CA MAS-I Chapter 1

1.3.0 Overview

5m

It's important for actuaries to understand the likelihood of extreme losses, as well as the average amounts of those losses. Without a good understanding of extreme losses, an insurance company would likely struggle to stay solvent if large insurance payouts were required, like with the occurrence of a natural disaster.

For claim severity distributions, extreme losses occur in the right tail. If a distribution has a high probability of extreme losses, it is said to have a ***heavy***, ***thick***, or ***fat tail***. In this section, we will cover 2 measures that are used to examine extreme losses. Then, we will look at 4 ways to compare the tails of distributions.

1.3.1 Conditional Tail Expectation

30m

The q *quantile* of a random variable is just another name for the $100q^{\text{th}}$ percentile, which we denote as π_q.

For $0 < q < 1$, if X is continuous, then

$$\pi_q = F_X^{-1}(q) \tag{1.3.1.1}$$

Note that $F_X^{-1}(\cdot)$ can be referred to as the quantile function.

The exam table provides the CDF formulas for most distributions. So, to find π_q, simply invert the CDF.

Let's demonstrate:

The annual loss amount, X, follows a Weibull distribution with parameters $\theta = 5{,}000$ and $\tau = 3$.

Calculate its 95th percentile, or $\pi_{0.95}$.

Refer to the exam table for the Weibull CDF, and set it equal to 0.95 to determine $\pi_{0.95}$.

$$\begin{aligned} F_X(\pi_{0.95}) &= 1 - e^{-(\pi_{0.95}\,/\,5{,}000)^3} \\ &= 0.95 \end{aligned}$$

$$e^{-(\pi_{0.95}\,/\,5{,}000)^3} = 0.05$$

$$\left(\frac{\pi_{0.95}}{5{,}000}\right)^3 = -\ln 0.05$$

$$\begin{aligned}\pi_{0.95} &= 5{,}000(-\ln 0.05)^{1/3} \\ &= \mathbf{7{,}207.83}\end{aligned}$$

Actuaries use percentiles to get a better understanding of the likelihood of extreme losses. For instance, using the example above, we calculated the 95$^{\text{th}}$ percentile of the annual loss to be 7,207.83. This means the chance of observing a loss greater than 7,207.83 is 5%.

Example 1.3.1.1

Insurance company XYZ issues a new health insurance policy. The policyholders are classified into two categories: smoker and non-smoker. 70% of the policyholders are non-smokers.

The company models the annual claim size of each policyholder using exponential distributions.

Category	Mean
Smoker	300
Non-smoker	150

Determine

1. the 0.95 quantile for a non-smoker.
2. the 0.90 quantile for a randomly selected policyholder.

Solution to (1)

Let X_S and X_{NS} be the annual claim sizes for a smoker and non-smoker, respectively.

$$X_S \sim \text{Exponential}\,(300)$$

$$X_{NS} \sim \text{Exponential}\,(150)$$

We want to determine $\pi_{0.95}$ for X_{NS}. For this, invert the CDF formula on the exam table.

$$\begin{aligned} F_{X_{NS}}(\pi_{0.95}) &= 1 - e^{-\pi_{0.95}/150} \\ &= 0.95 \\ \\ e^{-\pi_{0.95}/150} &= 0.05 \\ \\ \pi_{0.95} &= -150 \ln 0.05 \\ &= \mathbf{449.36} \end{aligned}$$

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Solution to (2)

