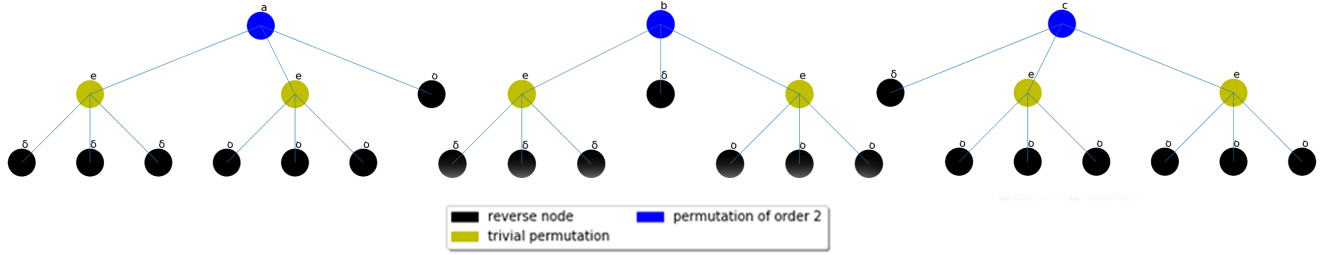# 1 The worst case

Here should be definition of the group G
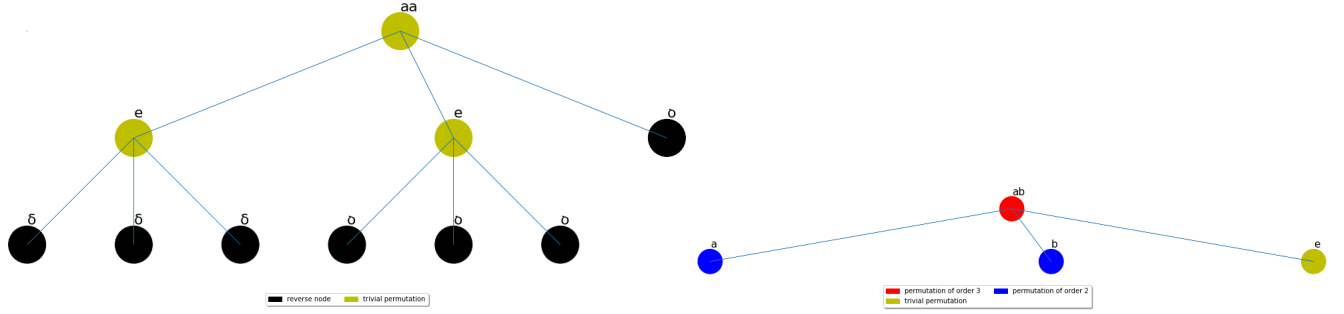
Definition 1.1. *Function $\textbf{size} : G \to \mathbf{N} \cup \{0\}$ is defined recursively:*

$$size(w = \pi(w_1, w_2, w_3)) = \begin{cases} 0, & w = e \\ 0, & w = \delta \quad \text{(reverse node)} \\ |w| + size(w_1) + size(w_2) + size(w_3), & \text{otherwise} \end{cases} \tag{1}$$

Thus, $size(a) = size(b) = size(c) = 1$ by definition.

Similarly, $size(aa) = size(bb) = size(cc) = 2, \quad size(ab) = size(ac) = ... = size(bc) = 4$

Definition 1.2. *$\textbf{abc-subset}$ $X \subset \{a, b, c\}^*$ is such subset:*

$$X = \left\{ (a^\pi b^\pi c^\pi)^k, \ (a^\pi b^\pi c^\pi)^k a^\pi, \ (a^\pi b^\pi c^\pi)^k a^\pi b^\pi \,|\, k \in \mathbf{N} \cup \{0\}, \ \pi \in S_3 \right\} \tag{2}$$

