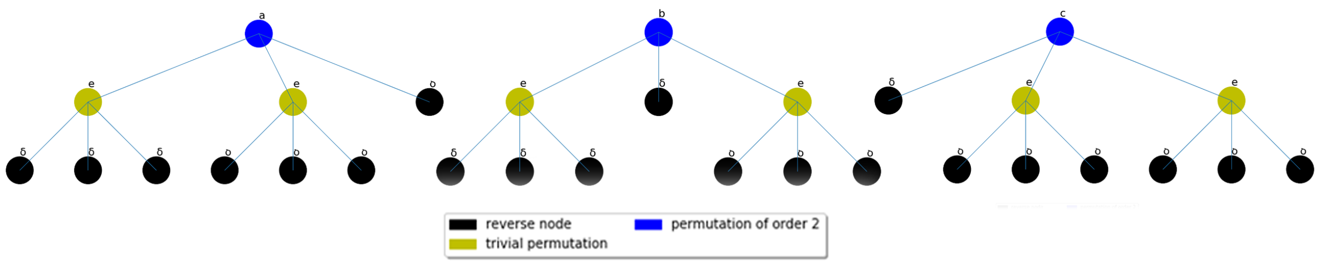
1 The worst case

Here should be definition of the group G

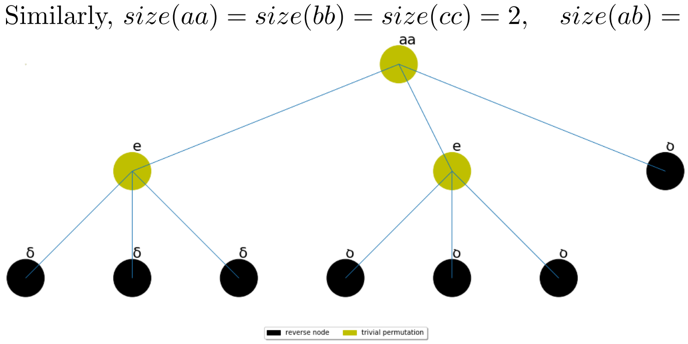
Definition 1.1. *Function **size** $: G \to \mathbf{N} \cup \{0\}$ is defined recursively:*

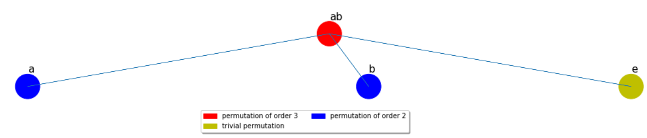
$$size(w = \pi(w_1, w_2, w_3)) = \begin{cases} 0, & w = e \\ 0, & w = \delta \quad (reverse\ node) \\ |w| + size(w_1) + size(w_2) + size(w_3), & otherwise \end{cases} \tag{1}$$

Thus, $size(a) = size(b) = size(c) = 1$ by definition.



Similarly, $size(aa) = size(bb) = size(cc) = 2, \quad size(ab) = size(ac) = ... = size(bc) = 4$





Definition 1.2. ***abc-subset*** *$X \subset \{a, b, c\}^*$ is such subset:*

$$X = \left\{(a^\pi b^\pi c^\pi)^k,\ (a^\pi b^\pi c^\pi)^k a^\pi,\ (a^\pi b^\pi c^\pi)^k a^\pi b^\pi \,|\, k \in \mathbf{N} \cup \{0\},\ \pi \in S_3\right\} \tag{2}$$

where a^π means S_3 group action on set $\{a, b, c\}$;
$(w_1 w_2 w_3)^k$ means repeating k times of the expression in the parenthesis.

