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Курсовий проект
на тему:

The word and order problems in Hanoi Tower Groups

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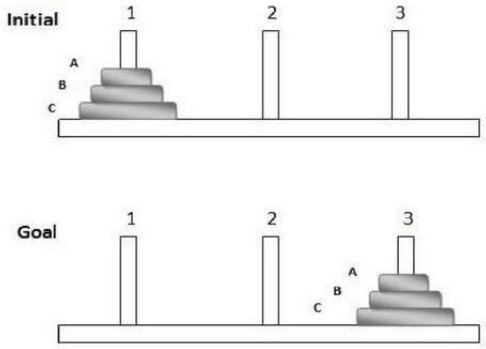
1. Introduction

This article is focused on combinatorial properties of some automaton groups, particularly of so-called Hanoi Tower Groups H_k that appear in various mathematical fields and could be defined in couple different ways. The majority of considered properties connected with **the word problem** that is one of three classical decision problems for finitely generated groups. There were proposed algorithms for solving the word and order problems in H_k, the correctness of working was proven and complexity in the worst case in H_3 was found.

Another goal of this work was to develop a decent Python framework for working with automaton groups at all and particularly with Hanoi Tower ones. More specifically, there were implemented classes for convenient work with elements of considered groups and set of tools to visualize some their properties, e.g. tree-like structure, process of algorithm solving the order problem etc. Should you want to see full implementation or try it for yourself, check https://github.com/davendiy/automata-groups.

2. General overview

2.1. **Hanoi Tower Game and Automaton Groups.** The Tower of Hanoi is a famous mathematical game that was invented by the French mathematician Édouard Lucas in 1883. A classical variant of the game is played with n disks placed on three pegs. The objective of this game is to move all discs from one peg to another obeying one simple rule - a bigger disk can't be placed upon smaller one. It is a well-known fact that the minimal number of moves required to solve the game is $2^n - 1$.



A general version of the game is played with n disks on k pegs and in this case the minimal number isn't as obvious as in the previous one. Although the three-peg version has a simple recursive algorithm long been known, the optimal solution for the Tower of Hanoi problem with even four pegs was not verified until 2014, by Bousch [6]. Besides, a Frame-Stewart algorithm has been known without proof of optimality since 1941. [7]

Hereby fix alphabet $\Sigma_k = \{0, 1, \dots k\}$. Each word $w = x_1 x_2 \dots x_n \in \Sigma_k^n$ represents correct configuration of the game with n disks on k pegs assuming $x_i \in \Sigma_k$ means the number of peg whereon the i-th smallest disk is placed.

Consider recursive functions $a_{(ij)} : \Sigma_k^* \to \Sigma_k^*$ as following (here $x \in \Sigma_k$ represents the first symbol):

$$a_{(ij)}(xw) = \begin{cases} iw, & x = j \\ jw, & x = i \\ xa_{(ij)}(w), & x \notin \{i, j\} \end{cases}$$

It's easy to check that aforementioned functions can be naturally interpreted as allowable single disk move from one peg to another. Since all of them are bijections they generate a so-called *Hanoi Tower Group* on k pegs H_k with identical function $e : \Sigma_k^* \to \Sigma_k^*$ as group's identity element and superposition as an action. [1]

The group H_k, $k \geq 3$ is also an example of an automaton group. In general, an *invertible automaton* is a quadruple $A = (S, X, \tau, \pi)$ where S is a finite set of states, X is a finite alphabet, $\tau : S \times X \to S$ a *transition function* and $\pi : S \times X \to X$ an *output function* such that, for each state $s \in S$, the restriction $\pi_s = \pi(s) : X \to X$ is a permutation in S_X (see [3]). If, for instance, the automaton is complete and invertible then its states generate a group.

Due to recursiveness of afore-defined functions elements of H_k also have tree-like structure, i.e. $\forall w \in H_k\, w$ can be represented as follows:

$$w = \pi(w_1, w_2, \dots w_k), \quad \pi \in Sym(\Sigma_k), w_i \in H_k \tag{1}$$

Therefore, elements of the group denote a k-ary tree automorphisms whereby Hanoi Tower Group was originally defined in [1]. In (1) π is called the *root permutation* of w while w_i are called the (first level) *sections* of w.

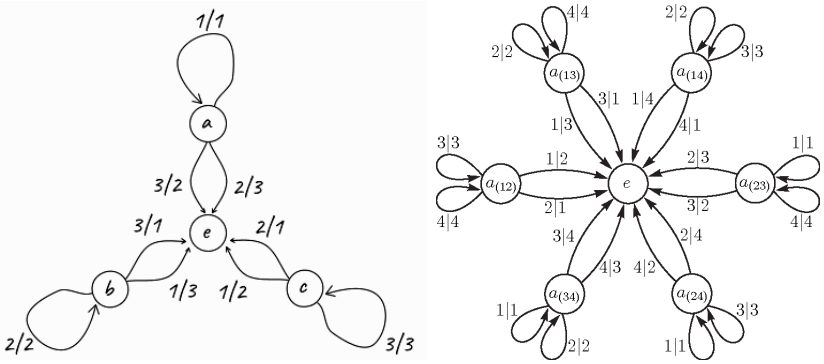


FIGURE 1. Automata for H_3 and H_4 respectively

Also, the multiplication rule is quite simple. Consider $g = \pi(g_1, g_2, \dots g_k)$, $h = \sigma(h_1, h_2, \dots h_k)$

$$g * h = \pi * \sigma(g_{\sigma(1)}, g_{\sigma(2)}, \dots g_{\sigma(k)})$$

where permutations multiplies in a way $(\pi * \sigma)(x) = \sigma(\pi(x))$. Hereafter, for simplicity, mark generators of H_3 as:

$$a := a_{(12)} = (12)(e, e, a_{(12)})$$

$$b := a_{(13)} = (13)(e, a_{(13)}, e)$$

$$c := a_{(23)} = (23)(a_{(23)}, e, e)$$

and generators of H_4:

$$a := a_{(23)},\ b := a_{(13)},\ c := a_{(12)}$$

$$d := a_{(34)},\ f := a_{(24)},\ g := a_{(14)}$$

Due to multiplication rule and defined generators it's obvious that H_k is so-called *contracting* group, i.e. $\forall w \in H_k$, $w = \pi(w_1, w_2, \dots, w_k) : |w_i| \leq |w|$ and thus, each element of H_k can be represented as finite k-ary tree. Hence, denote for every word w over generators its tree as T_w. Also define an operation called *slice* of the word w over generators recursively for any word over Σ_k (possibly infinite):

$$w|_x = w|_{x_0 x_1 \dots} = w_{x_0}|_{x_1 \dots}, \quad x \in \Sigma_k^*,\ x_i \in \Sigma_k$$

