Solving the word problem in a Hanoi Tower Group

David Zashkolny

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Abstract

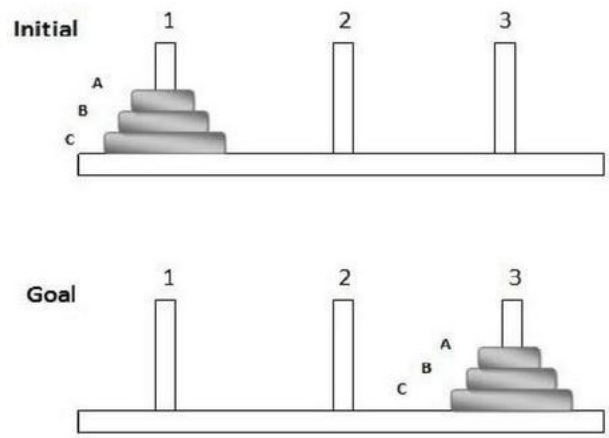
This article presents investigation of algorithm for solving word problem in the Hanoi Tower Group, its asymptotic in the worst case and the average one. Also some new abstraction and relations in the $H^{(3)}$ group are introduced here.

1 Introduction

2 Hanoi Tower Game and automaton groups

Let's firstly describe briefly the Hanoi Tower Game.

Fix an integer k, $k \geq 3$. The Hanoi Tower Game is played on k pegs, labeled by $0, 1, ...k-1$, with n disks labeled by $1, 2, ...n$. All the n disks have different size and the disks' labels reflect the relative size of each disk (the one labeled by 1 is the smallest disk, the one with 2 is the next smallest, etc.). A configuration is (by definition) any placement of the n disks on the k pegs in such a way that no disk is below a larger one (i.e. a size of the disks placed on any single peg is decreasing from the bottom to the top.). In a single step one may move the top disk from one peg to another as long as the newly obtained placement of disks is a configuration. Therefore, for any given two pegs x and y there is only one possible move which involves these two pegs (namely the smaller of the two top disks can be moved to the other peg). Initially all disks are on peg 0 and the object of the game is to move all of them to peg 1 in the smallest possible numbers of steps.



Now consider the free monoid X^* of words over the alphabet $X = \{0, ...k-1\}$. X has a k-regular rooted tree structure in which the empty word is the root and the words of length n constitute the level n in the tree. The k children of the vertex u are the vertices ux for $x = 0, ...k-1$. Denote this k-regular rooted tree by T Any automorphism g of T can be (uniquely) decomposed as $\pi_g(g_0, g_1, ...g_{k-1})$, where $\pi_g \in S_k$ is called *root permutation* of g and g_x, $x = 0, ...k-1$, are the tree automorphisms called the (first level) *sections* of g. Thee root permutation π_g and the sections g_i are determined uniquely by the relation $g(xw) = \pi_g(x)g_x(w)$, for all $x \in X$ and $w \in X^*$.

For any permutation π in S_k define a k-ary tree automorphism $a = a_\pi$ by $a = \pi(a_0, a_1, ..., a_{k-1})$, where a_i is the identity automorphism if i is in the support of π and $a_i = a$ otherwise. The action of the automorphism $a_{(ij)}$ on T is given (recursively) by

$$a_{(ij)}(iw) = jw, \quad a_{(ij)}(jw) = iw, \quad a_{(ij)}(xw) = xa_{(ij)}(w), \quad for\, x \notin \{i, j\}$$

Hanoi Towers group on k pegs, $k \geq 3$, is the group $H^{(k)} = \left\langle \left\{ a_{(ij)} | 0 \leq i < j \leq k-1 \right\} \right\rangle$ of k-ary tree automorphisms generated by the $a_{(ij)}$, $0 \leq i < j \leq k-1$, corresponding to the transpositions in S_k.