where a^π means S_3 group action on set $\{a, b, c\}$;
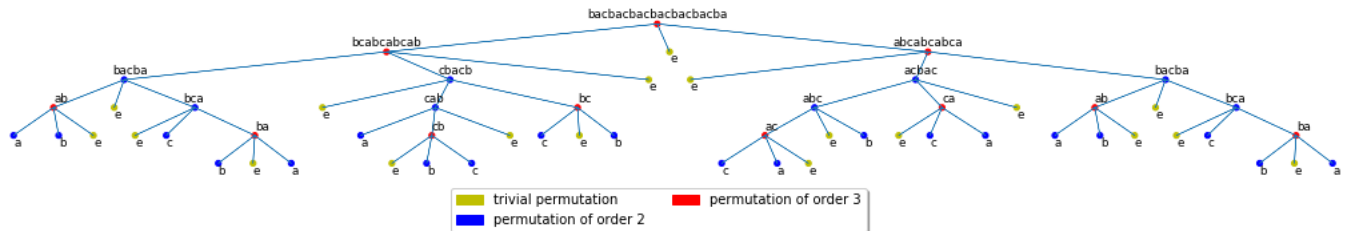*$(w_1 w_2 w_3)^k$ means repeating k times of the expression in the parenthesis.*

Lemma 1.1. *$\forall w \in X$ w will have one of the next possible structures:*

$$\pi(w_1, w_2, e), \quad \pi(w_1, e, w_2), \quad \pi(e, w_1, w_2),$$

where $w_1, w_2 \in X, \quad |w_1|, |w_2| \in \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right\}, \quad |w_1| + |w_2| = n, \quad n = |w|$

Proof. Follows from definition of multiplication. □

Example:

Therefore, here we have recursive formula for calculating function *size* **(1)** for elements from X:

$$size(w \in X) = a(n) = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right) + n, \quad a(1) = 1, \quad a(0) = 0, \quad n = |w|$$

Statement 1.1. *Exact form of function $a(n)$:*

$$a(n) = n\left(\lfloor \log_2 n + 1 \rfloor\right) + 2n - 2^{\lfloor \log_2 n \rfloor + 1} \tag{3}$$

Proof.

$$a(n+1) = n + 1 + a\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n+1}{2} \right\rceil\right) = n + 1 + a\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right)$$

$$\text{Let } b(n) := a(n+1) - a(n) = 1 + a\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \quad \text{- auxiliary recursion.}$$

$$b(n) = b\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1, \quad b(2) = 4 \quad \Rightarrow \quad b(n) = \lfloor \log_2 n \rfloor + 3$$

$$a(n+1) = b(n) + b(n-1) + \ldots + b(1) = \sum_{1 \le k \le n} \left(\lfloor \log_2 k + 3 \rfloor\right) \quad \Rightarrow \quad a(n) = 2(n-1) + \sum_{1 \le k \le n} \left(\lfloor \log_2 k + 1 \rfloor\right) =$$

$$= \begin{bmatrix} \text{amount of bits in all the} \\ \text{numbers from 1 to } N-1 \end{bmatrix} = 2(n-1) + \sum_{i=1}^{\lfloor \log_2 n \rfloor} (n - 2^i) = n\left(\lfloor \log_2 n + 1 \rfloor\right) + 2n - 2^{\lfloor \log_2 n \rfloor + 1}$$

$\square$

Theorem 1.2. *Elements from abc-subset X have maximum size, or in other words*

$$a(n) = \max(size(w) \,|\, w \in \{a, b, c\}^*, \, |w| = n)$$

Proof. $\square$

2   Generative function

$$T(z) = \sum_w size(w) z^{|w|} = \sum_{w_1} \sum_{w_2} \sum_{w_3} (s(w_1) + s(w_2) + s(w_3) + |w_1| + |w_2| + |w_3|) z^{|w_1| + |w_2| + |w_3|} =$$

$$= \sum_{w_1} \sum_{w_2} z^{|w_1| + |w_2|} \left((s(w_1) + s(w_2)) \sum_{w_3} z^{|w_3|} + \sum_{w_3} s(w_3) z^{|w_3|} + (|w_1| + |w_2|) \sum_{w_3} z^{|w_3|} + \sum_{w_3} |w_3| z^{|w_3|} \right) =$$

$$= \left[\sum_w z^{|w|} = F(z), \quad \sum_w |w| z^{|w|} = z \sum_w |w| z^{|w|-1} = zF'(z) \right] =$$

$$= \sum_{w_1} z^{|w_1|} \sum_{w_2} z^{|w_2|} \left((s(w_1) + s(w_2) + |w_1| + |w_2|) F(z) + T(z) + zF'(z) \right) =$$

