# Planar Groups

August 1, 2025

## Contents

# 1 Planar Groups

## 1.1 Group 1

Generators of group:

$$\left[\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

SNoT

$$\left[\begin{array}{c} \texttt{[1 0 0]} \\ \texttt{[0 1 0]} \\ \texttt{[0 0 1]} \end{array}\right]$$

$$[]$$

### 1.1.1 Dilation

Group 1 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

### 1.1.2 Self-replicating degrees.

## 1.2 Group 2

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

SNoT

$$\left[\begin{matrix} \texttt{[1 0 0]} \\ \texttt{[0 1 0]} \\ \texttt{[0 0 1]} \end{matrix}, \begin{matrix} \texttt{[-1 0 0]} \\ \texttt{[ 0 -1 0]} \\ \texttt{[ 0 0 1]} \end{matrix}\right]$$

$$[]$$

### 1.2.1 Dilation

Group 2 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

### 1.2.2 Self-replicating degrees.

## 1.3 Group 3

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

SNoT

$$\left[\begin{matrix} \texttt{[1 0 0]} \\ \texttt{[0 1 0]} \\ \texttt{[0 0 1]} \end{matrix}, \begin{matrix} \texttt{[-1 0 0]} \\ \texttt{[ 0 1 0]} \\ \texttt{[ 0 0 1]} \end{matrix}\right]$$

$$\left[\begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}\right]$$

### 1.3.1 Dilation

Group 3 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_1} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_1}, \frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0 - 1)(xx_1 - 1)}{x_0 x_1}$$

index of subgroup:

$$[G : H] = x_0 x_1$$

### 1.3.2 Self-replicating degrees.

1. 

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$$

Determinant:

$$x_0 x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0 x_1} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} - \frac{1}{x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_1}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

## 1.4 Group 4

Generators of group:

$$\left[\left(\begin{array}{ccc}-1 & 0 & 0\\ 0 & 1 & \frac{1}{2}\\ 0 & 0 & 1\end{array}\right), \left(\begin{array}{ccc}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right), \left(\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{array}\right)\right]$$

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$$\left[\begin{array}{c}\texttt{[1 0 0]}\\ \texttt{[0 1 0]}\\ \texttt{[0 0 1]}\end{array}, \begin{array}{c}\texttt{[ -1 0 0]}\\ \texttt{[ 0 1 -1/2]}\\ \texttt{[ 0 0 1]}\end{array}\right]$$

$$\left[\left(\begin{array}{cc}x_1 & 0\\ 0 & x_0\end{array}\right)\right]$$

### 1.4.1 Dilation

1. testing inverse A (should have integral entities):

$$A^{-1} = \left(\begin{array}{cc}x_1 & 0\\ 0 & x_0\end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ \hline 0 & 0 & 1\end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ \hline 0 & 0 & 1\end{array}\right) = \left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ \hline 0 & 0 & 1\end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c}-1 & 0 & 0\\ 0 & 1 & -\frac{1}{2}\\ \hline 0 & 0 & 1\end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c}-1 & 0 & 2\,a_0\\ 0 & 1 & -\frac{1}{2}\,x_0\\ \hline 0 & 0 & 1\end{array}\right) = \left(\begin{array}{cc|c}-1 & 0 & 0\\ 0 & 1 & -\frac{1}{2}\\ \hline 0 & 0 & 1\end{array}\right)$$

equations:

$$\left[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -m_1 + \frac{1}{2}\,x_0 - \frac{1}{2} = 0\right]$$

answer:

$$\left[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = \frac{1}{2}\,x_0 - \frac{1}{2}\right]$$

Simplicity

$$A = \left(\begin{array}{cc}\frac{1}{x_1} & 0\\ 0 & \frac{1}{x_0}\end{array}\right)$$

eigenvalues:

$$\left[\frac{1}{x_1}, \frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0 - 1)(xx_1 - 1)}{x_0 x_1}$$

index of subgroup:

$$[G : H] = x_0 x_1$$

### 1.4.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} x_1 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0 x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$\frac{1}{x_0 x_1} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} - \frac{1}{x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_1}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

## 1.5 Group 5

Generators of group:

$$\left[ \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

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$$\left[ \begin{matrix} \texttt{[1 0 0]} \\ \texttt{[0 1 0]} \\ \texttt{[0 0 1]} \end{matrix}, \begin{matrix} \texttt{[-1 0 0]} \\ \texttt{[ 1 1 0]} \\ \texttt{[ 0 0 1]} \end{matrix} \right]$$

$$\left[ \begin{pmatrix} -2\,x_0 + x_1 & 0 \\ x_0 & x_1 \end{pmatrix} \right]$$

### 1.5.1 Dilation

Group 5 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -2\,x_0 + x_1 & 0 \\ x_0 & x_1 \end{pmatrix}$$

   Simplicity

$$A = \begin{pmatrix} -\frac{1}{2\,x_0 - x_1} & 0 \\ \frac{x_0}{2\,x_0 x_1 - x_1^2} & \frac{1}{x_1} \end{pmatrix}$$

   eigenvalues:

$$\left[ -\frac{1}{2\,x_0 - x_1}, \frac{1}{x_1} \right]$$

   charpoly:

$$\frac{(2\,x x_0 - x x_1 + 1)(x x_1 - 1)}{(2\,x_0 - x_1) x_1}$$

   index of subgroup:

$$[G : H] = -(2\,x_0 - x_1) x_1$$

### 1.5.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} -2\,x_0 + x_1 & 0 \\ x_0 & x_1 \end{pmatrix}$$

Determinant:

$$-(2\,x_0 - x_1)x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{(2\,x_0 - x_1)x_1} \notin \mathbb{Z}$$

$$\frac{1}{2\,x_0 - x_1} - \frac{1}{x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2\,x_0 - x_1}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{x_1}, 1\right) \notin \mathbb{Z}$$