A "randomly selected policyholder" has a mixed distribution because this person has a 30% chance to be a smoker and a 70% chance to be a non-smoker. Therefore, the claim size, X, of this policyholder has a CDF of

$$F_X(x) = 0.3F_{X_S}(x) + 0.7F_{X_{NS}}(x)$$

Recall the 0.90 quantile is the value $\pi_{0.90}$ such that

$$F_X(\pi_{0.90}) = 0.9$$

$$\begin{aligned} 0.9 &= 0.3\left(1 - e^{-\pi_{0.90}\,/\,300}\right) + 0.7\left(1 - e^{-\pi_{0.90}\,/\,150}\right) \\ &= 1 - 0.3e^{-\pi_{0.90}\,/\,300} - 0.7e^{-\pi_{0.90}\,/\,150} \end{aligned}$$

$$\begin{aligned} 0 &= 0.3e^{-\pi_{0.90}\,/\,300} + 0.7e^{-\pi_{0.90}\,/\,150} - 0.1 \\ &= 0.7\left(e^{-\pi_{0.90}\,/\,300}\right)^2 + 0.3e^{-\pi_{0.90}\,/\,300} - 0.1 \end{aligned}$$

Let $u = e^{-\pi_{0.90}\,/\,300}$.

$$0 = 0.7u^2 + 0.3u - 0.1$$

$$\begin{aligned} u &= \frac{-(0.3) \pm \sqrt{0.3^2 - 4(0.7)(-0.1)}}{2(0.7)} \\ &= 0.220 \quad \text{or} \quad -0.649 \end{aligned}$$

Since $e^{-\pi_{0.90}\,/\,300}$ cannot be negative, we conclude that $e^{-\pi_{0.90}\,/\,300} = 0.220$.

$$e^{-\pi_{0.90}\,/\,300} = 0.220$$

$$\begin{aligned} \pi_{0.90} &= -300 \ln 0.220 \\ &= \mathbf{453.969} \end{aligned}$$

Coach's Remarks

When calculating a quantile of a discrete mixture, it is **incorrect** to compute the weighted average of the quantiles of the individual distributions. Specifically, the 0.90 quantile of X is not found by weighting the 0.90 quantiles of X_{NS} and X_S. Instead, use first principles, i.e. solve for the value of the mixture that produces the desired probability, as shown above.

For a continuous random variable X, the *conditional tail expectation (CTE)* with tolerance probability $1-q$ is X's conditional expected value given that X exceeds the q quantile, π_q.

$$\begin{aligned} \mathrm{CTE}_q(X) &= \mathrm{E}[X \mid X > \pi_q] \\ &= \frac{\displaystyle\int_{\pi_q}^{\infty} x \cdot f(x)\,\mathrm{d}x}{\Pr(X > \pi_q)} \\ &= \frac{\displaystyle\int_{\pi_q}^{\infty} x \cdot f(x)\,\mathrm{d}x}{1-q} \end{aligned} \tag{1.3.1.2}$$

From the equation above, notice the numerator is expressed as an integral. To avoid integration, express the CTE as shown:

$$\begin{aligned}\mathrm{CTE}_q(X) &= \pi_q + \mathrm{E}[X - \pi_q \mid X > \pi_q] \\ &= \pi_q + e(\pi_q) \\ &= \pi_q + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \pi_q]}{1-q}\end{aligned} \tag{1.3.1.3}$$

Equation 1.3.1.3 will often come in handy because the limited moments formula for most distributions is included in the exam table.

Similar to quantiles, the CTE helps actuaries to study extreme losses. Specifically, the CTE gives actuaries information on the expected size of losses that are above the q quantile. For example, if the CTE with tolerance probability 0.05 is 5,000, then for the top 5% of losses, the average loss amount is 5,000.

Let's work on a few examples.

Example 1.3.1.2

Suppose X is a random variable that has the following density function:

$$f(x) = \frac{2}{3}x, \qquad 1 < x < 2$$

Determine

1. the 0.75 quantile of X.
2. the CTE with tolerance probability 0.25 of X.

