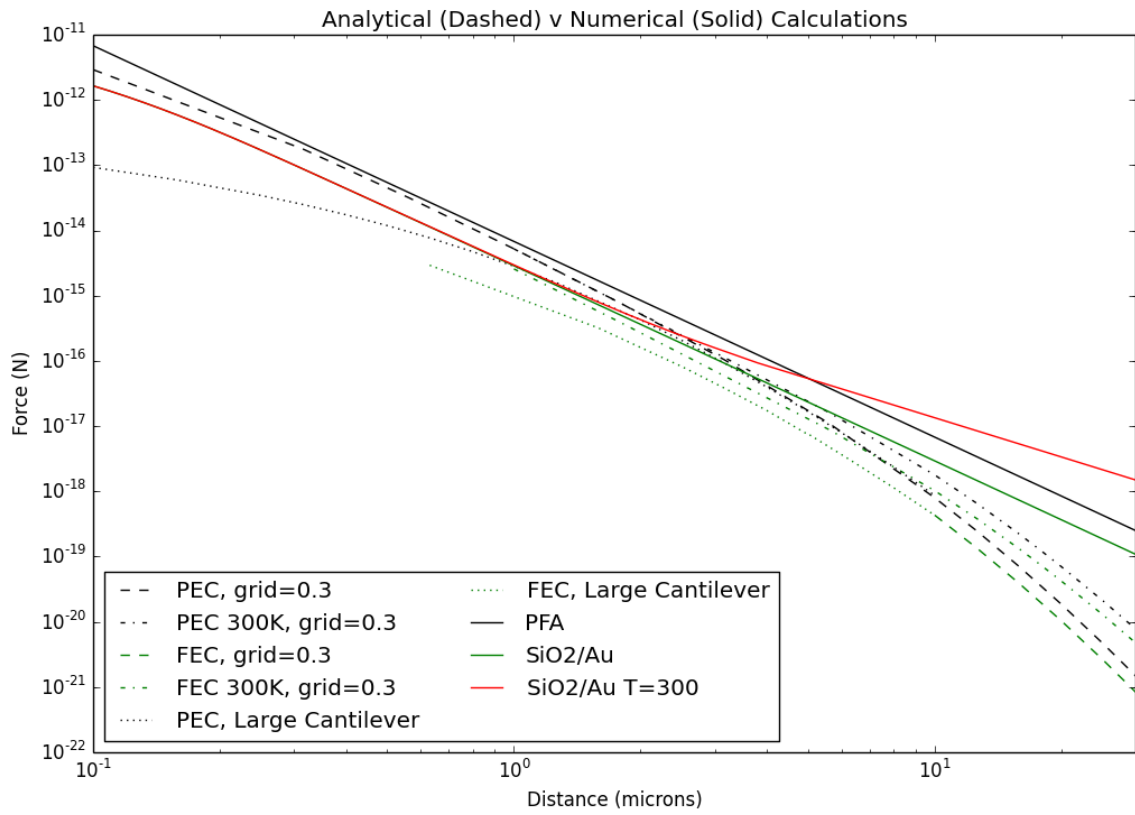


Figure 9: Earlier Casimir force estimates based on different geometries.