Note that the n-disk configurations are in bijective correspondence with the k^n words of length n over the alphabet $X = \{0, ..., k-1\}$. Namely, the word $x_1...x_n$ over X corresponds to the unique configuration in which the disk i, $i = 1, ..., n$, is placed on peg x_i. The action of the automorphism $a_{(ij)}$ corresponds to a move between the pegs i and j.

The group $H^{(k)}$, $k \geq 3$, is an example of an automaton group. In general, an *invertible* automaton is a quadruple $A = (S, X, \tau, \rho)$ in which S is a finite set of states, X a finite alphabet, $\tau : S \times X \to S$ a *transition function* and $\pi : S \times X \to X$ an *output function* such that, for each state $s \in S$, the restriction $\pi_s = \pi(s, \cdot) : X \to X$ is a permutation in S_X (see [3]). The states of A define recursively tree automorphisms by setting the permutation π_s to be the root permutation of s and the state $\tau(s, x)$ to be the section s_x of s at x. The group of tree automorphisms $G(A) = \langle s | s \in S \rangle$ generated by the automorphisms corresponding to the states of the invertible automaton A is called the *automaton group* of A. Invertible automata are often represented by diagrams such as the one on the left in Fig. 2 corresponding to $H^{(4)}$. Each state s is labeled by the permutation π_s and the labeled edges describe the transition function (if $\tau(s, x) = t$ then there is an edge labeled x connecting s to t).

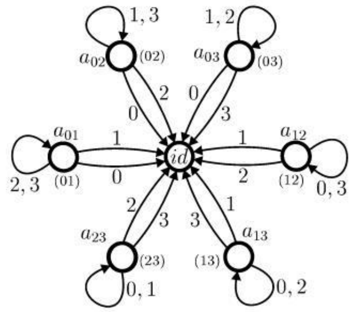


Figure 1: The automaton generating $H^{(4)}$

3 Trees

Now let's take a look at the $H^{(3)}$ group. Hereafter define $G := H^{(3)}$ and

$$a := a_{(12)} = (12)(e, e, a_{(12)})$$
$$b := a_{(13)} = (13)(e, a_{(13)}, e)$$
$$c := a_{(23)} = (23)(a_{(23)}, e, e)$$

Let's also introduce a new symbol δ which means a recursive call. Thus we have such representation of the G's atomic elements:

$$e = ()(\delta, \delta, \delta), \quad a = (12)(e, e, \delta), \quad b = (13)(e, \delta, e), \quad c = (23)(\delta, e, e)$$

Since now any element of the G can be represented as a finite rooted 3-regular tree.

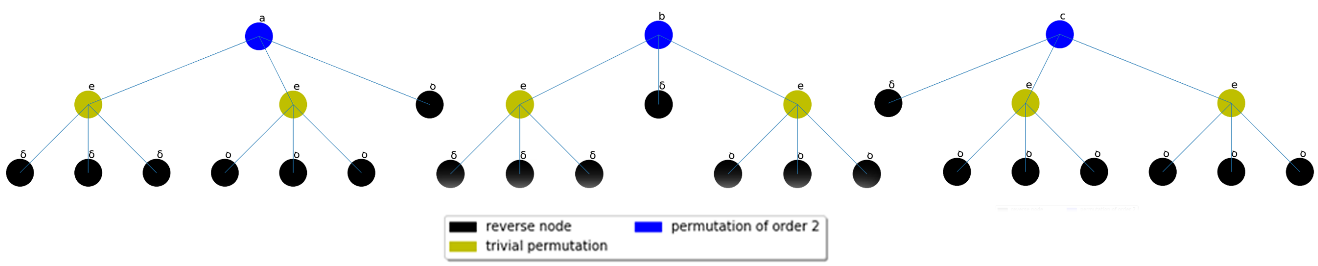


Figure 2: Corresponding trees for atomic elements.