Solution to (1)

Let $\pi_{0.75}$ be the 0.75 quantile of X.

$$\begin{aligned}0.75 &= \int_1^{\pi_{0.75}} \frac{2}{3} x \, \mathrm{d}x \\ &= \left[\frac{1}{3} x^2\right]_1^{\pi_{0.75}} \\ &= \frac{\pi_{0.75}^2 - 1}{3}\end{aligned}$$

$$\begin{aligned}\pi_{0.75} &= \sqrt{1 + 3(0.75)} \\ &= \mathbf{1.8028}\end{aligned}$$

$\blacksquare$

Solution to (2)

To determine $\mathrm{CTE}_{0.75}(X)$, use Equation 1.3.1.2.

$$\begin{aligned}\mathrm{CTE}_{0.75}(X) &= \mathrm{E}[X \mid X > \pi_{0.75}] \\ &= \mathrm{E}[X \mid X > 1.8028] \\ &= \frac{\int_{1.8028}^{2} x \cdot (2x \,/\, 3) \, \mathrm{d}x}{1 - 0.75} \\ &= \frac{\int_{1.8028}^{2} 2x^2 \,/\, 3 \, \mathrm{d}x}{0.25}\end{aligned}$$

$$\left[2x^3/9\right]^2$$

$$= \frac{\left[2x^3 / 9\right]_{1.8028}}{0.25}$$
$$= \mathbf{1.9031}$$

Coach's Remarks

The result above makes intuitive sense. Since X has an upper bound of 2 and is conditioned to be greater than 1.8028, its CTE with tolerance probability 25% should be a value between 1.8028 and 2. If the result falls outside of this range, check the calculations, including the limits of integration. This will help you catch mistakes and eliminate bad answer choices.

Example 1.3.1.3

Loss amount X follows a continuous distribution that has the following evaluated CDF values and limited expected values:

x	$F(x)$	$\mathrm{E}[X \wedge x]$
500	0.44	375
600	0.49	429
1,800	0.79	818
2,415	0.85	925

3,000	0.89	1,000
4,350	0.93	1,115
∞	1.00	1,500

Calculate the CTE with tolerance probability 0.15 of X.

Solution

The question does not provide the full distribution function of X, so Equation 1.3.1.2 cannot be used since it requires the PDF of X. Apply Equation 1.3.1.3 instead.

$$\begin{aligned}\text{CTE}_{0.85}(X) &= \pi_{0.85} + e[\pi_{0.85}] \\ &= \pi_{0.85} + \frac{\text{E}[X] - \text{E}[X \wedge \pi_{0.85}]}{1 - 0.85}\end{aligned}$$

Recall that $\pi_{0.85}$ is the value of X that produces $F(x) = 0.85$.

$$\begin{aligned}\pi_{0.85} &= F^{-1}(0.85) \\ &= 2{,}415\end{aligned}$$

Now, compute

$$\text{CTE}_{0.85}(X) = 2{,}415 + \frac{\text{E}[X] - \text{E}[X \wedge 2{,}415]}{1 - 0.85}$$

- $\text{E}[X] = \text{E}[X \wedge \infty] = 1{,}500$
- $\text{E}[X \wedge 2{,}415] = 925$

Therefore,

$$\begin{aligned}\mathrm{CTE}_{0.85}(X) &= 2{,}415 + \frac{1{,}500 - 925}{1 - 0.85} \\ &= \mathbf{6{,}248.33}\end{aligned}$$

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Example 1.3.1.4

Insurance company XYZ issues a new health insurance policy. The policyholders are classified into two categories: smoker and non-smoker. 70% of the policyholders are non-smokers.

The company models the annual claim size of each policyholder using exponential distributions.

Category	Mean
Smoker	300
Non-smoker	150

It is known that the 0.90 quantile of a randomly selected policyholder is 453.97.

Calculate the CTE with tolerance probability 0.10 for this policyholder.