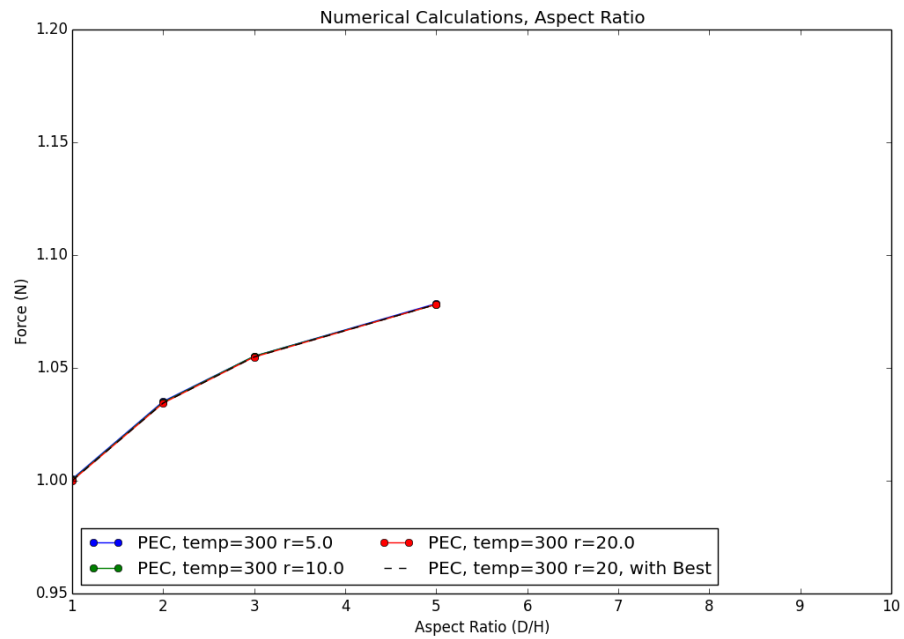
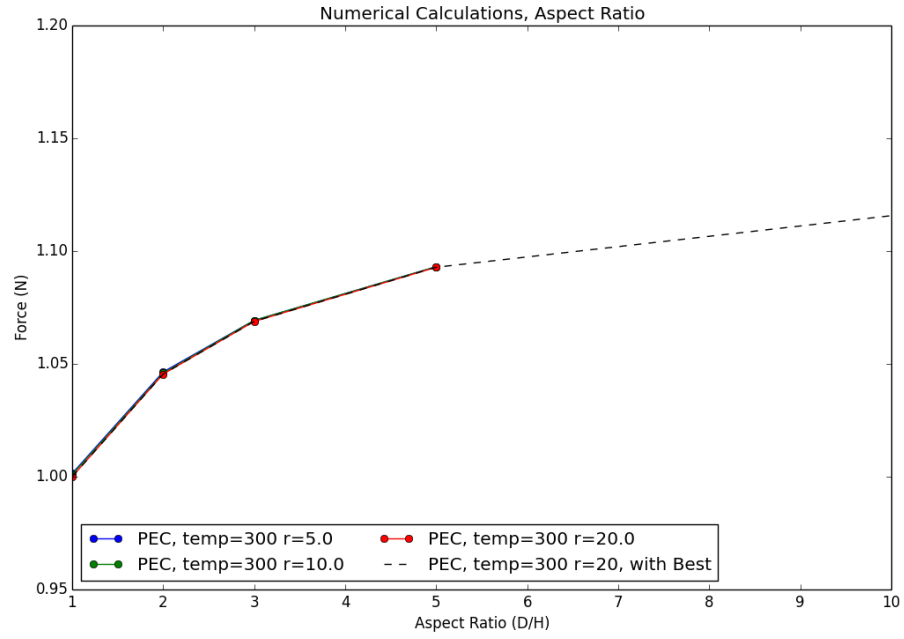


Figure 10: Depth versus relative grid size on the rear of the cantilever, showing relatively little dependence on relative grid size but large dependence on overall depth (fine grinding not as important as there being cells).

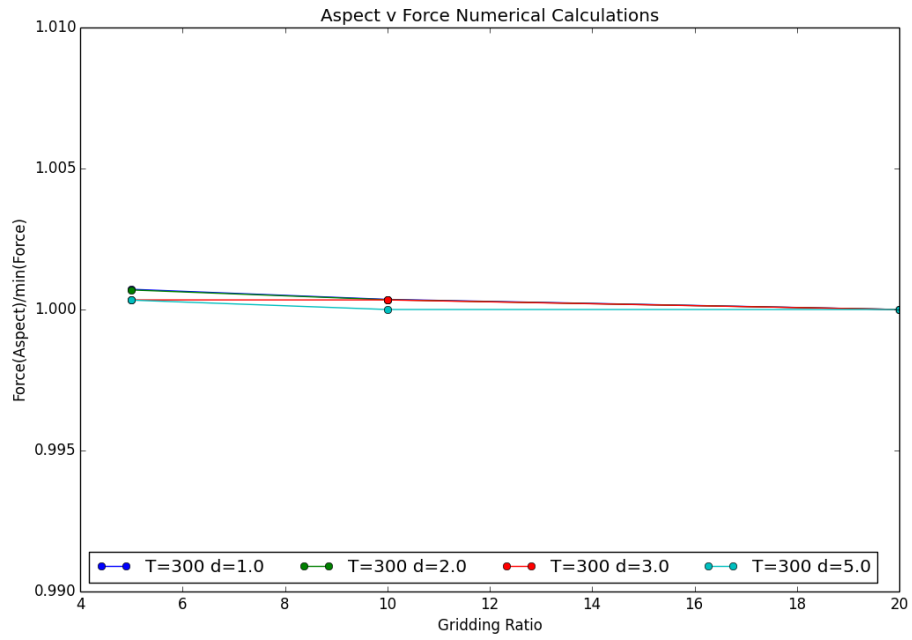
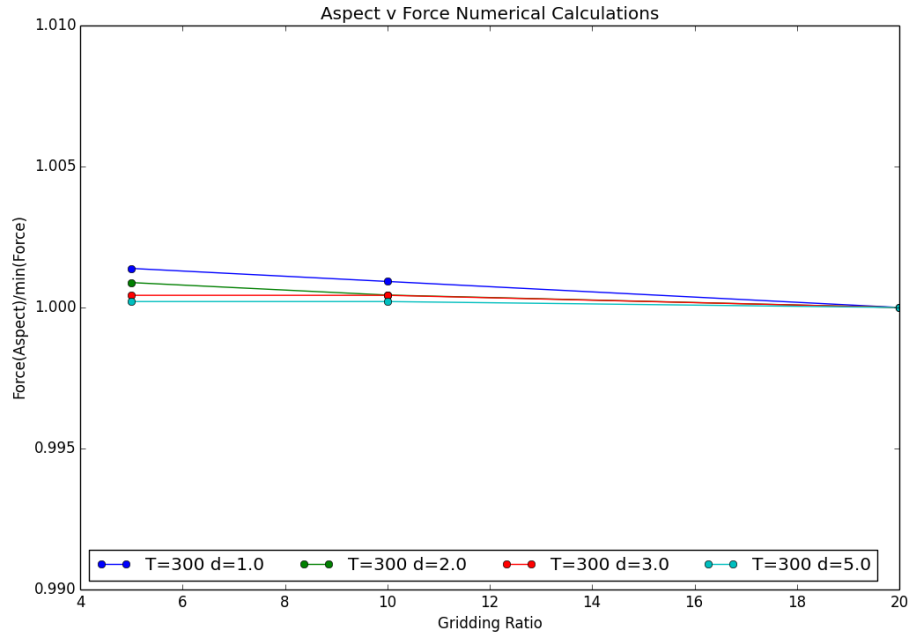