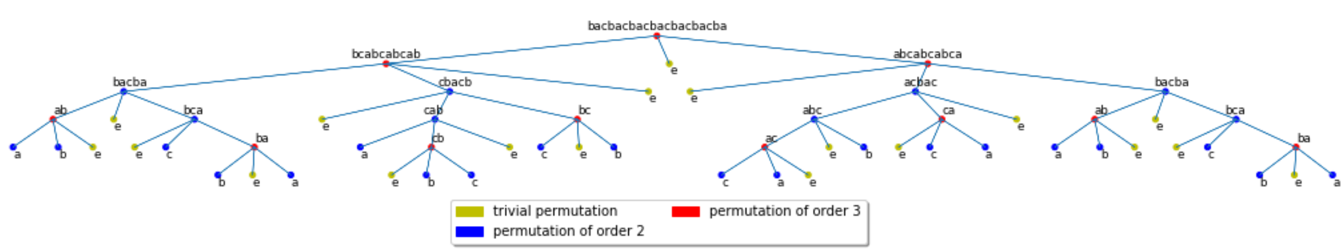
Lemma 1.1. *$\forall w \in X$ w will have one of the next possible structures:*

$$\pi(w_1, w_2, e), \quad \pi(w_1, e, w_2), \quad \pi(e, w_1, w_2),$$

where $w_1, w_2 \in X, \quad |w_1|, |w_2| \in \left\{\left\lfloor\frac{n}{2}\right\rfloor, \left\lceil\frac{n}{2}\right\rceil\right\}, \quad |w_1| + |w_2| = n, \quad n = |w|$

Proof. Follows from definition of multiplication. $\square$

Example:



Therefore, here we have recursive formula for calculating function $size$ **(1)** for elements from X:

$$size(w \in X) = a(n) = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right) + n, \quad a(1) = 1, \quad a(0) = 0, \quad n = |w|$$

Statement 1.1. *Exact form of function $a(n)$:*

$$a(n) = \begin{cases} n\left(\lfloor \log_2 n + 1 \rfloor\right) + 2n - 2^{\lfloor \log_2 n \rfloor + 1}, & n > 0 \\ a(0) = 0 \end{cases} \tag{3}$$

Proof.

$$a(n+1) = n + 1 + a\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n+1}{2} \right\rceil\right) = n + 1 + a\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right)$$

$$\text{Let } b(n) := a(n+1) - a(n) = 1 + a\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \text{ - auxiliary recursion.}$$

$$b(n) = b\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1, \quad b(2) = 4 \quad \Rightarrow \quad b(n) = \lfloor \log_2 n \rfloor + 3$$

$$a(n+1) = b(n) + b(n-1) + ... + b(1) = \sum_{1 \le k \le n} \left(\lfloor \log_2 k + 3 \rfloor\right) \quad \Rightarrow \quad a(n) = 2(n-1) + \sum_{1 \le k \le n} \left(\lfloor \log_2 k + 1 \rfloor\right) =$$

$$= \begin{bmatrix} \text{amount of bits in all the} \\ \text{numbers from } 1 \text{ to } N-1 \end{bmatrix} = 2(n-1) + \sum_{i=1}^{\lfloor \log_2 n \rfloor} (n - 2^i) = n\left(\lfloor \log_2 n + 1 \rfloor\right) + 2n - 2^{\lfloor \log_2 n \rfloor + 1}$$

□

Theorem 1.2. *Elements from abc-subset X have maximum size, or in other words*

$$a(n) = \max(size(w) \,|\, w \in \{a, b, c\}^n)$$

Proof. Induction on n.

1. Base: check examples of size calculation.
2. Induction step:
   Let $\forall k < n \quad a(k) = \max(size(w) \,|\, w \in \{a, b, c\}^k)$. Hereafter we consider any $w \in \{a, b, c\}^n$, $w = \pi(w_1, w_2, w_3)$, $w_1, w_2, w_3 \in \{a, b, c\}^* \cup \{e\}$. Now we need to show that $size(w) \le a(n)$

   Let $x_1 := |w_1|$, $x_2 := |w_2|$, $x_3 := |w_3|$. If w has maximum size than due to induction hypothesis

   $$size(w) \le a(x_1) + a(x_2) + a(x_3) + n \tag{4}$$

   (we don't know whether any w_1, w_2, w_3 from X could appear in w, so we use $\le$ symbol). Let's investigate how big this functional could be.