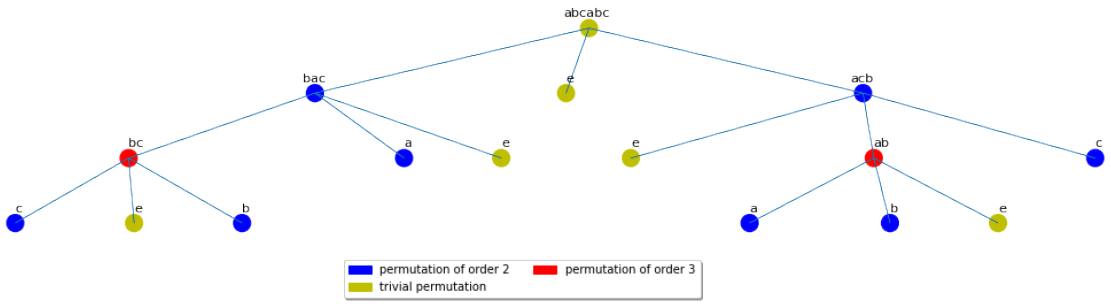


FIGURE 2. An example of tree-like structure in H_3

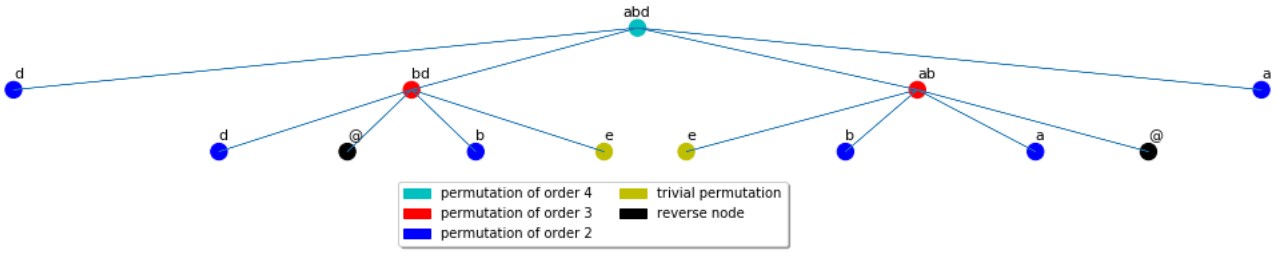


FIGURE 3. An example of tree-like structure in H_4

2.2. **Schreier Graphs.** Let finite generated group $G = \langle S \rangle$ act on a set X where $S = \{g_1, g_2, \dots g_n\}$ - generators. Consider oriented graph $\Gamma(G, X, S) = (V, E)$ where $V = X$ is a set of vertices and $E = \{(x, g(x)) | \, x \in X, g \in S\}$ is a set of edges. $\Gamma(G, X, S)$ is called *Schreier graph* of the group G generated by S acting on X.

This graph is not always simple and connected though. Particularly in case of H_k acting on Σ_k^* the correspondent Schreier graph $\Gamma(H_k, \Sigma_k^*, \{a_{(ij)}, \, 1 \leq i, j \leq k\})$ has loops and can be divided on infinite amount of connected components $\Gamma_n(H_k, \Sigma_k^n, \{a_{(ij)}, \, 1 \leq i, j \leq k\})$ (hereby denote them as H_k^n) that represent action of H_k on words over the alphabet Σ_k of length n. Therefore, it's quite natural to consider the sequence of connected graphs H_k^n instead of big one $\Gamma(H_k, \Sigma_k^*, \{a_{(ij)}, \, 1 \leq i, j \leq k\})$.

The sequence H_3^n is simple and therefore it's been well-studied ([8], [2]). Here are some famous facts:

- H_3 is a 3-regular graph (in general case, H_4^n is $\frac{k(k-1)}{2}$-regular)
- loops appear only on corner vertices (for $k \geq 4$ it is not true) that represent state in the game with all disks placed upon one peg
- adjacent vertices differ just with one letter, thus there is connection between H_3^n and Gray codes [9]
- the shortest path between the corners is just repetition of (abc) up to renaming of generators
- the repetition of just (ab) generates a Hamilton path from one corner to another
- the spectrum of the H_3^n can be found in [2]

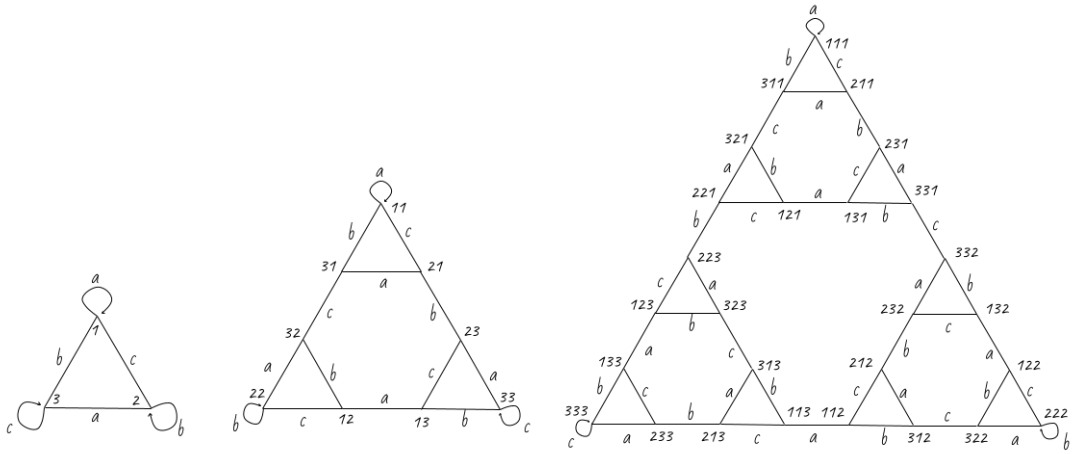


FIGURE 4. First three Schreier graphs for H_3

Nevertheless, for $k \geq 4$ the sequence H_k^n appears to be very complex. In [1] reader can found an asymptotic estimation of the H_k^n diameter. The explicit form of the shortest path or the spectrum is an open problem even for $k = 4$.

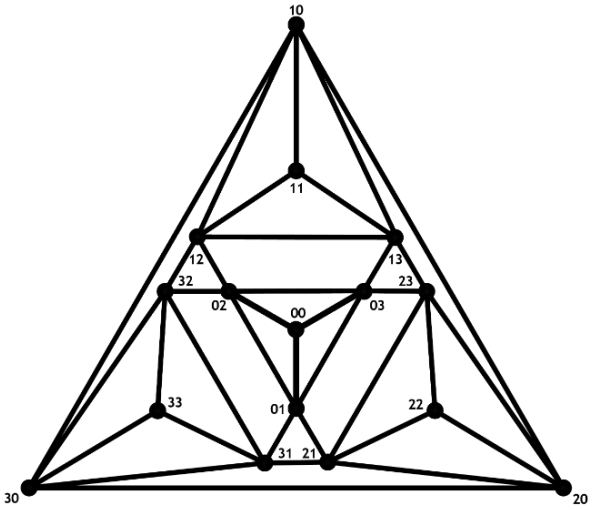


FIGURE 5. H_4^2 in the planar form (here disks enumerated with $\{0, 1, 2, 3\}$)