## 1.6 Group 6

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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```
[1 0 0]    [-1  0  0]    [-1  0  0]    [ 1  0  0]
[0 1 0] ,  [ 0 -1  0] ,  [ 0  1  0] ,  [ 0 -1  0]
[0 0 1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]
```

$$\left[\begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}, \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}\right]$$

### 1.6.1 Dilation

Group 6 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_1} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_1}, \frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0 - 1)(xx_1 - 1)}{x_0x_1}$$

index of subgroup:

$$[G : H] = x_0x_1$$

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ \frac{1}{x_1} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0x_1}}, \sqrt{\frac{1}{x_0x_1}}\right]$$

charpoly:

$$\frac{x^2x_0x_1 - 1}{x_0x_1}$$

index of subgroup:

$$[G : H] = -x_0x_1$$

### 1.6.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$$

Determinant:

$$x_0x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0x_1} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} - \frac{1}{x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_1}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}$$

Determinant:

$$-x_0x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0 x_1}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0 x_1}}, 1\right) \notin \mathbb{Z}$$

## 1.7 Group 7

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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$$\left[\begin{array}{c} \texttt{[1 0 0]} \\ \texttt{[0 1 0]} \\ \texttt{[0 0 1]} \end{array}, \begin{array}{c} \texttt{[-1 0 0]} \\ \texttt{[ 0 -1 0]} \\ \texttt{[ 0 0 1]} \end{array}, \begin{array}{c} \texttt{[ -1 0 1/2]} \\ \texttt{[ 0 1 0]} \\ \texttt{[ 0 0 1]} \end{array}, \begin{array}{c} \texttt{[ 1 0 -1/2]} \\ \texttt{[ 0 -1 0]} \\ \texttt{[ 0 0 1]} \end{array}\right]$$

$$\left[\begin{pmatrix} x_1 & 0 \\ 0 & x_0 \end{pmatrix}, \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}\right]$$

### 1.7.1 Dilation

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_1 & 0 \\ 0 & x_0 \end{pmatrix}$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 + \frac{1}{2}\,x_1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2}\,x_1 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\left[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -2\,a_0 - n_2 - \frac{1}{2}\,x_1 + \frac{1}{2} = 0, -m_2 = 0, -n_3 + \frac{1}{2}\,x\right.$$

answer:

$$\left[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -2\,a_0 - \frac{1}{2}\,x_1 + \frac{1}{2}, m_2 = 0, n_3 = \frac{1}{2}\,x_1 - \frac{1}{2}, m_3 = -2\,a_1\right.$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_1}, \frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0 - 1)(xx_1 - 1)}{x_0 x_1}$$

index of subgroup:

$$[G : H] = x_0 x_1$$

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 2\,a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & 1 & -\frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\left[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -n_2 - \frac{1}{2} = 0, -2\,a_1 - m_2 - \frac{1}{2}\,x_0 = 0, -2\,a_0 - n\right.$$

answer:

$$\left[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = \left(-\frac{1}{2}\right), m_2 = -2\,a_1 - \frac{1}{2}\,x_0, n_3 = -2\,a_0 + \frac{1}{2}, m_3 = \frac{1}{2}\,x\right.$$

[*] Contradiction: n2 == (-1/2)! A doens't form a virtual endomorphism.

### 1.7.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} x_1 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0 x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$-2\,a_0 - \frac{1}{2}\,x_1 + \frac{1}{2} \in \mathbb{Z}$$

$$\frac{1}{2}\,x_1 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

Coefficients of the charpoly:

$$\frac{1}{x_0 x_1} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} - \frac{1}{x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_1}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

## 1.8 Group 8

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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```
[ [1 0 0]     [-1  0  0]     [  -1     0  1/2]     [   1     0 -1/2] ]
[ [0 1 0]  ,  [ 0 -1  0]  ,  [   0     1 -1/2]  ,  [   0    -1 -1/2] ]
[ [0 0 1]     [ 0  0  1]     [   0     0    1]     [   0     0    1] ]
```

$$\left[\begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}, \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}\right]$$

### 1.8.1 Dilation

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 + \frac{1}{2}\,x_0 \\ 0 & 1 & -\frac{1}{2}\,x_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2}\,x_0 \\ 0 & -1 & 2\,a_1 - \frac{1}{2}\,x_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\left[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -2\,a_0 - n_2 - \frac{1}{2}\,x_0 + \frac{1}{2} = 0, -m_2 + \frac{1}{2}\,x_1 - \frac{1}{2} =\right.$$

answer:

$$\left[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -2\,a_0 - \frac{1}{2}\,x_0 + \frac{1}{2}, m_2 = \frac{1}{2}\,x_1 - \frac{1}{2}, n_3 = \frac{1}{2}\,x_0 - \frac{1}{2}, m_3 =\right.$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_1} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_1}, \frac{1}{x_0}\right]$$

charpoly:

$$\frac{(x x_0 - 1)(x x_1 - 1)}{x_0 x_1}$$

index of subgroup:

$$[G : H] = x_0 x_1$$

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2}\,x_1 \\ 0 & -1 & 2\,a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 - \frac{1}{2}\,x_1 \\ 0 & 1 & -\frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\left[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -n_2 + \frac{1}{2}\,x_1 - \frac{1}{2} = 0, -2\,a_1 - m_2 - \frac{1}{2}\,x_0 - \frac{1}{2} =\right.$$

answer:

$$\left[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = \frac{1}{2}\,x_1 - \frac{1}{2}, m_2 = -2\,a_1 - \frac{1}{2}\,x_0 - \frac{1}{2}, n_3 = -2\,a_0 + \frac{1}{2}\,x_1 +\right.$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ \frac{1}{x_1} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0x_1}}, \sqrt{\frac{1}{x_0x_1}}\right]$$

charpoly:

$$\frac{x^2x_0x_1 - 1}{x_0x_1}$$

index of subgroup:

$$[G : H] = -x_0x_1$$

### 1.8.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \end{pmatrix}$$

Determinant:

$$x_0x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$-2\,a_0 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$\frac{1}{2}\,x_1 - \frac{1}{2} \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 + \frac{1}{2}\,x_1 - \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$\frac{1}{x_0x_1} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} - \frac{1}{x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_1}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} 0 & x_1 \\ x_0 & 0 \end{pmatrix}$$

Determinant:

$$-x_0 x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$\frac{1}{2}\,x_1 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 - \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_0 + \frac{1}{2}\,x_1 + \frac{1}{2} \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$-\frac{1}{x_0 x_1} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0 x_1}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0 x_1}}, 1\right) \notin \mathbb{Z}$$

## 1.9 Group 9

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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$$\left[\begin{array}{c}\texttt{[1 0 0]}\\\texttt{[0 1 0]}\\\texttt{[0 0 1]}\end{array}, \begin{array}{c}\texttt{[-1  0  0]}\\\texttt{[ 0 -1  0]}\\\texttt{[ 0  0  1]}\end{array}, \begin{array}{c}\texttt{[-1  0  0]}\\\texttt{[ 1  1  0]}\\\texttt{[ 0  0  1]}\end{array}, \begin{array}{c}\texttt{[ 1  0  0]}\\\texttt{[-1 -1  0]}\\\texttt{[ 0  0  1]}\end{array}\right]$$

$$\left[\begin{pmatrix} -x_0 & -2\,x_0 \\ x_1 & x_0 \end{pmatrix}, \begin{pmatrix} x_0 - 2\,x_1 & 0 \\ x_1 & x_0 \end{pmatrix}\right]$$

### 1.9.1 Dilation

Group 9 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & -2\,x_0 \\ x_1 & x_0 \end{pmatrix}$$

   Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0 - 2\,x_1} & -\frac{2}{x_0 - 2\,x_1} \\ \frac{x_1}{x_0^2 - 2\,x_0 x_1} & \frac{1}{x_0 - 2\,x_1} \end{pmatrix}$$

   eigenvalues:

$$\left[-\frac{1}{\sqrt{x_0^2 - 2\,x_0 x_1}}, \frac{1}{\sqrt{x_0^2 - 2\,x_0 x_1}}\right]$$

   charpoly:

$$\frac{x^2 x_0^2 - 2\,x^2 x_0 x_1 - 1}{(x_0 - 2\,x_1)x_0}$$

   index of subgroup:

$$[G : H] = -x_0^2 + 2\,x_0 x_1$$

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 - 2\,x_1 & 0 \\ x_1 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0 - 2\,x_1} & 0 \\ -\frac{x_1}{x_0^2 - 2\,x_0 x_1} & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_0 - 2\,x_1}, \frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0 - 2\,xx_1 - 1)(xx_0 - 1)}{(x_0 - 2\,x_1)x_0}$$

index of subgroup:

$$[G : H] = (x_0 - 2\,x_1)x_0$$

### 1.9.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} -x_0 & -2\,x_0 \\ x_1 & x_0 \end{pmatrix}$$

Determinant:

$$-x_0^2 + 2\,x_0 x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{2\,x_1}{(x_0^2 - 2\,x_0 x_1)(x_0 - 2\,x_1)} - \frac{1}{(x_0 - 2\,x_1)^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{\sqrt{x_0^2 - 2\,x_0 x_1}}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{\sqrt{x_0^2 - 2\,x_0 x_1}}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} x_0 - 2\,x_1 & 0 \\ x_1 & x_0 \end{pmatrix}$$

Determinant:

$$(x_0 - 2\,x_1)x_0 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{(x_0 - 2\,x_1)x_0} \notin \mathbb{Z}$$

$$-\frac{1}{x_0 - 2\,x_1} - \frac{1}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_0 - 2\,x_1}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

## 1.10 Group 10

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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```
[ [1 0 0]    [-1  0  0]    [ 0 -1  0]    [ 0  1  0] ]
[ [0 1 0] ,  [ 0 -1  0] ,  [ 1  0  0] ,  [-1  0  0] ]
[ [0 0 1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1] ]
```

$$\left[\begin{pmatrix} -x_1 & x_0 \\ x_0 & x_1 \end{pmatrix}, \begin{pmatrix} x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}\right]$$

### 1.10.1 Dilation

Group 10 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_1 & x_0 \\ x_0 & x_1 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{x_1}{x_0^2+x_1^2} & \frac{x_0}{x_0^2+x_1^2} \\ \frac{x_0}{x_0^2+x_1^2} & \frac{x_1}{x_0^2+x_1^2} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{\sqrt{x_0^2+x_1^2}}, \frac{1}{\sqrt{x_0^2+x_1^2}}\right]$$

charpoly:

$$\frac{x^2x_0^2 + x^2x_1^2 - 1}{x_0^2 + x_1^2}$$

index of subgroup:

$$[G : H] = -x_0^2 - x_1^2$$

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{x_1}{x_0^2+x_1^2} & \frac{x_0}{x_0^2+x_1^2} \\ -\frac{x_0}{x_0^2+x_1^2} & \frac{x_1}{x_0^2+x_1^2} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{-i\,x_0 + x_1}{x_0^2 + x_1^2}, \frac{i\,x_0 + x_1}{x_0^2 + x_1^2}\right]$$

charpoly:

$$\frac{x^2x_0^2 + x^2x_1^2 - 2\,xx_1 + 1}{x_0^2 + x_1^2}$$

index of subgroup:

$$[G : H] = x_0^2 + x_1^2$$

### 1.10.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} -x_1 & x_0 \\ x_0 & x_1 \end{pmatrix}$$

Determinant:

$$-x_0^2 - x_1^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{x_0^2}{x_0^4 + 2\,x_0^2x_1^2 + x_1^4} - \frac{x_1^2}{\left(x_0^2 + x_1^2\right)^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{\sqrt{x_0^2 + x_1^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{\sqrt{x_0^2 + x_1^2}}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}$$

Determinant:

$$x_0^2 + x_1^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{x_0^2}{x_0^4 + 2\,x_0^2x_1^2 + x_1^4} + \frac{x_1^2}{\left(x_0^2 + x_1^2\right)^2} \notin \mathbb{Z}$$

$$-\frac{2\,x_1}{x_0^2+x_1^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{-i\,x_0+x_1}{x_0^2+x_1^2}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{i\,x_0+x_1}{x_0^2+x_1^2}, 1\right) \notin \mathbb{Z}$$

## 1.11 Group 11

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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```
[1 0 0]     [-1  0  0]     [ 0 -1  0]     [-1  0  0]     [ 0  1  0]     [ 1  0  0]     [ 0 -1  0]     [0
[0 1 0]  ,  [ 0 -1  0]  ,  [ 1  0  0]  ,  [ 0  1  0]  ,  [-1  0  0]  ,  [ 0 -1  0]  ,  [-1  0  0]  ,  [1
[0 0 1]     [ 0  0  1]     [ 0  0  1]     [ 0  0  1]     [ 0  0  1]     [ 0  0  1]     [ 0  0  1]     [0
```

$$\left[\begin{pmatrix} x_0 & x_0 \\ -x_0 & x_0 \end{pmatrix}, \begin{pmatrix} -x_0 & -x_0 \\ -x_0 & x_0 \end{pmatrix}, \begin{pmatrix} -x_0 & x_0 \\ x_0 & x_0 \end{pmatrix}, \begin{pmatrix} x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}, \begin{pmatrix} 0 & x_0 \\ -x_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}, \begin{pmatrix} -x_0 & 0 \\ 0 & x_0 \end{pmatrix},\right.$$

### 1.11.1 Dilation

Group 11 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{2\,x_0} & -\frac{1}{2\,x_0} \\ \frac{1}{2\,x_0} & \frac{1}{2\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}\right]$$

charpoly:

$$\frac{2\,x^2x_0^2 - 2\,xx_0 + 1}{2\,x_0^2}$$

index of subgroup:

$$[G:H] = 2\,x_0^2$$

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & -x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{2\,x_0} & -\frac{1}{2\,x_0} \\ -\frac{1}{2\,x_0} & \frac{1}{2\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}, \frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{2\,x^2 x_0^2 - 1}{2\,x_0^2}$$

index of subgroup:

$$[G : H] = -2\,x_0^2$$

3. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{2\,x_0} & \frac{1}{2\,x_0} \\ \frac{1}{2\,x_0} & \frac{1}{2\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}, \frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{2\,x^2 x_0^2 - 1}{2\,x_0^2}$$

index of subgroup:

$$[G : H] = -2\,x_0^2$$

4. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{2\,x_0} & \frac{1}{2\,x_0} \\ -\frac{1}{2\,x_0} & \frac{1}{2\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}\right]$$

charpoly:

$$\frac{2\,x^2 x_0^2 - 2\,x x_0 + 1}{2\,x_0^2}$$

index of subgroup:

$$[G:H] = 2\,x_0^2$$

5. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & -\frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[ -i\sqrt{\frac{1}{x_0^2}}, i\sqrt{\frac{1}{x_0^2}} \right]$$

charpoly:

$$\frac{x^2 x_0^2 + 1}{x_0^2}$$

index of subgroup:

$$[G:H] = x_0^2$$

6. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[ -\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}} \right]$$

charpoly:

$$\frac{(x x_0 + 1)(x x_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G:H] = -x_0^2$$

7. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[ -\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}} \right]$$

charpoly:

$$\frac{(xx_0 + 1)(xx_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G : H] = -x_0^2$$

8. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[ \frac{1}{x_0} \right]$$

charpoly:

$$\frac{(xx_0 - 1)^2}{x_0^2}$$

index of subgroup:

$$[G : H] = x_0^2$$

### 1.11.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} x_0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:

$$2\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{2\,x_0^2} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} -x_0 & -x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:

$$-2\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{2\,x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

3.
$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ x_0 & x_0 \end{pmatrix}$$

Determinant:
$$-2\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$-\frac{1}{2\,x_0^2} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$
$$\left(\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

4.
$$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}$$

Determinant:
$$2\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$\frac{1}{2\,x_0^2} \notin \mathbb{Z}$$
$$-\frac{1}{x_0} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$
$$\left(\frac{1}{2}i\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

5.
$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & 0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-i\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(i\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

6.
$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:

$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

7.
$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

8.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{2}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_0}, 2\right) \notin \mathbb{Z}$$

## 1.12 Group 12

Generators of group:

$$\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

SNoT

```
[ [1 0 0]   [-1  0  0]   [ 0 -1  0]   [  -1     0  1/2]   [ 0  1  0]   [   1     0 -1/2]   [
  [0 1 0] , [ 0 -1  0] , [ 1  0  0] , [   0     1 -1/2] , [-1  0  0] , [   0    -1 -1/2] , [
  [0 0 1]   [ 0  0  1]   [ 0  0  1]   [   0     0    1]   [ 0  0  1]   [   0     0    1]   [
```

$$\left[\begin{pmatrix} x_0 & x_0 \\ -x_0 & x_0 \end{pmatrix}, \begin{pmatrix} -x_0 & -x_0 \\ -x_0 & x_0 \end{pmatrix}, \begin{pmatrix} -x_0 & x_0 \\ x_0 & x_0 \end{pmatrix}, \begin{pmatrix} x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}, \begin{pmatrix} 0 & x_0 \\ -x_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}, \begin{pmatrix} -x_0 & 0 \\ 0 & x_0 \end{pmatrix}\right.$$

### 1.12.1 Dilation

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 \\ -1 & 0 & a_0 + a_1 - x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 \\ -1 & 0 & a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 - x_0 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & -1 & 2\,a_1 + x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} -1 & 0 & 2\,a_0 + x_0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\Big[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -a_0 - a_1 - n_2 = 0, a_0 - a_1 - m_2 = 0, -a_0 - a_1$$

answer:

$$\Big[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -a_0 - a_1, m_2 = a_0 - a_1, n_3 = -a_0 - a_1 - \frac{1}{2}, m_3 = -a$$

[*] Contradiction: n6 == (-1/2)! A doens't form a virtual endomorphism.

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & -x_0 \\ -x_0 & x_0 \end{pmatrix}$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & 1 & a_0 - a_1 \\ -1 & 0 & a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & -1 & a_0 + a_1 \\ -1 & 0 & a_0 + a_1 - x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & -1 & a_0 + a_1 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & 1 & a_0 - a_1 + x_0 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & 1 & x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & -x_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\Big[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -a_0 + a_1 - n_2 = 0, -a_0 - a_1 - m_2 = 0, -a_0 - a$$

answer:

$$\Big[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -a_0 + a_1, m_2 = -a_0 - a_1, n_3 = -a_0 - a_1 - \frac{1}{2}, m_3 = -$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{2\,x_0} & -\frac{1}{2\,x_0} \\ -\frac{1}{2\,x_0} & \frac{1}{2\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}, \frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{2\,x^2x_0^2 - 1}{2\,x_0^2}$$

index of subgroup:

$$[G : H] = -2\,x_0^2$$

3. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ x_0 & x_0 \end{pmatrix}$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a)\left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa)=\left(\begin{array}{cc|c} 0 & 1 & a_0-a_1 \\ -1 & 0 & a_0+a_1 \\ \hline 0 & 0 & 1 \end{array}\right)=\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa)=\left(\begin{array}{cc|c} 0 & 1 & a_0-a_1-x_0 \\ 1 & 0 & -a_0+a_1 \\ \hline 0 & 0 & 1 \end{array}\right)=\left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa)=\left(\begin{array}{cc|c} 0 & -1 & a_0+a_1 \\ 1 & 0 & -a_0+a_1 \\ \hline 0 & 0 & 1 \end{array}\right)=\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa)=\left(\begin{array}{cc|c} 0 & -1 & a_0+a_1 \\ -1 & 0 & a_0+a_1-x_0 \\ \hline 0 & 0 & 1 \end{array}\right)=\left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa)=\left(\begin{array}{cc|c} 1 & 0 & x_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right)=\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa)=\left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & 1 & x_0 \\ \hline 0 & 0 & 1 \end{array}\right)=\left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\left[-n_0=0,-m_0=0,-2\,a_0-n_1=0,-2\,a_1-m_1=0,-a_0+a_1-n_2=0,-a_0-a_1-m_2=0,-a_0+a\right.$$

answer:

$$\left[n_0=0,m_0=0,n_1=-2\,a_0,m_1=-2\,a_1,n_2=-a_0+a_1,m_2=-a_0-a_1,n_3=-a_0+a_1+x_0+\frac{1}{2},m_3\right.$$

Simplicity

$$A=\begin{pmatrix} -\frac{1}{2\,x_0} & \frac{1}{2\,x_0} \\ \frac{1}{2\,x_0} & \frac{1}{2\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}},\frac{1}{2}\sqrt{2}\sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{2\,x^2x_0^2-1}{2\,x_0^2}$$

index of subgroup:

$$[G:H]=-2\,x_0^2$$

4. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 + x_0 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 \\ -1 & 0 & a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 \\ -1 & 0 & a_0 + a_1 - x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 - x_0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 2\,a_1 + x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\Big[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -a_0 - a_1 - n_2 = 0, a_0 - a_1 - m_2 = 0, -a_0 + a_1$$

answer:

$$\Big[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -a_0 - a_1, m_2 = a_0 - a_1, n_3 = -a_0 + a_1 - x_0 + \frac{1}{2}, m_3 =$$

[*] Contradiction: m6 == (-1/2)! A doens't form a virtual endomorphism.

5. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & 0 \end{pmatrix}$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2}\,x_0 \\ 0 & -1 & 2\,a_1 - \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 \\ -1 & 0 & a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 - \frac{1}{2}\,x_0 \\ 0 & 1 & \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 + \frac{1}{2}\,x_0 \\ 1 & 0 & -a_0 + a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 + \frac{1}{2}\,x_0 \\ -1 & 0 & a_0 + a_1 - \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\Big[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -a_0 - a_1 - n_2 = 0, a_0 - a_1 - m_2 = 0, -n_3 + \frac{1}{2}$$

answer:

$$\Big[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -a_0 - a_1, m_2 = a_0 - a_1, n_3 = \frac{1}{2}\,x_0 - \frac{1}{2}, m_3 = -2\,a_1 +$$

Simplicity

$$A = \begin{pmatrix} 0 & -\frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[ -i\sqrt{\frac{1}{x_0^2}}, i\sqrt{\frac{1}{x_0^2}} \right]$$

charpoly:

$$\frac{x^2 x_0^2 + 1}{x_0^2}$$

index of subgroup:

$$[G : H] = x_0^2$$

6. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 \\ -1 & 0 & a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2}\,x_0 \\ 0 & -1 & 2\,a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 - \frac{1}{2}\,x_0 \\ 0 & 1 & -\frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 + \frac{1}{2}\,x_0 \\ -1 & 0 & a_0 + a_1 - \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{cc|r} 0 & 1 & a_0 - a_1 + \frac{1}{2}\,x_0 \\ 1 & 0 & -a_0 + a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\Big[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -a_0 + a_1 - n_2 = 0, -a_0 - a_1 - m_2 = 0, -n_3 +$$

answer:

$$\Big[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -a_0 + a_1, m_2 = -a_0 - a_1, n_3 = \frac{1}{2}\,x_0 - \frac{1}{2}, m_3 = -2\,a_1$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(xx_0+1)(xx_0-1)}{x_0^2}$$

index of subgroup:

$$[G:H] = -x_0^2$$

7. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

$$(A^{-1},a)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & 1 & a_0 - a_1 \\ -1 & 0 & a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} -1 & 0 & 2\,a_0 - \frac{1}{2}\,x_0 \\ 0 & 1 & -\frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & -1 & a_0 + a_1 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 1 & 0 & \frac{1}{2}\,x_0 \\ 0 & -1 & 2\,a_1 - \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & 1 & a_0 - a_1 + \frac{1}{2}\,x_0 \\ 1 & 0 & -a_0 + a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1},a)\left(\begin{array}{rr|r} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)(A,-Aa) = \left(\begin{array}{rr|r} 0 & -1 & a_0 + a_1 - \frac{1}{2}\,x_0 \\ -1 & 0 & a_0 + a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{rr|r} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\Big[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -a_0 + a_1 - n_2 = 0, -a_0 - a_1 - m_2 = 0, -2\,a_0 -$$

answer:

$$\Big[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -a_0 + a_1, m_2 = -a_0 - a_1, n_3 = -2\,a_0 + \frac{1}{2}\,x_0 + \frac{1}{2}, m_3 =$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(xx_0 + 1)(xx_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G : H] = -x_0^2$$

8. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 \\ 0 & -1 & 2\,a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 \\ 1 & 0 & -a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} -1 & 0 & 2\,a_0 + \frac{1}{2}\,x_0 \\ 0 & 1 & -\frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 \\ -1 & 0 & a_0 + a_1 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2}\,x_0 \\ 0 & -1 & 2\,a_1 - \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & -1 & a_0 + a_1 - \frac{1}{2}\,x_0 \\ -1 & 0 & a_0 + a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & -1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(A^{-1}, a) \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right) (A, -Aa) = \left(\begin{array}{cc|c} 0 & 1 & a_0 - a_1 + \frac{1}{2}\,x_0 \\ 1 & 0 & -a_0 + a_1 + \frac{1}{2}\,x_0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 1 \end{array}\right)$$

equations:

$$\left[-n_0 = 0, -m_0 = 0, -2\,a_0 - n_1 = 0, -2\,a_1 - m_1 = 0, -a_0 - a_1 - n_2 = 0, a_0 - a_1 - m_2 = 0, -2\,a_0 - n\right.$$

answer:

$$\left[n_0 = 0, m_0 = 0, n_1 = -2\,a_0, m_1 = -2\,a_1, n_2 = -a_0 - a_1, m_2 = a_0 - a_1, n_3 = -2\,a_0 - \frac{1}{2}\,x_0 + \frac{1}{2}, m_3 =\right.$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0 - 1)^2}{x_0^2}$$

index of subgroup:

$$[G : H] = x_0^2$$

### 1.12.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} -x_0 & -x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:

$$-2\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$-a_0 + a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 + x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$a_0 - a_1 \in \mathbb{Z}$$

$$-a_0 + a_1 - x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$a_0 - a_1 + \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 - \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$-\frac{1}{2\,x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2}\,\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2}\,\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ x_0 & x_0 \end{pmatrix}$$

Determinant:

$$-2\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$-a_0 + a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$-a_0 + a_1 + x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$a_0 - a_1 + \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$a_0 - a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 + x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-x_0 - \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$-\frac{1}{2\,x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2}\,\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2}\,\sqrt{2}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

3.

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & 0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$a_0 - a_1 \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 + \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 + a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$-2\,a_0 + \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 + a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$a_0 - a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 - \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 + \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-i\,\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(i\,\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

4.

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:

$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$-a_0 + a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 - \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$a_0 - a_1 \in \mathbb{Z}$$

$$-2\,a_0 + \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 - \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 + \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 + a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$a_0 - a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

5.

$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$

$$-2\,a_1 \in \mathbb{Z}$$

$$-a_0 + a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$-2\,a_0 + \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$a_0 - a_1 \in \mathbb{Z}$$
$$-\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$
$$-2\,a_1 + \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$
$$-a_0 + a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$
$$a_0 - a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$
$$-a_0 - a_1 + \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$
$$-a_0 - a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$
$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

6.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings):

$$-2\,a_0 \in \mathbb{Z}$$
$$-2\,a_1 \in \mathbb{Z}$$
$$-a_0 - a_1 \in \mathbb{Z}$$
$$a_0 - a_1 \in \mathbb{Z}$$
$$-2\,a_0 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$
$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 + a_1 \in \mathbb{Z}$$

$$-a_0 - a_1 \in \mathbb{Z}$$

$$\frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-2\,a_1 + \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 + \frac{1}{2}\,x_0 - \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 - a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$-a_0 + a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

$$a_0 - a_1 - \frac{1}{2}\,x_0 + \frac{1}{2} \in \mathbb{Z}$$

Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{2}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_0}, 2\right) \notin \mathbb{Z}$$

## 1.13 Group 13

Generators of group:

$$\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

SNoT

```
[1 0 0]    [ 0 -1  0]    [-1  1  0]
[0 1 0] ,  [ 1 -1  0] ,  [-1  0  0]
[0 0 1]    [ 0  0  1]    [ 0  0  1]
```

$$\left[\begin{pmatrix} -x_1 & x_0 + x_1 \\ x_0 & x_1 \end{pmatrix}, \begin{pmatrix} x_0 + x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}\right]$$