4 Word problem and an algorithm description

5 The worst case

Here should be definition of the group G

Definition 5.1. *Function* ***size*** $: G \to \mathbf{N} \cup \{0\}$ *is defined recursively:*

$$size(w = \pi(w_1, w_2, w_3)) = \begin{cases} 0, & w = e \\ 0, & w = \delta \quad \textit{(reverse node)} \\ |w| + size(w_1) + size(w_2) + size(w_3), & \textit{otherwise} \end{cases} \tag{1}$$

Thus, $size(a) = size(b) = size(c) = 1$ by definition (Fig 2).
Similarly, $size(aa) = size(bb) = size(cc) = 2, \quad size(ab) = size(ac) = ... = size(bc) = 4$

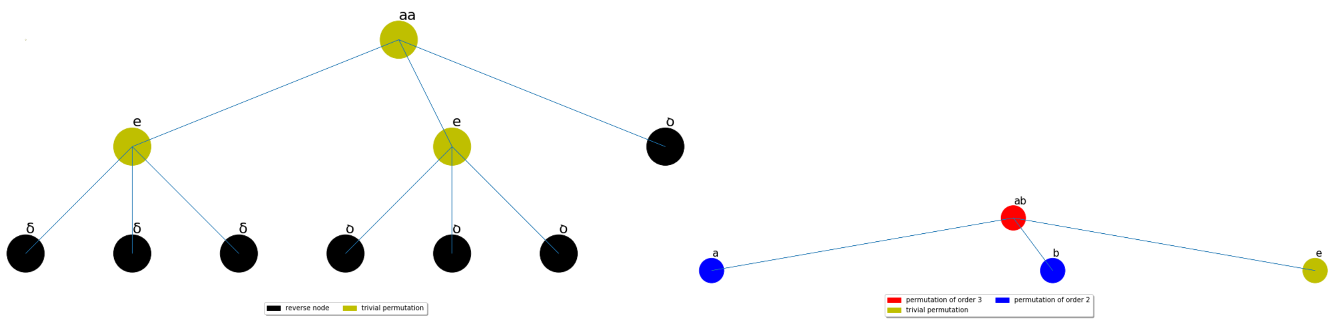


Figure 3: The words from $\{a, b, c\}^2$ could be divided into 2 classes.

Definition 5.2. ***abc-subset*** $X \subset \{a, b, c\}^*$ *is such subset:*

$$X = \left\{ (a^\pi b^\pi c^\pi)^k,\ (a^\pi b^\pi c^\pi)^k a^\pi,\ (a^\pi b^\pi c^\pi)^k a^\pi b^\pi \,|\, k \in \mathbf{N} \cup \{0\},\ \pi \in S_3 \right\} \tag{2}$$

where a^π *means* S_3 *group action on set* $\{a, b, c\}$*;*
$(w_1 w_2 w_3)^k$ *means repeating* k *times of the expression in the parenthesis.*

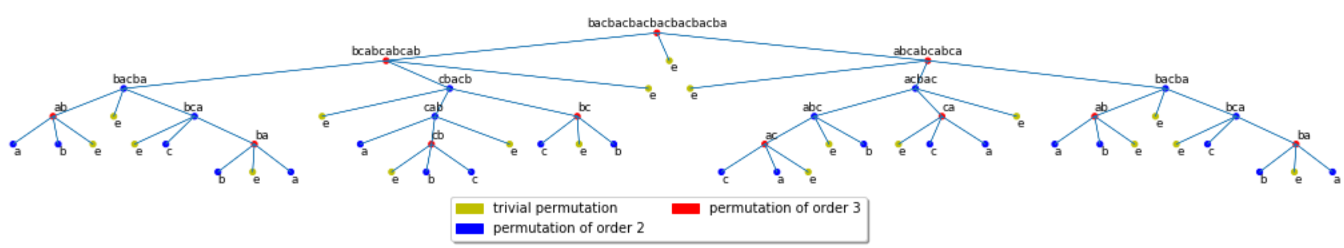
Lemma 5.1. $\forall w \in X$ w *will have one of the next possible structures:*

$$\pi(w_1, w_2, e), \quad \pi(w_1, e, w_2), \quad \pi(e, w_1, w_2),$$

where $w_1, w_2 \in X, \quad |w_1|, |w_2| \in \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right\}, \quad |w_1| + |w_2| = n, \quad n = |w|$

