The word and order problems in Hanoi Tower Groups

Zashkolnyi D.O.

The Tower of Hanoi is a well-known mathematical game. The classical variant of this game is played with n discs placed on three pegs. The objective of the game is to move all discs from one peg to another obeying certain simple rules. The minimal number of moves required to solve the game is $2^n - 1$.

In [1], it was shown that the Hanoi Tower game on k pegs can be modeled by a $\frac{k(k-1)}{2}$-generated group, called the Hanoi Tower group H_k, acting on words over the alphabet $\Sigma_k = \{1, 2, \ldots, k\}$. The action of generators corresponds to a singe disk move between two pegs. Hence, each strategy of the game corresponds to a word in group generators.

We study the word problem in the group H_3. The word problem is one of three classical decision problems for finitely generate groups. It is formulated as follows: *determine, given a word over group generators, whether it represents the identity element of the group.*

Since the Hanoi Tower groups are examples of so-called Automaton Groups, there is a common algorithm solving the word problem for them. We have proven that its complexity is $O(n \log n)$ in the worst case for the group H_3 and have found explicit elements that cause this case.

We also study the order problem in the groups H_3 and H_4. This problem is formulated similarly: *determine, given a word over group generators, whether it represents an element of finite order.* We proved that the order problem is decidable in the group H_3 and designed the corresponding algorithm. We proposed an algorithm for solving the order problem in the group H_4; however, it remains an open problem whether it always stops. We also have explored different variations of this algorithm and found counterexamples for many of them. Furthermore, we have implemented a fully-worked framework to work with Automaton Groups which you can find on https://github.com/davendiy/automata-groups

Список літератури

[1] R.I. Grigorchuk, Z. Sunik, Asymptotic aspects of Schreier graphs and Hanoi Towers groups, C. R. Math. Acad. Sci. Paris, 342 (2006), no. 8, 545–550., https://arxiv.org/abs/math/0601592

Authors

Davyd Zashkolnyi — 4-th year Bachelor's degree student, Mechanics and Mathematics Faculty, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine; E-mail: *davendiy@gmail.com*