Figure 11: Depth versus relative grid size on the rear of the cantilever, showing relatively little dependence on relative grid size but large dependence on overall depth (fine grinding not as important as there being cells).

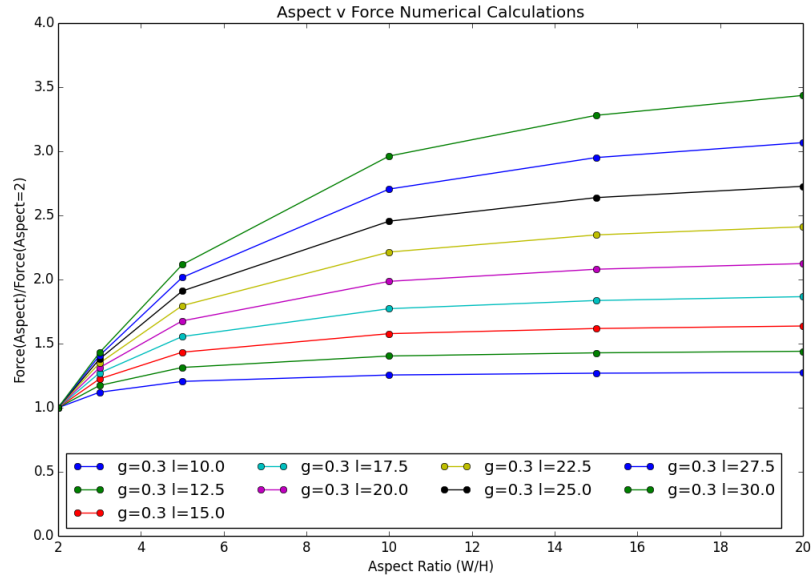
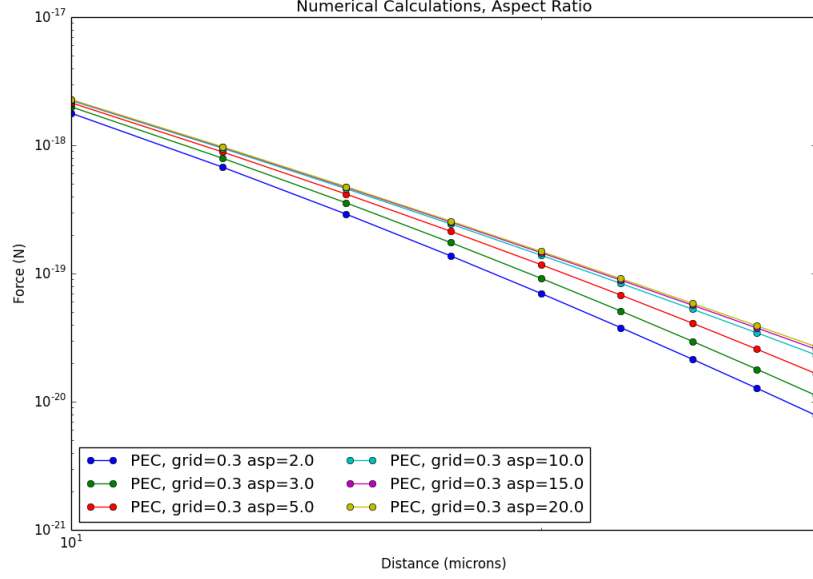


Figure 12: Width (in terms of aspect ratio) versus distance from the cantilever, showing increased variation of value from that at aspect ratio of 2 as separation increases (which makes sense geometrically). The cantilever we utilize corresponds to an aspect ratio of 10.