Solution

Apply Equation 1.3.1.3 and substitute $\pi_{0.90} = 453.97$.

$$\begin{aligned}\text{CTE}_{0.9}(X) &= 453.97 + e[453.97] \\ &= 453.97 + \frac{\text{E}[X] - \text{E}[X \wedge 453.97]}{1 - 0.9}\end{aligned}$$

Calculate each component. To determine the expected value of a mixture, apply the Law of Total Expectation.

$$\begin{aligned}\text{E}[X] &= 0.3\text{E}[X_S] + 0.7\text{E}[X_{NS}] \\ &= 0.3(300) + 0.7(150) \\ &= 195\end{aligned}$$

$$\begin{aligned}\text{E}[X \wedge 453.97] &= 0.3\text{E}[X_S \wedge 453.97] + 0.7\text{E}[X_{NS} \wedge 453.97] \\ &= 0.3\left(300\left[1 - e^{-453.97\,/\,300}\right]\right) + 0.7\left(150\left[1 - e^{-453.97\,/\,150}\right]\right) \\ &= 170.091\end{aligned}$$

Putting the pieces together,

$$\begin{aligned}\text{CTE}_{0.9}(X) &= 453.97 + \frac{195 - 170.091}{1 - 0.9} \\ &= \mathbf{703.058}\end{aligned}$$

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Coach's Remarks

Similar to quantiles, it is **incorrect** to weight the CTEs of the individual distributions to determine the CTE of the mixture.

$$\mathrm{CTE}_{0.9}(X) \neq 0.3\mathrm{CTE}_{0.9}(X_S) + 0.7\mathrm{CTE}_{0.9}(X_{NS})$$

Equation 1.3.1.3 can be used to derive a shortcut for the CTE for a lognormal random variable.

If X is lognormal with parameters μ and σ^2, then

$$\mathrm{CTE}_q(X) = \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_q)}{1-q}\right] \qquad (1.3.1.4)$$

The derivation is included in the appendix at the end of this section.

Coach's Remarks

The advantage of using Equation 1.3.1.4 is that this formula does not require the calculation of π_q. However, Equation 1.3.1.3 can still be used instead, so learning Equation 1.3.1.4 is optional.

A shortcut formula for the CTE of a normal random variable can also be derived.

If X is normal with parameters μ and σ^2, then

$$\mathrm{CTE}_q(X) = \mu + \sigma\left[\frac{\phi(z_q)}{\quad}\right] \qquad (1.3.1.5)$$

where $\phi(x)$ is the PDF of a standard normal distribution.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Do not confuse $\phi(x)$ with $\Phi(x)$, which is the CDF of a standard normal.

The derivation is included in the appendix at the end of this section.

Example 1.3.1.5

X follows a distribution with the density function:

$$f(x) = \frac{1}{x \cdot 3\sqrt{2\pi}} e^{-(\ln x - 2)^2/[2(9)]}, \qquad x > 0$$

Calculate $\mathrm{CTE}_{0.85}(X)$.

Solution

Based on the PDF, X follows a lognormal distribution with parameters $\mu = 2$ and $\sigma^2 = 9$. Applying Equation 1.3.1.4,

$$\mathrm{CTE}_{0.85}(X) = \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_{0.85})}{1 - 0.85}\right]$$

- $\mathrm{E}[X] = e^{2+(9/2)} = 665.142$

- $z_{0.85} = \Phi^{-1}(0.85) = 1.036$

Calculate the CTE:

$$\begin{aligned}\text{CTE}_{0.85}(X) &= 665.142 \cdot \left[\frac{\Phi(3 - 1.036)}{1 - 0.85}\right] \\ &= 665.142\left(\frac{0.975}{0.15}\right) \\ &= \mathbf{4{,}323.421}\end{aligned}$$

Alternative Solution

As previously mentioned, this question can be solved using Equation 1.3.1.3.

$$\text{CTE}_{0.85}(X) = \pi_{0.85} + \frac{\text{E}[X] - \text{E}[X \wedge \pi_{0.85}]}{1 - 0.85}$$

First, solve for $\pi_{0.85}$.