   Consider function $f(x) = x \log_2 x$ instead of $a(x)$. We can do it because both $f(x)$ and $a(x)$ are monotonically increasing on $x \ge 1$. Thus, they are maximizing in similar way.
   Now we need to solve optimization problem

   $$\max(f(x_1) + f(x_2) + f(x_3) \,|x_1 > 1,\, x_2 > 1,\, x_3 > 1,\, x_1 + x_2 + x_3 = n)$$

   $$F = f(x_1) + f(x_2) + f(x_3) = x_1 \log_2 x_1 + x_2 \log_2 x_2 + x_3 \log_3 x_3 = \left[x_3 = n - x_1 - x_2\right] =$$

   $$= x_1 \log_2 x_1 + x_2 \log_2 x_2 + (n - x_1 - x_2) \log_2 (n - x_1 - x_2) =$$

   $$= x_1 \log_2 x_1 + x_2 \log_2 x_2 + n \log_2 (n - x_1 - x_2) - x_1 \log_2 (n - x_1 - x_2) - x_2 \log_2 (n - x_1 - x_2)$$

   $$\frac{\partial F}{\partial x_1} = \log_2 x_1 + \frac{1}{\ln 2} - \frac{n}{(n - x_1 - x_2) \ln 2} - \log_2 (n - x_1 - x_2) + \frac{x_1}{(n - x_1 - x_2) \ln 2} + \frac{x_2}{(n - x_1 - x_2) \ln 2}$$

$$\frac{\partial F}{\partial x_2} = \log_2 x_2 + \frac{1}{\ln 2} - \frac{n}{(n-x_1-x_2)\ln 2} - \log_2(n-x_1-x_2) + \frac{x_1}{(n-x_1-x_2)\ln 2} + \frac{x_2}{(n-x_1-x_2)\ln 2}$$

If both $\frac{\partial F}{\partial x_1}$ and $\frac{\partial F}{\partial x_2}$ equal to zero then $\log_2 x_1 = \log_2 x_2 \Rightarrow x_1$ should be equal to x_2.
Let's substitute $x = x_1 = x_2$ to F and differentiate it:

$$\frac{dF}{dx} = 2\log_2 x + \frac{2}{\ln 2} - \frac{2n}{(n-2x)\ln 2} - 2\log_2(n-2x) + \frac{4x}{(n-2x)\ln 2} = 0$$

Hence, $x = \frac{n}{3}$ - the single solution (Wolfram). Substituting it to F we can find out that $x_1 = x_2 = x_3 = \frac{n}{3}$ - global minimum. Thus, F (therefore right part of **(4)**) reaches its maximum value on the bounds of its domain.
Now we are allowed to assume that $\exists i\, w_i = e$, because we know that otherwise

$$size(w) \le a(x_1) + a(x_2) + a(x_3) + n \le a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right) + n$$

It's easy to check that there are only 2 possible types of w with one or more trivial element below: familiar to us $w \in X$ or such $w = z_1 z_2 z_3 ... z_n$ where there are at least one i such that $z_i = z_{i+1}$, $\Rightarrow$ w could be reduced $\Rightarrow size(w) < a(n)$ due to definition.

□

2 Generative function

$$T(z) = \sum_w size(w) z^{|w|} = \sum_{w_1}\sum_{w_2}\sum_{w_3} \left(s(w_1)+s(w_2)+s(w_3)+|w_1|+|w_2|+|w_3|\right) z^{|w_1|+|w_2|+|w_3|} =$$

$$= \sum_{w_1}\sum_{w_2} z^{|w_1|+|w_2|}\left((s(w_1)+s(w_2))\sum_{w_3} z^{|w_3|} + \sum_{w_3} s(w_3) z^{|w_3|} + (|w_1|+|w_2|)\sum_{w_3} z^{|w_3|} + \sum_{w_3} |w_3| z^{|w_3|}\right) =$$

$$= \left[\sum_w z^{|w|} = F(z), \quad \sum_w |w| z^{|w|} = z\sum_w |w| z^{|w|-1} = zF'(z)\right] =$$

$$= \sum_{w_1} z^{|w_1|}\sum_{w_2} z^{|w_2|}\left((s(w_1)+s(w_2)+|w_1|+|w_2|)F(z) + T(z) + zF'(z)\right) =$$