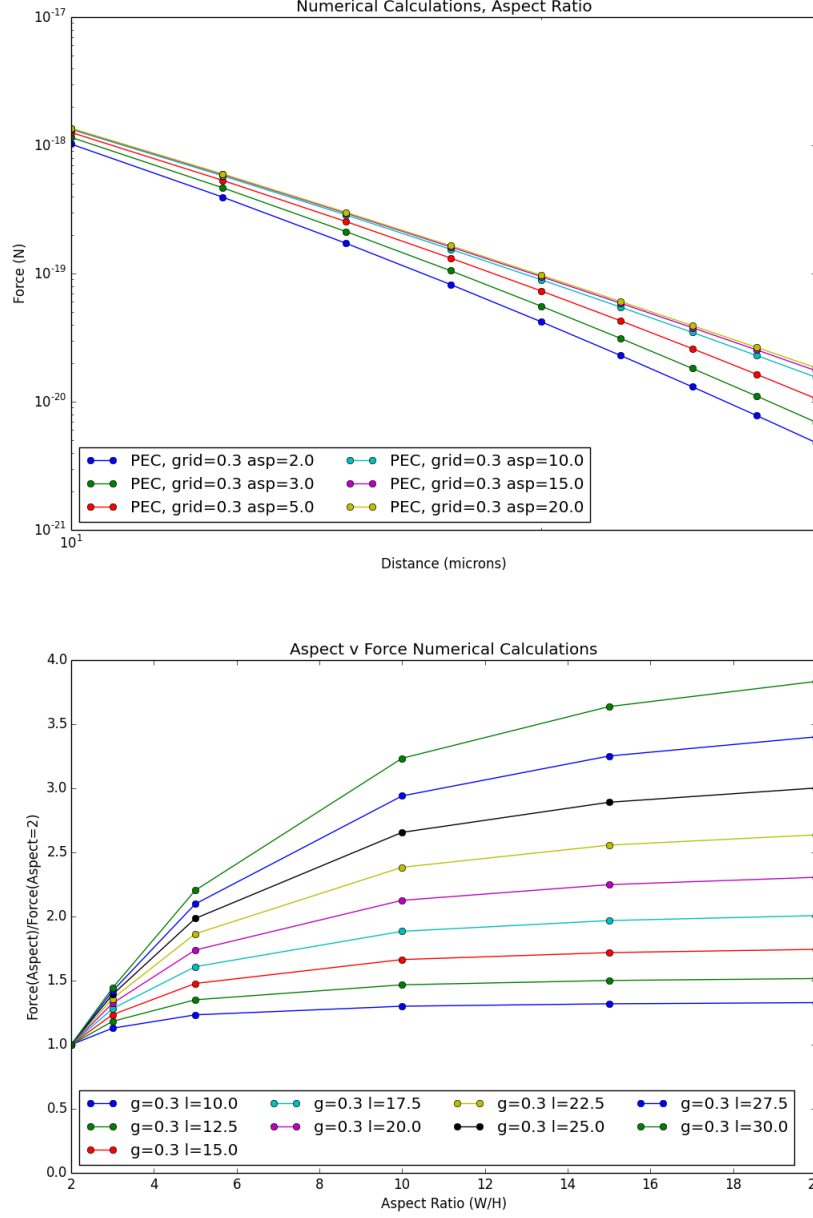


Figure 13: Width (in terms of aspect ratio) versus distance from the cantilever for finite conductivity, showing increased variation of value from that at aspect ratio of 2 as separation increases (which makes sense geometrically). The cantilever we utilize corresponds to an aspect ratio of 10.