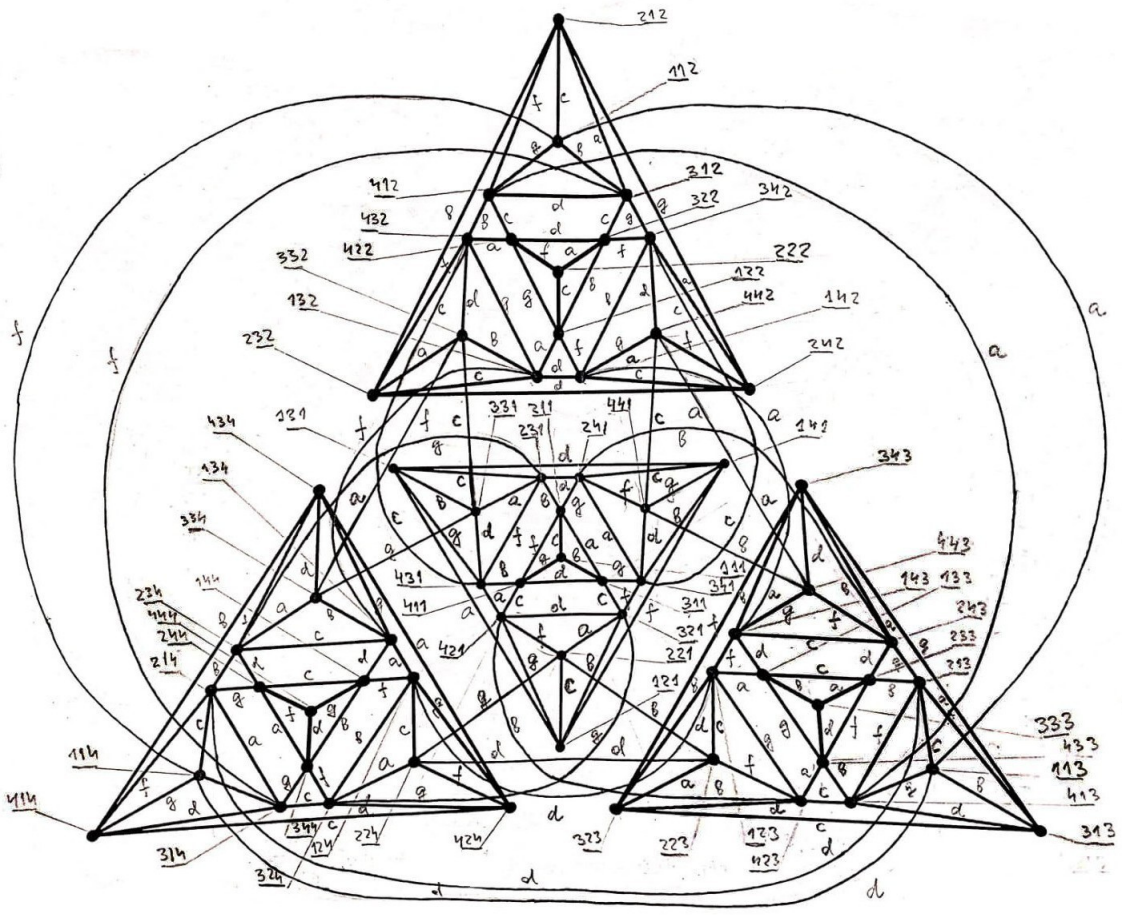


FIGURE 6. H_4^3 without loops

3. Word Problem

3.1. **Description, algorithm, correctness.** There are three classical decision problems for finitely-generated groups:

- **The Word problem** - *determine, given an arbitrary word over group generators, whether or not it represents the identity element of the group*
- **The Conjugacy problem** - *determine, given two words w_1 and w_2 over group generators, whether or not they represent conjugate elements in the group*
- **The Isomorphism Problem** - *determine, given two presentation, whether or not groups they represent are isomorphic.*

In this work only the first one is considered. In [10] it was shown that there is an algorithm solving the word problem for $H_k, k \geq 3$ in subexponential time $exp(O(\log^{m-2} n))$. The algorithm depends on fact that each H_k is contracting and therefore for every $w \in H_k$ it's enough just to check whether all the permutations in its tree-like structure are identical.

Although it's obvious that $a_{(ij)}a_{(ij)} = e$, the general portrait of identical elements even in H_3 is implicit and very difficult to construct. Nevertheless, despite the fact that it is not applicable for solving the word problem, it should be mentioned that in [4] a full representation of H_3 is obtained:

$$H^{(3)} = \left\langle a, b, c, \mid a^2, b^2, c^2, \tau^n(w_1), \tau^n(w_2), \tau^n(w_3), \tau^n(w_4)\, \forall n \geq 0 \right\rangle$$

where τ is an endomorphism of H_3 defined by the substitution

$$a \mapsto a, \quad b \mapsto b^c, \quad c \mapsto c^b$$

and where

$$\begin{aligned} w_1 &= [b,a][b,c][c,a][a,c]^b[a,b]^c[c,b] \\ w_2 &= [b,c]^a[c,b][b,a][c,a][a,b][a,c]^b \\ w_3 &= [c,b][a,b][b,c]^a[c,b]^2[b,a][b,c]^a[b,c]^a \\ w_4 &= [b,c]^a[a,b]^c[b,a]^2[a,c][a,b]^c[c,a][c,b] \end{aligned}$$

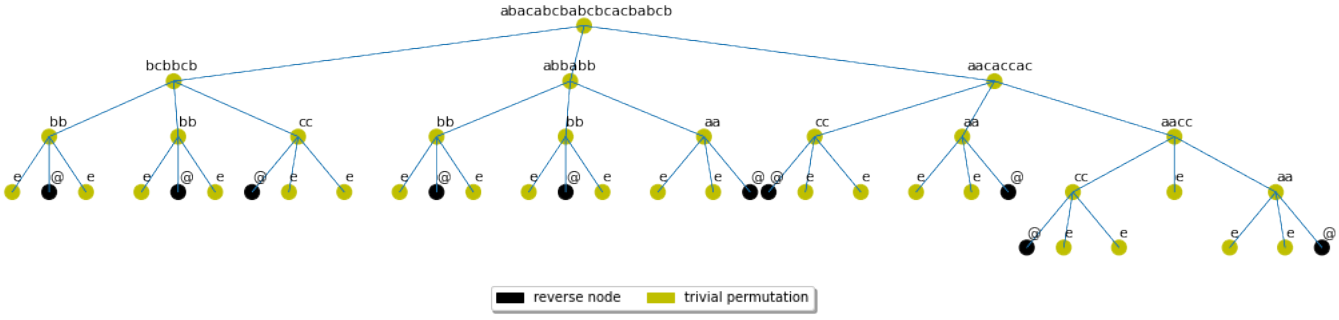


Figure 7. Example of non-trivial identity element in H_3

Since all elements w_x from the T_w are being checked for their permutation, it makes sense to consider functional $size(T_w)$ that is sum of word lengths within the tree. In other words:

$$size(w = \pi(w_1, w_2, \dots w_k)) = \begin{cases} 0, & w \in \{e, \delta\} \\ |w| + \sum_{i=1}^{k} size(w_i), & \text{otherwise} \end{cases} \tag{2}$$

where δ is auxiliary symbol that means recursive repetition of the word on the first level. Henceforth, for simplicity, notation $size(w)$ will be used as well as $size(T_w)$ with the same meaning. As an example:

$$size(a_{(ij)}) = 1, \quad \forall (ij) \in Sym(\Sigma_k)$$
$$size(a_{(ij)}a_{(ij)} = (e, \dots, a_{(ij)}a_{(ij)}, \dots e) = 2$$

Proposition 1. *Complexity of the algorithm $t(w)$ solving the word problem for $w \in S_k^*$ is $O(size(T_w))$. Here S_k means set of generators a_{ij}.*

Proof. For every vertex $w_x \in T_w$ the root permutation and its first-level sections can be computed in $O(|w_x|)$ operations. Thus,

$$t(w) = \sum_{w_x \in T_w} O(|w_x|) = O(size(T_w)), \text{ by definition (2).}$$

□

3.2. **The worst case complexity in H_3.** Define $X \subset H_3$ to be a set of all the elements that can be represented as periodic words over generators with 3-length word without repetitions as the period. More precisely:

$$X = \left\{ (xyz)^k, (xyz)^k x, (xyz)^k xy \middle| \, k \in \mathbb{N} \cup \{0\}, \, x, y, z \in S_3, x \neq y \neq z \neq x \right\} \tag{3}$$

As it was already mentioned, these elements also can be interpreted as the optimal strategies in the game.