$$= \sum_{w_1} z^{|w_1|}\left((T(z)+zF'(z))\sum_{w_2} z^{|w_2|} + F(z)\left((s(w_1)+|w_1|)\sum_{w_2} z^{|w_2|} + \sum_{w_2} s(w_2) z^{|w_2|} + \sum_{w_2} |w_2| z^{|w_2|}\right)\right) =$$

$$= \sum_{w_1} z^{|w_1|}\left[(T(z)+zF'(z))F(z) + F(z)((s(w_1)+|w_1|)F(z) + T(z) + zF'(z))\right] =$$

$$= \sum_{w_1} z^{|w_1|}\left[T(z)F(z) + zF(z)F'(z) + F^2(z)(s(w_1)+|w_1|) + F(z)T(z) + zF(z)F'(z)\right] =$$

$$= \sum_{w_1} z^{|w_1|}\left[2T(z)F(z) + 2zF(z)F'(z) + F^2(z)(s(w_1)+|w_1|)\right] =$$

$$= 2T(z)F^2(z) + 2zF^2(z)F'(z) + F^2(z)T(z) + zF^2(z)F'(z) = 3T(z)F^2(z) + 3zF^2(z)F'(z)$$

$$\Rightarrow T(z) = \frac{3zF^2(z)F'(z)}{1-3F^2(z)}, \quad F(z) = \sum_w z^{|w|} = \sum_{n=1}^{\infty} x_n z^n$$

1. $x_n = 3^n$

$$F(z) = \sum_{n=1}^{\infty} 3^n z^n = \sum_{n=1}^{\infty} (3z)^n = \frac{1}{1-3z} \quad \Rightarrow \quad T(z) = \frac{3z\frac{1}{(1-3z)^2}\frac{3}{(1-3z)^2}}{1-3\frac{1}{(1-3z)^2}} = \frac{9z}{81z^4-108z^3+27z^2+6z-2} = \frac{P(z)}{Q(z)}$$

Roots of Q(z): $\dfrac{1}{3}, \quad \dfrac{1}{3}, \quad \dfrac{1}{3} - \dfrac{1}{\sqrt{3}}, \quad \dfrac{1}{3} + \dfrac{1}{\sqrt{3}}$

The root $\dfrac{1}{3} - \dfrac{1}{\sqrt{3}}$ has the least absolute value (~ 0.24). Thus, by theorem of asymptotic growth of the rational generative functions' coefficients

$$[z^n]T(z) \sim C\left(\frac{3\sqrt{3}}{\sqrt{3}-3}\right)^n \quad \Rightarrow \quad \frac{[z^n]T(z)}{3^n} \sim C\left(\frac{\sqrt{3}}{\sqrt{3}-3}\right)^n$$

2. $x_n = 3 * 2^{n-1}$

$$F(z) = \sum_{n=1}^{\infty} 3 * 2^{n-1} z^n = \frac{3}{2}\sum_{n=1}^{\infty}(2z)^n = \frac{3}{2-4z}, \quad \Rightarrow, \quad T(z) = \frac{3z\left(\frac{3}{2-4z}\right)^2 \frac{d}{dz}\left(\frac{3}{2-4z}\right)}{1-3\left(\frac{3}{2-4z}\right)^2} =$$

$$= \frac{81z}{64z^4 - 128z^3 - 12z^2 + 76z - 23} = \frac{P(z)}{Q(z)}$$

Roots of Q(z): $\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2} - \dfrac{3\sqrt{3}}{4}, \dfrac{1}{2} + \dfrac{3\sqrt{3}}{4}$

The root $\dfrac{1}{2}$ of multiplicity 2 has the least absolute value. Thus, by theorem of asymptotic growth of the rational generative functions' coefficients

$$[z^n]T(z) \sim C(2)^n n, \quad C = 2\frac{(-2)^2 * P(1/2)}{Q''(1/2)} = \frac{8 * 81/2}{24(32 * 1/4 - 32 * 1/2 - 1)} = -\frac{81}{42}$$

$$\Rightarrow \quad \frac{[z^n]T(z)}{3 * 2^{n-1}} \sim \frac{-81 * 2^n * n}{42 * 3 * 2^{n-1}} = -\frac{9}{7}n$$