$$0.85 = \Phi\left(\frac{\ln\left[\pi_{0.85}\right] - 2}{\sqrt{9}}\right)$$

$$\begin{aligned}\ln\left[\pi_{0.85}\right] &= 2 + \Phi^{-1}(0.85) \cdot \sqrt{9} \\ &= 5.108\end{aligned}$$

$$\begin{aligned}\pi_{0.85} &= e^{5.108} \\ &= 165.339\end{aligned}$$

Thus, the goal is

$$\text{CTE}_{0.85}(X) = 165.339 + \frac{\text{E}[X] - \text{E}[X \wedge 165.339]}{1 - 0.85}$$

Compute each component using the exam table formulas:

- As calculated in the first solution, $\text{E}[X] = 665.142$.
- To calculate $\text{E}[X \wedge 165.339]$,

$$\begin{aligned}\text{E}[X \wedge 165.339] &= e^{2+(9/2)} \cdot \Phi\left[\frac{\ln(165.339) - 2 - 9}{\sqrt{9}}\right] + 165.339\,[1 - F(165.339)] \\ &= e^{6.5} \cdot \Phi(-1.964) + 165.339\,(0.15) \\ &= 665.142\,(1 - 0.975) + 24.80 \\ &= 41.429\end{aligned}$$

Therefore,

$$\begin{aligned}\text{CTE}_{0.85}(X) &= 165.339 + \frac{665.142 - 41.429}{1 - 0.85} \\ &= \mathbf{4{,}323.421}\end{aligned}$$

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1.3.2 Tail Weight

15m

The *tail weight* of loss distributions describes the likelihood of extreme losses occurring. In this lesson, "tail" refers to the **right tail** of a distribution. Therefore, a heavier tail weight implies that large claims are more likely to occur, compared to a distribution with a lighter tail weight.

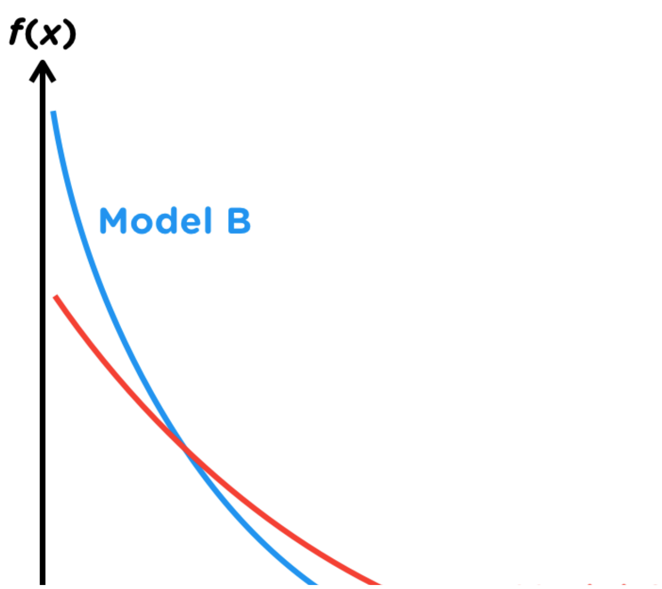
Consider the two models below:

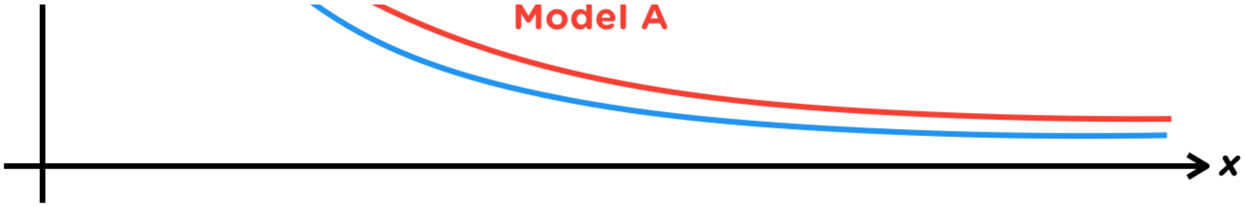
$$\text{Model A: Pareto } (3,\, 60)$$

$$\text{Model B: Exponential } (30)$$

Determine which model has a heavier tail.