Proof. Follows from definition of multiplication. □

Example:



Therefore, here we have recursive formula for calculating function *size* **(1)** for elements from X:

$$size(w \in X) = a(n) = a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right) + n, \quad a(1) = 1, \quad a(0) = 0, \quad n = |w|$$

Statement 5.1. *Exact form of function $a(n)$:*

$$a(n) = \begin{cases} n\left(\lfloor \log_2 n + 1 \rfloor\right) + 2n - 2^{\lfloor \log_2 n \rfloor + 1}, & n > 0 \\ a(0) = 0 \end{cases} \tag{3}$$

Proof.

$$a(n+1) = n + 1 + a\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n+1}{2} \right\rceil\right) = n + 1 + a\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right)$$

$$\text{Let } b(n) := a(n+1) - a(n) = 1 + a\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \text{ - auxiliary recursion.}$$

$$b(n) = b\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1, \quad b(2) = 4 \quad \Rightarrow \quad b(n) = \lfloor \log_2 n \rfloor + 3$$

$$a(n+1) = b(n) + b(n-1) + ... + b(1) = \sum_{1 \le k \le n} \left(\lfloor \log_2 k + 3 \rfloor\right) \quad \Rightarrow \quad a(n) = 2(n-1) + \sum_{1 \le k \le n} \left(\lfloor \log_2 k + 1 \rfloor\right) =$$

$$= \begin{bmatrix} \text{amount of bits in all the} \\ \text{numbers from } 1 \text{ to } N-1 \end{bmatrix} = 2(n-1) + \sum_{i=1}^{\lfloor \log_2 n \rfloor} (n - 2^i) = n\left(\lfloor \log_2 n + 1 \rfloor\right) + 2n - 2^{\lfloor \log_2 n \rfloor + 1}$$

□

Theorem 5.2. *Elements from abc-subset X have maximum size, or in other words*

$$a(n) = \max(size(w) \,|\, w \in \{a, b, c\}^n)$$

Proof. Induction on n.

1. Base: check examples of size calculation.
2. Induction step:
   Let $\forall k < n \quad a(k) = \max(size(w) \,|\, w \in \{a, b, c\}^k)$. Hereafter we consider any $w \in \{a, b, c\}^n$, $w = \pi(w_1, w_2, w_3)$, $w_1, w_2, w_3 \in \{a, b, c\}^* \cup \{e\}$. Now we need to show that $size(w) \le a(n)$

   Let $x_1 := |w_1|$, $x_2 := |w_2|$, $x_3 := |w_3|$. If w has maximum size than due to induction hypothesis

   $$size(w) \le a(x_1) + a(x_2) + a(x_3) + n \tag{4}$$

   (we don't know whether any w_1, w_2, w_3 from X could appear in w, so we use $\le$ symbol). Let's investigate how big this functional could be.

   Consider function $f(x) = x \log_2 x$ instead of $a(x)$. We can do it because both $f(x)$ and $a(x)$ are monotonically increasing on $x \ge 1$. Thus, they are maximizing in similar way.
   Now we need to solve optimization problem

   $$\max(f(x_1) + f(x_2) + f(x_3) \,|x_1 > 1,\, x_2 > 1,\, x_3 > 1,\, x_1 + x_2 + x_3 = n)$$

   $$F = f(x_1) + f(x_2) + f(x_3) = x_1 \log_2 x_1 + x_2 \log_2 x_2 + x_3 \log_3 x_3 = \Big[x_3 = n - x_1 - x_2\Big] =$$