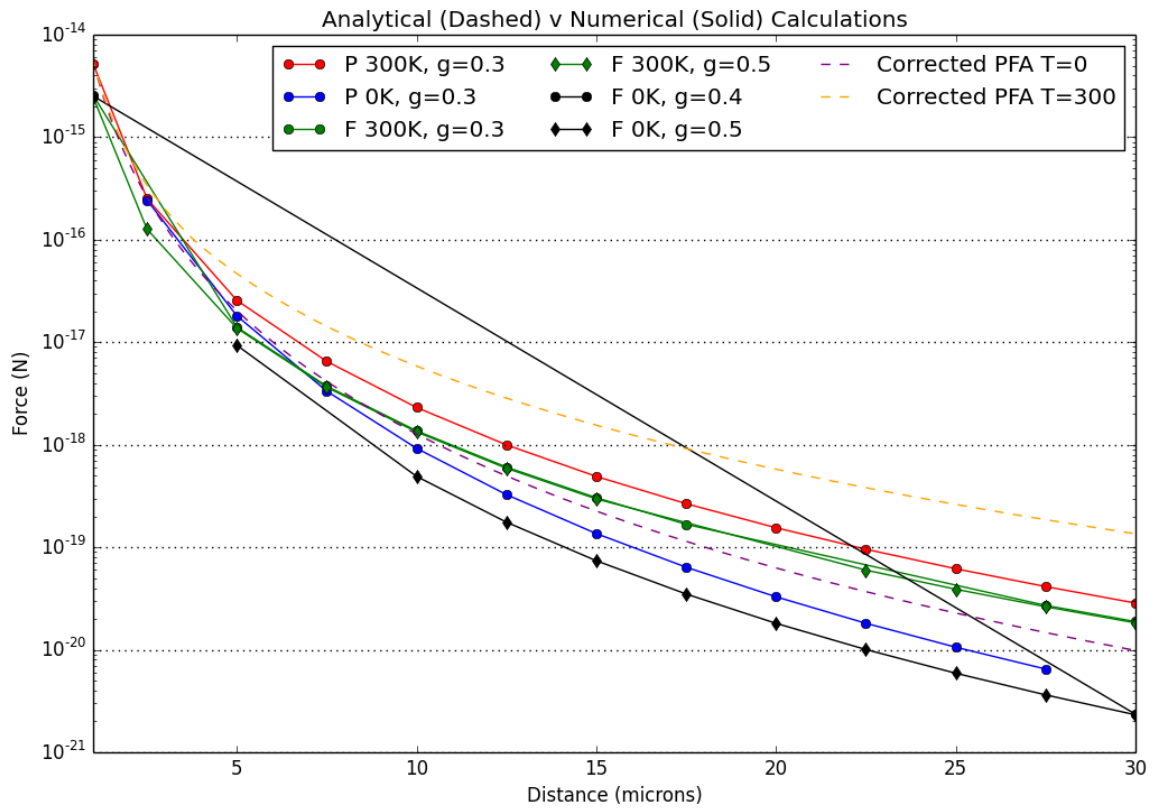


Figure 14: Casimir force estimate based on extended geometry.

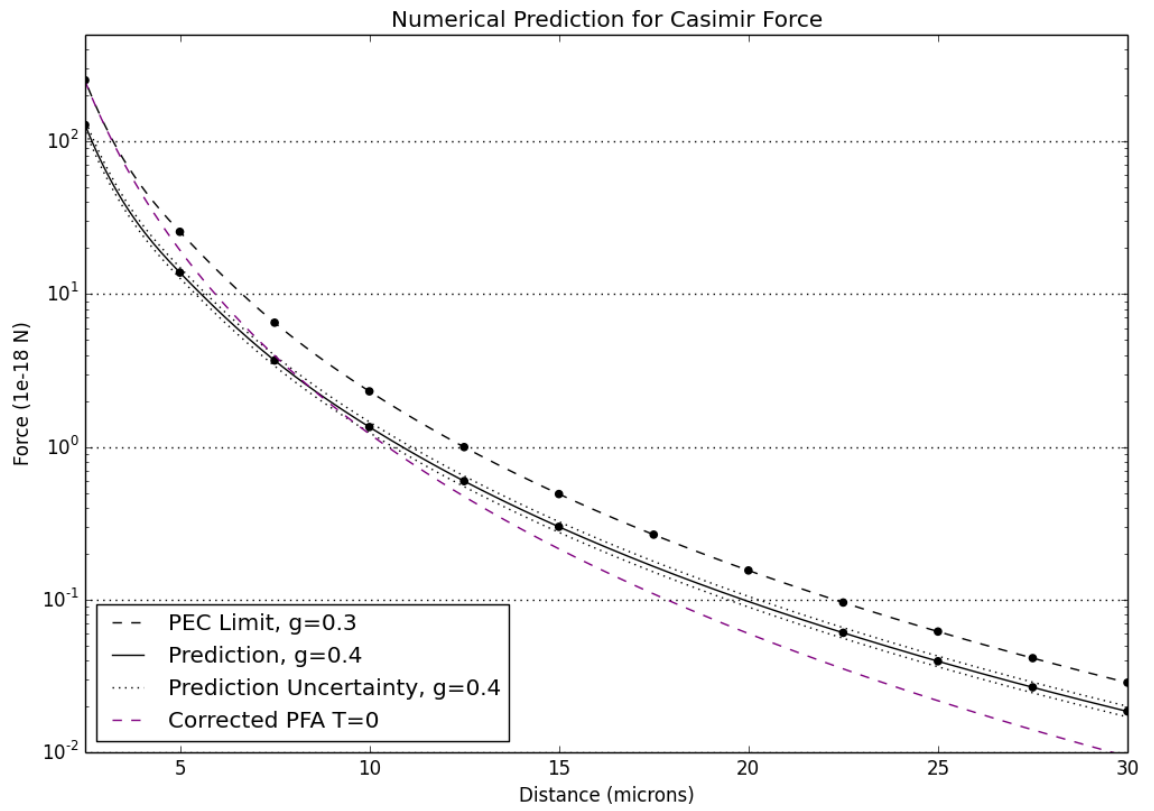


Figure 15: Best prediction for Casimir force at 300K with finite conductivity correction.