Lemma 1. *Any $w \in X$ will have one of the following forms:*

$$\pi(w_1, w_2, e), \quad \pi(w_1, e, w_2), \quad \pi(e, w_1, w_2),$$
$$\text{where } w_1, w_2 \in X, \quad |w_1|, |w_2| \in \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right\}, \quad |w_1| + |w_2| = n, \quad n = |w|$$

Proof. Follows from the definition of multiplication. □

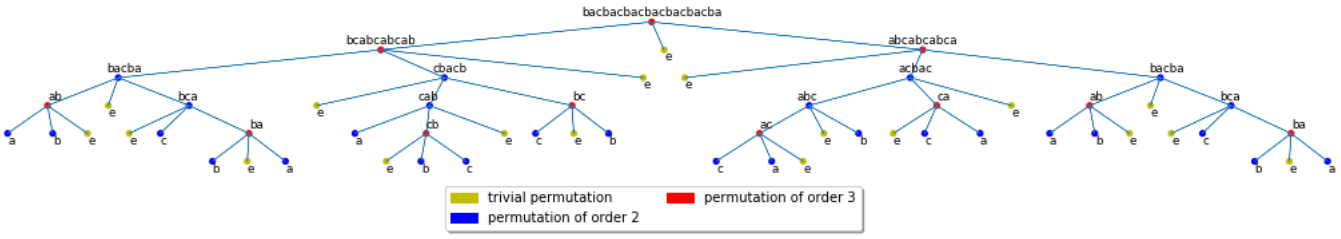


FIGURE 8. Example of an element from X

Therefore, here is a recursive formula for calculating function *size* (2) for elements from X:

$$size(w \in X) =: a(n) = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right) + n, \quad a(1) = 1, \quad a(0) = 0, \quad n = |w|$$

Theorem 1. *Exact form of function $a(n)$:*

$$a(n) = \begin{cases} n\left(\lfloor \log_2 n + 1 \rfloor\right) + 2n - 2^{\lfloor \log_2 n \rfloor + 1}, & n > 0 \\ a(0) = 0 \end{cases} \tag{4}$$

Proof.

$$a(n+1) = n+1+a\left(\left\lfloor\frac{n+1}{2}\right\rfloor\right)+a\left(\left\lceil\frac{n+1}{2}\right\rceil\right) = n+1+a\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)+a\left(\left\lceil\frac{n}{2}\right\rceil\right)$$

Let $b(n) := a(n+1)-a(n) = 1+a\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)-a\left(\left\lfloor\frac{n}{2}\right\rfloor\right)$ - auxiliary recursion.

$$b(n) = b\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+1, \quad b(2)=4 \quad \Rightarrow \quad b(n) = \lfloor\log_2 n\rfloor+3 \tag{5}$$

$$a(n+1) = b(n)+b(n-1)+...+b(1) = \sum_{1\le k\le n}(\lfloor\log_2 k+3\rfloor) \quad \Rightarrow \quad a(n) = 2(n-1)+\sum_{1\le k\le n}(\lfloor\log_2 k+1\rfloor) =$$

$$= \begin{bmatrix}\text{amount of bits in all the}\\ \text{numbers from 1 to } N-1\end{bmatrix} = 2(n-1)+\sum_{i=1}^{\lfloor\log_2 n\rfloor}(n-2^i) = n\,(\lfloor\log_2 n+1\rfloor)+2n-2^{\lfloor\log_2 n\rfloor+1}$$

□

Theorem 2. *Elements from X have maximum size among elements of length n, i.e.*

$$a(n) = \max(size(w)\,|\,w\in S_3^n)$$

Proof. Induction on n.

(1) Base: check examples of size calculation.

(2) Induction step:
Let $\forall k < n \quad a(k) = \max(size(w)\,|\,w\in S_3^k)$. Consider any $w\in S_3^n$, $w = \pi(w_1,w_2,w_3)$. Now it shall be shown that $size(w)\le a(n)$

Let $x_1 := |w_1|$, $x_2 := |w_2|$, $x_3 := |w_3|$. If w has maximum size then, due to induction hypothesis

$$size(w) \le a(x_1)+a(x_2)+a(x_3)+n \tag{6}$$

(we don't know what triplet (w_1,w_2,w_3) could appear in w, so the $\le$ symbol is used here).

Since $b(x) = a(x+1)-a(x)$ is a monotonically increasing function (5) value of functional (6) grows as long as the difference between w_i. Thus, we are allowed to assume that $\exists i,\ w_i = e$, because otherwise

$$size(w) \le a(x_1)+a(x_2)+a(x_3)+n \le a\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+a\left(\left\lceil\frac{n}{2}\right\rceil\right)+n$$

It's easy to check that there are only 2 possible forms of w with one or more identical element at the first level: $w\in X$ or such $w = z_1z_2z_3...z_n$ that has at least one i with $z_i = z_{i+1}$, $\Rightarrow$ w could be reduced $\Rightarrow$ $size(w) < a(n)$ due to definition.

□

3.3. **Average case complexity in H_3.** Due to aforementioned theorems it's guaranteed that the word problem is solvable in $O(n\log(n))$ time, where n is the word's length. However, in the real life algorithm works faster. Therefore, there is some sense to consider an average complexity instead of the worst one.

Nevertheless, this problem is much more difficult than the worst case and the exact distribution of *size* over all the n-length words is still an open problem. However, randomized experiments shows that distribution approximates $Binom(f(n),p)$ (figures 10, 9)