   $$= x_1 \log_2 x_1 + x_2 \log_2 x_2 + (n - x_1 - x_2) \log_2(n - x_1 - x_2) =$$

   $$= x_1 \log_2 x_1 + x_2 \log_2 x_2 + n \log_2(n - x_1 - x_2) - x_1 \log_2(n - x_1 - x_2) - x_2 \log_2(n - x_1 - x_2)$$

   $$\frac{\partial F}{\partial x_1} = \log_2 x_1 + \frac{1}{\ln 2} - \frac{n}{(n - x_1 - x_2)\ln 2} - \log_2(n - x_1 - x_2) + \frac{x_1}{(n - x_1 - x_2)\ln 2} + \frac{x_2}{(n - x_1 - x_2)\ln 2}$$

   $$\frac{\partial F}{\partial x_2} = \log_2 x_2 + \frac{1}{\ln 2} - \frac{n}{(n - x_1 - x_2)\ln 2} - \log_2(n - x_1 - x_2) + \frac{x_1}{(n - x_1 - x_2)\ln 2} + \frac{x_2}{(n - x_1 - x_2)\ln 2}$$

   If both $\dfrac{\partial F}{\partial x_1}$ and $\dfrac{\partial F}{\partial x_2}$ equal to zero then $\log_2 x_1 = \log_2 x_2 \Rightarrow x_1$ should be equal to x_2.
   Let's substitute $x = x_1 = x_2$ to F and differentiate it:

$$\frac{dF}{dx} = 2\log_2 x + \frac{2}{\ln 2} - \frac{2n}{(n-2x)\ln 2} - 2\log_2(n-2x) + \frac{4x}{(n-2x)\ln 2} = 0$$

Hence, $x = \dfrac{n}{3}$ - the single solution (Wolfram). Substituting it to F we can find out that $x_1 = x_2 = x_3 = \dfrac{n}{3}$ - global minimum. Thus, F (therefore right part of **(4)**) reaches its maximum value on the bounds of its domain.
Now we are allowed to assume that $\exists i\, w_i = e$, because we know that otherwise

$$size(w) \le a(x_1) + a(x_2) + a(x_3) + n \le a\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n}{2} \right\rceil\right) + n$$

It's easy to check that there are only 2 possible types of w with one or more trivial element below: familiar to us $w \in X$ or such $w = z_1 z_2 z_3 ... z_n$ where there are at least one i such that $z_i = z_{i+1}$, $\Rightarrow$ w could be reduced $\Rightarrow size(w) < a(n)$ due to definition.

□

6 Dual automaton

7 Average case

8 Hypotheses and comments

9 Generative function

$$T(z) = \sum_{w} size(w) z^{|w|} = \sum_{w_1}\sum_{w_2}\sum_{w_3} \left(s(w_1) + s(w_2) + s(w_3) + |w_1| + |w_2| + |w_3|\right) z^{|w_1|+|w_2|+|w_3|} =$$

$$= \sum_{w_1}\sum_{w_2} z^{|w_1|+|w_2|} \left((s(w_1) + s(w_2)) \sum_{w_3} z^{|w_3|} + \sum_{w_3} s(w_3) z^{|w_3|} + (|w_1| + |w_2|) \sum_{w_3} z^{|w_3|} + \sum_{w_3} |w_3| z^{|w_3|} \right) =$$

$$= \left[\sum_{w} z^{|w|} = F(z), \quad \sum_{w} |w| z^{|w|} = z \sum_{w} |w| z^{|w|-1} = zF'(z) \right] =$$

$$= \sum_{w_1} z^{|w_1|} \sum_{w_2} z^{|w_2|} \left((s(w_1) + s(w_2) + |w_1| + |w_2|) F(z) + T(z) + zF'(z) \right) =$$

$$= \sum_{w_1} z^{|w_1|} \left((T(z) + zF'(z)) \sum_{w_2} z^{|w_2|} + F(z) \left((s(w_1) + |w_1|) \sum_{w_2} z^{|w_2|} + \sum_{w_2} s(w_2) z^{|w_2|} + \sum_{w_2} |w_2| z^{|w_2|} \right) \right) =$$