Evaluate the graph below:





According to the diagram, Model A (Pareto) is more likely to produce larger values than Model B (exponential) because it has a greater density function in its tail. Therefore, **Model A** has a heavier tail than Model B.

However, constructing every PDF by hand during the exam is impractical. Fortunately, there are four tests that can be used to compare the tails of distributions.

1. **Number of positive raw moments**
   For greater tail weights, it becomes less likely that the integral of $x^k \cdot f(x)$ over all possible values will converge. If that integral does not converge for higher values of k, then these higher moment don't exist. So, the fewer positive raw moments that exist, the greater the tail weight.

2. **Ratio of the survival functions (or densities)**
   The faster the survival function approaches zero, the thinner the tail is. So, if the ratio of the survival functions approaches infinity as x approaches infinity, the model in the numerator has a heavier tail. The limit of the ratio of the survival functions can be computed using the l'Hôpital rule. So, this also allows us to use the PDFs in place of the survival functions since $S'(x) = -f(x)$.

3. **Hazard rate function**
   If the hazard rate function decreases as x increases, this indicates that there is a higher probability of extreme losses, which means the distribution has a heavy tail.

4. **CTEs (or quantiles)**
   The larger a given CTE (or quantile) is for a distribution, the larger the extreme losses are. So, the distribution with a larger CTE (or quantile) for a given value of q has a greater tail weight.

As practice, apply the tests on the models above.

Example 1.3.3.1

Example 1.3.2.1

Consider the following models:

- Model A: Pareto distribution with $\alpha = 3$ and $\theta = 60$.
- Model B: Exponential distribution with $\theta = 30$.

Determine the model with the heavier tail using

1. the number of positive raw moments.
2. the ratio of the survival functions/density functions.
3. the hazard rate test.
4. the CTEs.

Solution to (1)

From the exam table,

- the Pareto distribution only has valid positive moments for $-1 < k < 3$ (because $\alpha = 3$).
- the exponential distribution has an infinite number of positive moments.

Model A has a heavier tail because it has fewer positive moments.

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Solution to (2)

Calculate the survival functions from the CDFs provided in the exam table.

- Model A: $S_A(x) = \left(\dfrac{60}{x+60}\right)^3$
- Model B: $S_B(x) = e^{-x\,/\,30}$

Take the ratio and set x to infinity:

$$\lim_{x\to\infty} \frac{S_A(x)}{S_B(x)} = \lim_{x\to\infty} \frac{\left(\dfrac{60}{x+60}\right)^3}{e^{-x\,/\,30}} = \lim_{x\to\infty} \frac{60^3 \cdot e^{\,x\,/\,30}}{(x+60)^3}$$

Because an exponential expression goes to infinity faster than a polynomial, the fraction above diverges to infinity. Therefore, **Model A** has a heavier tail. We will get the same result by evaluating the limit of the ratio of the density functions. Another way to evaluate the limit is to apply l'Hôpital's rule.

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Solution to (3)

Compute the hazard functions of both models.

- Model A

$$f_A(x)$$

$$\begin{aligned} h_A(x) &= \frac{f_A(x)}{S_A(x)} \\ &= \frac{3 \cdot 60^3 \cdot (x+60)^{-4}}{60^3 \cdot (x+60)^{-3}} \\ &= \frac{3}{x+60} \end{aligned}$$

- Model B

$$\begin{aligned} h_B(x) &= \frac{f_B(x)}{S_B(x)} \\ &= \frac{e^{-x/30}/\,30}{e^{-x/30}} \\ &= \frac{1}{30} \end{aligned}$$

Model A has a hazard function that decreases when x increases, while Model B has a hazard function that stays constant. Therefore, **Model A** has a heavier tail.