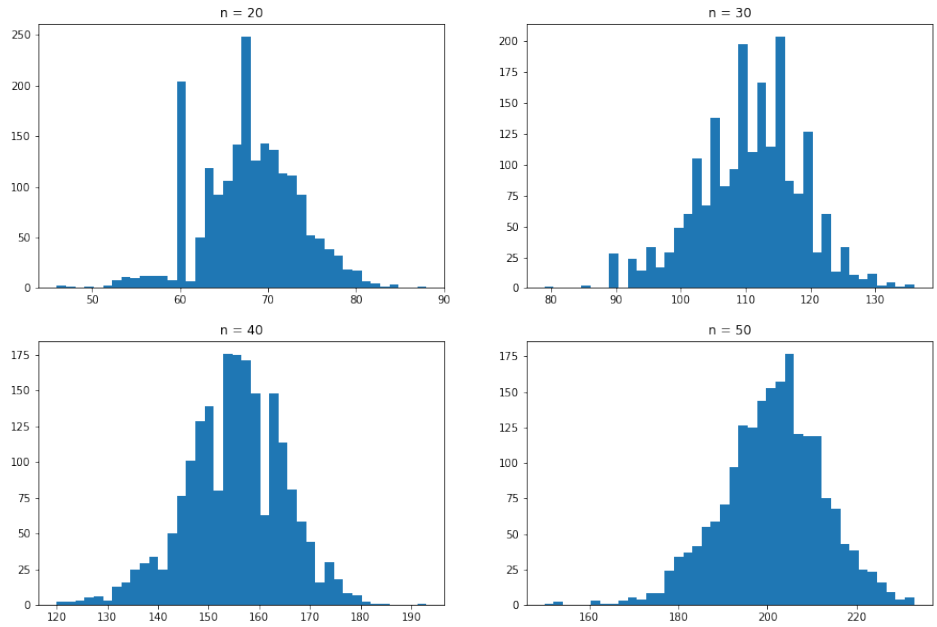


FIGURE 9. Distribution of *size* of 2000 random words for some big n

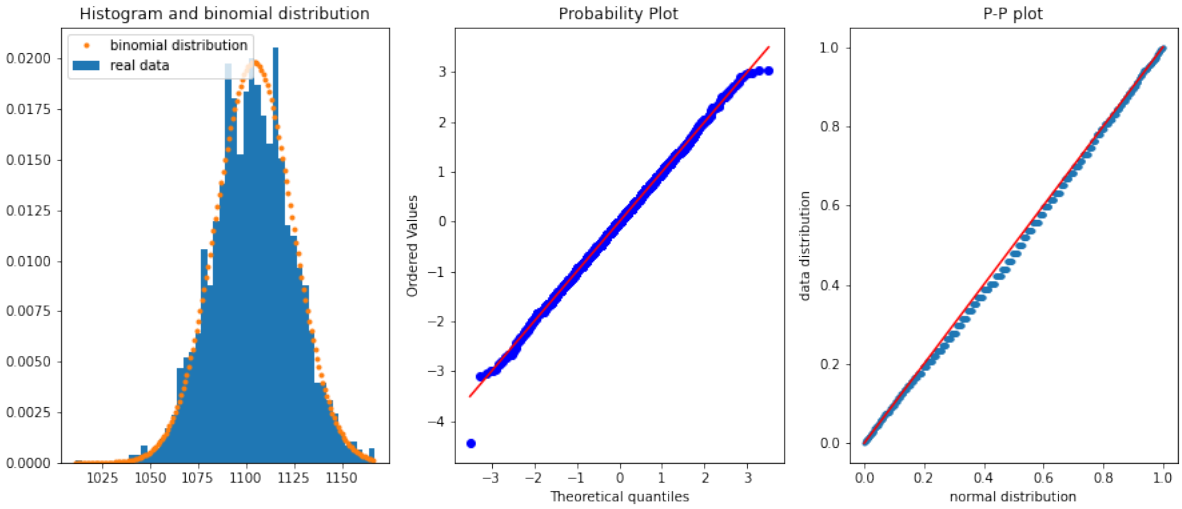


FIGURE 10. Comparing *size* of 3000 random words with $Binom(a(n), 0.633)$ and with standard normal distributions for $n = 200$

Conjecture 1. *Let ξ_n be the random word of length n over generators and $\theta_n \sim Binom(a(n), p)$, $p = const \approx 2/3$ Then*

$$||size(\xi_n) - \theta_n||_2 \to 0,\ n \to \infty$$

If this hypothesis is true then average size of n-length word $\approx E(Binom(a(n), p)) = pa(n) = O(n \log n)$.

4. Order Problem

Another goal of this work is to explore elements of H_k of finite order.

4.1. **Description, algorithm, correctness.** The order problem is formulated as follows: *determine, given an arbitrary word over group generators, whether or not it represents an element of finite order in the group*

To construct an algorithm solving this problem consider any element $w \in H_k$, $w = \pi(w_1, w_2, \dots w_k)$.

Proposition 2.

$$order(w) = order(\pi) * lcm(order(s_i), 1 \leq i \leq k), \tag{7}$$

where $w^{order(\pi)} = (s_1, s_2, \dots s_k)$ *and lcm means the least common multiplier. Note that* $order(s_i)$ *can be infinite.*

Proof. Due to the multiplication rule

$$(w^{order(\pi)})^n = (s_1^n, s_2^n, \dots s_k^n)$$

Assuming that $order(s_i) < \infty$ for all i it's enough to take $lcm(s_i, 1 \leq i \leq k)$ as n and get the entire first level from identities. Otherwise, none of n will reduce the element of infinite order. □

Corollary 1. *The order problem is solvable in* H_3.

Proof. Construct an algorithm that checks whether order of w is finite. It just computes function 7 recursively.

More precisely, given $w = \pi(w_1, w_2, \dots w_k)$:

(1) set $N := 1$
(2) get w, set $N_w := N$
(3) calculate $w^{order(\pi)} = (s_1, s_2, \dots s_k)$, update $N = N * order(\pi)$
(4) for every word s_i:
- if s_i is e, skip it. If though s_i is one of the atoms, return that the order is ∞
- check whether s_i has already appeared at the some level before. If so then calculate the value N/N_{s_i}. Hereupon it will be called *length of cycle*. If it is 1 then we call it as *cycle of unitary length* and just skip it. Otherwise we call it as *cycle of positive length* and return that the order is ∞
- check whether s_i is finite, passing it to the step (2) as the new w. If so, continue, else return that the order is ∞

(5) return that the order is finite

Cycle of positive length means that the order will increase forever and thus, it is ∞. At the other hand, cycle of unitary length just means recursive repetition of itself at some level without raising to some power. It means that the root permutation is identical too. Now it shall be shown that the considered algorithm always stops in H_3. Due to the multiplication rule

$$s_i = \prod_{j=1}^{order(\pi)} w_{x_j}, \quad x_j \in orbit_\pi(i) \tag{8}$$

and to the definition of atoms

$$|s_i| \leq \sum_{j=1}^{k} |w_j| = (k-2)|w|$$

Therefore, the length of s_i can't increase and algorithm will eventually encounter atoms or cycle of any length.

□

As a result, there is an algorithm solving the order problem for the H_3. It is still an open problem whether the proposed algorithm always stops in $H_k, k \geq 4$ processing elements in the breadth-first order , i.e. whether each element $w \in H_k$ of infinite order has a cycle of positive length. Nevertheless, it can be improved:

- due to the equation 8, s_i will contain only elements w_j, $j \in orbit_i(\pi)$. Moreover, two different elements s_i, s_j, $j \in orbit_i(\pi)$ are identical up to cyclic shifting and therefore, since every atom has order 2, they are conjugated. Finally, it means that s_i and s_j have the same order and thus, only one element s_i, $i \in orbit_i(\pi)$ should be checked
- likewise during the step 4 of the algorithm it's enough to check whether any cyclic shifts of s_i has already appeared at some level before. Moreover, it also can be checked solving the word problem for every word has already appeared and all cyclic shifts of s_i whether any pair represent the same element of the H_k (despite the complexity of this checking)

However, even with these improvements the finiteness of algorithm's work is still an open problem.