$$= \sum_{w_1} z^{|w_1|} \left[(T(z) + zF'(z)) F(z) + F(z) ((s(w_1) + |w_1|) F(z) + T(z) + zF'(z)) \right] =$$

$$= \sum_{w_1} z^{|w_1|} \left[T(z)F(z) + zF(z)F'(z) + F^2(z)(s(w_1) + |w_1|) + F(z)T(z) + zF(z)F'(z) \right] =$$

$$= \sum_{w_1} z^{|w_1|} \left[2T(z)F(z) + 2zF(z)F'(z) + F^2(z)(s(w_1) + |w_1|) \right] =$$

$$= 2T(z)F^2(z) + 2zF^2(z)F'(z) + F^2(z)T(z) + zF^2(z)F'(z) = 3T(z)F^2(z) + 3zF^2(z)F'(z)$$

$$\Rightarrow T(z) = \frac{3zF^2(z)F'(z)}{1 - 3F^2(z)}, \quad F(z) = \sum_{w} z^{|w|} = \sum_{n=1}^{\infty} x_n z^n$$

1. $x_n = 3^n$

$$F(z) = \sum_{n=1}^{\infty} 3^n z^n = \sum_{n=1}^{\infty} (3z)^n = \frac{1}{1-3z} \quad \Rightarrow \quad T(z) = \frac{3z\dfrac{1}{(1-3z)^2}\dfrac{3}{(1-3z)^2}}{1-3\dfrac{1}{(1-3z)^2}} = \frac{9z}{81z^4 - 108z^3 + 27z^2 + 6z - 2} = \frac{P(z)}{Q(z)}$$

Roots of Q(z): $\frac{1}{3}, \quad \frac{1}{3}, \quad \frac{1}{3} - \frac{1}{\sqrt{3}}, \quad \frac{1}{3} + \frac{1}{\sqrt{3}}$

The root $\frac{1}{3} - \frac{1}{\sqrt{3}}$ has the least absolute value (~ 0.24). Thus, by theorem of asymptotic growth of the rational generative functions' coefficients

$$[z^n]T(z) \sim C\left(\frac{3\sqrt{3}}{\sqrt{3}-3}\right)^n \quad \Rightarrow \quad \frac{[z^n]T(z)}{3^n} \sim C\left(\frac{\sqrt{3}}{\sqrt{3}-3}\right)^n$$

2. $x_n = 3 * 2^{n-1}$

$$F(z) = \sum_{n=1}^{\infty} 3 * 2^{n-1} z^n = \frac{3}{2}\sum_{n=1}^{\infty} (2z)^n = \frac{3}{2-4z}, \quad \Rightarrow, \quad T(z) = \frac{3z\left(\dfrac{3}{2-4z}\right)^2 \dfrac{d}{dz}\left(\dfrac{3}{2-4z}\right)}{1-3\left(\dfrac{3}{2-4z}\right)^2} =$$

$$= \frac{81z}{64z^4 - 128z^3 - 12z^2 + 76z - 23} = \frac{P(z)}{Q(z)}$$

Roots of Q(z): $\frac{1}{2}, \frac{1}{2}, \frac{1}{2} - \frac{3\sqrt{3}}{4}, \frac{1}{2} + \frac{3\sqrt{3}}{4}$

The root $\frac{1}{2}$ of multiplicity 2 has the least absolute value. Thus, by theorem of asymptotic growth of the rational generative functions' coefficients

$$[z^n]T(z) \sim C(2)^n n, \quad C = 2\frac{(-2)^2 * P(1/2)}{Q''(1/2)} = \frac{8 * 81/2}{24(32 * 1/4 - 32 * 1/2 - 1)} = -\frac{81}{42}$$

$$\Rightarrow \quad \frac{[z^n]T(z)}{3 * 2^{n-1}} \sim \frac{-81 * 2^n * n}{42 * 3 * 2^{n-1}} = -\frac{9}{7}n$$

References

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