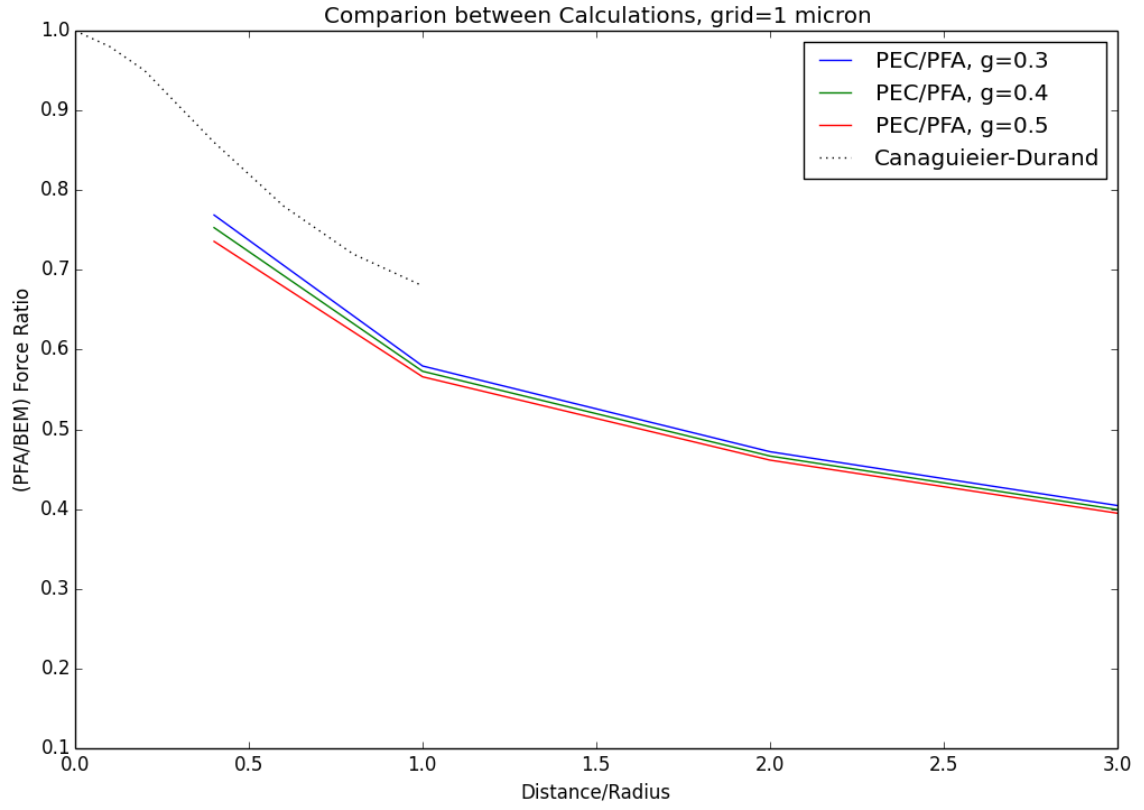


Figure 16: Casimir force estimates compared to PFA, based on large geometry. Compared to correction for infinite plane.