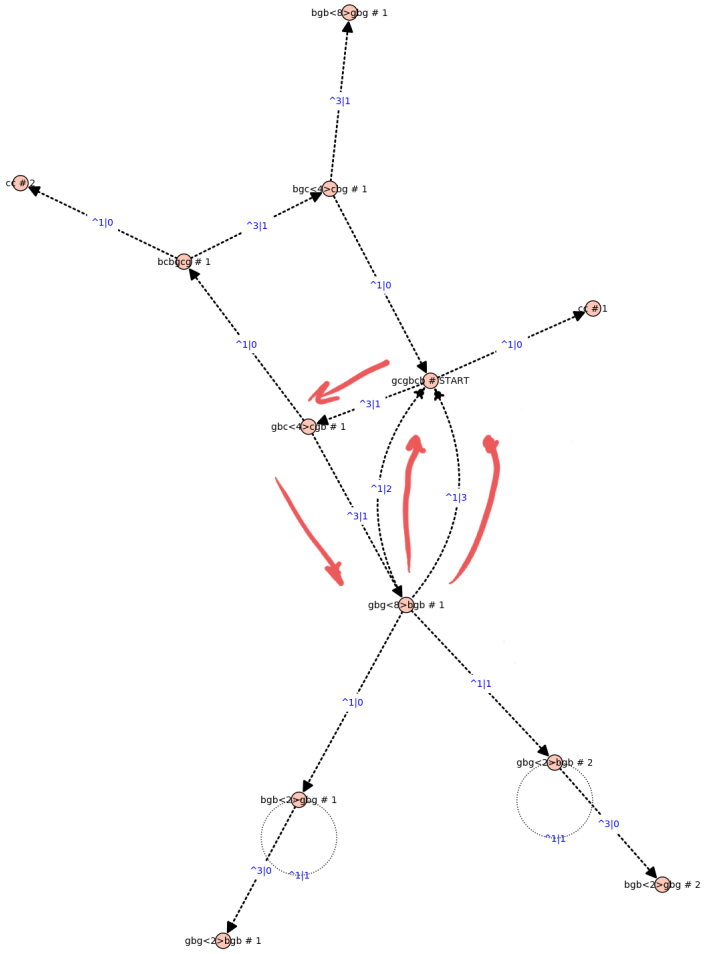


FIGURE 11. Graph that represents work of the algorithm. Vertex with label <START> means the given word w. Edges $\hat{}x|y$ mean raising to the power x and making slice towards y

5. Implementation

In this part some specifics of Python implementation of automaton groups will be described. Full code and documentation reader can find in https://github.com/davendiy/automata-groups.

5.1. **Overview.** Here is some part of documentation, that describes all the implemented classes and functions. Algorithmic specifics (structure of elements, tree drawing, multiplication) will be described after it.

class **Permutation***(form: list/tuple/int)*
Pure Python version of sympy.combinatorics.Permutation.
Implements an element of the Symmetric group S_n.
———————

def *array_form()*
Return an array form of Permutation.
def *cyclic_form()*
Return representation of the permutation as list of disjoint cycles.
def *order()*
Return order of the Permutation.
attr *size*
Return size of the Permutation, i.e. n.
———————
Examples

```
>>> pi = Permutation([1, 2, 0, 3])
>>> pi.cyclic_form()
>>> pi.cyclic_form
[[0, 1, 2]]
>>> pi.array_form
[1, 2, 0, 3]
>>> sigma = Permutation([1, 0, 3, 4, 2])
>>> sigma.cyclic_form
[[0, 1], [2, 3, 4]]
>>> sigma
Permutation([1, 0, 3, 4, 2])
>>> str(sigma)
'(0 1)(2 3 4)'
>>> pi * sigma
Permutation([0, 3, 1, 4, 2])
>>> str(pi * sigma)
'(1 3 4 2)'
>>> (pi * sigma).size
5
>>> pi.size
4
>>> sigma.size
5

```

class **TriedDict**(***words*)

Implementation of Dictionary based on trie a.k.a. prefix tree as a recursive dictionary. Note that the interface of Python dictionary is fully available and only new methods will be mentioned.

———————

Parameters

– ***words* : elements of the result dictionary

———————

def *max_prefix(string: str)*

Get prefix of the given string with maximum length that contains in the TriedDict and corresponding value.

————————

Returns

(*found_prefix*, *left_string*, *value*)

———————

Examples

```
>>> tr_dict = TriedDict(test=2, test2=3)
>>> tr_dict['hello there'] = 4
>>> print(tr_dict)
TriedDict({'t': {'e': {'s': {'t': {'': 2, '2': {'':
 3}}}}}, 'h': {'e': {'l': {'l': {'o': {' ': {'t': {'h':
  {'e': {'r': {'e': {'': 4}}}}}}}}}}}})
>>> 'hello there' in tr_dict
True
>>> 'test3' in tr_dict
False
>>> del tr_dict['hello there']
>>> 'hello there' in tr_dict
False
>>> tr_dict.max_prefix('test4')
('test', '4', 2)
>>> list(tr_dict.items())
[('test', 2), ('test2', 3)]
>>> list(tr_dict.keys())
['test', 'test2']
>>> list(tr_dict.values())
[2, 3]

```

class **Trie**(**words: str)

Implementation of Trie a.k.a prefix tree as a recursive dictionary. The interface of Python *set* is fully available.

———————

Parameters

– **words* : elements for the result trie

———————

Examples

```

```

```
>>> trie = Trie("word", "word2", "word3", "w", "wo", "
 next")
>>> print(trie)
Trie({'w': {'o': {'r': {'d': {'': '', '2': {'': ''}, '3
 ': {'': ''}}}, '': ''}, '': ''}, 'n': {'e': {'x': {'t'
 : {'': ''}}}}})
>>> 'word' in trie
True
>>> 'word123' in trie
False
>>> 'wor' in trie
False
>>> trie.add('word1231')
>>> 'word1231' in trie
True
>>> trie.remove('w')
>>> 'w' in trie
False
>>> list(trie)
['word', 'word2', 'word3', 'word1231', 'wo', 'next']
>>> trie.max_prefix('word123next')
('word', '123next')

```

Raises

– *TypeError* : when called methods __getitem__, __setitem__, __delitem__

Notes

Trie is based on TriedDict, however it doesn't support container protocol and therefore doesn't have attributes keys, values and items.

class **Tree***(value, children)*

Recursive tree structure, capable to be drawn in matplotlib.

Object of this class represents a tree node with value and any amount of children.

Parameters

- *value*: str, value that will be shown on the plot
- *children*: iterable of Tree objects

---

def *copy()*

Create copy of the entire tree recursively.

def *add_child(child, position=None)*

Add copy of the child to the tree.