### 1.13.1 Dilation

Group 13 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_1 & x_0 + x_1 \\ x_0 & x_1 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{x_1}{x_0^2+x_0x_1+x_1^2} & \frac{x_0+x_1}{x_0^2+x_0x_1+x_1^2} \\ \frac{x_0}{x_0^2+x_0x_1+x_1^2} & \frac{x_1}{x_0^2+x_0x_1+x_1^2} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{\sqrt{x_0^2 + x_0x_1 + x_1^2}}, \frac{1}{\sqrt{x_0^2 + x_0x_1 + x_1^2}}\right]$$

charpoly:

$$\frac{x^2x_0^2 + x^2x_0x_1 + x^2x_1^2 - 1}{x_0^2 + x_0x_1 + x_1^2}$$

index of subgroup:

$$[G : H] = -(x_0 + x_1)x_0 - x_1^2$$

2. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 + x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{x_1}{x_0^2+x_0x_1+x_1^2} & \frac{x_0}{x_0^2+x_0x_1+x_1^2} \\ -\frac{x_0}{x_0^2+x_0x_1+x_1^2} & \frac{x_0+x_1}{x_0^2+x_0x_1+x_1^2} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{x_0\left(-i\sqrt{3}+1\right)+2\,x_1}{2\,(x_0^2+x_0x_1+x_1^2)}, \frac{x_0\left(i\sqrt{3}+1\right)+2\,x_1}{2\,(x_0^2+x_0x_1+x_1^2)}\right]$$

charpoly:

$$\frac{x^2x_0^2+x^2x_0x_1+x^2x_1^2-xx_0-2\,xx_1+1}{x_0^2+x_0x_1+x_1^2}$$

index of subgroup:

$$[G:H] = x_0^2+(x_0+x_1)x_1$$

### 1.13.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} -x_1 & x_0+x_1 \\ x_0 & x_1 \end{pmatrix}$$

Determinant:

$$-(x_0+x_1)x_0-x_1^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{x_0^2}{x_0^4+2\,x_0^3x_1+3\,x_0^2x_1^2+2\,x_0x_1^3+x_1^4}-\frac{x_0x_1}{x_0^4+2\,x_0^3x_1+3\,x_0^2x_1^2+2\,x_0x_1^3+x_1^4}-\frac{x_1^2}{(x_0^2+x_0x_1+x_1^2)^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{\sqrt{x_0^2+x_0x_1+x_1^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{\sqrt{x_0^2+x_0x_1+x_1^2}}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} x_0+x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}$$

Determinant:

$$x_0^2+(x_0+x_1)x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{x_0^2}{x_0^4 + 2\,x_0^3 x_1 + 3\,x_0^2 x_1^2 + 2\,x_0 x_1^3 + x_1^4} + \frac{x_0 x_1}{\left(x_0^2 + x_0 x_1 + x_1^2\right)^2} + \frac{x_1^2}{\left(x_0^2 + x_0 x_1 + x_1^2\right)^2} \notin \mathbb{Z}$$

$$-\frac{x_0}{x_0^2 + x_0 x_1 + x_1^2} - \frac{2\,x_1}{x_0^2 + x_0 x_1 + x_1^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{x_0\left(-i\,\sqrt{3} + 1\right) + 2\,x_1}{2\,(x_0^2 + x_0 x_1 + x_1^2)}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{x_0\left(i\,\sqrt{3} + 1\right) + 2\,x_1}{2\,(x_0^2 + x_0 x_1 + x_1^2)}, 1\right) \notin \mathbb{Z}$$

## 1.14 Group 14

Generators of group:

$$\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

SNoT

```
[1 0 0]    [ 0 -1  0]    [ 0 -1  0]    [-1  1  0]    [ 1  0  0]    [-1  1  0]
[0 1 0] ,  [ 1 -1  0] ,  [-1  0  0] ,  [-1  0  0] ,  [ 1 -1  0] ,  [ 0  1  0]
[0 0 1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]
```

$$\left[\begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}, \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}, \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}, \begin{pmatrix} x_0 & 0 \\ x_0 & -x_0 \end{pmatrix}, \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}, \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}\right]$$

### 1.14.1 Dilation

Group 14 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

   Simplicity

   $$A = \begin{pmatrix} \frac{1}{x_0} & -\frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

   eigenvalues:

   $$\left[-\frac{1}{2}i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2}i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}\right]$$

   charpoly:

   $$\frac{x^2x_0^2 - xx_0 + 1}{x_0^2}$$

   index of subgroup:

   $$[G : H] = x_0^2$$

2. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ -\frac{1}{x_0} & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}\right]$$

charpoly:

$$\frac{x^2 x_0^2 - x x_0 + 1}{x_0^2}$$

index of subgroup:

$$[G : H] = x_0^2$$

3. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0} & \frac{1}{x_0} \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(x x_0 + 1)(x x_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G : H] = -x_0^2$$

4. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ x_0 & -x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ \frac{1}{x_0} & -\frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(xx_0+1)(xx_0-1)}{x_0^2}$$

index of subgroup:

$$[G:H] = -x_0^2$$

5. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(xx_0+1)(xx_0-1)}{x_0^2}$$

index of subgroup:

$$[G:H] = -x_0^2$$

6. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0-1)^2}{x_0^2}$$

index of subgroup:

$$[G:H] = x_0^2$$

### 1.14.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2} i\,\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2} i\,\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2} i\,\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2} i\,\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

3.

$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

4.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ x_0 & -x_0 \end{pmatrix}$$

Determinant:

$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

5.

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:

$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

6.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{2}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_0}, 2\right) \notin \mathbb{Z}$$

## 1.15 Group 15

Generators of group:

$$\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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$$\left[\begin{matrix} \texttt{[1 0 0]} \\ \texttt{[0 1 0]} \\ \texttt{[0 0 1]} \end{matrix}, \begin{matrix} \texttt{[ 0 -1 0]} \\ \texttt{[ 1 -1 0]} \\ \texttt{[ 0 0 1]} \end{matrix}, \begin{matrix} \texttt{[0 1 0]} \\ \texttt{[1 0 0]} \\ \texttt{[0 0 1]} \end{matrix}, \begin{matrix} \texttt{[-1 1 0]} \\ \texttt{[-1 0 0]} \\ \texttt{[ 0 0 1]} \end{matrix}, \begin{matrix} \texttt{[-1 0 0]} \\ \texttt{[-1 1 0]} \\ \texttt{[ 0 0 1]} \end{matrix}, \begin{matrix} \texttt{[ 1 -1 0]} \\ \texttt{[ 0 -1 0]} \\ \texttt{[ 0 0 1]} \end{matrix}\right]$$

$$\left[\begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}, \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}, \begin{pmatrix} -x_0 & 0 \\ -x_0 & x_0 \end{pmatrix}, \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}, \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}, \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}\right]$$

### 1.15.1 Dilation

Group 15 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

   Simplicity

   $$A = \begin{pmatrix} \frac{1}{x_0} & -\frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

   eigenvalues:

   $$\left[-\frac{1}{2} i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2} i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}\right]$$

   charpoly:

   $$\frac{x^2 x_0^2 - x x_0 + 1}{x_0^2}$$

   index of subgroup:

   $$[G : H] = x_0^2$$

2. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ -\frac{1}{x_0} & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[ -\frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0} \right]$$

charpoly:

$$\frac{x^2 x_0^2 - x x_0 + 1}{x_0^2}$$

index of subgroup:

$$[G : H] = x_0^2$$

3. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ -x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0} & 0 \\ -\frac{1}{x_0} & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[ -\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}} \right]$$

charpoly:

$$\frac{(x x_0 + 1)(x x_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G : H] = -x_0^2$$

4. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0} & \frac{1}{x_0} \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[ -\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}} \right]$$

charpoly:

$$\frac{(xx_0+1)(xx_0-1)}{x_0^2}$$

index of subgroup:

$$[G:H]=-x_0^2$$

5. testing inverse A (should have integral entities):

$$A^{-1}=\begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A=\begin{pmatrix} 0 & \frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(xx_0+1)(xx_0-1)}{x_0^2}$$

index of subgroup:

$$[G:H]=-x_0^2$$

6. testing inverse A (should have integral entities):

$$A^{-1}=\begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A=\begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0-1)^2}{x_0^2}$$

index of subgroup:

$$[G:H]=x_0^2$$

### 1.15.2 Self-replicating degrees.

1.
$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:
$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$\frac{1}{x_0^2} \notin \mathbb{Z}$$
$$-\frac{1}{x_0} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\frac{1}{2}i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$
$$\left(\frac{1}{2}i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

2.
$$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:
$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$\frac{1}{x_0^2} \notin \mathbb{Z}$$
$$-\frac{1}{x_0} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\frac{1}{2}i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$
$$\left(\frac{1}{2}i\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

3.
$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:
$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$
$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

4.
$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:
$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$
$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

5.
$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:
$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

6.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{2}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_0}, 2\right) \notin \mathbb{Z}$$

## 1.16 Group 16

Generators of group:

$$\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

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```
[1 0 0]    [ 0 -1  0]    [-1  0  0]    [-1  1  0]    [ 0  1  0]    [ 1 -1  0]
[0 1 0] ,  [ 1 -1  0] ,  [ 0 -1  0] ,  [-1  0  0] ,  [-1  1  0] ,  [ 1  0  0]
[0 0 1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]
```

$$\left[\begin{pmatrix} -x_1 & x_0 + x_1 \\ x_0 & x_1 \end{pmatrix}, \begin{pmatrix} x_0 + x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}\right]$$

### 1.16.1 Dilation

Group 16 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} -x_1 & x_0 + x_1 \\ x_0 & x_1 \end{pmatrix}$$

   Simplicity

   $$A = \begin{pmatrix} -\frac{x_1}{x_0^2+x_0x_1+x_1^2} & \frac{x_0+x_1}{x_0^2+x_0x_1+x_1^2} \\ \frac{x_0}{x_0^2+x_0x_1+x_1^2} & \frac{x_1}{x_0^2+x_0x_1+x_1^2} \end{pmatrix}$$

   eigenvalues:

   $$\left[-\frac{1}{\sqrt{x_0^2 + x_0x_1 + x_1^2}}, \frac{1}{\sqrt{x_0^2 + x_0x_1 + x_1^2}}\right]$$

   charpoly:

   $$\frac{x^2x_0^2 + x^2x_0x_1 + x^2x_1^2 - 1}{x_0^2 + x_0x_1 + x_1^2}$$

   index of subgroup:

   $$[G : H] = -(x_0 + x_1)x_0 - x_1^2$$

2. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} x_0 + x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{x_1}{x_0^2+x_0x_1+x_1^2} & \frac{x_0}{x_0^2+x_0x_1+x_1^2} \\ -\frac{x_0}{x_0^2+x_0x_1+x_1^2} & \frac{x_0+x_1}{x_0^2+x_0x_1+x_1^2} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{x_0\left(-i\sqrt{3}+1\right)+2\,x_1}{2\left(x_0^2+x_0x_1+x_1^2\right)}, \frac{x_0\left(i\sqrt{3}+1\right)+2\,x_1}{2\left(x_0^2+x_0x_1+x_1^2\right)}\right]$$

charpoly:

$$\frac{x^2x_0^2+x^2x_0x_1+x^2x_1^2-xx_0-2\,xx_1+1}{x_0^2+x_0x_1+x_1^2}$$

index of subgroup:

$$[G:H] = x_0^2+(x_0+x_1)x_1$$

### 1.16.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} -x_1 & x_0+x_1 \\ x_0 & x_1 \end{pmatrix}$$

Determinant:

$$-(x_0+x_1)x_0-x_1^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{x_0^2}{x_0^4+2\,x_0^3x_1+3\,x_0^2x_1^2+2\,x_0x_1^3+x_1^4}-\frac{x_0x_1}{x_0^4+2\,x_0^3x_1+3\,x_0^2x_1^2+2\,x_0x_1^3+x_1^4}-\frac{x_1^2}{\left(x_0^2+x_0x_1+x_1^2\right)^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{\sqrt{x_0^2+x_0x_1+x_1^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{\sqrt{x_0^2+x_0x_1+x_1^2}}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} x_0+x_1 & -x_0 \\ x_0 & x_1 \end{pmatrix}$$

Determinant:

$$x_0^2+(x_0+x_1)x_1 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{x_0^2}{x_0^4 + 2\,x_0^3 x_1 + 3\,x_0^2 x_1^2 + 2\,x_0 x_1^3 + x_1^4} + \frac{x_0 x_1}{\left(x_0^2 + x_0 x_1 + x_1^2\right)^2} + \frac{x_1^2}{\left(x_0^2 + x_0 x_1 + x_1^2\right)^2} \notin \mathbb{Z}$$

$$-\frac{x_0}{x_0^2 + x_0 x_1 + x_1^2} - \frac{2\,x_1}{x_0^2 + x_0 x_1 + x_1^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{x_0\left(-i\sqrt{3} + 1\right) + 2\,x_1}{2\left(x_0^2 + x_0 x_1 + x_1^2\right)}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{x_0\left(i\sqrt{3} + 1\right) + 2\,x_1}{2\left(x_0^2 + x_0 x_1 + x_1^2\right)}, 1\right) \notin \mathbb{Z}$$

## 1.17 Group 17

Generators of group:

$$\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

SNoT