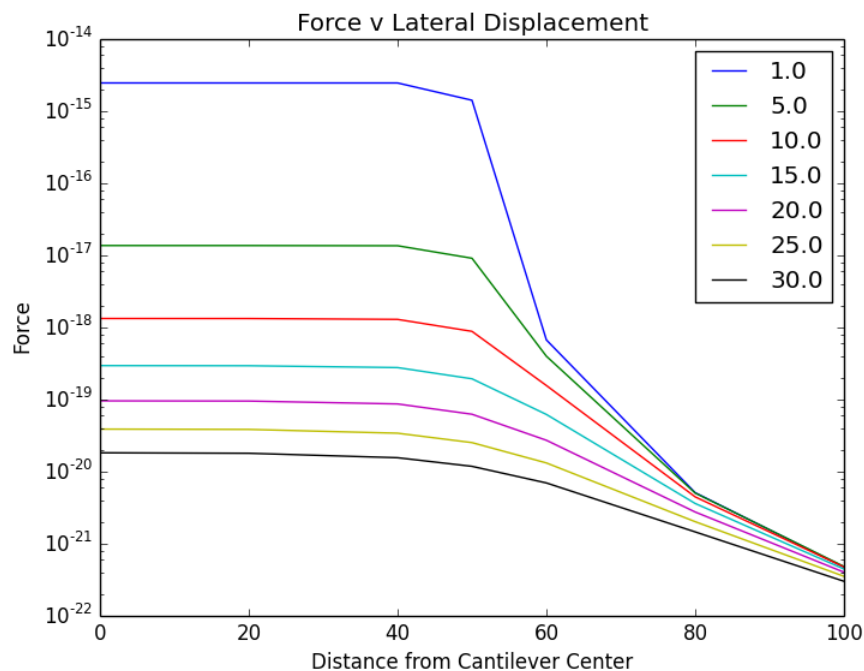
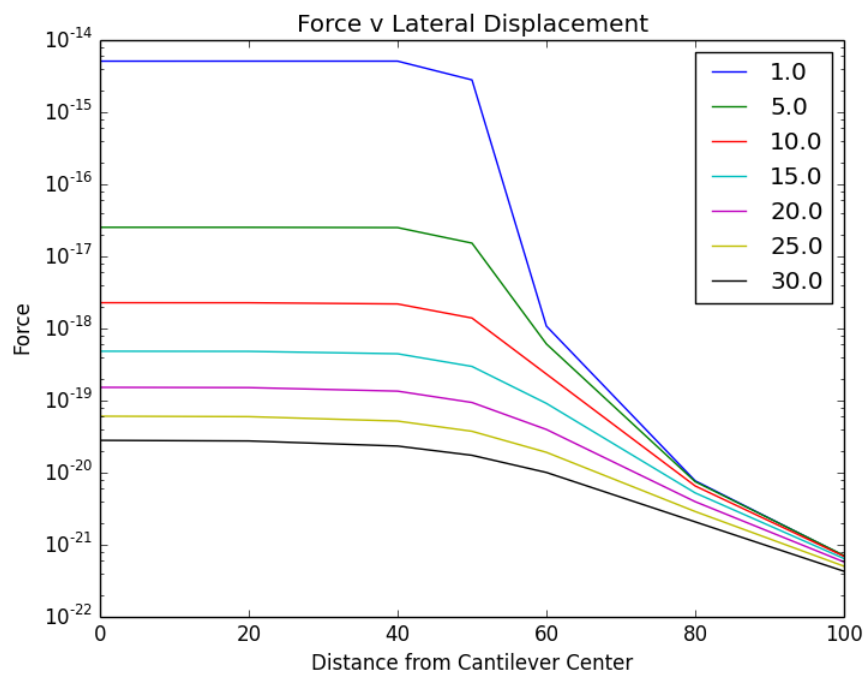


Figure 17: Force as a function of lateral displacement along the cantilever.

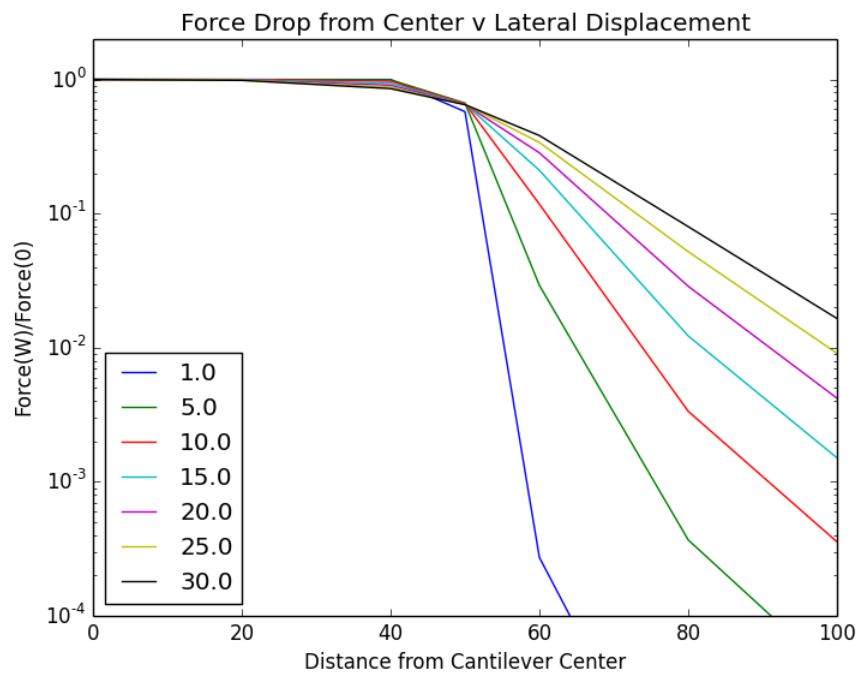
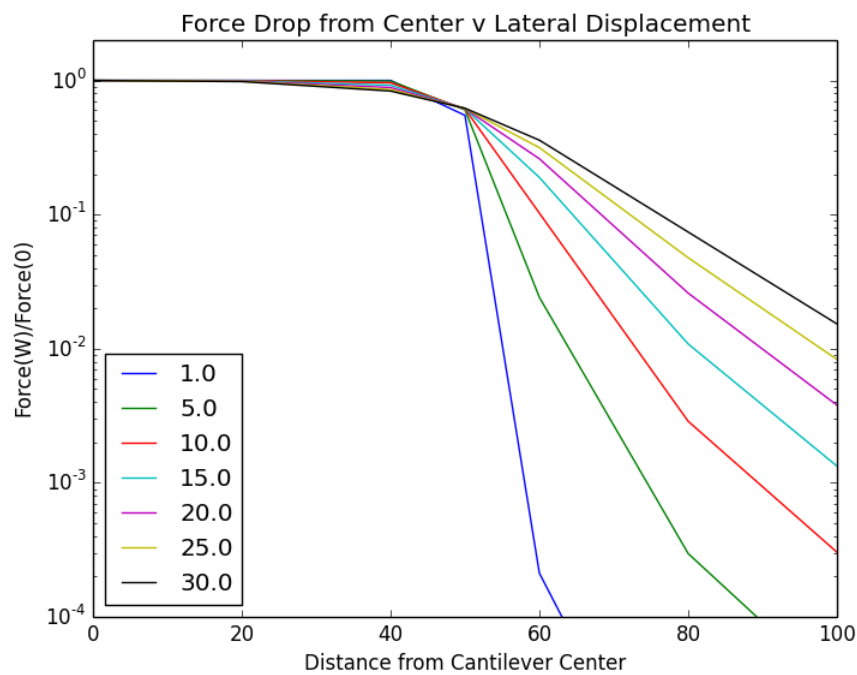