Parameters

- *child*: Tree or any another type
  If one has type of Tree, it will be copied and added (inserted) to the deque of children

Else new Tree with the given value will be created
- *position*

def *height*()
Just height of tree, calculated using dfs.
————————
Notes
Returned value isn't cached, therefore dfs will be run anyway.

def *vert_amount*()
Calculates the amount of all the vertices recursively using dfs.
————————
Notes
Returned value isn't cached, therefore dfs will be run anyway.

def *remove(child)*
Remove child from tree.
————————
Parameters
- *child*: Tree or value of tree
————————
Notes
If there are more than one children with same value, only the first one will be deleted if parameter child represents value.

def *draw(start_x=0, start_y=0, scale=10, radius=2, fontsize=10, save_filename='')*
Draw the tree in matplotlib.
————————
Parameters
- *start_x*: x coordinate of the start position on the plane
- *start_y*: y coordinate of the start position on the plane
- *scale*: length of one step of offset. Offset is measure of vertices' displacement relative to left upper corner. Distance between 2 generation == one step of offset.
- *radius*: radius of vertices
- *fontsize*: size of font (like fontsize in matplotlib.pyplot.text)
- *save_filename*: name of file, where it should be save. Default == '', means that the picture won't be saved

def *make_offsets()*
Calculates offsets of all the vertices relative to zero point - the left bound. Uses for drawing without overlaps.

class **AutomataGroupElement**(name: str, permutation: Permutation, children=None, is_atom=False, group=None)
Class that represents element of any Automaton group.
————————
Parameters
- *name*: symbol that represents one of generators (atoms) or word over generators
- *permutation*: the root permutation of element
- *children*: a list of strings that represent elements of the first row of the element
- *is_atom*: True if element is one of generators and its name should be in alphabet

- *group*: object AutomataGroup, which represents a group wherefrom the element is given. If this parameter isn't given a lot of methods can't be executed (see docs of methods)

———————

Attributes

- *name*: name, can't be set
- *permutation* : root permutation, can't be set
- *cardinality* : amount of children / size of the permutations, can't be set
- *parent_ group*: group wherefrom the element. Can be set, i.e. every element can be moved from one group to another

———————

def *dfs*()

Go through the entire tree portrait of element and return all the elements on vertices in deep-first order.

————————

Notes

Can't be executed if *parent_ group* isn't defined.

def *bfs*()

Go through the entire tree portrait of element and return all the elements on vertices in deep-first order

————————

Notes

Can't be executed if *parent_ group* isn't defined.

def *__getitem__(item)*

Assuming *item* be the word x over $\{0, \dots n\}$, where n is cardinality, return the slice towards this word $w|_x$

————————

Notes

Can't be executed if *parent_ group* isn't defined.

def *__call__(word)*

Apply the group action of the element on the given word.

————————

Notes

Can't be executed if *parent_ group* isn't defined.

def *__mul__(other)*

Multiply this element by other, applying Lempel-Ziv algorithm

————————

Notes

Can't be executed if *parent_ group* isn't defined.

def *__pow__(power)*

Binary raising to the power.

————————

Notes

Can't be executed if *parent_ group* isn't defined.

def *inverse()*

Get the inverse element of this one, i.e. raising to the (-1) power.

————————

Notes

Can't be executed if *parent_ group* isn't defined.

def *is_ one()*

Check whether the given element represents an identity of the group, solving the word problem with the proposed algorithm.

———————

Notes

Can't be executed if *parent_ group* isn't defined.

def *order(check_ finite=True)*

Return the order of the element solving the order problem with proposed algorithm. If the element has infinite order, return float('inf'). Check finiteness of element if the *check_ finite* is True.

———————

Notes

Can't be executed if *parent_ group* isn't defined.

def *is_ finite()*

Check whether the given element has finite order solving the order problem with proposed algorithm.

———————

Notes

Can't be executed if *parent_ group* isn't defined.

def *order_ graph(graph)*

Solve the order problem with proposed algorithm and build a directed graph of states while doing it. *graph* can be any type that implements digraph interface (e.g. sage.graphs, networkx, etc).

———————

Notes

Can't be executed if *parent_ group* isn't defined.

def *show(self, start_ x=0, start_ y=0, scale=10, radius=4, fontsize=50, save_ filename='', show_ full=False, y_ scale_ mul=3, lbn=False, show_ names=True)*

Draws the tree-like structure of element in matplotlib.

———————

Parameters

- *start_ x*: x coordinate of the start position on the plane
- *start_ y*: y coordinate of the start position on the plane
- *scale*: length of one step of offset. Offset is measure of vertices' displacement relative to left upper corner. Distance between 2 generation == one step of offset.
- *radius*: radius of vertices
- *fontsize*: size of font (like fontsize in matplotlib.pyplot.text)
- *save_ filename*: name of file, where it should be save. Default == '', means that the picture won't be saved
- *show_ full*: False if you want elements with attribute 'simplify' to draw as just one node with name.
- *y_ scale_ mul*: multiplier of y-axes scale It's used for bigger step between generations
- *lbn*: leaves belong names, True if you want to print names of leaves belong them
- *show_ names*: True if you want to plot the names of vertices

———————

Examples

```
>>> H3 = AutomataGroup.generate_H3()
>>> H3.gens
[H3(a = (2)(0 1) (e, e, a)), H3(b = (0 2) (e, b, e)),
 H3(c = (1 2) (c, e, e))]
>>> el = H3('abcabcabc')
```

```
>>> el
H3(abcabcabc = (0 2) (acbac, e, bacb))
>>> el.tree.size()
38
>>> el.order()
inf
>>> el.describe()
    ======H3(abcabcabc = (0 2) (acbac, e, bacb))======
Group:
AutomataGroup H3
over alphabet {'0', '1', '2'}
generated by <H3(a = (2)(0 1) (e, e, a)), H3(b = (0 2)
 (e, b, e)), H3(c = (1 2) (c, e, e))>.

size:       38
height:     7

Generation: 0, element: abcabcabc
Generation: 1, element: bacbacbac
Generation: 1, element: e
Generation: 2, element: abcabcabc
Found cycle between abcabcabc and abcabcabc of length
 4.0

is finite: False
order:     inf

Found cycle
start deep:   0
end deep:     2
start el:     abcabcabc
end el:       abcabcabc
start power:  1
end power:    4
cycle weight: 4.0
full path:    00
word with cycle orbit:  (00)
==========================================

>>> el.inverse()
H3(cbacbacba = (0 2) (bcab, e, cabca))
>>> el ** 3
H3(abcabcabcabcabcabcabcabcabc = (0 2) (acbacbacbacbac,
  e, bacbacbacbacb))
```

class **AutomataGroup**(*name, gens: AutomataGropElement-s, reduce_func=id_func, lempel_ziv=True*)

Class that represents finite-generated automaton group.

———————

Parameters

- *name*: name of the group. Note that only one group can have particular name
- *gens*: iterable of AutomataGroupElement-s, generators of the group
- *reduce_func*: function that takes word over generators and reduces it removing subwords that represent identical elements. In H_k it just removes repetitions, i.e. abaaba -> e.
- *lempel-ziv*: True if effective multiplication should be used

———————

Examples

```
>>> H3 = AutomataGroup.generate_H3()
>>> H3.gens
[H3(a = (2)(0 1) (e, e, a)), H3(b = (0 2) (e, b, e)),
 H3(c = (1 2) (c, e, e))]
>>> H3('abca')
H3(abca = (0 1 2) (e, ac, ba))
>>> H3.random_el(20)
H3(acbcabcbcacbcacacbac = (2) (cbccabc, cbabaca, abaccc
 ))
>>> H3.expected_words
{'0', '1', '2'}
>>> H3.alphabet
['a', 'b', 'c']
>>>

```

5.2. **Structure of elements.** It's a convenient way to represent elements of H_k as k-ary rooted trees. However, it isn't the best way of representation due to several reasons:

- for the most situations there is no need to have a full tree but getting the first level and the root permutation is crucial
- majority of automaton groups isn't contracting and therefore the corresponding tree will be infinite
- keeping the full tree while doing some calculations (especially if we talk about the order problem) requires a huge amount of memory.