```
[1 0 0]     [ 0 -1  0]    [-1  0  0]    [ 0 -1  0]    [-1  1  0]    [ 0  1  0]    [ 1  0  0]    [-
[0 1 0]  ,  [ 1 -1  0] ,  [ 0 -1  0] ,  [-1  0  0] ,  [-1  0  0] ,  [-1  1  0] ,  [ 1 -1  0] ,  [
[0 0 1]     [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [ 0  0  1]    [
```

$$\left[\begin{pmatrix} -x_0 & -x_0 \\ -2\,x_0 & x_0 \end{pmatrix}, \begin{pmatrix} 2\,x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}, \begin{pmatrix} -x_0 & 2\,x_0 \\ -2\,x_0 & x_0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\,x_0 & \frac{1}{2}\,x_0 \\ -\frac{1}{2}\,x_0 & x_0 \end{pmatrix}, \begin{pmatrix} -x_0 & \frac{1}{2}\,x_0 \\ -\frac{1}{2}\,x_0 & x_0 \end{pmatrix}, \begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix},\right.$$

### 1.17.1 Dilation

Group 17 is a semi-direct product, therefore the dilation part is trivial and only consists of integral vectors.

1. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} -x_0 & -x_0 \\ -2\,x_0 & x_0 \end{pmatrix}$$

   Simplicity

   $$A = \begin{pmatrix} -\frac{1}{3\,x_0} & -\frac{1}{3\,x_0} \\ -\frac{2}{3\,x_0} & \frac{1}{3\,x_0} \end{pmatrix}$$

   eigenvalues:

   $$\left[-\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}\right]$$

   charpoly:

   $$\frac{3\,x^2x_0^2 - 1}{3\,x_0^2}$$

   index of subgroup:

   $$[G : H] = -3\,x_0^2$$

2. testing inverse A (should have integral entities):

   $$A^{-1} = \begin{pmatrix} 2\,x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{3\,x_0} & \frac{1}{3\,x_0} \\ -\frac{1}{3\,x_0} & \frac{2}{3\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2}i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}\right]$$

charpoly:

$$\frac{3\,x^2x_0^2 - 3\,xx_0 + 1}{3\,x_0^2}$$

index of subgroup:

$$[G : H] = 3\,x_0^2$$

3. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & 2\,x_0 \\ -2\,x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{3\,x_0} & -\frac{2}{3\,x_0} \\ \frac{2}{3\,x_0} & -\frac{1}{3\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{3\,x^2x_0^2 + 1}{3\,x_0^2}$$

index of subgroup:

$$[G : H] = 3\,x_0^2$$

4. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} \frac{1}{2}\,x_0 & \frac{1}{2}\,x_0 \\ -\frac{1}{2}\,x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{4}{3\,x_0} & -\frac{2}{3\,x_0} \\ \frac{2}{3\,x_0} & \frac{2}{3\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{x_0}, i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{x_0}\right]$$

charpoly:

$$\frac{3\,x^2x_0^2 - 6\,xx_0 + 4}{3\,x_0^2}$$

index of subgroup:

$$[G:H] = \frac{3}{4}\,x_0^2$$

5. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & \frac{1}{2}\,x_0 \\ -\frac{1}{2}\,x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{4}{3\,x_0} & \frac{2}{3\,x_0} \\ -\frac{2}{3\,x_0} & \frac{4}{3\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-2\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 2\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{3\,x^2x_0^2 - 4}{3\,x_0^2}$$

index of subgroup:

$$[G:H] = -\frac{3}{4}\,x_0^2$$

6. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & -\frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[-\frac{1}{2}i\,\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2}i\,\sqrt{3}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}\right]$$

charpoly:

$$\frac{x^2x_0^2 - xx_0 + 1}{x_0^2}$$

index of subgroup:

$$[G:H] = x_0^2$$

7. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ -\frac{1}{x_0} & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[ -\frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, \frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0} \right]$$

charpoly:

$$\frac{x^2 x_0^2 - x x_0 + 1}{x_0^2}$$

index of subgroup:

$$[G : H] = x_0^2$$

8. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ -x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0} & 0 \\ -\frac{1}{x_0} & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[ -\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}} \right]$$

charpoly:

$$\frac{(x x_0 + 1)(x x_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G : H] = -x_0^2$$

9. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{x_0} & \frac{1}{x_0} \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(xx_0 + 1)(xx_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G : H] = -x_0^2$$

10. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} 0 & \frac{1}{x_0} \\ \frac{1}{x_0} & 0 \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{(xx_0 + 1)(xx_0 - 1)}{x_0^2}$$

index of subgroup:

$$[G : H] = -x_0^2$$

11. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} -x_0 & 2\,x_0 \\ x_0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} -\frac{1}{3\,x_0} & \frac{2}{3\,x_0} \\ \frac{1}{3\,x_0} & \frac{1}{3\,x_0} \end{pmatrix}$$

eigenvalues:

$$\left[-\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, \sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}\right]$$

charpoly:

$$\frac{3\,x^2x_0^2 - 1}{3\,x_0^2}$$

index of subgroup:

$$[G:H] = -3\,x_0^2$$

12. testing inverse A (should have integral entities):

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Simplicity

$$A = \begin{pmatrix} \frac{1}{x_0} & 0 \\ 0 & \frac{1}{x_0} \end{pmatrix}$$

eigenvalues:

$$\left[\frac{1}{x_0}\right]$$

charpoly:

$$\frac{(xx_0 - 1)^2}{x_0^2}$$

index of subgroup:

$$[G:H] = x_0^2$$

### 1.17.2 Self-replicating degrees.

1.

$$A^{-1} = \begin{pmatrix} -x_0 & -x_0 \\ -2\,x_0 & x_0 \end{pmatrix}$$

Determinant:

$$-3\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{3\,x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

2.

$$A^{-1} = \begin{pmatrix} 2\,x_0 & -x_0 \\ x_0 & x_0 \end{pmatrix}$$

Determinant:

$$3\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{3\,x_0^2} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2}i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2}i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{2\,x_0}, 1\right) \notin \mathbb{Z}$$

3.

$$A^{-1} = \begin{pmatrix} -x_0 & 2\,x_0 \\ -2\,x_0 & x_0 \end{pmatrix}$$

Determinant:

$$3\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{3\,x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

4.

$$A^{-1} = \begin{pmatrix} \frac{1}{2}\,x_0 & \frac{1}{2}\,x_0 \\ -\frac{1}{2}\,x_0 & x_0 \end{pmatrix}$$

Determinant:

$$\frac{3}{4}\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{4}{3\,x_0^2} \notin \mathbb{Z}$$

$$-\frac{2}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

$$\left(i\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}} + \frac{1}{x_0}, 1\right) \notin \mathbb{Z}$$

5.

$$A^{-1} = \begin{pmatrix} -x_0 & \frac{1}{2}\,x_0 \\ -\frac{1}{2}\,x_0 & x_0 \end{pmatrix}$$

Determinant:

$$-\frac{3}{4}\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{4}{3\,x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-2\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(2\,\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

6.

$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\, x_0}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\, x_0}, 1\right) \notin \mathbb{Z}$$

7.

$$A^{-1} = \begin{pmatrix} x_0 & -x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{1}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\, x_0}, 1\right) \notin \mathbb{Z}$$

$$\left(\frac{1}{2} i \sqrt{3} \sqrt{\frac{1}{x_0^2}} + \frac{1}{2\, x_0}, 1\right) \notin \mathbb{Z}$$

8.
$$A^{-1} = \begin{pmatrix} -x_0 & 0 \\ -x_0 & x_0 \end{pmatrix}$$

Determinant:
$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$
$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

9.
$$A^{-1} = \begin{pmatrix} -x_0 & x_0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:
$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:
$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$
$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:
$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$
$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

10.
$$A^{-1} = \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}$$

Determinant:
$$-x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

11.

$$A^{-1} = \begin{pmatrix} -x_0 & 2\,x_0 \\ x_0 & x_0 \end{pmatrix}$$

Determinant:

$$-3\,x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$-\frac{1}{3\,x_0^2} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(-\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

$$\left(\sqrt{\frac{1}{3}}\sqrt{\frac{1}{x_0^2}}, 1\right) \notin \mathbb{Z}$$

12.

$$A^{-1} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$$

Determinant:

$$x_0^2 \in \mathbb{Z}$$

Conditions on endomorphism (self-coverings): Coefficients of the charpoly:

$$\frac{1}{x_0^2} \notin \mathbb{Z}$$

$$-\frac{2}{x_0} \notin \mathbb{Z}$$

$$1 \notin \mathbb{Z}$$

Conditions of eigenvalues:

$$\left(\frac{1}{x_0}, 2\right) \notin \mathbb{Z}$$