Figure 18: Force ratio as a function of lateral displacement along the cantilever.

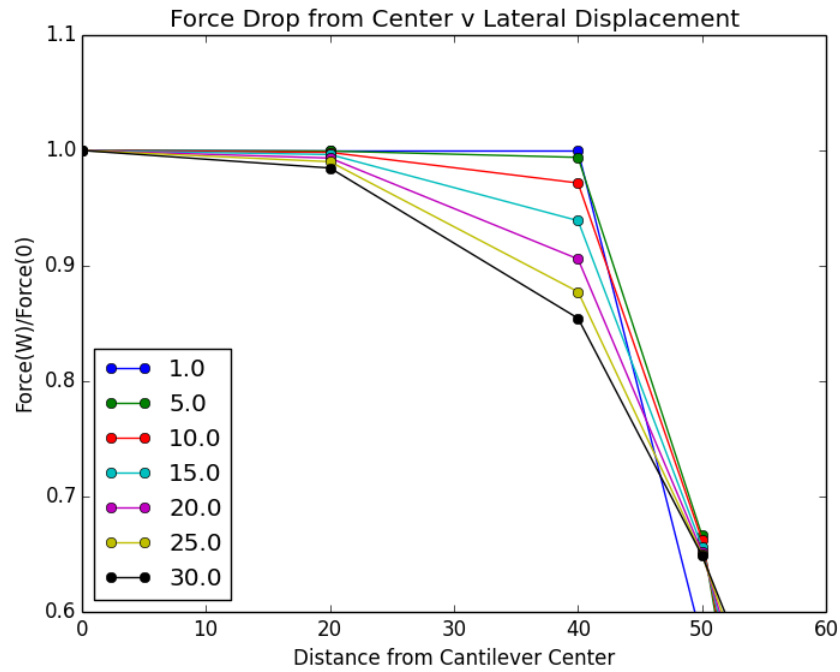
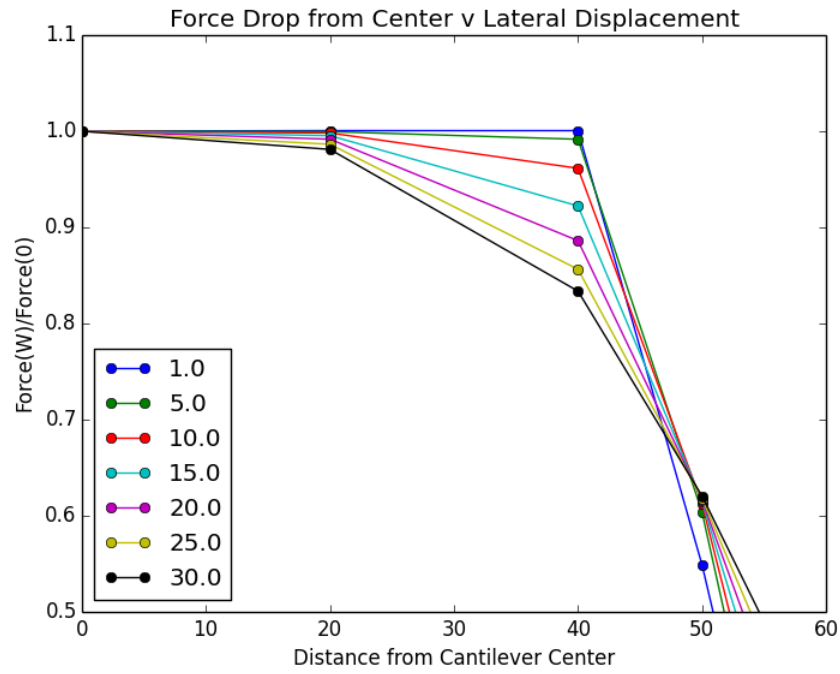


Figure 19: Force ratio as a function of lateral displacement along the cantilever.