Hence, it's quite natural to handle elements of an automaton group just as $w = \pi(w_1, w_2, \dots w_n)$ and calculate the entire tree *only in time of need*.

In fact, tree-like representation is not exactly a tree because it often has cycles. Particularly in H_k even atoms require some technical solution to avoid the infinite nature of tree-like structure, caused by recursive repetition of itself. For instance in H_3:

$$a := a_{(12)} = (12)(e, e, a_{(12)})$$
$$b := a_{(13)} = (13)(e, a_{(13)}, e)$$
$$c := a_{(23)} = (23)(a_{(23)}, e, e)$$

To handle this situation an auxiliary symbol δ is used (in the source code it is @ at this time), which means the recursive repetition of itself.

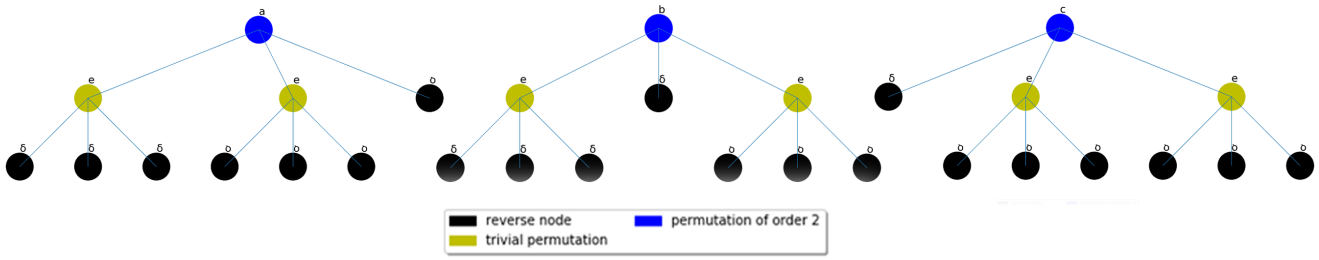


FIGURE 12. Atoms of H_3

Although, it should be noted, that this technical solution isn't the best and will be replaced soon, because it doesn't handle an issue with cycles of length > 1. As an example, consider famous Grigorchuk group generated by:

$$a = (12)(e, e), \quad b = (a, c)$$
$$c = (a, d), \quad d = (e, b)$$

Should we try to represent it with just δ we will get something like at Figure 13. You can observe that at the 4-th level element c was removed in order to stop the infinite recursion.

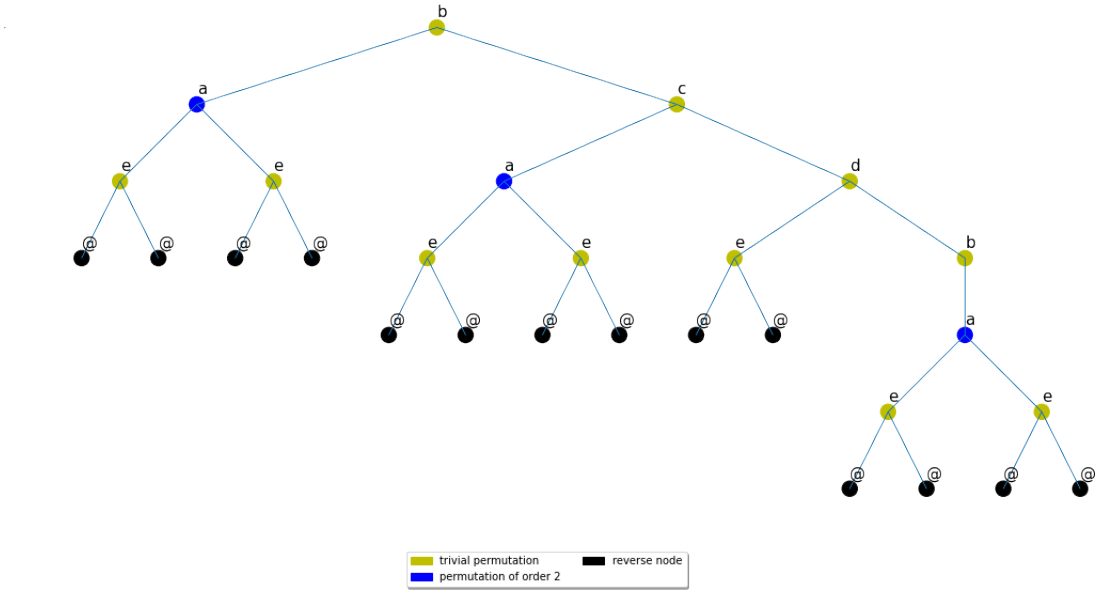


FIGURE 13. Tree-like structure of b atom of the Grigorchuk group

5.3. **Tree drawing.** It would be great to be able to draw a tree-like structure automatically for clarity. Since all the available open-source instruments (e.g. [12], [11]) don't provide labeling that differs from vertices' names, own Python version was implemented just for this purpose, results of which used in this work.

Simply, an algorithm works as follows:

```
Given tree T.
    1. draw each child of T recursively
    2. shift each child to the right position
    3. draw the root above just in the middle
```

The main problem is the 2-nd step because it's not completely clear what does *right position* mean. For this purpose such property about vertices is used.

Definition 1. ***Offset*** *of vertex $\phi : V \to \mathbb{R}+$ is defined recursively:*

- $\phi(v) = 0$ *for a tree with only 1 vertex*
- *given $T = w(T_1, T_2, \dots T_n)$, where w is the root and T_i are children. Consider ϕ_i as offset functions for each T_i separately and $x_i = max(\phi_i(v), v \in T_i)$. Then for $v \in T_i$*

$$\phi(v) = \sum_{j=1}^{i}(x_j + 1) + \phi_i(v)$$

and for the root:

$$\phi(w) = \frac{1}{n}\sum_{i=1}^{n}(x_i + 1)$$

Proposition 3. *If a tree is drawn using offset ϕ and deep as coordinates of each vertex (x, y) respectfully, there will be no intersections of edges.*

Applying different scales for (x, y) coordinates and using different radius of circles that represent vertices, we will get a result similar to the figures in this work.

5.4. **Effective multiplication.** Solving the order problem it often happens to handle huge words (up to 10^6 letters). Standard way to find out the permutation and the first level of a word is just to multiply all the letters-atoms. Therefore, complexity of the naive algorithm is $O(n)$, which could be very time consuming if you have to do hundreds of them. Besides, since the alphabet is small for the most cases, there are multitudes of repetitive patterns within the huge word (just like in DNA). Thus, an inspiration appears to use somehow Lempel-Ziv-like ([13]) algorithm herein.

Assuming every result of multiplication is being cached the algorithm can be described in the following way:

```
Input data: a huge word w
            a dictionary of already processed elements (with restriction on the maximum length)
Output: permutation and the first level of w

    1. set res := e — identical element
    2. set v := the longest prefix of w that is stored in the dictionary
    3. res := res * v  — multiplication with caching
    4. while w isn't empty go to (1)
    5. return res
```

Complexity of this algorithm is connected to the famous Lempel-Ziv complexity of finite words (check [14]) and it is out of scope of this work.

Algorithm will be even more effective, if the dictionary is implemented as trie data structure. [15]

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