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Solution to (4)

Recall that CTE_q can be found as $\pi_q + e(\pi_q)$. So, we'll need the inverted CDF formulas, as well as the mean excess loss formulas introduced in Section 1.2 for the Pareto and exponential distributions.

- Model A:

$$\pi_q = F_A^{-1}(q) = \frac{60}{(1-q)^{1/3}} - 60$$

$$(X_A - \pi_q \mid X_A > \pi_q) \sim \text{Pareto}\,(3,\ 60 + \pi_q)$$

$$e_A(\pi_q) = \mathrm{E}[X_A - \pi_q \mid X_A > \pi_q] = \frac{60 + \pi_q}{2}$$

$$\begin{aligned}\text{CTE}_q(A) &= \pi_q + \frac{60 + \pi_q}{2} \\ &= 1.5\pi_q + 30 \\ &= 1.5\left[\frac{60}{(1-q)^{1/3}} - 60\right] + 30\end{aligned}$$

- Model B:

$$\pi_q = F_B^{-1}(q) = -30\ln(1-q)$$

$$(X_B - \pi_q \mid X_B > \pi_q) \sim \text{Exponential}\,(30)$$

$$e_B(\pi_q) = \mathrm{E}[X_B - \pi_q \mid X_B > \pi_q] = 30$$

$$\begin{aligned}\text{CTE}_q(B) &= \pi_q + 30 \\ &= -30\ln(1-q) + 30\end{aligned}$$

We can check the CTE values for each model for several possible values of q:

We can check the CTE values for each model for several possible values of q.

q	$\text{CTE}_q(A)$	$\text{CTE}_q(B)$
0.75	82.8661	71.5888
0.80	93.8978	78.2831
0.85	109.3865	86.9136
0.90	133.8991	99.0776
0.95	184.2976	119.8720

Notice that the CTE for Model A is consistently larger than the CTE for Model B, and it is increasing at a faster rate. Therefore, **Model A** has a heavier tail.

1.3 Summary

5m

q quantile

For a continuous random variable X, the q quantile is:

$$\pi_q = F_X^{-1}(q)$$

Conditional Tail Expectation (CTE)

For a continuous random variable X, its CTE with tolerance probability $1-q$ is:

$$\begin{aligned} \mathrm{CTE}_q(X) &= \mathrm{E}[X \mid X > \pi_q] \\ &= \pi_q + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \pi_q]}{1-q} \end{aligned}$$

The CTE formulas for normal and lognormal distributions are:

	$\mathrm{CTE}_q(X)$
Normal	$\mu + \sigma\left[\frac{\phi(z_q)}{1-q}\right]$
Lognormal	$\mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_q)}{1-q}\right]$

Tail Weight

There are four methods to measure tail weight:

1. The **fewer** positive raw moments that exist, the **greater** the tail weight.
2. If the ratio of the survival functions or the density functions **approaches infinity** as x increases, the numerator has a **heavier** tail.
3. If the hazard rate function **decreases** with x, the distribution has a **heavy** tail.
4. The **larger** a given CTE or quantile is, the **greater** the tail weight.

Appendix

10m

CTE for Lognormal Distribution

Let

$$X \sim \text{Lognormal}\,(\mu,\, \sigma)$$

Recall Equation 1.3.1.3:

$$\text{CTE}_q(X) = \pi_q + \frac{\text{E}[X] - \text{E}[X \wedge \pi_q]}{1-q}$$

Also, we can derive the general form of the lognormal q quantile to be:

$$\begin{aligned} \pi_q &= F_X^{-1}(q) \\ &= \exp\left(\mu + z_q \sigma\right) \end{aligned}$$