$$= \sum_{w_1} z^{|w_1|} \left((T(z) + zF'(z)) \sum_{w_2} z^{|w_2|} + F(z) \left((s(w_1) + |w_1|) \sum_{w_2} z^{|w_2|} + \sum_{w_2} s(w_2) z^{|w_2|} + \sum_{w_2} |w_2| z^{|w_2|} \right) \right) =$$

$$= \sum_{w_1} z^{|w_1|} \left[(T(z) + zF'(z)) F(z) + F(z)((s(w_1) + |w_1|) F(z) + T(z) + zF'(z)) \right] =$$

$$= \sum_{w_1} z^{|w_1|} \left[T(z)F(z) + zF(z)F'(z) + F^2(z)(s(w_1) + |w_1|) + F(z)T(z) + zF(z)F'(z) \right] =$$

$$= \sum_{w_1} z^{|w_1|} \left[2T(z)F(z) + 2zF(z)F'(z) + F^2(z)(s(w_1) + |w_1|) \right] =$$

$$= 2T(z)F^2(z) + 2zF^2(z)F'(z) + F^2(z)T(z) + zF^2(z)F'(z) = 3T(z)F^2(z) + 3zF^2(z)F'(z)$$

$$\Rightarrow T(z) = \frac{3zF^2(z)F'(z)}{1 - 3F^2(z)}, \quad F(z) = \sum_w z^{|w|} = \sum_{n=1}^{\infty} x_n z^n$$

2

1. $x_n = 3^n$

$$F(z) = \sum_{n=1}^{\infty} 3^n z^n = \sum_{n=1}^{\infty} (3z)^n = \frac{1}{1-3z} \quad \Rightarrow \quad T(z) = \frac{3z\dfrac{1}{(1-3z)^2}\dfrac{3}{(1-3z)^2}}{1 - 3\dfrac{1}{(1-3z)^2}} = \frac{9z}{81z^4 - 108z^3 + 27z^2 + 6z - 2} = \frac{P(z)}{Q(z)}$$

Roots of Q(z): $\dfrac{1}{3}, \quad \dfrac{1}{3}, \quad \dfrac{1}{3} - \dfrac{1}{\sqrt{3}}, \quad \dfrac{1}{3} + \dfrac{1}{\sqrt{3}}$

The root $\dfrac{1}{3} - \dfrac{1}{\sqrt{3}}$ has the least absolute value (~ 0.24). Thus, by theorem of asymptotic growth of the rational generative functions' coefficients

$$[z^n]T(z) \sim C\left(\frac{3\sqrt{3}}{\sqrt{3}-3}\right)^n \quad \Rightarrow \quad \frac{[z^n]T(z)}{3^n} \sim C\left(\frac{\sqrt{3}}{\sqrt{3}-3}\right)^n$$

2. $x_n = 3 * 2^{n-1}$

$$F(z) = \sum_{n=1}^{\infty} 3 * 2^{n-1} z^n = \frac{3}{2}\sum_{n=1}^{\infty}(2z)^n = \frac{3}{2-4z}, \quad \Rightarrow, \quad T(z) = \frac{3z\left(\dfrac{3}{2-4z}\right)^2 \dfrac{d}{dz}\left(\dfrac{3}{2-4z}\right)}{1 - 3\left(\dfrac{3}{2-4z}\right)^2} =$$

$$= \frac{81z}{64z^4 - 128z^3 - 12z^2 + 76z - 23} = \frac{P(z)}{Q(z)}$$

Roots of Q(z): $\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2} - \dfrac{3\sqrt{3}}{4}, \dfrac{1}{2} + \dfrac{3\sqrt{3}}{4}$

The root $\dfrac{1}{2}$ of multiplicity 2 has the least absolute value. Thus, by theorem of asymptotic growth of the rational generative functions' coefficients

$$[z^n]T(z) \sim C(2)^n n, \quad C = 2\frac{(-2)^2 * P(1/2)}{Q''(1/2)} = \frac{8 * 81/2}{24(32 * 1/4 - 32 * 1/2 - 1)} = -\frac{81}{42}$$

$$\Rightarrow \quad \frac{[z^n]T(z)}{3 * 2^{n-1}} \sim \frac{-81 * 2^n * n}{42 * 3 * 2^{n-1}} = -\frac{9}{7}n$$