Using the exam table,

$$\begin{aligned} \text{E}[X \wedge \pi_q] &= \exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{\ln[\pi_q] - \mu - \sigma^2}{\sigma}\right) + \pi_q \cdot [1-q] \\ &= \text{E}[X] \cdot \Phi\left(\frac{\ln[\pi_q] - \mu - \sigma^2}{\sigma}\right) + \pi_q \cdot [1-q] \\ &= \text{E}[X] \cdot \Phi\left(\frac{\ln[\exp\left(\mu + z_q\sigma\right)] - \mu - \sigma^2}{\sigma}\right) + \pi_q \cdot [1-q] \\ &= \text{E}[X] \cdot \Phi\left(\frac{\mu + z_q\sigma - \mu - \sigma^2}{\sigma}\right) + \pi_q \cdot [1-q] \end{aligned}$$

$$= \mathrm{E}[X] \cdot \Phi(z_q - \sigma) + \pi_q \cdot [1 - q]$$

Then,

$$\begin{aligned}\frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \pi_q]}{1-q} &= \frac{\mathrm{E}[X] - (\mathrm{E}[X] \cdot \Phi(z_q - \sigma) + \pi_q \cdot [1 - q])}{1-q} \\ &= \mathrm{E}[X] \cdot \left[\frac{1 - \Phi(z_q - \sigma)}{1-q}\right] - \pi_q \\ &= \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_q)}{1-q}\right] - \pi_q\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{CTE}_q(X) &= \pi_q + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \pi_q]}{1-q} \\ &= \pi_q + \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_q)}{1-q}\right] - \pi_q \\ &= \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_q)}{1-q}\right]\end{aligned}$$

CTE for Normal Distribution

Let

$$X \sim \mathrm{Normal}\ (\mu,\ \sigma)$$

Then, $Z = \dfrac{X - \mu}{\sigma}$ is the standard normal random variable, with density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Also, denote the $100q^{\text{th}}$ percentile of X and Z as x_q and z_q, respectively.

$$\begin{aligned}
\mathrm{CTE}_q(X) &= \mathrm{E}\left[X \mid X > x_q\right] \\
&= \mathrm{E}\left[\mu + \sigma Z \mid \mu + \sigma Z > x_q\right] \\
&= \mu + \sigma \cdot \mathrm{E}\left[Z \mid Z > \frac{x_q - \mu}{\sigma}\right] \\
&= \mu + \sigma \cdot \mathrm{E}\left[Z \mid Z > z_q\right] \\
&= \mu + \sigma \cdot \int_{z_q}^{\infty} z \cdot \phi\left(z \mid Z > z_q\right) \mathrm{d}z \\
&= \mu + \sigma \cdot \int_{z_q}^{\infty} z \cdot \frac{\phi(z)}{\Pr(Z > z_q)} \, \mathrm{d}z \\
&= \mu + \sigma \cdot \int_{z_q}^{\infty} z \cdot \frac{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}{1 - q} \, \mathrm{d}z \\
&= \mu + \frac{\sigma}{(1 - q)\sqrt{2\pi}} \cdot \int_{z_q}^{\infty} z \cdot e^{-z^2/2} \, \mathrm{d}z \\
&= \mu + \frac{\sigma}{(1 - q)\sqrt{2\pi}} \cdot e^{-z_q^2/2} \\
&= \mu + \frac{\sigma}{1 - q} \cdot \frac{1}{\sqrt{2\pi}} e^{-z_q^2/2} \\
&= \mu + \frac{\sigma}{1 - q} \cdot \phi\left(z_q\right)
\end{aligned}$$

The substitution method was used to evaluate the integral $\int_{z_q}^{\infty} z \cdot e^{-z^2/2} \, \mathrm{d}z$. Let $u = \dfrac{z^2}{2}$. Then, $\mathrm{d}u = z \cdot \mathrm{d}z$. So,

$$\begin{aligned}\int_{z_q}^{\infty} z \cdot e^{-\frac{z^2}{2}}\,\mathrm{d}z &= \int_{z_q^2\,/\,2}^{\infty} e^{-u}\,\mathrm{d}u \\ &= \Big[-e^{-u}\Big]_{z_q^2\,/\,2}^{\infty} \\ &= e^{-z_q^2\,/\,2}\end{aligned}$$